Topics in Differentiation

Exercise Set 3.1

1. (a) $1 + y + x\dfrac{dy}{dx} - 6x^2 = 0$, $\dfrac{dy}{dx} = \dfrac{6x^2 - y - 1}{x}$.

(b) $y = \dfrac{2 + 2x^3 - x}{x} = \dfrac{2}{x} + 2x^2 - 1$, $\dfrac{dy}{dx} = -\dfrac{2}{x^2} + 4x$.

(c) From part (a), $\dfrac{dy}{dx} = 6x - \dfrac{1}{x} - \dfrac{1}{x}y = 6x - \dfrac{1}{x} - \dfrac{1}{x}\left(\dfrac{2}{x} + 2x^2 - 1\right) = 4x - \dfrac{2}{x^2}$.

2. (a) $\dfrac{1}{2}y^{-1/2}\dfrac{dy}{dx} - \cos x = 0$ or $\dfrac{dy}{dx} = 2\sqrt{y}\cos x$.

(b) $y = (2 + \sin x)^2 = 4 + 4\sin x + \sin^2 x$ so $\dfrac{dy}{dx} = 4\cos x + 2\sin x\cos x$.

(c) From part (a), $\dfrac{dy}{dx} = 2\sqrt{y}\cos x = 2\cos x(2 + \sin x) = 4\cos x + 2\sin x\cos x$.

3. $2x + 2y\dfrac{dy}{dx} = 0$ so $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

4. $3x^2 + 3y^2\dfrac{dy}{dx} = 3y^2 + 6xy\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{3y^2 - 3x^2}{3y^2 - 6xy} = \dfrac{y^2 - x^2}{y^2 - 2xy}$.

5. $x^2\dfrac{dy}{dx} + 2xy + 3x(3y^2)\dfrac{dy}{dx} + 3y^3 - 1 = 0$, $(x^2 + 9xy^2)\dfrac{dy}{dx} = 1 - 2xy - 3y^3$, so $\dfrac{dy}{dx} = \dfrac{1 - 2xy - 3y^3}{x^2 + 9xy^2}$.

6. $x^3(2y)\dfrac{dy}{dx} + 3x^2y^2 - 5x^2\dfrac{dy}{dx} - 10xy + 1 = 0$, $(2x^3y - 5x^2)\dfrac{dy}{dx} = 10xy - 3x^2y^2 - 1$, so $\dfrac{dy}{dx} = \dfrac{10xy - 3x^2y^2 - 1}{2x^3y - 5x^2}$.

7. $-\dfrac{1}{2x^{3/2}} - \dfrac{\frac{dy}{dx}}{2y^{3/2}} = 0$, so $\dfrac{dy}{dx} = -\dfrac{y^{3/2}}{x^{3/2}}$.

8. $2x = \dfrac{(x - y)(1 + dy/dx) - (x + y)(1 - dy/dx)}{(x - y)^2}$, $2x(x - y)^2 = -2y + 2x\dfrac{dy}{dx}$, so $\dfrac{dy}{dx} = \dfrac{x(x - y)^2 + y}{x}$.

9. $\cos(x^2y^2)\left[x^2(2y)\dfrac{dy}{dx} + 2xy^2\right] = 1$, so $\dfrac{dy}{dx} = \dfrac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2)}$.

10. $-\sin(xy^2)\left[y^2 + 2xy\dfrac{dy}{dx}\right] = \dfrac{dy}{dx}$, so $\dfrac{dy}{dx} = -\dfrac{y^2\sin(xy^2)}{2xy\sin(xy^2) + 1}$.

11. $3\tan^2(xy^2+y)\sec^2(xy^2+y)\left(2xy\dfrac{dy}{dx}+y^2+\dfrac{dy}{dx}\right)=1$, so $\dfrac{dy}{dx}=\dfrac{1-3y^2\tan^2(xy^2+y)\sec^2(xy^2+y)}{3(2xy+1)\tan^2(xy^2+y)\sec^2(xy^2+y)}$.

12. $\dfrac{(1+\sec y)[3xy^2(dy/dx)+y^3]-xy^3(\sec y\tan y)(dy/dx)}{(1+\sec y)^2}=4y^3\dfrac{dy}{dx}$, multiply through by $(1+\sec y)^2$ and solve for $\dfrac{dy}{dx}$ to get $\dfrac{dy}{dx}=\dfrac{y(1+\sec y)}{4y(1+\sec y)^2-3x(1+\sec y)+xy\sec y\tan y}$.

13. $4x-6y\dfrac{dy}{dx}=0$, $\dfrac{dy}{dx}=\dfrac{2x}{3y}$, $4-6\left(\dfrac{dy}{dx}\right)^2-6y\dfrac{d^2y}{dx^2}=0$, so $\dfrac{d^2y}{dx^2}=-\dfrac{3\left(\frac{dy}{dx}\right)^2-2}{3y}=\dfrac{2(3y^2-2x^2)}{9y^3}=-\dfrac{8}{9y^3}$.

14. $\dfrac{dy}{dx}=-\dfrac{x^2}{y^2}$, $\dfrac{d^2y}{dx^2}=-\dfrac{y^2(2x)-x^2(2ydy/dx)}{y^4}=-\dfrac{2xy^2-2x^2y(-x^2/y^2)}{y^4}=-\dfrac{2x(y^3+x^3)}{y^5}$, but $x^3+y^3=1$, so $\dfrac{d^2y}{dx^2}=-\dfrac{2x}{y^5}$.

15. $\dfrac{dy}{dx}=-\dfrac{y}{x}$, $\dfrac{d^2y}{dx^2}=-\dfrac{x(dy/dx)-y(1)}{x^2}=-\dfrac{x(-y/x)-y}{x^2}=\dfrac{2y}{x^2}$.

16. $y+x\dfrac{dy}{dx}+2y\dfrac{dy}{dx}=0$, $\dfrac{dy}{dx}=-\dfrac{y}{x+2y}$, $2\dfrac{dy}{dx}+x\dfrac{d^2y}{dx^2}+2\left(\dfrac{dy}{dx}\right)^2+2y\dfrac{d^2y}{dx^2}=0$, $\dfrac{d^2y}{dx^2}=\dfrac{2y(x+y)}{(x+2y)^3}$.

17. $\dfrac{dy}{dx}=(1+\cos y)^{-1}$, $\dfrac{d^2y}{dx^2}=-(1+\cos y)^{-2}(-\sin y)\dfrac{dy}{dx}=\dfrac{\sin y}{(1+\cos y)^3}$.

18. $\dfrac{dy}{dx}=\dfrac{\cos y}{1+x\sin y}$, $\dfrac{d^2y}{dx^2}=\dfrac{(1+x\sin y)(-\sin y)(dy/dx)-(\cos y)[(x\cos y)(dy/dx)+\sin y]}{(1+x\sin y)^2}=$ $-\dfrac{2\sin y\cos y+(x\cos y)(2\sin^2 y+\cos^2 y)}{(1+x\sin y)^3}$, but $x\cos y=y$, $2\sin y\cos y=\sin 2y$, and $\sin^2 y+\cos^2 y=1$, so $\dfrac{d^2y}{dx^2}=-\dfrac{\sin 2y+y(\sin^2 y+1)}{(1+x\sin y)^3}$.

19. By implicit differentiation, $2x+2y(dy/dx)=0$, $\dfrac{dy}{dx}=-\dfrac{x}{y}$; at $(1/2,\sqrt{3}/2)$, $\dfrac{dy}{dx}=-\sqrt{3}/3$; at $(1/2,-\sqrt{3}/2)$, $\dfrac{dy}{dx}=+\sqrt{3}/3$. Directly, at the upper point $y=\sqrt{1-x^2}$, $\dfrac{dy}{dx}=\dfrac{-x}{\sqrt{1-x^2}}=-\dfrac{1/2}{\sqrt{3/4}}=-1/\sqrt{3}$ and at the lower point $y=-\sqrt{1-x^2}$, $\dfrac{dy}{dx}=\dfrac{x}{\sqrt{1-x^2}}=+1/\sqrt{3}$.

20. If $y^2-x+1=0$, then $y=\sqrt{x-1}$ goes through the point $(10,3)$ so $dy/dx=1/(2\sqrt{x-1})$. By implicit differentiation $dy/dx=1/(2y)$. In both cases, $dy/dx|_{(10,3)}=1/6$. Similarly $y=-\sqrt{x-1}$ goes through $(10,-3)$ so $dy/dx=-1/(2\sqrt{x-1})=-1/6$ which yields $dy/dx=1/(2y)=-1/6$.

21. False; $x=y^2$ defines two functions $y=\pm\sqrt{x}$. See Definition 3.1.1.

22. True.

23. False; the equation is equivalent to $x^2=y^2$ which is satisfied by $y=|x|$.

24. True.

25. $4x^3+4y^3\dfrac{dy}{dx}=0$, so $\dfrac{dy}{dx}=-\dfrac{x^3}{y^3}=-\dfrac{1}{15^{3/4}}\approx-0.1312$.

26. $3y^2\dfrac{dy}{dx}+x^2\dfrac{dy}{dx}+2xy+2x-6y\dfrac{dy}{dx}=0$, so $\dfrac{dy}{dx}=-2x\dfrac{y+1}{3y^2+x^2-6y}=0$ at $x=0$.

27. $4(x^2+y^2)\left(2x+2y\dfrac{dy}{dx}\right)=25\left(2x-2y\dfrac{dy}{dx}\right)$, $\dfrac{dy}{dx}=\dfrac{x[25-4(x^2+y^2)]}{y[25+4(x^2+y^2)]}$; at $(3,1)$ $\dfrac{dy}{dx}=-9/13$.

28. $\dfrac{2}{3}\left(x^{-1/3}+y^{-1/3}\dfrac{dy}{dx}\right)=0$, $\dfrac{dy}{dx}=-\dfrac{y^{1/3}}{x^{1/3}}=\sqrt{3}$ at $(-1,3\sqrt{3})$.

29. $4a^3\dfrac{da}{dt}-4t^3=6\left(a^2+2at\dfrac{da}{dt}\right)$, solve for $\dfrac{da}{dt}$ to get $\dfrac{da}{dt}=\dfrac{2t^3+3a^2}{2a^3-6at}$.

30. $\dfrac{1}{2}u^{-1/2}\dfrac{du}{dv}+\dfrac{1}{2}v^{-1/2}=0$, so $\dfrac{du}{dv}=-\dfrac{\sqrt{u}}{\sqrt{v}}$.

31. $2a^2\omega\dfrac{d\omega}{d\lambda}+2b^2\lambda=0$, so $\dfrac{d\omega}{d\lambda}=-\dfrac{b^2\lambda}{a^2\omega}$.

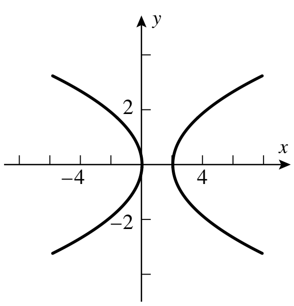
32. $1=(\cos x)\dfrac{dx}{dy}$, so $\dfrac{dx}{dy}=\dfrac{1}{\cos x}=\sec x$.

33. $2x+x\dfrac{dy}{dx}+y+2y\dfrac{dy}{dx}=0$. Substitute $y=-2x$ to obtain $-3x\dfrac{dy}{dx}=0$. Since $x=\pm 1$ at the indicated points, $\dfrac{dy}{dx}=0$ there.

34. (a) The equation and the point $(1,1)$ are both symmetric in x and y (if you interchange the two variables you get the same equation and the same point). Therefore the outcome "horizontal tangent at $(1,1)$" could be replaced by "vertical tangent at $(1,1)$", and these cannot both be the case.

(b) Implicit differentiation yields $\dfrac{dy}{dx}=\dfrac{2x-y}{x-2y}$, which is zero only if $y=2x$; coupled with the equation $x^2-xy+y^2=1$ we obtain $x^2-2x^2+4x^2=1$, or $3x^2=1, x=(\sqrt{3}/3,2\sqrt{3}/3)$ and $(-\sqrt{3}/3,-2\sqrt{3}/3)$.

35. (a)



(b) Implicit differentiation of the curve yields $(4y^3+2y)\dfrac{dy}{dx}=2x-1$, so $\dfrac{dy}{dx}=0$ only if $x=1/2$ but $y^4+y^2\geq 0$ so $x=1/2$ is impossible.

(c) $x^2-x-(y^4+y^2)=0$, so by the Quadratic Formula, $x=\dfrac{-1\pm\sqrt{(2y^2+1)^2}}{2}=1+y^2$ or $-y^2$, and we have the two parabolas $x=-y^2, x=1+y^2$.

36. By implicit differentiation, $2y(2y^2+1)\dfrac{dy}{dx}=2x-1$, $\dfrac{dx}{dy}=\dfrac{2y(2y^2+1)}{2x-1}=0$ only if $2y(2y^2+1)=0$, which can only hold if $y=0$. From $y^4+y^2=x(x-1)$, if $y=0$ then $x=0$ or 1, and so $(0,0)$ and $(1,0)$ are the two points where the tangent is vertical.

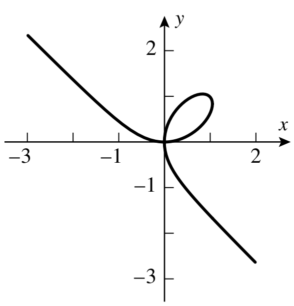
37. The point (1,1) is on the graph, so $1 + a = b$. The slope of the tangent line at (1,1) is $-4/3$; use implicit differentiation to get $\dfrac{dy}{dx} = -\dfrac{2xy}{x^2 + 2ay}$ so at (1,1), $-\dfrac{2}{1+2a} = -\dfrac{4}{3}$, $1+2a = 3/2$, $a = 1/4$ and hence $b = 1 + 1/4 = 5/4$.

38. The slope of the line $x + 2y - 2 = 0$ is $m_1 = -1/2$, so the line perpendicular has slope $m = 2$ (negative reciprocal). The slope of the curve $y^3 = 2x^2$ can be obtained by implicit differentiation: $3y^2\dfrac{dy}{dx} = 4x, \dfrac{dy}{dx} = \dfrac{4x}{3y^2}$. Set $\dfrac{dy}{dx} = 2$; $\dfrac{4x}{3y^2} = 2$, $x = (3/2)y^2$. Use this in the equation of the curve: $y^3 = 2x^2 = 2((3/2)y^2)^2 = (9/2)y^4, y = 2/9$, $x = \dfrac{3}{2}\left(\dfrac{2}{9}\right)^2 = \dfrac{2}{27}$.

39. We shall find when the curves intersect and check that the slopes are negative reciprocals. For the intersection solve the simultaneous equations $x^2 + (y-c)^2 = c^2$ and $(x-k)^2 + y^2 = k^2$ to obtain $cy = kx = \dfrac{1}{2}(x^2 + y^2)$. Thus $x^2 + y^2 = cy + kx$, or $y^2 - cy = -x^2 + kx$, and $\dfrac{y-c}{x} = -\dfrac{x-k}{y}$. Differentiating the two families yields (black) $\dfrac{dy}{dx} = -\dfrac{x}{y-c}$, and (gray) $\dfrac{dy}{dx} = -\dfrac{x-k}{y}$. But it was proven that these quantities are negative reciprocals of each other.

40. Differentiating, we get the equations (black) $x\dfrac{dy}{dx} + y = 0$ and (gray) $2x - 2y\dfrac{dy}{dx} = 0$. The first says the (black) slope is $-\dfrac{y}{x}$ and the second says the (gray) slope is $\dfrac{x}{y}$, and these are negative reciprocals of each other.

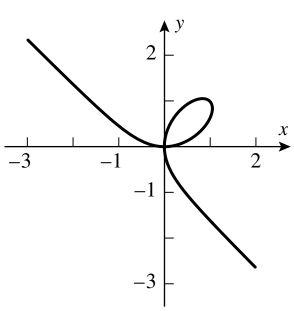
41. (a)



(b) $x \approx 0.84$

(c) Use implicit differentiation to get $dy/dx = (2y - 3x^2)/(3y^2 - 2x)$, so $dy/dx = 0$ if $y = (3/2)x^2$. Substitute this into $x^3 - 2xy + y^3 = 0$ to obtain $27x^6 - 16x^3 = 0, x^3 = 16/27$, $x = 2^{4/3}/3$ and hence $y = 2^{5/3}/3$.

42. (a)



(b) Evidently (by symmetry) the tangent line at the point $x = 1, y = 1$ has slope -1.

(c) Use implicit differentiation to get $dy/dx = (2y - 3x^2)/(3y^2 - 2x)$, so $dy/dx = -1$ if $2y - 3x^2 = -3y^2 + 2x, 2(y-x) + 3(y-x)(y+x) = 0$. One solution is $y = x$; this together with $x^3 + y^3 = 2xy$ yields $x = y = 1$.

For these values $dy/dx = -1$, so that $(1,1)$ is a solution. To prove that there is no other solution, suppose $y \neq x$. From $dy/dx = -1$ it follows that $2(y-x) + 3(y-x)(y+x) = 0$. But $y \neq x$, so $x + y = -2/3$, which is not true for any point in the first quadrant.

43. By the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx}$. Using implicit differentiation for $2y^3t + t^3y = 1$ we get $\dfrac{dy}{dt} = -\dfrac{2y^3+3t^2y}{6ty^2+t^3}$, but $\dfrac{dt}{dx} = \dfrac{1}{\cos t}$, so $\dfrac{dy}{dx} = -\dfrac{2y^3+3t^2y}{(6ty^2+t^3)\cos t}$.

44. Let $P(x_0, y_0)$ be a point where a line through the origin is tangent to the curve $2x^2 - 4x + y^2 + 1 = 0$. Implicit differentiation applied to the equation of the curve gives $dy/dx = (2-2x)/y$. At P the slope of the curve must equal the slope of the line so $(2-2x_0)/y_0 = y_0/x_0$, or $y_0^2 = 2x_0(1-x_0)$. But $2x_0^2 - 4x_0 + y_0^2 + 1 = 0$ because (x_0, y_0) is on the curve, and elimination of y_0^2 in the latter two equations gives $2x_0 = 4x_0 - 1$, $x_0 = 1/2$ which when substituted into $y_0^2 = 2x_0(1-x_0)$ yields $y_0^2 = 1/2$, so $y_0 = \pm\sqrt{2}/2$. The slopes of the lines are $(\pm\sqrt{2}/2)/(1/2) = \pm\sqrt{2}$ and their equations are $y = \sqrt{2}x$ and $y = -\sqrt{2}x$.

Exercise Set 3.2

1. $\dfrac{1}{5x}(5) = \dfrac{1}{x}$.

2. $\dfrac{1}{x/3}\dfrac{1}{3} = \dfrac{1}{x}$.

3. $\dfrac{1}{1+x}$.

4. $\dfrac{1}{2+\sqrt{x}}\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{1}{2\sqrt{x}(2+\sqrt{x})}$.

5. $\dfrac{1}{x^2-1}(2x) = \dfrac{2x}{x^2-1}$.

6. $\dfrac{3x^2-14x}{x^3-7x^2-3}$.

7. $\dfrac{d}{dx}\ln x - \dfrac{d}{dx}\ln(1+x^2) = \dfrac{1}{x} - \dfrac{2x}{1+x^2} = \dfrac{1-x^2}{x(1+x^2)}$.

8. $\dfrac{d}{dx}(\ln|1+x| - \ln|1-x|) = \dfrac{1}{1+x} - \dfrac{-1}{1-x} = \dfrac{2}{1-x^2}$.

9. $\dfrac{d}{dx}(2\ln x) = 2\dfrac{d}{dx}\ln x = \dfrac{2}{x}$.

10. $3(\ln x)^2\dfrac{1}{x}$.

11. $\dfrac{1}{2}(\ln x)^{-1/2}\left(\dfrac{1}{x}\right) = \dfrac{1}{2x\sqrt{\ln x}}$.

12. $\dfrac{d}{dx}\dfrac{1}{2}\ln x = \dfrac{1}{2x}$.

13. $\ln x + x\dfrac{1}{x} = 1 + \ln x$.

14. $x^3\left(\frac{1}{x}\right)+(3x^2)\ln x = x^2(1+3\ln x)$.

15. $2x\log_2(3-2x)+\frac{-2x^2}{(\ln 2)(3-2x)}$.

16. $\left[\log_2(x^2-2x)\right]^3+3x\left[\log_2(x^2-2x)\right]^2\frac{2x-2}{(x^2-2x)\ln 2}$.

17. $\frac{2x(1+\log x)-x/(\ln 10)}{(1+\log x)^2}$.

18. $1/[x(\ln 10)(1+\log x)^2]$.

19. $\frac{1}{\ln x}\left(\frac{1}{x}\right)=\frac{1}{x\ln x}$.

20. $\frac{1}{\ln(\ln(x))}\frac{1}{\ln x}\frac{1}{x}$.

21. $\frac{1}{\tan x}(\sec^2 x)=\sec x\csc x$.

22. $\frac{1}{\cos x}(-\sin x)=-\tan x$.

23. $-\sin(\ln x)\frac{1}{x}$.

24. $2\sin(\ln x)\cos(\ln x)\frac{1}{x}=\frac{\sin(2\ln x)}{x}=\frac{\sin(\ln x^2)}{x}$.

25. $\frac{1}{\ln 10\sin^2 x}(2\sin x\cos x)=2\frac{\cot x}{\ln 10}$.

26. $\frac{1}{\ln 10}\frac{d}{dx}\ln\cos^2 x=\frac{1}{\ln 10}\frac{-2\sin x\cos x}{\cos^2 x}=-\frac{2\tan x}{\ln 10}$.

27. $\frac{d}{dx}\left[3\ln(x-1)+4\ln(x^2+1)\right]=\frac{3}{x-1}+\frac{8x}{x^2+1}=\frac{11x^2-8x+3}{(x-1)(x^2+1)}$.

28. $\frac{d}{dx}[2\ln\cos x+\frac{1}{2}\ln(1+x^4)]=-2\tan x+\frac{2x^3}{1+x^4}$.

29. $\frac{d}{dx}\left[\ln\cos x-\frac{1}{2}\ln(4-3x^2)\right]=-\tan x+\frac{3x}{4-3x^2}$

30. $\frac{d}{dx}\left(\frac{1}{2}[\ln(x-1)-\ln(x+1)]\right)=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$.

31. True, because $\frac{dy}{dx}=\frac{1}{x}$, so as $x=a\to 0^+$, the slope approaches infinity.

32. False, e.g. $f(x)=\sqrt{x}$.

33. True; if $x>0$ then $\frac{d}{dx}\ln|x|=1/x$; if $x<0$ then $\frac{d}{dx}\ln|x|=1/x$.

34. False; $\dfrac{d}{dx}(\ln x)^2 = 2\dfrac{1}{x}\ln x \neq \dfrac{2}{x}$.

35. $\ln|y| = \ln|x| + \dfrac{1}{3}\ln|1+x^2|$, so $\dfrac{dy}{dx} = x\sqrt[3]{1+x^2}\left[\dfrac{1}{x} + \dfrac{2x}{3(1+x^2)}\right]$.

36. $\ln|y| = \dfrac{1}{5}[\ln|x-1| - \ln|x+1|]$, so $\dfrac{dy}{dx} = \dfrac{1}{5}\sqrt[5]{\dfrac{x-1}{x+1}}\left[\dfrac{1}{x-1} - \dfrac{1}{x+1}\right]$.

37. $\ln|y| = \dfrac{1}{3}\ln|x^2-8| + \dfrac{1}{2}\ln|x^3+1| - \ln|x^6-7x+5|$, so
$\dfrac{dy}{dx} = \dfrac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}\left[\dfrac{2x}{3(x^2-8)} + \dfrac{3x^2}{2(x^3+1)} - \dfrac{6x^5-7}{x^6-7x+5}\right]$.

38. $\ln|y| = \ln|\sin x| + \ln|\cos x| + 3\ln|\tan x| - \dfrac{1}{2}\ln|x|$, so $\dfrac{dy}{dx} = \dfrac{\sin x\cos x\tan^3 x}{\sqrt{x}}\left[\cot x - \tan x + \dfrac{3\sec^2 x}{\tan x} - \dfrac{1}{2x}\right]$

39. **(a)** $\log_x e = \dfrac{\ln e}{\ln x} = \dfrac{1}{\ln x}$, so $\dfrac{d}{dx}[\log_x e] = -\dfrac{1}{x(\ln x)^2}$.

(b) $\log_x 2 = \dfrac{\ln 2}{\ln x}$, so $\dfrac{d}{dx}[\log_x 2] = -\dfrac{\ln 2}{x(\ln x)^2}$.

40. **(a)** From $\log_a b = \dfrac{\ln b}{\ln a}$ for $a, b > 0$ it follows that $\log_{(1/x)} e = \dfrac{\ln e}{\ln(1/x)} = -\dfrac{1}{\ln x}$, so $\dfrac{d}{dx}\left[\log_{(1/x)} e\right] = \dfrac{1}{x(\ln x)^2}$.

(b) $\log_{(\ln x)} e = \dfrac{\ln e}{\ln(\ln x)} = \dfrac{1}{\ln(\ln x)}$, so $\dfrac{d}{dx}\log_{(\ln x)} e = -\dfrac{1}{(\ln(\ln x))^2}\dfrac{1}{x\ln x} = -\dfrac{1}{x(\ln x)(\ln(\ln x))^2}$.

41. $f'(x_0) = \dfrac{1}{x_0} = e,\ y - (-1) = e(x - x_0) = ex - 1,\ y = ex - 2$.

42. $y = \log x = \dfrac{\ln x}{\ln 10},\ y' = \dfrac{1}{x\ln 10},\ y_0 = \log 10 = 1,\ y - 1 = \dfrac{1}{10\ln 10}(x - 10)$.

43. $f(x_0) = f(-e) = 1,\ f'(x)|_{x=-e} = -\dfrac{1}{e},\ y - 1 = -\dfrac{1}{e}(x + e),\ y = -\dfrac{1}{e}x$.

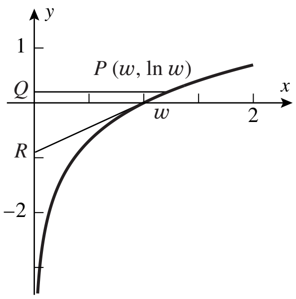
44. $y - \ln 2 = -\dfrac{1}{2}(x+2),\ y = -\dfrac{1}{2}x + \ln 2 - 1$.

45. **(a)** Let the equation of the tangent line be $y = mx$ and suppose that it meets the curve at (x_0, y_0). Then $m = \left.\dfrac{1}{x}\right|_{x=x_0} = \dfrac{1}{x_0}$ and $y_0 = mx_0 + b = \ln x_0$. So $m = \dfrac{1}{x_0} = \dfrac{\ln x_0}{x_0}$ and $\ln x_0 = 1, x_0 = e, m = \dfrac{1}{e}$ and the equation of the tangent line is $y = \dfrac{1}{e}x$.

(b) Let $y = mx + b$ be a line tangent to the curve at (x_0, y_0). Then b is the y-intercept and the slope of the tangent line is $m = \dfrac{1}{x_0}$. Moreover, at the point of tangency, $mx_0 + b = \ln x_0$ or $\dfrac{1}{x_0}x_0 + b = \ln x_0$, $b = \ln x_0 - 1$, as required.

46. Let $y(x) = u(x)v(x)$, then $\ln y = \ln u + \ln v$, so $y'/y = u'/u + v'/v$, or $y' = uv' + vu'$. Let $y = u/v$, then $\ln y = \ln u - \ln v$, so $y'/y = u'/u - v'/v$, or $y' = u'/v - uv'/v^2 = (u'v - uv')/v^2$. The logarithm of a product (quotient) is the sum (difference) of the logarithms.

47. The area of the triangle PQR is given by the formula $|PQ||QR|/2$. $|PQ| = w$, and, by Exercise 45 part (b), $|QR| = 1$, so the area is $w/2$.



48. Since $y = 2\ln x$, let $y = 2z$; then $z = \ln x$ and we apply the result of Exercise 45 to find that the area is, in the x-z plane, $w/2$. In the x-y plane, since $y = 2z$, the vertical dimension gets doubled, so the area is w.

49. If $x = 0$ then $y = \ln e = 1$, and $\dfrac{dy}{dx} = \dfrac{1}{x+e}$. But $e^y = x + e$, so $\dfrac{dy}{dx} = \dfrac{1}{e^y} = e^{-y}$.

50. If $x = 0$ then $y = -\ln e^2 = -2$, and $\dfrac{dy}{dx} = \dfrac{1}{e^2 - x}$. But $e^y = \dfrac{1}{e^2 - x}$, so $\dfrac{dy}{dx} = e^y$.

51. Let $y = \ln(x + a)$. Following Exercise 49 we get $\dfrac{dy}{dx} = \dfrac{1}{x+a} = e^{-y}$, and when $x = 0, y = \ln(a) = 0$ if $a = 1$, so let $a = 1$, then $y = \ln(x + 1)$.

52. Let $y = -\ln(a - x)$, then $\dfrac{dy}{dx} = \dfrac{1}{a-x}$. But $e^y = \dfrac{1}{a-x}$, so $\dfrac{dy}{dx} = e^y$. If $x = 0$ then $y = -\ln(a) = -\ln 2$ provided $a = 2$, so $y = -\ln(2 - x)$.

53. **(a)** Set $f(x) = \ln(1 + 3x)$. Then $f'(x) = \dfrac{3}{1+3x}$, $f'(0) = 3$. But $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x\to 0} \dfrac{\ln(1+3x)}{x}$.

(b) Set $f(x) = \ln(1 - 5x)$. Then $f'(x) = \dfrac{-5}{1-5x}$, $f'(0) = -5$. But $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x\to 0} \dfrac{\ln(1-5x)}{x}$.

54. **(a)** $f(x) = \ln x$; $f'(e^2) = \lim\limits_{\Delta x\to 0} \dfrac{\ln(e^2 + \Delta x) - 2}{\Delta x} = \dfrac{d}{dx}(\ln x)\Big|_{x=e^2} = \dfrac{1}{x}\Big|_{x=e^2} = e^{-2}$.

(b) $f(w) = \ln w$; $f'(1) = \lim\limits_{w\to 1} \dfrac{\ln w - \ln 1}{w - 1} = \lim\limits_{w\to 1} \dfrac{\ln w}{w-1} = \dfrac{1}{w}\Big|_{w=1} = 1$.

55. **(a)** Let $f(x) = \ln(\cos x)$, then $f(0) = \ln(\cos 0) = \ln 1 = 0$, so $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x\to 0} \dfrac{\ln(\cos x)}{x}$, and $f'(0) = -\tan 0 = 0$.

(b) Let $f(x) = x^{\sqrt{2}}$, then $f(1) = 1$, so $f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h\to 0} \dfrac{(1+h)^{\sqrt{2}} - 1}{h}$, and $f'(x) = \sqrt{2}x^{\sqrt{2}-1}, f'(1) = \sqrt{2}$.

56.
$$\begin{aligned}\frac{d}{dx}[\log_b x] &= \lim_{h\to 0}\frac{\log_b(x+h)-\log_b(x)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\log_b\left(\frac{x+h}{x}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\log_b\left(1+\frac{h}{x}\right)\\
&= \lim_{v\to 0}\frac{1}{vx}\log_b(1+v) && \text{Let } v=h/x \text{ and note that } v\to 0 \text{ as } h\to 0\\
&= \frac{1}{x}\lim_{v\to 0}\frac{1}{v}\log_b(1+v) && h \text{ and } v \text{ are variable, whereas } x \text{ is constant}\\
&= \frac{1}{x}\lim_{v\to 0}\log_b(1+v)^{1/v}\\
&= \frac{1}{x}\log_b \lim_{v\to 0}(1+v)^{1/v} && \text{Theorem 1.5.5}\\
&= \frac{1}{x}\log_b e = \frac{1}{x}\cdot\frac{\ln e}{\ln b} = \frac{1}{x\ln b}. && \text{Formula 7 of Section 1.3}\end{aligned}$$

57. Differentiating implicitly gives $0 = \dfrac{1}{p}\dfrac{dp}{dt} - \dfrac{1}{2.3-0.0046p}(-0.0046)\dfrac{dp}{dt} - 2.3$, from which $\dfrac{dp}{dt} = 0.0046p(500-p)$ as claimed.

58. Implicit differentiation yields $\dfrac{1}{y}\dfrac{dy}{dt} + \dfrac{1}{1-y}\dfrac{dy}{dt} = \alpha$, from which we obtain that $\dfrac{dy}{dt} = \alpha y(1-y)$. The right side is an inverted parabola, with a maximum value of $\alpha/4$ at $y = 1/2$. Thus y is growing most rapidly when half the population has the information.

Exercise Set 3.3

1. (a) $f'(x) = 5x^4 + 3x^2 + 1 \geq 1$ so f is increasing and one-to-one on $-\infty < x < +\infty$.

(b) $f(1) = 3$ so $1 = f^{-1}(3)$; $\dfrac{d}{dx}f^{-1}(x) = \dfrac{1}{f'(f^{-1}(x))}$, $(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{9}$.

2. (a) $f'(x) = 3x^2 + 2e^x$; $f'(x) > 0$ for all x (since $3x^2 \geq 0$ and $2e^x > 0$), so f is increasing and one-to-one on $-\infty < x < +\infty$.

(b) $f(0) = 2$ so $0 = f^{-1}(2)$; $\dfrac{d}{dx}f^{-1}(x) = \dfrac{1}{f'(f^{-1}(x))}$, $(f^{-1})'(2) = \dfrac{1}{f'(0)} = \dfrac{1}{2}$.

3. $f^{-1}(x) = \dfrac{2}{x} - 3$, so directly $\dfrac{d}{dx}f^{-1}(x) = -\dfrac{2}{x^2}$. Using Formula (2), $f'(x) = \dfrac{-2}{(x+3)^2}$, so $\dfrac{1}{f'(f^{-1}(x))} = -(1/2)(f^{-1}(x)+3)^2$, and $\dfrac{d}{dx}f^{-1}(x) = -(1/2)\left(\dfrac{2}{x}\right)^2 = -\dfrac{2}{x^2}$.

4. $f^{-1}(x) = \dfrac{e^x-1}{2}$, so directly, $\dfrac{d}{dx}f^{-1}(x) = \dfrac{e^x}{2}$. Next, $f'(x) = \dfrac{2}{2x+1}$, and using Formula (2), $\dfrac{d}{dx}f^{-1}(x) = \dfrac{2f^{-1}(x)+1}{2} = \dfrac{e^x}{2}$.

5. (a) $f'(x) = 2x + 8$; $f' < 0$ on $(-\infty, -4)$ and $f' > 0$ on $(-4, +\infty)$; not enough information. By inspection, $f(1) = 10 = f(-9)$, so not one-to-one.

(b) $f'(x) = 10x^4 + 3x^2 + 3 \geq 3 > 0$; $f'(x)$ is positive for all x, so f is one-to-one.

(c) $f'(x) = 2 + \cos x \geq 1 > 0$ for all x, so f is one-to-one.

(d) $f'(x) = -(\ln 2)\left(\frac{1}{2}\right)^x < 0$ because $\ln 2 > 0$, so f is one-to-one for all x.

6. **(a)** $f'(x) = 3x^2 + 6x = x(3x+6)$ changes sign at $x = -2, 0$, so not enough information; by observation (of the graph, and using some guesswork), $f(0) = -8 = f(-3)$, so f is not one-to-one.

(b) $f'(x) = 5x^4 + 24x^2 + 2 \geq 2 > 0$; f' is positive for all x, so f is one-to-one.

(c) $f'(x) = \dfrac{1}{(x+1)^2}$; f is one-to-one because:
if $x_1 < x_2 < -1$ then $f' > 0$ on $[x_1, x_2]$, so $f(x_1) \neq f(x_2)$
if $-1 < x_1 < x_2$ then $f' > 0$ on $[x_1, x_2]$, so $f(x_1) \neq f(x_2)$
if $x_1 < -1 < x_2$ then $f(x_1) > 1 > f(x_2)$ since $f(x) > 1$ on $(-\infty, -1)$ and $f(x) < 1$ on $(-1, +\infty)$

(d) Note that $f(x)$ is only defined for $x > 0$. $\dfrac{d}{dx}\log_b x = \dfrac{1}{x \ln b}$, which is always negative ($0 < b < 1$), so f is one-to-one.

7. $y = f^{-1}(x)$, $x = f(y) = 5y^3 + y - 7$, $\dfrac{dx}{dy} = 15y^2 + 1$, $\dfrac{dy}{dx} = \dfrac{1}{15y^2+1}$; check: $1 = 15y^2\dfrac{dy}{dx} + \dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{1}{15y^2+1}$.

8. $y = f^{-1}(x)$, $x = f(y) = 1/y^2$, $\dfrac{dx}{dy} = -2y^{-3}$, $\dfrac{dy}{dx} = -y^3/2$; check: $1 = -2y^{-3}\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = -y^3/2$.

9. $y = f^{-1}(x)$, $x = f(y) = 2y^5 + y^3 + 1$, $\dfrac{dx}{dy} = 10y^4 + 3y^2$, $\dfrac{dy}{dx} = \dfrac{1}{10y^4 + 3y^2}$; check: $1 = 10y^4\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{1}{10y^4+3y^2}$.

10. $y = f^{-1}(x)$, $x = f(y) = 5y - \sin 2y$, $\dfrac{dx}{dy} = 5 - 2\cos 2y$, $\dfrac{dy}{dx} = \dfrac{1}{5 - 2\cos 2y}$; check: $1 = (5 - 2\cos 2y)\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{1}{5 - 2\cos 2y}$.

11. Let $P(a, b)$ be given, not on the line $y = x$. Let Q_1 be its reflection across the line $y = x$, yet to be determined. Let Q have coordinates (b, a).

(a) Since P does not lie on $y = x$, we have $a \neq b$, i.e. $P \neq Q$ since they have different abscissas. The line $\vec{PQ}$ has slope $(b-a)/(a-b) = -1$ which is the negative reciprocal of $m = 1$ and so the two lines are perpendicular.

(b) Let (c, d) be the midpoint of the segment PQ. Then $c = (a+b)/2$ and $d = (b+a)/2$ so $c = d$ and the midpoint is on $y = x$.

(c) Let $Q(c, d)$ be the reflection of P through $y = x$. By definition this means P and Q lie on a line perpendicular to the line $y = x$ and the midpoint of P and Q lies on $y = x$.

(d) Since the line through P and Q is perpendicular to the line $y = x$ it is parallel to the line through P and Q_1; since both pass through P they are the same line. Finally, since the midpoints of P and Q_1 and of P and Q both lie on $y = x$, they are the same point, and consequently $Q = Q_1$.

12. Let (a, b) and (A, B) be points on a line with slope m. Then $m = (B-b)/(A-a)$. Consider the associated points (B, A) and (b, a). The line through these two points has slope $(A-a)/(B-b)$, which is the reciprocal of m. Thus (B, A) and (b, a) define the line with slope $1/m$.

13. If $x < y$ then $f(x) \le f(y)$ and $g(x) \le g(y)$; thus $f(x) + g(x) \le f(y) + g(y)$. Moreover, $g(x) \le g(y)$, so $f(g(x)) \le f(g(y))$. Note that $f(x)g(x)$ need not be increasing, e.g. $f(x) = g(x) = x$, both increasing for all x, yet $f(x)g(x) = x^2$, not an increasing function.

14. On $[0, 1]$ let $f(x) = x-2,\ g(x) = 2-x$, then f and g are one-to-one but $f+g$ is not. If $f(x) = x+1,\ g(x) = 1/(x+1)$ then f and g are one-to-one but fg is not. Finally, if f and g are one-to-one and if $f(g(x)) = f(g(y))$ then, because f is one-to-one, $g(x) = g(y)$, and since g is one-to-one, $x = y$, so $f(g(x))$ is one-to-one.

15. $\dfrac{dy}{dx} = 7e^{7x}$.

16. $\dfrac{dy}{dx} = -10xe^{-5x^2}$.

17. $\dfrac{dy}{dx} = x^3e^x + 3x^2e^x = x^2e^x(x+3)$.

18. $\dfrac{dy}{dx} = -\dfrac{1}{x^2}e^{1/x}$.

19. $\dfrac{dy}{dx} = \dfrac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \dfrac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} = 4/(e^x + e^{-x})^2$.

20. $\dfrac{dy}{dx} = e^x \cos(e^x)$.

21. $\dfrac{dy}{dx} = (x \sec^2 x + \tan x)e^{x \tan x}$.

22. $\dfrac{dy}{dx} = \dfrac{(\ln x)e^x - e^x(1/x)}{(\ln x)^2} = \dfrac{e^x(x \ln x - 1)}{x(\ln x)^2}$.

23. $\dfrac{dy}{dx} = (1 - 3e^{3x})e^{(x - e^{3x})}$.

24. $\dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{1}{\sqrt{1+5x^3}}15x^2 \exp(\sqrt{1+5x^3}) = \dfrac{15}{2}x^2(1+5x^3)^{-1/2} \exp(\sqrt{1+5x^3})$.

25. $\dfrac{dy}{dx} = \dfrac{(x-1)e^{-x}}{1 - xe^{-x}} = \dfrac{x-1}{e^x - x}$.

26. $\dfrac{dy}{dx} = \dfrac{1}{\cos(e^x)}[-\sin(e^x)]e^x = -e^x \tan(e^x)$.

27. $f'(x) = 2^x \ln 2$; $y = 2^x$, $\ln y = x \ln 2$, $\dfrac{1}{y}y' = \ln 2$, $y' = y \ln 2 = 2^x \ln 2$.

28. $f'(x) = -3^{-x} \ln 3$; $y = 3^{-x}$, $\ln y = -x \ln 3$, $\dfrac{1}{y}y' = -\ln 3$, $y' = -y \ln 3 = -3^{-x} \ln 3$.

29. $f'(x) = \pi^{\sin x}(\ln \pi) \cos x$; $y = \pi^{\sin x}$, $\ln y = (\sin x) \ln \pi$, $\dfrac{1}{y}y' = (\ln \pi) \cos x$, $y' = \pi^{\sin x}(\ln \pi) \cos x$.

30. $f'(x) = \pi^{x \tan x}(\ln \pi)(x \sec^2 x + \tan x)$; $y = \pi^{x \tan x}$, $\ln y = (x \tan x) \ln \pi$, $\dfrac{1}{y}y' = (\ln \pi)(x \sec^2 x + \tan x)$, $y' = \pi^{x \tan x}(\ln \pi)(x \sec^2 x + \tan x)$.

31. $\ln y = (\ln x)\ln(x^3-2x)$, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{3x^2-2}{x^3-2x}\ln x + \dfrac{1}{x}\ln(x^3-2x)$, $\dfrac{dy}{dx} = (x^3-2x)^{\ln x}\left[\dfrac{3x^2-2}{x^3-2x}\ln x + \dfrac{1}{x}\ln(x^3-2x)\right]$.

32. $\ln y = (\sin x)\ln x$, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{\sin x}{x} + (\cos x)\ln x$, $\dfrac{dy}{dx} = x^{\sin x}\left[\dfrac{\sin x}{x} + (\cos x)\ln x\right]$.

33. $\ln y = (\tan x)\ln(\ln x)$, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x\ln x}\tan x + (\sec^2 x)\ln(\ln x)$, $\dfrac{dy}{dx} = (\ln x)^{\tan x}\left[\dfrac{\tan x}{x\ln x} + (\sec^2 x)\ln(\ln x)\right]$.

34. $\ln y = (\ln x)\ln(x^2+3)$, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2x}{x^2+3}\ln x + \dfrac{1}{x}\ln(x^2+3)$, $\dfrac{dy}{dx} = (x^2+3)^{\ln x}\left[\dfrac{2x}{x^2+3}\ln x + \dfrac{1}{x}\ln(x^2+3)\right]$.

35. $\ln y = (\ln x)(\ln(\ln x))$, $\dfrac{dy/dx}{y} = (1/x)(\ln(\ln x)) + (\ln x)\dfrac{1/x}{\ln x} = (1/x)(1+\ln(\ln x))$, $dy/dx = \dfrac{1}{x}(\ln x)^{\ln x}(1+\ln\ln x)$.

36. **(a)** Because x^x is not of the form a^x where a is constant.

(b) $y = x^x$, $\ln y = x\ln x$, $\dfrac{1}{y}y' = 1+\ln x$, $y' = x^x(1+\ln x)$.

37. $\dfrac{dy}{dx} = (3x^2-4x)e^x + (x^3-2x^2+1)e^x = (x^3+x^2-4x+1)e^x$.

38. $\dfrac{dy}{dx} = (4x-2)e^{2x} + (2x^2-2x+1)2e^{2x} = 4x^2e^{2x}$.

39. $\dfrac{dy}{dx} = (2x + \dfrac{1}{2\sqrt{x}})3^x + (x^2+\sqrt{x})3^x\ln 3$.

40. $\dfrac{dy}{dx} = (3x^2 + \dfrac{1}{3}x^{-2/3})5^x + (x^3+\sqrt[3]{x})5^x\ln 5$.

41. $\dfrac{dy}{dx} = 4^{3\sin x - e^x}\ln 4(3\cos x - e^x)$.

42. $\dfrac{dy}{dx} = 2^{\cos x + \ln x}\ln 2(-\sin x + \dfrac{1}{x})$.

43. $\dfrac{dy}{dx} = \dfrac{3}{\sqrt{1-(3x)^2}} = \dfrac{3}{\sqrt{1-9x^2}}$.

44. $\dfrac{dy}{dx} = -\dfrac{1/2}{\sqrt{1-\left(\frac{x+1}{2}\right)^2}} = -\dfrac{1}{\sqrt{4-(x+1)^2}}$.

45. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-1/x^2}}(-1/x^2) = -\dfrac{1}{|x|\sqrt{x^2-1}}$.

46. $\dfrac{dy}{dx} = \dfrac{\sin x}{\sqrt{1-\cos^2 x}} = \dfrac{\sin x}{|\sin x|} = \begin{cases} 1, & \sin x > 0 \\ -1, & \sin x < 0 \end{cases}$.

47. $\dfrac{dy}{dx} = \dfrac{3x^2}{1+(x^3)^2} = \dfrac{3x^2}{1+x^6}$.

48. $\dfrac{dy}{dx} = \dfrac{5x^4}{|x^5|\sqrt{(x^5)^2-1}} = \dfrac{5}{|x|\sqrt{x^{10}-1}}$.

49. $y = 1/\tan x = \cot x$, $dy/dx = -\csc^2 x$.

50. $y = (\tan^{-1} x)^{-1}$, $dy/dx = -(\tan^{-1} x)^{-2}\left(\dfrac{1}{1+x^2}\right)$.

51. $\dfrac{dy}{dx} = \dfrac{e^x}{|x|\sqrt{x^2-1}} + e^x \sec^{-1} x$.

52. $\dfrac{dy}{dx} = -\dfrac{1}{(\cos^{-1} x)\sqrt{1-x^2}}$.

53. $\dfrac{dy}{dx} = 0$.

54. $\dfrac{dy}{dx} = \dfrac{3x^2(\sin^{-1} x)^2}{\sqrt{1-x^2}} + 2x(\sin^{-1} x)^3$.

55. $\dfrac{dy}{dx} = 0$.

56. $\dfrac{dy}{dx} = -1/\sqrt{e^{2x}-1}$.

57. $\dfrac{dy}{dx} = -\dfrac{1}{1+x}\left(\dfrac{1}{2}x^{-1/2}\right) = -\dfrac{1}{2(1+x)\sqrt{x}}$.

58. $\dfrac{dy}{dx} = -\dfrac{1}{2\sqrt{\cot^{-1} x}\,(1+x^2)}$.

59. False; $y = Ae^x$ also satisfies $\dfrac{dy}{dx} = y$.

60. False; $dy/dx = 1/x$ is rational, but $y = \ln x$ is not.

61. True; examine the cases $x > 0$ and $x < 0$ separately.

62. True; $\dfrac{d}{dx}\sin^{-1} x + \dfrac{d}{dx}\cos^{-1} x = 0$.

63. **(a)** Let $x = f(y) = \cot y$, $0 < y < \pi$, $-\infty < x < +\infty$. Then f is differentiable and one-to-one and $f'(f^{-1}(x)) = -\csc^2(\cot^{-1} x) = -x^2 - 1 \neq 0$, and $\dfrac{d}{dx}[\cot^{-1} x]\Big|_{x=0} = \lim_{x\to 0}\dfrac{1}{f'(f^{-1}(x))} = -\lim_{x\to 0}\dfrac{1}{x^2+1} = -1$.

(b) If $x \neq 0$ then, from Exercise 48(a) of Section 0.4, $\dfrac{d}{dx}\cot^{-1} x = \dfrac{d}{dx}\tan^{-1}\dfrac{1}{x} = -\dfrac{1}{x^2}\dfrac{1}{1+(1/x)^2} = -\dfrac{1}{x^2+1}$. For $x = 0$, part (a) shows the same; thus for $-\infty < x < +\infty$, $\dfrac{d}{dx}[\cot^{-1} x] = -\dfrac{1}{x^2+1}$.

(c) For $-\infty < u < +\infty$, by the chain rule it follows that $\dfrac{d}{dx}[\cot^{-1} u] = -\dfrac{1}{u^2+1}\dfrac{du}{dx}$.

64. **(a)** By the chain rule, $\dfrac{d}{dx}[\csc^{-1} x] = \dfrac{d}{dx}\sin^{-1}\dfrac{1}{x} = -\dfrac{1}{x^2}\dfrac{1}{\sqrt{1-(1/x)^2}} = \dfrac{-1}{|x|\sqrt{x^2-1}}$.

(b) By the chain rule, $\dfrac{d}{dx}[\csc^{-1} u] = \dfrac{du}{dx}\dfrac{d}{du}[\csc^{-1} u] = \dfrac{-1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}$.

(c) From Section 0.4 equation (11), $\sec^{-1}x+\csc^{-1}x=\pi/2$, so $\dfrac{d}{dx}\sec^{-1}x=-\dfrac{d}{dx}\csc^{-1}x=\dfrac{1}{|x|\sqrt{x^2-1}}$ by part (a).

(d) By the chain rule, $\dfrac{d}{dx}[\sec^{-1}u]=\dfrac{du}{dx}\dfrac{d}{du}[\sec^{-1}u]=\dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}$.

65. $x^3+x\tan^{-1}y=e^y$, $3x^2+\dfrac{x}{1+y^2}y'+\tan^{-1}y=e^yy'$, $y'=\dfrac{(3x^2+\tan^{-1}y)(1+y^2)}{(1+y^2)e^y-x}$.

66. $\sin^{-1}(xy)=\cos^{-1}(x-y)$, $\dfrac{1}{\sqrt{1-x^2y^2}}(xy'+y)=-\dfrac{1}{\sqrt{1-(x-y)^2}}(1-y')$, $y'=\dfrac{y\sqrt{1-(x-y)^2}+\sqrt{1-x^2y^2}}{\sqrt{1-x^2y^2}-x\sqrt{1-(x-y)^2}}$.

67. (a) $f(x)=x^3-3x^2+2x=x(x-1)(x-2)$ so $f(0)=f(1)=f(2)=0$ thus f is not one-to-one.

(b) $f'(x)=3x^2-6x+2$, $f'(x)=0$ when $x=\dfrac{6\pm\sqrt{36-24}}{6}=1\pm\sqrt{3}/3$. $f'(x)>0$ (f is increasing) if $x<1-\sqrt{3}/3$, $f'(x)<0$ (f is decreasing) if $1-\sqrt{3}/3<x<1+\sqrt{3}/3$, so $f(x)$ takes on values less than $f(1-\sqrt{3}/3)$ on both sides of $1-\sqrt{3}/3$ thus $1-\sqrt{3}/3$ is the largest value of k.

68. (a) $f(x)=x^3(x-2)$ so $f(0)=f(2)=0$ thus f is not one-to-one.

(b) $f'(x)=4x^3-6x^2=4x^2(x-3/2)$, $f'(x)=0$ when $x=0$ or $3/2$; f is decreasing on $(-\infty,3/2]$ and increasing on $[3/2,+\infty)$ so $3/2$ is the smallest value of k.

69. (a) $f'(x)=4x^3+3x^2=(4x+3)x^2=0$ only at $x=0$. But on $[0,2]$, f' has no sign change, so f is one-to-one.

(b) $F'(x)=2f'(2g(x))g'(x)$ so $F'(3)=2f'(2g(3))g'(3)$. By inspection $f(1)=3$, so $g(3)=f^{-1}(3)=1$ and $g'(3)=(f^{-1})'(3)=1/f'(f^{-1}(3))=1/f'(1)=1/7$ because $f'(x)=4x^3+3x^2$. Thus $F'(3)=2f'(2)(1/7)=2(44)(1/7)=88/7$. $F(3)=f(2g(3))=f(2\cdot 1)=f(2)=25$, so the line tangent to $F(x)$ at $(3,25)$ has the equation $y-25=(88/7)(x-3)$, $y=(88/7)x-89/7$.

70. (a) $f'(x)=-e^{4-x^2}\left(2+\dfrac{1}{x^2}\right)<0$ for all $x>0$, so f is one-to-one.

(b) By inspection, $f(2)=1/2$, so $2=f^{-1}(1/2)=g(1/2)$. By inspection, $f'(2)=-\left(2+\dfrac{1}{4}\right)=-\dfrac{9}{4}$, and $F'(1/2)=f'([g(x)]^2)\dfrac{d}{dx}[g(x)^2]\Big|_{x=1/2}=f'([g(x)]^2)2g(x)g'(x)\Big|_{x=1/2}=f'(2^2)2\cdot 2\dfrac{1}{f'(g(x))}\Big|_{x=1/2}=4\dfrac{f'(4)}{f'(2)}=4\dfrac{e^{-12}(2+\frac{1}{16})}{(2+\frac{1}{4})}=\dfrac{33}{9e^{12}}=\dfrac{11}{3e^{12}}$.

71. $y=Ae^{kt}$, $dy/dt=kAe^{kt}=k(Ae^{kt})=ky$.

72. $y=Ae^{2x}+Be^{-4x}$, $y'=2Ae^{2x}-4Be^{-4x}$, $y''=4Ae^{2x}+16Be^{-4x}$ so $y''+2y'-8y=(4Ae^{2x}+16Be^{-4x})+2(2Ae^{2x}-4Be^{-4x})-8(Ae^{2x}+Be^{-4x})=0$.

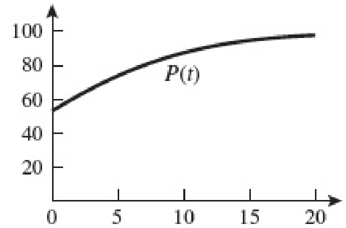
73. (a) $y'=-xe^{-x}+e^{-x}=e^{-x}(1-x)$, $xy'=xe^{-x}(1-x)=y(1-x)$.

(b) $y'=-x^2e^{-x^2/2}+e^{-x^2/2}=e^{-x^2/2}(1-x^2)$, $xy'=xe^{-x^2/2}(1-x^2)=y(1-x^2)$.

74. (a) Losing 15% of the value means the value after a year's depreciation is $V_{\text{new}}=0.85V_{\text{previous}}$. Using induction gives the formula.

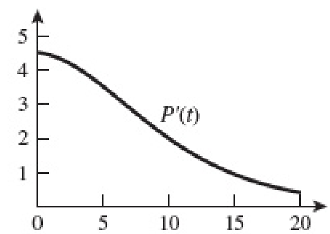
(b) Differentiating $V = 20000(0.85)^t$ gives $\dfrac{dV}{dt} = 20000 \ln 0.85 (0.85)^t$. Evaluating this at $t = 5$ gives $\dfrac{dV}{dt} \approx -1442$ dollars/year. Thus the car is losing value at about \$1442 dollars/year when it is 5 years old.

75. (a)

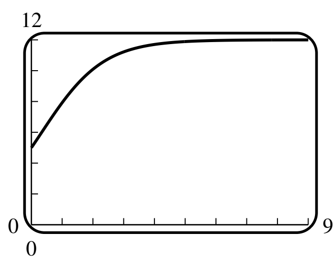


(b) The percentage converges to 100%, full coverage of broadband internet access. The limit of the expression in the denominator is clearly 53 as $t \to \infty$.

(c) The rate converges to 0 according to the graph.

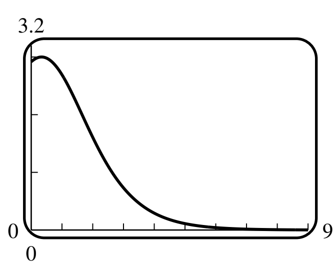


76. (a)



(b) P tends to 12 as t gets large; $\lim\limits_{t\to+\infty} P(t) = \lim\limits_{t\to+\infty} \dfrac{60}{5+7e^{-t}} = \dfrac{60}{5 + 7 \lim\limits_{t\to+\infty} e^{-t}} = \dfrac{60}{5} = 12.$

(c) The rate of population growth tends to zero.



77. $f(x) = e^{3x}, f'(0) = \lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = 3e^{3x}\Big|_{x=0} = 3.$

78. $f(x) = e^{x^2}, f'(0) = 2xe^{x^2}\Big|_{x=0} = 0.$

79. $\lim\limits_{h\to 0} \dfrac{10^h - 1}{h} = \dfrac{d}{dx} 10^x\Big|_{x=0} = \dfrac{d}{dx} e^{x \ln 10}\Big|_{x=0} = \ln 10.$

80. $\lim\limits_{h\to 0} \dfrac{\tan^{-1}(1+h) - \pi/4}{h} = \dfrac{d}{dx} \tan^{-1} x\Big|_{x=1} = \dfrac{1}{1+x^2}\Big|_{x=1} = \dfrac{1}{2}.$

81. $\lim_{\Delta x\to 0}\dfrac{9[\sin^{-1}(\frac{\sqrt{3}}{2}+\Delta x)]^2-\pi^2}{\Delta x}=\dfrac{d}{dx}(3\sin^{-1}x)^2\Big|_{x=\frac{\sqrt{3}}{2}}=2(3\sin^{-1}x)\dfrac{3}{\sqrt{1-x^2}}\Big|_{x=\frac{\sqrt{3}}{2}}=2(3\dfrac{\pi}{3})\dfrac{3}{\sqrt{1-(3/4)}}=12\pi.$

82. $\lim_{w\to 2}\dfrac{3\sec^{-1}w-\pi}{w-2}=\dfrac{d}{dx}3\sec^{-1}x\Big|_{x=2}=\dfrac{3}{|2|\sqrt{2^2-1}}=\dfrac{\sqrt{3}}{2}.$

83. $\lim_{k\to 0^+}9.8\dfrac{1-e^{-kt}}{k}=9.8\lim_{k\to 0^+}\dfrac{1-e^{-kt}}{k}=9.8\dfrac{d}{dk}(-e^{-kt})\big|_{k=0}=9.8\,t$, so if the fluid offers no resistance, then the speed will increase at a constant rate of 9.8 m/s^2.

Exercise Set 3.4

1. $\dfrac{dy}{dt}=3\dfrac{dx}{dt}$

(a) $\dfrac{dy}{dt}=3(2)=6.$ **(b)** $-1=3\dfrac{dx}{dt},\ \dfrac{dx}{dt}=-\dfrac{1}{3}.$

2. $\dfrac{dx}{dt}+4\dfrac{dy}{dt}=0$

(a) $1+4\dfrac{dy}{dt}=0$ so $\dfrac{dy}{dt}=-\dfrac{1}{4}$ when $x=2$. **(b)** $\dfrac{dx}{dt}+4(4)=0$ so $\dfrac{dx}{dt}=-16$ when $x=3$.

3. $8x\dfrac{dx}{dt}+18y\dfrac{dy}{dt}=0$

(a) $8\dfrac{1}{2\sqrt{2}}\cdot 3+18\dfrac{1}{3\sqrt{2}}\dfrac{dy}{dt}=0,\ \dfrac{dy}{dt}=-2.$ **(b)** $8\left(\dfrac{1}{3}\right)\dfrac{dx}{dt}-18\dfrac{\sqrt{5}}{9}\cdot 8=0,\ \dfrac{dx}{dt}=6\sqrt{5}.$

4. $2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt}=2\dfrac{dx}{dt}+4\dfrac{dy}{dt}$

(a) $2\cdot 3(-5)+2\cdot 1\dfrac{dy}{dt}=2(-5)+4\dfrac{dy}{dt},\ \dfrac{dy}{dt}=-10.$

(b) $2(1+\sqrt{2})\dfrac{dx}{dt}+2(2+\sqrt{3})\cdot 6=2\dfrac{dx}{dt}+4\cdot 6,\ \dfrac{dx}{dt}=-12\dfrac{\sqrt{3}}{2\sqrt{2}}=-3\sqrt{3}\sqrt{2}.$

5. **(b)** $A=x^2$.

(c) $\dfrac{dA}{dt}=2x\dfrac{dx}{dt}.$

(d) Find $\dfrac{dA}{dt}\Big|_{x=3}$ given that $\dfrac{dx}{dt}\Big|_{x=3}=2$. From part (c), $\dfrac{dA}{dt}\Big|_{x=3}=2(3)(2)=12$ ft^2/min.

6. **(b)** $A=\pi r^2$.

(c) $\dfrac{dA}{dt}=2\pi r\dfrac{dr}{dt}.$

(d) Find $\dfrac{dA}{dt}\Big|_{r=5}$ given that $\dfrac{dr}{dt}\Big|_{r=5}=2$. From part (c), $\dfrac{dA}{dt}\Big|_{r=5}=2\pi(5)(2)=20\pi$ cm^2/s.

7. **(a)** $V=\pi r^2h$, so $\dfrac{dV}{dt}=\pi\left(r^2\dfrac{dh}{dt}+2rh\dfrac{dr}{dt}\right).$

(b) Find $\left.\dfrac{dV}{dt}\right|_{\substack{h=6,\\r=10}}$ given that $\left.\dfrac{dh}{dt}\right|_{\substack{h=6,\\r=10}} = 1$ and $\left.\dfrac{dr}{dt}\right|_{\substack{h=6,\\r=10}} = -1$. From part (a), $\left.\dfrac{dV}{dt}\right|_{\substack{h=6,\\r=10}} = \pi[10^2(1)+2(10)(6)(-1)] = -20\pi$ in^3/s; the volume is decreasing.

8. (a) $\ell^2 = x^2 + y^2$, so $\dfrac{d\ell}{dt} = \dfrac{1}{\ell}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$.

(b) Find $\left.\dfrac{d\ell}{dt}\right|_{\substack{x=3,\\y=4}}$ given that $\dfrac{dx}{dt} = \dfrac{1}{2}$ and $\dfrac{dy}{dt} = -\dfrac{1}{4}$. From part (a) and the fact that $\ell = 5$ when $x = 3$ and $y = 4$, $\left.\dfrac{d\ell}{dt}\right|_{\substack{x=3,\\y=4}} = \dfrac{1}{5}\left[3\left(\dfrac{1}{2}\right) + 4\left(-\dfrac{1}{4}\right)\right] = \dfrac{1}{10}$ ft/s; the diagonal is increasing.

9. (a) $\tan\theta = \dfrac{y}{x}$, so $\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2}$, $\dfrac{d\theta}{dt} = \dfrac{\cos^2\theta}{x^2}\left(x\dfrac{dy}{dt} - y\dfrac{dx}{dt}\right)$.

(b) Find $\left.\dfrac{d\theta}{dt}\right|_{\substack{x=2,\\y=2}}$ given that $\left.\dfrac{dx}{dt}\right|_{\substack{x=2,\\y=2}} = 1$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=2,\\y=2}} = -\dfrac{1}{4}$. When $x = 2$ and $y = 2$, $\tan\theta = 2/2 = 1$ so $\theta = \dfrac{\pi}{4}$ and $\cos\theta = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$. Thus from part (a), $\left.\dfrac{d\theta}{dt}\right|_{\substack{x=2,\\y=2}} = \dfrac{(1/\sqrt{2})^2}{2^2}\left[2\left(-\dfrac{1}{4}\right) - 2(1)\right] = -\dfrac{5}{16}$ rad/s; θ is decreasing.

10. Find $\left.\dfrac{dz}{dt}\right|_{\substack{x=1,\\y=2}}$ given that $\left.\dfrac{dx}{dt}\right|_{\substack{x=1,\\y=2}} = -2$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=1,\\y=2}} = 3$. $\dfrac{dz}{dt} = 2x^3y\dfrac{dy}{dt} + 3x^2y^2\dfrac{dx}{dt}$, $\left.\dfrac{dz}{dt}\right|_{\substack{x=1,\\y=2}} = (4)(3) + (12)(-2) = -12$ units/s; z is decreasing.

11. Let A be the area swept out, and θ the angle through which the minute hand has rotated. Find $\dfrac{dA}{dt}$ given that $\dfrac{d\theta}{dt} = \dfrac{\pi}{30}$ rad/min; $A = \dfrac{1}{2}r^2\theta = 8\theta$, so $\dfrac{dA}{dt} = 8\dfrac{d\theta}{dt} = \dfrac{4\pi}{15}$ in^2/min.

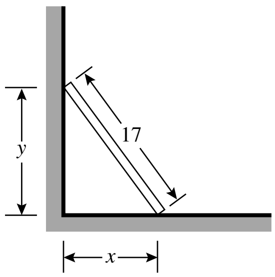
12. Let r be the radius and A the area enclosed by the ripple. We want $\left.\dfrac{dA}{dt}\right|_{t=10}$ given that $\dfrac{dr}{dt} = 3$. We know that $A = \pi r^2$, so $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$. Because r is increasing at the constant rate of 3 ft/s, it follows that $r = 30$ ft after 10 seconds so $\left.\dfrac{dA}{dt}\right|_{t=10} = 2\pi(30)(3) = 180\pi$ ft^2/s.

13. Find $\left.\dfrac{dr}{dt}\right|_{A=9}$ given that $\dfrac{dA}{dt} = 6$. From $A = \pi r^2$ we get $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$ so $\dfrac{dr}{dt} = \dfrac{1}{2\pi r}\dfrac{dA}{dt}$. If $A = 9$ then $\pi r^2 = 9$, $r = 3/\sqrt{\pi}$ so $\left.\dfrac{dr}{dt}\right|_{A=9} = \dfrac{1}{2\pi(3/\sqrt{\pi})}(6) = 1/\sqrt{\pi}$ mi/h.

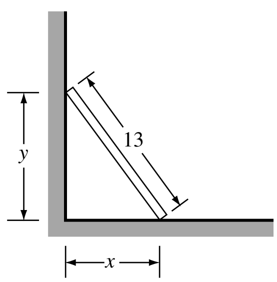
14. The volume V of a sphere of radius r is given by $V = \dfrac{4}{3}\pi r^3$ or, because $r = \dfrac{D}{2}$ where D is the diameter, $V = \dfrac{4}{3}\pi\left(\dfrac{D}{2}\right)^3 = \dfrac{1}{6}\pi D^3$. We want $\left.\dfrac{dD}{dt}\right|_{r=1}$ given that $\dfrac{dV}{dt} = 3$. From $V = \dfrac{1}{6}\pi D^3$ we get $\dfrac{dV}{dt} = \dfrac{1}{2}\pi D^2\dfrac{dD}{dt}$, $\dfrac{dD}{dt} = \dfrac{2}{\pi D^2}\dfrac{dV}{dt}$, so $\left.\dfrac{dD}{dt}\right|_{r=1} = \dfrac{2}{\pi(2)^2}(3) = \dfrac{3}{2\pi}$ ft/min.

15. Find $\left.\dfrac{dV}{dt}\right|_{r=9}$ given that $\dfrac{dr}{dt} = -15$. From $V = \dfrac{4}{3}\pi r^3$ we get $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$ so $\left.\dfrac{dV}{dt}\right|_{r=9} = 4\pi(9)^2(-15) = -4860\pi$. Air must be removed at the rate of 4860π cm^3/min.

16. Let x and y be the distances shown in the diagram. We want to find $\left.\dfrac{dy}{dt}\right|_{y=8}$ given that $\dfrac{dx}{dt} = 5$. From $x^2+y^2 = 17^2$ we get $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$, so $\dfrac{dy}{dt} = -\dfrac{x}{y}\dfrac{dx}{dt}$. When $y = 8$, $x^2 + 8^2 = 17^2$, $x^2 = 289 - 64 = 225$, $x = 15$ so $\left.\dfrac{dy}{dt}\right|_{y=8} = -\dfrac{15}{8}(5) = -\dfrac{75}{8}$ ft/s; the top of the ladder is moving down the wall at a rate of 75/8 ft/s.

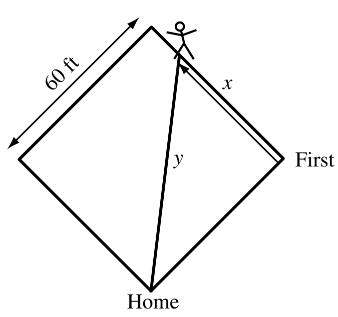


17. Find $\left.\dfrac{dx}{dt}\right|_{y=5}$ given that $\dfrac{dy}{dt} = -2$. From $x^2+y^2 = 13^2$ we get $2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt} = 0$ so $\dfrac{dx}{dt} = -\dfrac{y}{x}\dfrac{dy}{dt}$. Use $x^2+y^2 = 169$ to find that $x = 12$ when $y = 5$ so $\left.\dfrac{dx}{dt}\right|_{y=5} = -\dfrac{5}{12}(-2) = \dfrac{5}{6}$ ft/s.



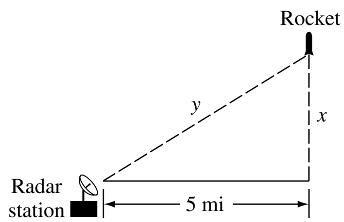
18. Let θ be the acute angle, and x the distance of the bottom of the plank from the wall. Find $\left.\dfrac{d\theta}{dt}\right|_{x=2}$ given that $\left.\dfrac{dx}{dt}\right|_{x=2} = -\dfrac{1}{2}$ ft/s. The variables θ and x are related by the equation $\cos\theta = \dfrac{x}{10}$ so $-\sin\theta\dfrac{d\theta}{dt} = \dfrac{1}{10}\dfrac{dx}{dt}$, $\dfrac{d\theta}{dt} = -\dfrac{1}{10\sin\theta}\dfrac{dx}{dt}$. When $x = 2$, the top of the plank is $\sqrt{10^2 - 2^2} = \sqrt{96}$ ft above the ground so $\sin\theta = \sqrt{96}/10$ and $\left.\dfrac{d\theta}{dt}\right|_{x=2} = -\dfrac{1}{\sqrt{96}}\left(-\dfrac{1}{2}\right) = \dfrac{1}{2\sqrt{96}} \approx 0.051$ rad/s.

19. Let x denote the distance from first base and y the distance from home plate. Then $x^2+60^2 = y^2$ and $2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$. When $x = 50$ then $y = 10\sqrt{61}$ so $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt} = \dfrac{50}{10\sqrt{61}}(25) = \dfrac{125}{\sqrt{61}}$ ft/s.

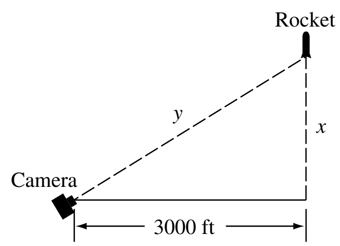


20. Find $\left.\dfrac{dx}{dt}\right|_{x=4}$ given that $\left.\dfrac{dy}{dt}\right|_{x=4} = 2000$. From $x^2 + 5^2 = y^2$ we get $2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$ so $\dfrac{dx}{dt} = \dfrac{y}{x}\dfrac{dy}{dt}$. Use $x^2 + 25 = y^2$

to find that $y = \sqrt{41}$ when $x = 4$ so $\left.\dfrac{dx}{dt}\right|_{x=4} = \dfrac{\sqrt{41}}{4}(2000) = 500\sqrt{41}$ mi/h.



21. Find $\left.\dfrac{dy}{dt}\right|_{x=4000}$ given that $\left.\dfrac{dx}{dt}\right|_{x=4000} = 880$. From $y^2 = x^2 + 3000^2$ we get $2y\dfrac{dy}{dt} = 2x\dfrac{dx}{dt}$ so $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt}$. If $x = 4000$, then $y = 5000$ so $\left.\dfrac{dy}{dt}\right|_{x=4000} = \dfrac{4000}{5000}(880) = 704$ ft/s.



22. Find $\left.\dfrac{dx}{dt}\right|_{\phi=\pi/4}$ given that $\left.\dfrac{d\phi}{dt}\right|_{\phi=\pi/4} = 0.2$. But $x = 3000\tan\phi$ so $\dfrac{dx}{dt} = 3000(\sec^2\phi)\dfrac{d\phi}{dt}$, $\left.\dfrac{dx}{dt}\right|_{\phi=\pi/4} = 3000\left(\sec^2\dfrac{\pi}{4}\right)(0.2) =$ 1200 ft/s.

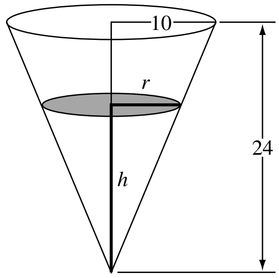
23. **(a)** If x denotes the altitude, then $r - x = 3960$, the radius of the Earth. $\theta = 0$ at perigee, so $r = 4995/1.12 \approx 4460$; the altitude is $x = 4460 - 3960 = 500$ miles. $\theta = \pi$ at apogee, so $r = 4995/0.88 \approx 5676$; the altitude is $x = 5676 - 3960 = 1716$ miles.

(b) If $\theta = 120°$, then $r = 4995/0.94 \approx 5314$; the altitude is $5314 - 3960 = 1354$ miles. The rate of change of the altitude is given by $\dfrac{dx}{dt} = \dfrac{dr}{dt} = \dfrac{dr}{d\theta}\dfrac{d\theta}{dt} = \dfrac{4995(0.12\sin\theta)}{(1 + 0.12\cos\theta)^2}\dfrac{d\theta}{dt}$. Use $\theta = 120°$ and $d\theta/dt = 2.7°/\text{min} = (2.7)(\pi/180)$ rad/min to get $dr/dt \approx 27.7$ mi/min.

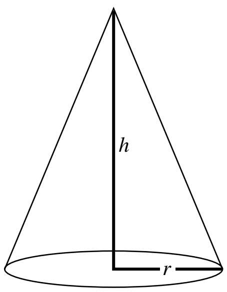
24. **(a)** Let x be the horizontal distance shown in the figure. Then $x = 4000\cot\theta$ and $\dfrac{dx}{dt} = -4000\csc^2\theta\dfrac{d\theta}{dt}$, so $\dfrac{d\theta}{dt} = -\dfrac{\sin^2\theta}{4000}\dfrac{dx}{dt}$. Use $\theta = 30°$ and $dx/dt = 300$ mi/h $= 300(5280/3600)$ ft/s $= 440$ ft/s to get $d\theta/dt = -0.0275$ rad/s $\approx -1.6°/\text{s}$; θ is decreasing at the rate of $1.6°/\text{s}$.

(b) Let y be the distance between the observation point and the aircraft. Then $y = 4000\csc\theta$ so $dy/dt = -4000(\csc\theta\cot\theta)(d\theta/dt)$. Use $\theta = 30°$ and $d\theta/dt = -0.0275$ rad/s to get $dy/dt \approx 381$ ft/s.

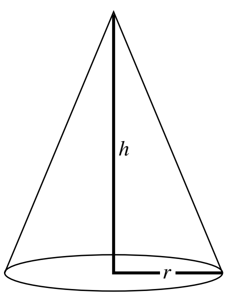
25. Find $\left.\dfrac{dh}{dt}\right|_{h=16}$ given that $\dfrac{dV}{dt} = 20$. The volume of water in the tank at a depth h is $V = \dfrac{1}{3}\pi r^2 h$. Use similar triangles (see figure) to get $\dfrac{r}{h} = \dfrac{10}{24}$ so $r = \dfrac{5}{12}h$ thus $V = \dfrac{1}{3}\pi\left(\dfrac{5}{12}h\right)^2 h = \dfrac{25}{432}\pi h^3$, $\dfrac{dV}{dt} = \dfrac{25}{144}\pi h^2\dfrac{dh}{dt}$; $\dfrac{dh}{dt} = \dfrac{144}{25\pi h^2}\dfrac{dV}{dt}$, $\left.\dfrac{dh}{dt}\right|_{h=16} = \dfrac{144}{25\pi(16)^2}(20) = \dfrac{9}{20\pi}$ ft/min.



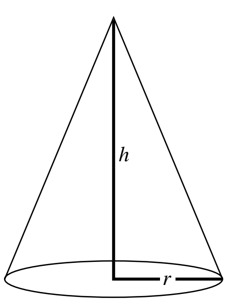
26. Find $\left.\dfrac{dh}{dt}\right|_{h=6}$ given that $\dfrac{dV}{dt} = 8$. $V = \dfrac{1}{3}\pi r^2 h$, but $r = \dfrac{1}{2}h$ so $V = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{1}{12}\pi h^3$, $\dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2 \dfrac{dh}{dt}$, $\dfrac{dh}{dt} = \dfrac{4}{\pi h^2}\dfrac{dV}{dt}$, $\left.\dfrac{dh}{dt}\right|_{h=6} = \dfrac{4}{\pi(6)^2}(8) = \dfrac{8}{9\pi}$ ft/min.



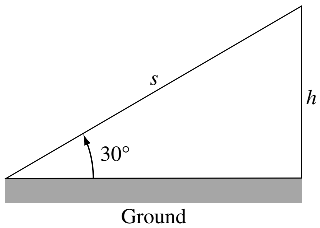
27. Find $\left.\dfrac{dV}{dt}\right|_{h=10}$ given that $\dfrac{dh}{dt} = 5$. $V = \dfrac{1}{3}\pi r^2 h$, but $r = \dfrac{1}{2}h$ so $V = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{1}{12}\pi h^3$, $\dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2 \dfrac{dh}{dt}$, $\left.\dfrac{dV}{dt}\right|_{h=10} = \dfrac{1}{4}\pi(10)^2(5) = 125\pi$ ft^3/min.



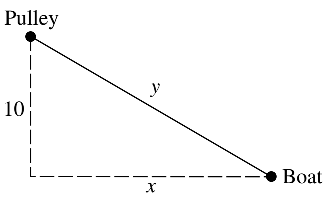
28. Let r and h be as shown in the figure. If C is the circumference of the base, then we want to find $\left.\dfrac{dC}{dt}\right|_{h=8}$ given that $\dfrac{dV}{dt} = 10$. It is given that $r = \dfrac{1}{2}h$, thus $C = 2\pi r = \pi h$ so $\dfrac{dC}{dt} = \pi\dfrac{dh}{dt}$. Use $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{12}\pi h^3$ to get $\dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2\dfrac{dh}{dt}$, so $\dfrac{dh}{dt} = \dfrac{4}{\pi h^2}\dfrac{dV}{dt}$. Substitution of $\dfrac{dh}{dt}$ into $\dfrac{dC}{dt}$ gives $\dfrac{dC}{dt} = \dfrac{4}{h^2}\dfrac{dV}{dt}$ so $\left.\dfrac{dC}{dt}\right|_{h=8} = \dfrac{4}{64}(10) = \dfrac{5}{8}$ ft/min.



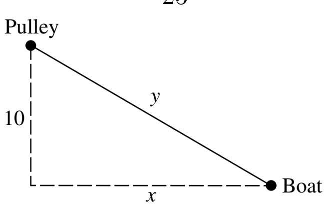
29. With s and h as shown in the figure, we want to find $\dfrac{dh}{dt}$ given that $\dfrac{ds}{dt} = 500$. From the figure, $h = s\sin 30° = \dfrac{1}{2}s$ so $\dfrac{dh}{dt} = \dfrac{1}{2}\dfrac{ds}{dt} = \dfrac{1}{2}(500) = 250$ mi/h.



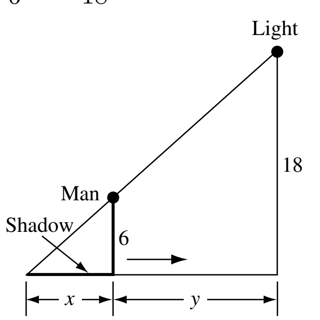
30. Find $\left.\dfrac{dx}{dt}\right|_{y=125}$ given that $\dfrac{dy}{dt} = -20$. From $x^2 + 10^2 = y^2$ we get $2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$ so $\dfrac{dx}{dt} = \dfrac{y}{x}\dfrac{dy}{dt}$. Use $x^2 + 100 = y^2$ to find that $x = \sqrt{15,525} = 15\sqrt{69}$ when $y = 125$ so $\left.\dfrac{dx}{dt}\right|_{y=125} = \dfrac{125}{15\sqrt{69}}(-20) = -\dfrac{500}{3\sqrt{69}}$. The boat is approaching the dock at the rate of $\dfrac{500}{3\sqrt{69}}$ ft/min.



31. Find $\dfrac{dy}{dt}$ given that $\left.\dfrac{dx}{dt}\right|_{y=125} = -12$. From $x^2 + 10^2 = y^2$ we get $2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$ so $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt}$. Use $x^2 + 100 = y^2$ to find that $x = \sqrt{15,525} = 15\sqrt{69}$ when $y = 125$ so $\dfrac{dy}{dt} = \dfrac{15\sqrt{69}}{125}(-12) = -\dfrac{36\sqrt{69}}{25}$. The rope must be pulled at the rate of $\dfrac{36\sqrt{69}}{25}$ ft/min.

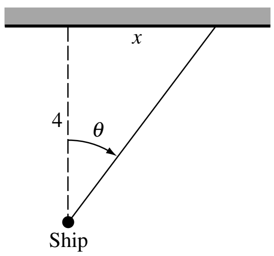


32. (a) Let x and y be as shown in the figure. It is required to find $\dfrac{dx}{dt}$, given that $\dfrac{dy}{dt} = -3$. By similar triangles, $\dfrac{x}{6} = \dfrac{x+y}{18}$, $18x = 6x + 6y$, $12x = 6y$, $x = \dfrac{1}{2}y$, so $\dfrac{dx}{dt} = \dfrac{1}{2}\dfrac{dy}{dt} = \dfrac{1}{2}(-3) = -\dfrac{3}{2}$ ft/s.

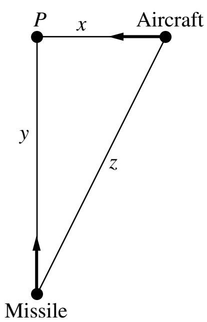


(b) The tip of the shadow is $z = x + y$ feet from the street light, thus the rate at which it is moving is given by $\dfrac{dz}{dt} = \dfrac{dx}{dt} + \dfrac{dy}{dt}$. In part (a) we found that $\dfrac{dx}{dt} = -\dfrac{3}{2}$ when $\dfrac{dy}{dt} = -3$ so $\dfrac{dz}{dt} = (-3/2) + (-3) = -9/2$ ft/s; the tip of the shadow is moving at the rate of 9/2 ft/s toward the street light.

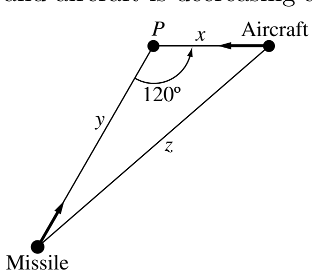
33. Find $\left.\dfrac{dx}{dt}\right|_{\theta=\pi/4}$ given that $\dfrac{d\theta}{dt}=\dfrac{2\pi}{10}=\dfrac{\pi}{5}$ rad/s. Then $x=4\tan\theta$ (see figure) so $\dfrac{dx}{dt}=4\sec^2\theta\dfrac{d\theta}{dt}$, $\left.\dfrac{dx}{dt}\right|_{\theta=\pi/4}=4\left(\sec^2\dfrac{\pi}{4}\right)\left(\dfrac{\pi}{5}\right)=8\pi/5$ km/s.



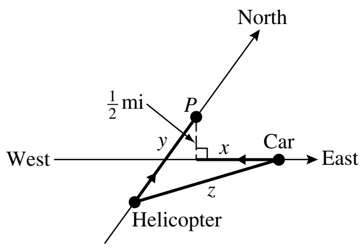
34. If x, y, and z are as shown in the figure, then we want $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\y=4}}$ given that $\dfrac{dx}{dt}=-600$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=2,\\y=4}}=-1200$. But $z^2=x^2+y^2$ so $2z\dfrac{dz}{dt}=2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt}$, $\dfrac{dz}{dt}=\dfrac{1}{z}\left(x\dfrac{dx}{dt}+y\dfrac{dy}{dt}\right)$. When $x=2$ and $y=4$, $z^2=2^2+4^2=20$, $z=\sqrt{20}=2\sqrt{5}$ so $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\y=4}}=\dfrac{1}{2\sqrt{5}}[2(-600)+4(-1200)]=-\dfrac{3000}{\sqrt{5}}=-600\sqrt{5}$ mi/h; the distance between missile and aircraft is decreasing at the rate of $600\sqrt{5}$ mi/h.

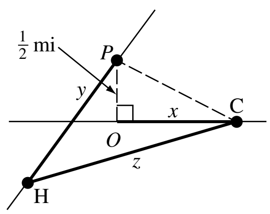


35. We wish to find $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\y=4}}$ given $\dfrac{dx}{dt}=-600$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=2,\\y=4}}=-1200$ (see figure). From the law of cosines, $z^2=x^2+y^2-2xy\cos 120^\circ=x^2+y^2-2xy(-1/2)=x^2+y^2+xy$, so $2z\dfrac{dz}{dt}=2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt}+x\dfrac{dy}{dt}+y\dfrac{dx}{dt}$, $\dfrac{dz}{dt}=\dfrac{1}{2z}\left[(2x+y)\dfrac{dx}{dt}+(2y+x)\dfrac{dy}{dt}\right]$. When $x=2$ and $y=4$, $z^2=2^2+4^2+(2)(4)=28$, so $z=\sqrt{28}=2\sqrt{7}$, thus $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\y=4}}=\dfrac{1}{2(2\sqrt{7})}[(2(2)+4)(-600)+(2(4)+2)(-1200)]=-\dfrac{4200}{\sqrt{7}}=-600\sqrt{7}$ mi/h; the distance between missile and aircraft is decreasing at the rate of $600\sqrt{7}$ mi/h.



36. (a) Let P be the point on the helicopter's path that lies directly above the car's path. Let x, y, and z be the distances shown in the first figure. Find $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\y=0}}$ given that $\dfrac{dx}{dt}=-75$ and $\dfrac{dy}{dt}=100$. In order to find an equation relating x, y, and z, first draw the line segment that joins the point P to the car, as shown in the second figure. Because triangle OPC is a right triangle, it follows that PC has length $\sqrt{x^2+(1/2)^2}$; but triangle HPC is also a right triangle so $z^2 = \left(\sqrt{x^2+(1/2)^2}\right)^2 + y^2 = x^2+y^2+1/4$ and $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt}+0$, $\dfrac{dz}{dt} = \dfrac{1}{z}\left(x\dfrac{dx}{dt}+y\dfrac{dy}{dt}\right)$. Now, when $x=2$ and $y=0$, $z^2 = (2)^2+(0)^2+1/4 = 17/4$, $z=\sqrt{17}/2$ so $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\y=0}} = \dfrac{1}{(\sqrt{17}/2)}[2(-75)+0(100)] = -300/\sqrt{17}$ mi/h.





(b) Decreasing, because $\dfrac{dz}{dt} < 0$.

37. (a) We want $\left.\dfrac{dy}{dt}\right|_{\substack{x=1,\\y=2}}$ given that $\left.\dfrac{dx}{dt}\right|_{\substack{x=1,\\y=2}} = 6$. For convenience, first rewrite the equation as $xy^3 = \dfrac{8}{5}+\dfrac{8}{5}y^2$ then $3xy^2\dfrac{dy}{dt}+y^3\dfrac{dx}{dt} = \dfrac{16}{5}y\dfrac{dy}{dt}$, $\dfrac{dy}{dt} = \dfrac{y^3}{\frac{16}{5}y-3xy^2}\dfrac{dx}{dt}$, so $\left.\dfrac{dy}{dt}\right|_{\substack{x=1,\\y=2}} = \dfrac{2^3}{\frac{16}{5}(2)-3(1)2^2}(6) = -60/7$ units/s.

(b) Falling, because $\dfrac{dy}{dt} < 0$.

38. Find $\left.\dfrac{dx}{dt}\right|_{(2,5)}$ given that $\left.\dfrac{dy}{dt}\right|_{(2,5)} = 2$. Square and rearrange to get $x^3 = y^2-17$, so $3x^2\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$, $\dfrac{dx}{dt} = \dfrac{2y}{3x^2}\dfrac{dy}{dt}$, $\left.\dfrac{dx}{dt}\right|_{(2,5)} = \left(\dfrac{5}{6}\right)(2) = \dfrac{5}{3}$ units/s.

39. The coordinates of P are $(x, 2x)$, so the distance between P and the point $(3,0)$ is $D = \sqrt{(x-3)^2+(2x-0)^2} = \sqrt{5x^2-6x+9}$. Find $\left.\dfrac{dD}{dt}\right|_{x=3}$ given that $\left.\dfrac{dx}{dt}\right|_{x=3} = -2$. $\dfrac{dD}{dt} = \dfrac{5x-3}{\sqrt{5x^2-6x+9}}\dfrac{dx}{dt}$, so $\left.\dfrac{dD}{dt}\right|_{x=3} = \dfrac{12}{\sqrt{36}}(-2) = -4$ units/s.

40. (a) Let D be the distance between P and $(2,0)$. Find $\left.\dfrac{dD}{dt}\right|_{x=3}$ given that $\left.\dfrac{dx}{dt}\right|_{x=3} = 4$. $D = \sqrt{(x-2)^2+y^2} = \sqrt{(x-2)^2+x} = \sqrt{x^2-3x+4}$, so $\dfrac{dD}{dt} = \dfrac{2x-3}{2\sqrt{x^2-3x+4}}\dfrac{dx}{dt}$; $\left.\dfrac{dD}{dt}\right|_{x=3} = \dfrac{3}{2\sqrt{4}}4 = 3$ units/s.

(b) Let θ be the angle of inclination. Find $\left.\dfrac{d\theta}{dt}\right|_{x=3}$ given that $\left.\dfrac{dx}{dt}\right|_{x=3} = 4$. $\tan\theta = \dfrac{y}{x-2} = \dfrac{\sqrt{x}}{x-2}$, so $\sec^2\theta\dfrac{d\theta}{dt} = -\dfrac{x+2}{2\sqrt{x}(x-2)^2}\dfrac{dx}{dt}$, $\dfrac{d\theta}{dt} = -\cos^2\theta\dfrac{x+2}{2\sqrt{x}(x-2)^2}\dfrac{dx}{dt}$. When $x=3$, $D=2$ so $\cos\theta = \dfrac{1}{2}$ and $\left.\dfrac{d\theta}{dt}\right|_{x=3} = -\dfrac{1}{4}\dfrac{5}{2\sqrt{3}}(4) = -\dfrac{5}{2\sqrt{3}}$ rad/s.

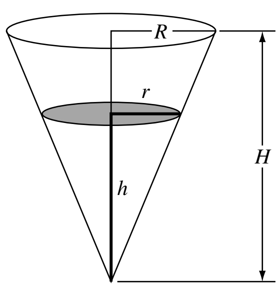
41. Solve $\dfrac{dx}{dt} = 3\dfrac{dy}{dt}$ given $y = x/(x^2+1)$. Then $y(x^2+1) = x$. Differentiating with respect to x, $(x^2+1)\dfrac{dy}{dx}+y(2x) = 1$. But $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1}{3}$ so $(x^2+1)\dfrac{1}{3}+2xy = 1$, $x^2+1+6xy = 3$, $x^2+1+6x^2/(x^2+1) = 3$, $(x^2+1)^2+6x^2-3x^2-3 =$

0, $x^4 + 5x^2 - 2 = 0$. By the quadratic formula applied to x^2 we obtain $x^2 = (-5 \pm \sqrt{25+8})/2$. The minus sign is spurious since x^2 cannot be negative, so $x^2 = (-5+\sqrt{33})/2$, and $x = \pm\sqrt{(-5+\sqrt{33})/2}$.

42. Since P is constant, differentiation yields $0 = \dfrac{dP}{dt} = 0.87(3l^2v^2\dfrac{dv}{dt} + 2lv^3\dfrac{dl}{dt})$. Substituting $l = 16$, $v = 4$, and $dv/dt = 0.01$ gives $0 = 3 \cdot 16^2 \cdot 4^2 \cdot 0.01 + 2 \cdot 16 \cdot 4^3\dfrac{dl}{dt}$. Solving for the rate of change of the blade length, we obtain $\dfrac{dl}{dt} = -\dfrac{122.88}{2048} = -0.06$ m/s.

43. Find $\left.\dfrac{dS}{dt}\right|_{s=10}$ given that $\left.\dfrac{ds}{dt}\right|_{s=10} = -2$. From $\dfrac{1}{s} + \dfrac{1}{S} = \dfrac{1}{6}$ we get $-\dfrac{1}{s^2}\dfrac{ds}{dt} - \dfrac{1}{S^2}\dfrac{dS}{dt} = 0$, so $\dfrac{dS}{dt} = -\dfrac{S^2}{s^2}\dfrac{ds}{dt}$. If $s = 10$, then $\dfrac{1}{10} + \dfrac{1}{S} = \dfrac{1}{6}$ which gives $S = 15$. So $\left.\dfrac{dS}{dt}\right|_{s=10} = -\dfrac{225}{100}(-2) = 4.5$ cm/s. The image is moving away from the lens.

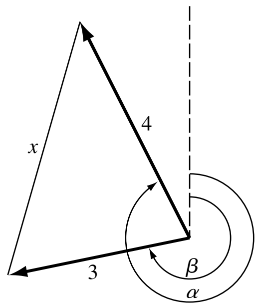
44. Suppose that the reservoir has height H and that the radius at the top is R. At any instant of time let h and r be the corresponding dimensions of the cone of water (see figure). We want to show that $\dfrac{dh}{dt}$ is constant and independent of H and R, given that $\dfrac{dV}{dt} = -kA$ where V is the volume of water, A is the area of a circle of radius r, and k is a positive constant. The volume of a cone of radius r and height h is $V = \dfrac{1}{3}\pi r^2 h$. By similar triangles $\dfrac{r}{h} = \dfrac{R}{H}$, $r = \dfrac{R}{H}h$ thus $V = \dfrac{1}{3}\pi\left(\dfrac{R}{H}\right)^2 h^3$, so $\dfrac{dV}{dt} = \pi\left(\dfrac{R}{H}\right)^2 h^2\dfrac{dh}{dt}$. But it is given that $\dfrac{dV}{dt} = -kA$ or, because $A = \pi r^2 = \pi\left(\dfrac{R}{H}\right)^2 h^2$, $\dfrac{dV}{dt} = -k\pi\left(\dfrac{R}{H}\right)^2 h^2$, which when substituted into the previous equation for $\dfrac{dV}{dt}$ gives $-k\pi\left(\dfrac{R}{H}\right)^2 h^2 = \pi\left(\dfrac{R}{H}\right)^2 h^2\dfrac{dh}{dt}$, and $\dfrac{dh}{dt} = -k$.



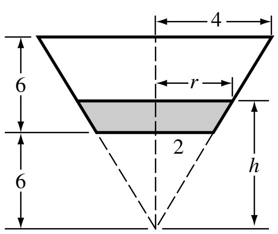
45. Let r be the radius, V the volume, and A the surface area of a sphere. Show that $\dfrac{dr}{dt}$ is a constant given that $\dfrac{dV}{dt} = -kA$, where k is a positive constant. Because $V = \dfrac{4}{3}\pi r^3$, $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. But it is given that $\dfrac{dV}{dt} = -kA$ or, because $A = 4\pi r^2$, $\dfrac{dV}{dt} = -4\pi r^2 k$ which when substituted into the previous equation for $\dfrac{dV}{dt}$ gives $-4\pi r^2 k = 4\pi r^2\dfrac{dr}{dt}$, and $\dfrac{dr}{dt} = -k$.

46. Let x be the distance between the tips of the minute and hour hands, and α and β the angles shown in the figure. Because the minute hand makes one revolution in 60 minutes, $\dfrac{d\alpha}{dt} = \dfrac{2\pi}{60} = \pi/30$ rad/min; the hour hand makes one revolution in 12 hours (720 minutes), thus $\dfrac{d\beta}{dt} = \dfrac{2\pi}{720} = \pi/360$ rad/min. We want to find $\left.\dfrac{dx}{dt}\right|_{\substack{\alpha=2\pi,\\ \beta=3\pi/2}}$ given that $\dfrac{d\alpha}{dt} = \pi/30$ and $\dfrac{d\beta}{dt} = \pi/360$. Using the law of cosines on the triangle shown in the figure, $x^2 = 3^2 + 4^2 - 2(3)(4)\cos(\alpha - \beta) = 25 - 24\cos(\alpha - \beta)$, so $2x\dfrac{dx}{dt} = 0 + 24\sin(\alpha - \beta)\left(\dfrac{d\alpha}{dt} - \dfrac{d\beta}{dt}\right)$,

$\frac{dx}{dt} = \frac{12}{x}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right)\sin(\alpha - \beta)$. When $\alpha = 2\pi$ and $\beta = 3\pi/2$, $x^2 = 25 - 24\cos(2\pi - 3\pi/2) = 25$, $x = 5$; so $\left.\frac{dx}{dt}\right|_{\substack{\alpha=2\pi,\\ \beta=3\pi/2}} = \frac{12}{5}(\pi/30 - \pi/360)\sin(2\pi - 3\pi/2) = \frac{11\pi}{150}$ in/min.



47. Extend sides of cup to complete the cone and let V_0 be the volume of the portion added, then (see figure) $V = \frac{1}{3}\pi r^2 h - V_0$ where $\frac{r}{h} = \frac{4}{12} = \frac{1}{3}$ so $r = \frac{1}{3}h$ and $V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h - V_0 = \frac{1}{27}\pi h^3 - V_0$, $\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$, $\frac{dh}{dt} = \frac{9}{\pi h^2}\frac{dV}{dt}$, $\left.\frac{dh}{dt}\right|_{h=9} = \frac{9}{\pi(9)^2}(20) = \frac{20}{9\pi}$ cm/s.



Exercise Set 3.5

1. (a) $f(x) \approx f(1) + f'(1)(x - 1) = 1 + 3(x - 1)$.

(b) $f(1 + \Delta x) \approx f(1) + f'(1)\Delta x = 1 + 3\Delta x$.

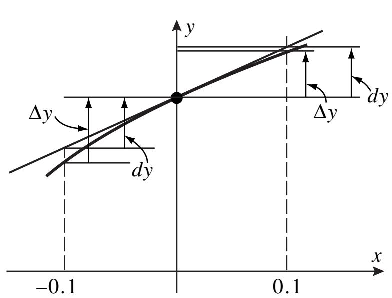
(c) From part (a), $(1.02)^3 \approx 1 + 3(0.02) = 1.06$. From part (b), $(1.02)^3 \approx 1 + 3(0.02) = 1.06$.

2. (a) $f(x) \approx f(2) + f'(2)(x - 2) = 1/2 + (-1/2^2)(x - 2) = (1/2) - (1/4)(x - 2)$.

(b) $f(2 + \Delta x) \approx f(2) + f'(2)\Delta x = 1/2 - (1/4)\Delta x$.

(c) From part (a), $1/2.05 \approx 0.5 - 0.25(0.05) = 0.4875$, and from part (b), $1/2.05 \approx 0.5 - 0.25(0.05) = 0.4875$.

3. (a) $f(x) \approx f(x_0) + f'(x_0)(x - x_0) = 1 + (1/(2\sqrt{1})(x - 0) = 1 + (1/2)x$, so with $x_0 = 0$ and $x = -0.1$, we have $\sqrt{0.9} = f(-0.1) \approx 1 + (1/2)(-0.1) = 1 - 0.05 = 0.95$. With $x = 0.1$ we have $\sqrt{1.1} = f(0.1) \approx 1 + (1/2)(0.1) = 1.05$.

(b) 

4. **(b)** The approximation is $\sqrt{x} \approx \sqrt{x_0} + \dfrac{1}{2\sqrt{x_0}}(x - x_0)$, so show that $\sqrt{x_0} + \dfrac{1}{2\sqrt{x_0}}(x - x_0) \geq \sqrt{x}$ which is equivalent to $g(x) = \sqrt{x} - \dfrac{x}{2\sqrt{x_0}} \leq \dfrac{\sqrt{x_0}}{2}$. But $g(x_0) = \dfrac{\sqrt{x_0}}{2}$, and $g'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2\sqrt{x_0}}$ which is negative for $x > x_0$ and positive for $x < x_0$. This shows that g has a maximum value at $x = x_0$, so the student's observation is correct.

5. $f(x) = (1+x)^{15}$ and $x_0 = 0$. Thus $(1+x)^{15} \approx f(x_0) + f'(x_0)(x - x_0) = 1 + 15(1)^{14}(x - 0) = 1 + 15x$.

6. $f(x) = \dfrac{1}{\sqrt{1-x}}$ and $x_0 = 0$, so $\dfrac{1}{\sqrt{1-x}} \approx f(x_0) + f'(x_0)(x - x_0) = 1 + \dfrac{1}{2(1-0)^{3/2}}(x - 0) = 1 + x/2$.

7. $\tan x \approx \tan(0) + \sec^2(0)(x - 0) = x$.

8. $\dfrac{1}{1+x} \approx 1 + \dfrac{-1}{(1+0)^2}(x - 0) = 1 - x$.

9. $x_0 = 0, f(x) = e^x, f'(x) = e^x, f'(x_0) = 1$, hence $e^x \approx 1 + 1 \cdot x = 1 + x$.

10. $x_0 = 0, f(x) = \ln(1+x), f'(x) = 1/(1+x), f'(x_0) = 1$, hence $\ln(1+x) \approx 0 + 1 \cdot (x - 0) = x$.

11. $x^4 \approx (1)^4 + 4(1)^3(x - 1)$. Set $\Delta x = x - 1$; then $x = \Delta x + 1$ and $(1 + \Delta x)^4 = 1 + 4\Delta x$.

12. $\sqrt{x} \approx \sqrt{1} + \dfrac{1}{2\sqrt{1}}(x - 1)$, and $x = 1 + \Delta x$, so $\sqrt{1 + \Delta x} \approx 1 + \Delta x/2$.

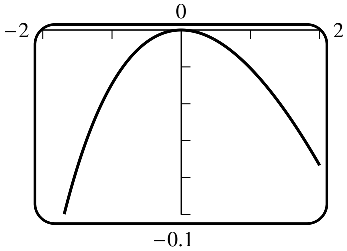
13. $\dfrac{1}{2+x} \approx \dfrac{1}{2+1} - \dfrac{1}{(2+1)^2}(x - 1)$, and $2 + x = 3 + \Delta x$, so $\dfrac{1}{3 + \Delta x} \approx \dfrac{1}{3} - \dfrac{1}{9}\Delta x$.

14. $(4+x)^3 \approx (4+1)^3 + 3(4+1)^2(x - 1)$ so, with $4 + x = 5 + \Delta x$ we get $(5 + \Delta x)^3 \approx 125 + 75\Delta x$.

15. Let $f(x) = \tan^{-1} x, f(1) = \pi/4, f'(1) = 1/2, \tan^{-1}(1 + \Delta x) \approx \dfrac{\pi}{4} + \dfrac{1}{2}\Delta x$.

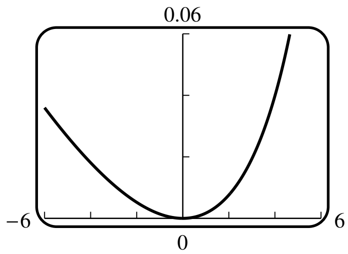
16. $f(x) = \sin^{-1}\left(\dfrac{x}{2}\right), \sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}, f'(x) = \dfrac{1/2}{\sqrt{1 - x^2/4}}, f'(1) = 1/\sqrt{3}$. $\sin^{-1}\left(\dfrac{1}{2} + \dfrac{1}{2}\Delta x\right) \approx \dfrac{\pi}{6} + \dfrac{1}{\sqrt{3}}\Delta x$.

17. $f(x) = \sqrt{x+3}$ and $x_0 = 0$, so $\sqrt{x+3} \approx \sqrt{3} + \dfrac{1}{2\sqrt{3}}(x - 0) = \sqrt{3} + \dfrac{1}{2\sqrt{3}}x$, and $\left|f(x) - \left(\sqrt{3} + \dfrac{1}{2\sqrt{3}}x\right)\right| < 0.1$ if $|x| < 1.692$.



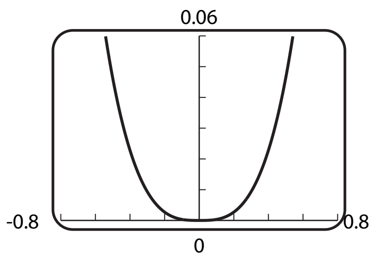
$\left|f(x) - \left(\sqrt{3} + \frac{1}{2\sqrt{3}}x\right)\right|$

18. $f(x) = \dfrac{1}{\sqrt{9-x}}$ so $\dfrac{1}{\sqrt{9-x}} \approx \dfrac{1}{\sqrt{9}} + \dfrac{1}{2(9-0)^{3/2}}(x - 0) = \dfrac{1}{3} + \dfrac{1}{54}x$, and $\left|f(x) - \left(\dfrac{1}{3} + \dfrac{1}{54}x\right)\right| < 0.1$ if $|x| < 5.5114$.



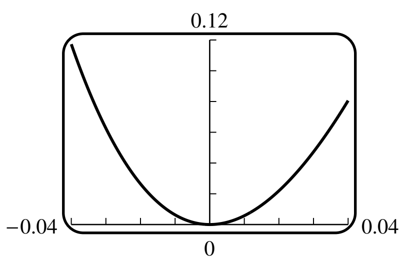
$\left|f(x) - \left(\frac{1}{3} + \frac{1}{54}x\right)\right|$

19. $\tan 2x \approx \tan 0 + (\sec^2 0)(2x - 0) = 2x$, and $|\tan 2x - 2x| < 0.1$ if $|x| < 0.3158$.



$|f(x) - 2x|$

20. $\dfrac{1}{(1+2x)^5} \approx \dfrac{1}{(1+2\cdot 0)^5} + \dfrac{-5(2)}{(1+2\cdot 0)^6}(x-0) = 1 - 10x$, and $|f(x) - (1-10x)| < 0.1$.



$|f(x) - (1 - 10x)|$

21. **(a)** The local linear approximation $\sin x \approx x$ gives $\sin 1^\circ = \sin(\pi/180) \approx \pi/180 = 0.0174533$ and a calculator gives $\sin 1^\circ = 0.0174524$. The relative error $|\sin(\pi/180) - (\pi/180)|/(\sin \pi/180) = 0.000051$ is very small, so for such a small value of x the approximation is very good.

(b) Use $x_0 = 45^\circ$ (this assumes you know, or can approximate, $\sqrt{2}/2$).

(c) $44^\circ = \dfrac{44\pi}{180}$ radians, and $45^\circ = \dfrac{45\pi}{180} = \dfrac{\pi}{4}$ radians. With $x = \dfrac{44\pi}{180}$ and $x_0 = \dfrac{\pi}{4}$ we obtain $\sin 44^\circ = \sin\dfrac{44\pi}{180} \approx \sin\dfrac{\pi}{4} + \left(\cos\dfrac{\pi}{4}\right)\left(\dfrac{44\pi}{180} - \dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\left(\dfrac{-\pi}{180}\right) = 0.694765$. With a calculator, $\sin 44^\circ = 0.694658$.

22. **(a)** $\tan x \approx \tan 0 + \sec^2 0(x - 0) = x$, so $\tan 2^\circ = \tan(2\pi/180) \approx 2\pi/180 = 0.034907$, and with a calculator $\tan 2^\circ = 0.034921$.

(b) Use $x_0 = \pi/3$ because we know $\tan 60^\circ = \tan(\pi/3) = \sqrt{3}$.

(c) With $x_0 = \dfrac{\pi}{3} = \dfrac{60\pi}{180}$ and $x = \dfrac{61\pi}{180}$ we have $\tan 61^\circ = \tan\dfrac{61\pi}{180} \approx \tan\dfrac{\pi}{3} + \left(\sec^2\dfrac{\pi}{3}\right)\left(\dfrac{61\pi}{180} - \dfrac{\pi}{3}\right) = \sqrt{3} + 4\dfrac{\pi}{180} = 1.8019$, and with a calculator $\tan 61^\circ = 1.8040$.

23. $f(x) = x^4$, $f'(x) = 4x^3$, $x_0 = 3$, $\Delta x = 0.02$; $(3.02)^4 \approx 3^4 + (108)(0.02) = 81 + 2.16 = 83.16$.

24. $f(x) = x^3,\ f'(x) = 3x^2,\ x_0 = 2,\ \Delta x = -0.03;\ (1.97)^3 \approx 2^3 + (12)(-0.03) = 8 - 0.36 = 7.64.$

25. $f(x) = \sqrt{x},\ f'(x) = \dfrac{1}{2\sqrt{x}},\ x_0 = 64,\ \Delta x = 1;\ \sqrt{65} \approx \sqrt{64} + \dfrac{1}{16}(1) = 8 + \dfrac{1}{16} = 8.0625.$

26. $f(x) = \sqrt{x},\ f'(x) = \dfrac{1}{2\sqrt{x}},\ x_0 = 25,\ \Delta x = -1;\ \sqrt{24} \approx \sqrt{25} + \dfrac{1}{10}(-1) = 5 - 0.1 = 4.9.$

27. $f(x) = \sqrt{x},\ f'(x) = \dfrac{1}{2\sqrt{x}},\ x_0 = 81,\ \Delta x = -0.1;\ \sqrt{80.9} \approx \sqrt{81} + \dfrac{1}{18}(-0.1) \approx 8.9944.$

28. $f(x) = \sqrt{x},\ f'(x) = \dfrac{1}{2\sqrt{x}},\ x_0 = 36,\ \Delta x = 0.03;\ \sqrt{36.03} \approx \sqrt{36} + \dfrac{1}{12}(0.03) = 6 + 0.0025 = 6.0025.$

29. $f(x) = \sin x,\ f'(x) = \cos x,\ x_0 = 0,\ \Delta x = 0.1;\ \sin 0.1 \approx \sin 0 + (\cos 0)(0.1) = 0.1.$

30. $f(x) = \tan x,\ f'(x) = \sec^2 x,\ x_0 = 0,\ \Delta x = 0.2;\ \tan 0.2 \approx \tan 0 + (\sec^2 0)(0.2) = 0.2.$

31. $f(x) = \cos x,\ f'(x) = -\sin x,\ x_0 = \pi/6,\ \Delta x = \pi/180;\ \cos 31^\circ \approx \cos 30^\circ + \left(-\dfrac{1}{2}\right)\left(\dfrac{\pi}{180}\right) = \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{360} \approx 0.8573.$

32. $f(x) = \ln x, x_0 = 1, \Delta x = 0.01, \ln x \approx \Delta x, \ln 1.01 \approx 0.01.$

33. $\tan^{-1}(1 + \Delta x) \approx \dfrac{\pi}{4} + \dfrac{1}{2}\Delta x, \Delta x = -0.01, \tan^{-1} 0.99 \approx \dfrac{\pi}{4} - 0.005 \approx 0.780398.$

34. **(a)** Let $f(x) = (1 + x)^k$ and $x_0 = 0$. Then $(1 + x)^k \approx 1^k + k(1)^{k-1}(x - 0) = 1 + kx$. Set $k = 37$ and $x = 0.001$ to obtain $(1.001)^{37} \approx 1.037$.

(b) With a calculator $(1.001)^{37} = 1.03767$.

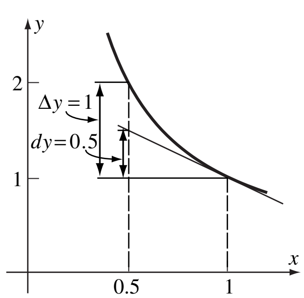
(c) It is the linear term of the expansion.

35. $\sqrt[3]{8.24} = 8^{1/3}\sqrt[3]{1.03} \approx 2(1 + \frac{1}{3}0.03) \approx 2.02$, and $4.08^{3/2} = 4^{3/2}1.02^{3/2} = 8(1 + 0.02(3/2)) = 8.24$.

36. $6^\circ = \pi/30$ radians; $h = 500\tan(\pi/30) \approx 500[\tan 0 + (\sec^2 0)\dfrac{\pi}{30}] = 500\pi/30 \approx 52.36$ ft.

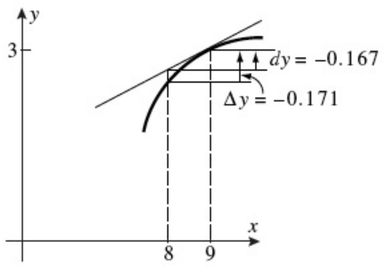
37. **(a)** $dy = (-1/x^2)dx = (-1)(-0.5) = 0.5$ and $\Delta y = 1/(x + \Delta x) - 1/x = 1/(1 - 0.5) - 1/1 = 2 - 1 = 1.$

(b)



38. **(a)** $dy = (1/2\sqrt{x})dx = (1/(2\cdot 3))(-1) = -1/6 \approx -0.167$ and $\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{9 + (-1)} - \sqrt{9} = \sqrt{8} - 3 \approx -0.172.$

(b)



39. $dy = 3x^2dx$; $\Delta y = (x + \Delta x)^3 - x^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$.

40. $dy = 8dx$; $\Delta y = [8(x + \Delta x) - 4] - [8x - 4] = 8\Delta x$.

41. $dy = (2x-2)dx$; $\Delta y = [(x+\Delta x)^2 - 2(x+\Delta x)+1] - [x^2 - 2x+1] = x^2 + 2x\ \Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1 = 2x\ \Delta x + (\Delta x)^2 - 2\Delta x$.

42. $dy = \cos x\, dx$; $\Delta y = \sin(x + \Delta x) - \sin x$.

43. **(a)** $dy = (12x^2 - 14x)dx$.

(b) $dy = x\, d(\cos x) + \cos x\, dx = x(-\sin x)dx + \cos x dx = (-x \sin x + \cos x)dx$.

44. **(a)** $dy = (-1/x^2)dx$.

(b) $dy = 5 \sec^2 x\, dx$.

45. **(a)** $dy = \left(\sqrt{1-x} - \dfrac{x}{2\sqrt{1-x}}\right) dx = \dfrac{2-3x}{2\sqrt{1-x}}dx$.

(b) $dy = -17(1 + x)^{-18}dx$.

46. **(a)** $dy = \dfrac{(x^3-1)d(1) - (1)d(x^3-1)}{(x^3-1)^2} = \dfrac{(x^3-1)(0) - (1)3x^2dx}{(x^3-1)^2} = -\dfrac{3x^2}{(x^3-1)^2}dx$.

(b) $dy = \dfrac{(2-x)(-3x^2)dx - (1-x^3)(-1)dx}{(2-x)^2} = \dfrac{2x^3 - 6x^2 + 1}{(2-x)^2}dx$.

47. False; $dy = (dy/dx)dx$.

48. True.

49. False; they are equal whenever the function is linear.

50. False; if $f'(x_0) = 0$ then the approximation is constant.

51. $dy = \dfrac{3}{2\sqrt{3x-2}}dx$, $x = 2$, $dx = 0.03$; $\Delta y \approx dy = \dfrac{3}{4}(0.03) = 0.0225$.

52. $dy = \dfrac{x}{\sqrt{x^2+8}}dx$, $x = 1$, $dx = -0.03$; $\Delta y \approx dy = (1/3)(-0.03) = -0.01$.

53. $dy = \dfrac{1-x^2}{(x^2+1)^2}dx$, $x = 2$, $dx = -0.04$; $\Delta y \approx dy = \left(-\dfrac{3}{25}\right)(-0.04) = 0.0048$.

54. $dy = \left(\dfrac{4x}{\sqrt{8x+1}} + \sqrt{8x+1}\right) dx$, $x = 3$, $dx = 0.05$; $\Delta y \approx dy = (37/5)(0.05) = 0.37$.

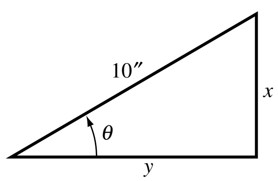
55. (a) $A = x^2$ where x is the length of a side; $dA = 2x\,dx = 2(10)(\pm 0.1) = \pm 2$ ft^2.

(b) Relative error in x is within $\dfrac{dx}{x} = \dfrac{\pm 0.1}{10} = \pm 0.01$ so percentage error in x is $\pm 1\%$; relative error in A is within $\dfrac{dA}{A} = \dfrac{2x\,dx}{x^2} = 2\dfrac{dx}{x} = 2(\pm 0.01) = \pm 0.02$ so percentage error in A is $\pm 2\%$.

56. (a) $V = x^3$ where x is the length of a side; $dV = 3x^2dx = 3(25)^2(\pm 1) = \pm 1875$ cm^3.

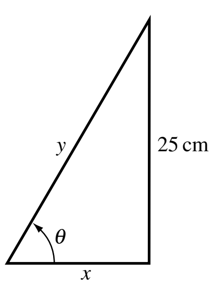
(b) Relative error in x is within $\dfrac{dx}{x} = \dfrac{\pm 1}{25} = \pm 0.04$ so percentage error in x is $\pm 4\%$; relative error in V is within $\dfrac{dV}{V} = \dfrac{3x^2dx}{x^3} = 3\dfrac{dx}{x} = 3(\pm 0.04) = \pm 0.12$ so percentage error in V is $\pm 12\%$.

57. (a) $x = 10\sin\theta$, $y = 10\cos\theta$ (see figure), $dx = 10\cos\theta d\theta = 10\left(\cos\dfrac{\pi}{6}\right)\left(\pm\dfrac{\pi}{180}\right) = 10\left(\dfrac{\sqrt{3}}{2}\right)\left(\pm\dfrac{\pi}{180}\right) \approx \pm 0.151$ in, $dy = -10(\sin\theta)d\theta = -10\left(\sin\dfrac{\pi}{6}\right)\left(\pm\dfrac{\pi}{180}\right) = -10\left(\dfrac{1}{2}\right)\left(\pm\dfrac{\pi}{180}\right) \approx \pm 0.087$ in.



(b) Relative error in x is within $\dfrac{dx}{x} = (\cot\theta)d\theta = \left(\cot\dfrac{\pi}{6}\right)\left(\pm\dfrac{\pi}{180}\right) = \sqrt{3}\left(\pm\dfrac{\pi}{180}\right) \approx \pm 0.030$, so percentage error in x is $\approx \pm 3.0\%$; relative error in y is within $\dfrac{dy}{y} = -\tan\theta d\theta = -\left(\tan\dfrac{\pi}{6}\right)\left(\pm\dfrac{\pi}{180}\right) = -\dfrac{1}{\sqrt{3}}\left(\pm\dfrac{\pi}{180}\right) \approx \pm 0.010$, so percentage error in y is $\approx \pm 1.0\%$.

58. (a) $x = 25\cot\theta$, $y = 25\csc\theta$ (see figure); $dx = -25\csc^2\theta d\theta = -25\left(\csc^2\dfrac{\pi}{3}\right)\left(\pm\dfrac{\pi}{360}\right) = -25\left(\dfrac{4}{3}\right)\left(\pm\dfrac{\pi}{360}\right) \approx \pm 0.291$ cm, $dy = -25\csc\theta\cot\theta d\theta = -25\left(\csc\dfrac{\pi}{3}\right)\left(\cot\dfrac{\pi}{3}\right)\left(\pm\dfrac{\pi}{360}\right) = -25\left(\dfrac{2}{\sqrt{3}}\right)\left(\dfrac{1}{\sqrt{3}}\right)\left(\pm\dfrac{\pi}{360}\right) \approx \pm 0.145$ cm.



(b) Relative error in x is within $\dfrac{dx}{x} = -\dfrac{\csc^2\theta}{\cot\theta}d\theta = -\dfrac{4/3}{1/\sqrt{3}}\left(\pm\dfrac{\pi}{360}\right) \approx \pm 0.020$, so percentage error in x is $\approx \pm 2.0\%$; relative error in y is within $\dfrac{dy}{y} = -\cot\theta d\theta = -\dfrac{1}{\sqrt{3}}\left(\pm\dfrac{\pi}{360}\right) \approx \pm 0.005$, so percentage error in y is $\approx \pm 0.5\%$.

59. $\dfrac{dR}{R} = \dfrac{(-2k/r^3)dr}{(k/r^2)} = -2\dfrac{dr}{r}$, but $\dfrac{dr}{r} = \pm 0.05$ so $\dfrac{dR}{R} = -2(\pm 0.05) = \pm 0.10$; percentage error in R is $\pm 10\%$.

60. $F = 36/L$, thus $\Delta F \approx dF = -36L^{-2}dL = -36 \cdot 18^{-2} \cdot 0.9 = -0.1$. Hence the value of F decreases by about 0.1 microtesla.

61. $A = \frac{1}{4}(4)^2 \sin 2\theta = 4 \sin 2\theta$ thus $dA = 8 \cos 2\theta d\theta$ so, with $\theta = 30^\circ = \pi/6$ radians and $d\theta = \pm 15' = \pm 1/4^\circ = \pm\pi/720$ radians, $dA = 8 \cos(\pi/3)(\pm\pi/720) = \pm\pi/180 \approx \pm 0.017$ cm^2.

62. $A = x^2$ where x is the length of a side; $\frac{dA}{A} = \frac{2x\,dx}{x^2} = 2\frac{dx}{x}$, but $\frac{dx}{x} = \pm 0.01$, so $\frac{dA}{A} = 2(\pm 0.01) = \pm 0.02$; percentage error in A is $\pm 2\%$

63. $V = x^3$ where x is the length of a side; $\frac{dV}{V} = \frac{3x^2 dx}{x^3} = 3\frac{dx}{x}$, but $\frac{dx}{x} = \pm 0.02$, so $\frac{dV}{V} = 3(\pm 0.02) = \pm 0.06$; percentage error in V is $\pm 6\%$.

64. $\frac{dV}{V} = \frac{4\pi r^2 dr}{4\pi r^3/3} = 3\frac{dr}{r}$, but $\frac{dV}{V} = \pm 0.03$ so $3\frac{dr}{r} = \pm 0.03$, $\frac{dr}{r} = \pm 0.01$; maximum permissible percentage error in r is $\pm 1\%$.

65. $A = \frac{1}{4}\pi D^2$ where D is the diameter of the circle; $\frac{dA}{A} = \frac{(\pi D/2)dD}{\pi D^2/4} = 2\frac{dD}{D}$, but $\frac{dA}{A} = \pm 0.01$ so $2\frac{dD}{D} = \pm 0.01$, $\frac{dD}{D} = \pm 0.005$; maximum permissible percentage error in D is $\pm 0.5\%$.

66. $V = x^3$ where x is the length of a side; approximate ΔV by dV if $x = 1$ and $dx = \Delta x = 0.02$, $dV = 3x^2 dx = 3(1)^2(0.02) = 0.06$ in^3.

67. V = volume of cylindrical rod $= \pi r^2 h = \pi r^2(15) = 15\pi r^2$; approximate ΔV by dV if $r = 2.5$ and $dr = \Delta r = 0.1$. $dV = 30\pi r\,dr = 30\pi(2.5)(0.1) \approx 23.5619$ cm^3.

68. $P = \frac{2\pi}{\sqrt{g}}\sqrt{L}$, $dP = \frac{2\pi}{\sqrt{g}}\frac{1}{2\sqrt{L}}dL = \frac{\pi}{\sqrt{g}\sqrt{L}}dL$, $\frac{dP}{P} = \frac{1}{2}\frac{dL}{L}$ so the relative error in $P \approx \frac{1}{2}$ the relative error in L. Thus the percentage error in P is $\approx \frac{1}{2}$ the percentage error in L.

69. Differentiating $R = \log_{10}(A/A_0)$, we obtain $\frac{dR}{dA} = \frac{1}{A \ln 10}$. Thus $dR = \frac{1}{\ln 10}\frac{dA}{A}$, and $\Delta R \approx dR \approx 0.4343\frac{dA}{A}$.

70. Differentiation gives $dT = -\frac{10 \ln 2}{(\ln V_1 - \ln 20)^2}\frac{dV_1}{V_1}$. Using $V_1 = 33$ and $dV_1 = \Delta V_1 = \pm 0.4$, we obtain that $\Delta T \approx dT = -\frac{10 \ln 2}{(\ln 33 - \ln 20)^2}\frac{\pm 0.4}{33} \approx \pm 0.3$ days.

Exercise Set 3.6

1. (a) $\lim_{x\to 2} \frac{x^2 - 4}{x^2 + 2x - 8} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x+4)(x-2)} = \lim_{x\to 2} \frac{x+2}{x+4} = \frac{2}{3}$ or, using L'Hôpital's rule, $\lim_{x\to 2} \frac{x^2 - 4}{x^2 + 2x - 8} = \lim_{x\to 2} \frac{2x}{2x+2} = \frac{2}{3}$.

(b) $\lim_{x\to+\infty} \frac{2x-5}{3x+7} = \frac{2 - \lim_{x\to+\infty} \frac{5}{x}}{3 + \lim_{x\to+\infty} \frac{7}{x}} = \frac{2}{3}$ or, using L'Hôpital's rule, $\lim_{x\to+\infty} \frac{2x-5}{3x+7} = \lim_{x\to+\infty} \frac{2}{3} = \frac{2}{3}$.

2. (a) $\frac{\sin x}{\tan x} = \cos x$ so $\lim_{x\to 0} \frac{\sin x}{\tan x} = \lim_{x\to 0} \cos x = 1$ or, using L'Hôpital's rule, $\lim_{x\to 0} \frac{\sin x}{\tan x} = \lim_{x\to 0} \frac{\cos x}{\sec^2 x} = 1$.

(b) $\frac{x^2 - 1}{x^3 - 1} = \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)} = \frac{x+1}{x^2+x+1}$ so $\lim_{x\to 1} \frac{x^2-1}{x^3-1} = \frac{2}{3}$ or, using L'Hôpital's rule,

$$\lim_{x\to 1}\frac{x^2-1}{x^3-1}=\lim_{x\to 1}\frac{2x}{3x^2}=\frac{2}{3}.$$

3. True; $\ln x$ is not defined for negative x.

4. True; apply L'Hôpital's rule n times, where $n = \deg\, p(x)$.

5. False; apply L'Hôpital's rule n times.

6. True; the logarithm of the expression approaches $-\infty$.

7. $\displaystyle\lim_{x\to 0}\frac{e^x}{\cos x}=1.$

8. $\displaystyle\lim_{x\to 0}\frac{2\cos 2x}{5\cos 5x}=\frac{2}{5}.$

9. $\displaystyle\lim_{\theta\to 0}\frac{\sec^2\theta}{1}=1.$

10. $\displaystyle\lim_{t\to 0}\frac{te^t+e^t}{-e^t}=-1.$

11. $\displaystyle\lim_{x\to \pi^+}\frac{\cos x}{1}=-1.$

12. $\displaystyle\lim_{x\to 0^+}\frac{\cos x}{2x}=+\infty.$

13. $\displaystyle\lim_{x\to +\infty}\frac{1/x}{1}=0.$

14. $\displaystyle\lim_{x\to +\infty}\frac{3e^{3x}}{2x}=\lim_{x\to +\infty}\frac{9e^{3x}}{2}=+\infty.$

15. $\displaystyle\lim_{x\to 0^+}\frac{-\csc^2 x}{1/x}=\lim_{x\to 0^+}\frac{-x}{\sin^2 x}=\lim_{x\to 0^+}\frac{-1}{2\sin x\cos x}=-\infty.$

16. $\displaystyle\lim_{x\to 0^+}\frac{-1/x}{(-1/x^2)e^{1/x}}=\lim_{x\to 0^+}\frac{x}{e^{1/x}}=0.$

17. $\displaystyle\lim_{x\to +\infty}\frac{100x^{99}}{e^x}=\lim_{x\to +\infty}\frac{(100)(99)x^{98}}{e^x}=\cdots=\lim_{x\to +\infty}\frac{(100)(99)(98)\cdots(1)}{e^x}=0.$

18. $\displaystyle\lim_{x\to 0^+}\frac{\cos x/\sin x}{\sec^2 x/\tan x}=\lim_{x\to 0^+}\cos^2 x=1.$

19. $\displaystyle\lim_{x\to 0}\frac{2/\sqrt{1-4x^2}}{1}=2.$

20. $\displaystyle\lim_{x\to 0}\frac{1-\dfrac{1}{1+x^2}}{3x^2}=\lim_{x\to 0}\frac{1}{3(1+x^2)}=\frac{1}{3}.$

21. $\displaystyle\lim_{x\to +\infty}xe^{-x}=\lim_{x\to +\infty}\frac{x}{e^x}=\lim_{x\to +\infty}\frac{1}{e^x}=0.$

22. $\lim_{x\to\pi}(x-\pi)\tan(x/2) = \lim_{x\to\pi}\dfrac{x-\pi}{\cot(x/2)} = \lim_{x\to\pi}\dfrac{1}{-(1/2)\csc^2(x/2)} = -2.$

23. $\lim_{x\to+\infty} x\sin(\pi/x) = \lim_{x\to+\infty}\dfrac{\sin(\pi/x)}{1/x} = \lim_{x\to+\infty}\dfrac{(-\pi/x^2)\cos(\pi/x)}{-1/x^2} = \lim_{x\to+\infty}\pi\cos(\pi/x) = \pi.$

24. $\lim_{x\to0^+}\tan x\ln x = \lim_{x\to0^+}\dfrac{\ln x}{\cot x} = \lim_{x\to0^+}\dfrac{1/x}{-\csc^2 x} = \lim_{x\to0^+}\dfrac{-\sin^2 x}{x} = \lim_{x\to0^+}\dfrac{-2\sin x\cos x}{1} = 0.$

25. $\lim_{x\to(\pi/2)^-}\sec 3x\cos 5x = \lim_{x\to(\pi/2)^-}\dfrac{\cos 5x}{\cos 3x} = \lim_{x\to(\pi/2)^-}\dfrac{-5\sin 5x}{-3\sin 3x} = \dfrac{-5(+1)}{(-3)(-1)} = -\dfrac{5}{3}.$

26. $\lim_{x\to\pi}(x-\pi)\cot x = \lim_{x\to\pi}\dfrac{x-\pi}{\tan x} = \lim_{x\to\pi}\dfrac{1}{\sec^2 x} = 1.$

27. $y=(1-3/x)^x,\ \lim_{x\to+\infty}\ln y = \lim_{x\to+\infty}\dfrac{\ln(1-3/x)}{1/x} = \lim_{x\to+\infty}\dfrac{-3}{1-3/x} = -3,\ \lim_{x\to+\infty} y = e^{-3}.$

28. $y=(1+2x)^{-3/x},\ \lim_{x\to0}\ln y = \lim_{x\to0}-\dfrac{3\ln(1+2x)}{x} = \lim_{x\to0}-\dfrac{6}{1+2x} = -6,\ \lim_{x\to0} y = e^{-6}.$

29. $y=(e^x+x)^{1/x},\ \lim_{x\to0}\ln y = \lim_{x\to0}\dfrac{\ln(e^x+x)}{x} = \lim_{x\to0}\dfrac{e^x+1}{e^x+x} = 2,\ \lim_{x\to0} y = e^2.$

30. $y=(1+a/x)^{bx},\ \lim_{x\to+\infty}\ln y = \lim_{x\to+\infty}\dfrac{b\ln(1+a/x)}{1/x} = \lim_{x\to+\infty}\dfrac{ab}{1+a/x} = ab,\ \lim_{x\to+\infty} y = e^{ab}.$

31. $y=(2-x)^{\tan(\pi x/2)},\ \lim_{x\to1}\ln y = \lim_{x\to1}\dfrac{\ln(2-x)}{\cot(\pi x/2)} = \lim_{x\to1}\dfrac{2\sin^2(\pi x/2)}{\pi(2-x)} = 2/\pi,\ \lim_{x\to1} y = e^{2/\pi}.$

32. $y=[\cos(2/x)]^{x^2},\ \lim_{x\to+\infty}\ln y = \lim_{x\to+\infty}\dfrac{\ln\cos(2/x)}{1/x^2} = \lim_{x\to+\infty}\dfrac{(-2/x^2)(-\tan(2/x))}{-2/x^3} = \lim_{x\to+\infty}\dfrac{-\tan(2/x)}{1/x} =$
$\lim_{x\to+\infty}\dfrac{(2/x^2)\sec^2(2/x)}{-1/x^2} = -2,\ \lim_{x\to+\infty} y = e^{-2}.$

33. $\lim_{x\to0}\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right) = \lim_{x\to0}\dfrac{x-\sin x}{x\sin x} = \lim_{x\to0}\dfrac{1-\cos x}{x\cos x+\sin x} = \lim_{x\to0}\dfrac{\sin x}{2\cos x - x\sin x} = 0.$

34. $\lim_{x\to0}\dfrac{1-\cos 3x}{x^2} = \lim_{x\to0}\dfrac{3\sin 3x}{2x} = \lim_{x\to0}\dfrac{9}{2}\cos 3x = \dfrac{9}{2}.$

35. $\lim_{x\to+\infty}\dfrac{(x^2+x)-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to+\infty}\dfrac{x}{\sqrt{x^2+x}+x} = \lim_{x\to+\infty}\dfrac{1}{\sqrt{1+1/x}+1} = 1/2.$

36. $\lim_{x\to0}\dfrac{e^x-1-x}{xe^x-x} = \lim_{x\to0}\dfrac{e^x-1}{xe^x+e^x-1} = \lim_{x\to0}\dfrac{e^x}{xe^x+2e^x} = 1/2.$

37. $\lim_{x\to+\infty}[x-\ln(x^2+1)] = \lim_{x\to+\infty}[\ln e^x - \ln(x^2+1)] = \lim_{x\to+\infty}\ln\dfrac{e^x}{x^2+1},\ \lim_{x\to+\infty}\dfrac{e^x}{x^2+1} = \lim_{x\to+\infty}\dfrac{e^x}{2x} = \lim_{x\to+\infty}\dfrac{e^x}{2} = +\infty,$
so $\lim_{x\to+\infty}[x-\ln(x^2+1)] = +\infty$

38. $\lim_{x\to+\infty}\ln\dfrac{x}{1+x} = \lim_{x\to+\infty}\ln\dfrac{1}{1/x+1} = \ln(1) = 0.$

39. $y = x^{\sin x}, \ln y = \sin x \ln x, \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)(-\tan x) = 1(-0) = 0$, so $\lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} y = e^0 = 1$.

40. $y = (e^{2x}-1)^x, \ln y = x \ln(e^{2x}-1), \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(e^{2x}-1)}{1/x} = \lim_{x\to 0^+} \frac{2e^{2x}}{e^{2x}-1}(-x^2) =$
$= \lim_{x\to 0^+} \frac{x}{e^{2x}-1} \lim_{x\to 0^+} (-2xe^{2x}) = \lim_{x\to 0^+} \frac{1}{2e^{2x}} \lim_{x\to 0^+} (-2xe^{2x}) = \frac{1}{2}\cdot 0 = 0, \lim_{x\to 0^+} y = e^0 = 1$.

41. $y = \left[-\frac{1}{\ln x}\right]^x, \ln y = x \ln\left[-\frac{1}{\ln x}\right], \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln\left[-\frac{1}{\ln x}\right]}{1/x} = \lim_{x\to 0^+} \left(-\frac{1}{x\ln x}\right)(-x^2) = -\lim_{x\to 0^+} \frac{x}{\ln x} = 0$, so $\lim_{x\to 0^+} y = e^0 = 1$.

42. $y = x^{1/x}, \ln y = \frac{\ln x}{x}, \lim_{x\to +\infty} \ln y = \lim_{x\to +\infty} \frac{\ln x}{x} = \lim_{x\to +\infty} \frac{1/x}{1} = 0$, so $\lim_{x\to +\infty} y = e^0 = 1$.

43. $y = (\ln x)^{1/x}, \ln y = (1/x)\ln\ln x, \lim_{x\to +\infty} \ln y = \lim_{x\to +\infty} \frac{\ln\ln x}{x} = \lim_{x\to +\infty} \frac{1/(x\ln x)}{1} = 0$, so $\lim_{x\to +\infty} y = 1$.

44. $y = (-\ln x)^x, \ln y = x\ln(-\ln x), \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \ln(-\ln x)/(1/x) = \lim_{x\to 0^+} \frac{(1/(x\ln x)}{(-1/x^2)} = \lim_{x\to 0^+} (-\frac{x}{\ln x}) = 0$, so $\lim_{x\to 0^+} y = 1$.

45. $y = (\tan x)^{\pi/2-x}, \ln y = (\pi/2-x)\ln\tan x, \lim_{x\to(\pi/2)^-} \ln y = \lim_{x\to(\pi/2)^-} \frac{\ln\tan x}{1/(\pi/2-x)} = \lim_{x\to(\pi/2)^-} \frac{(\sec^2 x/\tan x)}{1/(\pi/2-x)^2} =$
$\lim_{x\to(\pi/2)^-} \frac{(\pi/2-x)}{\cos x}\frac{(\pi/2-x)}{\sin x} = \lim_{x\to(\pi/2)^-} \frac{(\pi/2-x)}{\cos x} \lim_{x\to(\pi/2)^-} \frac{(\pi/2-x)}{\sin x} = 1\cdot 0 = 0$, so $\lim_{x\to(\pi/2)^-} y = 1$.

46. **(a)** $\lim_{x\to +\infty} \frac{\ln x}{x^n} = \lim_{x\to +\infty} \frac{1/x}{nx^{n-1}} = \lim_{x\to +\infty} \frac{1}{nx^n} = 0$.

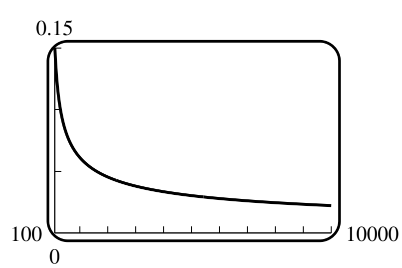
(b) $\lim_{x\to +\infty} \frac{x^n}{\ln x} = \lim_{x\to +\infty} \frac{nx^{n-1}}{1/x} = \lim_{x\to +\infty} nx^n = +\infty$.

47. **(a)** L'Hôpital's rule does not apply to the problem $\lim_{x\to 1} \frac{3x^2-2x+1}{3x^2-2x}$ because it is not an indeterminate form.

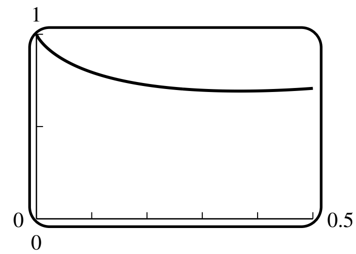
(b) $\lim_{x\to 1} \frac{3x^2-2x+1}{3x^2-2x} = 2$.

48. **(a)** L'Hôpital's rule does not apply to the problem $\lim_{x\to 2} \frac{e^{3x^2-12x+12}}{x^4-16}$, because it is not an indeterminate form.

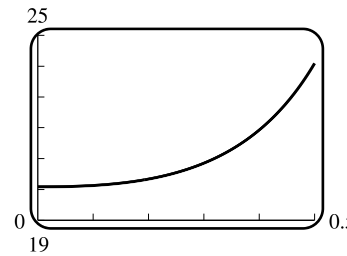
(b) $\lim_{x\to 2^-}$ and $\lim_{x\to 2^+}$ exist, with values $-\infty$ if x approaches 2 from the left and $+\infty$ if from the right. The general limit $\lim_{x\to 2}$ does not exist.



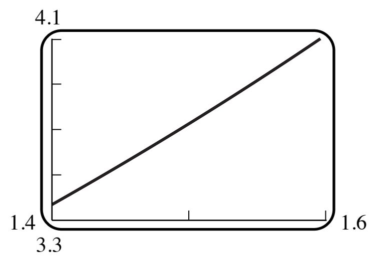
49. $\lim_{x\to +\infty} \frac{1/(x\ln x)}{1/(2\sqrt{x})} = \lim_{x\to +\infty} \frac{2}{\sqrt{x}\ln x} = 0$.



50. $y = x^x$, $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \dfrac{\ln x}{1/x} = \lim\limits_{x\to 0^+} -x = 0$, $\lim\limits_{x\to 0^+} y = 1$.

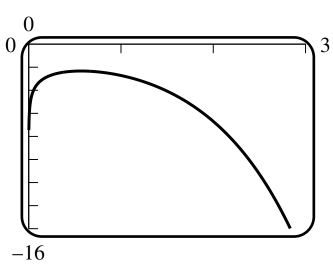


51. $y = (\sin x)^{3/\ln x}$, $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \dfrac{3\ln\sin x}{\ln x} = \lim\limits_{x\to 0^+} (3\cos x)\dfrac{x}{\sin x} = 3$, $\lim\limits_{x\to 0^+} y = e^3$.

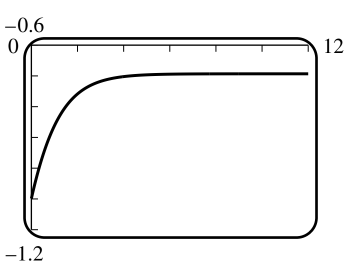


52. $\lim\limits_{x\to \pi/2^-} \dfrac{4\sec^2 x}{\sec x \tan x} = \lim\limits_{x\to \pi/2^-} \dfrac{4}{\sin x} = 4$.

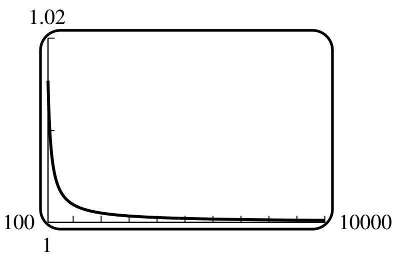
53. $\ln x - e^x = \ln x - \dfrac{1}{e^{-x}} = \dfrac{e^{-x}\ln x - 1}{e^{-x}}$; $\lim\limits_{x\to+\infty} e^{-x}\ln x = \lim\limits_{x\to+\infty} \dfrac{\ln x}{e^x} = \lim\limits_{x\to+\infty} \dfrac{1/x}{e^x} = 0$ by L'Hôpital's rule, so $\lim\limits_{x\to+\infty} [\ln x - e^x] = \lim\limits_{x\to+\infty} \dfrac{e^{-x}\ln x - 1}{e^{-x}} = -\infty$; no horizontal asymptote.



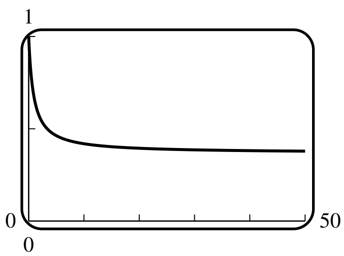
54. $\lim\limits_{x\to+\infty} [\ln e^x - \ln(1+2e^x)] = \lim\limits_{x\to+\infty} \ln \dfrac{e^x}{1+2e^x} = \lim\limits_{x\to+\infty} \ln \dfrac{1}{e^{-x}+2} = \ln \dfrac{1}{2}$; horizontal asymptote $y = -\ln 2$. Also, $\lim\limits_{x\to-\infty} \ln \dfrac{e^x}{1+2e^x} = -\infty$.



55. $y = (\ln x)^{1/x}$, $\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty} \dfrac{\ln(\ln x)}{x} = \lim_{x\to+\infty} \dfrac{1}{x\ln x} = 0$; $\lim_{x\to+\infty} y = 1$, $y = 1$ is the horizontal asymptote.



56. $y = \left(\dfrac{x+1}{x+2}\right)^x$, $\lim_{x\to\pm\infty} \ln y = \lim_{x\to\pm\infty} \dfrac{\ln\dfrac{x+1}{x+2}}{1/x} = \lim_{x\to\pm\infty} \dfrac{-x^2}{(x+1)(x+2)} = -1$; $\lim_{x\to\pm\infty} y = e^{-1}$ is the horizontal asymptote.



57. **(a)** 0 **(b)** $+\infty$ **(c)** 0 **(d)** $-\infty$ **(e)** $+\infty$ **(f)** $-\infty$

58. **(a)** Type 0^0; $y = x^{(\ln a)/(1+\ln x)}$; $\lim_{x\to0^+} \ln y = \lim_{x\to0^+} \dfrac{(\ln a)\ln x}{1+\ln x} = \lim_{x\to0^+} \dfrac{(\ln a)/x}{1/x} = \lim_{x\to0^+} \ln a = \ln a$, so we obtain that $\lim_{x\to0^+} y = e^{\ln a} = a$.

(b) Type ∞^0; same calculation as part (a) with $x \to +\infty$.

(c) Type 1^∞; $y = (x+1)^{(\ln a)/x}$, $\lim_{x\to0} \ln y = \lim_{x\to0} \dfrac{(\ln a)\ln(x+1)}{x} = \lim_{x\to0} \dfrac{\ln a}{x+1} = \ln a$, so $\lim_{x\to0} y = e^{\ln a} = a$.

59. $\lim_{x\to+\infty} \dfrac{1+2\cos 2x}{1}$ does not exist, nor is it $\pm\infty$; $\lim_{x\to+\infty} \dfrac{x+\sin 2x}{x} = \lim_{x\to+\infty} \left(1 + \dfrac{\sin 2x}{x}\right) = 1$.

60. $\lim_{x\to+\infty} \dfrac{2-\cos x}{3+\cos x}$ does not exist, nor is it $\pm\infty$; $\lim_{x\to+\infty} \dfrac{2x-\sin x}{3x+\sin x} = \lim_{x\to+\infty} \dfrac{2-(\sin x)/x}{3+(\sin x)/x} = \dfrac{2}{3}$.

61. $\lim_{x\to+\infty} (2 + x\cos 2x + \sin 2x)$ does not exist, nor is it $\pm\infty$; $\lim_{x\to+\infty} \dfrac{x(2+\sin 2x)}{x+1} = \lim_{x\to+\infty} \dfrac{2+\sin 2x}{1+1/x}$, which does not exist because $\sin 2x$ oscillates between -1 and 1 as $x \to +\infty$.

62. $\lim_{x\to+\infty} \left(\dfrac{1}{x} + \dfrac{1}{2}\cos x + \dfrac{\sin x}{2x}\right)$ does not exist, nor is it $\pm\infty$; $\lim_{x\to+\infty} \dfrac{x(2+\sin x)}{x^2+1} = \lim_{x\to+\infty} \dfrac{2+\sin x}{x+1/x} = 0$.

63. $\lim_{R\to0^+} \dfrac{\frac{Vt}{L}e^{-Rt/L}}{1} = \dfrac{Vt}{L}$.

64. **(a)** $\lim_{x\to\pi/2} (\pi/2 - x)\tan x = \lim_{x\to\pi/2} \dfrac{\pi/2 - x}{\cot x} = \lim_{x\to\pi/2} \dfrac{-1}{-\csc^2 x} = \lim_{x\to\pi/2} \sin^2 x = 1$.

(b) $\lim\limits_{x\to\pi/2}\left(\dfrac{1}{\pi/2-x}-\tan x\right)=\lim\limits_{x\to\pi/2}\left(\dfrac{1}{\pi/2-x}-\dfrac{\sin x}{\cos x}\right)=\lim\limits_{x\to\pi/2}\dfrac{\cos x-(\pi/2-x)\sin x}{(\pi/2-x)\cos x}=$
$=\lim\limits_{x\to\pi/2}\dfrac{-(\pi/2-x)\cos x}{-(\pi/2-x)\sin x-\cos x}=\lim\limits_{x\to\pi/2}\dfrac{(\pi/2-x)\sin x+\cos x}{-(\pi/2-x)\cos x+2\sin x}=0$ (by applying L'H's rule twice).

(c) $1/(\pi/2-1.57)\approx 1255.765534, \tan 1.57\approx 1255.765592;\ 1/(\pi/2-1.57)-\tan 1.57\approx 0.000058.$

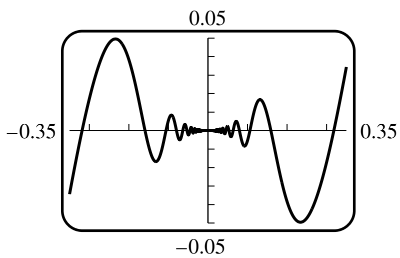
65. **(b)** $\lim\limits_{x\to+\infty}x(k^{1/x}-1)=\lim\limits_{t\to0^+}\dfrac{k^t-1}{t}=\lim\limits_{t\to0^+}\dfrac{(\ln k)k^t}{1}=\ln k.$

(c) $\ln 0.3=-1.20397,\ 1024\left(\sqrt[1024]{0.3}-1\right)=-1.20327;\ \ln 2=0.69315,\ 1024\left(\sqrt[1024]{2}-1\right)=0.69338.$

66. If $k\neq-1$ then $\lim\limits_{x\to0}(k+\cos\ell x)=k+1\neq0$, so $\lim\limits_{x\to0}\dfrac{k+\cos\ell x}{x^2}=\pm\infty$. Hence $k=-1$, and by the rule
$\lim\limits_{x\to0}\dfrac{-1+\cos\ell x}{x^2}=\lim\limits_{x\to0}\dfrac{-\ell\sin\ell x}{2x}=\lim\limits_{x\to0}\dfrac{-\ell^2\cos\ell x}{2}=-\dfrac{\ell^2}{2}=-4$ if $\ell=\pm2\sqrt{2}$.

67. **(a)** No; $\sin(1/x)$ oscillates as $x\to0$.

(b)



(c) For the limit as $x\to0^+$ use the Squeezing Theorem together with the inequalities $-x^2\le x^2\sin(1/x)\le x^2$. For $x\to0^-$ do the same; thus $\lim\limits_{x\to0}f(x)=0$.

68. **(a)** Apply the rule to get $\lim\limits_{x\to0}\dfrac{-\cos(1/x)+2x\sin(1/x)}{\cos x}$ which does not exist (nor is it $\pm\infty$).

(b) Rewrite as $\lim\limits_{x\to0}\left[\dfrac{x}{\sin x}\right][x\sin(1/x)]$, but $\lim\limits_{x\to0}\dfrac{x}{\sin x}=\lim\limits_{x\to0}\dfrac{1}{\cos x}=1$ and $\lim\limits_{x\to0}x\sin(1/x)=0$, thus
$\lim\limits_{x\to0}\left[\dfrac{x}{\sin x}\right][x\sin(1/x)]=(1)(0)=0.$

69. $\lim\limits_{x\to0^+}\dfrac{\sin(1/x)}{(\sin x)/x}$, $\lim\limits_{x\to0^+}\dfrac{\sin x}{x}=1$ but $\lim\limits_{x\to0^+}\sin(1/x)$ does not exist because $\sin(1/x)$ oscillates between -1 and 1 as $x\to+\infty$, so $\lim\limits_{x\to0^+}\dfrac{x\sin(1/x)}{\sin x}$ does not exist.

70. Since $f(a)=g(a)=0$, then for $x\neq a$, $\dfrac{f(x)}{g(x)}=\dfrac{(f(x)-f(a)/(x-a)}{(g(x)-g(a))/(x-a)}$. Now take the limit: $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{(f(x)-f(a)/(x-a)}{(g(x)-g(a))/(x-a)}=\dfrac{f'(a)}{g'(a)}.$

Chapter 3 Review Exercises

1. **(a)** $3x^2+x\dfrac{dy}{dx}+y-2=0,\ \dfrac{dy}{dx}=\dfrac{2-y-3x^2}{x}.$

(b) $y=(1+2x-x^3)/x=1/x+2-x^2, dy/dx=-1/x^2-2x.$

(c) $\dfrac{dy}{dx} = \dfrac{2 - (1/x + 2 - x^2) - 3x^2}{x} = -1/x^2 - 2x.$

2. (a) $xy = x - y,\ x\dfrac{dy}{dx} + y = 1 - \dfrac{dy}{dx},\ \dfrac{dy}{dx} = \dfrac{1-y}{x+1}.$

(b) $y(x+1) = x, y = \dfrac{x}{x+1}, y' = \dfrac{1}{(x+1)^2}.$

(c) $\dfrac{dy}{dx} = \dfrac{1-y}{x+1} = \dfrac{1 - \frac{x}{x+1}}{1+x} = \dfrac{1}{(x+1)^2}.$

3. $-\dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{1}{x^2} = 0$ so $\dfrac{dy}{dx} = -\dfrac{y^2}{x^2}.$

4. $3x^2 - 3y^2\dfrac{dy}{dx} = 6(x\dfrac{dy}{dx} + y),\ -(3y^2 + 6x)\dfrac{dy}{dx} = 6y - 3x^2$ so $\dfrac{dy}{dx} = \dfrac{x^2 - 2y}{y^2 + 2x}.$

5. $\left(x\dfrac{dy}{dx} + y\right)\sec(xy)\tan(xy) = \dfrac{dy}{dx},\ \dfrac{dy}{dx} = \dfrac{y\sec(xy)\tan(xy)}{1 - x\sec(xy)\tan(xy)}.$

6. $2x = \dfrac{(1+\csc y)(-\csc^2 y)(dy/dx) - (\cot y)(-\csc y\cot y)(dy/dx)}{(1+\csc y)^2}$, $2x(1+\csc y)^2 = -\csc y(\csc y + \csc^2 y - \cot^2 y)\dfrac{dy}{dx}$, but $\csc^2 y - \cot^2 y = 1$, so $\dfrac{dy}{dx} = -\dfrac{2x(1+\csc y)}{\csc y}.$

7. $\dfrac{dy}{dx} = \dfrac{3x}{4y},\ \dfrac{d^2y}{dx^2} = \dfrac{(4y)(3) - (3x)(4dy/dx)}{16y^2} = \dfrac{12y - 12x(3x/(4y))}{16y^2} = \dfrac{12y^2 - 9x^2}{16y^3} = \dfrac{-3(3x^2 - 4y^2)}{16y^3}$, but $3x^2 - 4y^2 = 7$ so $\dfrac{d^2y}{dx^2} = \dfrac{-3(7)}{16y^3} = -\dfrac{21}{16y^3}.$

8. $\dfrac{dy}{dx} = \dfrac{y}{y-x},\ \dfrac{d^2y}{dx^2} = \dfrac{(y-x)(dy/dx) - y(dy/dx - 1)}{(y-x)^2} = \dfrac{(y-x)\left(\dfrac{y}{y-x}\right) - y\left(\dfrac{y}{y-x} - 1\right)}{(y-x)^2} = \dfrac{y^2 - 2xy}{(y-x)^3}$, but $y^2 - 2xy = -3$, so $\dfrac{d^2y}{dx^2} = -\dfrac{3}{(y-x)^3}.$

9. $\dfrac{dy}{dx} = \tan(\pi y/2) + x(\pi/2)\dfrac{dy}{dx}\sec^2(\pi y/2),\ \left.\dfrac{dy}{dx}\right|_{y=1/2} = 1 + (\pi/4)\left.\dfrac{dy}{dx}\right|_{y=1/2}(2),\ \left.\dfrac{dy}{dx}\right|_{y=1/2} = \dfrac{2}{2-\pi}.$

10. Let $P(x_0, y_0)$ be the required point. The slope of the line $4x - 3y + 1 = 0$ is $4/3$ so the slope of the tangent to $y^2 = 2x^3$ at P must be $-3/4$. By implicit differentiation $dy/dx = 3x^2/y$, so at P, $3x_0^2/y_0 = -3/4$, or $y_0 = -4x_0^2$. But $y_0^2 = 2x_0^3$ because P is on the curve $y^2 = 2x^3$. Elimination of y_0 gives $16x_0^4 = 2x_0^3$, $x_0^3(8x_0 - 1) = 0$, so $x_0 = 0$ or $1/8$. From $y_0 = -4x_0^2$ it follows that $y_0 = 0$ when $x_0 = 0$, and $y_0 = -1/16$ when $x_0 = 1/8$. It does not follow, however, that $(0,0)$ is a solution because $dy/dx = 3x^2/y$ (the slope of the curve as determined by implicit differentiation) is valid only if $y \neq 0$. Further analysis shows that the curve is tangent to the x-axis at $(0,0)$, so the point $(1/8, -1/16)$ is the only solution.

11. Substitute $y = mx$ into $x^2 + xy + y^2 = 4$ to get $x^2 + mx^2 + m^2x^2 = 4$, which has distinct solutions $x = \pm 2/\sqrt{m^2 + m + 1}$. They are distinct because $m^2 + m + 1 = (m + 1/2)^2 + 3/4 \geq 3/4$, so $m^2 + m + 1$ is never zero. Note that the points of intersection occur in pairs (x_0, y_0) and $(-x_0, -y_0)$. By implicit differentiation, the slope of the tangent line to the ellipse is given by $dy/dx = -(2x + y)/(x + 2y)$. Since the slope is unchanged if we replace (x, y) with $(-x, -y)$, it follows that the slopes are equal at the two point of intersection. Finally we must examine the special case $x = 0$ which cannot be written in the form $y = mx$. If $x = 0$ then $y = \pm 2$, and the formula for dy/dx gives $dy/dx = -1/2$, so the slopes are equal.

12. By implicit differentiation, $3x^2 - y - xy' + 3y^2y' = 0$, so $y' = (3x^2 - y)/(x - 3y^2)$. This derivative is zero when $y = 3x^2$. Substituting this into the original equation $x^3 - xy + y^3 = 0$, one has $x^3 - 3x^3 + 27x^6 = 0$, $x^3(27x^3 - 2) = 0$. The unique solution in the first quadrant is $x = 2^{1/3}/3, y = 3x^2 = 2^{2/3}/3$.

13. By implicit differentiation, $3x^2 - y - xy' + 3y^2y' = 0$, so $y' = (3x^2 - y)/(x - 3y^2)$. This derivative exists except when $x = 3y^2$. Substituting this into the original equation $x^3 - xy + y^3 = 0$, one has $27y^6 - 3y^3 + y^3 = 0$, $y^3(27y^3 - 2) = 0$. The unique solution in the first quadrant is $y = 2^{1/3}/3, x = 3y^2 = 2^{2/3}/3$

14. By implicit differentiation, $dy/dx = k/(2y)$ so the slope of the tangent to $y^2 = kx$ at (x_0, y_0) is $k/(2y_0)$ if $y_0 \neq 0$. The tangent line in this case is $y - y_0 = \dfrac{k}{2y_0}(x - x_0)$, or $2y_0y - 2y_0^2 = kx - kx_0$. But $y_0^2 = kx_0$ because (x_0, y_0) is on the curve $y^2 = kx$, so the equation of the tangent line becomes $2y_0y - 2kx_0 = kx - kx_0$ which gives $y_0y = k(x + x_0)/2$. If $y_0 = 0$, then $x_0 = 0$; the graph of $y^2 = kx$ has a vertical tangent at $(0, 0)$ so its equation is $x = 0$, but $y_0y = k(x + x_0)/2$ gives the same result when $x_0 = y_0 = 0$.

15. $y = \ln(x + 1) + 2\ln(x + 2) - 3\ln(x + 3) - 4\ln(x + 4)$, $dy/dx = \dfrac{1}{x+1} + \dfrac{2}{x+2} - \dfrac{3}{x+3} - \dfrac{4}{x+4}$.

16. $y = \dfrac{1}{2}\ln x + \dfrac{1}{3}\ln(x + 1) - \ln\sin x + \ln\cos x$, so $\dfrac{dy}{dx} = \dfrac{1}{2x} + \dfrac{1}{3(x+1)} - \dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x} = \dfrac{5x+3}{6x(x+1)} - \cot x - \tan x$.

17. $\dfrac{dy}{dx} = \dfrac{1}{2x}(2) = 1/x$.

18. $\dfrac{dy}{dx} = 2(\ln x)\left(\dfrac{1}{x}\right) = \dfrac{2\ln x}{x}$.

19. $\dfrac{dy}{dx} = \dfrac{1}{3x(\ln x + 1)^{2/3}}$.

20. $y = \dfrac{1}{3}\ln(x + 1)$, $y' = \dfrac{1}{3(x+1)}$.

21. $\dfrac{dy}{dx} = \log_{10}\ln x = \dfrac{\ln\ln x}{\ln 10}, y' = \dfrac{1}{(\ln 10)(x\ln x)}$.

22. $y = \dfrac{1 + \ln x/\ln 10}{1 - \ln x/\ln 10} = \dfrac{\ln 10 + \ln x}{\ln 10 - \ln x}$, $y' = \dfrac{(\ln 10 - \ln x)/x + (\ln 10 + \ln x)/x}{(\ln 10 - \ln x)^2} = \dfrac{2\ln 10}{x(\ln 10 - \ln x)^2}$.

23. $y = \dfrac{3}{2}\ln x + \dfrac{1}{2}\ln(1 + x^4)$, $y' = \dfrac{3}{2x} + \dfrac{2x^3}{(1 + x^4)}$.

24. $y = \dfrac{1}{2}\ln x + \ln\cos x - \ln(1 + x^2)$, $y' = \dfrac{1}{2x} - \dfrac{\sin x}{\cos x} - \dfrac{2x}{1+x^2} = \dfrac{1 - 3x^2}{2x(1+x^2)} - \tan x$.

25. $y = x^2 + 1$ so $y' = 2x$.

26. $y = \ln\dfrac{(1 + e^x + e^{2x})}{(1 - e^x)(1 + e^x + e^{2x})} = -\ln(1 - e^x)$, $\dfrac{dy}{dx} = \dfrac{e^x}{1 - e^x}$.

27. $y' = 2e^{\sqrt{x}} + 2xe^{\sqrt{x}}\dfrac{d}{dx}\sqrt{x} = 2e^{\sqrt{x}} + \sqrt{x}e^{\sqrt{x}}$.

28. $y' = \dfrac{abe^{-x}}{(1 + be^{-x})^2}$.

29. $y' = \dfrac{2}{\pi(1+4x^2)}$.

30. $y = e^{(\sin^{-1} x)\ln 2}$, $y' = \dfrac{\ln 2}{\sqrt{1-x^2}} 2^{\sin^{-1} x}$.

31. $\ln y = e^x \ln x$, $\dfrac{y'}{y} = e^x\left(\dfrac{1}{x} + \ln x\right)$, $\dfrac{dy}{dx} = x^{e^x} e^x \left(\dfrac{1}{x} + \ln x\right) = e^x\left[x^{e^x - 1} + x^{e^x}\ln x\right]$.

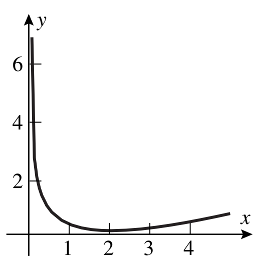
32. $\ln y = \dfrac{\ln(1+x)}{x}$, $\dfrac{y'}{y} = \dfrac{x/(1+x) - \ln(1+x)}{x^2} = \dfrac{1}{x(1+x)} - \dfrac{\ln(1+x)}{x^2}$, $\dfrac{dy}{dx} = \dfrac{1}{x}(1+x)^{(1/x)-1} - \dfrac{(1+x)^{(1/x)}}{x^2}\ln(1+x)$.

33. $y' = \dfrac{2}{|2x+1|\sqrt{(2x+1)^2-1}}$.

34. $y' = \dfrac{1}{2\sqrt{\cos^{-1} x^2}}\dfrac{d}{dx}\cos^{-1} x^2 = -\dfrac{1}{\sqrt{\cos^{-1} x^2}}\dfrac{x}{\sqrt{1-x^4}}$.

35. $\ln y = 3\ln x - \dfrac{1}{2}\ln(x^2+1)$, $y'/y = \dfrac{3}{x} - \dfrac{x}{x^2+1}$, $y' = \dfrac{3x^2}{\sqrt{x^2+1}} - \dfrac{x^4}{(x^2+1)^{3/2}}$.

36. $\ln y = \dfrac{1}{3}(\ln(x^2-1) - \ln(x^2+1))$, $\dfrac{y'}{y} = \dfrac{1}{3}\left(\dfrac{2x}{x^2-1} - \dfrac{2x}{x^2+1}\right) = \dfrac{4x}{3(x^4-1)}$ so $y' = \dfrac{4x}{3(x^4-1)}\sqrt[3]{\dfrac{x^2-1}{x^2+1}}$.

37. **(b)** 

(c) $\dfrac{dy}{dx} = \dfrac{1}{2} - \dfrac{1}{x}$, so $\dfrac{dy}{dx} < 0$ at $x = 1$ and $\dfrac{dy}{dx} > 0$ at $x = e$.

(d) The slope is a continuous function which goes from a negative value to a positive value; therefore it must take the value zero between, by the Intermediate Value Theorem.

(e) $\dfrac{dy}{dx} = 0$ when $x = 2$.

38. $\beta = 10\log I - 10\log I_0$, $\dfrac{d\beta}{dI} = \dfrac{10}{I\ln 10}$.

(a) $\left.\dfrac{d\beta}{dI}\right|_{I=10I_0} = \dfrac{1}{I_0\ln 10}$ dB/(W/m^2).

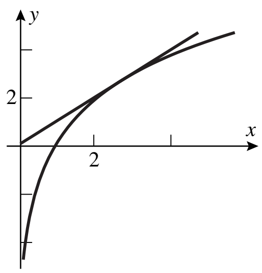
(b) $\left.\dfrac{d\beta}{dI}\right|_{I=100I_0} = \dfrac{1}{10I_0\ln 10}$ dB/(W/m^2).

(c) $\left.\dfrac{d\beta}{dI}\right|_{I=100I_0} = \dfrac{1}{1000I_0\ln 10}$ dB/(W/m^2).

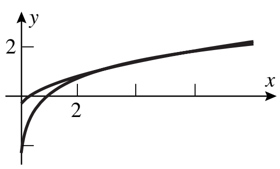
39. Solve $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ given $y = x\ln x$. Then $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = (1+\ln x)\dfrac{dx}{dt}$, so $1 + \ln x = 3$, $\ln x = 2$, $x = e^2$.

40. $x = 2, y = 0; y' = -2x/(5 - x^2) = -4$ at $x = 2$, so $y - 0 = -4(x - 2)$ or $y = -4x + 8$.

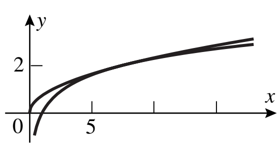
41. Set $y = \log_b x$ and solve $y' = 1$: $y' = \dfrac{1}{x \ln b} = 1$ so $x = \dfrac{1}{\ln b}$. The curves intersect when (x, x) lies on the graph of $y = \log_b x$, so $x = \log_b x$. From Formula (8), Section 1.6, $\log_b x = \dfrac{\ln x}{\ln b}$ from which $\ln x = 1$, $x = e$, $\ln b = 1/e$, $b = e^{1/e} \approx 1.4447$.



42. **(a)** Find the point of intersection: $f(x) = \sqrt{x} + k = \ln x$. The slopes are equal, so $m_1 = \dfrac{1}{x} = m_2 = \dfrac{1}{2\sqrt{x}}$, $\sqrt{x} = 2$, $x = 4$. Then $\ln 4 = \sqrt{4} + k$, $k = \ln 4 - 2$.



(b) Since the slopes are equal $m_1 = \dfrac{k}{2\sqrt{x}} = m_2 = \dfrac{1}{x}$, so $k\sqrt{x} = 2$. At the point of intersection $k\sqrt{x} = \ln x$, $2 = \ln x$, $x = e^2$, $k = 2/e$.



43. As long as $f' \neq 0$, g must be differentiable; this can be inferred from the graphs. Note that if $f' = 0$ at a point then g' cannot exist (infinite slope). (For example, $f(x) = x^3$ at $x = 0$).

44. **(a)** $f'(x) = -3/(x+1)^2$. If $x = f(y) = 3/(y+1)$ then $y = f^{-1}(x) = (3/x) - 1$, so $\dfrac{d}{dx} f^{-1}(x) = -\dfrac{3}{x^2}$; and $\dfrac{1}{f'(f^{-1}(x))} = -\dfrac{(f^{-1}(x)+1)^2}{3} = -\dfrac{(3/x)^2}{3} = -\dfrac{3}{x^2}$.

(b) $f(x) = e^{x/2}$, $f'(x) = \frac{1}{2}e^{x/2}$. If $x = f(y) = e^{y/2}$ then $y = f^{-1}(x) = 2\ln x$, so $\dfrac{d}{dx} f^{-1}(x) = \dfrac{2}{x}$; and $\dfrac{1}{f'(f^{-1}(x))} = 2e^{-f^{-1}(x)/2} = 2e^{-\ln x} = 2x^{-1} = \dfrac{2}{x}$.

45. Let $P(x_0, y_0)$ be a point on $y = e^{3x}$ then $y_0 = e^{3x_0}$. $dy/dx = 3e^{3x}$ so $m_{\tan} = 3e^{3x_0}$ at P and an equation of the tangent line at P is $y - y_0 = 3e^{3x_0}(x - x_0)$, $y - e^{3x_0} = 3e^{3x_0}(x - x_0)$. If the line passes through the origin then $(0, 0)$ must satisfy the equation so $-e^{3x_0} = -3x_0 e^{3x_0}$ which gives $x_0 = 1/3$ and thus $y_0 = e$. The point is $(1/3, e)$.

46. $\ln y = \ln 5000 + 1.07x$; $\dfrac{dy/dx}{y} = 1.07$, or $\dfrac{dy}{dx} = 1.07y$.

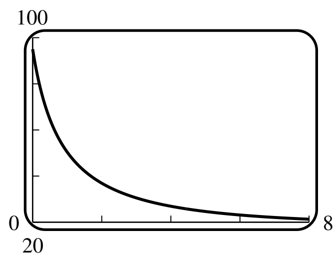
47. $\ln y = 2x \ln 3 + 7x \ln 5$; $\dfrac{dy/dx}{y} = 2\ln 3 + 7\ln 5$, or $\dfrac{dy}{dx} = (2\ln 3 + 7\ln 5)y$.

48. $\dfrac{dk}{dT} = k_0 \exp\left[-\dfrac{q(T-T_0)}{2T_0T}\right]\left(-\dfrac{q}{2T^2}\right) = -\dfrac{qk_0}{2T^2}\exp\left[-\dfrac{q(T-T_0)}{2T_0T}\right]$.

49. $y' = ae^{ax}\sin bx + be^{ax}\cos bx$, and $y'' = (a^2-b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx$, so
$y'' - 2ay' + (a^2+b^2)y = (a^2-b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx - 2a(ae^{ax}\sin bx + be^{ax}\cos bx) + (a^2+b^2)e^{ax}\sin bx = 0.$

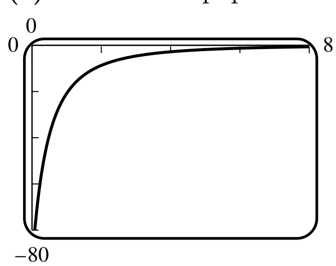
50. $\sin(\tan^{-1}x) = x/\sqrt{1+x^2}$ and $\cos(\tan^{-1}x) = 1/\sqrt{1+x^2}$, and $y' = \dfrac{1}{1+x^2}$, $y'' = \dfrac{-2x}{(1+x^2)^2}$, hence $y'' + 2\sin y\cos^3 y = \dfrac{-2x}{(1+x^2)^2} + 2\dfrac{x}{\sqrt{1+x^2}}\dfrac{1}{(1+x^2)^{3/2}} = 0.$

51. (a)



(b) As t tends to $+\infty$, the population tends to 19: $\lim\limits_{t\to+\infty} P(t) = \lim\limits_{t\to+\infty}\dfrac{95}{5-4e^{-t/4}} = \dfrac{95}{5-4\lim\limits_{t\to+\infty}e^{-t/4}} = \dfrac{95}{5} = 19.$

(c) The rate of population growth tends to zero.



52. (a) $y = (1+x)^\pi$, $\lim\limits_{h\to0}\dfrac{(1+h)^\pi - 1}{h} = \dfrac{d}{dx}(1+x)^\pi\Big|_{x=0} = \pi(1+x)^{\pi-1}\Big|_{x=0} = \pi.$

(b) Let $y = \dfrac{1-\ln x}{\ln x}$. Then $y(e) = 0$, and $\lim\limits_{x\to e}\dfrac{1-\ln x}{(x-e)\ln x} = \dfrac{dy}{dx}\Big|_{x=e} = -\dfrac{1/x}{(\ln x)^2} = -\dfrac{1}{e}.$

53. In the case $+\infty - (-\infty)$ the limit is $+\infty$; in the case $-\infty - (+\infty)$ the limit is $-\infty$, because large positive (negative) quantities are added to large positive (negative) quantities. The cases $+\infty - (+\infty)$ and $-\infty - (-\infty)$ are indeterminate; large numbers of opposite sign are subtracted, and more information about the sizes is needed.

54. (a) When the limit takes the form $0/0$ or ∞/∞.

(b) Not necessarily; only if $\lim\limits_{x\to a} f(x) = 0$. Consider $g(x) = x$; $\lim\limits_{x\to0} g(x) = 0$. Then $\lim\limits_{x\to0}\dfrac{\cos x}{x}$ is not indeterminate, whereas $\lim\limits_{x\to0}\dfrac{\sin x}{x}$ is indeterminate.

55. $\lim\limits_{x\to+\infty}(e^x - x^2) = \lim\limits_{x\to+\infty} x^2(e^x/x^2 - 1)$, but $\lim\limits_{x\to+\infty}\dfrac{e^x}{x^2} = \lim\limits_{x\to+\infty}\dfrac{e^x}{2x} = \lim\limits_{x\to+\infty}\dfrac{e^x}{2} = +\infty$, so $\lim\limits_{x\to+\infty}(e^x/x^2 - 1) = +\infty$ and thus $\lim\limits_{x\to+\infty} x^2(e^x/x^2-1) = +\infty$.

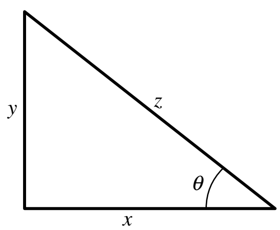
56. $\lim\limits_{x\to1}\dfrac{\ln x}{x^4-1} = \lim\limits_{x\to1}\dfrac{1/x}{4x^3} = \dfrac{1}{4}$; $\lim\limits_{x\to1}\sqrt{\dfrac{\ln x}{x^4-1}} = \sqrt{\lim\limits_{x\to1}\dfrac{\ln x}{x^4-1}} = \dfrac{1}{2}.$

57. $\lim_{x\to 0} \dfrac{x^2 e^x}{\sin^2 3x} = \left[\lim_{x\to 0} \dfrac{3x}{\sin 3x}\right]^2 \left[\lim_{x\to 0} \dfrac{e^x}{9}\right] = \dfrac{1}{9}$.

58. $\lim_{x\to 0} a^x \ln a = \ln a$.

59. The boom is pulled in at the rate of 5 m/min, so the circumference $C = 2r\pi$ is changing at this rate, which means that $\dfrac{dr}{dt} = \dfrac{dC}{dt} \cdot \dfrac{1}{2\pi} = -5/(2\pi)$. $A = \pi r^2$ and $\dfrac{dr}{dt} = -5/(2\pi)$, so $\dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dt} = 2\pi r(-5/2\pi) = -250$, so the area is shrinking at a rate of 250 m^2/min.

60. Find $\left.\dfrac{d\theta}{dt}\right|_{\substack{x=1\\y=1}}$ given $\dfrac{dz}{dt} = a$ and $\dfrac{dy}{dt} = -b$. From the figure $\sin\theta = y/z$; when $x = y = 1$, $z = \sqrt{2}$. So $\theta = \sin^{-1}(y/z)$ and $\dfrac{d\theta}{dt} = \dfrac{1}{\sqrt{1 - y^2/z^2}}\left(\dfrac{1}{z}\dfrac{dy}{dt} - \dfrac{y}{z^2}\dfrac{dz}{dt}\right) = -b - \dfrac{a}{\sqrt{2}}$ when $x = y = 1$.



61. **(a)** $\Delta x = 1.5 - 2 = -0.5$; $dy = \dfrac{-1}{(x-1)^2}\Delta x = \dfrac{-1}{(2-1)^2}(-0.5) = 0.5$; and $\Delta y = \dfrac{1}{(1.5-1)} - \dfrac{1}{(2-1)} = 2 - 1 = 1$.

(b) $\Delta x = 0 - (-\pi/4) = \pi/4$; $dy = \left(\sec^2(-\pi/4)\right)(\pi/4) = \pi/2$; and $\Delta y = \tan 0 - \tan(-\pi/4) = 1$.

(c) $\Delta x = 3 - 0 = 3$; $dy = \dfrac{-x}{\sqrt{25 - x^2}} = \dfrac{-0}{\sqrt{25-(0)^2}}(3) = 0$; and $\Delta y = \sqrt{25 - 3^2} - \sqrt{25 - 0^2} = 4 - 5 = -1$.

62. $\cot 46^\circ = \cot\dfrac{46\pi}{180}$; let $x_0 = \dfrac{\pi}{4}$ and $x = \dfrac{46\pi}{180}$. Then

$\cot 46^\circ = \cot x \approx \cot\dfrac{\pi}{4} - \left(\csc^2\dfrac{\pi}{4}\right)\left(x - \dfrac{\pi}{4}\right) = 1 - 2\left(\dfrac{46\pi}{180} - \dfrac{\pi}{4}\right) = 0.9651$; with a calculator, $\cot 46^\circ = 0.9657$.

63. **(a)** $h = 115\tan\phi$, $dh = 115\sec^2\phi\, d\phi$; with $\phi = 51^\circ = \dfrac{51}{180}\pi$ radians and $d\phi = \pm 0.5^\circ = \pm 0.5\left(\dfrac{\pi}{180}\right)$ radians, $h \pm dh = 115(1.2349) \pm 2.5340 = 142.0135 \pm 2.5340$, so the height lies between 139.48 m and 144.55 m.

(b) If $|dh| \le 5$ then $|d\phi| \le \dfrac{5}{115}\cos^2\dfrac{51}{180}\pi \approx 0.017$ radian, or $|d\phi| \le 0.98^\circ$.

Chapter 3 Making Connections

1. **(a)** If $t > 0$ then $A(-t)$ is the amount K there was t time-units ago in order that there be 1 unit now, i.e. $K \cdot A(t) = 1$, so $K = \dfrac{1}{A(t)}$. But, as said above, $K = A(-t)$. So $A(-t) = \dfrac{1}{A(t)}$.

(b) If s and t are positive, then the amount 1 becomes $A(s)$ after s seconds, and that in turn is $A(s)A(t)$ after another t seconds, i.e. 1 becomes $A(s)A(t)$ after $s + t$ seconds. But this amount is also $A(s+t)$, so $A(s)A(t) = A(s+t)$. Now if $0 \le -s \le t$ then $A(-s)A(s+t) = A(t)$. From the first case, we get $A(s+t) = A(s)A(t)$. If $0 \le t \le -s$ then $A(s+t) = \dfrac{1}{A(-s-t)} = \dfrac{1}{A(-s)A(-t)} = A(s)A(t)$ by the previous cases. If s and t are both negative then by the first case, $A(s+t) = \dfrac{1}{A(-s-t)} = \dfrac{1}{A(-s)A(-t)} = A(s)A(t)$.

(c) If $n > 0$ then $A\left(\dfrac{1}{n}\right)A\left(\dfrac{1}{n}\right)\ldots A\left(\dfrac{1}{n}\right) = A\left(n\dfrac{1}{n}\right) = A(1)$, so $A\left(\dfrac{1}{n}\right) = A(1)^{1/n} = b^{1/n}$ from part (b). If $n < 0$ then by part (a), $A\left(\dfrac{1}{n}\right) = \dfrac{1}{A\left(-\frac{1}{n}\right)} = \dfrac{1}{A(1)^{-1/n}} = A(1)^{1/n} = b^{1/n}$.

(d) Let m, n be integers. Assume $n \neq 0$ and $m > 0$. Then $A\left(\dfrac{m}{n}\right) = A\left(\dfrac{1}{n}\right)^m = A(1)^{m/n} = b^{m/n}$.

(e) If f, g are continuous functions of t and f and g are equal on the rational numbers $\left\{\dfrac{m}{n} : n \neq 0\right\}$, then $f(t) = g(t)$ for all t. Because if x is irrational, then let t_n be a sequence of rational numbers which converges to x. Then for all $n > 0, f(t_n) = g(t_n)$ and thus $f(x) = \lim_{n\to+\infty} f(t_n) = \lim_{n\to+\infty} g(t_n) = g(x)$.

2. **(a)** From Figure 1.3.4 it is evident that $(1+h)^{1/h} < e < (1-h)^{-1/h}$ provided $h > 0$, and $(1-h)^{-1/h} < e < (1+h)^{1/h}$ for $h < 0$.

(b) Suppose $h > 0$. Then $(1+h)^{1/h} < e < (1-h)^{-1/h}$. Raise to the power h: $1 + h < e^h < 1/(1-h); h < e^h - 1 < h/(1-h); 1 < \dfrac{e^h - 1}{h} < 1/(1-h)$; use the Squeezing Theorem as $h \to 0^+$. Use a similar argument in the case $h < 0$.

(c) The quotient $\dfrac{e^h - 1}{h}$ is the slope of the secant line through $(0, 1)$ and (h, e^h), and this secant line converges to the tangent line as $h \to 0$.

(d) $\dfrac{d}{dx}e^x = \lim_{h\to 0} \dfrac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0} \dfrac{e^h - 1}{h} = e^x$ from part (b).