Chapter 2

Second-Order Differential Equations

2.1 The Linear Second-Order Equation

1. It is a routine exercise in differentiation to show that $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation, while $y_p(x)$ is a solution of the nonhomogeneous equation. The Wronskian of $y_1(x)$ and $y_2(x)$ is

$$W(x) = \begin{vmatrix} \sin(6x) & \cos(6x) \\ 6\cos(6x) & -6\sin(6x) \end{vmatrix} = -6\sin^2(x) - 6\sin^2(x) = -6,$$

and this is nonzero for all x, so these solutions are linearly independent on the real line. The general solution of the nonhomogeneous differential equation is

$$y = c_1 \sin(6x) + c_2 \cos(6x) + \frac{1}{36}(x-1).$$

For the initial value problem, we need

$$y(0) = c_2 - \frac{1}{36} = -5$$

so $c_2 = -179/36$. And

$$y'(0) = 2 = 6c_1 + \frac{1}{36}$$

so $c_1 = 71/216$. The unique solution of the initial value problem is

$$y(x) = \frac{71}{216}\sin(6x) - \frac{179}{36}\cos(6x) + \frac{1}{36}(x-1).$$

2. The Wronskian of e^{4x} and e^{-4x} is

$$W(x) = \begin{vmatrix} e^{4x} & e^{-4x} \\ 4e^{4x} & -4e^{-4x} \end{vmatrix} = -8 \neq 0$$

so these solutions of the associated homogeneous equation are independent. With the particular solution $y_p(x)$ of the nonhomogeneous equation, this equation has general solution

$$y(x) = c_1e^{4x} + c_2e^{-4x} - \frac{1}{4}x^2 - \frac{1}{32}.$$

From the initial conditions we obtain

$$y(0) = c_1 + c_2 - \frac{1}{32} = 12$$

and

$$y'(0) = 4c_1 - 4c_2 = 3.$$

Solve these to obtain $c_1 = 409/64$ and $c_2 = 361/64$ to obtain the solution

$$y(x) = \frac{409}{64}e^{4x} + \frac{361}{64}e^{-4x} - \frac{1}{4}x^2 - \frac{1}{32}.$$

3. The associated homogeneous equation has solutions e^{-2x} and e^{-x}. Their Wronskian is

$$W(x) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = e^{-3x}$$

and this is nonzero for all x. The general solution of the nonhomogeneous differential equation is

$$y(x) = c_1e^{-2x} + c_2e^{-x} + \frac{15}{2}.$$

For the initial value problem, solve

$$y(0) = -3 = c_1 + c_2 + \frac{15}{2}$$

and

$$y'(0) = -1 = -2c_1 - c_2$$

to get $c_1 = 23/2, c_2 = -22$. The initial value problem has solution

$$y(x) = \frac{23}{2}e^{-2x} - 22e^{-x} + \frac{15}{2}.$$

4. The associated homogeneous equation has solutions

$$y_1(x) = e^{3x}\cos(2x), y_2(x) = e^{3x}\sin(2x).$$

The Wronskian of these solutions is

$$W(x) = \begin{vmatrix} e^{3x}\cos(2x) & e^{3x}\sin(2x) \\ 3e^{3x}\cos(2x) - 2e^{3x}\sin(2x) & 3e^{3x}\sin(2x) + 2e^{3x}\cos(2x) \end{vmatrix} = e^{6x} \neq 0$$

for all x. The general solution of the nonhomogeneous equation is

$$y(x) = c_1 e^{3x} \cos(2x) + c_2 e^{3x} \sin(2x) - \frac{1}{8}e^x.$$

To satisfy the initial conditions, it is required that

$$y(0) = -1 = c_1 - \frac{1}{8}$$

and

$$3c_1 + 2c_2 - \frac{1}{8} = 1.$$

Solve these to obtain $c_1 = -7/8$ and $c_2 = 15/8$. The solution of the initial value problem is

$$y(x) = -\frac{7}{8}e^{3x} \cos(2x) + \frac{15}{8}e^{3x} \sin(2x) - \frac{1}{8}e^x.$$

5. The associated homogeneous equation has solutions

$$y_1(x) = e^x \cos(x), y_2(x) = e^x \sin(x).$$

These have Wronskian

$$W(x) = \begin{vmatrix} e^x \cos(x) & e^x \sin(x) \\ e^x \cos(x) - e^x \sin(x) & e^x \sin(x) + e^x \cos(x) \end{vmatrix} = e^{2x} \neq 0$$

so these solutions are independent. The general solution of the nonhomogeneous differential equation is

$$y(x) = c_1 e^x \cos(x) + c_2 e^x \sin(x) - \frac{5}{2}c^2 - 5x - \frac{5}{2}.$$

We need

$$y(0) = c_1 - \frac{5}{2} = 6$$

and

$$y'(0) = 1 = c_1 + c_2 - 5.$$

Solve these to get $c_1 = 17/2$ and $c_2 = -5/2$ to get the solution

$$y(x) = \frac{17}{2}e^x \cos(x) - \frac{5}{2}e^x \sin(x) - \frac{5}{2}x^2 - 5x - \frac{5}{2}.$$

6. Suppose y_1 and y_2 are solutions of the homogeneous equation (2.2). Then

$$y_1'' + py_1' + qy_1 = 0$$

and

$$y_2'' + py_2' + qy_2 = 0.$$

Multiply the first equation by y_2 and the second by $-y_1$ and add the resulting equations to obtain

$$y_1''y_2 - y_2''y_1 + p(y_1'y_2 - y_2'y_1) = 0.$$

We want to relate this equation to the Wronskian of these solutions, which is

$$W = y_1y_2' - y_2y_1'.$$

Now

$$W' = y_1y_2'' - y_2y_1''.$$

Then

$$W' + pW = 0.$$

This is a linear first-order differential equation for W. Multiply this equation by the integrating factor

$$e^{\int p(x)\,dx}$$

to obtain

$$We^{\int p(x)\,dx} + pWe^{\int p(x)\,dx} = 0,$$

which we can write as

$$\left(We^{\int p(x)\,dx}\right)' = 0.$$

Integrate this to obtain

$$We^{\int p(x)\,dx} = k,$$

with k constant. Then

$$W(x) = ke^{-\int p(x)\,dx}.$$

This shows that $W(x) = 0$ for all x (if $k = 0$), and $W(x) \neq 0$ for all x (if $k \neq 0$).

Now suppose that y_1 and y_2 are independent and observe that

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_1y_2' - y_2y_1'}{y_1^2} = \frac{1}{y_1^2}W(x).$$

If $k = 0$, then $W(x) = 0$ for all x and the quotient y_2/y_1 has zero derivative and so is constant:

$$\frac{y_2}{y_1} = c$$

for some constant c. But then $y_2(x) = cy_1(x)$, contradicting the assumption that these solutions are linearly independent. Therefore $k \neq 0$ and $W(x) \neq -$ for all x, as was to be shown.

7. The Wronskian of x^2 and x^3 is
$$W(x) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4.$$
Then $W(0) = 0$, while $W(x) \neq 0$ if $x \neq 0$. This is impossible if x^2 and x^3 are solutions of equation (2.2) for some functions $p(x)$ and $q(x)$. We conclude that these functions are not solutions of equation (2.2).

8. It is routine to verify that $y_1(x)$ and $y_2(x)$ are solutions of the differential equation. Compute
$$W(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2.$$
Then $W(0) = 0$ but $W(x) > 0$ if $x \neq 0$. However, to write the differential equation in the standard form of equation (2.2), we must divide by x^2 to obtain
$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0.$$
This is undefined at $x = 0$, which is in the interval $-1 << 1$, so the theorem does not apply.

9. If $y_2(x)$ and $y_2(x)$ both have a relative extremum (max or min) at some x_0 within (a, b), then
$$y'(x_0) = y_2'(x_0) = 0.$$
But then the Wronskian of these functions vanishes at 0, and these solutions must be independent.

10. By assumption, $\varphi(x)$ is the unique solution of the initial value problem
$$y'' + py' + qy = 0; y(x_0) = 0.$$
But the function that is identically zero on I is also a solution of this initial value problem. Therefore these solutions are the same, and $\varphi(x) = 0$ for all x in I.

11. If $y_1(x_0) = y_2(x_0) = 0$, then the Wronskian of $y_1(x)$ and $y_2(x)$ is zero at x_0, and these two functions must be linearly dependent.

2.2 The Constant Coefficient Homogeneous Equation

1. From the differential equation we read the characteristic equation
$$\lambda^2 - \lambda - 6 = 0,$$
which has roots -2 and 3. The general solution is
$$y(x) = c_1 e^{-2x} + c_2 e^{3x}.$$

2. The characteristic equation is

$$\lambda^2 - 2\lambda + 10 = 0$$

with roots $1 \pm 3i$. We can write a general solution

$$y(x) = c_1 e^x \cos(3x) + c_2 e^x \sin(3x).$$

3. The characteristic equation is

$$\lambda^2 + 6\lambda + 9 = 0$$

with repeated roots $-3, -3$. Then

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$$

is a general solution.

4. The characteristic equation is

$$\lambda^2 - 3\lambda = 0$$

with roots $0, 3$, and

$$y(x) = c_1 + c_2 e^{3x}$$

is a general solution.

5. characteristic equation $\lambda^2 + 10\lambda + 26 = 0$, with roots $-5 \pm i$; general solution

$$y(x) = c_1 e^{-5x} \cos(x) + c_2 e^{-5x} \sin(x).$$

6. characteristic equation $\lambda^2 + 6\lambda - 40 = 0$, with roots $4, -10$; general solution

$$y(x) = c_1 e^{4x} + c_2 e^{-10x}.$$

7. characteristic equation $\lambda^2 + 3\lambda + 18 = 0$, with roots $-3/2 \pm 3\sqrt{7}i/2$; general solution

$$y(x) = c_2 e^{-3x/2} \cos\left(\frac{3\sqrt{7}x}{2}\right) + c_2 e^{-3x/2} \sin\left(\frac{3\sqrt{7}x}{2}\right).$$

8. characteristic equation $\lambda^2 + 16\lambda + 64 = 0$, with repeated roots $-8, -8$; general solution

$$y(x) = e^{-8x}(c_1 + c_2 x).$$

9. characteristic equation $\lambda^2 - 14\lambda + 49 = 0$, with repeated roots $7, 7$; general solution

$$y(x) = e^{7x}(c_1 + c_2 x).$$

10. characteristic equation $\lambda^2 - 6\lambda + 7 = 0$, with roots $3 \pm \sqrt{2}i$; general solution
$$y(x) = c_1 e^{3x} \cos(\sqrt{2}x) + c_2 e^{3x} \sin(\sqrt{2}x).$$

In each of Problems 11–20 the solution is found by finding a general solution of the differential equation and then using the initial conditions to find the particular solution of the initial value problem.

11. The differential equation has characteristic equation $\lambda^2 + 3\lambda = 0$, with roots $0, -3$. The general solution is
$$y(x) = c_1 + c_2 e^{-3x}.$$
Choose c_1 and c_2 to satisfy:
$$y(0) = c_1 + c_2 = 3,$$
$$y'(0) = -3c_2 = 6.$$
Then $c_2 = -2$ and $c_1 = 5$, so the unique solution of the initial value problem is
$$y(x) = 5 - 2e^{-3x}.$$

12. characteristic equation $\lambda^2 + 2\lambda - 3 = 0$, with roots $1, -3$; general solution
$$y(x) = c_1 e^{x} + c_2 e^{-3x}.$$
Solve
$$y(0) = c_1 + c_2 = 6, y'(0) = c_1 - 3c_2 = -2$$
to get $c_1 = 4$ and $c_2 = 2$. The solution is
$$y(x) = 4e^{x} + 2e^{-3x}.$$

13. The initial value problem has the solution $y(x) = 0$ for all x. This can be seen by inspection or by finding the general solution of the differential equation and then solving for the constants to satisfy the initial conditions.

14. $y(x) = e^{2x}(3 - x)$

15. characteristic equation $\lambda^2 + \lambda - 12 = 0$, with roots $3, -4$. The general solution is
$$y(x) = c_1 e^{3x} + c_2 e^{-4x}.$$
We need
$$y(2) = c_1 e^{6} + c_2 e^{-8} = 2$$
and
$$y'(2) = 3c_1 e^{6} - 4c_2 e^{-8} = -1.$$
Solve these to obtain
$$c_1 = e^{-6}, c_2 = e^{8}.$$

The solution of the initial value problem is

$$y(x) = e^{-6}e^{3x} + e^{8}e^{-4x}.$$

This can also be written

$$y(x) = e^{3(x-2)} + e^{-4(x-2)}.$$

16.

$$y(x) = \frac{\sqrt{6}}{4}e^{x}\left(e^{\sqrt{6}x} - e^{-\sqrt{6}x}\right)$$

17. $y(x) = e^{x-1}(29 - 17x)$

18.

$$\begin{aligned} y(x) = {} & \frac{8}{e^5\sqrt{23}}\sin(\sqrt{23})e^{5x/2}\cos(\sqrt{23}x/2) \\ & - \frac{8}{e^5\sqrt{23}}\cos(\sqrt{23})e^{5x/2}\sin(\sqrt{23}x/2) \end{aligned}$$

19.

$$\begin{aligned} y(x) = e^{(x+2)/2} \Big[& \cos(\sqrt{15}(x+2)/2) \\ & + \frac{5}{\sqrt{15}}\sin(\sqrt{15}(x+2)/2)\Big] \end{aligned}$$

20.

$$y(x) = ae(-1+\sqrt{5})x/2 + be^{(-1-\sqrt{5})x/2},$$

where

$$a = \left(\frac{9+7\sqrt{5}}{2\sqrt{5}}\right)e^{-2+\sqrt{5}} \text{ and } b = \left(\frac{7\sqrt{5}-9}{2\sqrt{5}}\right)e^{-2-\sqrt{5}}$$

21. (a) The characteristic equation is $\lambda^2 - 2\alpha\lambda + \alpha^2 = 0$, with α as a repeated root. The general solution is

$$y(x) = (c_1 + c_2 x)e^{\alpha x}.$$

(b) The characteristic equation is $\lambda^2 - 2\alpha\lambda + (\alpha^2 - \epsilon^2) = 0$, with roots $\alpha + \epsilon, \alpha - \epsilon$. The general solution is

$$y_\epsilon(x) = c_1 e^{(\alpha+\epsilon)x} + c_2 e^{(\alpha-\epsilon)x}.$$

We can also write

$$y_\epsilon(x) = \left(c_1 e^{\epsilon x} + c_2 e^{-\epsilon x}\right)e^{\alpha x}.$$

In general,

$$\lim_{\epsilon\to 0} y_\epsilon(x) = (c_1 + c_2)e^{\alpha x} \neq y(x).$$

Note, however, that the coefficients in the differential equations in (a) and (b) can be made arbitrarily close by choosing ϵ sufficiently small.

22. With $a^2 = 4b$, one solution is $y_1(x) = e^{-ax/2}$. Attempt a second solution $y_2(x) = u(x)e^{-ax/2}$. Substitute this into the differential equation to get

$$\left[u'' - au' + \frac{a^2}{4}u + a\left(u' - \frac{a}{2}u\right) + bu\right] e^{-ax/2} = 0.$$

Because $a^2 - 4b = 0$, this reduces to

$$u''(x) = 0.$$

Then $u(x) = cx + d$, with c and d arbitrary constants, and the functions $(cx + d)e^{-ax/2}$ are also solutions of the differential equation. If we choose $c = 1$ and $d = 0$, we obtain $y_2(x) = xe^{-ax/2}$ as a second solution. Further, this solution is independent from $y_1(x)$, because the Wronskian of these solutions is

$$W(x) = \begin{vmatrix} e^{-ax/2} & xe^{-ax/2} \\ -(a/2)e^{-ax/2} & e^{-ax/2} - (a/2)xe^{-ax/2} \end{vmatrix} = e^{-ax},$$

and this is nonzero.

23. The roots of the characteristic equation are

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \text{ and } \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

Because $a^2 - 4b < a^2$ by assumption, λ_1 and λ_2 are both negative (if $a^2 - 4b \geq 0$), or complex conjugates (if $a^2 - 4b < 0$). There are three cases.

Case 1 - Suppose λ_1 and λ_2 are real and unequal. Then the general solution is

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

and this has limit zero as $x \to \infty$ because λ_1 and λ_2 are negative.

Case 2 - Suppose $\lambda_1 = \lambda_2$. Now the general solution is

$$y(x) = (c_1 + c_2 x)e^{\lambda_1 x},$$

and this also has limit zero as $x \to \infty$.

Case 3 - Suppose λ_1 and λ_2 are complex. Now the general solution is

$$y(x) = \left[c_1 \cos(\sqrt{4b - a^2}x/2) + c_2 \sin(\sqrt{4b - a^2}x/2)\right] e^{-ax/2},$$

and this has limit zero as $x \to \infty$ because $a > 0$.

If, for example, $a = 1$ and $b = -1$, then one solution is $e^{(-1+\sqrt{5})x/2}$, and this tends to ∞ as $x \to \infty$.

2.3 Particular Solutions of the Nonhomogeneous Equation

1. Two independent solutions of $y'' + y = 0$ are $y_1(x) = \cos(x)$ and $y_2(x) = \sin(x)$, with Wronskian

$$W(x) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1.$$

Let $f(x) = \tan(x)$ and use equations (2.7) and (2.8). First,

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x)f(x)}{W(x)} = -\int \tan(x)\sin(x)\,dx \\ &= -\int \frac{\sin^2(x)}{\cos(x)}\,dx \\ &= -\int \frac{1-\cos^2(x)}{\cos(x)}\,dx \\ &= \int \cos(x)\,dx - \int \sec(x)\,dx \\ &= \sin(x) - \ln|\sec(x) + \tan(x)|. \end{aligned}$$

Next,

$$\begin{aligned} u_2(x) &= \int \frac{y_1(x)f(x)}{W(x)}\,dx = \int \cos(x)\tan(x)\,dx \\ &= \int \sin(x)\,dx = -\cos(x). \end{aligned}$$

The general solution is

$$\begin{aligned} y(x) &= c_1\cos(x) + c_2\sin(x) + u_1(x)y_1(x) + u_2(x)y_2(x) \\ &= c_1\cos(x) + c_2\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|. \end{aligned}$$

2. Two independent solutions of the associated homogeneous equation are $y_1(x) = e^{3x}$ and $y_2(x) = e^x$. Their Wronskian is $W(x) = -2e^{4x}$. Compute

$$\begin{aligned} u_1(x) &= -\int \frac{2e^x\cos(x+3)}{-2e^{4x}}\,dx \\ &= \int e^{-3x}\cos(x+3)\,dx \\ &= -\frac{3}{10}e^{-3x}\cos(x+3) + \frac{1}{10}e^{-3x}\sin(x+3) \end{aligned}$$

and

$$\begin{aligned} u_2(x) &= \int \frac{2e^{3x}\cos(x+3)}{-2e^{4x}}\,dx \\ &= \int e^{-x}\cos(x+3)\,dx \\ &= \frac{1}{2}e^{-x}\cos(x+3) - \frac{1}{2}e^{-x}\cos(x+3). \end{aligned}$$

The general solution is

$$\begin{aligned} y(x) = c_1 e^{3x} &+ c_2 e^{x} \\ &- \frac{3}{10}\cos(x+3) + \frac{1}{10}\sin(x+3) \\ &+ \frac{1}{2}\cos(x+3) - \frac{1}{2}\sin(x+3). \end{aligned}$$

More compactly, the general solution is

$$y(x) = c_1 e^{3x} + c_2 e^{x} + \frac{1}{5}\cos(x+3) - \frac{2}{5}\sin(x+3).$$

For Problems 3–6, some details of the calculations are omitted.

3. The associated homogeneous equation has independent solutions $y_1(x) = \cos(3x)$ and $y_2(x) = \sin(3x)$, with Wronskian 3. The general solution is

$$y(x) = c_1\cos(3x) + c_2\sin(3x) + 4x\sin(3x) + \frac{4}{3}\cos(3x)\ln|\cos(3x)|.$$

4. $y_1(x) = e^{3x}$ and $y_2(x) = e^{-x}$, with $W(x) = -4e^{-2x}$. With

$$f(x) = 2\sin^2(x) = 1 - \cos(2x)$$

we find the general solution

$$y(x) = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{3} + \frac{7}{65}\cos(2x) + \frac{4}{65}\sin(2x).$$

5. $y_1(x) = e^{x}$ and $y_2(x) = e^{2x}$, with Wronskian $W(x) = e^{3x}$. With $f(x) = \cos(e^{-x})$, we find the general solution

$$y(x) = c_1 e^{x} + c_2 e^{2x} - e^{2x}\cos(e^{-x}).$$

6. $y_1(x) = e^{3x}$ and $y_2(x) = e^{2x}$, with Wronskian $W(x) = e^{-5x}$. Use the identity

$$8\sin^2(4x) = 4\cos(8x) - 1$$

in determining $u_1(x)$ and $u_2(x)$ to write the general solution

$$y(x) = c_1 e^{3x} + c_2 e^{2x} + \frac{2}{3} + \frac{58}{1241}\cos(8x) + \frac{40}{1241}\sin(8x).$$

In Problems 7–16 the method of undetermined coefficients is used to find a particular solution of the nonhomogeneous equation. Details are included for Problems 7 and 8, and solutions are outlined for the remainder of these problems.

7. The associated homogeneous equation has independent solutions $y_1(x) = e^{2x}$ and e^{-x}. Because $2x^2+5$ is a polynomial of degree 2, attempt a second degree polynomial
$$y_p(x) = Ax^2 + Bx + C$$
for the nonhomogeneous equation. Substitute $y_p(x)$ into this nonhomogeneous equation to obtain
$$2A - (2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 + 5.$$
Equating coefficients of like powers of x on the left and right, we have the equations
$$\begin{aligned} -2A &= 2(\text{ coefficients of } x^2) \\ -2A - 2B &- 0(\text{ coefficients of } x \\ 2A - 2B - 2C &= 5(\text{ constant term.}) \end{aligned}$$
Then $A = -1, B = 1$ and $C = -4$. Then
$$y_p(x) = -x^2 + x - 4$$
and a general solution of the (nonhomogeneous) equation is
$$y = c_1e^{2x} + c_2e^{-x} - x^2 + x - 4.$$

8. $y_1(x) = e^{3x}$ and $y_2(x) = e^{-2x}$ are independent solutions of the associated homogeneous equation. Because e^{2x} is not a solution of the homogeneous equation, attempt a particular solution $y_p(x) = Ae^{2x}$ of the nonhomogeneous equation. Substitute this into the differential equation to get
$$4A - 2A - 6A = 8,$$
so $A = -2$ and a general solution is
$$y(x) = c_1e^{3x} + c_2e^{-2x} - 2e^{2x}.$$

9. $y_1(x) = e^x \cos(3x)$ and $y_2(x) = e^x \sin(3x)$ are independent solutions of the associated homogeneous equation. Try a particular solution $y_p(x) = Ax^2 + Bx + C$ to obtain the general solution
$$y(x) = c_1e^x \cos(3x) + c_2e^x \sin(3x) + 2x^2 + x - 1.$$

10. For the associated homogeneous equation, $y_1(x) = e^{2x}\cos(x)$ and $y_2(x) = e^{2x}\sin(x)$. Try $y_p(x) = Ae^{2x}$ to get $A = 21$ and obtain the general solution

$$y(x) = c_1 e^{2x}\cos(x) + c_2 e^{2x}\sin(x) + 21e^{2x}.$$

11. For the associated homogeneous equation, $y_1(x) = e^{2x}$ and $y_2(x) = e^{4x}$. Because e^x is not a solution of the homogeneous equation, attempt a particular solution of the nonhomogeneous equation of the form $y_p(x) = Ae^x$. We get $A = 1$, so a general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{4x} + e^x.$$

12. $y_1(x) = e^{-3x}$ and $y_2(x) = e^{-3x}$. Because $f(x) = 9\cos(3x)$ (which is not a solution of the associated homogeneous equation), attempt a particular solution

$$y_p(x) = A\cos(3x) + B\sin(3x).$$

This attempt includes both a sine and cosine term even though $f(x)$ has only a cosine term, because both terms may be needed to find a particular solution. Substitute this into the nonhomogeneous equation to obtain $A = 0$ and $B = 1/2$, so a general solution is

$$y(x) = (c_1 + c_2 x)e^{-3x} + \frac{1}{2}\sin(3x).$$

In this case $y_p(x)$ contains only a sine term, although $f(x)$ has only the cosine term.

13. $y_1(x) = e^x$ and $y_2(x) = e^{2x}$. Because $f(x) = 10\sin(x)$, attempt

$$y_p(x) = A\cos(x) + B\sin(x).$$

Substitute this into the (nonhomogeneous) equation to find that $A = 3$ and $B = 1$. A general solution is

$$y(x) = c_1 e^x + c_2 e^{2x} + 3\cos(x) + \sin(x).$$

14. $y_1(x) = 1$ and $y_2(x) = e^{-4x}$. Finding a particular solution $y_p(x)$ for this problem requires some care. First, $f(x)$ contains a polynomial term and an exponential term, so we are tempted to try $y_p(x)$ as a second degree polynomial $Ax^2 + Bx + C$ plus an exponential term De^{3x} to account for the exponential term in the equation. However, note that $y_1(x) = 1$, a constant solution, is one term of the proposed polynomial part, so multiply this part by x to try

$$y_p(x) = Ax^3 + Bx^2 + Cx + De^{3x}.$$

Substitute this into the nonhomogeneous differential equation to get

$$6Ax + 2B + 9De^{3x} - 4(3Ax^2 + 2Bx + C + 3De^{3x}) = 8x^2 + 2e^{3x}.$$

Matching coefficients of like terms, we conclude that

$$\begin{aligned} 2B-4C &= 0(\text{ from the constant terms}) \\ 6A-8B &= 0(\text{ from the } x \text{ terms}), \\ -12A &= 8(\text{ from the } x^2 \text{ terms}) \\ -3D &= 2. \end{aligned}$$

Then $D=-\frac{2}{3}, A=-\frac{2}{3}, B=-\frac{1}{2}$ and $C=-\frac{1}{4}$. Then

$$y_p(x)=-\frac{2}{3}x^3-\frac{1}{2}x^2-\frac{1}{4}x-\frac{2}{3}e^{3x}$$

and a general solution of the nonhomogeneous equation is

$$y(x)=c_1+c_2e^{-4x}-\frac{2}{3}x^3-\frac{1}{2}x^2-\frac{1}{4}x-\frac{2}{3}e^{3x}.$$

15. $y_1(x)=e^{2x}\cos(3x)$ and $y_2(x)=e^{2x}\sin(3x)$. Try

$$y_px=Ae^{2x}+Be^{3x}.$$

This will work because neither e^{2x} nor e^{3x} is a solution of the associated homogeneous equation. Substitute $y_p(x)$ into the differential equation and obtain $A=1/3, B=-1/2$. The differential equation has general solution

$$y(x)=[c_1\cos(3x)+c_2\sin(3x)]e^{2x}+\frac{1}{3}e^{2x}-\frac{1}{2}e^{3x}.$$

16. $y_1(x)=e^x$ and $y_2(x)=xe^x$. Try

$$y_p(x)=Ax+B+C\cos(3x)+D\sin(3x).$$

This leads to the general solution

$$y(x)=(c_1+c_2x)e^x+3x+6+\frac{3}{2}\cos(3x)-2\sin(3x).$$

In Problems 17–24 the strategy is to first find a general solution of the differential equation, then solve for the constants to find a solution satisfying the initial conditions. Problems 17–22 are well suited to the use of undetermined coefficients, while Problems 23 and 24 can be solved fairly directly using variation of parameters.

17. $y_1(x)=e^{2x}$ and $y_2(x)=e^{-2x}$. Because e^{2x} is a solution of the associated homogeneous equation, use xe^{2x} in the method of undetermined coefficients, attempting

$$y_p(x)=Axe^{2x}+Bx+C.$$

Substitute this into the nonhomogeneous differential equation to obtain

$$4Axe^{2x} + 4Axe^{2x} - 4Axe^{2x} - 4Bx - 4C = -7e^{2x} + x.$$

Then $A = -7/4, B = -1/4$ and $C = 0$, so the differential equation has the general solution

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{7}{4}xe^{2x} - \frac{1}{4}x.$$

We need

$$y(0) = c_1 + c_2 = 1$$

and

$$y'(0) = 2c_1 - 2c_2 - \frac{7}{4} - \frac{1}{4} = 3.$$

Then $c_1 = 7/4$ and $c_2 = -3/4$. The initial value problem has the unique solution

$$y(x) = \frac{7}{4}e^{2x} - \frac{3}{4}e^{-2x} - \frac{7}{4}xe^{2x} - \frac{1}{4}x.$$

18. $y_1 = 1$ and $y_2(x) = e^{-4x}$ are independent solutions of the associated homogeneous equation. For $y_p(x)$, try

$$y_p(x) = Ax + B\cos(x) + C\sin(x),$$

with the term Ax because 1 is a solution of the homogeneous equation. This leads to the general solution

$$y(x) = c_1 + c_2 e^{-4x} + 2x - 2\cos(x) + 8\sin(x).$$

Now we need

$$y(0) = c_1 + c_2 - 2 = 3$$

and

$$y'(0) = -4c_2 + 2 + 8 = 2.$$

Then $c_1 = 3$ and $c_2 = 2$, so the initial value problem has the solution

$$y(x) = 3 + 2e^{-4x} + 2x - 2\cos(x) + 8\sin(x).$$

19. We find the general solution

$$y(x) = c_1 e^{-2x} + c_2 e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}.$$

The solution of the initial value problem is

$$y(x) = \frac{3}{8}e^{-2x} - \frac{19}{120}e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}.$$

20. 1 and e^{3x} are independent solutions of the associated homogeneous equation. Attempt a particular solution

$$y_p(x) = Ae^{2x}\cos(x) + Be^{2x}\sin(x)$$

of the nonhomogeneous equation to find the general solution

$$y(x) = c_1 + c_2e^{3x} - \frac{1}{5}e^{2x}(\cos(x) + 3\sin(x)).$$

The solution of the initial value problem is

$$y(x) = \frac{1}{5} + e^{3x} - \frac{1}{5}(\cos(x) + 3\sin(x)).$$

21. e^{4x} and e^{-2x} are independent solutions of the associated homogeneous equation. The nonhomogeneous equation has general solution

$$y(x) = c_1e^{4x} + c_2e^{-2x} - 2e^{-x} - e^{2x}.$$

The solution of the initial value problem is

$$y(x) = 2e^{4x} + 2e^{-2x} - 2e^{-x} - e^{2x}.$$

22. The general solution of the differential equation is

$$y(x) = e^{x/2}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + 1.$$

It is easier to fit the initial conditions specified at $x = 1$ if we write this general solution as

$$y(x) = e^{x/2}\left[d_1\cos\left(\frac{\sqrt{3}}{2}(x-1)\right) + d_2\sin\left(\frac{\sqrt{3}}{2}(x-1)\right)\right] + 1.$$

Now

$$y(1) = e^{1/2}d_1 + 1 = 4 \text{ and } y'(1) = \frac{1}{2}e^{1/2}d_1 + \frac{\sqrt{3}}{2}e^{1/2}d_2 = -2.$$

Solve these to obtain $d_1 = 3e^{-1/2}$ and $d_2 = -7e^{-1/2}/\sqrt{3}$. The solution of the initial value problem is

$$y(x) = e^{(x-1)/2}\left[3\cos\left(\frac{\sqrt{3}}{2}(x-1)\right) - \frac{7}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}(x-1)\right)\right] + 1.$$

23. The differential equation has general solution

$$y(x) = c_1e^{x} + c_2e^{-x} - \sin^2(x) - 2.$$

The solution of the initial value problem is

$$y(x) = 4e^{-x} - \sin^2(x) - 2.$$

24. The general solution is

$$y(x) = c_1\cos(x) + c_2\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|,$$

and the solution of the initial value problem is

$$y(x) = 4\cos(x) + 4\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|.$$

2.4 The Euler Differential Equation

Details are included with solutions for Problems 1–2, while just the solutions are given for Problems 3–10. These solutions are for $x > 0$.

1. Read from the differential equation that the characteristic equation is

$$r^2 + r - 6 = 0$$

with roots $2, -3$. The general solution is

$$y(x) = c_1x^2 + c_2x^{-3}.$$

2. The characteristic equation is

$$r^2 + 2r + 1 = 0$$

with repeated root $-1, -1$. The general solution is

$$y(x) = c_1x^{-1} + c_2x^{-1}\ln(x).$$

We can also write

$$y(x) = \frac{1}{x}\left(c_1 + c_2\ln(x)\right)$$

for $x > 0$.

3.
$$y(x) = c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))$$

4.
$$y(x) = c_1x^2 + c_1\frac{1}{x^2}$$

5.
$$y(x) = c_1x^2 + c_1\frac{1}{x^4}$$

6.
$$y(x) = \frac{1}{x^2}\left(c_2\cos(3\ln(x)) + c_2\sin(3\ln(x))\right)$$

7.
$$y(x) = c_1\frac{1}{x^2} + c_2\frac{1}{x^3}$$

8.
$$y(x) = x^2(c_1 \cos(7\ln(x)) + c_2 \sin(7\ln(x)))$$

9.
$$y(x) = \frac{1}{x^{12}}(c_1 + c_2 \ln(x))$$

10.
$$y(x) = c_1 x^7 + c_2 x^5$$

11. The general solution of the differential equation is
$$y(x) = c_1 x^3 + c_2 x^{-7}.$$
From the initial conditions, we need
$$y(2) = 8c_1 + 2^{-7}c_2 = 1 \text{ and } y'(2) = 3c_1 2^2 - 7c_2 2^{-8} = 0.$$
Solve for c_1 and c_2 to obtain the solution of the initial value problem
$$y(x) = \frac{7}{10}\left(\frac{x}{2}\right)^3 + \frac{3}{10}\left(\frac{x}{2}\right)^{-7}.$$

12. The initial value problem has the solution
$$y(x) = -3 + 2x^2.$$

13. $y(x) = x^2(4 - 3\ln(x))$

14. $y(x) = -4x^{-12}(1 + 12\ln(x))$

15. $y(x) = 3x^6 - 2x^4$

16.
$$y(x) = \frac{11}{4}x^2 + \frac{17}{4}x^{-2}$$

17. With $Y(t) = y(e^t)$, use the chain rule to get
$$y'(x) = \frac{dY}{dt}\frac{dt}{dx} = \frac{1}{x}Y'(t)$$
and then
$$\begin{aligned} y''(x) &= \frac{d}{dx}\left(\frac{1}{x}Y'(t)\right) \\ &= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{d}{dx}(Y'(t)) \\ &= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{dY'}{dt}\frac{dt}{dx} \\ &= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{1}{x}Y''(t) \\ &= \frac{1}{x^2}(Y''(t) - Y'(t)). \end{aligned}$$

Then

$$x^2y''(x) = Y''(t) - Y'(t).$$

Substitute these into Euler's equation to get

$$Y''(t) + (A-1)Y'(t) + BY(t) = 0.$$

This is a constant coefficient second-order homogeneous differential equation for $Y(t)$, which we know how to solve.

18. If $x < 0$, let $t = \ln(-x) = \ln|x|$. We can also write $x = -e^t$. Note that

$$\frac{dt}{dx} = \frac{1}{-x}(-1) = \frac{1}{x}$$

just as in the case $x > 0$. Now let $y(x) = y(-e^t) = Y(t)$ and proceed with chain rule differentiations as in the solution of Problem 17. First,

$$y'(x) = \frac{dY}{dt}\frac{dt}{dx} = \frac{1}{x}Y'(t)$$

and

$$\begin{aligned} y''(x) &= \frac{d}{dx}\left(\frac{1}{x}Y'(t)\right) \\ &= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{dt}{dx}Y''(t) \\ &= -\frac{1}{x^2}Y'(t) + \frac{1}{x^2}Y''(t). \end{aligned}$$

Then

$$x^2y''(x) = Y''(t) - Y'(t)$$

just as we saw with $x > 0$. Now Euler's equation transforms to

$$Y'' + (A-1)Y' + BY = 0.$$

We obtain the solution in all cases by solving this linear constant coefficient second-order equation. Omitting all the details, we obtain the solution of Euler's equation for negative x by replacing x with $|x|$ in the solution for positive x. For example, suppose we want to solve

$$x^2y'' + xy' + y = 0$$

for $x < 0$. Solve this for $x > 0$ to get

$$y(x) = c_1\cos(\ln(x)) + c_2\sin(\ln(x)).$$

The solution for $x < 0$ is

$$y(x) = c_1\cos(\ln|x|) + c_2\sin(\ln|x|).$$

19. The problem to solve is

$$x^2y'' - 5dxy' + 10y = 0; y(1) = 4, y'(1) = -6.$$

We know how to solve this problem. Here is an alternative method, using the transformation $x = e^t$, or $t = \ln(x)$ for $x > 0$ (since the initial conditions are specified at $x = 1$). Euler's equation transforms to

$$Y'' - 6Y' + 10Y = 0.$$

However, also transform the initial conditions:

$$Y(0) = y(1) = 4, Y'(0) = (1)y'(1) = -6.$$

This differential equation for $Y(t)$ has general solution

$$Y(t) = c_1e^{3t}\cos(t) + c_2e^{3t}\sin(t).$$

Now

$$Y(0) = c_2 = 4$$

and

$$Y'(0) = 3c_1 + c_2 = -6,$$

so $c_2 = -18$. The solution of the transformed initial value problem is

$$Y(t) = 4e^{3t}\cos(t) - 18e^{3t}\sin(t).$$

The original initial value problem therefore has the solution

$$y(x) = 4x^3\cos(\ln(x)) - 19x^3\sin(\ln(x))$$

for $x > 0$. The new twist here is that the entire initial value problem (including initial conditions) was transformed in terms of t and solved for $Y(t)$, then this solution $Y(t)$ in terms of t was transformed back to the solution $y(x)$ in terms of x.

20. Suppose

$$x^2y'' + Axy' + By = 0$$

has repeated roots. Then the characteristic equation

$$r^2 + (A-1)r + B = 0$$

has $(1-A)/2$ as a repeated root, and we have only one solution $y_1(x) = x^{(1-A)/2}$ so far. For another solution, independent from y_1, look for a solution of the form $y_2(x) = u(x)y_1(x)$. Then

$$y_2' = u'y_1 + uy_1'$$

and

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''.$$

Substitute y_2 into the differential equation to get

$$x^2(u''y_1 + 2u'y_1' + uy_1'') + Ax(u'y_1 + uy_1') + Buy_1 = 0.$$

Three terms in this equation cancel, because

$$u(x^2y_1'' + Axy_1' + By_1) = 0$$

by virtue of y_1 being a solution. This leaves

$$x^2u''y_1 + 2x^2u'y_1' + Axu'y_1 = 0.$$

Assuming that $x > 0$, divide by x to get

$$xu''y_1 + 2xu'y_1' + Au'y_1 = 0.$$

Substitute $y_1(x) = x^{(1-A)/2}$ into this to obtain

$$xu''x^{(1-A)/2} + 2xu'\left(\frac{1-A}{2}\right)x^{(-1-A)/2} + Au'x^{(1-A)/2} = 0.$$

Divide this by $x^{(1-A)/2}$ to get

$$xu'' + (1-A)u' + Au' = 0,$$

and this reduces to

$$xu'' + u' = 0.$$

Let $z = u'$ to obtain

$$xz' + z = 0,$$

or

$$(xz)' = 0.$$

Then $xz = c$, constant, so

$$z = u' = \frac{c}{x}.$$

Then $u(x) = c\ln(x) + d$. We only need one second solution, so let $c = 1$ and $d = 0$ to get $u(x) = \ln(x)$. A second solution, independent from $y_1(x)$, is

$$y_2(x) = y_1(x)\ln(x),$$

as given without derivation in the chapter.

2.5 Series Solutions

2.5.1 Power Series Solutions

1. Put $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation to obtain

$$\begin{aligned} y' - xy &= \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=2}^{\infty} a_{n-2} x^{n-1} \\ &= a_1 + (2a_2 - a_0)x + \sum_{n=3}^{\infty} (n a_n - a_{n-2}) x^{n-1} \\ &= 1 - x. \end{aligned}$$

Then a_0 is arbitrary, $a_1 = 1$, $2a_2 - a_0 = -1$, and

$$a_n = \frac{1}{n} a_{n-2} \text{ for } n = 3, 4, \cdots .$$

This is the recurrence relation. If we set $a_0 = c_0 + 1$, we obtain the coefficients

$$a_2 = \frac{1}{2} c_0, a_4 = \frac{1}{2 \cdot 4} c_0, a_6 = \frac{1}{2 \cdot 4 \cdot 6} c_0,$$

and so on. Further,

$$a_1 = 1, a_3 = \frac{1}{3}, a_5 = \frac{1}{3 \cdot 5}, a_7 = \frac{1}{3 \cdot 5 \cdot 7}$$

and so on. The solution can be written

$$\begin{aligned} y(x) = 1 &+ \sum_{n=0}^{\infty} \frac{1}{3 \cdot 5 \cdots 2n+1} x^{2n+1} \\ &+ c_0 \left(1 + \sum_{n=1}^{\infty} \frac{1}{2 \cdot 4 \cdots 2n} x^{2n} \right). \end{aligned}$$

2. Write

$$\begin{aligned} y' - x^3 y &= \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+3} \\ &= a_1 + 2a_2 x + 3a_3 x^2 + \sum_{n=4}^{\infty} (n a_n - a_{n-4}) x^{n-1} = 4. \end{aligned}$$

The recurrence relation is

$$a_n = \frac{1}{n} a_{n-4} \text{ for } n = 4, 5, \cdots ,$$

with a_0 arbitrary, $a_1 = 4$ and $a_2 = a_3 = 0$. This yields the solution

$$y(x) = 4\sum_{n=0}^{\infty} \frac{1}{1 \cdot 5 \cdot 9 \cdots (4n+1)} x^{4n+1} + a_0\left(1 + \sum_{n=1}^{\infty} \frac{1}{4 \cdot 8 \cdot 12 \cdots 4n} x^{4n}\right).$$

3. Write

$$\begin{aligned} y' + (1-x^2)y &= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= (a_1 + a_0) + (2a_2 + a_1)x + \sum_{n=3}^{\infty}(n a_n + a_{n-1} - a_{n-3})x^{n-1} \\ &= x. \end{aligned}$$

The recurrence relation is

$$n a_n + a_{n-1} - a_{n-3} = 0 \text{ for } n = 3, 4, \cdots .$$

Here a_0 is arbitrary, $a_1 + a_0 = 0$ and $2a_2 + a_1 = 1$. This gives us the solution

$$y(x) = a_0\left(1 - x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{7}{4!}x^4 + \frac{19}{5!}x^5 + \cdots\right) + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{11}{5!}x^5 - \frac{31}{6!}x^6 + \cdots .$$

4. Begin with

$$\begin{aligned} y'' + 2y' - xy &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= (2a_2 + 2a_1) + (3 \cdot 2a_3 + 2 \cdot 2a_2 + a_0)x \\ &+ \sum_{n=1}^{\infty}(n(n-1)a_n + 2(n-1)a_{n-1} + a_{n-2})x^{n-2} = 0. \end{aligned}$$

The recurrence relation is

$$n(n-1)a_n + 2(n-1)a_{n-1} + a_{n-2} = 0 \text{ for } n = 4, 5, \cdots .$$

Further, a_0 and a_1 are arbitrary, $a_2 = -A_1$ and

$$6a_3 + 4a_2 + a_0 = 0.$$

Taking $a_0 = 1, a_1 = 0$, we obtain the solution

$$y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{30}x^5 + \frac{1}{60}x^6 + \cdots .$$

With $a_0 = 0, a_1 = 1$ we get a second, linearly independent solution

$$y_2(x) = x - x^2 + \frac{2}{3}x^3 - \frac{5}{12}x^4 + \frac{7}{60}x^5 + \cdots .$$

5. Write

$$y'' - xy' + y = \sum_{n=2}^{\infty} n(n-1)_n x^{n-2} - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n$$
$$= 2a_2 + a_0 + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 3.$$

Here a_0 and a_1 are arbitrary and $a_2 = (3 - a_0)/2$. The recurrence relation is

$$a_{n+2} = \frac{n-1}{(n+2)(n+1)} \text{ for } n = 1, 2, \cdots .$$

This yields the general solution

$$y(x) = a_0 + a_1 x + \frac{3-a_0}{2}x^2 + \frac{3-a_0}{4!}x^4$$
$$+ \frac{3(3-a_0)}{6!}x^6 + \frac{3 \cdot 5(3-a_0)}{8!}x^8 + \cdots .$$

6. Begin with

$$y'' + xy' + xy = \sum_{n=2}^{\infty} n(n-1)ax_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$= 2a_2 + \sum_{n=3}^{\infty}(n(n-1)a_n + (n-2)a_{n-2} + a_{n-3})x^{n-2} = 0.$$

Here a_0 and a_1 are arbitrary and $a_2 = 0$. The recurrence relation is

$$a_n = -\frac{(n-2)a_{n-2} - a_{n-3}}{n(n-1)} \text{ for } n = 3, 4, \cdots .$$

With $a_0 = 1$ and $a_1 = 0$, we obtain one solution

$$y_1(x) = 1 - \frac{2}{3}x^3 + \frac{3}{2 \cdot 3 \cdot 4 \cdot 5}x^5$$
$$+ \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}x^6 - \frac{3 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}x^7 + \cdots .$$

With $a_0 = 0$ and $a_1 = 1$, we obtain a second, linearly independent solution

$$y_2(x) = x - \frac{1}{2 \cdot 3}x^3 - \frac{1}{3 \cdot 4}x^4$$
$$+ \frac{3}{2 \cdot 3 \cdot 4 \cdot 5}x^5 + \frac{3 \cdot 5}{2 \cdot 3 \cdot 5 \cdot 6}x^6 + \cdots .$$

7. We have

$$y'' - x^2y' + 2y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$
$$- \sum_{n=1}^{\infty} na_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^n$$
$$= 2a_2 + 2a_0 + (6a_3 + 2a_1)x$$
$$+ \sum_{n=1}^{\infty}(n(n-1)a_n - (n-3)a_{n-3} + 2a_{n-2})x^{n-2} = x.$$

Then a_0 and a_1 are arbitrary, $a_2 = -a_0$, and $6a_3+2a_1 = 1$. The recurrence relation is

$$a_n = \frac{(n-3)a_{n-3} - 2a_{n-2}}{n(n-1)}$$

for $n = 4, 5, \cdots$. The general solution has the form

$$y(x) = a_0\left[1 - x^2 + \frac{1}{6}x^4 - \frac{1}{10}x^5 - \frac{1}{90}x^6 + \cdots\right]$$
$$+ a_1\left[x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{30}z^5 - \frac{7}{180}x^6 + \cdots\right]$$
$$+ \frac{1}{6}x^3 - \frac{1}{6}x^5 + \frac{1}{60}x^6 + \frac{1}{1260}x^7 - \frac{1}{480}x^8 + \cdots.$$

Note that $a_0 = y(0)$ and $a_1 = y'(0)$. The third series represents the solution obtained subject to $y(0) = y'(0) = 0$.

8. Using the Maclaurin expansion for $\cos(x)$, we have

$$y' + xy = \sum_{n=1}^{\infty} na_n xn - 1 + \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$= a_1 + \sum_{n=0}^{\infty}(2na_{2n} + a_{2n-2})x^{2n-1}$$
$$+ \sum_{n=1}^{\infty}((2n+1)a_{2n+1} + a_{2n-1})x^{2n}$$
$$= \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}.$$

a_0 is arbitrary and $a_1 = 1$. The recurrence relation is

$$a_{2n} = -\frac{1}{2n}a_{2n-2} \text{ and } a_{2n+1} = \frac{-a_{2n-1} + (-1)^n/((2n)!)}{2n+1}$$

for $n = 1, 2, \cdots$. This yields the solution

$$y(x) = a_0 \left[1 - \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 - \frac{2 \cdot 4 \cdot 6^{\,6}}{x} + \cdots \right] + \left[x - \frac{3}{3!}x^3 + \frac{13}{5!}x^5 - \frac{79}{7!}x^7 + \frac{633}{9!}x^9 + \cdots \right].$$

9. We have

$$y'' + (1-x)y' + 2y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=1}^{\infty} na_n x^n + 2\sum_{n=0}^{\infty} a_n x^n$$
$$= (2a_2 + a_1 + 2a_0) + \sum_{n=3}^{\infty} (n(n-1)a_n + (n-1)a_{n-1} - (n-4)a_{2n-2})x^{n-2}$$
$$= 1 - x^2.$$

Then a_0 and a_1 are arbitrary, $2a_2 + a_1 + 2a_0 = 1$, $6a_3 + 2a_2 + a_1 = 0$, and $12a_4 + 3a_3 = -1$. The recurrence relation is

$$a_n = \frac{-(n-1)a_{n-1} + a_{n-4}a_{n-2}}{n(n-1)}$$

for $n = 5, 6, \cdots$. The general solution is

$$y(x) = a_0 \left[1 - x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{30}x^5 - \cdots \right] + a_1 \left(x - \frac{1}{2}x^2 \right) + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{360}x^6 + \frac{1}{2520}x^7 + \cdots.$$

Here $a_0 = y(0)$ and $a_1 = y'(0)$.

10. Using the Maclaurin expansion of e^x, we have

$$y'' + xy' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n$$
$$= 2a_2 + \sum_{n=3}^{\infty} (n(n-1)a_n + (n-2)a_{n-2})x^{n-2}$$
$$= -\sum_{n=3}^{\infty} \frac{1}{(n-2)!} x^{n-2}.$$

Then a_0 and a_1 are arbitrary, $a_2 = 0$ and

$$a_n = \frac{-(n-2)a_{n-2} - 1/(n-2)!}{n(n-1)}$$

for $n = 3, 4, \cdots$. This leads to the solution

$$y(x) = a_0 + a_1\left[x - \frac{1}{3!}x^3 + \frac{3}{5!}x^5 - \frac{15}{7!}x^7 + \frac{105}{9!}x^9 + \cdots\right]$$
$$+\left[-\frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{2}{5!}x^5 + \frac{3}{6!}x^6 - \frac{11}{7!}x^7 + \frac{19}{8!}x^8 + \cdots\right].$$

Note that $a_0 = y(0)$ and $a_1 = y'(0)$.

2.5.2 Frobenius Solutions

1. Substitute $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation to get

$$xy'' + (1-x)y' + y = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$
$$+\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r}$$
$$= r^2 c_0 x^{r-1} + \sum_{n=1}^{\infty}((n+r)^2 c_n - (n+r-2)c_{n-1})x^{n+r-1}$$
$$= 0.$$

Because c_0 is assumed to be nonzero, r must satisfy the indicial equation $r^2 = 0$, so $r_1 = r_2 = 0$. One solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n,$$

while a second solution has the form

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n.$$

For the first solution, choose the coefficients to satisfy $c_0 = 1$ and

$$c_n = \frac{n-2}{n^2}c_{n-1} \text{ for } n = 1, 2, \cdots .$$

This yields the solution $y_1(x) = 1 - x$. The second solution is therefore

$$y_2(x) = (1-x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n.$$

Substitute this into the differential equation to obtain

$$\begin{aligned}
&x\left[-\frac{2}{x}-\frac{1-x}{x^2}\right]+(1-x)\left[-\ln(x)+\frac{1-x}{x}\right]\\
&+(1-x)\ln(x)+\sum_{n=2}^{\infty}n(n-1)c_n^*x^{n-1}+(1-x)\sum_{n=1}^{\infty}c_n^*x^{n-1}\\
&+\sum_{n=1}^{\infty}c_n^*x^n\\
&=(-3+c_1^*)+(1+4c_2^*)x+\sum_{n=3}^{\infty}(n^2c_n^*-(n-2)c_{n-1}^*)x^{n-2}\\
&=0.
\end{aligned}$$

The coefficients are determined by $c_1^*=3$, $c_2^*=-1/4$, and

$$c_n^*=\frac{n-2}{n^2} \text{ for } n=3,4,\cdots.$$

A second solution is

$$y_2(x)=(1-x)\ln(x)+3x-\sum_{n=2}^{\infty}\frac{1}{n(n-1)}x^n.$$

2. Omitting some routine details, the indicial equation is $r(r-1)=0$, so $r_1=1$ and $r_2=0$. There are solutions

$$y_1(x)=\sum_{n=0}^{\infty}c_nx^{n+1} \text{ and } y_2(x)=ky_1(x)\ln(x)\sum_{n=0}^{\infty}c_n^*x^n.$$

For y_1, the recurrence relation is

$$c_n=\frac{2(n+r-2)}{(n+r)(n+r-1)}c_{n-1}$$

for $n=1,2,\cdots$. With $r=1$ and $c_0=1$, this yields

$$y_1(x)=x,$$

a solution that can be seen by inspection from the differential equation. For the second solution, substitute $y_2(x)$ into the differential equation to get

$$\begin{aligned}
&(2c_0^*+k)+2(c_2^*-k)x\\
&+\sum_{n=1}^{\infty}(n(n-1)c_n^*-2(n-2)c_{n-1}^*)x^{n-1}=0.
\end{aligned}$$

Choose $c_0^* = 1$ to obtain $k = -2$. c_1^* is arbitrary, and we will take $c_1^* = 0$. Finally, $c_2^* = -2$ and

$$c_n^* = \frac{2(n-2)}{n(n-1)} c_{n-1}^* \text{ for } n = 3, 4, \cdots .$$

This yields the second solution

$$y_2(x) = -2x\ln(x) + 1 - \sum_{n=2}^{\infty} \frac{2^n}{n!(n-1)} x^n.$$

3. The indicial equation is $r^2 - 4r = 0$, so $r_1 = 4$ and $r_2 = 0$. There are solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+4} \text{ and } y_2(x) = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n.$$

With $r = 4$ the recurrence relation is

$$c_n = \frac{n+1}{n} c_{n-1} \text{ for } n = 1, 2, \cdots .$$

Then

$$y_1(x) = x^4(1 + 2x + 3x^2 + 4x^3 + \cdots).$$

Using the geometric series, we can observe that

$$\begin{aligned} y_1(x) &= x^4 \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots) \\ &= x^4 \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x^4}{(1-x)^2}. \end{aligned}$$

This gives us the second solution

$$y_2(x) = \frac{3-4x}{(1-x)^2}.$$

4. The indicial equation is $4r^2 - 9 = 0$, with roots $r_1 = 3/2$ and $r_2 = -3/2$. There are solutions

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+3/2} \text{ and } y_2(x) = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n-3/2}.$$

Upon substituting these into the differential equation, we obtain

$$y_1(x) = x^{3/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!(5 \cdot 7 \cdot 9 \cdots (2n+3))} x^{2n}\right]$$

and

$$y_2(x) = x^{-3/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n+1} n!(3) \cdots (2n-3)} x^{2n}\right].$$

5. The indicial equation is $4r^2 - 2r = 0$, with roots $r_1 = 1/2$ and $r_2 = 0$. There are solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1/2} \text{ and } y_2(x) = \sum_{n=0}^{\infty} c_n^* x^n.$$

Substitute these into the differential equation to get

$$\begin{aligned} y_1(x) &= x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!(3\cdot 5\cdot 7\cdots(2n+1))} x^n\right] \\ &= x^{1/2}\left[1 - \frac{1}{6}x + \frac{1}{120}x^2 - \frac{1}{5040}x^3 + \frac{1}{362880}x^4 + \cdots\right] \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!(1\cdot 3\cdot 5\cdots(2n-1))} x^n \\ &= 1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 + \frac{1}{40320}x^4 + \cdots . \end{aligned}$$

6. The indicial equation is $4r^2 - 1 = 0$, with roots $r_1 = 1/2$ and $r_2 = -1/2$. There are solutions

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1/2} \text{ and } y_2(x) = k y_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n-1/2}.$$

After substituting these into the differential equation, we obtain the simple solutions

$$y_1(x) = x^{1/2} \text{ and } y_2(x) = x^{-1/2}.$$

These solutions are consistent with the observation that, upon division by 4, the differential equation is an Euler equation.

7. The indicial equation is $r^2 - 3r + 2 = 0$, with roots $r_1 = 2$ and $r_2 = 1$. There are solutions

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+2} \text{ and } \sum_{n=0}^{\infty} c_n^* x^{n-2}.$$

Substitute these in turn into the differential equation to obtain the solutions

$$y_1(x) = x^2 + \frac{1}{3!}x^4 + \frac{1}{5!}x^6 + \frac{1}{7!}x^8 + \cdots$$

and

$$y_2(x) = x - x^2 + \frac{1}{2!}x^3 - \frac{1}{3!}x^4 + \frac{1}{4!}x^5 - \cdots .$$

We can recognize these series as

$$y_1(x) = x\sinh(x) \text{ and } y_2(x) = xe^{-x}.$$

8. The indicial equation is $r^2 - 2r = 0$, with roots $r_1 = 2, r_2 = 0$. There are solutions

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+2} \text{ and } y_2(x) = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n.$$

The recurrence relation for the $c_n's$ is

$$c_n = \frac{-2}{n(n-2)} \text{ for } n = 3, 4, \cdots$$

and we obtain, with $c_0 = 1$,

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{n(n+2)} x^{n+2} \\ &= x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{1}{45}x^5 + \frac{1}{540}x^6 + \cdots . \end{aligned}$$

For the second solution, substitute $y_2(x)$ into the differential equation to get

$$2c_0^* - c_1^* + \sum_{n=1}^{\infty} \left[n(n-2)c_n^* + c_{n-1}^* + \frac{(-1)^n 2^n k}{n((n-2)!)^2} \right] x^{n-1} = 0.$$

Setting $c_0^* = 1$ for simplicity, we obtain $c_1^* = 2, k = -2$, c_2^* arbitrary (we take this to be zero), and the recurrence relation

$$c_n^* = -\frac{1}{n(n-2)} \left[2c_{n-1}^* + \frac{(-1)^n 2^{n+1}}{n((n-2)!)^2} \right]$$

for $n = 3, 4, \cdots$. We obtain the second solution

$$y_2(x) = -2y_1 \ln(x) + 1 + 2x + \frac{16}{9}x^3 - \frac{25}{36}x^4 + \frac{157}{1350}x^6 - \cdots .$$

9. The indicial equation is $2r^2 = 0$, with roots $r_1 = r_2 = 0$. There are solutions

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n \text{ and } y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n.$$

Upon substituting these into the differential equation, we obtain the independent solutions

$$y_1(x) = 1 - x$$

and

$$y_2(x) = (1-x)\ln\left(\frac{x}{x-2}\right) - 2.$$

10. The indicial equation is $r^2 - 1 = 0$, with roots $r_1 = 1$ and $r_2 = -1$. There are solutions

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+1} \text{ and } y_2(x) = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n-1}.$$

Substitute each of these into the differential equation to get

$$y_1(x) = x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n(1 \cdot 4 \cdot 7 \cdots (3n-2))}{3^n n!(5 \cdot 8 \cdot 11 \cdots (3n+2))}\right] x^{3n}$$

and

$$y_2(x) = \frac{1}{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1 \cdot 2 \cdot 5 \cdots (3n-1))}{3^n n!(4 \cdot 7 \cdots (3n-2))}\right] x^{3n}.$$

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

Table 1: Sums of pairs of dice, Problem 1, Section 2.

3. The mean is

$$\overline{x} = \frac{4(-3) + 2(-1) + 6(0) + 4(1) + 12(3) + 3(4)}{31} = 1.2258.$$

The median is the sixteenth number from the left, or 1 (the last 1 to the right in the ordered list).

The standard deviation is

$$s = \sqrt{\frac{151.42}{30}} = 2.2466.$$

Section 2 Random Variables and Probability Distributions

1. If we roll two dice, there are thirty-six possible outcomes. The sums of the numbers that can come up on the two dice are listed in Table 1.

For example, if o is the outcome that one die comes up 2 and the other 3, then the sum of the dice is 5, so $X(o) = 5$.

The table gives all of the values that $X(o)$ can take on, over all outcomes o of the experiment. Each value is listed as often as it occurs as a value of X. For example, 4 occurs three times, because $X(o) = 4$ for three different outcomes (namely $(2, 2)$, $(1, 3)$ and $(3, 1)$).

Define a probability distribution P on X by by letting $P(x)$ be the probability of x, for each value x that X can assume.

For example, since 2 occurs once out of 36 entries in this table, assign to this value of X the probability

$$P(2) = \frac{1}{36}.$$

Similarly, 11 occurs twice, so give the value 11 of X the probability

$$P(11) = \frac{2}{36}.$$

Since 3 occurs twice in the table, $P(3) = 2/36$, and so on.

Calculating $P(n)$ for each n in the table, we obtain

$$P(2) = P(12) = \frac{1}{36}, P(3) = P(11) = \frac{2}{36},$$
$$P(4) = P(10) = \frac{3}{36}, P(5) = P(9) = \frac{4}{36},$$
$$P(6) = P(8) = \frac{5}{36}, P(7) = \frac{6}{36}.$$

Notice that $\sum_x P(x) = 1$, as required for a probability function.

The mean of X is

$$\begin{aligned}
\mu &= \sum_x xP(x) \\
&= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) \\
&+ 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right) \\
&= 7.
\end{aligned}$$

This is interpreted to mean that, on average, we expect to come up with a seven if we roll two dice. This is a reasonable expectation in view of the fact that there are more ways to roll 7 than any other sum with two dice.

The standard deviation is

$$\sigma = \sqrt{\sum_x (x-7)^2 P(x)}.$$

To compute this, first compute

$$\begin{aligned}
&\sum_x (x-7)^2 P(x) \\
&= (2-7)^2\left(\frac{1}{36}\right) + (3-7)^2\left(\frac{2}{36}\right) + (4-7)^2\left(\frac{3}{36}\right) \\
&+ (5-7)^2\left(\frac{4}{36}\right) + (6-7)^2\left(\frac{5}{36}\right) + (7-7)^2\left(\frac{6}{36}\right) \\
&+ (8-7)^2\left(\frac{5}{36}\right) + (9-7)^2\left(\frac{4}{36}\right) + (10-7)^2\left(\frac{3}{36}\right) \\
&+ (11-7)^2\left(\frac{2}{36}\right) + (12-7)^2\left(\frac{1}{36}\right) \\
&= 5.8333.
\end{aligned}$$

Then

$$\sigma = \sqrt{5.8333} = 2.4152.$$

2. Flip four coins, with sixteen possible outcomes. If o is an outcome, $X(o)$ can have only two values, namely 1 if two, three, or four tails are in o, or 3 otherwise (one tail or no tails in o). There are five outcomes with one tail or no tail, and eleven with two or more tails, so

$$P(1) = \frac{11}{16} \text{ and } P(3) = \frac{5}{16}.$$

The mean is

$$\mu = \sum_x xP(x) = 1\left(\frac{11}{16}\right) + 3\left(\frac{5}{16}\right) = \frac{26}{16} = 1.625.$$

For the standard deviation of X, compute

$$\begin{aligned}
&\sum_x (x-\mu)^2 P(x) \\
&= (1-1.625)^2\left(\frac{11}{16}\right) + (3-1.625)^2\left(\frac{5}{16}\right) \\
&= 0.85938.
\end{aligned}$$

Then

$$\sigma = \sqrt{0.85938} = 0.92703.$$

3. We have

$$\begin{aligned}
&X(1) = 0, \\
&X(2) = X(3) = X(5) = X(7) = X(11) = X(13) = X(17) = X(19) = 1, \\
&X(4) = X(6) = X(9) = X(10) = X(14) = X(15) = 2, \\
&X(8) = X(12) = X(18) = X(20) = 3, \\
&X(16) = 4.
\end{aligned}$$

The values assumed by X are $0, 1, 2, 3, 4$. From the list of values, we get

$$P(0) = \frac{1}{20}, P(1) = \frac{8}{20}, P(2) = \frac{6}{20}, P(3) = \frac{4}{20}, P(4) = \frac{1}{20}.$$

These are the probabilities of the values of the random variable X.

The mean of X is

$$\begin{aligned}
\mu &= \sum_x xP(x) \\
&= 0\left(\frac{1}{20}\right) + 1\left(\frac{8}{20}\right) + 2\left(\frac{6}{20}\right) \\
&+ 3\left(\frac{4}{20}\right) + 4\left(\frac{1}{20}\right) \\
&= 1.8.
\end{aligned}$$

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Table 2: Outcomes of rolling two dice, Problem 4, Section 2.

For the standard deviation of X, first compute

$$\begin{aligned}&\sum_x (x-\mu)^2 P(x)\\ &= (0-1.8)^2\left(\frac{1}{20}\right) + (1-1.8)^2\left(\frac{8}{20}\right)\\ &+ (2-1.8)^2\left(\frac{6}{20}\right) + (3-1.8)^2\left(\frac{4}{20}\right)\\ &+ (4-1.8)^2\left(\frac{1}{20}\right) = 0.9600.\end{aligned}$$

Then

$$\sigma = \sqrt{(0.9600)} = 0.9798.$$

4. The outcomes of two rolls of the dice are displayed in Table 2.

Then

$$\begin{aligned}&X(n,n) = n \text{ for } n = 1,2,3,4,5,6,\\ &X(1,2) = X(2,1) = X(2,4) = X(4,2) = X(3,6) = X(6,3) = 2,\\ &X(1,3) = X(3,1) = X(2,6) = X(6,2) = 3,\\ &X(1,4) = X(4,1) = 4, X(1,5) = X(5,1) = 5, X(1,6) = X(6,1) = 6,\\ &X(2,3) = X(3,2) = X(4,6) = X(6,4) = 3/2,\\ &X(2,5) = X(5,2) = 5/2, X(3,4) = X(4,3) = 4/3,\\ &X(5,3) = X(3,5) = 5/3, X(4,5) = X(5,4) = 5/4,\\ &X(5,6) = X(6,5) = 6/5.\end{aligned}$$

Using this list to compute probabilities, we find that

$$\begin{aligned}P(1) &= \frac{1}{36}, P(2) = \frac{7}{36}, P(3) = \frac{5}{36},\\ P(4) &= \frac{3}{36}, P(5) = \frac{3}{36}, P(6) = \frac{3}{36}.\end{aligned}$$

The mean of X is

$$\mu = \sum_x P(x)$$
$$= 1\left(\frac{1}{36}\right) + 2\left(\frac{7}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{3}{36}\right) + 6\left(\frac{3}{36}\right)$$
$$+ \frac{3}{2}\left(\frac{4}{36}\right) + \frac{5}{2}\left(\frac{2}{36}\right) + \frac{4}{3}\left(\frac{2}{36}\right) + \frac{5}{3}\left(\frac{2}{36}\right) + \frac{5}{4}\left(\frac{2}{36}\right) + \frac{6}{5}\left(\frac{2}{36}\right)$$
$$= 2.6916.$$

Using μ, we can compute the standard deviation of X:

$$\sum_x (x-\mu)^2 P(x)$$
$$= (1-2.6916)^2\left(\frac{1}{36}\right) + (2-2.6916)^2\left(\frac{7}{36}\right) + (3-2.6916)^2\left(\frac{5}{36}\right)$$
$$+ (4-2.6916)^2\left(\frac{3}{36}\right) + (5-2.6916)^2\left(\frac{3}{36}\right) + (6-2.6916)^2\left(\frac{3}{36}\right)$$
$$+ (3/2-2.6916)^2\left(\frac{4}{36}\right) + (5/2-2.6916)^2\left(\frac{2}{36}\right) + (4/3-2.6916)^2\left(\frac{2}{36}\right)$$
$$+ (5/3-2.6916)^2\left(\frac{2}{36}\right) + (5/4-2.6916)^2\left(\frac{2}{36}\right) + (6/5-2.6916)^2\left(\frac{21}{36}\right)$$
$$= 2.2442.$$

Then

$$\sigma = \sqrt{2.2442} = 1.4981.$$

5. Draw cards from a (fifty-two card) deck. There are ${}_{52}C_2$ ways to do this, disregarding order. If o is an outcome in which both cards are numbered, then $X(o)$ equals the sum of the numbers on the cards. If exactly one of the cards is a face card or ace, then $X(o) = 11$, and if both cards are chosen from the face cards or aces, then $X(o) = 12$. Therefore the values of $X(o)$ for all possible outcomes are $4, 5, \cdots, 20$. By a routine but tedious counting of the ways the numbered cards can take on various possible totals, we obtain

$$P(4) = \frac{6}{1326}, P(5) = \frac{16}{1326}, P(6) = \frac{22}{1326}, P(7) = \frac{32}{1326},$$
$$P(8) = \frac{38}{1326}, P(9) = \frac{48}{1326}, P(10) = \frac{54}{1326}, P(11) = \frac{640}{1326},$$
$$P(12) = \frac{190}{1326}, P(13) = \frac{64}{1326}, P(14) = \frac{54}{1326}, P(15) = \frac{48}{1326},$$
$$P(16) = \frac{38}{1326}, P(17) = \frac{32}{1326}, P(18) = \frac{22}{1326}, P(19) = \frac{16}{1326},$$
$$P(20) = \frac{6}{1326}.$$

Using these, compute the mean of X:

$$\mu = \sum_x xP(x) = 11.566$$

and the standard deviation of X:

$$\sigma = \sqrt{\sum_x (x-\mu)^2 P(x)} = \sqrt{5.4289} = 2.33.$$

6. X takes on values $1, 2, \pi$. In particular,

$$\begin{aligned} X(1) &= X(5) = X(7) = X(11) = X(13) = X(17) \\ &= X(19) = X(23) = X(25) = X(29) = \pi, \\ X(2) &= X(4) = \cdots = X(\text{even integer}) = 1, \\ X(3) &= X(9) = X(15) = X(21) = X(27) = 2. \end{aligned}$$

These enable us to write the probability distribution

$$P(1) = \frac{15}{30} = \frac{1}{2}, P(2) = \frac{5}{30} = \frac{1}{6}, P(\pi) = \frac{10}{30} = \frac{1}{3}.$$

The mean of X is

$$\mu = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{6}\right) + \pi\left(\frac{1}{3}\right) = \frac{5}{6} + \frac{\pi}{3},$$

and this is approximately 1.8805.

The standard deviation of X is the square root of

$$\begin{aligned} &\sum_x (x-\mu)^2 P(x) \\ &= (1-1.8805)^2\left(\frac{1}{2}\right) + (2-1.8805)^2\left(\frac{1}{6}\right) + (\pi - 1.8805)^2\left(\frac{1}{3}\right), \end{aligned}$$

which is approximately 0.92014. Then

$$\sigma = \sqrt{92014} = 0.95924.$$

Section 3 <u>The Binomial and Poisson Distributions</u>

1. (a)

$$P(2) = \binom{8}{2}(0.43)^2(1-0.43)^6 = 0.17756.$$

Since x is H or T in each, there are $8+6+4+2=20$ outcomes of this experiment. Then

$$\Pr(\text{dice total at least 9}) = \frac{20}{72} = \frac{5}{18}.$$

(b) Suppose the dice total at least 9 and the coin comes up heads. Now (from (a)) there are 10 outcomes, so

$$\Pr(\text{dice total at least 9, coin is } H) = \frac{10}{72} = \frac{5}{36}.$$

(c) Now suppose the coin is a tail and both dice come up the same. Now there are six outcomes, namely

$$(1,1,T),(2,2,T),\cdots,(6,6,T)$$

so

$$\Pr(\text{tail, both dice the same}) = \frac{6}{72} = \frac{1}{12}.$$

(d) Suppose the dice both come up even and the coin is a tail. Now the outcomes have the appearance (x,y,T), where x and y can independently be 2, 4 or 6. There are 9 outcomes of this form, so

$$\Pr(\text{tail, both dice even}) = \frac{9}{72} = \frac{1}{8}.$$

Section 2 Four Counting Principles

1. The fact that these are letters of the alphabet that we are arranging in order is irrelevant. The issue is that there are nine distinct objects. The number of arrangements is 9!, which is $362,880$.

2. There are 26 letters in the English slphabet. The problem is one of determining the number of ways of choosing 17 objects from 26 objects, with order taken into account. This is

$$\begin{aligned} {}_{26}P_{17} &= \frac{26!}{(26-17)!} = \frac{26!}{9!} \\ &= (10)(11)(12)\cdots(23)(24)(25) = 156,000. \end{aligned}$$

3. Since any of the nine integers can be used in any of the nine places of the ID number, there are 9 ways the first digit can be chosen, 9 ways the second digit can be chosen, and so on. The total number of codes is 9^9, or

$$3.87420489(10)^8.$$

4. The first plan (all different symbols in the different places) allows for $5! = 120$ passwords. The second plan (choosing with replacement) allows for $5^5 = 3,125$ passwords.

5. These 7 symbols have $7! = 5,040$ permutations or arrangements.

   If a is fixed as the first symbol, then there are six symbols to choose in any order for the other six places, and the number of choices is $6! = 720$.

   If a is fixed as the first symbol, and g as the fifth, then we have five symbols left to choose in any order for the remaining five places. The number of ways of doing this is $5! = 120$.

6. We want to find n so that $n! \geq 20,000$. Clearly there are infinitely many such integers n, but we would like the smallest possible n. With a little experimentation, we find that $8! = 40,320$, more than enough, while $7! = 5040$ is too small, so choose $n = 8$.

   For $n! \geq 100,000$, try some values of n. We find that $10! = 3,628,800$, more than enough, while $9! = 362,880$ is too small. The most efficient choice in this case is $n = 10$.

7. There are $12! = 479,001,600$ outcomes.

8. There are 10 letters from a through l, inclusive. If three of the places are fixed (it does not matter which three), there are seven letters available to put in any order into the remaining seven places. There are $7! = 5,040$ ways to do this.

9. We want to pick 7 objects from 25, taking order into account. The number of ways to do this is

$$_{25}P_7 = \frac{25!}{18!} = (19)(20)(21)(22)(23)(24)(25) = 2,422,728,000.$$

10. The number of ballots is

$$_{16}P_5 = \frac{16!}{11!} = 12(13)(14)(15)(16) = 524,160.$$

11. Because order is important, the number of possibilities is

$$_{22}P_6 = \frac{22!}{16!} = 53,721,360.$$

12. (a) The number of choices is

$$_{20}P_3 = \frac{20!}{17!} = 6,840.$$

   (b) If the list begins with 4, there are only two numbers to choose from the remaining nineteen numbers. There are

$$_{19}P_2 = \frac{19!}{17!} = (18)(19) = 342$$

ways to do this. This result does not depend on which number is fixed in the first position. The percentage of choices beginning with 4 is $100(342/6,840)$, or 5 percent. This is reasonable from a common sense point of view, since, with 20 number to choose from, we would expect 5 percent to begin with any particular one of the numbers.

(c) This question is really the same as the question in (b), since it does not matter which number is fixed. The answer is that 5 percent of the choices ends in 9.

(d) If two places are fixed at 3 and 15, then there are eighteen numbers left, from which we want to choose one. There are 18 such choices.

13. Without order (and, we assume, without replacement), the number of ten card hands is

$$_{52}C_{10} = \frac{52!}{10!42!} = 15,820,024,220.$$

14. Disregarding order, the number of nine man lineups that can be formed from a seventeen person roster is

$$_{17}C_9 = \frac{17!}{8!9!} = 24,310.$$

If the order makes a difference, then the number is

$$_{17}P_9 = \frac{17!}{9!} = 8!_{17}C_9 = 8,821,612,800.$$

15. The number of combinations is

$$_{20}C_4 = \frac{20!}{4!16!} = 4,845.$$

16. The number is

$$_{40}C_{12} = \frac{40!}{12!28!} = 5,586,853,480.$$

17. The number of outcomes of flipping five coins is $2^5 = 32$.

(a) The number of ways of getting exactly two heads from the five flips is $_5C_2 = 5!/(2!3!) = 10$. The probability of getting exactly two heads (or exactly two tails) is

$$\Pr(\text{exactly two heads}) = \frac{10}{32} = \frac{5}{16}.$$

(b) We get at least two heads if we get exactly two, or exactly three, or exactly four, or exactly five heads. The sum of the number of ways of doing each of these is

$$_5C_2 +_5 C_3 +_5 C_4 +_5 C_5 = 10 + 10 + 5 + 1 = 26.$$

Therefore

$$\Pr(\text{at least two heads}) = \frac{26}{32} = \frac{13}{16}.$$

18. If four dice are rolled, the number of outcomes is $6^4 = 1,296$.

(a) We need the number of ways the four dice can come up with exactly two of the dice showing 4. We get exactly two dice showing 4 by choosing any two of the four dice to come up 4, and allowing the other two dice to come up any of $1, 2, 3, 5, 6$, five possibilities for the other two. The number of ways this can happen, by the multiplication principle, is

$$(5)(5)_4C_2 = 25\frac{4!}{2!2!} = 25\frac{4!}{2!2!} = 150.$$

The probability of getting exactly two four's is

$$\Pr(\text{exactly two four's}) = \frac{150}{1296} = \frac{23}{216}.$$

(b) The number of ways of getting exactly three four's is $5({}_4C_1) = 20$, so

$$\Pr(\text{exactly three four's}) = \frac{20}{1296} = \frac{5}{324}.$$

(c) We get at least two four's if we get exactly two four's, exactly three four's, or all four tosses coming up 4. There are 150 ways of getting exactly two, and 20 ways of getting exactly three four's. Clearly in four tosses there is one way of getting four four's. Therefore

$$\Pr(\text{at least two four's}) = \frac{150 + 20 + 1}{1296} = \frac{19}{144}.$$

(d) To total 22, the dice could come up $6, 6, 6, 4$, in any order (four ways), or $6, 6, 5, 5$ in any order (six ways). Therefore

$$\Pr(\text{dice total } 22) = \frac{10}{1296} = \frac{5}{648}.$$

19. The number of ways of drawing two cards out of 52 cards, without regard to order, is ${}_{52}C_2 = 1,326$.

(a) We get two kings if we happen to get two of the four kings, and there are ${}_4C_2 = 6$ ways of doing this. Then

$$\Pr(\text{two kings are drawn}) = \frac{6}{1326} = \frac{1}{221}.$$

(b) The aces and face cards constitute 16 of the 52 cards. If none of the two cards is drawn from these sixteen cards, then the two cards are drawn from the remaining 36 cards. Disregarding order, there are ${}_{36}C_2 = 630$ ways to do this. Therefore

$$\Pr(\text{no ace or face card}) = \frac{630}{1326} = \frac{105}{221}.$$

20. The number of ways of choosing four letters from 26 letters, with order, is

$$ {}_{26}P_4 = \frac{26!}{(26-4)!} = \frac{26!}{22!} = (23)(24)(25)(26) = 14,950. $$

(a) If the first letter is set at q, then we are left to choose, with order, three letters from the remaining 25 letters. There are ${}_{25}P_3 = 2,300$ ways to do this, so

$$ \Pr(\text{first letter is q}) = \frac{2300}{14950} = \frac{2}{13}. $$

(b) Suppose a and b are two of the letters. How many ways can this occur? Imagine a string of four boxes. Pick any two, and put a and b in these boxes. There are ${}_4P_2 = 12$ ways to do this. For the other two boxes, put (keeping track of the order) any two of the remaining 24 letters. There are ${}_{24}P_2$ ways to do this. The number of ordered strings of four letters with a and b two of the letters, is

$$ ({}_4P_2)({}_{24}P_2) = \frac{4!}{2!}\frac{24!}{22!} = 564. $$

Then

$$ \Pr(a \text{ and } b \text{ were chosen}) = \frac{560}{14950} = \frac{7}{187}. $$

(c) The probability of $abdz$ occurring is 1/14950, since this string can occur in exactly one way.

21. The number of ways of choosing, without order, three of the eight bowling balls is ${}_8C_3 = 56$.

(a) For none of the balls to be defective, they had to come from the six nondefective ones. There are ${}_6C_3 = 20$ ways to do this. Then

$$ \Pr(\text{none defective}) = \frac{20}{56} = \frac{5}{14}. $$

(b) There are two ways to take one defective ball. The other two would have to be taken from the six nondefective ones, which can be done in ${}_6C_2 = 15$ ways. Then

$$ \Pr(\text{exactly one defective ball}) = \frac{30}{56} = \frac{15}{28}. $$

(c) In choosing three bowling balls, there are 3 ways of picking the two defective ones. Then there are six ways of choosing the third ball as nondefective. Therefore

$$ \Pr(\text{both defective balls are chosen}) = \frac{6(3)}{56} = \frac{9}{28}. $$

22. This problem involves tossing dice, except now each die has only four faces, numbered $1, 2, 3, 4$. The number of outcomes is $4^7 = 16,384$, since each of the seven tosses has four possibilities.

(a) There is one way all seven dice can come up 3, so

$$\Pr(\text{all come up 3}) = \frac{1}{16384}.$$

(b) There are ${}_7C_5 = 21$ ways of picking five of the dice and imagining they come up 1, and imagining the other two come up 4. Therefore

$$\Pr(\text{five ones, two fours}) = \frac{21}{16384}.$$

(c) The only ways to roll a total of 26 are to roll (in any order),

$$4, 4, 4, 4, 4, 4, 2$$

and there are seven ways to do this (seven possible locations for the 2), or to roll

$$4, 4, 4, 4, 4, 3, 3$$

and there are ${}_7C_2 = 21$ ways to do this. Therefore

$$\Pr(\text{total 26}) = \frac{7 + 21}{16384} = \frac{7}{4096}.$$

(d) The sum is at least 26 if the sum is 26, 27, or, the largest possible, 28. We already know from (c) that there are 28 ways to total exactly 26. To roll a 27, we must get (in any order),

$$4, 4, 4, 4, 4, 4, 3$$

and there are 7 ways to do this. To roll 28, we must get all four's, and there is one way to do this. Therefore

$$\Pr(\text{total at least 26}) = \frac{28 + 7 + 1}{16384} = \frac{9}{4096}.$$

23. Taking order into account, there are

$${}_{20}P_5 = \frac{20!}{15!} = (16)(17)(18)(19)(20) = 1,860,480$$

ways to choose 5 of the balls.

(a) There is only one way to choose the balls numbered $1, 2, 3, 4, 5$ in this order. the probability is

$$\Pr(\text{select } 1, 2, 3, 4, 5 \text{ in this order}) = \frac{1}{1860480}.$$

(b) We want the probability that ball number 3 was drawn (somewhere in the five drawings). We therefore need the number of ordered choices of five of the twenty numbers, that include the number 3. We can think of this as choosing, in order, four of the nineteen numbers $1, 2, 4, 5, \cdots, 18, 19, 20$, and then inserting 3 in any of the positions from the first through fifth numbers of the drawing. This will result in all ordered sequences of length five from the twenty numbers, and containing the number 3 in some position. There are therefore $5({}_{19}P_4) = 465,120$ such sequences. Therefore

$$\Pr(\text{selecting a } 3) = \frac{465120}{1860480} = \frac{1}{4}.$$

(c) We must count all the drawings (sequences) that contain at least one even number. This could mean the sequence contains one, two, three, four or five even numbers. This is a complicated counting problem if approached directly. It is easier to count the sequences that have no even number, hence are formed just from the ten even integers from 1 through 20. If this number is N_0, then the number X of sequences having an even number is

$$X = 1,860,480 - N_0.$$

Now N_0 is easy to compute, since this is the number of ordered five-term sequences of the ten odd numbers. Thus

$$N_0 =_{10} P_5 = 30,240.$$

Then X = 1860480 - 30240 = 1830240.

Then

$$\Pr(\text{an even number was drawn}) = \frac{1830240}{1860480}.$$

This is approximately 0.984, so it is very likely that an even numbered ball was drawn.

24. We need to be clear what an outcome of this experiment looks like. An outcome can be written as a string abc, with each letter representing (in some way) a chosen drawer out of the nine available drawers. The number of outcomes, without regard to order, is ${}_9C_3 = 84$.

(a) A person gets at least 1,000 dollars by picking exactly one drawer with the thousand dollar bill and two without. There are $2({}_7C_2)$ ways to do this. Or, we could pick both drawers with the thousand dollar bill and one of the others. There are 7 ways to do this. The number of ways of getting at least a thousand dollars is therefore $42 + 7$, and

$$\Pr(\text{getting at least one thousand dollars}) = \frac{49}{84} = \frac{7}{12}.$$

(b) A person cannot end up with less than one dollar. This is not an outcome in the sample space. We may also assign a probability of zero to this proposed outcome.

(c) The payoff is 1.50 exactly when the person chooses three drawers, each containing fifty cents. The number of ways this can happen is ${}_7C_3 = 35$, so

$$\Pr(1.50 \text{ payoff}) = \frac{35}{84} = \frac{5}{12}.$$

Notice that the sum of the probabilities of (a) and (c) is 1. This is because, in choosing three drawers, the person must either choose all drawers having fifty cents, or must choose at least one of the drawers having a thousand dollar bill.

25. The number of five-card hands (disregarding the order of the deal) is ${}_{52}C_5 = 2,598,960$.

(a) The number of hands containing exactly one jack and exactly one king is

$$4(4)({}_{44}C_3) = 211,904.$$

This is because there are four ways of getting one jack, four ways of getting one king, and then we choose (without order) three cards from the remaining 44 cards. Thus

$$\Pr(\text{exactly one jack and exactly one king}) = \frac{211904}{2598960}.$$

This is approximately 0.082, so this event is quite likely.

(b) The hand will contain at least two aces if it has exactly two aces, exactly three aces, or exactly four aces. The number of such hands is

$$({}_4C_2)({}_{48}C_3) + ({}_4C_3)({}_{48}C_2) + ({}_4C_4)({}_{48}C_1) = 108,336.$$

For the first term, choose two aces out of four, then three cards from the remaining forty-eight cards, and similarly for the other two terms for drawing three aces or drawing four aces. Then

$$\Pr(\text{draw at least two aces}) = \frac{108336}{2598960}.$$

This is approximately 0.042. As we might expect, a hand with at least two aces is not very likely.

26. There are 101 numbers to choose from, and

$${}_{101}C_2 = 668,324,943,343,021,950,370$$

ways to do this (without regard to order).

(a) There are twenty-one numbers in the 80 to 100 range, inclusive. The number of ways of choosing twenty of these (without order) is ${}_{21}C_{20} = 21$. Therefore

$$\Pr(\text{all numbers are larger than 79}) = \frac{21}{668324943343021950370}.$$

This is very close to zero. In the real world no sane person would bet on drawing all the twenty numbers from the eighty to one hundred range.

(b) There are

$$_{100}C_{19} = 132,341,52,939,212,267,400$$

ways to choose a five: choose one five, then nineteen numbers from the remaining one hundred numbers. Therefore

$$\Pr(\text{choose a five}) = \frac{132341572939212267400}{668324943343021950370}.$$

This is approximately 0.190. Since twenty numbers is nearly 1/5 the 101 numbers, it is not surprising that the probability of getting any particular number is close to 1/5.

Section 3 Complementary Events

1. There are many ways seven dice can come up with at least two fours (call this event E). We can count these, but it may be easier to look at the complementary event E^C, which is that fewer than two dice come up 4. This is the event that exactly one die comes up 4, or none of them do. If no 4 comes up, then each of the seven dice has five possible numbers showing, for 5^7 possibilities. If exactly one die comes up with a four, then the other dice have five possible numbers showing, and this can occur in $7(5^6)$ ways. This means that E^C has

   $$5^7 + 7(5^6) = 187,500$$

   outcomes. Then, since the total number of outcomes of seven rolls is $6^7 = 279,939$, we have

   $$\Pr(E^C) = \frac{187500}{279936}.$$

   The probability we are interested in is

   $$\Pr(E) = \Pr(\text{at least one 4}) = 1 - \frac{187500}{279936}.$$

   This is approximately 0.330.

2. In fourteen coin tosses, there are $2^{14} = 16,384$ outcomes.

   Now let E be the event that at least three of the fourteen coins come up heads. E has many outcomes in it. It may be easier to deal with E^C, which is the event that fewer than three of the fourteen coins come up heads. E^C consists of the events: no head comes up, one head comes up, or two heads come up.

Competing Species Population Models

In a competing species model, two populations compete with each other for the same resource. We will develop two models for analyzing the behavior of these populations.

A Simple Competing Species Model

A relatively simple competing species model can be obtained by assuming that an increase in either population causes a reduction in the resource for both populations, hence should contribute to a decline of both populations. The effect of this interaction is modeled as the product of the populations, leading to the system

$$x'(t) = ax(t) - bx(t)y(t),$$
$$y'(t) = ky(t) - cx(t)y(t),$$

in which $x(t)$ and $y(t)$ are the populations at time t and the coefficients are positive constants. In matrix form,

$$\mathbf{X}' = \begin{pmatrix} ax - bxy \\ ky - cxy \end{pmatrix}.$$

The critical points are $(0, 0)$ and $(k/c, a/b)$.

Example 1 We will analyze the competing species model

$$\mathbf{X}' = \begin{pmatrix} 2x - 0.3xy \\ 4y - 0.7xy \end{pmatrix}.$$

The critical points are $(0, 0)$ and $(\frac{40}{7}, \frac{20}{3})$.

First look at $(0, 0)$. The matrix of the linearized system is

$$\mathbf{A}_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

with eigenvalues $2, 4$. The origin is an unstable nodal source for this system. Figure 1 is a phase portrait showing trajectories moving out of and away from the origin.

At the other critical point, the matrix of the linearized system is

$$\mathbf{A}_{(40/7,20/3)} = \begin{pmatrix} 0 & -\frac{12}{7} \\ -\frac{14}{3} & 0 \end{pmatrix},$$

with eigenvalues $\pm 2\sqrt{2}$. This critical point is an unstable saddle point of the linearized model, hence also of the original competing species model. Figure 2 is a phase portrait of this competing species system, showing some trajectories

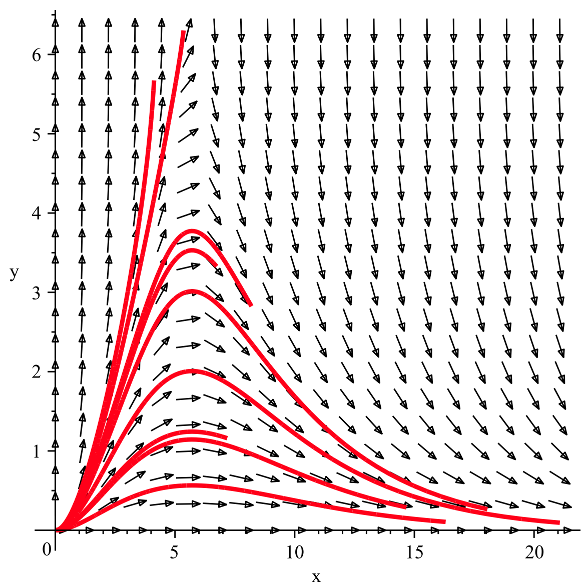

Figure 1: Trajectories near the origin in Example 1.

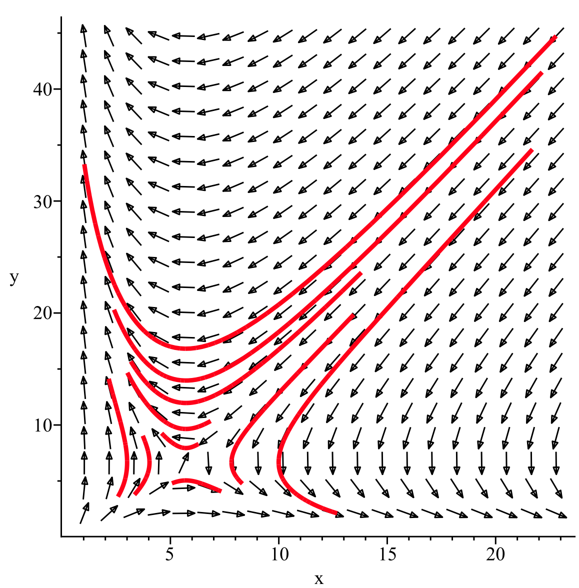

Figure 2: Trajectories near $(40/7, 20/3)$ in Example 1.

near the critical point $(\frac{40}{7}, \frac{20}{3})$. If the initial conditions are "near" this critical point, say

$$x(0) = 15, y(0) = 5$$

then the $x-$ population prospers and grows indefinitely as t increases, while the $y-$ population dies out. $\diamond$

This first competing species model can be reduced to a single differential equation in terms of x and y. For the system of Example 1, first write

$$\frac{dy}{dx} = \frac{4y - 0.7xy}{2x - 0.3xy} = \frac{y(40 - 7x)}{x(20 - 3y)},$$

and the variables separate to yield

$$\frac{20 - 3y}{y}\, dy = \frac{40 - 7x}{x}\, dx.$$

Integrate and rearrange terms to obtain the implicitly defined general solution

$$\frac{y^{20}}{x^{40}} e^{7x-3y} = k.$$

Initial conditions $x(0)$ and $y(0)$ determine k and the trajectory through this point.

In general, for this competing species model, the asymptotes of the trajectories separate the first quadrant into four regions, labeled *1, 2, 3, 4* in Figure 3. If the initial population point $(x(0), y(0))$ is in region *1* or *4*, then the $x-$ population will increase in time while the $y-$ population shrinks to zero as $t \to \infty$. If the initial population point is in region *2* or *3*, then the $y-$ population survives and the $x-$ population becomes extinct.

An Extended Competing Species Model

The competing species model just considered leaves no room for compromise or diplomacy - one species survives, the other dies. We would like a more sophisticated model allowing a greater variety of outcomes (as occur in the real world).

One way to do this is to add a term to each equation that accounts for factors within each population that might limit its growth,independent of interactions with the other population, which are already taken into account by the xy terms. Assuming that such factors are proportional to the square of the population, we have the competing species model

$$\mathbf{X}' = \begin{pmatrix} Gx - Dx^2 - Cxy, \\ gy - dy^2 - cxy \end{pmatrix}.$$

Unlike the previous model, these equations do not admit exponential growth or decay in the absence of the other population. Think of D and d as the internal growth-limiting factors, while C and c are the competition factors.

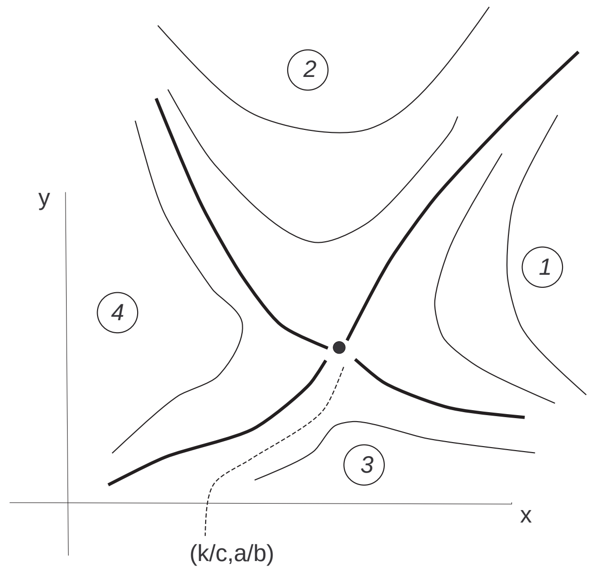

Figure 3: Phase portrait of a simple competing species model.

We will look at two examples before making a general analysis of this model.

Example 2 The model

$$\mathbf{X}' = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} \frac{3}{5}x - \frac{1}{6}x^2 - \frac{1}{4}xy \\ y - \frac{1}{4}y^2 - \frac{1}{2}xy \end{pmatrix}$$

has four critical points in the first quadrant:

$$(0,0), (0,4), (\frac{18}{5}, 0), \text{ and } (\frac{6}{5}, \frac{8}{5}).$$

For behavior of trajectories near $(0,0)$, look at

$$\mathbf{A}_{(0,0)} = \begin{pmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & 0 \\ 0 & 1 \end{pmatrix}.$$

This has eigenvalues 1 and $\frac{3}{5}$ so the origin is an unstable nodal source.

For $(0,4)$, form

$$\mathbf{A}_{(0,4)} = \begin{pmatrix} -\frac{2}{5} & 0 \\ -2 & -1 \end{pmatrix}.$$

This has eigenvalues $-\frac{2}{5}, -1$, so $(0,4)$ is an asymptotically stable nodal sink. Solutions beginning "near" $(0,4)$ tend toward $(0,4)$ as t increases. Because we are only interested in integer values of $x(t)$ and $y(t)$ as population counts, this

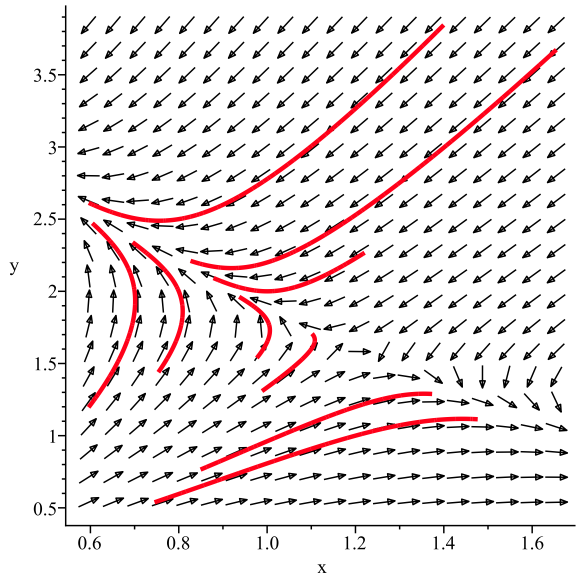

Figure 4: Phase portrait in Example 2.

is the obvious result that, if $x(0) = 0$, and $y(0) = 4$, then the populations tend toward $x = 0, y = 4$ as t increases.

For $(\frac{18}{5}, 0)$, compute

$$\mathbf{A}_{(18/5,0)} = \begin{pmatrix} -\frac{3}{5} & -\frac{9}{10} \\ 0 & -\frac{4}{5} \end{pmatrix}.$$

This has eigenvalues $-\frac{3}{5}, -\frac{4}{5}$ and this critical point is also an asymptotically stable nodal sink.

Finally, for the critical point $(\frac{6}{5}, \frac{8}{5})$, we find that

$$\mathbf{A}_{(6/5,8/5)} = \begin{pmatrix} -\frac{1}{5} & -\frac{3}{10} \\ -\frac{4}{5} & -\frac{7}{5} \end{pmatrix}$$

with eigenvalues $-\frac{4}{5}, \frac{1}{5}$. This critical point is an unstable saddle point.

Figure 4 shows a phase portrait for this system. Trajectories flow outward from the origin and, depending on where they start at time zero, flow toward the critical point $(0, 4)$ (so y survives and x becomes extinct), or toward $(\frac{18}{5}, 0)$ (so x survives and y becomes extinct). Some of these trajectories also suggest the behavior of the system near the saddle point $(\frac{6}{5}, \frac{8}{5})$. Because this point is unstable, it is possible to find initial points close to this critical point from which the $x-$ species survives and the $y-$ species does not, or from which the $y-$ species survives and the $x-$ species dies out. $\diamond$

Example 3 Contrast the outcomes of Example 2 with those of the model

$$\mathbf{X}' = \begin{pmatrix} x(3 - x - \frac{1}{4}y) \\ y(2 - \frac{1}{2}y - \frac{1}{6}x) \end{pmatrix}.$$

The critical points are

$$(0,0), (0,4), (3,0) \text{ and } (\frac{24}{11}, \frac{36}{11}).$$

Analyze each critical point as follows.

For $(0,0)$, compute

$$\mathbf{A}_{(0,0)} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},$$

with eigenvalues $3, 2$. The origin is an unstable nodal source.

For $(0,4)$,

$$\mathbf{A}_{(0,4)} = \begin{pmatrix} 2 & 0 \\ -\frac{2}{3} & -2 \end{pmatrix},$$

with eigenvalues $-2, 2$. Then $(0,4)$ is an unstable saddle point.

For $(3,0)$, we have

$$\mathbf{A}_{(3,0)} = \begin{pmatrix} -3 & -\frac{3}{4} \\ 0 & \frac{3}{2} \end{pmatrix},$$

with eigenvalues $-3, \frac{3}{2}$. Then $(3,0)$ is also an unstable saddle point.

Finally, at $(\frac{24}{11}, \frac{36}{11})$,

$$\mathbf{A}_{(24/11,36/11)} = \begin{pmatrix} -\frac{24}{11} & -\frac{6}{11} \\ -\frac{6}{11} & -\frac{18}{11} \end{pmatrix},$$

with eigenvalues $\frac{-21 \pm 3\sqrt{5}}{11}$. These numbers are both negative, so this critical point is an asymptotically stable nodal sink. Figure 5 near $(6/5, 8/5)$ shows a phase portrait for this system with some trajectories near this critical point. ◇

It is possible to carry out a general analysis for the competing species model

$$\mathbf{X}' = \begin{pmatrix} Gx - Dx^2 - Cxy, \\ gy - dy^2 - cxy \end{pmatrix}.$$

First look at the critical points from a geometric point of view. These critical points are simultaneous solutions of

$$x(G - Dx - Cy) = 0,$$
$$y(g - cx - dy) = 0,$$

Solutions for x and y are coordinates of points of intersection of pairs of lines, namely the $x-$, axis, the $y-$ axis, and the lines

$$Dx + Cy = G$$
$$cx + dy = g.$$

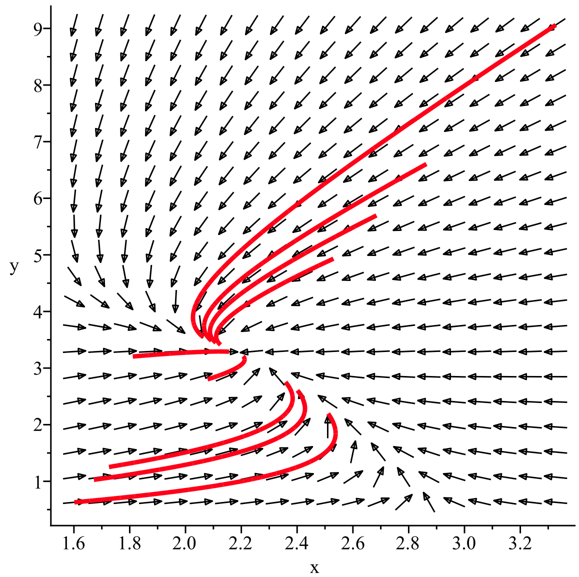

Figure 5: Phase portrait in Example 3.

Figure 6 shows the four possible relative positions of the last two lines in the first octant. The critical points are

$$(0,0), (\frac{G}{D}, 0), (0, \frac{g}{d}), \text{ and } \left(\frac{Gd - Cg}{dD - cD}, \frac{Dg - cG}{cD - cD}\right).$$

Now consider the critical points in turn.

$(0,0)$ - Write the system as

$$\mathbf{A}_{(0,0)} = \begin{pmatrix} G & 0 \\ 0 & K \end{pmatrix} \mathbf{X} + \begin{pmatrix} -Dx^2 - Cxy \\ -dy^2 - cxy \end{pmatrix}.$$

The matrix of the linear part has eigenvalues G, g, both positive. If these are unequal then the origin is an unstable node. If $G = g$ then the origin is an unstable node.

$(\frac{G}{D}, 0)$ - Now compute

$$\mathbf{A}_{(G/D,0)} = \begin{pmatrix} -G & -CG/D \\ 0 & -cG/D \end{pmatrix}$$

with eigenvalues $-g, g - \frac{cG}{D}$. Certainly $-G < 0$. If $\frac{g}{c} > \frac{G}{D}$ then the second eigenvalue is positive and this critical point is an unstable node. If $\frac{g}{c} < \frac{G}{D}$ then both eigenvalues are negative. If they are distinct, then the critical point is an asymptotically stable node. If they are equal, then the almost linear system has an asymptotically stable node or spiral point at $(\frac{G}{D}, 0)$.

$(0, \frac{g}{d})$ - Now we find that

$$\mathbf{A}_{(0,g/d)} = \begin{pmatrix} G - \frac{Cg}{d} & 0 \\ -\frac{cg}{d} & -g \end{pmatrix},$$

with eigenvalues $-g, G - \frac{Cg}{d}$. If $G/C > g/d$ then the second eigenvalue is positive and $(0, g/d)$ is an unstable node. If $G/C < g/d$ then both eigenvalues are negative. If these eigenvalues are distinct, then $(0, g/d)$ is an asymptotically stable node. If these eigenvalues are equal, then the almost linear system as an asymptotically stable node or spiral point at $(0, g/d)$.

$((Gd - Cg)/(dD - cC), (Dg - cG)/(dD - cD))$ - Denote this point $(\tilde{x}, \tilde{y})$. This is the point of intersection of the lines

$$Dx + Cy = G, cx + dy = g.$$

In the present context, look at the case that the point of intersection falls in the first quadrant (Figures 6(3) and Figure 6(4)). In Figure 6(3),

$$\frac{G}{D} > \frac{g}{c} \text{ and } \frac{g}{d} > \frac{G}{C},$$

while in Figure 6(4),

$$\frac{g}{c} > \frac{G}{D} \text{ and } \frac{G}{C} > \frac{g}{d}.$$

Now compute

$$\mathbf{A}_{(\tilde{x},\tilde{y})} = \begin{pmatrix} G - 2D\tilde{x} - C\tilde{y} & -C\tilde{x} \\ -c\tilde{y} & g - 2d\tilde{y} - cx\tilde{x} \end{pmatrix}.$$

But recall that

$$G = D\tilde{x} + C\tilde{y} \text{ and } g = c\tilde{x} + d\tilde{y}.$$

Then

$$\mathbf{A}_{(\tilde{x},\tilde{y})} = \begin{pmatrix} -D\tilde{x} & -C\tilde{x} \\ -c\tilde{y} & -d\tilde{y} \end{pmatrix},$$

with eigenvalues

$$\frac{1}{2}\left(-(D\tilde{x} + d\tilde{y}) \pm \sqrt{(D\tilde{x} + d\tilde{y})^2 - 4(Dd - Cc)\tilde{x}\tilde{y}}\right).$$

These eigenvalues can be written

$$\frac{1}{2}\left(-(D\tilde{x} + d\tilde{y}) \pm \sqrt{(D\tilde{x} - d\tilde{y})^2 + 4Cc\tilde{x}\tilde{y}}\right).$$

This formulation makes it clears that these eigenvalues are real. Two cases occur.

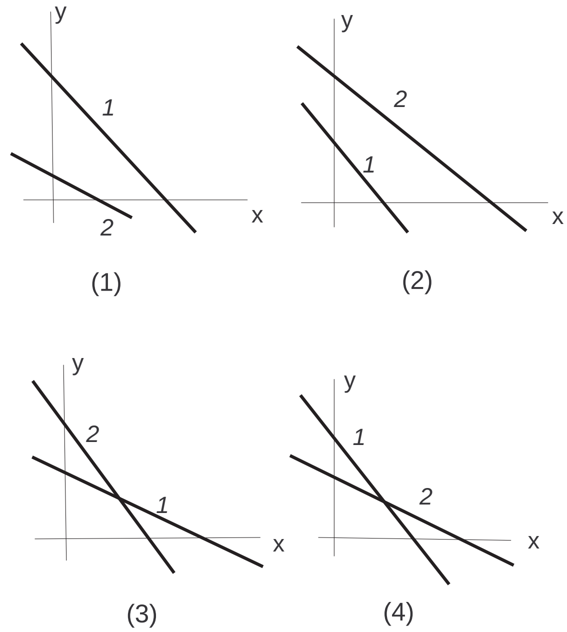

Figure 6: Relative positions of lines (*1*) $Dx + Cy = 0$ and (*2*) $cx + dy = 0$.

If $Dd = Cc < 0$, then one eigenvalue is positive and the other negative. In this case $(\tilde{x}, \tilde{y})$ is an unstable saddle point. If the population point $(x(0), y(0))$ starts near enough to this critical point, one population will die out with time and the other will survive. This case corresponds to Figure 6(3). The condition $Dd < Cd)$ can be interpreted to mean that the product of the internal limiting factors is less than the product of the competition factors. The competition factors tending to increase the populations are dominant in the model and only one population survives.

If $Dd - Cd > 0$ then both eigenvalues are negative and $(\tilde{x}, \tilde{y})$ is an asymptotically stable node (Figure 6(4)). If $(x(0), y(0)$ is sufficiently close to this node, trajectories through this initial point approach the node in the limit and both species survive (coexistence). Now $Cd < Dd$ and the competition factors are less important than internal limiting factors. With competition playing less of a role in the population, mutual survival can occur.

Problems

Each of Problems 1–6 deals with the simple competing species model treated first in this section. In each, (a) determine the critical points in the first octant and classify their type and stability properties, (b) draw a phase portrait for the system, (c) interpret the survival prospects for each species, as dictated by the model.

1. $x' = x - 4xy, y^{prime} = 3y - 6xy$
2. $x' = 3x - xy, y' = 4y - 10xy$
3. $x' = 3x - 2xy, y' = 6y - 2xy$
4. $x' = 8x - 3xy, y' = 2y - 7xy$
5. $x' = 3x - 7xy, y' = y - 4xy$
6. $x' = 4x - 9xy, y' = 5y - 2xy$

Problems 7–12 deal with the extended competing species model. In each, (a) determine the critical points in the first octant and the types and stability properties, (b) draw a phase portrait for the system, (c) interpret the survival prospects for each species, as dictated by the model.

7. $x' = 7x - 5x^2 - 2xy, y' = 3y - 4y^2 - xy$
8. $x' = 4x - 7x^2 - xy, y' = y - 2y^2 - 3xy$
9. $x' = 2x - 3x^2 - xy, y' = 3y - y^2 - 2xy$
10. $x' = x - 9x^2 - 3xy, y' = y - y^2 - xy$
11. $x' = 3x - x^2 - 2xy, y' = y - 2y^2 - xy$
12. $x' = x - 6x^2 - 3xy, y' = 5y - 2y^2 - 8xy$