Advanced Calculus
A Transition to Analysis
Instructor's Solutions Manual

Joseph B. Dence
St. Louis, MO

Thomas P. Dence
Ashland University Ashland, OH

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Sets, Numbers, and Functions

1.1. No, because $T \wedge F$ is F.

1.2. Yes, because $F \vee T$ is T. If $\sim (p \wedge q)$ is $T(F)$, then so is $(\sim p \vee \sim q)$.

1.3. **(a)**

p	q	$\sim p$	$\sim p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

(b)

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

(c)

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

1.4. (a), (b), (f) hold.

1.5. A Right Distributive Law is already implied by R2(b), R4: $(y+z)x = x(y+z) = xy + xz = yx + zx$. Statements analogous to field axioms R2(b), R6(b) would fail for 3×3 matrices.

1.6. **(a)** For $<\mathbf{Z}_5, \oplus, \otimes>$, axioms analogous to R1 – R5 are inherited from $\mathbf{R}$. The additive inverses of $0, 1, 2, 3, 4$ are $0, 4, 3, 2, 1$, respectively,

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and the multiplicative inverses of 1, 2, 3, 4, are 1, 3, 2, 4, respectively. Hence, an axiom analogous to R6 holds, so $<\mathbf{Z}_5, \oplus, \otimes>$ is a field.

(b) For $\mathbf{Z}_5$ suppose that $\mathbf{P} \subseteq \{1, 2, 3, 4\}$ is nonempty; let $x \in \mathbf{P}$. Addition of x to itself a sufficient number of times produces all members of $\{1, 2, 3, 4\}$, so $\mathbf{P} = \{1, 2, 3, 4\}$. But then $-x \in \mathbf{P}$, which is not allowed by Axiom R7(b), so $\mathbf{P} = \emptyset$.

(c) If $\oplus, \otimes$ are defined modularly as with $<\mathbf{Z}_5, \oplus, \otimes>$ at the start of Exercise 1.6, then $<\mathbf{Z}_7, \oplus, \otimes>$ and $<\mathbf{Z}_{11}, \oplus, \otimes>$ are found to be finite fields. But the set $\mathbf{Z}_6$ does not produce a field where addition and multiplication are modular because, for example, 2 then has no multiplicative inverse. CONJECTURE: $<\mathbf{Z}_p, \oplus, \otimes>$ is a field iff p is prime.

1.7. (a) If $0, 0'$ are distinct additive identities, we interpret "distinct" to mean that their difference is nonzero. Let $0 + (-0') = c$, where $c \neq 0, 0'$. Post-addition of $0'$ to both sides gives from Axiom R3(a)

$$0 + \left[-0' + 0'\right] = c + 0'. \qquad (*)$$

In the brackets, let 0 be the zero resulting from addition of the number $0'$ to its additive inverse $-0'$. On the right-hand side of $(*)$, let $0'$ be a zero as in Axiom R5(b). We obtain

$$0 + 0 = c,$$

so from R5(b) again we have $0 = c$, which is not allowed. The difficulty can be removed if $0, 0'$ are not distinct.

(b) Interpret "$1, 1'$ being distinct" to mean that $1 \cdot (1')^{-1}$ is neither 1 nor $1'$. Let $1 \cdot (1')^{-1} = c$. Post-multiplication of both sides by $1'$ gives from Axiom R3(b)

$$1 \cdot \left[(1')^{-1} \cdot 1'\right] = c \cdot 1'. \qquad (*)$$

In the brackets, let 1 be the multiplicative identity resulting from multiplication of the nonzero number $1'$ by its multiplicative inverse $(1')^{-1}$. On the right-hand side of $(*)$, apply Axiom R5(b); we obtain

$$1 = 1 \cdot 1 = c,$$

which is not allowed. The difficulty can be removed if $1, 1'$ are not distinct.

1.8. If $x \neq y$, then there is a nonzero $c \in \mathbf{R}$ such that $x = y + c$. Pre-addition of $(-y)$ to both sides gives, from Axiom R3(a),

$$(-y) + x = [(-y) + y] + c$$

and then
$$x + (-y) = 0 + c = c \neq 0,$$

from Axioms R2(a), R6(a), R5(b). This is a direct proof.

The inequality $y + (-x) \neq 0$ follows analogously if x and y are interchanged in the steps above.

1.9. (a) $(-a) + [a + b] = (-a) + [a + c]$ gives, from Axiom R3(a), $[(-a) + a] + b = [(-a) + a] + c$, and $b = c$ then follows from Axioms R2(a), R6(a), R5(b).

(b) By Axiom R5(b), $y + 0 = y$ for any $y \in \mathbf{R}$. Pre-multiplication by any $x \in \mathbf{R}$ gives $x \cdot (y + 0) = x \cdot y$, and use of Axioms R4 and R2(b) gives

$$x \cdot y + x \cdot 0 = x \cdot y.$$

Since $x \cdot y$ is in $\mathbf{R}$ (Axiom R1), it has an additive inverse, $-(x \cdot y)$ (Axiom R6(a)). Pre-addition of $-(x \cdot y)$ to both sides of the equation gives, from Axiom R3(a),

$$[-(x \cdot y) + x \cdot y] + x \cdot 0 = -(x \cdot y) + x \cdot y,$$

and then from Axioms R6(a) and R5(b) we obtain $x \cdot 0 = 0$.

1.10. By Axiom R6(a) we have $1 + (-1) = 0$. Pre-multiplication of both sides by any $x \in \mathbf{R}$ and use of Axioms R4, R2(b), R5(b) give $x + (-1) \cdot x = 0 \cdot x = 0$, from Exercise 1.9(b). Finally, pre-addition of $-x$ to both sides of $x + (-1) \cdot x = 0$ and use of Axioms R3(a), R6(a) and R5(b) give

$$0 + (-1) \cdot x = -x,$$

which reduces to $(-1) \cdot x = (-x)$, by R5(b) a second time.

If $x = -1$, then the right-hand side is the additive inverse of -1, which is 1, so we obtain $(-1) \cdot (-1) = 1$.

1.11. On the left-hand side of the tautology

$$x + [(-x) + y] = x + [(-x) + y]$$

replace $(-x)$ by $(-1) \cdot x$ and y by $(-1) \cdot (-y)$ (Exercise 1.10):

$$x + [(-1) \cdot x + (-1) \cdot (-y)] = x + [(-x) + y],$$

and then
$$x + (-1)\,[x + (-y)] = x + [(-x) + y], \qquad (*)$$

by Axiom R4. Finally, application of Exercise 1.9(a) to the left-hand side of (*) and use of Exercise 1.10 a second time give

$$-[x + (-y)] = (-x) + y.$$

1.12. If $x > x$ were to hold for some $x \in \mathbf{R}$, then by definition of $>$ one would have $[x + (-x)] \in \mathbf{P}$. But by Axiom R6(a), $x + (-x) = 0$ for all $x \in \mathbf{R}$, and by definition of $\mathbf{P}$, $0 \notin \mathbf{P}$. Hence, $x > x$ cannot be true, and so $>$ is Nonreflexive.

If $x > y$ and $y > z$ hold, then by definition of $>$

$$\begin{cases} [x + (-y)] \in \mathbf{P} \\ [y + (-z)] \in \mathbf{P}. \end{cases}$$

Addition and use of Axiom R3(a) give $[x + (-z)] \in \mathbf{P}$, from the definition of $\mathbf{P}$. Finally, by definition of $>$ again, we have $x > z$. Hence, $>$ is Transitive.

1.13. By hypothesis, $[b + (-a)] \in \mathbf{P}$ and $c \in \mathbf{P}$. Hence, by definition of $\mathbf{P}$, $c \cdot [b + (-a)] = [cb + c(-a)] \in \mathbf{P}$, from Axiom R4. From Exercise 1.10 we replace $c(-a)$ by $c[(-1)(a)]$, and then by $[c(-1)] \cdot a$ (Axiom R3(b)). Finally, this is $-(c \cdot a)$ (Axioms R2(b), R3(b), and Exercise 1.10). Thus, $[cb + (-(ca))] \in \mathbf{P}$, and this is equivalent to $cb > ca$.

1.14. (a)

k	x_k	x_k^2	k	x_k	x_k^2
1	1	1	10	1.731830	2.999236
2	7/5	1.96	11	1.731964	2.999698
3	1.592593	2.536351	12	1.732016	2.999880
4	1.675497	2.807290	13	1.732037	2.999953
5	1.709452	2.922225	14	1.732045	2.999981
6	1.723074	2.968984	15	1.732049	2.999993
7	1.728493	2.987690	16	1.732050	2.999997
8	1.730642	2.995123	17	1.732050	2.999999
9	1.731493	2.998069	18	1.732051	2.999999

(b) $x_{k+1} = 4 - \frac{13}{4+x_k} \to 3 - x_{k+1}^2 = \frac{13(3-x_k^2)}{(4+x_k)^2} > 0$ if $x_k^2 < 3$. Since $x_k^2 < 3$ is true for $k = 1$, then $3 - x_{k+1}^2 > 0$ is true, so for all $k \in \mathbf{N}$ by mathematical induction $x_k^2 < 3$ holds.

Similarly, $x_{k+1} - x_k = 4 - \frac{13}{4+x_k} - x_k = \frac{3-x_k^2}{4+x_k}$, and since $3 - x_k^2 > 0$ for all $k \in \mathbf{N}$, then by mathematical induction $x_{k+1} - x_k > 0$ for all $k \in \mathbf{N}$.

(c)

k	x_k	x_k^2	k	x_k	x_k^2
1	2	4	9	1.732200	3.000517
2	11/6	3.361111	10	1.732110	3.000205
3	1.771429	3.137959	11	1.732074	3.000081
4	1.747525	3.053843	12	1.732060	3.000032
5	1.738157	3.021189	13	1.732054	3.000011
6	1.734464	3.008366	14	1.732052	3.000005
7	1.733005	3.003307	15	1.732051	3.000002
8	1.732428	3.001308	16	1.732051	3.000001

(d) It appears that $\sup \mathbf{S}_1 = \inf \mathbf{S}_2 = \sqrt{3}$. These should exist by the Axiom of Completeness because $\mathbf{S}_1$ is bounded from above and $\mathbf{S}_2$ is bounded from below.

1.15. Suppose that the nonempty set $\mathbf{S}$ of real numbers were alleged to have two suprema, U_1 and U_2, and that $U_2 > U_1$. But this is silly because if U_1 is truly a supremum, then U_2 is merely an upper bound. And if U_2 were truly a supremum, then it is the *smallest* number such that $U_2 \geq x$ for all $x \in \mathbf{S}$. Hence, U_1 cannot even be just an upper bound of $\mathbf{S}$. As $\mathbf{S}$ is stated to have a supremum, it can have only one.

1.16. By hypothesis, $l \leq x$ for every $x \in \mathbf{S}$. Define $\mathbf{S}'$ to be the set of additive inverses of all the elements in $\mathbf{S}$, that is, $\mathbf{S}' = \{y : y = -x, x \in \mathbf{S}\}$. Then $-l \geq y$ for every $y \in \mathbf{S}'$. By Axiom R8 there is a smallest number U such that $U \geq y$. Hence, $-U \geq l$ is the largest number L such that $L \leq x$ for every $x \in \mathbf{S}$, that is, $-U = \inf \mathbf{S}$.

1.17. (a)

k	x_k	k	x_k	k	x_k
0	0	5	121/81	10	1.499975
1	1	6	364/243	11	1.499992
2	4/3	7	1.499314	12	1.499997
3	13/9	8	1.499771		
4	40/27	9	1.499924		

(b) CONJECTURE: $\sup \mathbf{S} = 3/2$.

(c) Let $x_k = N_k/D_k$; it appears that $N_{k+1} = 3N_k + 1$ and $D_k = 3^{k-1}$, $k \geq 1$.

Iterating on $N_{k+1} = 3N_k + 1$, it also appears that $N_k = \sum_{j=0}^{k-1} 3^j = (3^k - 1)/2$. Hence, $x_k = (3/2) - \left[2(3^{k-1})\right]^{-1}$, and so all x_k's are bounded above by $3/2$; $\sup \mathbf{S}$ should exist.

1.18. The proof reproduces the core of that in Exercise 1.13, but with $x, y \in \mathbf{R}$ entirely arbitrary:

$$x \cdot (-y) = x \cdot [(-1) \cdot y] = [x \cdot (-1)] \cdot y = [(-1) \cdot x] \cdot y$$
$$= (-1) \cdot (x \cdot y) = -(x \cdot y).$$

1.19. **(a)** $x > y$ and $z < 0$ imply that $[x + (-y)] \in \mathbf{P}$ and $-z \in \mathbf{P}$. Hence, by definition of P, we have

$$(-z)\,[x + (-y)] \in \mathbf{P}$$

or

$$[(-z)(x) + (-z)(-y)] \in \mathbf{P}.$$

Using Exercises 1.11, 1.15, we obtain

$$[-(zx) + \{(-1)(-1)\}(zy)] = [-(zx) + zy] \in \mathbf{P},$$

and this is equivalent to $zy > zx$.

(b) $xy < 0$ is equivalent to $-(xy) \in \mathbf{P}$, that is, $x(-y) \in \mathbf{P}$. If $x > 0$, so $x \in \mathbf{P}$, then $-y \in \mathbf{P}$ (and, hence, $y < 0$) will guarantee that the product $x(-y) = -(x \cdot y)$ will be in $\mathbf{P}$. For if $-y \notin \mathbf{P}$, then by Axiom R7(b) $y \in \mathbf{P}$ and so $xy \in \mathbf{P}$, which contradicts $xy < 0$.

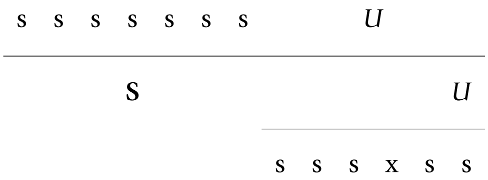
(c) If $x > 0$, then $x \in \mathbf{P}$ and $x^4 = [(x)(x)]\,[(x)(x)] \in \mathbf{P}$ by a 3-fold application of the definition of $\mathbf{P}$. If $x < 0$, then $x \notin \mathbf{P}$ and by Axiom R7(b) and Exercise 1.10, $(-1) \cdot x = -x \in \mathbf{P}$; hence, by Axiom R7(c), $[(-1) \cdot x]\,[(-1) \cdot x] = [(-1) \cdot (-1)] \cdot [x \cdot x] = 1 \cdot x^2 = x^2 \in \mathbf{P}$. Finally, by Axiom R7(b) again, $x^4 = (x^2) \cdot (x^2) \in \mathbf{P}$, that is, $x^4 > 0$.

1.20. Let $\mathbf{S}' = \{y : y = -x \text{ iff } x \in \mathbf{S}\}$; additionally, let $L = \inf \mathbf{S}$ and let l be any lower bound of $\mathbf{S}$. Then $x \in \mathbf{S}$ implies $x \geq L \geq l$, so for any $y \in \mathbf{S}'$ one has $y \leq -L \leq -l$. Now suppose that $l \in \mathbf{S}$; then $-l \in \mathbf{S}'$ and by Theorem 1.3 $-l$ must be $\sup \mathbf{S}'$. It follows from Exercise 1.16 that $-(-l)$ is $\inf \mathbf{S}$, that is $l = L$.

1.21. SHORT ANSWER: Assume x_0 is the smallest, positive real number. Then $0 < \frac{1}{3} \cdot x_0 < x_0$, a contradiction. LONGER ANSWER: We accept (although a proof is easy) that $1 \in \mathbf{P}$. Axiom R7(c) gives $2, 3 \in \mathbf{P}$. Now assume that $\frac{1}{3} \notin \mathbf{P}$, so $-\frac{1}{3} \in \mathbf{P}$. Then $\left(-\frac{1}{3}\right) \cdot 3 = (-1)\left[\frac{1}{3} \cdot 3\right] = (-1) \cdot 1 = -1 \in \mathbf{P}$, by definition of P. But $1 \in \mathbf{P}$ implies $-1 \notin \mathbf{P}$, a contradiction. Hence, $\frac{1}{3} \in \mathbf{P}$ and consequently, also, $\frac{2}{3} = 2 \cdot \frac{1}{3} \in \mathbf{P}$. Now assume that x_0 is the smallest, positive real number. Then $x_0 - x_0 \cdot \frac{2}{3} = x_0 \cdot 1 - x_0 \cdot \frac{2}{3} = x_0 \cdot \left(1 - \frac{2}{3}\right) = x_0 \cdot \frac{1}{3} > 0$, so $x_0 > x_0 \cdot \frac{2}{3} > 0$, a contradiction. Hence, x_0 does not exist.

1.22. $\sqrt{xy} \neq \frac{x+y}{2} \rightarrow 2\sqrt{xy} \neq x + y \rightarrow 4(xy) \neq x^2 + 2xy + y^2 \rightarrow 0 \neq x^2 - 2xy + y^2 \rightarrow 0 \neq (x - y)^2 \rightarrow 0 \neq x - y \rightarrow y \neq x$. The second implication holds because both sides of $2\sqrt{xy} \neq x + y$ are positive.

1.23. Suppose that there were an $x < U$ such that no $s \in \mathbf{S}$ satisfies $x < s$. For all $s \in \mathbf{S}$ one would then have $s \leq x < U$. This contradicts U being $\sup \mathbf{S}$, so no such x can exist.



1.24. Let $x = U - \varepsilon$, where $0 < \varepsilon < U$. The situation is then identical to that described in Exercise 1.23, so corresponding to any $\varepsilon > 0$, there is an $s \in \mathbf{S}$ such that $x < s \leq U$, that is, $U - \varepsilon < s \leq U$.

1.25. Assume that there is an $x \in \mathbf{Q}^+$ such that $x^2 = 5$; let $x = \frac{a}{b}$, $a, b \in \mathbf{N}$. We stipulate at the outset that $\frac{a}{b}$ has been reduced to lowest terms, that is, the largest common divisor of a, b is 1. Then $\frac{a^2}{b^2} = 5$, or $a^2 = 5b^2$. By the Fundamental Theorem of Arithmetic, the prime factorizations of $a^2, 5b^2$ must be the same. Hence, since 5 divides $5b^2$, then 5 divides a^2. As 5 is prime (not factorable into 2 factors, each larger than 1), so 5 must divide a. Thus, $a = 5k$ and $25k^2 = a^2$, or $5k^2 = b^2$. The same argument implies that 5 must divide b. This is now a contradiction, since a, b were stated to have no common divisor larger than 1. We conclude that no such $x \in \mathbf{Q}^+$, as assumed, can exist.

1.26. By Axiom R8, in the form of Exercise 1.16, both $\inf \mathbf{S}$ and $\inf \mathbf{T}$ exist. For each $y \in \mathbf{T}$ one has $y \geq \inf \mathbf{T}$. But $\mathbf{S} \subseteq \mathbf{T}$, so each $x \in \mathbf{S}$ is a $y \in \mathbf{T}$; hence, $\mathbf{S}$ is bounded from below by $L = \inf \mathbf{T}$. By the definition of infimum (Section 1.3), $\inf \mathbf{S} \geq L$ then follows.

1.27. By Theorem 1.5 there is a natural number N such that $N(y - x) > 3$. We now seek an integer M such that $x < \frac{M}{N} < x + \frac{3}{N}$. This will hold iff $Nx < M < Nx + 3$. But in the open interval $(Nx, Nx + 3)$ there are always 2 or 3 integers (depending upon whether Nx is, or is not, integral). Hence, an M exists and we have $x < \frac{M}{N} < x + \frac{3}{N} < y$.

1.28. A convex polygon of $k + 2$ sides has $k + 2$ vertices. The number of diagonals that can be drawn to a given vertex is $(k + 2) - 3 = k - 1$. As there are $k + 2$ vertices, then the total number of diagonals might be $(k + 2)(k - 1)$. But this counts each diagonal twice; hence, the correct number of diagonals is $(k + 2)(k - 1)/2$, $k \in \mathbf{N}$.

1.29. If $\mathbf{A}, \mathbf{B}$ are two bounded subsets of $\mathbf{R}$, then $x \in \mathbf{A} \cup \mathbf{B}$ means $x \in \mathbf{A}$ or $x \in \mathbf{B}$. Then $x \in \mathbf{A}$ implies $x \leq \sup \mathbf{A}$ and $x \in \mathbf{B}$ implies $x \leq \sup \mathbf{B}$. Hence, for any $x \in \mathbf{A} \cup \mathbf{B}$ one must have $x \leq \max\{\sup \mathbf{A}, \sup \mathbf{B}\}$.

Now consider the union

$$\bigcup_{k=1}^{n} \mathbf{S}_k = \begin{cases} \mathbf{S}_1 & n = 1 \\ \mathbf{S}_1 \cup \mathbf{S}_2 & n = 2 \\ \{x : x \in \bigcup_{k=1}^{n-1} \mathbf{S}_k \text{ or } x \in \mathbf{S}_n\} & n > 2. \end{cases}$$

We have $\sup \bigcup_{k=1}^{1} \mathbf{S}_k = \sup \mathbf{S}_1$, $\sup \bigcup_{k=1}^{2} \mathbf{S}_k = \max\{\sup \mathbf{S}_1, \sup \mathbf{S}_2\}$. Assume that for an arbitrary $n = K$ one has

$$\sup \bigcup_{k=1}^{K} \mathbf{S}_k = \max\{\sup \mathbf{S}_1, \sup \mathbf{S}_2, \cdots, \sup \mathbf{S}_K\}.$$

Next, for $n = K + 1$ let $\mathbf{T} = \left(\bigcup_{k=1}^{K} \mathbf{S}_k\right) \cup \mathbf{S}_{K+1}$; then from the initial lemma

$$\begin{aligned} \sup \mathbf{T} &= \max\left\{\sup\left(\bigcup_{k=1}^{K} \mathbf{S}_k\right), \sup \mathbf{S}_{K+1}\right\} \\ &= \max\{\max\{\sup \mathbf{S}_1, \sup \mathbf{S}_2, \ldots, \sup \mathbf{S}_K\}, \sup \mathbf{S}_{K+1}\} \\ &= \max\{\sup \mathbf{S}_1, \sup \mathbf{S}_2, \ldots, \sup \mathbf{S}_{K+1}\}. \end{aligned}$$

It follows by Mathematical Induction that for any $n \in \mathbf{N}$ one has

$$\sup \bigcup_{k=1}^{n} \mathbf{S}_k = \max\{\sup \mathbf{S}_1, \sup \mathbf{S}_2, \ldots, \sup \mathbf{S}_n\}.$$

1.30. (a) Yes (b) No; the addition of two invertible 3×3 matrices does not necessarily give another invertible 3×3 matrix.

1.31. SYMMETRY: $\mathbf{x} * \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3 = y_1x_1 + y_2x_2 + y_3x_3 = \mathbf{y} * \mathbf{x}$;
POSITIVITY: $\mathbf{x} * \mathbf{x} = x_1^2 + x_2^2 + x_3^2 > 0$ if at least one of x_1, x_2, x_3 is unequal to 0; otherwise, $\mathbf{0} * \mathbf{0} = 0^2 + 0^2 + 0^2 = 0$;
LINEARITY: $(k \cdot \mathbf{x}) * \mathbf{y} = (kx_1)y_1 + (kx_2)y_2 + (kx_3)y_3 = k(x_1y_1 + x_2y_2 + x_3y_3) = k \cdot (\mathbf{x} * \mathbf{y})$;

$$\begin{aligned} (\mathbf{x} \oplus \mathbf{y}) * \mathbf{z} &= (x_1 + y_1)z_1 + (x_2 + y_2)z_2 + (x_3 + y_3)z_3 \\ &= (x_1z_1 + x_2z_2 + x_3z_3) + (y_1z_1 + y_2z_2 + y_3z_3) \\ &= \mathbf{x} * \mathbf{z} + \mathbf{y} * \mathbf{z}. \end{aligned}$$

1.32. $\mathbf{z} = \frac{\sqrt{53}}{53} \cdot (-6, 1, -4)$.

1.33. (a) $x = (x - y) + y$, so from the Triangle Inequality

$$|x| = |(x - y) + y| \le |x - y| + |y|, \text{ and}$$

$$|x| - |y| \le |x - y|. \qquad (*)$$

Interchanging x and y, we also have

$$|y| - |x| \le |y - x| = |x - y|. \qquad (**)$$

Inequalities (*, **) together are equivalent to $||x| - |y|| \le |x - y|$.

(b) $|xy| = |x||y|$ is trivially true if either $x = 0$ or $y = 0$, for the equation reduces to $0 = 0$. If $x < 0, y > 0$, then $xy < 0$ and $|xy| = (-x)y = |x|y = |x||y|$; If $x < 0, y < 0$, then $xy > 0$ and $|xy| = xy = (-|x|)(-|y|) = |x||y|$; If $x > 0, y > 0$, then $xy > 0$ and $|xy| = xy = |x||y|$.

1.34. Let $S_k = x_1 + x_2 + \cdots + x_k$; by the Triangle Inequality we have $|S_2| = |x_1 + x_2| \le |x_1| + |x_2|$. Assume that for arbitrary $k < n$, one has $|S_k| \le \sum_{j=1}^{k} |x_j|$. Then for S_{k+1} we obtain

$$|S_{k+1}| = |S_k + x_{k+1}| \le |S_k| + |x_{k+1}| \le \sum_{j=1}^{k} |x_j| + |x_{k+1}| = \sum_{j=1}^{k+1} |x_j|.$$

Hence, by Mathematical Induction $|S_k| \le \sum_{j=1}^{k} |x_j|$ is true for all $k = 1, 2, 3, \ldots, n$.

1.35. (a) If either $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$, then $\mathbf{x}^* \mathbf{y} = 0$ and $|\mathbf{x}^* \mathbf{y}| = 0$. Suppose, without loss of generality, that $\mathbf{x} = \mathbf{0}$; then $\|\mathbf{x}\| = 0$, and $\|\mathbf{x}\|\,\|\mathbf{y}\| = 0 \cdot \|\mathbf{y}\| = 0$, so $|\mathbf{x}^* \mathbf{y}| = \|\mathbf{x}\|\,\|\mathbf{y}\|$.

(b) $\mathbf{P} = \mathbf{x} \oplus (c \cdot \mathbf{y}) : \mathbf{P}^* \mathbf{P} > 0$ by Positivity. Expansion of the inner product, using both left and right linearity, gives

$$\begin{aligned}
[\mathbf{x} \oplus (c\mathbf{y})]^* [\mathbf{x} \oplus (c\mathbf{y})] &= \left\{[\mathbf{x} \oplus (c\mathbf{y})]^* \mathbf{x}\right\} + \left\{[\mathbf{x} \oplus (c\mathbf{y})]^* (c\mathbf{y})\right\} \\
&= \left\{(\mathbf{x}^* \mathbf{x}) + (c\mathbf{y})^* \mathbf{x}\right\} + \left\{\mathbf{x}^* (c\mathbf{y}) + (c\mathbf{y})^* (c\mathbf{y})\right\} \\
&= \left\{\|\mathbf{x}\|^2 + c(\mathbf{y}^* \mathbf{x})\right\} + \left\{c(\mathbf{x}^* \mathbf{y}) + c(\mathbf{y}^* (c\mathbf{y}))\right\} \\
&= \|\mathbf{x}\|^2 + 2c(\mathbf{x}^* \mathbf{y}) + c^2 \|\mathbf{y}\|^2 \\
&> 0.
\end{aligned}$$

(c) Viewing the left-hand side of the inequality in (b) as a quadratic in c, we see that it has no real roots. The discriminant D, formed from the coefficients, must be negative, that is,

$$\mathbf{D} = [2(\mathbf{x}^* \mathbf{y})]^2 - 4\|\mathbf{y}\|^2 \|\mathbf{x}\|^2 < 0,$$

or

$$|\mathbf{x}^* \mathbf{y}| < \|\mathbf{x}\|\,\|\mathbf{y}\|.$$

Combination of this strict inequality with the particular result in part (a) gives for all $\mathbf{x}, \mathbf{y}$

$$|\mathbf{x}^* \mathbf{y}| < \|\mathbf{x}\|\,\|\mathbf{y}\|.$$

1.36. $\|\mathbf{x} \oplus \mathbf{y}\|^2 = (\mathbf{x} \oplus \mathbf{y})^* (\mathbf{x} \oplus \mathbf{y}) = \mathbf{x}^* \mathbf{x} + \mathbf{x}^* \mathbf{y} + \mathbf{y}^* \mathbf{x} + \mathbf{y}^* \mathbf{y} = \|\mathbf{x}\|^2 + 2(\mathbf{x}^* \mathbf{y}) + \|\mathbf{y}\|^2 \le \|\mathbf{x}\|^2 + 2|\mathbf{x}^* \mathbf{y}| + \|\mathbf{y}\|^2 \le \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2 = \left(\|\mathbf{x}\| + \|\mathbf{y}\|\right)^2$. Then taking the positive square roots, we obtain $\|\mathbf{x} \oplus \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

1.37. Suppose that $\mathbf{p}$ is distinct from $\mathbf{a}$; then $\|\mathbf{a} - \mathbf{p}\| > 0$ will hold. Now choose $\varepsilon = \frac{1}{2}\|\mathbf{a} - \mathbf{p}\| > 0$. Every point $\mathbf{q}$ in $\mathbf{B}_n(\mathbf{a}; \boldsymbol{\varepsilon})$ is of distance $d_n(\mathbf{q}, \mathbf{a}) < \frac{1}{2}\|\mathbf{a} - \mathbf{p}\|$ from $\mathbf{a}$. Since $\mathbf{p}$ is of distance $\|\mathbf{a} - \mathbf{p}\|$ from $\mathbf{a}$, then $\mathbf{p}$ is not in $\mathbf{B}_n(\mathbf{a}; \varepsilon)$ for at least one choice of ε. The statement of the theorem follows by the Law of Contraposition.

$\mathbf{B}_n(\mathbf{a}; \boldsymbol{\varepsilon})$

$\mathbf{p} \quad \mathbf{a} - \boldsymbol{\varepsilon} \quad \mathbf{q} \quad \mathbf{a} \quad \mathbf{q} \quad \mathbf{a} + \boldsymbol{\varepsilon}$

$\leftarrow |\mathbf{a} - \mathbf{p}| \rightarrow$

1.38. 60 elements in $\mathbf{S} \times \mathbf{T} \times \mathbf{W}$; 2^{60} subsets of $\mathbf{S} \times \mathbf{T} \times \mathbf{W}$.

1.39. **(a)** $f = \{(x, y) : y = 2x + 1, x \in \mathbf{R}^1\}, \mathbf{I} = [0, 1], f(\mathbf{I}) = [1, 3]$;

$$\begin{aligned} f^{-1}(f(\mathbf{I})) &= f^{-1}([1, 3]) = \{x : (x, y) \in f, y \in [1, 3]\} \\ &= \{x : y = 2x + 1, y \in [1, 3]\} \\ &= \left\{x : x = \frac{1}{2}(y - 1), y \in [1, 3]\right\} \\ &= [0, 1]. \end{aligned}$$

(b) Let $x \in \mathbf{I} \subseteq \mathbf{D}(f)$; then $y = f(x)$ is an element of $f(\mathbf{I})$. Since f is an injection, this implies that f^{-1} is actually a function. Thus, $f^{-1}(f(x)) = f^{-1}(y) = x$ uniquely, and as $x \in \mathbf{I}$ was arbitrary, then $x \in f^{-1}(f(\mathbf{I}))$. Hence, $\mathbf{I} \subseteq f^{-1}(f(\mathbf{I}))$.
The proper set inclusion $\mathbf{I} \subset f^{-1}(f(\mathbf{I}))$ would imply that there is an $x \notin \mathbf{I}$ and an $x' \in \mathbf{I}$ such that $x \ne x'$ but $f(x) = f(x')$. But this cannot be since f is an injection. It follows that $\mathbf{I} = f^{-1}(f(\mathbf{I}))$.

(c) $f = \{(x, y) : y = x^2 + 1, x \in \mathbf{R}^1\}, \mathbf{I} = [0, 1], f(\mathbf{I}) = [1, 2]$. But $1 \in \mathbf{I}$ and, yet, $f^{-1}(f(1)) = f^{-1}(2) = \{-1, 1\}$ and $-1 \notin \mathbf{I}$, so $f^{-1}(f(\mathbf{I})) \ne \mathbf{I}$. We conclude from part (b) that the present f is not an injection.

(d) Consistent with the remarks in (b) we have that $f^{-1}(f(\mathbf{I})) \supset \mathbf{I}$ when $f^{-1}(f(\mathbf{I})) \ne \mathbf{I}$. This is also suggested by the example in (c).

1.40. **(a)** Let $y \in f(f^{-1}(\mathbf{H})) \subseteq \mathbf{S}$. Since f is a surjection, there is an $x \in f^{-1}(\mathbf{H})$ such that $(x, y) \in f$. But by definition of $f^{-1}, x \in f^{-1}(\mathbf{H})$ iff $y = f(x)$ belongs to $\mathbf{H}$. As $y \in f(f^{-1}(\mathbf{H}))$ was arbitrary, then $f(f^{-1}(\mathbf{H})) \subseteq \mathbf{H}$.

The proper set inclusion $f(f^{-1}(\mathbf{H})) \subset \mathbf{H}$ could mean either (a) there is a $y_0 \in \mathbf{H} \backslash f(f^{-1}(\mathbf{H}))$ such that $f^{-1}(y_0) \in f^{-1}(\mathbf{H})$ and $f(f^{-1}(y_0) = y_0)$, or (b) there is a $y_1 \in \mathbf{H} \backslash f(f^{-1}(\mathbf{H}))$ such that y_1 has *no* inverse image in $f^{-1}(\mathbf{H})$, or possibly even *none* in $\mathbf{D}(f)$. But (a) cannot be, for y_0 then should have been included in $f(f^{-1}(\mathbf{H}))$. And (b) cannot hold because f being onto $\mathbf{S}$ implies that it is onto $\mathbf{H}$, so y_1 must have an inverse image in $f^{-1}(\mathbf{H})$. It follows that $f(f^{-1}(\mathbf{H})) = \mathbf{H}$.

(b) $f = \{(x, y) : y = x^2 + 1, x \in \mathbf{R}^1\}$, $\mathbf{H} = [1/2, 2], \mathbf{S} = \mathbf{R}^1$. We have $f^{-1}(\mathbf{H}) = [-1, 1]$, and the points in $\mathbf{T} = [1/2, 1)$, a subset of $\mathbf{H}$, have no inverse images in $f^{-1}(\mathbf{H})$, or even in $\mathbf{D}(f)$. Then $f(f^{-1}(\mathbf{T})) = f(\emptyset) = \emptyset$ and, therefore, $f(f^{-1}(\mathbf{H})) = [1, 2] \neq \mathbf{H}$. We conclude from part (a) that the present f is not a surjection.

(c) Consistent with the remarks in (a), as well as the example in (b), we have that $f(f^{-1}(\mathbf{H})) \subset \mathbf{H}$ when $f(f^{-1}(\mathbf{H})) \neq \mathbf{H}$.

1.41. $x \in \mathbf{I} \cup \mathbf{J} \to x \in \mathbf{I}$ or $x \in \mathbf{J} \to f(x) \in f(\mathbf{I})$ or $f(x) \in f(\mathbf{J})$

$$\to f(x) \in f(\mathbf{I}) \cup f(\mathbf{J})$$

Example: $f(x) = x^2, \mathbf{I} = \{1, 2\}, \mathbf{J} = \{1, 3\}; f(\mathbf{I} \cup \mathbf{J}) = f(\{1, 2, 3\}) = \{1, 4, 9\}$, and

$$f(\mathbf{I}) \cup f(\mathbf{J}) = \{1, 4\} \cup \{1, 9\} = \{1, 4, 9\} = f(\mathbf{I} \cup \mathbf{J}).$$

We assume that $f(\mathbf{I}), f(\mathbf{J})$ are defined for any x in either $\mathbf{I}$ or $\mathbf{J}$ and, therefore, f may be taken as onto $f(\mathbf{I} \cup \mathbf{J})$; f is then a surjection with respect to any subset of $\mathbf{I} \cup \mathbf{J}$. Then

$$y \in f(\mathbf{I} \cap \mathbf{J}) \to f^{-1}(y) = x \in \mathbf{I} \cap \mathbf{J} \to x \in \mathbf{I} \text{ and } x \in \mathbf{J}$$
$$\to f(x) \in f(\mathbf{I}) \text{ and } f(x) \in f(\mathbf{J})$$
$$\to y \in f(\mathbf{I}) \cap f(\mathbf{J}).$$

The direction of the implications only permits $f(\mathbf{I} \cap \mathbf{J}) \subseteq f(\mathbf{I}) \cap f(\mathbf{J})$. Example: $f(x) = x^2 - x, \mathbf{I} = \{0, 2\}, \mathbf{J} = \{-1, 1, 2\}; f(\mathbf{I} \cap \mathbf{J}) = f(\{2\}) = \{2\}$, and $f(\mathbf{I}) \cap f(\mathbf{J}) = \{0, 2\} \cap \{0, 2\} = \{0, 2\}$, so $f(\mathbf{I}) \cap f(\mathbf{J}) \supset f(\mathbf{I} \cap \mathbf{J})$.

1.42. **(a)** Both $f[g]$ and $g[f]$ make sense;
(b) $f[g]$ makes sense; $g[f]$ does not;
(c) Neither makes sense.

1.43. **(a)** The inverse *relation* derived from the mapping $f : \mathbf{D}(f) \to \mathbf{S}$ is the set $f^{-1} = \{(y, x) : (x, y) \in f\}$. Let $y \in \mathbf{R}(f)$ be arbitrary; then if f is an injection and $(x_1, y), (x_2, y) \in f$, we have $x_1 = x_2$. Hence, given