Solutions Manual for

AUCTION THEORY*

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*V. Krishna, *Auction Theory* (2nd. Ed.), Elsevier, 2009.

2 Private Value Auctions: A First Look

Problem 2.1 *(Power distribution) Suppose there are two bidders with private values that are distributed independently according to the distribution* $F(x) = x^a$ *over* $[0,1]$ *where* $a > 0$. *Find symmetric equilibrium bidding strategies in a first-price auction.*

Solution. Since $N = 2$, $G(x) = F(x) = x^a$. Thus, using the formula on page 16 of the text,

$$\beta^{\mathrm{I}}(x) = x - \int_0^x \frac{G(y)}{G(x)} dy = x - \int_0^x \frac{y^a}{x^a} dy = \frac{a}{1+a} x$$

■

Problem 2.2 *(Pareto distribution) Suppose there are two bidders with private values that are distributed independently according to a Pareto distribution* $F(x) = 1 - (x+1)^{-2}$ *over* $[0,\infty)$. *Find symmetric equilibrium bidding strategies in a first-price auction. Show by direct computation that the expected revenues in a first- and second-price auction are the same.*

Solution. Again, since $N = 2$, $G(x) = F(x) = 1 - (x+1)^{-2}$. Thus,

$$\begin{aligned}
\beta^{\mathrm{I}}(x) &= x - \int_0^x \frac{G(y)}{G(x)} dy \\
&= x - \int_0^x \frac{1-(y+1)^{-2}}{1-(x+1)^{-2}} dy \\
&= \frac{x}{x+2}
\end{aligned}$$

In the first-price auction, the expected revenue of the seller is

$$\begin{aligned}
E\left[R^{\mathrm{I}}\right] &= 2E\left[m^I(x)\right] \\
&= 2E\left[G(x) \times \beta^{\mathrm{I}}(x)\right] \\
&= 2\int_0^\infty \left(1-(x+1)^{-2}\right) \frac{x}{x+2} 2(x+1)^{-3} dx \\
&= 1/3
\end{aligned}$$

Let Y_2 be the second highest value, and its density is $f_2(y) = 2(1-F(y))g(y)$ (see Appendix C).

In a second-price auction, the expected revenue of the seller is

$$\begin{aligned}
E\left[R^{\mathrm{II}}\right] &= E[Y_2] \\
&= \int_0^\infty y2(y+1)^{-2} 2(y+1)^{-3} dy \\
&= 1/3
\end{aligned}$$

Therefore, the expected revenues in the two auctions are the same. ■

Problem 2.3 *(Stochastic dominance) Consider an N-bidder first-price auction with independent private values. Let β be the symmetric equilibrium bidding strategy when which each bidder's value is distributed according to F on $[0, \omega]$. Similarly, let β^* be the equilibrium strategy when each bidder's value distribution is F^* on $[0, \omega^*]$.*

a. *Show that if F^* dominates F in termsof the reverse hazard rate (see Appendix B for a definition) then for all $x \in [0, \omega]$, $\beta^*(x) \geq \beta(x)$.*

b. *By considering $F(x) = 3x - x^2$ on $[0, \frac{1}{2}(3 - \sqrt{5})]$ and $F^*(x) = 3x - 2x^2$ on $\left[0, \frac{1}{2}\right]$, show that the condition that F^* first-order stochastically dominates F is not sufficient to guarantee that $\beta^*(x) \geq \beta(x)$.*

Solution. Part a. Because $G(x) = F(x)^{N-1}$ and $g(x) = (N-1) F(x)^{N-2} f(x)$, the symmetric equilibrium in Proposition 2.2 could be rewritten as follows

$$\begin{aligned}
\beta(x) &= \frac{1}{G(x)} \int_0^x y g(y)\, dy \\
&= \frac{1}{[F(x)]^{N-1}} \int_0^x y (N-1) F(x)^{N-2} f(x)\, dy \\
&= (N-1) \int_0^x y \frac{f(y)}{F(y)} dy \\
&= (N-1) \int_0^x y \sigma(y)\, dy
\end{aligned}$$

where $\sigma(x)$ is the reverse hazard rate. Similarly, we have

$$\beta^*(x) = (N-1) \int_0^x y \sigma^*(y)\, dy$$

So it is easy to see that if F^* dominates F in terms of reverse hazard rate, then $\sigma^*(y) \geq \sigma(y)$ for all $y \in [0, \omega]$. Therefore $\beta^*(x) \geq \beta(x)$ for all $x \in [0, \omega]$.

Part b. Obviously, $F^*(x) \leq F(x)$, so F^* stochastically dominates F. The distributions F and F^* are illustrated in Figure S2.1, where the solid line represents F and the dashed line represents F^*.

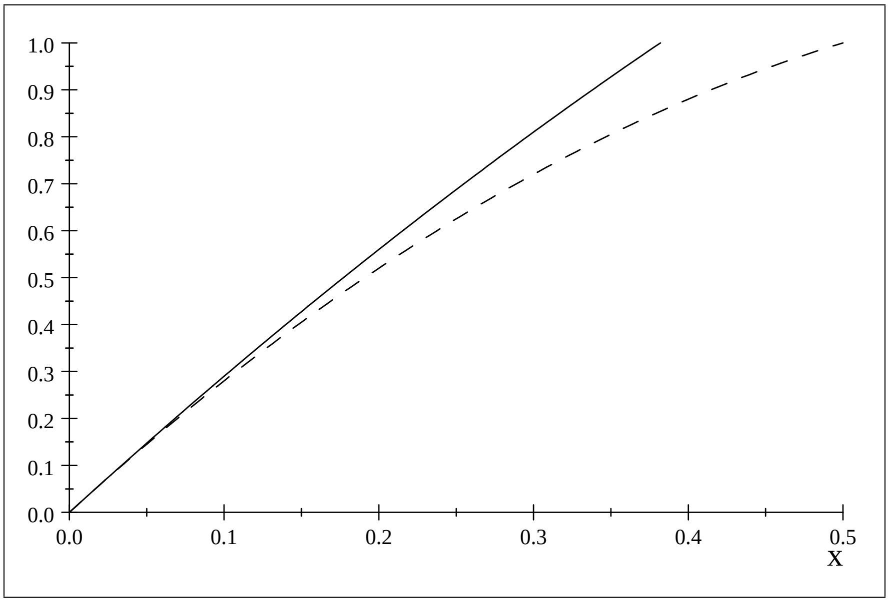

Figure S2.1

Suppose there are two bidders, then

$$\begin{aligned}\beta(x) &= x - \int_0^x \frac{G(y)}{G(x)} dy \\ &= x - \int_0^x \frac{3y - y^2}{3x - x^2} dy \\ &= \frac{1}{6} x \frac{2x - 9}{x - 3}\end{aligned}$$

for $x \in \left[0, \frac{1}{2}\left(3 - \sqrt{5}\right)\right]$. Similarly,

$$\begin{aligned}\beta^*(x) &= x - \int_0^x \frac{3y - 2y^2}{3x - 2x^2} dy \\ &= \frac{1}{6} \frac{x}{2x - 3} (8x - 9)\end{aligned}$$

for $x \in \left[0, \frac{1}{2}\right]$. It is easy to see that $\beta^*(x) < \beta(x)$ for $x \in (0, \frac{1}{2}\left(3 - \sqrt{5}\right)]$. The bidding strategies β and β^* are plotted in Figure S2.2, where β is the solid line and β^* is the dashed line.

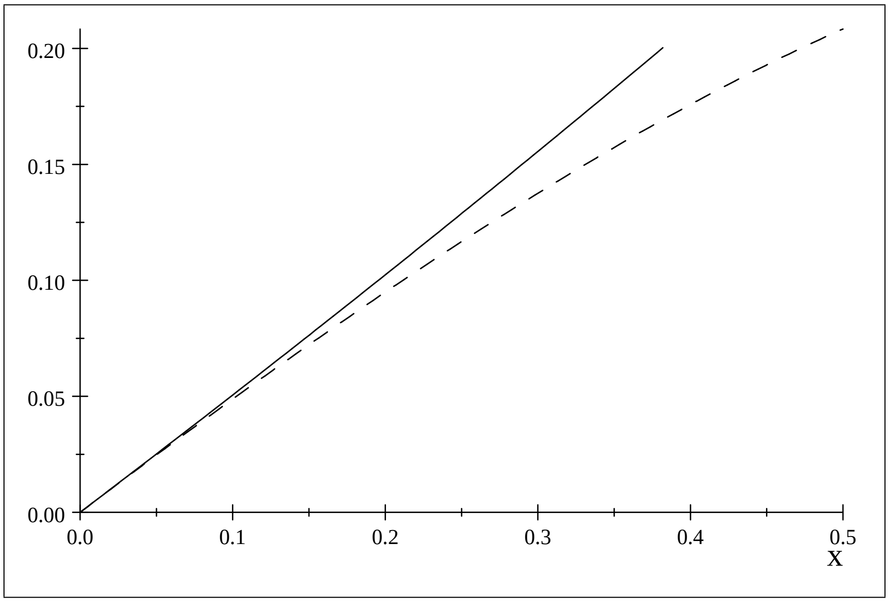

Figure S2.2

■

Problem 2.4 *(Mixed auction) Consider an N-bidder auction which is a "mixture" of a first- and second-price auction in the sense that the highest bidder wins and pays a convex combination of his own bid and the second-highest bid. Precisely, there is a fixed $\alpha \in (0,1)$ such that upon winning, bidder i pays $\alpha b_i + (1-\alpha)(\max_{j \neq i} b_j)$. Find a symmetric equilibrium bidding strategy in such an auction when all bidders' values are distributed according to F.*

Solution. Suppose all bidders other than 1 follow the strategy β. The expected payoff of bidder i from bidding b when his value is x is

$$\begin{aligned}
\Pi(b,x)) &= G\left(\beta^{-1}(b)\right)[x - \alpha b - (1-\alpha)E[\beta(Y_1) \mid \beta(Y_1) \leq b_1]] \\
&= G\left(\beta^{-1}(b)\right)\left[x - \alpha b - (1-\alpha)\frac{\int_0^{\beta^{-1}(b)} \beta(y)g(y)dy}{G(\beta^{-1}(b))}\right] \\
&= G\left(\beta^{-1}(b)\right)(x - \alpha b) - (1-\alpha)\int_0^{\beta^{-1}(b)} \beta(y)g(y)dy
\end{aligned}$$

Maximizing this with respect to b yields the first-order condition:

$$\begin{aligned}
0 &= \frac{g\left(\beta^{-1}(b)\right)}{\beta'\left(\beta^{-1}(b)\right)}(x - \alpha b) - \alpha G\left(\beta^{-1}(b)\right) \\
&\quad - (1-\alpha)\, b g(\beta^{-1}(b))\frac{1}{\beta'\left(\beta^{-1}(b)\right)}
\end{aligned}$$

At a symmetric equilibrium, $b = \beta(x)$, so the first-order condition becomes

$$G(x)\beta'(x) + \frac{1}{\alpha}g(x)\beta(x) = \frac{1}{\alpha}xg(x)$$

Using $G(x)^{(1/\alpha)-1}$ as the integrating factor, the solution to the above differential equation is easily seen to be

$$\beta(x) = \frac{1}{G_\alpha(x)} \int_0^x y g_\alpha(y)\, dy$$

where $G_\alpha \equiv G^{1/\alpha}$ and $g_\alpha = G'_\alpha$. ■

Problem 2.5 *(Resale) Consider a two-bidder first-price auction in which bidders' values are distributed according to F. Let β be the symmetric equilibrium (as derived in Proposition 2.2). Now suppose that after the auction is over, both the losing and winning bids are publicly announced. In addition, there is the possibility of post-auction* resale*: The winner of the auction may, if he so wishes, offer the object to the other bidder at a fixed "take-it-or-leave-it" price of p. If the other bidder agrees, then the object changes hands and the losing bidder pays the winning bidder p. Otherwise, the object stays with the winning bidder and no money changes hands. The possibility of post-auction resale in this manner is commonly known to both bidders prior to participating in the auction. Show that β remains an equilibrium even if resale is allowed. In particular, show that a bidder with value x cannot gain by bidding an amount $b > \beta(x)$ even when he has the option of reselling the object to the other bidder.*

Solution. First, let us consider the resale stage. Suppose bidder 1 wins the auction and the announced bids are b_1 and b_2. Hence bidder 1 can recover bidder 2's private value by $x_2 = \beta_2^{-1}(b_2)$.Therefore bidder 1 suggests the price x_2 which extracts all the surplus from bidder 2 if $x_2 \geq x_1$, and does not offer otherwise. Then bidder 1's payoff is $\max(x_1 - b_1, x_2 - b_1)$. If bidder 1 loses the auction, he gets zero payoff because bidder 2 offers price x_1 to him and extracts all the surplus.

Second, now we move to the auction stage. Let $\beta(x) = \frac{1}{F(x)} \int_0^x y f(y) dy$ be the symmetric equilibrium without resale. We are going to show that any deviation of bidder 1 from $\beta(x)$ is not profitable. Suppose bidder 1 deviates by bidding $\beta(z)$ when his private value is x, while bidder 2 still plays $\beta(x_2)$. Bidder 1's *ex ante* expected payoff is

$$\Pi_1(z, x) = \begin{cases} (x - \beta(z))\, F(z) \text{ if } x \geq z \\ (x - \beta(z))\, F(x) + \int_x^z (y - \beta(z))\, f(y) dy \text{ if } x < z \end{cases}$$

If $x_2 < z \leq x$ there is no resale. If $z > x \geq x_2$, bidder 1 does not offer to bidder 2 and his payoff remains the same. If $z \geq x_2 \geq x$, bidder 1 sells to bidder 2 and the payoff after resale is $x_2 - \beta(z)$. Note that

$$\begin{aligned}
& (x - \beta(z))\, F(x) + \int_x^z (y - \beta(z))\, f(y)dy \\
= & \; xF(x) + \int_x^z yf(y)dy - \beta(z)F(z) \\
= & \; xF(x) + \int_x^z yf(y)dy - \int_0^z yf(y)dy \\
= & \; xF(x) - F(x)\frac{1}{F(x)} \int_0^x yf(y)dy \\
= & \; (x - \beta(x))\, F(x)
\end{aligned}$$

so we have

$$\Pi_1(z, x) = \begin{cases} (x - \beta(z))\, F(z) \text{ if } x \geq z \\ (x - \beta(x))\, F(x) \text{ if } x < z \end{cases}$$

which is not more than $\Pi_1(x, x)$. So no deviation strictly increases a bidder's payoff and $\beta\,(x)$ is still an equilibrium in the presence of resale. ■