# CHAPTER TWO

## Solutions for Section 2.1

**1.** **(a)** The function $N = f(t)$ is decreasing when $t = 1950$. Therefore, $f'(1950)$ is negative. That means that the number of farms in the US was decreasing in 1950.

**(b)** The function $N = f(t)$ is decreasing in 1960 as well as in 1980 but it is decreasing faster in 1960 than in 1980. Therefore, $f'(1960)$ is more negative than $f'(1980)$.

**2.** **(a)** The average rate of change is the slope of the secant line in Figure 2.1, which shows that this slope is positive.

**(b)** The instantaneous rate of change is the slope of the graph at $x = 3$, which we see from Figure 2.2 is negative.

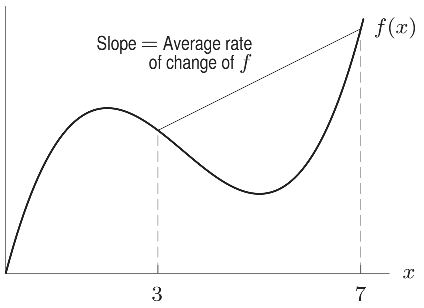



**Figure 2.1**

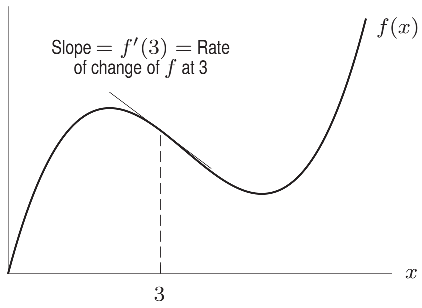



**Figure 2.2**

**3.** **(a)** For the interval $0 \le t \le 0.2$, we have

$$\begin{pmatrix}\text{Average velocity}\\ 0 \le t \le 0.2\end{pmatrix} = \frac{s(0.2) - s(0)}{0.2 - 0} = \frac{0.5}{0.2} = 2.5 \text{ ft/sec.}$$

**(b)** For the interval $0.2 \le t \le 0.4$, we have

$$\begin{pmatrix}\text{Average velocity}\\ 0.2 \le t \le 0.4\end{pmatrix} = \frac{s(0.4) - s(0.2)}{0.4 - 0.2} = \frac{1.3}{0.2} = 6.5 \text{ ft/sec.}$$

**(c)** To estimate the instantaneous velocity at $t = 0.2$, we can average the average velocities found on the left and the right of $t = 0.2$. So a reasonable estimate of the velocity at $t = 0.2$ is $\frac{1}{2}(6.5 + 2.5) = 4.5$ ft/sec.

**4.** **(a)** Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 1$ and $t = 1.1$. We have

$$\text{Average velocity} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{7.84 - 7}{0.1} = 8.4 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{7.0804 - 7}{0.01} = 8.04 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{7.008004 - 7}{0.001} = 8.004 \text{ m/sec.}$$

**(b)** We see in part (a) that as we choose a smaller and smaller interval around $t = 1$ the average velocity appears to be getting closer and closer to 8, so we estimate the instantaneous velocity at $t = 1$ to be 8 m/sec.

**5.** **(a)** The average rate of change of a function over an interval is represented graphically as the slope of the secant line to its graph over the interval. See Figure 2.3. Segment $AB$ is the secant line to the graph in the interval from $x = 0$ to $x = 3$ and segment $BC$ is the secant line to the graph in the interval from $x = 3$ to $x = 5$.

We can easily see that slope of $AB >$ slope of $BC$. Therefore, the average rate of change between $x = 0$ and $x = 3$ is greater than the average rate of change between $x = 3$ and $x = 5$.

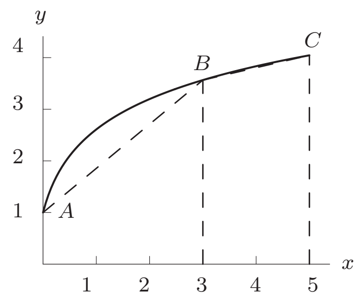


**Figure 2.3**

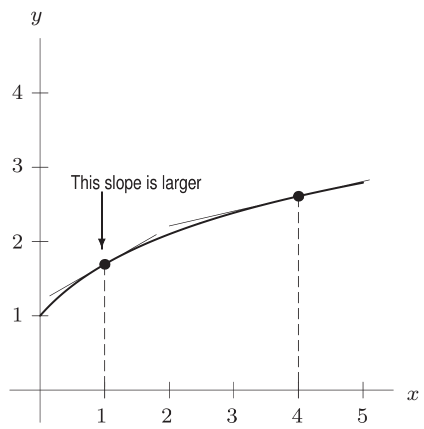


**Figure 2.4**

**(b)** We can see from the graph in Figure 2.4 that the function is increasing faster at $x = 1$ than at $x = 4$. Therefore, the instantaneous rate of change at $x = 1$ is greater than the instantaneous rate of change at $x = 4$.

**(c)** The units of rate of change are obtained by dividing units of cost by units of product: thousands of dollars/kilogram.

**6.** **(a)** The average velocity between $t = 2$ and $t = 5$ is

$$\frac{\text{Distance}}{\text{Time}} = \frac{s(5) - s(2)}{5 - 2} = \frac{25 - 4}{3} = \frac{21}{3} = 7 \text{ ft/sec.}$$

**(b)** Using an interval of size 0.1, we have

$$\left(\begin{array}{c}\text{Instantaneous velocity} \\ \text{at } t = 2\end{array}\right) \approx \frac{s(2.1) - s(2)}{2.1 - 2} = \frac{4.41 - 4}{0.1} = 4.1.$$

Using an interval of size 0.01, we have

$$\left(\begin{array}{c}\text{Instantaneous velocity} \\ \text{at } t = 2\end{array}\right) \approx \frac{s(2.01) - s(2)}{2.01 - 2} = \frac{4.0401 - 4}{0.01} = 4.01.$$

From this we guess that the instantaneous velocity at $t = 2$ is about 4 ft/sec.

**7.** **(a)** We have

$$\text{Rate of change of population} = \frac{6.89 - 4.08}{2010 - 1975} = 0.080 \text{ billion people per year.}$$

**(b)** To estimate $f'(2010)$, we use the rate of change formula on an interval containing 2010:

$$f'(2010) \approx \frac{6.89 - 6.45}{2010 - 2005} = 0.088 \text{ billion people per year.}$$

**8.** **(a)** The size of the tumor when $t = 0$ months is $S(0) = 2^0 = 1$ cubic millimeter. The size of the tumor when $t = 6$ months is $S(6) = 2^6 = 64$ cubic millimeters. The total change in the size of the tumor is $S(6) - S(0) = 64 - 1 = 63$ mm$^3$.

**(b)** The average rate of change in the size of the tumor during the first six months is:

$$\text{Average rate of change} = \frac{S(6) - S(0)}{6 - 0} = \frac{64 - 1}{6} = \frac{63}{6} = 10.5 \text{ cubic millimeters/month.}$$

**(c)** We will consider intervals to the right of $t = 6$:

| $t$ (months) | 6 | 6.001 | 6.01 | 6.1 |
|---|---|---|---|---|
| $S$ (cubic millimeters) | 64 | 64.0444 | 64.4452 | 68.5935 |

$$\text{Average rate of change} = \frac{68.5935 - 64}{6.1 - 6} = \frac{4.5935}{0.1} = 45.935$$
$$\text{Average rate of change} = \frac{64.4452 - 64}{6.01 - 6} = \frac{0.4452}{0.01} = 44.52$$
$$\text{Average rate of change} = \frac{64.0444 - 64}{6.001 - 6} = \frac{0.0444}{0.001} = 44.4$$

We can continue taking smaller intervals but the value of the average rate will not change much. Therefore, we can say that a good estimate of the growing rate of the tumor at $t = 6$ months is about 44.4 cubic millimeters/month.

**9.** We use the interval $x = 1$ to $x = 1.01$:

$$g'(1) \approx \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{4^{1.01} - 4^1}{0.01} = \frac{4.05583 - 4}{0.01} = 5.583.$$

For greater accuracy, we can use the smaller interval $x = 1$ to $x = 1.001$:

$$g'(1) \approx \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{4^{1.001} - 4^1}{0.001} = \frac{4.005549 - 4}{0.001} = 5.549.$$

**10. (a)** Figure 2.5 shows that for $t = 2$ the function $g(t) = (0.8)^t$ is decreasing. Therefore, $g'(2)$ is negative.

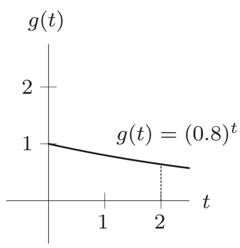



**Figure 2.5**

**(b)** To estimate $g'(2)$ we take the small interval between $t = 2$ and $t = 2.001$ to the right of $t = 2$.

$$g'(2) \approx \frac{g(2.001) - g(2)}{2.001 - 2} = \frac{0.8^{2.001} - 0.8^2}{0.001} = \frac{0.6399 - 0.64}{0.001} = \frac{-0.0001}{0.001} = -0.1$$

**11.** The slope is positive at $A$ and $D$; negative at $C$ and $F$. The slope is most positive at $A$; most negative at $F$.

**12.**

| Slope | $-3$ | $-1$ | 0 | 1/2 | 1 | 2 |
|---|---|---|---|---|---|---|
| Point | $F$ | $C$ | $E$ | $A$ | $B$ | $D$ |

**13.** Using the interval $1 \leq x \leq 1.001$, we estimate

$$f'(1) \approx \frac{f(1.001) - f(1)}{0.001} = \frac{3.0033 - 3.0000}{0.001} = 3.3$$

The graph of $f(x) = 3^x$ is concave up so we expect our estimate to be greater than $f'(1)$.

**14. (a)** Since the values of $P$ go up as $t$ goes from 2 to 4 to 6, we see that $f'(4)$ appears to be positive. The percent of households with cable television is increasing at $t = 4$.

**(b)** We estimate $f'(2)$ using the difference quotient for the interval to the right of $t = 2$, as follows:

$$f'(2) \approx \frac{\Delta P}{\Delta t} = \frac{66.732 - 66.25}{4-2} = \frac{0.48}{2} = 0.24.$$

The fact that $f'(2) = 0.24$ tells us that the percent of households with cable television in the United States was increasing at a rate of $0.24$ million per year when $t = 2$ (that means 2000).

Similarly:

$$f'(10) \approx \frac{\Delta P}{\Delta t} = \frac{60.958 - 64.874}{12 - 10} = \frac{-3.916}{2} = -1.958.$$

The fact that $f'(10) = -1.958$ tells us that the percent of households in the United States with cable television was decreasing at a rate of $1.958$ million per year when $t = 10$ (that means 2008).

**15. (a)** Since $75.2$ percent live in the city in 1990 and $35.1$ percent in 1890, we have

$$\text{Average rate of change} = \frac{75.2 - 35.1}{1990 - 1890} = \frac{40.1}{100} = 0.401.$$

Thus the average rate of change is $0.401$ percent/year.

**(b)** By looking at the population in 1990 and 2000 we see that

$$\text{Average rate of change} = \frac{79.0 - 75.2}{2000 - 1990} = \frac{3.8}{10} = 0.38$$

which gives an average rate of change of $0.38$ percent/year between 1990 and 2000. Alternatively, we look at the population in 1980 and 1990 and see that

$$\text{Average rate of change} = \frac{75.2 - 73.7}{1990 - 1980} = \frac{1.5}{10} = 0.15$$

giving an average rate of change of $0.15$ percent/year between 1980 and 1990. We see that the rate of change in the year 1990 is somewhere between $0.38$ and $0.15$ percent/year. A good estimate is $(0.38 + 0.15)/2 = 0.265$ percent per year. In fact, the definition of urban area was changed for the 2000 data, so this estimate should be used with care.

**(c)** By looking at the population in 1830 and 1860 we see that

$$\text{Average rate of change} = \frac{19.8 - 9.0}{1860 - 1830} = \frac{10.8}{30} = 0.36,$$

giving an average rate of change of $0.36$ percent/year between 1830 and 1860. Alternatively we can look at the population in 1800 and 1830 and see that

$$\text{Average rate of change} = \frac{9.0 - 6.0}{1830 - 1800} = \frac{3.0}{30} = 0.10.$$

giving an average rate of change of $0.10$ percent/year between 1800 and 1830 . We see that the rate of change at the year 1830 is somewhere between $0.10$ and $0.36$. This tells us that in the year 1830 the percent of the population in urban areas is changing by a rate somewhere between $0.10$ percent/year and $0.36$ percent/year.

**16.**

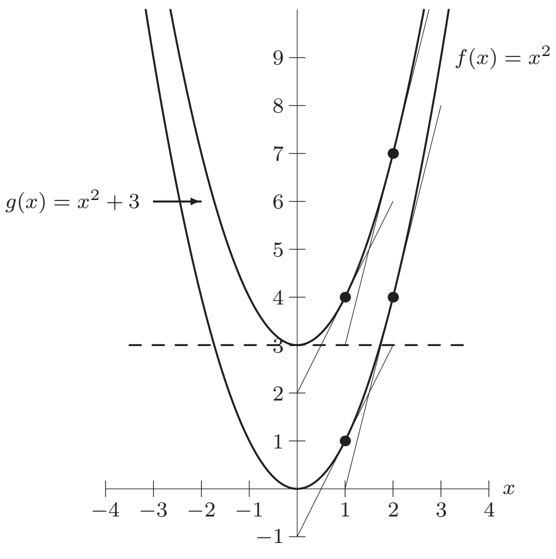



Figure 2.6

**(a)** The tangent line to the graph of $f(x) = x^2$ at $x = 0$ coincides with the $x$-axis and therefore is horizontal (slope $= 0$). The tangent line to the graph of $g(x) = x^2 + 3$ at $x = 0$ is the dashed line indicated in the figure and it also has a slope equal to zero. Therefore both tangent lines at $x = 0$ are parallel.

We see in Figure 2.6 that the tangent lines at $x = 1$ appear parallel, and the tangent lines at $x = 2$ appear parallel. The slopes of the tangent lines at any value $x = a$ will be equal.

**(b)** Adding a constant shifts the graph vertically, but does not change the slope of the curve.

**17.** The coordinates of $A$ are $(4, 25)$. See Figure 2.7. The coordinates of $B$ and $C$ are obtained using the slope of the tangent line. Since $f'(4) = 1.5$, the slope is 1.5

From $A$ to $B$, $\Delta x = 0.2$, so $\Delta y = 1.5(0.2) = 0.3$. Thus, at $C$ we have $y = 25 + 0.3 = 25.3$. The coordinates of $B$ are $(4.2, 25.3)$.

From $A$ to $C$, $\Delta x = -0.1$, so $\Delta y = 1.5(-0.1) = -0.15$. Thus, at $C$ we have $y = 25 - 0.15 = 24.85$. The coordinates of $C$ are $(3.9, 24.85)$.

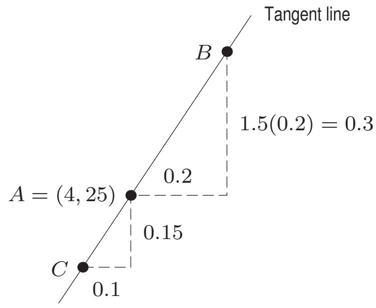



**Figure 2.7**

**18.** **(a)** Since the point $A = (7, 3)$ is on the graph of $f$, we have $f(7) = 3$.

**(b)** The slope of the tangent line touching the curve at $x = 7$ is given by

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{3.8 - 3}{7.2 - 7} = \frac{0.8}{0.2} = 4.$$

Thus, $f'(7) = 4$.

**19.** The answers to parts (a)–(d) are shown in Figure 2.8.

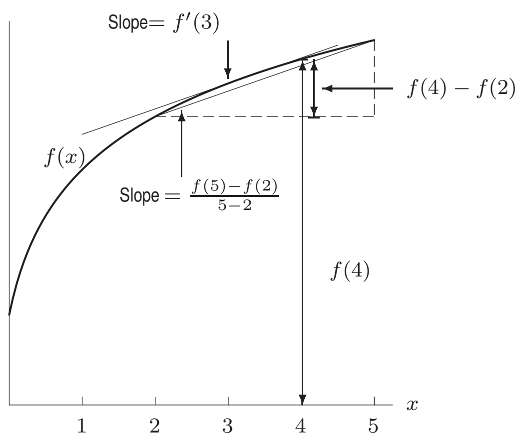



**Figure 2.8**

**20.** **(a)** Since $f$ is increasing, $f(4) > f(3)$.
**(b)** From Figure 2.9, it appears that $f(2) - f(1) > f(3) - f(2)$.
**(c)** The quantity $\dfrac{f(2) - f(1)}{2 - 1}$ represents the slope of the secant line connecting the points on the graph at $x = 1$ and $x = 2$. This is greater than the slope of the secant line connecting the points at $x = 1$ and $x = 3$ which is $\dfrac{f(3) - f(1)}{3 - 1}$.
**(d)** The function is steeper at $x = 1$ than at $x = 4$ so $f'(1) > f'(4)$.

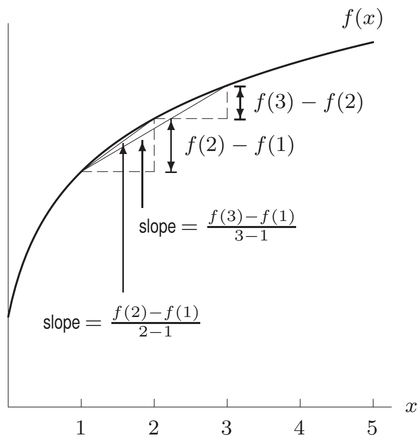



**Figure 2.9**

**21.** Using a difference quotient with $h = 0.001$, say, we find

$$f'(1) \approx \frac{1.001\ln(1.001) - 1\ln(1)}{1.001 - 1} = 1.0005$$
$$f'(2) \approx \frac{2.001\ln(2.001) - 2\ln(2)}{2.001 - 2} = 1.6934$$

The fact that $f'$ is larger at $x = 2$ than at $x = 1$ suggests that $f$ is concave up between $x = 1$ and $x = 2$.

**22.** The quantity $f(0)$ represents the population on October 17, 2006, so $f(0) = 300$ million.
The quantity $f'(0)$ represents the rate of change of the population (in millions per year). Since

$$\frac{1 \text{ person}}{11 \text{ seconds}} = \frac{1/10^6 \text{ million people}}{11/(60 \cdot 60 \cdot 24 \cdot 365) \text{ years}} = 2.867 \text{ million people/year},$$

so we have $f'(0) = 2.867$.

**23.** **(a)** $f'(t)$ is negative or zero, because the average number of hours worked a week has been decreasing or constant over time.
$g'(t)$ is positive, because hourly wage has been increasing.
$h'(t)$ is positive, because average weekly earnings has been increasing.
**(b)** We use a difference quotient to the right for our estimates.
(i)

$$f'(1970) \approx \frac{36.0 - 37.0}{1975 - 1970} = -0.2 \quad \text{hours/year}$$
$$f'(1995) \approx \frac{34.3 - 34.3}{2000 - 1995} = 0 \quad \text{hours/year.}$$

In 1970, the average number of hours worked by a production worker in a week was decreasing at the rate of 0.2 hours per year. In 1995, the number of hours was not changing.

(ii)

$$g'(1970) \approx \frac{4.73 - 3.40}{1975 - 1970} = \$0.27 \quad \text{per year}$$

$$g'(1995) \approx \frac{14.00 - 11.64}{2000 - 1995} = \$0.47 \quad \text{per year.}$$

In 1970, the hourly wage was increasing at a rate of $0.27 per year. In 1995, the hourly wage was increasing at a rate of $0.47 per year.

(iii)

$$h'(1970) \approx \frac{170.28 - 125.80}{1975 - 1970} = \$8.90 \quad \text{per year}$$

$$h'(1995) \approx \frac{480.41 - 399.53}{2000 - 1995} = \$16.18 \quad \text{per year.}$$

In 1970, average weekly earnings were increasing at a rate of $8.90 a year. In 1995, weekly earnings were increasing at a rate of $16.18 a year.

## Solutions for Section 2.2

**1.** The graph is that of the line $y = -2x + 2$. The slope, and hence the derivative, is $-2$. See Figure 2.10.

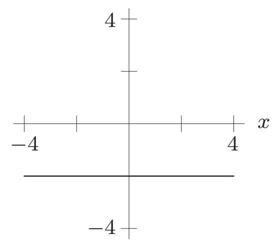



Figure 2.10

**2.** See Figure 2.11.

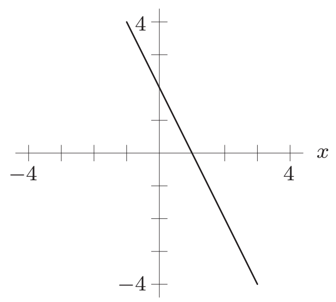



Figure 2.11

**3.** See Figure 2.12.

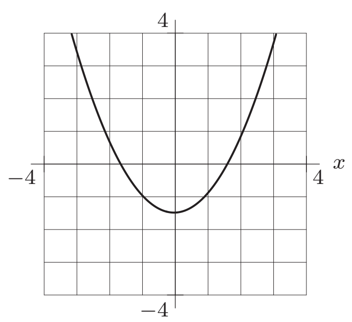



Figure 2.12

**4.** The slope of this curve is approximately $-1$ at $x = -4$ and at $x = 4$, approximately $0$ at $x = -2.5$ and $x = 1.5$, and approximately $1$ at $x = 0$. See Figure 2.13.

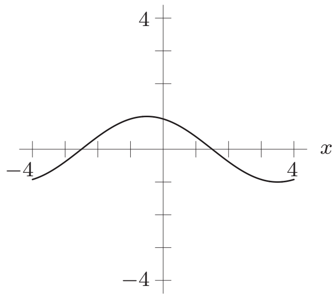



Figure 2.13

**5.** See Figure 2.14.

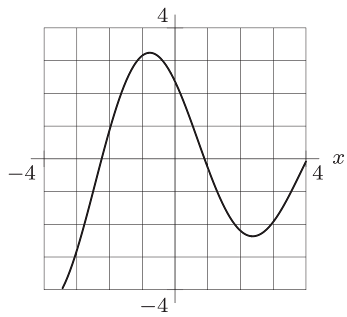



Figure 2.14

**6.** See Figure 2.15.

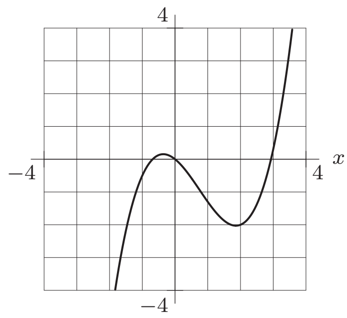



Figure 2.15

**7.** Estimating the slope of the lines in Figure 2.16, we find that $f'(-2) \approx 1.0$, $f'(-1) \approx 0.3$, $f'(0) \approx -0.5$, and $f'(2) \approx -1$.

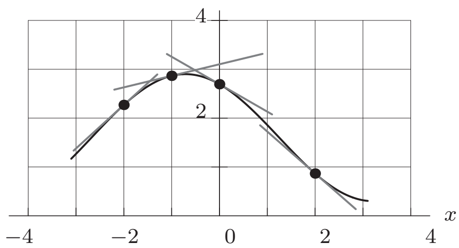



Figure 2.16

**8.** We visualize tangent lines on the graph at the given points, and estimate the slopes of the tangent lines. We find that $f'(1) \approx -2$, $f'(2) \approx -1$, $f'(3) \approx 0$, $f'(4) \approx 1$, and $f'(2) \approx 2$.

**9.** (a) $x_3$ (b) $x_4$ (c) $x_5$ (d) $x_3$

**10.** For $x = 0$, 5, 10, and 15, we use the interval to the right to estimate the derivative. For $x = 20$, we use the interval to the left. For $x = 0$, we have

$$f'(0) \approx \frac{f(5) - f(0)}{5 - 0} = \frac{70 - 100}{5 - 0} = \frac{-30}{5} = -6.$$

Similarly, we find the other estimates in Table 2.1.

**Table 2.1**

| $x$ | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| $f'(x)$ | $-6$ | $-3$ | $-1.8$ | $-1.2$ | $-1.2$ |

**11.** The value of $R'(0)$ is the derivative of the function $R(x) = 100(1.1)^x$ at $x = 0$. This is the same as the rate of change of $R(x)$ at $x = 0$. We estimate this by computing the average rate of change over intervals near $x = 0$.

If we use the intervals $-0.001 \leq x \leq 0$ and $0 \leq x \leq 0.001$, we see that:

$$\begin{pmatrix}\text{Average rate of change} \\ \text{on } -0.001 \leq x \leq 0\end{pmatrix} = \frac{100(1.1)^0 - 100(1.1)^{-0.001}}{0 - (-0.001)} = \frac{100 - 99.990469}{0.001} = 9.531,$$

$$\begin{pmatrix}\text{Average rate of change} \\ \text{on } 0 \leq x \leq 0.001\end{pmatrix} = \frac{100(1.1)^{0.001} - 100(1.1)^0}{0.001 - 0} = \frac{100.009531 - 100}{0.001} = 9.531.$$

It appears that the rate of change of $R(x)$ at $x = 0$ is 9.531, so we estimate $R'(0) = 9.531$.

**12.** See Figure 2.17.


Figure 2.17

**13.** This is a line with slope 1, so the derivative is the constant function $f'(x) = 1$. The graph is the horizontal line $y = 1$. See Figure 2.18.

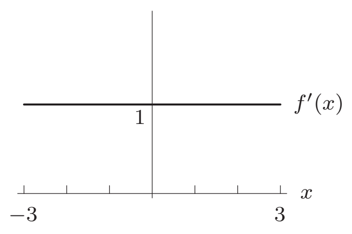



Figure 2.18

**14.** See Figure 2.19.

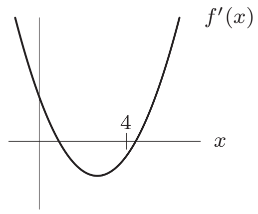



**Figure 2.19**

**15.** See Figure 2.20.

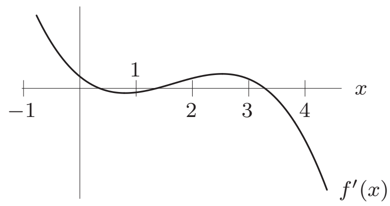



**Figure 2.20**

**16.** See Figure 2.21.

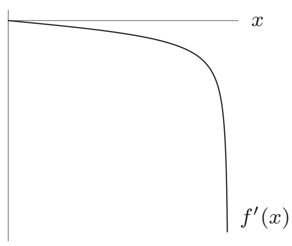



**Figure 2.21**

**17.** See Figure 2.22.

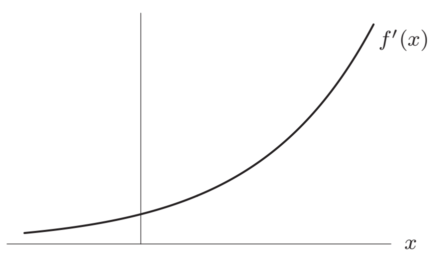



**Figure 2.22**

**18.** The function is decreasing for $x < -2$ and $x > 2$, and increasing for $-2 < x < 2$. The matching derivative must be negative (below the $x$-axis) for $x < -2$ and $x > 2$, positive (above the $x$-axis) for $-2 < x < 2$, and zero (on the $x$-axis) for $x = -2$ and $x = 2$. The matching derivative is in graph VIII.

**19.** The function is a line with negative slope, so $f'(x)$ is a negative constant, and the graph of $f'(x)$ is a horizontal line below the $x$-axis. The matching derivative is in graph IV.

**20.** The function is increasing for $x < 2$ and decreasing for $x > 2$. The corresponding derivative is positive (above the $x$-axis) for $x < 2$, negative (below the $x$-axis) for $x > 2$, and zero at $x = 2$. The matching derivative is in graph II.

**21.** The function is increasing for $x < -2$ and decreasing for $x > -2$. The corresponding derivative is positive (above the $x$-axis) for $x < -2$, negative (below the $x$-axis) for $x > -2$, and zero at $x = -2$. The derivatives in graphs VI and VII both satisfy these requirements. To decide which is correct, consider what happens as $x$ gets large. The graph of $f(x)$ approaches an asymptote, gets more and more horizontal, and the slope gets closer and closer to zero. The derivative in graph VI meets this requirement and is the correct answer.

**22.** The graph is increasing for $0 < t < 15$ and is decreasing for $15 < t < 30$. One possible graph is shown in Figure 2.23. The units on the horizontal axis are years and the units on the vertical axis are people.

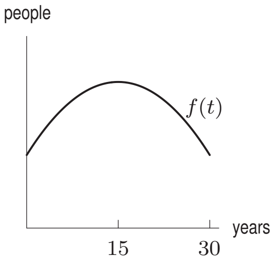



**Figure 2.23**

The derivative is positive for $0 < t < 15$ and negative for $15 < t < 30$. Two possible graphs are shown in Figure 2.24. The units on the horizontal axes are years and the units on the vertical axes are people per year.

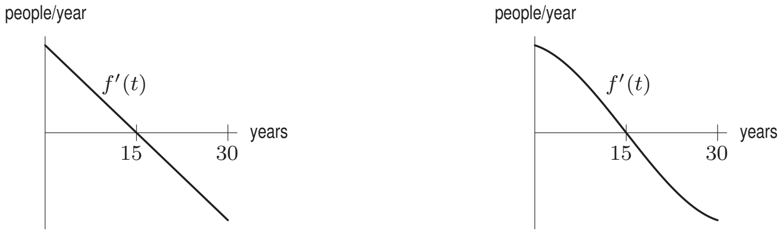



**Figure 2.24**

**23.** The value of $g(x)$ is increasing at a decreasing rate for $2.7 < x < 4.2$ and increasing at an increasing rate for $x > 4.2$.

$$\frac{\Delta y}{\Delta x} = \frac{7.4 - 6.0}{5.2 - 4.7} = 2.8 \qquad \text{between } x = 4.7 \text{ and } x = 5.2$$
$$\frac{\Delta y}{\Delta x} = \frac{9.0 - 7.4}{5.7 - 5.2} = 3.2 \qquad \text{between } x = 5.2 \text{ and } x = 5.7$$

Thus $g'(x)$ should be close to 3 near $x = 5.2$.

**24.** Since $f'(x) > 0$ for $x < -1$, $f(x)$ is increasing on this interval.
Since $f'(x) < 0$ for $x > -1$, $f(x)$ is decreasing on this interval.
Since $f'(x) = 0$ at $x = -1$, the tangent to $f(x)$ is horizontal at $x = -1$.
One possible shape for $y = f(x)$ is shown in Figure 2.25.

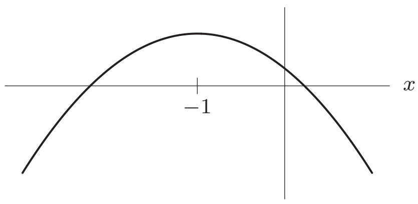



**Figure 2.25**

**25.** Since $f'(x) > 0$ for $1 < x < 3$, we see that $f(x)$ is increasing on this interval.
Since $f'(x) < 0$ for $x < 1$ and for $x > 3$, we see that $f(x)$ is decreasing on these intervals.
Since $f'(x) = 0$ for $x = 1$ and $x = 3$, the tangent to $f(x)$ will be horizontal at these $x$'s.
One of many possible shapes of $y = f(x)$ is shown in Figure 2.26.

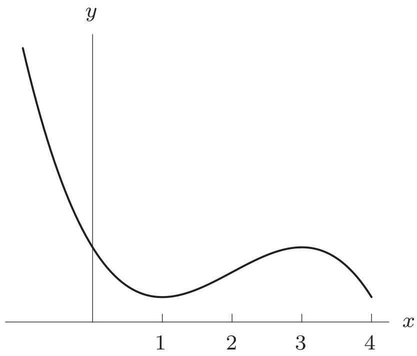



**Figure 2.26**

**26.** **(a)** Graph II
**(b)** Graph I
**(c)** Graph III

**27.** **(a)** $f'(1) \approx \dfrac{f(1.1) - f(1)}{0.1} = \dfrac{\ln(1.1) - \ln(1)}{0.1} \approx 0.95.$

$f'(2) \approx \dfrac{f(2.1) - f(2)}{0.1} = \dfrac{\ln(2.1) - \ln(2)}{0.1} \approx 0.49.$

$f'(3) \approx \dfrac{f(3.1) - f(3)}{0.1} = \dfrac{\ln(3.1) - \ln(3)}{0.1} \approx 0.33.$

$f'(4) \approx \dfrac{f(4.1) - f(4)}{0.1} = \dfrac{\ln(4.1) - \ln(4)}{0.1} \approx 0.25.$

$f'(5) \approx \dfrac{f(5.1) - f(5)}{0.1} = \dfrac{\ln(5.1) - \ln(5)}{0.1} \approx 0.20.$

**(b)** It looks like the derivative of $\ln(x)$ is $1/x$.

**28.**

**Table 2.2**

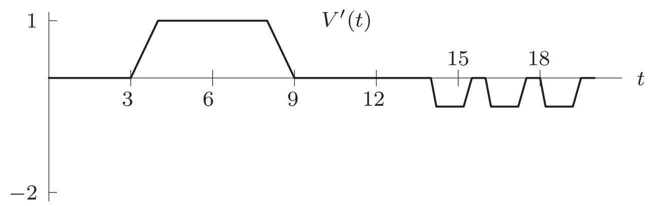
| $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|---|---|
| 1.998 | 2.6587 | 2.998 | 8.9820 | 3.998 | 21.3013 |
| 1.999 | 2.6627 | 2.999 | 8.9910 | 3.999 | 21.3173 |
| 2.000 | 2.6667 | 3.000 | 9.0000 | 4.000 | 21.3333 |
| 2.001 | 2.6707 | 3.001 | 9.0090 | 4.001 | 21.3493 |
| 2.002 | 2.6747 | 3.002 | 9.0180 | 4.002 | 21.3653 |

Near 2, the values of $f(x)$ seem to be increasing by 0.004 for each increase of 0.001 in $x$, so the derivative appears to be $\frac{0.004}{0.001} = 4$. Near 3, the values of $f(x)$ are increasing by 0.009 for each step of 0.001, so the derivative appears to be 9. Near 4, $f(x)$ increases by 0.016 for each step of 0.001, so the derivative appears to be 16. The pattern seems to be, then, that at a point $x$, the derivative of $f(x) = \frac{1}{3}x^3$ is $f'(x) = x^2$.

**29.** **(a)** The only graph in which the slope is 1 for all $x$ is Graph (III).
**(b)** The only graph in which the slope is positive for all $x$ is Graph (III).
**(c)** Graphs where the slope is 1 at $x = 2$ are Graphs (III) and (IV).
**(d)** Graphs where the slope is 2 at $x = 1$ are Graphs (II) and (IV).

**30.** **(a)** $t = 3$
**(b)** $t = 9$
**(c)** $t = 14$

**(d)**




## Solutions for Section 2.3

**1.** In Leibniz notation the derivative is $dD/dt$ and the units are feet per minute.

**2.** In Leibniz notation the derivative is $dC/dW$ and the units are dollars per pound.

**3.** In Leibniz notation the derivative is $dN/dD$ and the units are gallons per mile.

**4.** In Leibniz notation the derivative is $dP/dH$ and the units are dollars per hour.

**5.** **(a)** The statement $f(5) = 18$ means that when 5 milliliters of catalyst are present, the reaction will take 18 minutes. Thus, the units for 5 are ml while the units for 18 are minutes.

**(b)** As in part (a), 5 is measured in ml. Since $f'$ tells how fast $T$ changes per unit $a$, we have $f'$ measured in minutes/ml. If the amount of catalyst increases by 1 ml (from 5 to 6 ml), the reaction time decreases by about 3 minutes.

**6.** **(a)** If the price is \$150, then 2000 items will be sold.

**(b)** If the price goes up from \$150 by \$1 per item, about 25 fewer items will be sold. Equivalently, if the price is decreased from \$150 by \$1 per item, about 25 more items will be sold.

**7.** **(a)** The 12 represents the weight of the chemical; therefore, its units are pounds. The 5 represents the cost of the chemical; therefore, its units are dollars. The statement $f(12) = 5$ means that when the weight of the chemical is 12 pounds, the cost is 5 dollars.

**(b)** We expect the derivative to be positive since we expect the cost of the chemical to increase when the weight bought increases.

**(c)** Again, 12 is the weight of the chemical in pounds. The units of the $0.4$ are dollars/pound since it is the rate of change of the cost as a function of the weight of the chemical bought. The statement $f'(12) = 0.4$ means that the cost is increasing at a rate of $0.4$ dollars per pound when the weight is 12 pounds, or that an additional pound will cost about an extra $40$ cents.

**8.** We use the interval from $t = 6$ to $t = 8$ to estimate $f'(6)$,

$$f'(6) \approx \frac{f(8) - f(6)}{8 - 6} = \frac{200 - 150}{2} = 25.$$

Thus, we have $f'(6) \approx 25$ megawatts per year. In 1996, the world solar energy output was increasing at a rate of about 25 megawatts per year.

**9.** **(a)** The units of compliance are units of volume per units of pressure, or liters per centimeter of water.

**(b)** The increase in volume for a 5 cm reduction in pressure is largest between 10 and 15 cm. Thus, the compliance appears maximum between 10 and 15 cm of pressure reduction. The derivative is given by the slope, so

$$\text{Compliance} \approx \frac{0.70 - 0.49}{15 - 10} = 0.042 \text{ liters per centimeter.}$$

**(c)** When the lung is nearly full, it cannot expand much more to accommodate more air.

**10.** Let $p$ be the rating points earned by the CBS Evening News, let $R$ be the revenue earned in millions of dollars, and let $R = f(p)$. When $p = 4.3$,

$$\text{Rate of change of revenue} \approx \frac{\$5.5 \text{ million}}{0.1 \text{ point}} = 55 \text{ million dollars/point.}$$

Thus

$$f'(4.3) \approx 55.$$

**11.** **(a)** The yam is cooling off so $T$ is decreasing and $f'(t)$ is negative.
**(b)** Since $f(t)$ is measured in degrees Fahrenheit and $t$ is measured in minutes, $df/dt$ must be measured in units of $^\circ$F/min.

**12.** The units of $f'(x)$ are feet/mile. The derivative, $f'(x)$, represents the rate of change of elevation with distance from the source, so if the river is flowing downhill everywhere, the elevation is always decreasing and $f'(x)$ is always negative. (In fact, there may be some stretches where the elevation is more or less constant, so $f'(x) = 0$.)

**13.** **(a)** The units of lapse rate are the same as for the derivative $dT/dz$, namely (units of $T$)/(units of $z$) = $^\circ C/\text{km}$.
**(b)** Since the lapse rate is 6.5, the derivative of $T$ with respect to $z$ is $dT/dz = -6.5^\circ C/\text{km}$. The air temperature drops about $6.5^\circ$ for one more kilometer you go up.

**14.** **(a)** This means that investing the \$1000 at 5% would yield \$1649 after 10 years.
**(b)** Writing $g'(r)$ as $dB/dr$, we see that the units of $dB/dr$ are dollars per percent (interest). We can interpret $dB$ as the extra money earned if interest rate is increased by $dr$ percent. Therefore $g'(5) = \frac{dB}{dr}|_{r=5} \approx 165$ means that the balance, at 5% interest, would increase by about \$165 if the interest rate were increased by 1%. In other words, $g(6) \approx g(5) + 165 = 1649 + 165 = 1814$.

**15.** **(a)** The derivative $f'(t)$ appears to be negative for most of the period 2005-2009, because according to the table, gold production is decreasing. There appears to have been a recovery during 2008–2009 because production has increased so $f'(t)$ appears to be positive during this period.
**(b)** The derivative (or rate of change) appears to be greatest between 2008 and 2009.
**(c)** We have

$$f'(2009) \approx \frac{2450 - 2290}{2009 - 2008} = 160 \text{ metric tons/year.}$$

In 2009, gold production was increasing at a rate of approximately 160 metric tons per year.
**(d)** In 2009, gold production was 2450 metric tons and was increasing at a rate of 160 metric tons each year. Therefore, in 2010 (one year later), we have

$$f(2010) \approx 2450 + 160 = 2610 \text{ metric tons,}$$

and in 2015 (6 years later), we have

$$f(2015) \approx 2450 + 160(6) = 3410 \text{ metric tons.}$$

We estimate that gold production in 2010 is 2450 metric tons and gold production in 2015 is 3410 metric tons.

**16.** **(a)** Since $W = f(c)$ where $W$ is weight in pounds and $c$ is the number of Calories consumed per day:

| | | |
|---|---|---|
| $f(1800) = 155$ | means that | consuming 1800 Calories per day results in a weight of 155 pounds. |
| $f'(2000) = 0$ | means that | consuming 2000 Calories per day causes neither weight gain nor loss. |

**(b)** The units of $dW/dc$ are pounds/(Calories/day).

**17.** **(a)** The statement $f(200) = 1300$ means that it costs \$1300 to produce 200 gallons of the chemical.
**(b)** The statement $f'(200) = 6$ means that when the number of gallons produced is 200, costs are increasing at a rate of \$6 per gallon. In other words, it costs about \$6 to produce the next (the $201^{\text{st}}$) gallon of the chemical.

**18.** **(a)** An additional dollar per year of government purchases increases national output for the year by about \$0.60. The derivative is called a fiscal policy multiplier because if government purchases increase by $x$ dollars per year, then national output increases by about $0.60x$ dollars per year.
**(b)** An additional tax dollar collected per year decreases national output for the year by about \$0.26. The derivative is called a fiscal policy multiplier because if government tax revenues increase by $x$ dollars per year, then national output decreases by about $0.26x$ dollars per year.

**19.** **(a)** Positive, since weight increases as the child gets older.
**(b)** $f(8) = 45$ tells us that when the child is 8 years old, the child weighs 45 pounds.
**(c)** The units of $f'(a)$ are lbs/year. The value of $f'(a)$ gives approximate increase in weight for a 1 year increase in age.
**(d)** $f'(8) = 4$ tells us that an 8-year-old child weighs about 4 more pounds after the next year.
**(e)** As $a$ increases, $f'(a)$ will decrease since the rate of growth slows down as the child grows up.

**20.** Let $f(t)$ be the age of onset of Alzheimer's, in weeks, of a typical man who retires at age $t$ years. The derivative $f'(t)$ has units weeks/year. The study reports that each additional year of employment is associated with about a six week later age of onset, that is, $f'(t) \approx 6$.

**21.** **(a)** Since $f'(c)$ is negative, the function $P = f(c)$ is decreasing: on average, pelican eggshells are thinner if the PCB concentration, $c$, in the eggshell is higher.

**(b)** The statement $f(200) = 0.28$ means that the thickness of pelican eggshells is 0.28 mm when the concentration of PCBs in the eggshell is 200 parts per million (ppm).

The statement $f'(200) = -0.0005$ means that when the PCB concentration is 200 ppm, a 1 ppm increase in the concentration typically corresponds to about a 0.0005 mm decrease in eggshell thickness.

**22.** Since we do not have information beyond $t = 25$, we will assume that the function will continue to change at the same rate. Therefore,

$$\begin{aligned} f(26) &\approx f(25) + f'(25) \\ &= 3.6 + (-0.2) = 3.4. \end{aligned}$$

Since $30 = 25 + 5$, then

$$\begin{aligned} f(30) &\approx f(25) + f'(25)(5) \\ &= 3.6 + (-0.2)(5) = 2.6. \end{aligned}$$

**23.** Using $\Delta x = 1$, we can say that

$$\begin{aligned} f(21) &= f(20) + \text{change in } f(x) \\ &\approx f(20) + f'(20)\Delta x \\ &= 68 + (-3)(1) \\ &= 65. \end{aligned}$$

Similarly, using $\Delta x = -1$,

$$\begin{aligned} f(19) &= f(20) + \text{change in } f(x) \\ &\approx f(20) + f'(20)\Delta x \\ &= 68 + (-3)(-1) \\ &= 71. \end{aligned}$$

Using $\Delta x = 5$, we can write

$$\begin{aligned} f(25) &= f(20) + \text{change in } f(x) \\ &\approx f(20) + (-3)(5) \\ &= 68 - 15 \\ &= 53. \end{aligned}$$

**24.** **(a)** kilograms per week

**(b)** At week 24 the fetus is growing at a rate of 0.096 kg/week, or the fetus will weigh about 0.096 kilograms more in one week.

**25.** **(a)** The tangent line to the weight graph is steeper at 36 weeks then at 20 weeks, so $g'(36)$ is greater than $g'(20)$.

**(b)** The fetus increases its weight more rapidly at week 36 than at week 20.

**26.** Compare the secant line to the graph from week 0 to week 40 to the tangent lines at week 20 and week 36.

**(a)** At week 20 the secant line is steeper than the tangent line. The instantaneous weight growth rate is less than the average.

**(b)** At week 36 the tangent line is steeper than the secant line. The instantaneous weight growth rate is greater than the average.

**27.** **(a)** We use the interval from $t = 20$ to $t = 24$ to estimate $g'(20)$,

$$g'(20) \approx \frac{g(24) - g(20)}{24 - 20} = \frac{0.60 - 0.25}{4} = 0.0875 \text{ kg/week.}$$

**(b)** We use the interval from $t = 36$ to $t = 40$ to estimate $g'(36)$,

$$g'(36) \approx \frac{g(40) - g(36)}{40 - 36} = \frac{3.1 - 2.3}{4} = 0.20 \text{ kg/week.}$$

**(c)** The average rate of change is the slope of the secant line from $(0, 0)$ to $(40, 3.1)$. Thus,

$$\text{Average rate of change} = \frac{3.1 - 0}{40 - 0} = 0.078 \text{ kg/week.}$$

**28. (a)** The statement $f(8) = 5.1$ means that annual net sales for the Hershey Company were $5.1$ billion dollars in 2008. The statement $f'(8) = 0.22$ tells us that in 2008, annual net sales increase by about $0.22$ billion dollars in the next year.

**(b)** Since sales were $5.1$ billion in 2008 and increasing at $0.22$ billion dollars per year, we estimate that, four years later in 2012,

$$\text{Sales in 2012 } \approx 5.1 + 4(0.22) = 5.98 \text{ billion dollars.}$$

Thus $f(12) = 5.98$, so annual net sales for the Hershey Company are projected to be $5.98$ billion dollars in 2012. This prediction assumes that the growth rate remains constant.

**29. (a)** The statement $f(140) = 120$ means that a patient weighing $140$ pounds should receive a dose of $120$ mg of the painkiller. The statement $f'(140) = 3$ tells us that if the weight of a patient increases by one pound (from $140$ pounds), the dose should be increased by about $3$ mg.

**(b)** Since the dose for a weight of 140 lbs is 120 mg and at this weight the dose goes up by about 3 mg for one pound, a 145 lb patient should get about an additional $3(5) = 15$ mg. Thus, for a 145 lb patient, the correct dose is approximately

$$f(145) \approx 120 + 3(5) = 135 \text{ mg.}$$

**30. (a)** The statement $f(10) = 92.63$ tells us that meat production was $92.63$ million metric tons in 2010. The statement $f'(5) = 0.64$ means that in 2010, US meat production increased by about $0.64$ million metric tons in the next year.

**(b)** We assume that the growth rate remains constant until $t = 15$. Since production in 2010 is $92.63$ million metric tons and is increasing at $0.64$ million metric tons a year, we expect production in 2015 to be approximately

$$f(15) \approx 92.63 + 5 \cdot 0.64 = 92.63 + 3.2 = 95.83 \text{ million metric tons.}$$

**31. (a)** The statement $f(20) = 0.36$ means that 20 minutes after smoking a cigarette, there will be $0.36$ mg of nicotine in the body. The statement $f'(20) = -0.002$ means that 20 minutes after smoking a cigarette, about $0.002$ mg of nicotine leaves the body in the next minute. The units are 20 minutes, $0.36$ mg, and $-0.002$ mg/minute.

**(b)**

$$\begin{aligned} f(21) &\approx f(20) + \text{ change in } f \text{ in one minute} \\ &= 0.36 + (-0.002) \\ &= 0.358 \\ f(30) &\approx f(20) + \text{change in } f \text{ in 10 minutes} \\ &= 0.36 + (-0.002)(10) \\ &= 0.36 - 0.02 \\ &= 0.34 \end{aligned}$$

**32. (a)** Since $t$ represents the number of days from now, we are told $f(0) = 80$ and $f'(0) = 0.50$.

**(b)**

$$\begin{aligned} f(10) &\approx \text{value now} + \text{change in value in 10 days} \\ &= 80 + 0.50(10) \\ &= 80 + 5 \\ &= 85. \end{aligned}$$

In 10 days, we expect that the mutual fund will be worth about \$85 a share.

**33. (a)** The slope of the tangent line at 2 kg can be approximated by the slope of the secant line passing through the points $(2, 6)$ and $(3, 2)$. So

$$\text{Slope of tangent line} \approx \frac{v(3) - v(2)}{3 - 2} = \frac{2 - 6}{1} = -4 \text{ (cm/sec)/kg.}$$

**(b)** Since 50 grams $= 0.050$ kg, the contraction velocity changes by about $-4$(cm/sec)/kg $\cdot\ 0.050$kg $= -0.20$ cm/sec. The velocity is reduced by about $0.20$ cm/sec or $2.0$ mm/sec.

**(c)** Since $v(x)$ is the contraction velocity in cm/sec with a load of $x$ kg, we have $v'(2) = -4$.

**34.** **(a)** The slope of the tangent line at 2 hours can be approximated by the slope of the secant line passing through the points $(2, 4.2)$ and $(2.5, 4.75)$. So

$$\text{Slope of tangent line} \approx \frac{g(2.5) - g(2)}{2.5 - 2} = \frac{4.75 - 4.2}{0.5} = 1.1 \text{ (liters/minute)/hour.}$$

**(b)** The rate of change of the pumping rate is the slope of the tangent line. One minute $= 1/60$ hour, so in one minute the

$$\text{Pumping rate increases by about } 1.1 \frac{\text{(liter/minute)}}{\text{hour}} \cdot \frac{1}{60} \text{ hour } = 0.018 \text{ liter/minute.}$$

**(c)** Since $g(t)$ is the pumping rate in liters/minute at time $t$ hours, we have $g'(2) = 1.1$.

**35.** Units of $C'(r)$ are dollars/percent. Approximately, $C'(r)$ means the additional amount needed to pay off the loan when the interest rate is increased by $1\%$. The sign of $C'(r)$ is positive, because increasing the interest rate will increase the amount it costs to pay off a loan.

**36.** The consumption rates (kg/week) are the rates at which the quantities are decreasing, that is, $-1$ times the derivatives of the storage functions. To compare rates at a given time, compare the steepness of the tangent lines to the graphs at that time.

**(a)** At 3 weeks, the tangent line to the fat storage graph is steeper than the tangent line to the protein storage graph. During the third week, fat is consumed at a greater rate than protein.

**(b)** At 7 weeks, the protein storage graph is steeper than the fat storage graph. During the seventh week, protein is consumed at a greater rate than fat.

**37.** Where the graph is linear, the derivative of the fat storage function is constant. The derivative gives the rate of fat consumption (kg/week). Thus, for the first four weeks the body burns fat at a constant rate.

**38.** The fat consumption rate (kg/week) is the rate at which the quantity of fat is decreasing, that is, $-1$ times the derivative of the fat storage function. We estimate the derivatives at 3, 6, and 8 weeks by calculating the slope of the secant line.

**(a)** The tangent line at 3 weeks can be approximated by the secant line containing $(3, 6)$ and $(4, 4)$. So

$$\text{Slope of the tangent line } \approx \frac{4 - 12}{4 - 0} = -2.0 \text{ kg/week.}$$

The consumption rate is $2.0$ kg/week.

**(b)** Two points on the secant line are $(6, 1.2)$ and $(7, 0.8)$. So

$$\text{Slope of the tangent line} = \frac{0.8 - 1.2}{7 - 6} = -0.4 \text{ kg/week.}$$

The consumption rate is $0.4$ kg/week.

**(c)** Two points on the secant line are $(8, 0.5)$ and $(7, 0.8)$. So

$$\text{Slope of the tangent line} = \frac{0.8 - 0.5}{7 - 8} = -0.3 \text{ kg/week.}$$

The consumption rate is $0.3$ kg/week.

**39.** The body changes from burning more fat than protein to burning more protein. This is done by reducing the rate at which it burns fat and simultaneously increasing the rate at which it burns protein. The physiological reason is that the body has begun to run out of fat.

**40.** The graph of fat storage is linear for four weeks, then becomes concave up. Thus, the derivative of fat storage is constant for four weeks, then increases. This matches graph I.

The graph of protein storage is concave up for three weeks, then becomes concave down. Thus, the derivative of protein storage is increasing for three weeks and then becomes decreasing. This matches graph II.

**41.** **(a)** See part (b).

**(b)** See Figure 2.27.

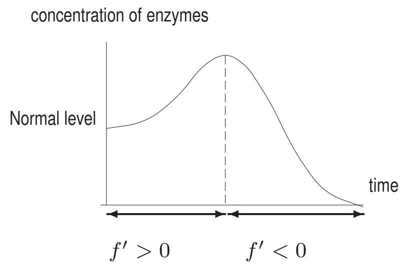



**Figure 2.27**

**(c)** The derivative, $f'$, is the rate at which the concentration is increasing or decreasing. We see that $f'$ is positive at the start of the disease and negative toward the end. In practice, of course, $f'$ cannot be measured directly. Checking the value of $C$ in blood samples taken on consecutive days allows us to estimate $f'(t)$:

$$f'(t) \approx f(t+1) - f(t) = \frac{f(t+1) - f(t)}{(t+1) - t}.$$

**42. (a)** The company hopes that increased advertising always brings in more customers instead of turning them away. Therefore, it hopes $f'(a)$ is always positive.

**(b)** If $f'(100) = 2$, it means that if the advertising budget is \$100,000, an extra dollar spent on advertising will bring in about \$2 worth of sales. If $f'(100) = 0.5$, an extra dollar above \$100 thousand spent on advertising will bring in about \$0.50 worth of sales.

**(c)** If $f'(100) = 2$, then as we saw in part (b), spending slightly more than \$100,000 will increase revenue by an amount greater than the additional expense, and thus more should be spent on advertising. If $f'(100) = 0.5$, then the increase in revenue is less than the additional expense, hence too much is being spent on advertising. The optimum amount to spend, $a$, is an amount that makes $f'(a) = 1$. At this point, the increases in advertising expenditures just pay for themselves. If $f'(a) < 1$, too much is being spent; if $f'(a) > 1$, more should be spent.

**43. (a)** If $f'(80{,}000) = 2$, it means that if the budget for materials is \$80,000, another dollar spent on materials will bring in about \$2 more in revenue. If $f'(80{,}000) = 0.5$, another dollar spent on materials will bring in about \$0.50 more in revenue.

**(b)** If $f'(100) = 2$, then as we saw in part (b), spending slightly more than \$80,000 will increase revenue by an amount greater than the additional expense, and thus more should be spent on materials. If $f'(100) = 0.5$, then the increase in revenue is less than the additional expense, and it does not make sense to spend more on materials.

**44. (a)** If $dA/dg = 1.3$, then producing 1 more gallon of biofuel requires about 1.3 more gallons of gasoline. This is not sustainable since we are using more gallons than we are producing.

**(b)** If $dA/dg = 0.2$, then producing 1 more gallon of biofuel requires about 0.2 more gallons of gasoline. This might be sustainable since we are able to use the gasoline to produce many more gallons of biofuel.

**45. (a)** Let $f(t)$ be the volume, in cubic km, of the Greenland Ice Sheet $t$ years since 2011 (Alternatively, in year $t$). We are given information about $f'(t)$, which has unit km$^3$ per year.

**(b)** If $t$ is in years since 2011, we know $f'(0)$ is between $-224$ and $-82$ cubic km/year. (Alternatively, $f'(2011)$ is between $-224$ and $-82$.)

**46. (a)** The statement $f(9) = 740$ tells us that there were 740 million acres of rain forest in Brazil in 2009. The derivative $f'(9) = -2.7$ tells us that in 2009, after one more year Brazil's rain forests will have shrunk by about 2.7 million acres.

**(b)** We have

$$\text{Relative rate of change } = \frac{f'(9)}{f(9)} = \frac{-2.7}{740} = -0.00365.$$

Brazil's rain forests are shrinking at a continuous rate of 0.365% per year.

**47.** Since $t = 4$ is the end of April 2009, we have $f(4) = 200$ million users. We use a difference quotient with $t = 2$ for the end of February 2009 and $t = 4$ for the end of April 2009 to estimate the derivative:

$$f'(4) \approx \frac{f(4) - f(2)}{4 - 2} = \frac{200 - 175}{4 - 2} = 12.5 \text{ million users per month.}$$

To estimate the relative rate of change, we use

$$\text{Relative rate of change } = \frac{f'(4)}{f(4)} = \frac{12.5}{200} = 0.0625 \text{ per month.}$$

The number of active Facebook users at the end of April 2009 was 200 million. The number was increasing at 12.5 million users per month, which represents a relative growth rate of 6.25% per month.

**48.** Estimating the relative rate of change using $\Delta t = 0.01$, we have

$$\frac{1}{f}\frac{\Delta f}{\Delta t} \approx \frac{1}{f(4)}\frac{f(4.01)-f(4)}{0.01} = \frac{1}{4^2}\frac{4.01^2-4^2}{0.01} = 0.50.$$

**49.** Estimating the relative rate of change using $\Delta t = 0.01$, we have

$$\frac{1}{f}\frac{\Delta f}{\Delta t} \approx \frac{1}{f(10)}\frac{f(10.01)-f(10)}{0.01} = \frac{1}{10^2}\frac{10.01^2-10^2}{0.01} = 0.20.$$

**50.** Let $P = f(t)$. In 2020 we have $t = 7$. The relative rate of change of $f$ in 2020 is $f'(7)/f(7)$. We estimate $f'(7)$ using a difference quotient.

**(a)** Estimating the relative rate of change using $\Delta t = 1$ at $t = 7$, we have

$$\frac{dP/dt}{P} = \frac{f'(7)}{f(7)} \approx \frac{1}{f(7)}\frac{f(8)-f(7)}{1} = 0.01106 = 1.106\% \text{ per year.}$$

**(b)** With $\Delta t = 0.1$ and $t = 7$, we have

$$\frac{dP/dt}{P} = \frac{f'(7)}{f(7)} \approx \frac{1}{f(7)}\frac{f(7.1)-f(7)}{0.1} = 0.01101 = 1.101\% \text{ per year.}$$

**(c)** With $\Delta t = 0.01$ and $t = 7$, we have

$$\frac{dP/dt}{P} = \frac{f'(7)}{f(7)} \approx \frac{1}{f(7)}\frac{f(7.01)-f(7)}{0.01} = 0.01100 = 1.100\% \text{ per year.}$$

The relative rate of change is approximately $1.1\%$ per year. This is as we would expect since the function $P = 7.1e^{0.011t}$ has a continuous rate of change of $1.1\%$ per year for all $t$.

**51.** **(a)** At 2 1/2 months, the baby weighs 5.67 kilograms.

**(b)** At 2 1/2 months, the baby's weight is increasing at a relative rate of 13% per month.

**52.** **(a)** The statement $g(36) = 15$ tells us that 15 billion Apps had been downloaded by June 2011 (36 months after June, 2008). The statement $g'(36) = 0.93$ tells us that in June 2011 the number of downloaded Apps was increasing at a rate of 0.93 billion Apps per month.

**(b)** We have

$$\text{Relative rate of change of downloaded Apps } = \frac{g'(36)}{g(36)} = \frac{0.93}{15} = 0.062.$$

In June 2011, monthly downloaded Apps were increasing at a continuous rate of $6.2\%$ per month.

**53.** December 2012 corresponds to $t = 3$. Let $B = f(t)$. The relative rate of change of $f$ at $t = 3$ is $f'(3)/f(3)$. We estimate $f'(3)$ using a difference quotient.

**(a)** Estimating the relative rate of change using $\Delta t = 1$ at $t = 3$, we have

$$\frac{dB/dt}{B} = \frac{f'(3)}{f(3)} \approx \frac{1}{f(3)}\frac{f(4)-f(3)}{1} = 0.061 = 6.1\% \text{ per month}$$

**(b)** With $\Delta t = 0.1$ and $t = 3$, we have

$$\frac{dB/dt}{B} = \frac{f'(3)}{f(3)} \approx \frac{1}{f(3)}\frac{f(3.1)-f(3)}{0.1} = 0.059 = 5.9\% \text{ per month}$$

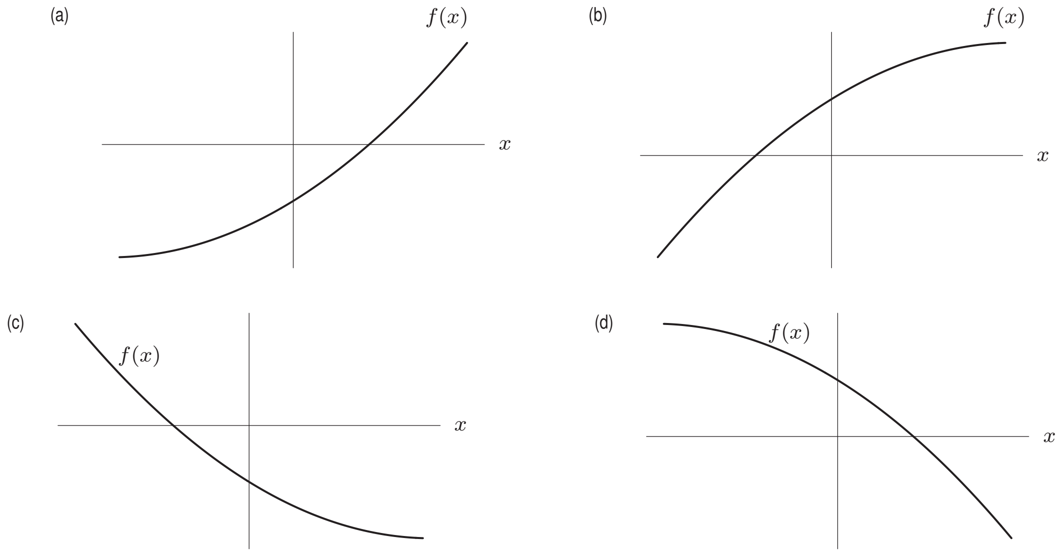
**(c)** With $\Delta t = 0.01$ and $t = 3$, we have

$$\frac{dB/dt}{B} = \frac{f'(3)}{f(3)} \approx \frac{1}{f(3)}\frac{f(3.01)-f(3)}{0.01} = 0.059 = 5.9\% \text{ per month}$$

The relative rate of change is approximately $5.9\%$ per month.

**54.** Since $O'(2000) = -1, 25$, we know the ODGI is decreasing at $1.25$ units per year. To reduce the ODGI from 95 to 0 will take $95/1.25 = 76$ years. Thus the ozone hole is predicted to recover by 2076.

## Solutions for Section 2.4

1. **(a)** Since $g(x)$ is decreasing at $x = 0$, the value of $g'(0)$ is negative.
   **(b)** Since $g(x)$ is concave down at $x = 0$, the value of $g''(0)$ is negative.
2. At $E$ both $dy/dx$ and $d^2y/dx^2$ could be negative because $y$ is decreasing and the graph is concave down there. At all the other points one or both of the derivatives could not be negative.
3. 


4. $f'(x) > 0$
   $f''(x) > 0$
5. $f'(x) = 0$
   $f''(x) = 0$
6. $f'(x) < 0$
   $f''(x) = 0$
7. $f'(x) < 0$
   $f''(x) > 0$
8. $f'(x) > 0$
   $f''(x) < 0$
9. $f'(x) < 0$
   $f''(x) < 0$
10. The derivative is positive on those intervals where the function is increasing and negative on those intervals where the function is decreasing. Therefore, the derivative is positive on the intervals $0 < t < 0.4$ and $1.7 < t < 3.4$, and negative on the intervals $0.4 < t < 1.7$ and $3.4 < t < 4$.

    The second derivative is positive on those intervals where the graph of the function is concave up and negative on those intervals where the graph of the function is concave down. Therefore, the second derivative is positive on the interval $1 < t < 2.6$ and negative on the intervals $0 < t < 1$ and $2.6 < t < 4$.
11. The derivative is positive on those intervals where the function is increasing and negative on those intervals where the function is decreasing. Therefore, the derivative is positive on the interval $-2.3 < t < -0.5$ and negative on the interval $-0.5 < t < 4$.

    The second derivative is positive on those intervals where the graph of the function is concave up and negative on those intervals where the graph of the function is concave down. Therefore, the second derivative is positive on the interval $0.5 < t < 4$ and negative on the interval $-2.3 < t < 0.5$.
12. The derivative of $w(t)$ appears to be negative since the function is decreasing over the interval given. The second derivative, however, appears to be positive since the function is concave up, i.e., it is decreasing at a decreasing rate.

**13.** The derivative, $s'(t)$, appears to be positive since $s(t)$ is increasing over the interval given. The second derivative also appears to be positive or zero since the function is concave up or possibly linear between $t = 1$ and $t = 3$, i.e., it is increasing at a non-decreasing rate.

**14.** The graph must be everywhere decreasing and concave up on some intervals and concave down on other intervals. One possibility is shown in Figure 2.28.

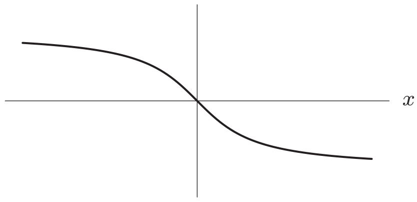



**Figure 2.28**

**15.** Since all advertising campaigns are assumed to produce an increase in sales, a graph of sales against time would be expected to have a positive slope.

A positive second derivative means the rate at which sales are increasing is increasing. If a positive second derivative is observed during a new campaign, it is reasonable to conclude that this increase in the rate sales are increasing is caused by the new campaign–which is therefore judged a success. A negative second derivative means a decrease in the rate at which sales are increasing, and therefore suggests the new campaign is a failure.

**16.** **(a)** The function appears to be decreasing and concave down, and so we conjecture that $f'$ is negative and that $f''$ is negative.

**(b)** We use difference quotients to the right:
$f'(2) \approx \frac{137-145}{4-2} = -4$
$f'(8) \approx \frac{56-98}{10-8} = -21.$

**17.** **(a)** The derivative, $f'(t)$, appears to be positive since the number of cars is increasing. The second derivative, $f''(t)$, appears to be negative during the period 1975–1990 because the rate of change is increasing. For example, between 1975 and 1980, the rate of change is $(121.6 - 106.7)/5 = 2.98$ million cars per year, while between 1985 and 1990, the rate of change is $1.16$ million cars per year.

**(b)** The derivative, $f'(t)$, appears to be negative between 1990 and 1995 since the number of cars is decreasing, but increasing between 1995 and 2000. The second derivative, $f''(t)$, appears to be positive during the period 1990–2000 because the rate of change is increasing. For example, between 1990 and 1995, the rate of change is $(128.4 - 133.7)/5 = -1.06$ million cars per year, while between 1995 and 2000, the rate of change is $1.04$ million cars per year.

**(c)** To estimate $f'(2005)$ we consider the interval 2000–2005

$$f'(2005) \approx \frac{f(2005) - f(2000)}{2005 - 2000} \approx \frac{136.6 - 133.6}{5} = \frac{3}{5} = 0.6.$$

We estimate that $f'(2005) \approx 0.6$ million cars per year. The number of passenger cars in the US was increasing at a rate of about 600,000 cars per year in 2005.

**18.** This graph is increasing for all $x$, and is concave down to the left of 2 and concave up to the right of 2. One possible answer is shown in Figure 2.29:

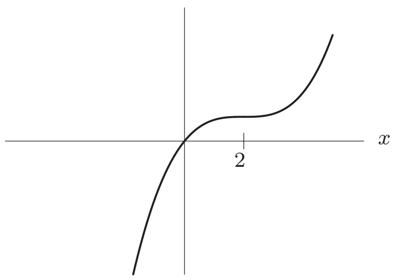



**Figure 2.29**

**19.** Since $f(2) = 5$, the graph goes through the point $(2, 5)$. Since $f'(2) = 1/2$, the slope of the curve is $1/2$ when it passes through this point. Since $f''(2) > 0$, the graph is concave up at this point. One possible graph is shown in Figure 2.30. Many other answers are also possible.

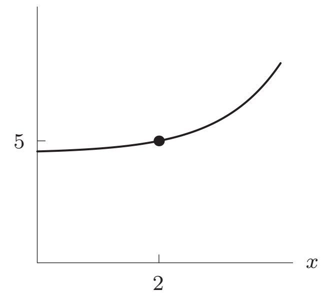



**Figure 2.30**

**20.** The two points at which $f' = 0$ are $A$ and $B$. Since $f'$ is nonzero at $C$ and $D$ and $f''$ is nonzero at all four points, we get the completed Table 2.3:

**Table 2.3**

| Point | $f$ | $f'$ | $f''$ |
|---|---|---|---|
| $A$ | $-$ | 0 | $+$ |
| $B$ | $+$ | 0 | $-$ |
| $C$ | $+$ | $-$ | $-$ |
| $D$ | $-$ | $+$ | $+$ |

**21.** (b). The positive first derivative tells us that the temperature is increasing; the negative second derivative tells us that the rate of increase of the temperature is slowing.

**22.** (e). Since the smallest value of $f'(t)$ was $2°\text{C/hour}$, we know that $f'(t)$ was always positive. Thus, the temperature rose all day.

**23.** To the right of $x = 5$, the function starts by increasing, since $f'(5) = 2 > 0$ (though $f$ may subsequently decrease) and is concave down, so its graph looks like the graph shown in Figure 2.31. Also, the tangent line to the curve at $x = 5$ has slope 2 and lies above the curve for $x > 5$. If we follow the tangent line until $x = 7$, we reach a height of 24. Therefore, $f(7)$ must be smaller than 24, meaning 22 is the only possible value for $f(7)$ from among the choices given.

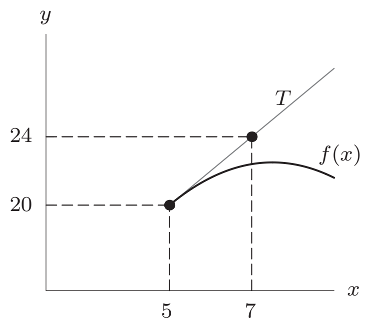



**Figure 2.31**

**24.** **(a)** The EPA will say that the rate of discharge is still rising. The industry will say that the rate of discharge is increasing less quickly, and may soon level off or even start to fall.

**(b)** The EPA will say that the rate at which pollutants are being discharged is leveling off, but not to zero—so pollutants will continue to be dumped in the lake. The industry will say that the rate of discharge has decreased significantly.

**25.** **(a)** Let $N(t)$ be the number of people below the poverty line. See Figure 2.32.

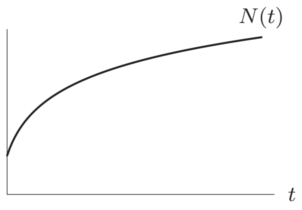
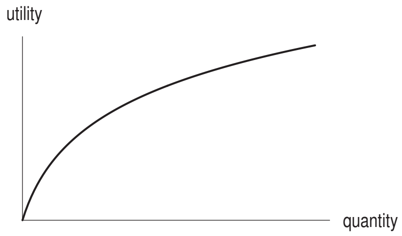



**Figure 2.32**

**(b)** $dN/dt$ is positive, since people are still slipping below the poverty line. $d^2N/dt^2$ is negative, since the rate at which people are slipping below the poverty line, $dN/dt$, is decreasing.

**26.** **(a)**




**(b)** As a function of quantity, utility is increasing but at a decreasing rate; the graph is increasing but concave down. So the derivative of utility is positive, but the second derivative of utility is negative.

**27.** **(a)** $dP/dt > 0$ and $d^2P/dt^2 > 0$.

**(b)** $dP/dt < 0$ and $d^2P/dt^2 > 0$ (but $dP/dt$ is close to zero).

**28.** (a) IV, (b) III, (c) II, (d) I, (e) IV, (f) II

**29.** **(a)** Since the sea level is rising, we know that $a'(t) > 0$ and $m'(t) > 0$. Since the rate is accelerating, we know that $a''(t) > 0$ and $m''(t) > 0$.

**(b)** The rate of change of sea level for the mid-Atlantic states is between 2 and 4, we know $2 < a'(t) < 4$. (Possibly also $a'(t) = 2$ or $a'(t) = 4$.)
Similarly, $2 < m'(t) < 10$. (Possibly also $m'(t) = 2$ or $m'(t) = 10$.)

**(c)** (i) If $a'(t) = 2$, then sea level rise $= 2 \cdot 100 = 200$ mm.
If $a'(t) = 4$, then sea level rise $= 4 \cdot 100 = 400$ mm.
So sea level rise is between 200 mm and 400 mm.

(ii) The shortest amount of time for the sea level in the Gulf of Mexico to rise 1 meter occurs when the rate is largest, 10 mm per year. Since 1 meter $= 1000$ mm,
shortest time to rise 1 meter $= 1000/10 = 100$ years.

**30.** **(a)** For each time interval we can calculate the average rate of change of the number of Facebook subscribers per month over this interval. For example, between March 2011 and June 2011, a period of 3 months,

$$\text{Average rate of change} = \frac{\Delta N}{\Delta t} = \frac{710.7 - 664.0}{3} = \frac{46.7}{3} = 15.57.$$

Thus, between March 2011 and June 2011, there were approximately 15,570,000 more subscribers each month. Values of $\Delta N/\Delta t$ are listed in Table 2.4.

**Table 2.4**

| Time interval | Mar 2011–Jun2011 | Jun 2011–Sep 2011 | Sep 2011–Dec 2011 | Dec 2011–Mar 2012 |
|---|---|---|---|---|
| Average rate of change, $\Delta N/\Delta t$ (millions/month) | 15.57 | 15.4 | 14.07 | 12.13 |

**(b)** We assume the data lies on a smooth curve. Since the values of $\Delta N/\Delta t$ are decreasing for March 2011–March 2012, this suggests that $dN/dt$ also decreases, so $d^2N/dt^2$ is negative for this period.

**31.** **(a)** For each time interval we can calculate the average rate of change of the number of yeast population per hour over this interval. For example, between 0 and 2 hours

$$\text{Average rate of change} = \frac{\Delta P}{\Delta t} = \frac{29.0 - 9.6}{2 - 0} = 9.7.$$

Values of $\Delta P/\Delta t$ are listed in Table 2.5.

**Table 2.5**

| Time interval | 0–2 | 2–4 | 4–6 | 6–8 | 8–10 | 10–12 | 12–14 | 14–16 | 16–18 |
|---|---|---|---|---|---|---|---|---|---|
| Average rate of change, $\Delta P/\Delta t$ | 9.7 | 21.1 | 51.8 | 88.1 | 81.3 | 40.8 | 23.0 | 7.6 | 2.9 |

**(b)** We assume the data lies on a smooth curve. Since the values of $\Delta P/\Delta t$ are increasing for $0 < t < 8$, this suggests that $dP/dt$ also increases, so $d^2P/dt^2$ is positive for this period. Since the values of $\Delta P/\Delta t$ are decreasing for $8 < t < 18$, this suggests that $dP/dt$ also decreases, so $d^2P/dt^2$ is negative for this period.

## Solutions for Section 2.5

**1.** The marginal cost is approximated by the difference quotient

$$MC \approx \frac{\Delta C}{\Delta q} = \frac{4830 - 4800}{1305 - 1295} = 3.$$

The marginal cost is approximately $3 per item.

**2.** **(a)** Marginal cost is the derivative $C'(q)$, so its units are dollars/barrel.
**(b)** It costs about $3 more to produce 101 barrels of olive oil than to produce 100 barrels.

**3.** Drawing in the tangent line at the point $(600, R(600))$, we get Figure 2.33.

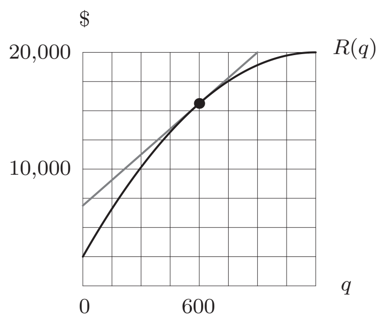



**Figure 2.33**

We see that each vertical increase of 2500 in the tangent line gives a corresponding horizontal increase of roughly 150. The marginal revenue at the production level of 600 units is

$$R'(600) = \frac{\text{Slope of tangent line}}{\text{to } R(q) \text{ at } q = 600} = \frac{2500}{150} = 16.67.$$

This tells us that after producing 600 units, the revenue for producing the $601^{\text{st}}$ product will be roughly $16.67.

**4.** Marginal cost $= C'(q)$. Therefore, marginal cost at $q$ is the slope of the graph of $C(q)$ at $q$. We can see that the slope at $q = 5$ is greater than the slope at $q = 30$. Therefore, marginal cost is greater at $q = 5$. At $q = 20$, the slope is small, whereas at $q = 40$ the slope is larger. Therefore, marginal cost at $q = 40$ is greater than marginal cost at $q = 20$.

**5.** Drawing in the tangent line at the point $(10000, C(10000))$ we get Figure 2.34.

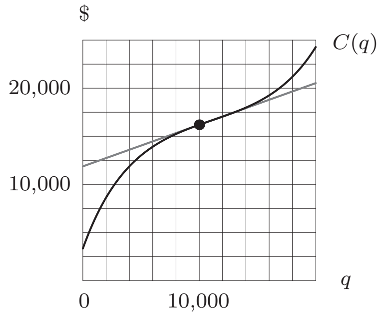



**Figure 2.34**

We see that each vertical increase of 2500 in the tangent line gives a corresponding horizontal increase of roughly 6000. Thus the marginal cost at the production level of 10,000 units is

$$C'(10{,}000) = \begin{array}{c}\text{Slope of tangent line}\\ \text{to } C(q) \text{ at } q = 10{,}000\end{array} = \frac{2500}{6000} = 0.42.$$

This tells us that after producing 10,000 units, it will cost roughly \$0.42 to produce one more unit.

**6.** **(a)** For $q = 500$

$$\text{Profit } = \pi(500) = R(500) - C(500) = 9400 - 7200 = 2200 \text{ dollars.}$$

**(b)** As production increases from $q = 500$ to $q = 501$,

$$\Delta R \approx R'(500)\Delta q = 20 \cdot 1 = 20 \text{ dollars,}$$
$$\Delta C \approx C'(500)\Delta q = 15 \cdot 1 = 15 \text{ dollars,}$$

Thus

$$\text{Change in profit } = \Delta\pi = \Delta R - \Delta C = 20 - 15 = 5 \text{ dollars.}$$

**7.** We have

$$C'(2000) \approx \frac{C(2500) - C(2000)}{2500 - 2000} = \frac{3825 - 3640}{500} = \$0.37/\text{ton}.$$

This means that recycling the 2001st ton of paper will cost around \$0.37. The marginal cost is smallest at the point where the derivative of the function is smallest. Thus the marginal cost appears to be smallest on the interval $2500 \leq q \leq 3000$.

**8.** The slope of the revenue curve is greater than the slope of the cost curve at both $q_1$ and $q_2$, so the marginal revenue is greater at both production levels.

**9.** **(a)** We can approximate $C(16)$ by adding $C'(15)$ to $C(15)$, since $C'(15)$ is an estimate of the cost of the $16^{\text{th}}$ item.

$$C(16) \approx C(15) + C'(15) = \$2300 + \$108 = \$2408.$$

**(b)** We approximate $C(14)$ by subtracting $C'(15)$ from $C(15)$, where $C'(15)$ is an approximation of the cost of producing the $15^{\text{th}}$ item.

$$C(14) \approx C(15) - C'(15) = \$2300 - \$108 = \$2192.$$

**10.** We know $MC \approx C(1{,}001) - C(1{,}000)$. Therefore, $C(1{,}001) \approx C(1{,}000) + MC$ or $C(1{,}001) \approx 5000 + 25 = 5025$ dollars.

Since we do not know $MC(999)$, we will assume that $MC(999) = MC(1{,}000)$. Therefore:

$$MC(999) = C(1{,}000) - C(999).$$

Then:

$$C(999) \approx C(1{,}000) - MC(999) = 5{,}000 - 25 = 4{,}975 \text{ dollars.}$$

Alternatively, we can reason that

$$MC(1{,}000) \approx C(1{,}000) - C(999),$$

so

$$C(999) \approx C(1{,}000) - MC(1{,}000) = 4{,}975 \text{ dollars.}$$

Now for $C(1{,}000)$, we have

$$C(1{,}100) \approx C(1{,}000) + MC \cdot 100.$$

Since $1{,}100 - 1{,}000 = 100$,

$$C(1{,}100) \approx 5{,}000 + 25 \times 100 = 5{,}000 + 2{,}500 = 7{,}500 \text{ dollars.}$$

**11.** **(a)** The cost to produce 50 units is \$4300 and the marginal cost to produce additional items is about \$24 per unit. Producing two more units (from 50 to 52) increases cost by \$48. We have

$$C(52) \approx 4300 + 24(2) = \$4348.$$

**(b)** When $q = 50$, the marginal cost is \$24 per item and the marginal revenue is \$35 per item. The profit on the $51^{\text{st}}$ item is approximately $35 - 24 = \$11$.

**(c)** When $q = 100$, the marginal cost is \$38 per item and the marginal revenue is \$35 per item, so the company loses about \$3 by producing the $101^{\text{st}}$ item. Since the company will lose money, it should not produce the $101^{\text{st}}$ item.

**12.** At $q = 50$, the slope of the revenue is larger than the slope of the cost. Thus, at $q = 50$, marginal revenue is greater than marginal cost and the $50^{\text{th}}$ bus should be added. At $q = 90$ the slope of revenue is less than the slope of cost. Thus, at $q = 90$ the marginal revenue is less than marginal cost and the $90^{\text{th}}$ bus should not be added.

**13.** **(a)** At $q = 2.1$ million,

$$\text{Profit } = \pi(2.1) = R(2.1) - C(2.1) = 6.9 - 5.1 = 1.8 \text{ million dollars.}$$

**(b)** If $\Delta q = 0.04$,

$$\text{Change in revenue, } \Delta R \approx R'(2.1)\Delta q = 0.7(0.04) = 0.028 \text{ million dollars } = \$28{,}000.$$

Thus, revenues increase by about \$28,000.

**(c)** If $\Delta q = -0.05$,

$$\text{Change in revenue, } \Delta R \approx R'(2.1)\Delta q = 0.7(-0.05) = -0.035 \text{ million dollars } = -\$35{,}000.$$

Thus, revenues decrease by about \$35,000.

**(d)** We find the change in cost by a similar calculation. For $\Delta q = 0.04$,

$$\text{Change in cost, } \Delta C \approx C'(2.1)\Delta q = 0.6(0.04) = 0.024 \text{ million dollars } = \$24{,}000$$
$$\text{Change in profit, } \Delta \pi \approx \$28{,}000 - \$24{,}000 = \$4000.$$

Thus, increasing production 0.04 million units increases profits by about \$4000.

For $\Delta q = -0.05$,

$$\text{Change in cost, } \Delta C \approx C'(2.1)\Delta q = 0.6(-0.05) = -0.03 \text{ million dollars } = -\$30{,}000$$
$$\text{Change in profit, } \Delta \pi \approx -\$35{,}000 - (-\$30{,}000) = -\$5000.$$

Thus, decreasing production 0.05 million units decreases profits by about \$5000.

**14.** **(a)** At $q = 2000$, we have

$$\text{Profit } = R(2000) - C(2000) = 7780 - 5930 = 1850 \text{ dollars.}$$

**(b)** If $q$ increases from 2000 to 2001,

$$\Delta R \approx R'(2000) \cdot \Delta q = 2.5 \cdot 1 = 2.5 \text{ dollars,}$$
$$\Delta C \approx C'(2000) \cdot \Delta q = 2.1 \cdot 1 = 2.1 \text{ dollars,}$$

Thus,

$$\text{Change in profit } = \Delta R - \Delta C \approx 2.5 - 2.1 = 0.4 \text{ dollars.}$$

Since increasing production increases profit, the company should increase production.

**(c)** By a calculation similar to that in part (b), as $q$ increases from 2000 to 2001,

$$\text{Change in profit } \approx 4.32 - 4.77 = -0.45 \text{ dollars.}$$

Since increasing production reduces the profit, the company should decrease production.

**15.** For each $q$, we calculate the average rate of change of the cost and the revenue over the interval to the right. For example, for $q = 0$

$$C'(0) \approx \begin{matrix}\text{Average rate}\\ \text{of change of } C(q)\end{matrix} = \frac{C(1) - C(0)}{1 - 0} = \frac{10 - 9}{1 - 0} = 1,$$

while

$$R'(0) \approx \begin{matrix}\text{Average rate}\\ \text{of change of } R(q)\end{matrix} = \frac{R(1) - R(0)}{1 - 0} = \frac{5 - 0}{1 - 0} = 5.$$

Estimates for $C'(q)$ and $R'(q)$ are in Table 2.6.

**Table 2.6**

| $q$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $C'(q)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $R'(q)$ | 5 | 5 | 5 | 5 | 5 | 5 | 5 |

**16.** For each value of $q$, we calculate the average rate of change of the cost and the revenue over the interval to the right. For example, for $q = 1$,

$$MC(1) \approx \begin{matrix}\text{Average rate}\\ \text{of change of } C(q)\end{matrix} = \frac{C(2) - C(1)}{2 - 1} = \frac{60 - 20}{2 - 1} = 40,$$

while

$$MR(1) \approx \begin{matrix}\text{Average rate}\\ \text{of change of } R(q)\end{matrix} = \frac{R(2) - R(1)}{2 - 1} = \frac{220 - 100}{2 - 1} = 120.$$

Estimates for $MC(q)$ and $MR(q)$ are in Table 2.7.

**Table 2.7**

| $q$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $MC(q)$ | 40 | 60 | 80 | 100 | 120 |
| $MR(q)$ | 120 | 110 | 80 | 40 | 30 |

## Solutions for Chapter 2 Review

**1. (a)** Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 1$ and $t = 1.1$. We have

$$\text{Average velocity} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{3.63 - 3}{0.1} = 6.3 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{3.0603 - 3}{0.01} = 6.03 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{3.006003 - 3}{0.001} = 6.003 \text{ m/sec.}$$

**(b)** We see in part (a) that as we choose a smaller and smaller interval around $t = 1$ the average velocity appears to be getting closer and closer to 6, so we estimate the instantaneous velocity at $t = 1$ to be 6 m/sec.

**2.** Since $f'(x) = 0$ where the graph is horizontal, $f'(x) = 0$ at $x = d$. The derivative is positive at points $b$ and $c$, but the graph is steeper at $x = c$. Thus $f'(x) = 0.5$ at $x = b$ and $f'(x) = 2$ at $x = c$. Finally, the derivative is negative at points $a$ and $e$ but the graph is steeper at $x = e$. Thus, $f'(x) = -0.5$ at $x = a$ and $f'(x) = -2$ at $x = e$. See Table 2.8.

Thus, we have $f'(d) = 0, f'(b) = 0.5, f'(c) = 2, f'(a) = -0.5, f'(e) = -2$.

**Table 2.8**

| $x$ | $f'(x)$ |
|---|---|
| $d$ | 0 |
| $b$ | 0.5 |
| $c$ | 2 |
| $a$ | $-0.5$ |
| $e$ | $-2$ |

**3.** **(a)** From Figure 2.35 we can see that for $x = 1$ the value of the function is decreasing. Therefore, the derivative of $f(x)$ at $x = 1$ is negative.

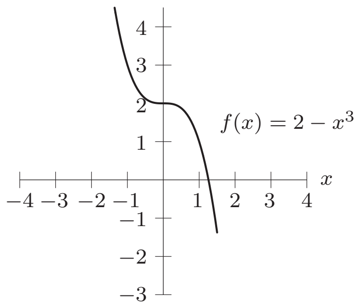



**Figure 2.35**

**(b)** $f'(1)$ is the derivative of the function at $x = 1$. This is the rate of change of $f(x) = 2 - x^3$ at $x = 1$. We estimate this by computing the average rate of change of $f(x)$ over intervals near $x = 1$.

Using the intervals $0.999 \leq x \leq 1$ and $1 \leq x \leq 1.001$, we see that

$$\begin{pmatrix}\text{Average rate of change}\\ \text{on } 0.999 \leq x \leq 1\end{pmatrix} = \frac{[2-1^3]-[2-0.999^3]}{1-0.999} = \frac{1-1.002997}{0.001} = -2.997,$$

$$\begin{pmatrix}\text{Average rate of change}\\ \text{on } 1 \leq x \leq 1.001\end{pmatrix} = \frac{[2-1.001^3]-[2-1^3]}{1.001-1} = \frac{0.996997-1}{0.001} = -3.003.$$

It appears that the rate of change of $f(x)$ at $x = 1$ is approximately $-3$, so we estimate $f'(1) = -3$.

**4.** We estimate $f'(2)$ using the average rate of change formula on a small interval around 2. We use the interval $x = 2$ to $x = 2.001$. (Any small interval around 2 gives a reasonable answer.) We have

$$f'(2) \approx \frac{f(2.001) - f(2)}{2.001 - 2} = \frac{3^{2.001} - 3^2}{2.001 - 2} = \frac{9.00989 - 9}{0.001} = 9.89.$$

**5.** **(a)** $f'(x)$ is negative when the function is decreasing and positive when the function is increasing. Therefore, $f'(x)$ is positive at $C$ and $G$. $f'(x)$ is negative at $A$ and $E$. $f'(x)$ is zero at $B$, $D$, and $F$.

**(b)** $f'(x)$ is the largest when the graph of the function is increasing the fastest (i.e. the point with the steepest positive slope). This occurs at point $G$. $f'(x)$ is the most negative when the graph of the function is decreasing the fastest (i.e. the point with the steepest negative slope). This occurs at point $A$.

**6.** **(a)** Since the point $B = (2, 5)$ is on the graph of $g$, we have $g(2) = 5$.

**(b)** The slope of the tangent line touching the graph at $x = 2$ is given by

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{5 - 5.02}{2 - 1.95} = \frac{-0.02}{0.05} = -0.4.$$

Thus, $g'(2) = -0.4$.

**7. (a)** Let $s = f(t)$.

(i) We wish to find the average velocity between $t = 1$ and $t = 1.1$. We have

$$\text{Average velocity} = \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{0.808496 - 0.909297}{0.1} = -1.00801 \text{ m/sec.}$$

(ii) We have

$$\text{Average velocity} = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{0.900793 - 0.909297}{0.01} = -0.8504 \text{ m/sec.}$$

(iii) We have

$$\text{Average velocity} = \frac{f(1.001) - f(1)}{1.001 - 1} = \frac{0.908463 - 0.909297}{0.001} = -0.834 \text{ m/sec.}$$

**(b)** We see in part (a) that as we choose a smaller and smaller interval around $t = 1$ the average velocity appears to be getting closer and closer to $-0.83$, so we estimate the instantaneous velocity at $t = 1$ to be $-0.83$ m/sec. In this case, more estimates with smaller values of $h$ would be very helpful in making a better estimate.

**8. (a)** The average velocity between $t = 3$ and $t = 5$ is

$$\frac{\text{Distance}}{\text{Time}} = \frac{s(5) - s(3)}{5 - 3} = \frac{25 - 9}{2} = \frac{16}{2} = 8 \text{ ft/sec.}$$

**(b)** Using an interval of size 0.1, we have

$$\begin{pmatrix} \text{Instantaneous velocity} \\ \text{at } t = 3 \end{pmatrix} \approx \frac{s(3.1) - s(3)}{3.1 - 3} = \frac{9.61 - 9}{0.1} = 6.1.$$

Using an interval of size 0.01, we have

$$\begin{pmatrix} \text{Instantaneous velocity} \\ \text{at } t = 3 \end{pmatrix} \approx \frac{s(3.01) - s(3)}{3.01 - 3} = \frac{9.0601 - 9}{0.01} = 6.01.$$

From this we guess that the instantaneous velocity at $t = 3$ is about 6 ft/sec.

**9.** We use the interval $x = 2$ to $x = 2.01$:

$$f'(2) \approx \frac{f(2.01) - f(2)}{2.01 - 2} = \frac{5^{2.01} - 5^2}{0.01} = \frac{25.4056 - 25}{0.01} = 40.56.$$

For greater accuracy, we can use the smaller interval $x = 2$ to $x = 2.001$:

$$f'(2) \approx \frac{f(2.001) - f(2)}{2.001 - 2} = \frac{5^{2.001} - 5^2}{0.001} = \frac{25.040268 - 25}{0.001} = 40.268.$$

**10.** $P'(0)$ is the derivative of the function $P(t) = 200(1.05)^t$ at $t = 0$. This is the same as the rate of change of $P(t)$ at $t = 0$. We estimate this by computing the average rate of change over intervals near $t = 0$.

If we use the intervals $-0.001 \le t \le 0$ and $0 \le t \le 0.001$, we see that:

$$\begin{pmatrix} \text{Average rate of change} \\ \text{on } -0.001 \le t \le 0 \end{pmatrix} = \frac{200(1.05)^0 - 200(1.05)^{-0.001}}{0 - (-0.001)} = \frac{200 - 199.990242}{0.001} = 9.758,$$

$$\begin{pmatrix} \text{Average rate of change} \\ \text{on } 0 \le t \le 0.001 \end{pmatrix} = \frac{200(1.05)^{0.001} - 200(1.05)^0}{0.001 - 0} = \frac{200.009758 - 200}{0.001} = 9.758.$$

It appears that the rate of change of $P(t)$ at $t = 0$ is 9.758, so we estimate $P'(0) = 9.758$.

**11.** See Figure 2.36.

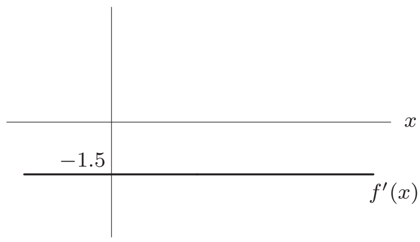



**Figure 2.36**

**12.** This is a line with slope $-2$, so the derivative is the constant function $f'(x) = -2$. The graph is a horizontal line at $y = -2$. See Figure 2.37.

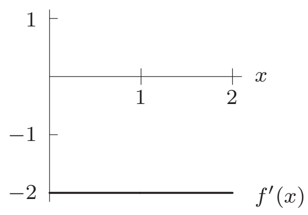



Figure 2.37

**13.** See Figure 2.38.

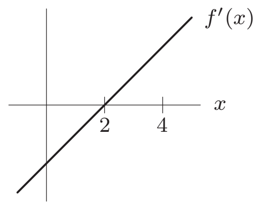



Figure 2.38

**14.** See Figure 2.39.

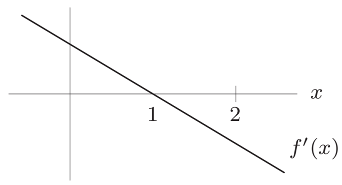



Figure 2.39

**15.** See Figure 2.40.

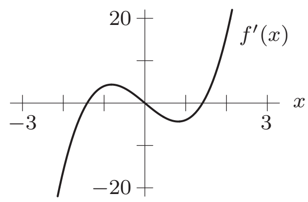



Figure 2.40

**16.** See Figure 2.41.

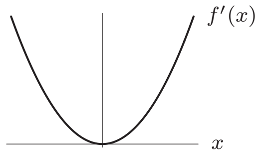



Figure 2.41

**17.** See Figure 2.42.

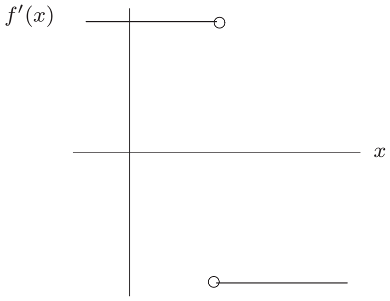



**Figure 2.42**

**18.** See Figure 2.43.

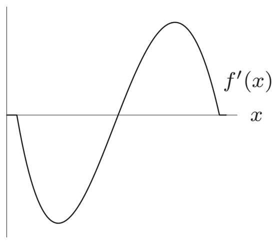



**Figure 2.43**

**19.** One possible graph is shown in Figure 2.44. Notice that $f(x)$ is increasing for $x < 1$ and decreasing for $x > 1$, so $f'(x)$ is above the $x$-axis to the left of 1 and below the $x$-axis to the right of 1. Also as $x$ gets large, the graph of $f(x)$ gets more and more horizontal. Thus, as $x$ gets large, $f'(x)$ gets closer and closer to 0, which means the graph gets closer and closer to the $x$-axis.

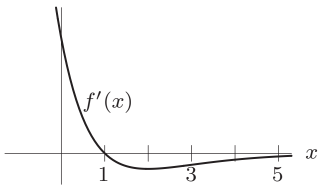



**Figure 2.44**

**20.** One possible graph is shown in Figure 2.45. Notice that $f(x)$ is linear with negative slope for $x < 0$, so $f'(x)$ is constant and negative to the left of 0. We see that $f(x)$ is increasing and getting steeper to the right of 0, so $f'(x)$ is positive and increasing there.

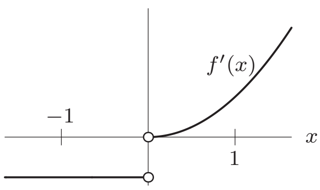



**Figure 2.45**

**21.** **(a)** We use the interval to the right of $x = 2$ to estimate the derivative. (Alternately, we could use the interval to the left of 2, or we could use both and average the results.) We have

$$f'(2) \approx \frac{f(4) - f(2)}{4 - 2} = \frac{24 - 18}{4 - 2} = \frac{6}{2} = 3.$$

We estimate $f'(2) \approx 3$.

**(b)** We know that $f'(x)$ is positive when $f(x)$ is increasing and negative when $f(x)$ is decreasing, so it appears that $f'(x)$ is positive for $0 < x < 4$ and is negative for $4 < x < 12$.

**22.** **(a)** The function $f$ is increasing where $f'$ is positive, so for $x_1 < x < x_3$.

**(b)** The function $f$ is decreasing where $f'$ is negative, so for $0 < x < x_1$ or $x_3 < x < x_5$.

**23.** kilograms/meter

**24.** The statement $f(20) = 57$ means that when $t = 20$, we have $P = 57$. This tells us that in 2002, 57% of households had a personal computer. The statement $f'(20) = 3$ tells us that in 2002, the percent of households with a personal computer is increasing at a rate of about 3% a year.

**25.** (Note that we are considering the average temperature of the yam, since its temperature is different at different points inside it.)

**(a)** It is positive, because the temperature of the yam increases the longer it sits in the oven.

**(b)** The units of $f'(20)$ are °F/min. The statement $f'(20) = 2$ means that at time $t = 20$ minutes, the temperature $T$ would increase by approximately 2°F if the yam is in the oven an additional minute.

**26.** **(a)** The statement $f(15) = 200$ tells us that when the price is \$15, we sell about 200 units of the product.

**(b)** The statement $f'(15) = -25$ tells us that if we increase the price by \$1 (from 15), we will sell about 25 fewer units of the product.

**27.** The derivative $f'(10)$ is the slope of the tangent line to the curve at $t = 10$. See Figure 2.46. Taking two points on the tangent line, we calculate its slope:

$$\text{Slope} \approx \frac{100 - 70}{5} = 6.$$

Since the slope is about 6, we have $f'(10) \approx 6$ cm/yr. At $t = 10$, the sturgeon was growing in length at a rate of about 6 centimeters a year.

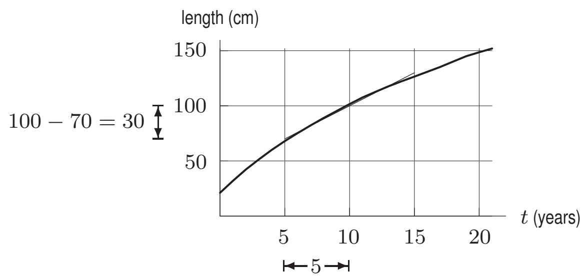



Figure 2.46

**28.** Using the approximation $\Delta y \approx f'(x)\Delta x$ with $\Delta x = 2$, we have $\Delta y \approx f'(20) \cdot 2 = 6 \cdot 2$, so

$$f(22) \approx f(20) + f'(20) \cdot 2 = 345 + 6 \cdot 2 = 357.$$

**29.** Moving away slightly from the center of the hurricane from a point 15 kilometers from the center moves you to a point with stronger winds. For example, the wind is stronger at 15.1 kilometers from the center of the hurricane than it is at 15 kilometers from the center.

**30.** After falling 20 meters the speed of the rock is increasing at a rate of 0.5 meters/second per meter.

**31.** Since $f(t) = 1.34(1.004)^t$, we have

$$f(9) = 1.34(1.004)^9 = 1.389.$$

To estimate $f'(9)$, we use a small interval around 9:

$$f'(9) \approx \frac{f(9.001) - f(9)}{9.001 - 9} = \frac{1.34(1.004)^{9.001} - 1.34(1.004)^9}{0.001} = 0.0055.$$

We see that $f(9) = 1.389$ billion people and $f'(9) = 0.0055$ billion (that is, 5.5 million) people per year. Since $t = 9$ in 2020, this model predicts that the population of China will be about 1,389,000,000 people in 2009 and growing at a rate of about 5,500,000 people per year at that time.

**32.** Since $B$ is measured in dollars and $t$ is measured in years, $dB/dt$ is measured in dollars per year. We can interpret $dB$ as the extra money added to your balance in $dt$ years. Therefore $dB/dt$ represents how fast your balance is growing, in units of dollars/year.

**33.** **(a)** If $f'(t) > 0$, the depth of the water is increasing. If $f'(t) < 0$, the depth of the water is decreasing.
**(b)** The depth of the water is increasing at 20 cm/min when $t = 30$ minutes.
**(c)** We use 1 meter = 100 cm, 1 hour = 60 min. At time $t = 30$ minutes

$$\text{Rate of change of depth} = 20\frac{\text{cm}}{\text{min}} = 20\frac{\text{cm}}{\text{min}} \cdot \frac{60 \text{ min}}{1 \text{ hr}} \cdot \frac{1 \text{ m}}{100 \text{ cm}} = 12 \text{ meters/hour}.$$

**34.** **(a)** Since the graph is below the $x$-axis at $x = 2$, the value of $f(2)$ is negative.
**(b)** Since $f(x)$ is decreasing at $x = 2$, the value of $f'(2)$ is negative.
**(c)** Since $f(x)$ is concave up at $x = 2$, the value of $f''(2)$ is positive.

**35.** At $B$ both $dy/dx$ and $d^2y/dx^2$ could be positive because $y$ is increasing and the graph is concave up there. At all the other points one or both of the derivatives could not be positive.

**36.** We have $\Delta h = h(6003) - h(6000) \approx h'(6000)\Delta x = (0.5)(3) = 1.5$ meters. The elevation increases approximately 1.5 meters as the climber moves from a position 6000 meters from the start of the trail to a position 6003 meters from the start. Thus the climber's elevation increases from 8000 meters to about 8001.5 meters. The new elevation is about 8001.5 meters above sea level.

**37.** The fact that $f(80) = 0.05$ means that when the car is moving at 80 km/hr is it using 0.05 liter of gasoline for each kilometer traveled.

The derivative $f'(v)$ is the rate of change of gasoline consumption with respect to speed. That is, $f'(v)$ tells us how the consumption of gasoline changes as speeds vary. We are told that $f'(80) = 0.0005$. This means that a 1-kilometer increase in speed results in an increase in consumption of about 0.0005 liter per km. At higher speeds, the vehicle burns more gasoline per km traveled than at lower speeds.

**38.** **(a)** Since the traffic flow is the number of cars per hour, it is the slope of the graph of $C(t)$. It is greatest where the graph of the function $C(t)$ is the steepest and increasing. This happens at approximately $t = 3$ hours, or 7am.
**(b)** By reading the values of $C(t)$ from the graph we see:

$$\begin{array}{c}\text{Average rate of change}\\ \text{on } 1 \le t \le 2\end{array} = \frac{C(2) - C(1)}{2 - 1} = \frac{1000 - 400}{1} = \frac{600}{1} = 600 \text{ cars/hour},$$

$$\begin{array}{c}\text{Average rate of change}\\ \text{on } 2 \le t \le 3\end{array} = \frac{C(3) - C(2)}{3 - 2} = \frac{2000 - 1000}{1} = \frac{1000}{1} = 1000 \text{ cars/hour}.$$

A good estimate of $C'(2)$ is the average of the last two results. Therefore:

$$C'(2) \approx \frac{(600 + 1000)}{2} = \frac{1600}{2} = 800 \text{ cars/hour}.$$

**(c)** Since $t = 2$ is 6 am, the fact that $C'(2) \approx 800$ cars/hour means that the traffic flow at 6 am is about 800 cars/hour.

**39.** **(a)** As the cup of coffee cools, the temperature decreases, so $f'(t)$ is negative.
**(b)** Since $f'(t) = dH/dt$, the units are degrees Celsius per minute. The quantity $f'(20)$ represents the rate at which the coffee is cooling, in degrees per minute, 20 minutes after the cup is put on the counter.

**40.** The statements $f(100) = 35$ and $f'(100) = 3$ tell us that at $x = 100$, the value of the function is 35 and the function is increasing at a rate of 3 units for a unit increase in $x$. Since we increase $x$ by 2 units in going from 100 to 102, the value of the function goes up by approximately $2 \cdot 3 = 6$ units, so

$$f(102) \approx 35 + 2 \cdot 3 = 35 + 6 = 41.$$

**41.** Units of $P'(t)$ are dollars/year. The practical meaning of $P'(t)$ is the rate at which the monthly payments change as the duration of the mortgage increases. Approximately, $P'(t)$ represents the change in the monthly payment if the duration is increased by one year. $P'(t)$ is negative because increasing the duration of a mortgage decreases the monthly payments.

**42.** Let $P = f(t)$.

**(a)** Estimating the relative rate of change using $\Delta t = 1$, we have

$$\frac{1}{P}\frac{\Delta P}{\Delta t} = \frac{1}{f(3)}\frac{f(4) - f(3)}{1} = 0.0356197 = 3.56\% \text{ per year}$$

**(b)** With $\Delta t = 0.1$ we have

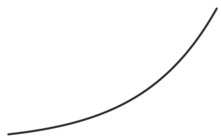
$$\frac{1}{P}\frac{\Delta P}{\Delta t} = \frac{1}{f(3)}\frac{f(3.1) - f(3)}{0.1} = 0.0350613 = 3.51\% \text{ per year}$$

**(c)** With $\Delta t = 0.01$ we have

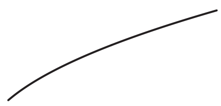
$$\frac{1}{P}\frac{\Delta P}{\Delta t} = \frac{1}{f(3)}\frac{f(3.01) - f(3)}{0.01} = 0.0350061 = 3.50\% \text{ per year}$$

In fact, the relative rate of change at $t = 3$ of the population for this city is exactly $0.035$.

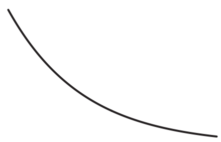
**43.** **(a)** This needs to be increasing, concave up everywhere.



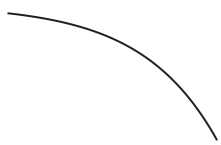
**(b)** This needs to be increasing, concave down everywhere.



**(c)** This needs to be decreasing, concave up everywhere.



**(d)** This needs to be decreasing, concave down everywhere.



**44.** **(a)** This function is increasing, so $f' > 0$. It appears to be increasing by increasing amounts, so $f'' > 0$.
**(b)** This function is decreasing, so $f' < 0$. It appears to be decreasing faster and faster, so $f'' < 0$.
**(c)** This function is increasing, so $f' > 0$. It appears to be increasing by decreasing amounts, so $f'' < 0$.

**45.** **(a)** minutes/kilometer.
**(b)** minutes/kilometer$^2$.

**46.** The function is everywhere increasing and concave up. One possible graph is shown in Figure 2.47.

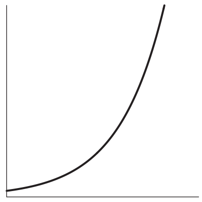



**Figure 2.47**

**47.** **(a)** At $x_4$ and $x_5$, because the graph is below the $x$-axis there.
**(b)** At $x_3$ and $x_4$, because the graph is sloping down there.
**(c)** At $x_3$ and $x_4$, because the graph is sloping down there. This is the same condition as part (b).
**(d)** At $x_2$ and $x_3$, because the graph is bending downward there.
**(e)** At $x_1$, $x_2$, and $x_5$, because the graph is sloping upward there.
**(f)** At $x_1$, $x_4$, and $x_5$, because the graph is bending upward there.

**48.** Since velocity is positive and acceleration is negative, we have $f' > 0$ and $f'' < 0$, and so the graph is increasing and concave down. See Figure 2.48.

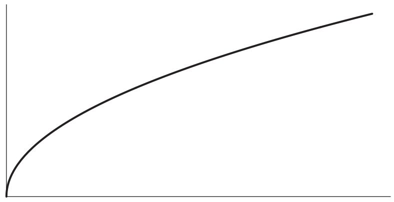



**Figure 2.48**

**49.** **(a)** At $t_3$, $t_4$, and $t_5$, because the graph is above the $t$-axis there.
**(b)** At $t_2$ and $t_3$, because the graph is sloping up there.
**(c)** At $t_1$, $t_2$, and $t_5$, because the graph is concave up there
**(d)** At $t_1$, $t_4$, and $t_5$, because the graph is sloping down there.
**(e)** At $t_3$ and $t_4$, because the graph is concave down there.

**50.** **(a)** For the three years ending in 1998 we have

$$\frac{\Delta P}{\Delta t} = \frac{54.1 - 62.4}{1998 - 1995} = \frac{-8.3}{3} \approx -2.77 \text{ \%/year}.$$

For the three years ending in 2001 we have

$$\frac{\Delta P}{\Delta t} = \frac{48.0 - 54.1}{2001 - 1998} = \frac{-6.1}{3} \approx -2.03 \text{ \%/year}.$$

For the three years ending in 2004 we have

$$\frac{\Delta P}{\Delta t} = \frac{43.5 - 48.0}{2004 - 2001} = \frac{-4.5}{3} \approx -1.50 \text{ \%/year}.$$

For the three years ending in 2007 we have

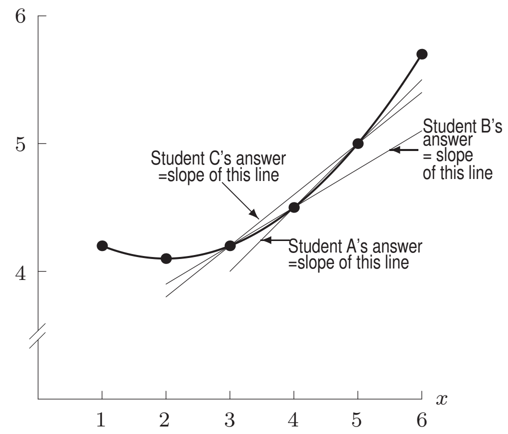
$$\frac{\Delta P}{\Delta t} = \frac{41.8 - 43.5}{2007 - 2004} = \frac{-1.7}{3} \approx -0.57 \text{ \%/year}.$$

**(b)** The fact that $\frac{\Delta P}{\Delta t}$ is increasing from 1995 to 2007 suggests that $\frac{d^2P}{dt^2}$ is positive.

**(c)** The values of $P$ and $\frac{\Delta P}{\Delta t}$ are troublesome because they indicate that the percent of students graduating is low, and that the number is getting smaller each year.

**(d)** Since $\frac{d^2P}{dt^2}$ is positive, the percent of students graduating is not decreasing as fast as it once was. Also, in 2007 the magnitude of $\Delta P/\Delta t$ is less than 1% a year, so the level of drop-outs does in fact seem to be hitting its minimum at around 40%.

**51.** **(a)**




**(b)** The slope of $f$ appears to be somewhere between student A's answer and student B's, so student C's answer, halfway in between, is probably the most accurate.

**52.** **(a)** The rate of energy consumption required when $v = 0$ is the vertical intercept, about 1.8 joules/sec.

**(b)** The graph shows $f(v)$ first decreases and then increases as $v$ increases. This tells us that the bird expends more energy per second to remain still than to travel at slow speeds (say 0.5 to 1 meter/sec), but that the rate of energy consumption required increases again at speeds beyond 1 meter/sec. The upward concavity of the graph tells us that as the bird speeds up, it uses energy at a faster and faster rate.

**(c)** Figure 2.49 shows a possible graph of the derivative $f'(v)$. Other answers are possible.

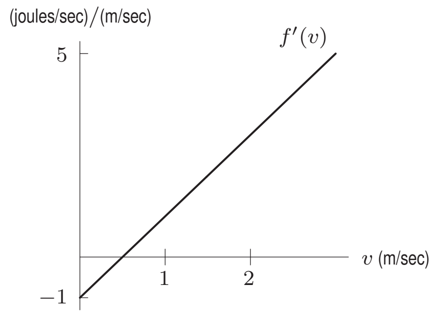



**Figure 2.49**

## STRENGTHEN YOUR UNDERSTANDING

**1.** True, this is the definition of the derivative.

**2.** True, one way to see is that the graph of $f(x)$ is increasing for all $x > 0$. Since the graph is increasing at $x = 1$, the instantaneous rate of change, $f'(1)$, is positive.

**3.** True, since $g'(1) \approx (g(1.001) - g(1))/(1.001 - 1) = (2^{1.001} - 2)/(1.001 - 1)$.

**4.** False, since the function $H(x) = \sqrt{x}$ is increasing, the slope is always positive.

**5.** False, the function $f(x) = x^2$ is such a function, with $a = 0$.

**6.** False, the average rate of change between 0 and 1 is given by the slope of the line connecting the points $(0, f(0))$ and $(1, f(1))$. So, for example, the graph of $f(x) = x^2$ has slope 0 at $x = 0$, but the average rate of change between 0 and 1 is $(f(1) - f(0))/(1 - 0) = 1$.

**7.** False, the function $r$ appears to be increasing, and therefore would have a positive derivative.

**8.** False, the derivative of $f$ at 3 is approximated by a difference quotient on a small interval containing 3, and is not the same as the average rate of change on the interval $x = 0$ to $x = 3$.

**9.** False, $R(w)$ is increasing for all $w$ so the derivative can not be negative at any point.

**10.** True, consider any constant function.

**11.** True.

**12.** False, the opposite is true: If the derivative of $f$ is negative on an interval, then $f$ is decreasing on that interval.

**13.** False. a function can be increasing while its slope is decreasing.

**14.** False, since $g$ could be less than $f$ at $x = 2$ but increasing faster than $f$ near $x = 2$. For example, $f(x) = 5$ and $g(x) = x^2$ has $f(2) = 5 > 4 = g(2)$, but $f'(2) = 0$ since $f$ is constant, and $g'(2) > 0$ since $g$ is increasing near $x = 2$.

**15.** True, the derivative is the slope of the curve, which is always 0 for a horizontal line.

**16.** False, since a function $f(x)$ can be decreasing and concave up, which means the function is decreasing less rapidly as $x \to \infty$. Therefore, the derivative of the function is less and less negative, so the derivative is increasing.

**17.** False, the function $\ln(t)$ is an increasing function, and therefore has positive derivative.

**18.** False, if we let $f(x) = 3 - x$, then $f(3) = 0$, but $f$ is linear, and so the slope is the derivative, which is $-1$.

**19.** True.

**20.** True.

**21.** True, since the derivative is the limit of the difference quotient $\frac{f(x+h)-f(x)}{h}$, the units of the derivative are the units of the numerator over the units of the denominator, which is dollars per student.

**22.** True, as long as it is understood that $x$ is the independent variable, then the two quantities are equal.

**23.** True, this interpretation is as specified in the text.

**24.** True, the local linear approximation is given by $f(10) + f'(10)(1) = 20 + 3(1) = 23$.

**25.** True, $dA/dB$ represents the change in $A$ with respect to a change in $B$, and so the units are the units of $A$ divided by the units of $B$.

**26.** False, since $f'(B) = dA/dB$, the units of $f'(B)$ are the units of $A$ divided by the units of $B$.

**27.** True, since the derivative is the approximate change in $f$ when the independent variable is increased by one.

**28.** True, since $f'(D) = dt/dD$ the units of $f'(D)$ are the units of $t$ divided by the units of $D$, which is minutes per milligram.

**29.** True, since for the first 10 years, height is an increasing function of age.

**30.** True, since if the derivative is negative, then $W$ is a decreasing function of $R$.

**31.** False. If we let $f(x) = x^2$ then $f''(x) > 0$ because $f$ is concave up, but $f(x)$ is decreasing for $x < 0$.

**32.** True, if $f'' < 0$, then the derivative is decreasing, which means that the graph is concave down.

**33.** False, since $f'' > 0$ means only that the derivative is increasing, that is, the graph of $f$ is concave up. However, $f$ can still be decreasing, for example $f(x) = 1/x$ for $x > 0$.

**34.** True, if $f'$ is decreasing, then $f'' < 0$ which means that $f$ is concave down.

**35.** True. If the car is slowing down, then the derivative is decreasing, which means the second derivative is negative.

**36.** True. The function $f(x) = e^x$ is positive, and since it is increasing and concave up, $f'$ and $f''$ are both positive.

**37.** True, any decaying exponential function has these properties, for example $f(x) = e^{-x}$.

**38.** True, any linear function $f$ has $f' = constant$ and so also has $f'' = 0$.

**39.** True, since $e^x$ is concave up everywhere.

**40.** False, since the graph of $f(x) = \ln(x)$ is always concave down, so $f''(x) < 0$.

**41.** True.

**42.** True.

**43.** True, if marginal revenue is greater than marginal cost, then the amount of revenue earned on producing an additional unit will be more than the amount it costs to produce, and so it will increase the profit.

**44.** True, since the total cost function never decreases, so its derivative is never negative.

**45.** False, if the revenue function is linear, then the marginal revenue is constant slope of the line, or 5.

**46.** False, the units of marginal cost are the units of cost divided by the units of quantity, which is dollars/unit.

**47.** True, both the marginal revenue and the marginal cost have dollars as the dependent variable and units as the dependent variable, so their units are the same.

**48.** False, most firms operate at a profit maximizing level. At this level, $P = R - C$ where $P$ is profit, $R$ is revenue, and $C$ is cost. If the profit is maximal at $q^*$, the the derivative of the profit function must be zero at $q^*$. This gives $P'(q^*) = 0 = R(q^*) - C(q^*)$ which says that $R(q^*) = C(q^*)$.

**49.** False, the cost and revenue functions can be equal at $q^*$ but have different slopes.

**50.** True. If two graphs intersect, then the values of the functions are equal at the intersection point.

**51.** True. If $P = f(t)$, then the relative rate of change is

$$\frac{f'(t)}{f(t)} = \frac{1}{P} \cdot \frac{dP}{dt}.$$

**52.** True, since the relative rate of change is $f'/f = 2/10 = 0.20 = 20\%$ per minute.

**53.** True, since 3% of 100 is 3 and the quantity is decreasing.

**54.** False. The rate of change is about 100 animals per month. To find the *relative* rate of change, we divide by the size of the population. The relative rate of change is $-100/500 = 0.20 = 20\%$ per month.

**55.** True. This is one of the reasons that linear functions and exponential functions are so important.

## PROJECTS FOR CHAPTER TWO

**1. (a)** A possible graph is shown in Figure 2.50. At first, the yam heats up very quickly, since the difference in temperature between it and its surroundings is so large. As time goes by, the yam gets hotter and hotter, its rate of temperature increase slows down, and its temperature approaches the temperature of the oven as an asymptote. The graph is thus concave down. (We are considering the average temperature of the yam, since the temperature in its center and on its surface will vary in different ways.)

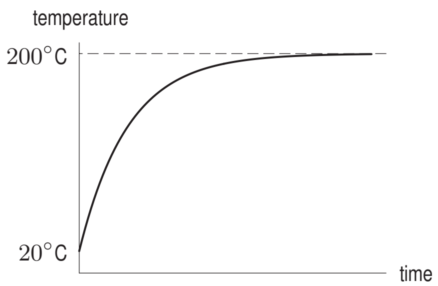



**Figure 2.50**

**(b)** If the rate of temperature increase were to remain 2°/min, in ten minutes the yam's temperature would increase $20°$, from $120°$ to $140°$. Since we know the graph is not linear, but concave down, the actual temperature is between $120°$ and $140°$.

**(c)** In 30 minutes, we know the yam increases in temperature by $45°$ at an average rate of $45/30 = 1.5°$/min. Since the graph is concave down, the temperature at $t = 40$ is therefore between $120 + 1.5(10) = 135°$ and $140°$.

**(d)** If the temperature increases at $2°$/minute, it reaches $150°$ after 15 minutes, at $t = 45$. If the temperature increases at $1.5°$/minute, it reaches $150°$ after 20 minutes, at $t = 50$. So $t$ is between 45 and 50 mins.

**2. (a)** Since illumination is concave up and temperature is concave down, the graph on the left side corresponds to the graph of illumination as a function of distance and the graph on the right side corresponds to the graph of temperature as a function of distance.

**(b)** The illumination drops from $75\%$ at $d = 2$ to $56\%$ at $d = 5$. Since $T = 47$ when $d = 5$ and $T = 53.5$ when $d = 2$,

$$\text{Average rate of change of temperature} = \frac{47 - 53.5}{5 - 2} = \frac{-6.5}{3} = -2.17° \text{ per foot.}$$

**(c)** A good estimate of the illumination when the distance is 3.5 feet is the average of the values of the illumination at 3 feet and at 4 feet. Therefore:

$$\text{Illumination at 3.5 feet} = \frac{67+60}{2} = 63.5\% < 65\%.$$

Since illumination is concave up, the $63.5\%$ is likely to be an overestimate, so you are not likely to be able to read the watch.

**(d)** Let's represent the illumination as a function of the distance by $I(d)$ and the temperature as a function of the distance by $T(d)$. Therefore:

$$\begin{aligned} I(7) &= I(6) + \text{change in } I(d) \\ &\approx I(6) + \text{Average rate of change of } I(d) \\ &= 53\% + (-3\%) = 50\%. \end{aligned}$$

And

$$\begin{aligned} T(7) &= T(6) + \text{ change in } T(d) \\ &\approx T(6) + \text{Average rate of change of } T(d) \\ &= 43.5^\circ + (-4.5^\circ) = 39^\circ F. \end{aligned}$$

**(e)** Let's calculate the distance when $T(d) = 40$ (when we are cold):

$$\begin{aligned} T(d) &= T(6) + T'(6)\cdot(d-6) \\ 40 &= 43.5 + (-4.5)(d-6) \\ 40 &= 43.5 - 4.5d + 27 \\ 40 &= 70.5 - 4.5d \\ -30.5 &= -4.5d \\ d &= \frac{-30.5}{-4.5} = 6.78 \text{ feet.} \end{aligned}$$

From part (d) we know that the illumination is $50\%$ (darkness) when the distance is 7 feet. Therefore, as we walk away from the candle, we first get cold and then we are in darkness.

**3. (a)** From the information given, $C(1994) = 3200$ ppt and $C(2010) = 2750$ ppt.

**(b)** Since the change has been approximately linear, the rate of change is constant:

$$C'(1994) = C'(2010) = \frac{2750-3200}{2010-1994} = -28.125 \text{ ppt per year}.$$

**(c)** The slope is $-28.125$ and $C(1994) = 3200$.
If $t$ is the year, we have

$$C(t) = 3200 - 28.125(t-1994).$$

**(d)** We solve

$$\begin{aligned} 1850 &= 3200 - 28.125(t-1994) \\ 28.125(t-1994) &= 3200 - 1850 \\ t &= 1994 + \frac{3200-1850}{28.125} = 2042. \end{aligned}$$

The CFC level in the atmosphere above the US is predicted to return to the original level in 2042.

**(e)** Since $C''(t) > 0$, the graph bends upward, so the answer to part (d) is too early. The CFCs are expected to reach their original level later than 2042.

## Solutions to Problems on Limits and the Definition of the Derivative

**1.** The answers to parts (a)–(f) are marked in Figure 2.51.

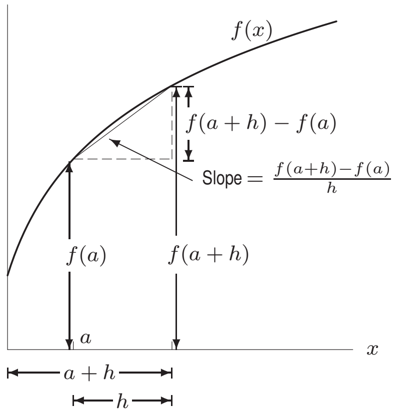



**Figure 2.51**

**2.** The answers to parts (a)–(f) are marked in Figure 2.52.

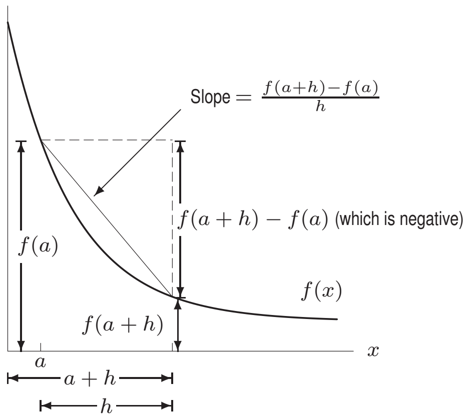



**Figure 2.52**

**3. (a)** When we substitute $h = 0$,we get $0/0$. The expression is undefined at $h = 0$.
**(b)** We have

$$\text{Substituting } h = 0.1\text{:} \quad \frac{e^{5(0.1)} - 1}{0.1} = 6.487.$$
$$\text{Substituting } h = 0.01\text{:} \quad \frac{e^{5(0.01)} - 1}{0.01} = 5.127.$$
$$\text{Substituting } h = 0.001\text{:} \quad \frac{e^{5(0.001)} - 1}{0.001} = 5.013.$$
$$\text{Substituting } h = 0.0001\text{:} \quad \frac{e^{5(0.0001)} - 1}{0.0001} = 5.001.$$

**(c)** It appears that

$$\lim_{h\to 0} \frac{e^{5h} - 1}{h} \approx 5.$$

**4.** **(a)** When we substitute $h = 0$,we get $0/0$. The expression is undefined at $h = 0$.
**(b)** We have

$$\text{Substituting } h = 0.1: \quad \frac{2e^{3(0.1)} - 2}{0.1} = 6.997.$$
$$\text{Substituting } h = 0.01: \quad \frac{2e^{3(0.01)} - 2}{0.01} = 6.091.$$
$$\text{Substituting } h = 0.001: \quad \frac{2e^{3(0.001)} - 2}{0.001} = 6.009.$$
$$\text{Substituting } h = 0.0001: \quad \frac{2e^{3(0.0001)} - 2}{0.0001} = 6.001.$$

**(c)** It appears that

$$\lim_{h\to 0} \frac{2e^{3h} - 2}{h} \approx 6.$$

**5.** **(a)** When we substitute $h = 0$,we get $0/0$. The expression is undefined at $h = 0$.
**(b)** We have

$$\text{Substituting } h = 0.1: \quad \frac{\ln(1.1)}{0.1} = 0.953.$$
$$\text{Substituting } h = 0.01: \quad \frac{\ln(1.01)}{0.01} = 0.995.$$
$$\text{Substituting } h = 0.001: \quad \frac{\ln(1.001)}{0.001} = 0.9995.$$
$$\text{Substituting } h = 0.0001: \quad \frac{\ln(1.0001)}{0.0001} = 0.99995.$$

**(c)** It appears that

$$\lim_{h\to 0} \frac{\ln(h+1)}{h} \approx 1.$$

**6.** **(a)** When we substitute $h = 0$,we get $0/0$. The expression is undefined at $h = 0$.
**(b)** We have

$$\text{Substituting } h = 0.1: \quad \frac{\sin(3(0.1))}{0.1} = 2.955.$$
$$\text{Substituting } h = 0.01: \quad \frac{\sin(3(0.01))}{0.01} = 2.9996.$$
$$\text{Substituting } h = 0.001: \quad \frac{\sin(3(0.001))}{0.001} = 2.999996.$$
$$\text{Substituting } h = 0.0001: \quad \frac{\sin(3(0.0001))}{0.0001} = 2.99999996.$$

**(c)** It appears that

$$\lim_{h\to 0} \frac{\sin(3h)}{h} \approx 3.$$

**7.** Figure 2.53 shows that as $x$ approaches $0$ from either side, the values of $\dfrac{5^x - 1}{x}$ appear to approach $1.6$, suggesting that

$$\lim_{x\to 0} \frac{5^x - 1}{x} \approx 1.6.$$

Zooming in on the graph near $x = 0$ provides further support for this conclusion. Notice that $\dfrac{5^x - 1}{x}$ is undefined at $x = 0$.

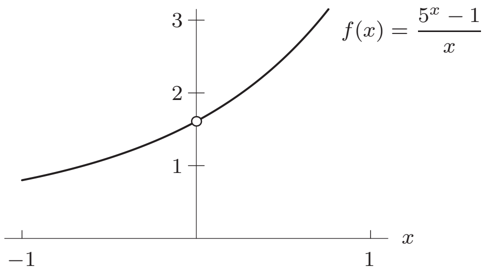



**Figure 2.53**

**8.** Figure 2.54 shows that as $x$ approaches 0 from either side, the value of $\frac{\sin x}{x}$ appears to approach 1, suggesting that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Zooming in on the graph near $x = 0$ provides further support for this conclusion. Notice that $\frac{\sin x}{x}$ is undefined at $x = 0$.

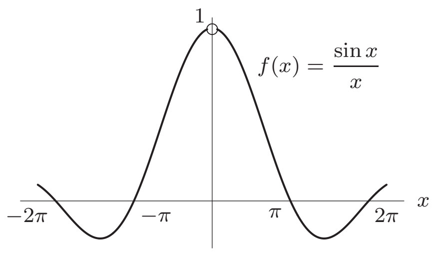



**Figure 2.54**

**9.** Using $h = 0.1, 0.01, 0.001$, we see

$$\frac{7^{0.1} - 1}{0.1} = 2.148$$
$$\frac{7^{0.01} - 1}{0.01} = 1.965$$
$$\frac{7^{0.001} - 1}{0.001} = 1.948$$
$$\frac{7^{0.0001} - 1}{0.0001} = 1.946.$$

This suggests that $\lim_{h\to 0} \frac{7^h - 1}{h} \approx 1.9$.

**10.** Using $h = 0.1, 0.01, 0.001$, we see

$$\frac{(3+0.1)^3 - 27}{0.1} = 27.91$$
$$\frac{(3+0.01)^3 - 27}{0.01} = 27.09$$
$$\frac{(3+0.001)^3 - 27}{0.001} = 27.009.$$

These calculations suggest that $\lim_{h\to 0} \frac{(3+h)^3 - 27}{h} = 27$.

**11.** Using radians,

| $h$ | $(\cos h - 1)/h$ |
|---|---|
| 0.01 | $-0.005$ |
| 0.001 | $-0.0005$ |
| 0.0001 | $-0.00005$ |

These values suggest that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$.

**12.** Using $h = 0.1, 0.01, 0.001$, we see

| $h$ | $(e^{1+h} - e)/h$ |
|---|---|
| 0.01 | 2.7319 |
| 0.001 | 2.7196 |
| 0.0001 | 2.7184 |

These values suggest that $\lim_{h\to 0} \frac{e^{1+h} - e}{h} = 2.7$. In fact, this limit is $e$.

**13.** No, $f(x)$ is not continuous on $0 \le x \le 2$, but it is continuous on the interval $0 \le x \le 0.5$.

**14.** Yes, $f(x)$ is continuous on $0 \le x \le 2$.

**15.** No, $f(x)$ is not continuous on $0 \le x \le 2$, but it is continuous on $0 \le x \le 0.5$.

**16.** No, $f(x)$ is not continuous on $0 \le x \le 2$, but it is continuous on $0 \le x \le 0.5$.

**17.** **(a)** Yes, $f(x)$ is continuous on $1 \le x \le 3$.
**(b)** Yes, $f(x)$ is continuous on $0.5 \le x \le 1.5$.
**(c)** No, $f(x)$ is not continuous on $3 \le x \le 5$ because of the jump at $x = 4$.
**(d)** No, $f(x)$ is not continuous on $2 \le x \le 6$ because of the jump at $x = 4$.

**18.** **(a)** No, $f(x)$ is not continuous on $1 \le x \le 3$ because of the jump at $x = 2$.
**(b)** No, $f(x)$ is not continuous on $0.5 \le x \le 1.5$ because of the jump at $x = 2$.
**(c)** Yes, $f(x)$ is continuous on $3 \le x \le 5$.
**(d)** Yes, $f(x)$ is continuous on $4 \le x \le 5$.

**19.** We have

$$\lim_{h\to 0} \frac{(5 + 2h) - 5}{h} = \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2.$$

**20.** We have

$$\lim_{h\to 0} \frac{(3h - 7) + 7}{h} = \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3.$$

**21.** We have

$$\lim_{h\to 0} \frac{(h + 1)^2 - 1}{h} = \lim_{h\to 0} \frac{(h^2 + 2h + 1) - 1}{h} = \lim_{h\to 0} \frac{h^2 + 2h}{h} = \lim_{h\to 0} (h + 2) = 0 + 2 = 2.$$

**22.** We have

$$\lim_{h\to 0} \frac{(h - 2)^2 - 4}{h} = \lim_{h\to 0} \frac{(h^2 - 4h + 4) - 4}{h} = \lim_{h\to 0} \frac{h^2 - 4h}{h} = \lim_{h\to 0} (h - 4) = 0 - 4 = -4.$$

**23.** Yes: $f(x) = x + 2$ is continuous for all values of $x$.

**24.** Yes: $f(x) = 2^x$ is continuous function for all values of $x$.

**25.** Yes: $f(x) = x^2 + 2$ is a continuous function for all values of $x$.

**26.** Yes: $f(x) = 1/(x - 1)$ is a continuous function on any interval that does not contain $x = 1$.

**27.** No: $f(x) = 1/(x - 1)$ is not continuous on any interval containing $x = 1$.

**28.** Yes: $f(x) = 1/(x^2 + 1)$ is continuous for all values of $x$ because the denominator is never $0$.

**29.** This function is not continuous. Each time someone is born or dies, the number jumps by one.

**30.** Even though the car is stopping and starting, the distance traveled is a continuous function of time.

**31.** Continuous

**32.** Since we can't make a fraction of a pair of pants, the number increases in jumps, so the function is not continuous.

**33.** The time is not a continuous function of position as distance from your starting point, because every time you cross from one time zone into the next, the time jumps by $1$ hour.

**34.** Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{5(x+h)-5x}{h} \\ &= \lim_{h\to 0} \frac{5x+5h-5x}{h} \\ &= \lim_{h\to 0} \frac{5h}{h}. \end{aligned}$$

As long as we let $h$ get close to zero without actually equaling zero, we can cancel the $h$ in the numerator and denominator, and we are left with $f'(x) = 5$.

**35.** Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(3(x+h)-2)-(3x-2)}{h} \\ &= \lim_{h\to 0} \frac{3x+3h-2-3x+2}{h} \\ &= \lim_{h\to 0} \frac{3h}{h}. \end{aligned}$$

As $h$ gets very close to zero without actually equaling zero, we can cancel the $h$ in the numerator and denominator to obtain

$$f'(x) = \lim_{h\to 0}(3) = 3.$$

**36.** Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{((x+h)^2+4)-(x^2+4)}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2+4-x^2-4}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h}. \end{aligned}$$

As $h$ gets very close to zero without actually equaling zero, we can cancel the $h$ in the numerator and denominator to obtain

$$f'(x) = \lim_{h\to 0}(2x+h) = 2x.$$

**37.** Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)^2-3x^2}{h} \\ &= \lim_{h\to 0} \frac{3(x^2+2xh+h^2)-3x^2}{h} \\ &= \lim_{h\to 0} \frac{3x^2+6xh+3h^2-3x^2}{h} \\ &= \lim_{h\to 0} \frac{6xh+3h^2}{h} = \lim_{h\to 0} \frac{h(6x+3h)}{h}. \end{aligned}$$

As $h$ gets very close to zero (but not equal to zero), we can cancel the $h$ in the numerator and denominator to leave the following:

$$f'(x) = \lim_{h\to 0}(6x + 3h).$$

As $h \to 0$, we have $f'(x) = 6x$.

**38.** Using the definition of the derivative, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{((x+h) - (x+h)^2) - (x - x^2)}{h} \\
&= \lim_{h\to 0} \frac{(x + h - (x^2 + 2xh + h^2)) - x + x^2}{h} \\
&= \lim_{h\to 0} \frac{x + h - x^2 - 2xh - h^2 - x + x^2}{h} \\
&= \lim_{h\to 0} \frac{h - 2xh - h^2}{h}.
\end{aligned}$$

As long as we let $h$ get close to zero without actually equaling zero, we can cancel the $h$ in the numerator and denominator, and we are left with $1 - 2x - h$. Taking the limit as $h$ goes to zero, we get $f'(x) = 1 - 2x$ since the other term goes to zero.

**39.** Using the definition of the derivative, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(5(x+h)^2 + 1) - (5x^2 + 1)}{h} \\
&= \lim_{h\to 0} \frac{5(x^2 + 2xh + h^2) + 1 - 5x^2 - 1}{h} \\
&= \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 + 1 - 5x^2 - 1}{h} \\
&= \lim_{h\to 0} \frac{10xh + 5h^2}{h} \\
&= \lim_{h\to 0} \frac{h(10x + 5h)}{h}.
\end{aligned}$$

As $h$ gets very close to zero without actually equaling zero, we can cancel the $h$ in the numerator and denominator to obtain

$$f'(x) = \lim_{h\to 0}(10x + 5h) = 10x.$$

**40.** Using the definition of the derivative, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(2(x+h)^2 + (x+h)) - (2x^2 + x)}{h} \\
&= \lim_{h\to 0} \frac{2(x^2 + 2xh + h^2) + (x+h) - 2x^2 - x}{h} \\
&= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 + x + h - 2x^2 - x}{h} \\
&= \lim_{h\to 0} \frac{4xh + 2h^2 + h}{h} \\
&= \lim_{h\to 0} \frac{h(4x + 2h + 1)}{h}.
\end{aligned}$$

As $h$ gets very close to zero without actually equaling zero, we can cancel the $h$ in the numerator and denominator to obtain

$$f'(x) = \lim_{h\to 0}(4x + 2h + 1) = 4x + 1.$$

**41.** Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0}\frac{-2(x+h)^3 - (-2x^3)}{h} \\ &= \lim_{h\to 0}\frac{-2(x^3 + 3x^2h + 3xh^2 + h^3) + 2x^3}{h} \\ &= \lim_{h\to 0}\frac{-2x^3 - 6x^2h - 6xh^2 - 2h^3 + 2x^3}{h} \\ &= \lim_{h\to 0}\frac{-6x^2h - 6xh^2 - 2h^3}{h}. \end{aligned}$$

As long as we let $h$ get close to zero without actually equaling zero, we can cancel the $h$ in the numerator and denominator, and we are left with $-6x^2 - 6xh - 2h^2$. Taking the limit as $h$ goes to zero, we get $f'(x) = -6x^2$ since the other two terms go to zero.

**42.** Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0}\frac{(1 - (x+h)^3) - (1 - x^3)}{h} \\ &= \lim_{h\to 0}\frac{(1 - (x^3 + 3x^2h + 3xh^2 + h^3)) - 1 + x^3}{h} \\ &= \lim_{h\to 0}\frac{1 - x^3 - 3x^2h - 3xh^2 - h^3 - 1 + x^3}{h} \\ &= \lim_{h\to 0}\frac{-3x^2h - 3xh^2 - h^3}{h}. \end{aligned}$$

As long as we let $h$ get close to zero without actually equaling zero, we can cancel the $h$ in the numerator and denominator, and we are left with $-3x^2 - 3xh - h^2$. Taking the limit as $h$ goes to zero, we get $f'(x) = -3x^2$ since the other two terms go to zero.

**43.** Using the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0}\frac{1/(x+h) - 1/x}{h}. \end{aligned}$$

Writing the numerator over a common denominator and simplifying, we get

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{(x - (x+h))/(x(x+h))}{h} \\ &= \lim_{h\to 0}\frac{-h/(x(x+h))}{h} \\ &= \lim_{h\to 0}\frac{-h}{hx(x+h)}. \end{aligned}$$

As long as we let $h$ get close to zero without actually equaling zero, we can cancel the $h$ in the numerator and denominator, and we are left with $-1/(x(x+h))$. Taking the limit as $h$ goes to zero, we get $f'(x) = -1/x^2$ since $h$ goes to zero.