INSTRUCTOR'S SOLUTIONS MANUAL
MULTIVARIABLE

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CALCULUS
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AND
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EARLY TRANSCENDENTALS
SECOND EDITION

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Contents

Chapter 8

Sequences and Infinite Series

8.1 An Overview

8.1.1 A *sequence* is an ordered list of numbers $a_1, a_2, a_3, \ldots$, often written $\{a_1, a_2, \ldots\}$ or $\{a_n\}$. For example, the natural numbers $\{1, 2, 3, ...\}$ are a sequence where $a_n = n$ for every n.

8.1.2 $a_1 = \frac{1}{1} = 1$; $a_2 = \frac{1}{2}$; $a_3 = \frac{1}{3}$; $a_4 = \frac{1}{4}$; $a_5 = \frac{1}{5}$.

8.1.3 $a_1 = 1$ (given); $a_2 = 1 \cdot a_1 = 1$; $a_3 = 2 \cdot a_2 = 2$; $a_4 = 3 \cdot a_3 = 6$; $a_5 = 4 \cdot a_4 = 24$.

8.1.4 A *finite sum* is the sum of a finite number of items, for example the sum of a finite number of terms of a sequence.

8.1.5 An *infinite series* is an infinite sum of numbers. Thus if $\{a_n\}$ is a sequence, then $a_1 + a_2 + \cdots = \sum_{k=1}^{\infty} a_k$ is an infinite series. For example, if $a_k = \frac{1}{k}$, then $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$ is an infinite series.

8.1.6 $S_1 = \sum_{k=1}^{1} k = 1$; $S_2 = \sum_{k=1}^{2} k = 1 + 2 = 3$; $S_3 = \sum_{k=1}^{3} k = 1 + 2 + 3 = 6$; $S_4 = \sum_{k=1}^{4} k = 1 + 2 + 3 + 4 = 10$.

8.1.7 $S_1 = \sum_{k=1}^{1} k^2 = 1$; $S_2 = \sum_{k=1}^{2} k^2 = 1 + 4 = 5$; $S_3 = \sum_{k=1}^{3} k^2 = 1 + 4 + 9 = 14$; $S_4 = \sum_{k=1}^{4} k^2 = 1 + 4 + 9 + 16 = 30$.

8.1.8 $S_1 = \sum_{k=1}^{1} \frac{1}{k} = \frac{1}{1} = 1$; $S_2 = \sum_{k=1}^{2} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$; $S_3 = \sum_{k=1}^{3} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$; $S_4 = \sum_{k=1}^{4} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$.

8.1.9 $a_1 = \dfrac{1}{10}; a_2 = \dfrac{1}{100}; a_3 = \dfrac{1}{1000}; a_4 = \dfrac{1}{10000}$.

8.1.10 $a_1 = 3(1) + 1 = 4$. $a_2 = 3(2) + 1 = 7$, $a_3 = 3(3) + 1 = 10$, $a_4 = 3(4) + 1 = 13$.

8.1.11 $a_1 = \frac{-1}{2}$, $a_2 = \frac{1}{2^2} = \frac{1}{4}$. $a_3 = \frac{-2}{2^3} = \frac{-1}{8}$, $a_4 = \frac{1}{2^4} = \frac{1}{16}$.

8.1.12 $a_1 = 2 - 1 = 1$. $a_2 = 2 + 1 = 3$, $a_3 = 2 - 1 = 1$, $a_4 = 2 + 1 = 3$.

8.1.13 $a_1 = \frac{2^2}{2+1} = \frac{4}{3}$. $a_2 = \frac{2^3}{2^2+1} = \frac{8}{5}$. $a_3 = \frac{2^4}{2^3+1} = \frac{16}{9}$. $a_4 = \frac{2^5}{2^4+1} = \frac{32}{17}$.

8.1.14 $a_1 = 1 + \frac{1}{1} = 2$; $a_2 = 2 + \frac{1}{2} = \frac{5}{2}$; $a_3 = 3 + \frac{1}{3} = \frac{10}{3}$; $a_4 = 4 + \frac{1}{4} = \frac{17}{4}$.

8.1.15 $a_1 = 1 + \sin(\pi/2) = 2$; $a_2 = 1 + \sin(2\pi/2) = 1 + \sin\pi = 1$; $a_3 = 1 + \sin(3\pi/2) = 0$; $a_4 = 1 + \sin(4\pi/2) = 1 + \sin 2\pi = 1$.

8.1.16 $a_1 = 2 \cdot 1^2 - 3 \cdot 1 + 1 = 0$; $a_2 = 2 \cdot 2^2 - 3 \cdot 2 + 1 = 3$; $a_3 = 2 \cdot 3^2 - 3 \cdot 3 + 1 = 10$; $a_4 = 2 \cdot 4^2 - 3 \cdot 4 + 1 = 21$.

8.1.17 $a_1 = 2$, $a_2 = 2 \cdot 2 = 4$, $a_3 = 2(4) = 8$, $a_4 = 2 \cdot 8 = 16$.

8.1.18 $a_1 = 32$, $a_2 = 32/2 = 16$, $a_3 = 16/2 = 8$, $a_4 = 8/2 = 4$.

8.1.19 $a_1 = 10$ (given); $a_2 = 3 \cdot a_1 - 12 = 30 - 12 = 18$; $a_3 = 3 \cdot a_2 - 12 = 54 - 12 = 42$; $a_4 = 3 \cdot a_3 - 12 = 126 - 12 = 114$.

8.1.20 $a_1 = 1$ (given); $a_2 = a_1^2 - 1 = 0$; $a_3 = a_2^2 - 1 = -1$; $a_4 = a_3^2 - 1 = 0$.

8.1.21 $a_1 = 0$ (given); $a_2 = 3 \cdot a_1^2 + 1 + 1 = 2$; $a_3 = 3 \cdot a_2^2 + 2 + 1 = 15$; $a_4 = 3 \cdot a_3^2 + 3 + 1 = 679$.

8.1.22 $a_0 = 1$ (given); $a_1 = 1$ (given); $a_2 = a_1 + a_0 = 2$; $a_3 = a_2 + a_1 = 3$; $a_4 = a_3 + a_2 = 5$.

8.1.23

a. $\frac{1}{32}, \frac{1}{64}$.

b. $a_1 = 1$; $a_{n+1} = \frac{a_n}{2}$.

c. $a_n = \frac{1}{2^{n-1}}$.

8.1.24

a. $-6, 7$.

b. $a_1 = 1$; $a_{n+1} = (-1)^n(|a_n| + 1)$.

c. $a_n = (-1)^{n+1}n$.

8.1.25

a. $-5, 5$.

b. $a_1 = -5$, $a_{n+1} = -a_n$.

c. $a_n = (-1)^n \cdot 5$.

8.1.26

a. 14, 17.

b. $a_1 = 2$; $a_{n+1} = a_n + 3$.

c. $a_n = -1 + 3n$.

8.1.27

a. 32, 64.

b. $a_1 = 1$; $a_{n+1} = 2a_n$.

c. $a_n = 2^{n-1}$.

8.1.28

a. 36, 49.

b. $a_1 = 1$; $a_{n+1} = (\sqrt{a_n} + 1)^2$.

c. $a_n = n^2$.

8.1.29

a. 243, 729.

b. $a_1 = 1$; $a_{n+1} = 3a_n$.

c. $a_n = 3^{n-1}$.

8.1.30

a. 2, 1.

b. $a_1 = 64$; $a_{n+1} = \frac{a_n}{2}$.

c. $a_n = \frac{64}{2^{n-1}} = 2^{7-n}$.

8.1.31 $a_1 = 9$, $a_2 = 99$, $a_3 = 999$, $a_4 = 9999$. This sequence diverges, because the terms get larger without bound.

8.1.32 $a_1 = 2$, $a_2 = 17$, $a_3 = 82$, $a_4 = 257$. This sequence diverges, because the terms get larger without bound.

8.1.33 $a_1 = \frac{1}{10}$, $a_2 = \frac{1}{100}$, $a_3 = \frac{1}{1000}$, $a_4 = \frac{1}{10{,}000}$. This sequence converges to zero.

8.1.34 $a_1 = \frac{1}{10}$, $a_2 = \frac{1}{100}$, $a_3 = \frac{1}{1000}$, $a_4 = \frac{1}{10{,}000}$. This sequence converges to zero.

8.1.35 $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = -\frac{1}{8}$, $a_4 = \frac{1}{16}$. This sequence converges to 0 because each term is smaller in absolute value than the preceding term and they get arbitrarily close to zero.

8.1.36 $a_1 = 0.9$, $a_2 = 0.99$, $a_3 = 0.999$, $a_4 = .9999$. This sequence converges to 1.

8.1.37 $a_1 = 1+1 = 2$, $a_2 = 1+1 = 2$, $a_3 = 2$, $a_4 = 2$. This constant sequence converges to 2.

8.1.38 $a_1 = 9 + \frac{9}{10} = 9.9$, $a_2 = 9 + \frac{9.9}{10} = 9.99$, $a_3 = 9 + \frac{9.99}{10} = 9.999$, $a_4 = 9 + \frac{9.999}{10} = 9.9999$. This sequence converges to 10.

8.1.39 $a_1 = \frac{50}{11} + 50 \approx 54.545$, $a_2 = \frac{54.545}{11} + 50 \approx 54.959$, $a_3 = \frac{54.959}{11} + 50 \approx 54.996$, $a_4 = \frac{54.996}{11} + 50 \approx 55.000$. This sequence converges to 55.

8.1.40 $a_1 = 0 - 1 = -1$. $a_2 = -10 - 1 = -11$, $a_3 = -110 - 1 = -111$, $a_4 = -1110 - 1 = -1111$. This sequence diverges.

8.1.41

n	1	2	3	4	4	6	7	8	9	10
a_n	0.4636	0.2450	0.1244	0.0624	0.0312	0.0156	0.0078	0.0039	0.0020	0.0010

This sequence appears to converge to 0.

8.1.42

n	1	2	3	4	5	6	7	8	9	10
a_n	3.1396	3.1406	3.1409	3.1411	3.1412	3.1413	3.1413	3.1413	3.1414	3.1414

This sequence appears to converge to π.

8.1.43

n	1	2	3	4	5	6	7	8	9	10
a_n	0	2	6	12	20	30	42	56	72	90

This sequence appears to diverge.

8.1.44

n	1	2	3	4	5	6	7	8	9	10
a_n	9.9	9.95	9.9667	9.975	9.98	9.9833	9.9857	9.9875	9.9889	9.99

This sequence appears to converge to 10.

8.1.45

n	1	2	3	4	5	6	7	8	9	10
a_n	0.83333	0.96154	0.99206	0.99840	0.99968	0.99994	0.99999	1.0000	1.0000	1.0000

This sequence appears to converge to 1.

8.1.46

n	1	2	3	4	5	6	7	8	9	10	11
a_n	0.9589	0.9896	0.9974	0.9993	0.9998	1.000	1.000	1.0000	1.000	1.000	1.000

This sequence converges to 1.

8.1.47

a. 2.5, 2.25, 2.125, 2.0625.

b. The limit is 2.

8.1.48

a. 1.33333, 1.125, 1.06667, 1.04167.

b. The limit is 1.

8.1.49

n	0	1	2	3	4	5	6	7	8	9	10
a_n	3	3.500	3.750	3.875	3.938	3.969	3.984	3.992	3.996	3.998	3.999

This sequence converges to 4.

8.1.50

n	0	1	2	3	4	5	6	7	8	9
a_n	1	−2.75	−3.688	−3.922	−3.981	−3.995	−3.999	−4.000	−4.000	−4.000

This sequence converges to −4.

8.1.51

n	0	1	2	3	4	5	6	7	8	9	10
a_n	0	1	3	7	15	31	63	127	255	511	1023

This sequence diverges.

8.1.52

n	0	1	2	3	4	5	6	7	8	9	10
a_n	10	4	3.4	3.34	3.334	3.333	3.333	3.333	3.333	3.333	3.333

This sequence converges to $\frac{10}{3}$.

8.1.53

n	0	1	2	3	4	5	6	7	8	9
a_n	1000	18.811	5.1686	4.1367	4.0169	4.0021	4.0003	4.0000	4.0000	4.0000

This sequence converges to 4.

8.1.54

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	1.4212	1.5538	1.5981	1.6119	1.6161	1.6174	1.6179	1.6180	1.6180	1.6180

This sequence converges to $\frac{1+\sqrt{5}}{2} \approx 1.618$.

8.1.55

a. 20, 10, 5, 2.5.

b. $h_n = 20(0.5)^n$.

8.1.56

a. 10, 9, 8.1, 7.29.

b. $h_n = 10(0.9)^n$.

8.1.57

a. 30, 7.5, 1.875, 0.46875.

b. $h_n = 30(0.25)^n$.

8.1.58

a. 20, 15, 11.25, 8.438

b. $h_n = 20(0.75)^n$.

8.1.59 $S_1 = 0.3$, $S_2 = 0.33$, $S_3 = 0.333$, $S_4 = 0.3333$. It appears that the infinite series has a value of $0.3333\ldots = \frac{1}{3}$.

8.1.60 $S_1 = 0.6$, $S_2 = 0.66$, $S_3 = 0.666$, $S_4 = 0.6666$. It appears that the infinite series has a value of $0.6666\ldots = \frac{2}{3}$.

8.1.61 $S_1 = 4$, $S_2 = 4.9$, $S_3 = 4.99$, $S_4 = 4.999$. The infinite series has a value of $4.999\cdots = 5$.

8.1.62 $S_1 = 1$, $S_2 = \frac{3}{2} = 1.5$, $S_3 = \frac{7}{4} = 1.75$, $S_4 = \frac{15}{8} = 1.875$. The infinite series has a value of 2.

8.1.63

a. $S_1 = \frac{2}{3}$, $S_2 = \frac{4}{5}$, $S_3 = \frac{6}{7}$, $S_4 = \frac{8}{9}$.

b. It appears that $S_n = \frac{2n}{2n+1}$.

c. The series has a value of 1 (the partial sums converge to 1).

8.1.64

a. $S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$, $S_4 = \frac{15}{16}$.

b. $S_n = 1 - \frac{1}{2^n}$.

c. The partial sums converge to 1, so that is the value of the series.

8.1.65

a. $S_1 = \frac{1}{3}$, $S_2 = \frac{2}{5}$, $S_3 = \frac{3}{7}$, $S_4 = \frac{4}{9}$.

b. $S_n = \frac{n}{2n+1}$.

c. The partial sums converge to $\frac{1}{2}$, which is the value of the series.

8.1.66

a. $S_1 = \frac{2}{3}$, $S_2 = \frac{8}{9}$, $S_3 = \frac{26}{27}$, $S_4 = \frac{80}{81}$.

b. $S_n = 1 - \frac{1}{3^n}$.

c. The partial sums converge to 1, which is the value of the series.

8.1.67

a. True. For example, $S_2 = 1 + 2 = 3$, and $S_4 = a_1 + a_2 + a_3 + a_4 = 1 + 2 + 3 + 4 = 10$.

b. False. For example, $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \cdots$ where $a_n = 1 - \frac{1}{2^n}$ converges to 1, but each term is greater than the previous one.

c. True. In order for the partial sums to converge, they must get closer and closer together. In order for this to happen, the difference between successive partial sums, which is just the value of a_n, must approach zero.

8.1.68 The height at the n^{th} bounce is given by the recurrence $h_n = r \cdot h_{n-1}$; an explicit form for this sequence is $h_n = h_0 \cdot r^n$. The distance traveled by the ball between the n^{th} and the $(n+1)^{\text{st}}$ bounce is thus $2h_n = 2h_0 \cdot r^n$, so that $S_{n+1} = \sum_{i=0}^{n} 2h_0 \cdot r^i$.

a. Here $h_0 = 20$, $r = 0.5$, so $S_1 = 40$, $S_2 = 40 + 40 \cdot 0.5 = 60$, $S_3 = S_2 + 40 \cdot (0.5)^2 = 70$, $S_4 = S_3 + 40 \cdot (0.5)^3 = 75$, $S_5 = S_4 + 40 \cdot (0.5)^4 = 77.5$

b.

n	1	2	3	4	5	6
a_n	40	60	70	75	77.5	78.75
n	7	8	9	10	11	12
a_n	79.375	79.688	79.844	79.922	79.961	79.980
n	13	14	15	16	17	18
a_n	79.990	79.995	79.998	79.999	79.999	80.000
n	19	20	21	22	23	24
a_n	80.000	80.000	80.000	80.000	80.000	80.000

The sequence converges to 80.

8.1.69 Using the work from the previous problem:

a. Here $h_0 = 20$, $r = 0.75$, so $S_1 = 40$, $S_2 = 40 + 40 \cdot 0.75 = 70$, $S_3 = S_2 + 40 \cdot (0.75)^2 = 92.5$, $S_4 = S_3 + 40 \cdot (0.75)^3 = 109.375$, $S_5 = S_4 + 40 \cdot (0.75)^4 = 122.03125$

b.

n	1	2	3	4	5	6
a_n	40	70	92.5	109.375	122.031	131.523
n	7	8	9	10	11	12
a_n	138.643	143.982	147.986	150.990	153.242	154.932
n	13	14	15	16	17	18
a_n	156.199	157.149	157.862	158.396	158.797	159.098
n	19	20	21	22	23	24
a_n	159.323	159.493	159.619	159.715	159.786	159.839

The sequence converges to 160.

8.1.70

a. $s_1 = -1$, $s_2 = 0$, $s_3 = -1$, $s_4 = 0$.

b. The limit does not exist.

8.1.71

a. 0.9, 0.99, 0.999, .9999.

b. The limit is 1.

8.1.72

a. 1.5, 3.75, 7.125, 12.1875.

b. The limit does not exist.

8.1.73

a. $\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}$.

b. The limit is 1/2.

8.1.74

a. 1, 3, 6, 10.

b. The limit does not exist.

8.1.75

a. $-1, 0, -1, 0$.

b. The limit does not exist.

8.1.76

a. $-1, 1, -2, 2$.

b. The limit does not exist.

8.1.77

a. $\frac{3}{10} = 0.3$, $\frac{33}{100} = 0.33$, $\frac{333}{1000} = 0.333$, $\frac{3333}{10000} = 0.3333$.

b. The limit is 1/3.

8.1.78

a. $p_0 = 250$, $p_1 = 250 \cdot 1.03 = 258$, $p_2 = 250 \cdot 1.03^2 = 265$, $p_3 = 250 \cdot 1.03^3 = 273$, $p_4 = 250 \cdot 1.03^4 = 281$.

b. The initial population is 250, so that $p_0 = 250$. Then $p_n = 250 \cdot (1.03)^n$, because the population increases by 3 percent each month.

c. $p_{n+1} = p_n \cdot 1.03$.

d. The population increases without bound.

8.1.79

a. $M_0 = 20$, $M_1 = 20 \cdot 0.5 = 10$, $M_2 = 20 \cdot 0.5^2 = 5$, $M_3 = 20 \cdot 0.5^3 = 2.5$, $M_4 = 20 \cdot 0.5^4 = 1.25$

b. $M_n = 20 \cdot 0.5^n$.

c. The initial mass is $M_0 = 20$. We are given that 50% of the mass is gone after each decade, so that $M_{n+1} = 0.5 \cdot M_n$, $n \geq 0$.

d. The amount of material goes to 0.

8.1.80

a. $c_0 = 100$, $c_1 = 103$, $c_2 = 106.09$, $c_3 = 109.27$, $c_4 = 112.55$.

b. $c_n = 100(1.03)^n$ for $n \geq 0$.

c. We are given that $c_0 = 100$ (where year 0 is 1984); because it increases by 3% per year, $c_{n+1} = 1.03 \cdot c_n$.

d. The sequence diverges.

8.1.81

a. $d_0 = 200$, $d_1 = 200 \cdot .95 = 190$, $d_2 = 200 \cdot .95^2 = 180.5$, $d_3 = 200 \cdot .95^3 = 171.475$, $d_4 = 200 \cdot .95^4 = 162.90125$.

b. $d_n = 200(0.95)^n$, $n \geq 0$.

c. We are given $d_0 = 200$; because 5% of the drug is washed out every hour, that means that 95% of the preceding amount is left every hour, so that $d_{n+1} = 0.95 \cdot d_n$.

d. The sequence converges to 0.

8.1.82

a. Using the recurrence $a_{n+1} = \frac{1}{2}\left(a_n + \frac{10}{a_n}\right)$, we build a table:

n	0	1	2	3	4	5
a_n	10	5.5	3.659090909	3.196005081	3.162455622	3.162277665

The true value is $\sqrt{10} \approx 3.162277660$, so the sequence converges with an error of less than 0.01 after only 4 iterations, and is within 0.0001 after only 5 iterations.

b. The recurrence is now $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$

c	$\sqrt{c}$	0	1	2	3	4	5	6
2	1.414	2	1.5	1.417	1.414	1.414	1.414	1.414
3	1.732	3	2	1.750	1.732	1.732	1.732	1.732
4	2.000	4	2.5	2.050	2.001	2.000	2.000	2.000
5	2.236	5	3	2.333	2.238	2.236	2.236	2.236
6	2.449	6	3.6	2.607	2.454	2.449	2.449	2.449
7	2.646	7	4	2.875	2.655	2.646	2.646	2.646
8	2.828	8	4.5	3.139	2.844	2.828	2.828	2.828
9	3.000	9	5.0	3.400	3.024	3.000	3.000	3.000
10	3.162	10	5.5	3.659	3.196	3.162	3.162	3.162

For $c = 2$ the sequence converges to within 0.01 after two iterations.
For $c = 3$, 4, 5, 6, and 7 the sequence converges to within 0.01 after three iterations.
For $c = 8$, 9, and 10 it requires four iterations.

8.2 Sequences

8.2.1 There are many examples; one is $a_n = \frac{1}{n}$. This sequence is nonincreasing (in fact, it is decreasing) and has a limit of 0.

8.2.2 Again there are many examples; one is $a_n = \ln(n)$. It is increasing, and has no limit.

8.2.3 There are many examples; one is $a_n = \frac{1}{n}$. This sequence is nonincreasing (in fact, it is decreasing), is bounded above by 1 and below by 0, and has a limit of 0.

8.2.4 For example, $a_n = (-1)^n$. For all values of n we have $|a_n| = 1$, so it is bounded. All the odd terms are -1 and all the even terms are 1, so the sequence does not have a limit.

8.2.5 $\{r^n\}$ converges for $-1 < r \le 1$. It diverges for all other values of r (see Theorem 8.3).

8.2.6 By Theorem 8.1, if we can find a function $f(x)$ such that $f(n) = a_n$ for all positive integers n, then if $\lim_{x\to\infty} f(x)$ exists and is equal to L, we then have $\lim_{n\to\infty} a_n$ exists and is also equal to L. This means that we can apply function-oriented limit methods such as L'Hôpital's rule to determine limits of sequences.

8.2.7 $\{e^{n/100}\}$ grows faster than $\{n^{100}\}$ as $n \to \infty$.

8.2.8 The definition of the limit of a sequence involves only the behavior of the n^{th} term of a sequence as n gets large (see the Definition of Limit of a Sequence). Thus suppose a_n, b_n differ in only finitely many terms, and that M is large enough so that $a_n = b_n$ for $n > M$. Suppose a_n has limit L. Then for $\varepsilon > 0$, if N is such that $|a_n - L| < \varepsilon$ for $n > N$, first increase N if required so that $N > M$ as well. Then we also have $|b_n - L| < \varepsilon$ for $n > N$. Thus a_n and b_n have the same limit. A similar argument applies if a_n has no limit.

8.2.9 Divide numerator and denominator by n^4 to get $\lim_{n\to\infty} \frac{1/n}{1+\frac{1}{n^4}} = 0$.

8.2.10 Divide numerator and denominator by n^{12} to get $\lim_{n\to\infty} \frac{1}{3+\frac{4}{n^{12}}} = \frac{1}{3}$.

8.2.11 Divide numerator and denominator by n^3 to get $\lim_{n\to\infty} \frac{3-n^{-3}}{2+n^{-3}} = \frac{3}{2}$.

8.2.12 Divide numerator and denominator by e^n to get $\lim_{n\to\infty} \frac{2+(1/e^n)}{1} = 2$.

8.2.13 Divide numerator and denominator by 3^n to get $\lim_{n\to\infty} \frac{3+(1/3^{n-1})}{1} = 3$.

8.2.14 Divide numerator by k and denominator by $k = \sqrt{k^2}$ to get $\lim_{k\to\infty} \frac{1}{\sqrt{9+(1/k^2)}} = \frac{1}{3}$.

8.2.15 $\lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$.

8.2.16 Multiply by $\dfrac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$ to obtain

$$\lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right) = \lim_{n\to\infty} \frac{\left(\sqrt{n^2+1}-n\right)\left(\sqrt{n^2+1}+n\right)}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} = 0.$$

8.2.17 Because $\lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$, $\lim_{n\to\infty} \frac{\tan^{-1} n}{n} = 0$.

8.2.18 Let $y = n^{2/n}$. Then $\ln y = \frac{2\ln n}{n}$. By L'Hôpital's rule we have $\lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{2}{x} = 0$, so $\lim_{n\to\infty} n^{2/n} = e^0 = 1$.

8.2.19 Find the limit of the logarithm of the expression, which is $n\ln\left(1+\frac{2}{n}\right)$. Using L'Hôpital's rule:

$$\lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{2}{n}\right)}{1/n} = \lim_{n\to\infty}\frac{\frac{1}{1+(2/n)}\left(\frac{-2}{n^2}\right)}{-1/n^2} = \lim_{n\to\infty}\frac{2}{1+(2/n)} = 2.$$

Thus the limit of the original expression is e^2.

8.2.20 Take the logarithm of the expression and use L'Hôpital's rule:

$$\lim_{n\to\infty} n\ln\left(\frac{n}{n+5}\right) = \lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+5}\right)}{1/n} = \lim_{n\to\infty}\frac{\frac{n+5}{n}\cdot\frac{5}{(n+5)^2}}{-1/n^2} = \lim_{n\to\infty}\frac{-5n}{n+5} = -5.$$

Thus the original limit is e^{-5}.

8.2.21 Take the logarithm of the expression and use L'Hôpital's rule:

$$\lim_{n\to\infty}\frac{n}{2}\ln\left(1+\frac{1}{2n}\right) = \lim_{n\to\infty}\frac{\ln(1+(1/2n))}{2/n} = \lim_{n\to\infty}\frac{\frac{1}{1+(1/2n)}\cdot\frac{-1}{2n^2}}{-2/n^2} = \lim_{n\to\infty}\frac{1}{4(1+(1/2n))} = \frac{1}{4}.$$

Thus the original limit is $e^{1/4}$.

8.2.22 Find the limit of the logarithm of the expression, which is $3n\ln\left(1+\frac{4}{n}\right)$. Using L'Hôpital's rule:

$$\lim_{n\to\infty} 3n\ln\left(1+\frac{4}{n}\right) = \lim_{n\to\infty}\frac{3\ln\left(1+\frac{4}{n}\right)}{1/n} = \lim_{n\to\infty}\frac{\frac{1}{1+(4/n)}\left(\frac{-12}{n^2}\right)}{-1/n^2} = \lim_{n\to\infty}\frac{12}{1+(4/n)} = 12.$$

Thus the limit of the original expression is e^{12}.

8.2.23 Using L'Hôpital's rule: $\lim_{n\to\infty}\frac{n}{e^n+3n} = \lim_{n\to\infty}\frac{1}{e^n+3} = 0$.

8.2.24 $\ln\frac{1}{n} = -\ln n$, so this is $-\lim_{n\to\infty}\frac{\ln n}{n}$. By L'Hôpital's rule, we have $-\lim_{n\to\infty}\frac{\ln n}{n} = -\lim_{n\to\infty}\frac{1}{n} = 0$.

8.2.25 Taking logs, we have $\lim_{n\to\infty}\frac{1}{n}\ln(1/n) = \lim_{n\to\infty} -\frac{\ln n}{n} = \lim_{n\to\infty}\frac{-1}{n} = 0$ by L'Hôpital's rule. Thus the original sequence has limit $e^0 = 1$.

8.2.26 Find the limit of the logarithm of the expression, which is $n\ln\left(1-\frac{4}{n}\right)$, using L'Hôpital's rule: $\lim_{n\to\infty} n\ln\left(1-\frac{4}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1-\frac{4}{n}\right)}{1/n} = \lim_{n\to\infty}\frac{\frac{1}{1-(4/n)}\left(\frac{4}{n^2}\right)}{-1/n^2} = \lim_{n\to\infty}\frac{-4}{1-(4/n)} = -4$. Thus the limit of the original expression is e^{-4}.

8.2.27 Except for a finite number of terms, this sequence is just $a_n = ne^{-n}$, so it has the same limit as this sequence. Note that $\lim_{n\to\infty}\frac{n}{e^n} = \lim_{n\to\infty}\frac{1}{e^n} = 0$, by L'Hôpital's rule.

8.2.28 $\ln(n^3+1) - \ln(3n^3+10n) = \ln\left(\frac{n^3+1}{3n^3+10n}\right) = \ln\left(\frac{1+n^{-3}}{3+10n^{-2}}\right)$, so the limit is $\ln(1/3) = -\ln 3$.

8.2.29 $\ln(\sin(1/n)) + \ln n = \ln(n\sin(1/n)) = \ln\left(\frac{\sin(1/n)}{1/n}\right)$. As $n\to\infty$, $\sin(1/n)/(1/n)\to 1$, so the limit of the original sequence is $\ln 1 = 0$.

8.2.30 Using L'Hôpital's rule:

$$\lim_{n\to\infty} n(1-\cos(1/n)) = \lim_{n\to\infty}\frac{1-\cos(1/n)}{1/n} = \lim_{n\to\infty}\frac{-\sin(1/n)(-1/n^2)}{-1/n^2} = -\sin(0) = 0.$$

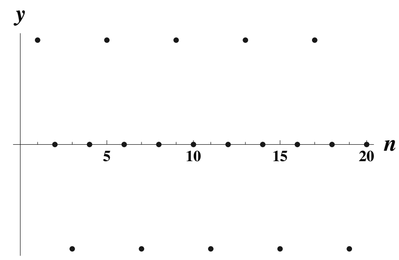
8.2.31 $\lim_{n\to\infty} n\sin(6/n) = \lim_{n\to\infty}\frac{\sin(6/n)}{1/n} = \lim_{n\to\infty}\frac{\frac{-6\cos(6/n)}{n^2}}{(-1/n^2)} = \lim_{n\to\infty} 6\cos(6/n) = 6\cdot\cos 0 = 6.$

8.2.32 Because $-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$, and because both $-\frac{1}{n}$ and $\frac{1}{n}$ have limit 0 as $n \to \infty$, the limit of the given sequence is also 0 by the Squeeze Theorem.

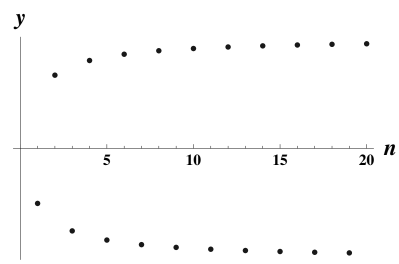
8.2.33 The terms with odd-numbered subscripts have the form $-\frac{n}{n+1}$, so they approach -1, while the terms with even-numbered subscripts have the form $\frac{n}{n+1}$ so they approach 1. Thus, the sequence has no limit.

8.2.34 Because $\frac{-n^2}{2n^3+n} \leq \frac{(-1)^{n+1}n^2}{2n^3+n} \leq \frac{n^2}{2n^3+n}$, and because both $\frac{-n^2}{2n^3+n}$ and $\frac{n^2}{2n^3+n}$ have limit 0 as $n \to \infty$, the limit of the given sequence is also 0 by the Squeeze Theorem. Note that $\lim_{n\to\infty} \frac{n^2}{2n^3+n} = \lim_{n\to\infty} \frac{1/n}{2+1/n^2} = \frac{0}{2} = 0$.

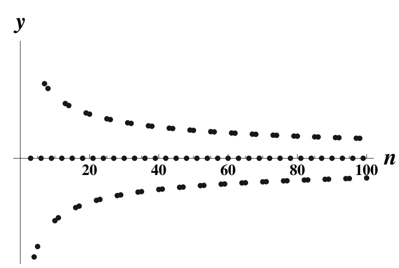
8.2.35 When n is an integer, $\sin\left(\frac{n\pi}{2}\right)$ oscillates between the values ± 1 and 0, so this sequence does not converge.



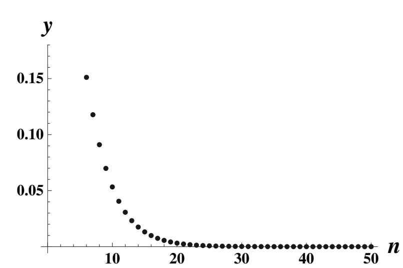
8.2.36 The even terms form a sequence $b_{2n} = \frac{2n}{2n+1}$, which converges to 1 (e.g. by L'Hôpital's rule); the odd terms form the sequence $b_{2n+1} = -\frac{n}{n+1}$, which converges to -1. Thus the sequence as a whole does not converge.



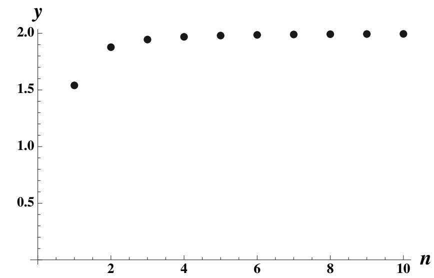
8.2.37 The numerator is bounded in absolute value by 1, while the denominator goes to ∞, so the limit of this sequence is 0.



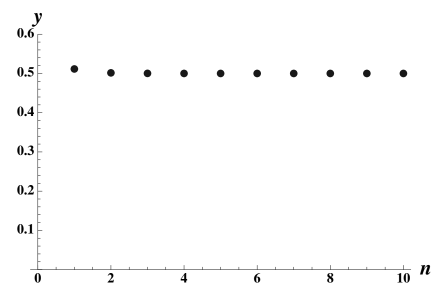
8.2.38 The reciprocal of this sequence is $b_n = \frac{1}{a_n} = 1 + \left(\frac{4}{3}\right)^n$, which increases without bound as $n \to \infty$. Thus a_n converges to zero.



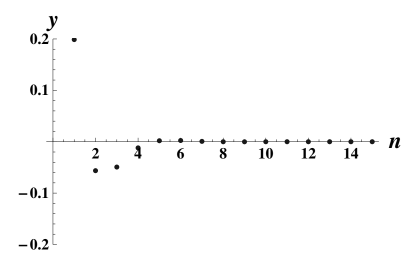
8.2.39 $\lim_{n\to\infty}(1+\cos(1/n)) = 1+\cos(0) = 2.$



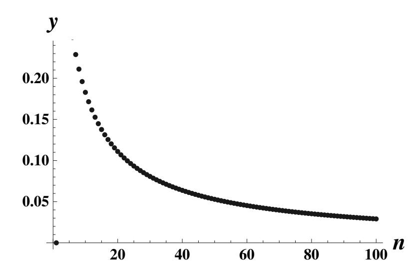
8.2.40 By L'Hôpital's rule we have: $\lim_{n\to\infty}\frac{e^{-n}}{2\sin(e^{-n})} = \lim_{n\to\infty}\frac{-e^{-n}}{2\cos(e^{-n})(-e^{-n})} = \frac{1}{2\cos 0} = \frac{1}{2}.$



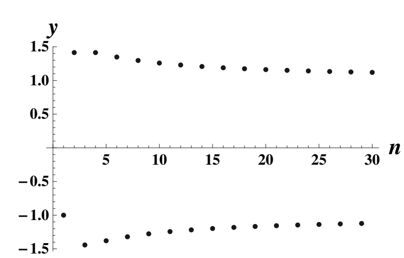
8.2.41 This is the sequence $\frac{\cos n}{e^n}$; the numerator is bounded in absolute value by 1 and the denominator increases without bound, so the limit is zero.



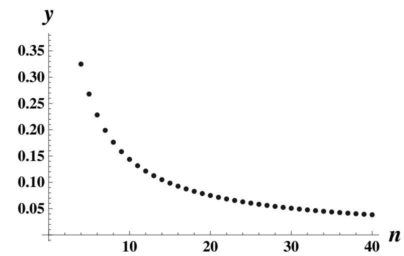
8.2.42 Using L'Hôpital's rule, we have $\lim_{n\to\infty}\frac{\ln n}{n^{1.1}} = \lim_{n\to\infty}\frac{1/n}{(1.1)n^{.1}} = \lim_{n\to\infty}\frac{1}{(1.1)n^{1.1}} = 0.$



8.2.43 Ignoring the factor of $(-1)^n$ for the moment, we see, taking logs, that $\lim_{n\to\infty}\frac{\ln n}{n} = 0$, so that $\lim_{n\to\infty}\sqrt[n]{n} = e^0 = 1$. Taking the sign into account, the odd terms converge to -1 while the even terms converge to 1. Thus the sequence does not converge.



8.2.44 $\lim_{n\to\infty} \frac{n\pi}{2n+2} = \frac{\pi}{2}$, using L'Hôpital's rule. Thus the sequence converges to $\cot(\pi/2) = 0$.



8.2.45 Because $0.2 < 1$, this sequence converges to 0. Because $0.2 > 0$, the convergence is monotone.

8.2.46 Because $1.2 > 1$, this sequence diverges monotonically to ∞.

8.2.47 Because $|-0.7| < 1$, the sequence converges to 0; because $-0.7 < 0$, it does not do so monotonically. The sequence converges by oscillation.

8.2.48 Because $|-1.01| > 1$, the sequence diverges; because $-1.01 < 0$, the divergence is not monotone.

8.2.49 Because $1.00001 > 1$, the sequence diverges; because $1.00001 > 0$, the divergence is monotone.

8.2.50 This is the sequence

$$\frac{2^{n+1}}{3^n} = 2 \cdot \left(\frac{2}{3}\right)^n;$$

because $0 < \frac{2}{3} < 1$, the sequence converges monotonically to zero.

8.2.51 Because $|-2.5| > 1$, the sequence diverges; because $-2.5 < 0$, the divergence is not monotone. The sequence diverges by oscillation.

8.2.52 $|-0.003| < 1$, so the sequence converges to zero; because $-.003 < 0$, the convergence is not monotone.

8.2.53 Because $-1 \le \cos n \le 1$, we have $\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$. Because both $\frac{-1}{n}$ and $\frac{1}{n}$ have limit 0 as $n \to \infty$, the given sequence does as well.

8.2.54 Because $-1 \le \sin 6n \le 1$, we have $-\frac{1}{5n} \le \frac{\sin 6n}{5n} \le \frac{1}{5n}$. Because both $-\frac{1}{5n}$ and $\frac{1}{5n}$ have limit 0 as $n \to \infty$, the given sequence does as well.

8.2.55 Because $-1 \le \sin n \le 1$ for all n, the given sequence satisfies $-\frac{1}{2^n} \le \frac{\sin n}{2^n} \le \frac{1}{2^n}$, and because both $\pm\frac{1}{2^n} \to 0$ as $n \to \infty$, the given sequence converges to zero as well by the Squeeze Theorem.

8.2.56 Because $-1 \le \cos(n\pi/2) \le 1$ for all n, we have $\frac{-1}{\sqrt{n}} \le \frac{\cos(n\pi/2)}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$ and because both $\pm\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, the given sequence converges to 0 as well by the Squeeze Theorem.

8.2.57 The inverse tangent function takes values between $-\pi/2$ and $\pi/2$, so the numerator is always between $-\pi$ and π. Thus $\frac{-\pi}{n^3+4} \le \frac{2\tan^{-1} n}{n^3+4} \le \frac{\pi}{n^3+4}$, and by the Squeeze Theorem, the given sequence converges to zero.

8.2.58 This sequence diverges. To see this, call the given sequence a_n, and assume it converges to limit L. Then because the sequence $b_n = \frac{n}{n+1}$ converges to 1, the sequence $c_n = \frac{a_n}{b_n}$ would converge to L as well. But $c_n = \sin^3 \frac{\pi n}{2}$ doesn't converge (because it is $1, -1, 1, -1 \cdots$), so the given sequence doesn't converge either.

8.2.59

a. After the n^{th} dose is given, the amount of drug in the bloodstream is $d_n = 0.5 \cdot d_{n-1} + 80$, because the half-life is one day. The initial condition is $d_1 = 80$.

b. The limit of this sequence is 160 mg.

c. Let $L = \lim_{n\to\infty} d_n$. Then from the recurrence relation, we have $d_n = 0.5 \cdot d_{n-1} + 80$, and thus $\lim_{n\to\infty} d_n = 0.5 \cdot \lim_{n\to\infty} d_{n-1} + 80$, so $L = 0.5 \cdot L + 80$, and therefore $L = 160$.

8.2.60

a.

$$\begin{aligned}
B_0 &= \$20,000\\
B_1 &= 1.005 \cdot B_0 - \$200 = \$19,900\\
B_2 &= 1.005 \cdot B_1 - \$200 = \$19,799.50\\
B_3 &= 1.005 \cdot B_2 - \$200 = \$19,698.50\\
B_4 &= 1.005 \cdot B_3 - \$200 = \$19,596.99\\
B_5 &= 1.005 \cdot B_4 - \$200 = \$19,494.97
\end{aligned}$$

b. $B_n = 1.005 \cdot B_{n-1} - \$200$

c. Using a calculator or computer program, B_n becomes negative after the 139^{th} payment, so 139 months or almost 11 years.

8.2.61

a.

$$\begin{aligned}
B_0 &= 0\\
B_1 &= 1.0075 \cdot B_0 + \$100 = \$100\\
B_2 &= 1.0075 \cdot B_1 + \$100 = \$200.75\\
B_3 &= 1.0075 \cdot B_2 + \$100 = \$302.26\\
B_4 &= 1.0075 \cdot B_3 + \$100 = \$404.52\\
B_5 &= 1.0075 \cdot B_4 + \$100 = \$507.56
\end{aligned}$$

b. $B_n = 1.0075 \cdot B_{n-1} + \$100$.

c. Using a calculator or computer program, $B_n > \$5,000$ during the 43^{rd} month.

8.2.62

a. Let D_n be the *total number* of liters of alcohol in the mixture after the n^{th} replacement. At the next step, 2 liters of the 100 liters is removed, thus leaving $0.98 \cdot D_n$ liters of alcohol, and then $0.1 \cdot 2 = 0.2$ liters of alcohol are added. Thus $D_n = 0.98 \cdot D_{n-1} + 0.2$. Now, $C_n = D_n/100$, so we obtain a recurrence relation for C_n by dividing this equation by 100: $C_n = 0.98 \cdot C_{n-1} + 0.002$.

$$\begin{aligned}
C_0 &= 0.4\\
C_1 &= 0.98 \cdot 0.4 + 0.002 = 0.394\\
C_2 &= 0.98 \cdot C_1 + 0.002 = 0.38812\\
C_3 &= 0.98 \cdot C_2 + 0.002 = 0.38236\\
C_4 &= 0.98 \cdot C_3 + 0.002 = 0.37671\\
C_5 &= 0.98 \cdot C_4 + 0.002 = 0.37118
\end{aligned}$$

The rounding is done to five decimal places.

b. Using a calculator or a computer program, $C_n < 0.15$ after the 89^{th} replacement.

c. If the limit of C_n is L, then taking the limit of both sides of the recurrence equation yields $L = 0.98L + 0.002$, so $.02L = .002$, and $L = .1 = 10\%$.

8.2.63 Because $n! \ll n^n$ by Theorem 8.6, we have $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

8.2.64 $\{3^n\} \ll \{n!\}$ because $\{b^n\} \ll \{n!\}$ in Theorem 8.6. Thus, $\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

8.2.65 Theorem 8.6 indicates that $\ln^q n \ll n^p$, so $\ln^{20} n \ll n^{10}$, so $\lim_{n\to\infty} \frac{n^{10}}{\ln^{20} n} = \infty$.

8.2.66 Theorem 8.6 indicates that $\ln^q n \ll n^p$, so $\ln^{1000} n \ll n^{10}$, so $\lim_{n\to\infty} \frac{n^{10}}{\ln^{1000} n} = \infty$.

8.2.67 By Theorem 8.6, $n^p \ll b^n$, so $n^{1000} \ll 2^n$, and thus $\lim_{n\to\infty} \frac{n^{1000}}{2^n} = 0$.

8.2.68 Note that $e^{1/10} = \sqrt[10]{e} \approx 1.1$. Let $r = \frac{e^{1/10}}{2}$ and note that $0 < r < 1$. Thus $\lim_{n\to\infty} \frac{e^{n/10}}{2^n} = \lim_{n\to\infty} r^n = 0$.

8.2.69 Let $\varepsilon > 0$ be given and let N be an integer with $N > \frac{1}{\varepsilon}$. Then if $n > N$, we have $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon$.

8.2.70 Let $\varepsilon > 0$ be given. We wish to find N such that $|(1/n^2) - 0| < \varepsilon$ if $n > N$. This means that $\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} < \varepsilon$. So choose N such that $\frac{1}{N^2} < \varepsilon$, so that $N^2 > \frac{1}{\varepsilon}$, and then $N > \frac{1}{\sqrt{\varepsilon}}$. This shows that such an N always exists for each ε and thus that the limit is zero.

8.2.71 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left|\frac{3n^2}{4n^2+1} - \frac{3}{4}\right| = \left|\frac{-3}{4(4n^2+1)}\right| = \frac{3}{4(4n^2+1)} < \varepsilon$. But this means that $3 < 4\varepsilon(4n^2+1)$, or $16\varepsilon n^2 + (4\varepsilon - 3) > 0$. Solving the quadratic, we get $n > \frac{1}{4}\sqrt{\frac{3}{\varepsilon} - 4}$, provided $\varepsilon < 3/4$. So let $N = \frac{1}{4}\sqrt{\frac{3}{\varepsilon}}$ if $\epsilon < 3/4$ and let $N = 1$ otherwise.

8.2.72 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $|b^{-n} - 0| = b^{-n} < \varepsilon$, so that $-n \ln b < \ln \varepsilon$. So choose N to be any integer greater than $-\frac{\ln \varepsilon}{\ln b}$.

8.2.73 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left|\frac{cn}{bn+1} - \frac{c}{b}\right| = \left|\frac{-c}{b(bn+1)}\right| = \frac{c}{b(bn+1)} < \varepsilon$. But this means that $\varepsilon b^2 n + (b\varepsilon - c) > 0$, so that $N > \frac{c}{b^2\varepsilon}$ will work.

8.2.74 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \varepsilon$. Thus we want $n < \varepsilon(n^2+1)$, or $\varepsilon n^2 - n + \varepsilon > 0$. Whenever n is larger than the larger of the two roots of this quadratic, the desired inequality will hold. The roots of the quadratic are $\frac{1 \pm \sqrt{1-4\varepsilon^2}}{2\varepsilon}$, so we choose N to be any integer greater than $\frac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon}$.

8.2.75

a. True. See Theorem 8.2 part 4.

b. False. For example, if $a_n = 1/n$ and $b_n = e^n$, then $\lim_{n\to\infty} a_n b_n = \infty$.

c. True. The definition of the limit of a sequence involves only the behavior of the n^{th} term of a sequence as n gets large (see the Definition of Limit of a Sequence). Thus suppose a_n, b_n differ in only finitely many terms, and that M is large enough so that $a_n = b_n$ for $n > M$. Suppose a_n has limit L. Then for $\varepsilon > 0$, if N is such that $|a_n - L| < \varepsilon$ for $n > N$, first increase N if required so that $N > M$ as well. Then we also have $|b_n - L| < \varepsilon$ for $n > N$. Thus a_n and b_n have the same limit. A similar argument applies if a_n has no limit.

d. True. Note that a_n converges to zero. Intuitively, the nonzero terms of b_n are those of a_n, which converge to zero. More formally, given ϵ, choose N_1 such that for $n > N_1$, $a_n < \epsilon$. Let $N = 2N_1 + 1$. Then for $n > N$, consider b_n. If n is even, then $b_n = 0$ so certainly $b_n < \epsilon$. If n is odd, then $b_n = a_{(n-1)/2}$, and $(n-1)/2 > ((2N_1+1)-1)/2 = N_1$ so that $a_{(n-1)/2} < \epsilon$. Thus b_n converges to zero as well.

e. False. If $\{a_n\}$ happens to converge to zero, the statement is true. But consider for example $a_n = 2 + \frac{1}{n}$. Then $\lim_{n\to\infty} a_n = 2$, but $(-1)^n a_n$ does not converge (it oscillates between positive and negative values increasingly close to ± 2).

f. True. Suppose $\{0.000001a_n\}$ converged to L, and let $\epsilon > 0$ be given. Choose N such that for $n > N$, $|0.000001a_n - L| < \epsilon \cdot 0.000001$. Dividing through by 0.000001, we get that for $n > N$, $|a_n - 1000000L| < \epsilon$, so that a_n converges as well (to $1000000L$).

8.2.76 $\{2n-3\}_{n=3}^{\infty}$.

8.2.77 $\{(n-2)^2 + 6(n-2) - 9\}_{n=3}^{\infty} = \{n^2 + 2n - 17\}_{n=3}^{\infty}$.

8.2.78 If $f(t) = \int_1^t x^{-2}dx$, then $\lim_{t\to\infty} f(t) = \lim_{n\to\infty} a_n$. But

$$\lim_{t\to\infty} f(t) = \int_1^{\infty} x^{-2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\bigg|_1^b\right] = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1.$$

8.2.79 Evaluate the limit of each term separately: $\lim_{n\to\infty} \frac{75^{n-1}}{99^n} = \frac{1}{99}\lim_{n\to\infty}\left(\frac{75}{99}\right)^{n-1} = 0$, while $\frac{-5^n}{8^n} \le \frac{5^n \sin n}{8^n} \le \frac{5^n}{8^n}$, so by the Squeeze Theorem, this second term converges to 0 as well. Thus the sum of the terms converges to zero.

8.2.80 Because $\lim_{n\to\infty} \frac{10n}{10n+4} = 1$, and because the inverse tangent function is continuous, the given sequence has limit $\tan^{-1} 1 = \pi/4$.

8.2.81 Because $\lim_{n\to\infty} 0.99^n = 0$, and because cosine is continuous, the first term converges to $\cos 0 = 1$. The limit of the second term is $\lim_{n\to\infty} \frac{7^n+9^n}{63^n} = \lim_{n\to\infty}\left(\frac{7}{63}\right)^n + \lim_{n\to\infty}\left(\frac{9}{63}\right)^n = 0$. Thus the sum converges to 1.

8.2.82 Dividing the numerator and denominator by $n!$ gives $a_n = \frac{(4^n/n!)+5}{1+(2^n/n!)}$. By Theorem 8.6, we have $4^n \ll n!$ and $2^n \ll n!$. Thus, $\lim_{n\to\infty} a_n = \frac{0+5}{1+0} = 5$.

8.2.83 Dividing the numerator and denominator by 6^n gives $a_n = \frac{1+(1/2)^n}{1+(n^{100}/6^n)}$. By Theorem 8.6, $n^{100} \ll 6^n$. Thus $\lim_{n\to\infty} a_n = \frac{1+0}{1+0} = 1$.

8.2.84 Dividing the numerator and denominator by n^8 gives $a_n = \frac{1+(1/n)}{(1/n)+\ln n}$. Because $1 + (1/n) \to 1$ as $n \to \infty$ and $(1/n) + \ln n \to \infty$ as $n \to \infty$, we have $\lim_{n\to\infty} a_n = 0$.

8.2.85 We can write $a_n = \frac{(7/5)^n}{n^7}$. Theorem 8.6 indicates that $n^7 \ll b^n$ for $b > 1$, so $\lim_{n\to\infty} a_n = \infty$.

8.2.86 A graph shows that the sequence appears to converge. Assuming that it does, let its limit be L. Then $\lim_{n\to\infty} a_{n+1} = \frac{1}{2}\lim_{n\to\infty} a_n + 2$, so $L = \frac{1}{2}L + 2$, and thus $\frac{1}{2}L = 2$, so $L = 4$.

8.2.87 A graph shows that the sequence appears to converge. Let its supposed limit be L, then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}(2a_n(1-a_n)) = 2(\lim_{n\to\infty} a_n)(1-\lim_{n\to\infty} a_n)$, so $L = 2L(1-L) = 2L - 2L^2$, and thus $2L^2 - L = 0$, so $L = 0, \frac{1}{2}$. Thus the limit appears to be either 0 or 1/2; with the given initial condition, doing a few iterations by hand confirms that the sequence converges to 1/2: $a_0 = 0.3$; $a_1 = 2 \cdot 0.3 \cdot 0.7 = .42$; $a_2 = 2 \cdot 0.42 \cdot 0.58 = 0.4872$.

8.2.88 A graph shows that the sequence appears to converge, and to a value other than zero; let its limit be L. Then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + \frac{2}{a_n}) = \frac{1}{2}\lim_{n\to\infty} a_n + \frac{1}{\lim_{n\to\infty} a_n}$, so $L = \frac{1}{2}L + \frac{1}{L}$, and therefore $L^2 = \frac{1}{2}L^2 + 1$. So $L^2 = 2$, and thus $L = \sqrt{2}$.

8.2.89 Computing three terms gives $a_0 = 0.5, a_1 = 4 \cdot 0.5 \cdot 0.5 = 1, a_2 = 4 \cdot 1 \cdot (1-1) = 0$. All successive terms are obviously zero, so the sequence converges to 0.

8.2.90 A graph shows that the sequence appears to converge. Let its limit be L. Then $\lim_{n\to\infty} a_{n+1} = \sqrt{2 + \lim_{n\to\infty} a_n}$, so $L = \sqrt{2+L}$. Thus we have $L^2 = 2 + L$, so $L^2 - L - 2 = 0$, and thus $L = -1, 2$. A square root can never be negative, so this sequence must converge to 2.

8.2.91 For $b = 2$, $2^3 > 3!$ but $16 = 2^4 < 4! = 24$, so the crossover point is $n = 4$. For e, $e^5 \approx 148.41 > 5! = 120$ while $e^6 \approx 403.4 < 6! = 720$, so the crossover point is $n = 6$. For 10, $24! \approx 6.2 \times 10^{23} < 10^{24}$, while $25! \approx 1.55 \times 10^{25} > 10^{25}$, so the crossover point is $n = 25$.

8.2.92

a. Rounded to the nearest fish, the populations are

$$\begin{aligned} F_0 &= 4000 \\ F_1 &= 1.015F_0 - 80 = 3980 \\ F_2 &= 1.015F_1 - 80 \approx 3960 \\ F_3 &= 1.015F_2 - 80 \approx 3939 \\ F_4 &= 1.015F_3 - 80 \approx 3918 \\ F_5 &= 1.015F_4 - 80 \approx 3897 \end{aligned}$$

b. $F_n = 1.015F_{n-1} - 80$

c. The population decreases and eventually reaches zero.

d. With an initial population of 5500 fish, the population increases without bound.

e. If the initial population is less than 5333 fish, the population will decline to zero. This is essentially because for a population of less than 5333, the natural increase of 1.5% does not make up for the loss of 80 fish.

8.2.93

a. The profits for each of the first ten days, in dollars are:

n	0	1	2	3	4	5	6	7	8	9	10
h_n	130.00	130.75	131.40	131.95	132.40	132.75	133.00	133.15	133.20	133.15	133.00

b. The profit on an item is revenue minus cost. The total cost of keeping the heifer for n days is $.45n$, and the revenue for selling the heifer on the n^{th} day is $(200 + 5n) \cdot (.65 - .01n)$, because the heifer gains 5 pounds per day but is worth a penny less per pound each day. Thus the total profit on the n^{th} day is $h_n = (200 + 5n) \cdot (.65 - .01n) - .45n = 130 + 0.8n - 0.05n^2$. The maximum profit occurs when $-.1n + .8 = 0$, which occurs when $n = 8$. The maximum profit is achieved by selling the heifer on the 8^{th} day.

8.2.94

a. $x_0 = 7$, $x_1 = 6$, $x_2 = 6.5 = \frac{13}{2}$, $x_3 = 6.25$, $x_4 = 6.375 = \frac{51}{8}$, $x_5 = 6.3125 = \frac{101}{16}$, $x_6 = 6.34375 = \frac{203}{32}$.

b. For the formula given in the problem, we have $x_0 = \frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^0 = 7$, $x_1 = \frac{19}{3} + \frac{2}{3} \cdot \frac{-1}{2} = \frac{19}{3} - \frac{1}{3} = 6$, so that the formula holds for $n = 0, 1$. Now assume the formula holds for all integers $\le k$; then

$$\begin{aligned} x_{k+1} = \frac{1}{2}(x_k + x_{k-1}) &= \frac{1}{2}\left(\frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^k + \frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k-1}\right) \\ &= \frac{1}{2}\left(\frac{38}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k-1}\left(-\frac{1}{2} + 1\right)\right) \\ &= \frac{1}{2}\left(\frac{38}{3} + 4 \cdot \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1} \cdot \frac{1}{2}\right) \\ &= \frac{1}{2}\left(\frac{38}{3} + 2 \cdot \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1}\right) \\ &= \frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1}. \end{aligned}$$

c. As $n \to \infty$, $(-1/2)^n \to 0$, so that the limit is 19/3, or 6 1/3.

8.2.95 The approximate first few values of this sequence are:

n	0	1	2	3	4	5	6
c_n	.7071	.6325	.6136	.6088	.6076	.6074	.6073

The value of the constant appears to be around 0.607.

8.2.96 We first prove that d_n is bounded by 200. If $d_n \le 200$, then $d_{n+1} = 0.5 \cdot d_n + 100 \le 0.5 \cdot 200 + 100 \le 200$. Because $d_0 = 100 < 200$, all d_n are at most 200. Thus the sequence is bounded. To see that it is monotone, look at

$$d_n - d_{n-1} = 0.5 \cdot d_{n-1} + 100 - d_{n-1} = 100 - 0.5 d_{n-1}.$$

But we know that $d_{n-1} \le 200$, so that $100 - 0.5 d_{n-1} \ge 0$. Thus $d_n \ge d_{n-1}$ and the sequence is nondecreasing.

8.2.97

a. If we "cut off" the expression after n square roots, we get a_n from the recurrence given. We can thus *define* the infinite expression to be the limit of a_n as $n \to \infty$.

b. $a_0 = 1$, $a_1 = \sqrt{2}$, $a_2 = \sqrt{1 + \sqrt{2}} \approx 1.5538$, $a_3 \approx 1.5981$, $a_4 \approx 1.6118$, and $a_5 \approx 1.6161$.

c. $a_{10} \approx 1.618$, which differs from $\frac{1+\sqrt{5}}{2} \approx 1.61803394$ by less than .001.

d. Assume $\lim_{n\to\infty} a_n = L$. Then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1 + a_n} = \sqrt{1 + \lim_{n\to\infty} a_n}$, so $L = \sqrt{1 + L}$, and thus $L^2 = 1 + L$. Therefore we have $L^2 - L - 1 = 0$, so $L = \frac{1 \pm \sqrt{5}}{2}$.

Because clearly the limit is positive, it must be the positive square root.

e. Letting $a_{n+1} = \sqrt{p + \sqrt{a_n}}$ with $a_0 = p$ and assuming a limit exists we have $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{p + a_n}$ $= \sqrt{p + \lim_{n\to\infty} a_n}$, so $L = \sqrt{p + L}$, and thus $L^2 = p + L$. Therefore, $L^2 - L - p = 0$, so $L = \frac{1 \pm \sqrt{1+4p}}{2}$, and because we know that L is positive, we have $L = \frac{1+\sqrt{4p+1}}{2}$. The limit exists for all positive p.

8.2.98 Note that $1 - \frac{1}{i} = \frac{i-1}{i}$, so that the product is $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdots$, so that $a_n = \frac{1}{n}$ for $n \ge 2$. The sequence $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ has limit zero.

8.2.99

a. Define a_n as given in the problem statement. Then we can *define* the value of the continued fraction to be $\lim_{n\to\infty} a_n$.

b. $a_0 = 1$, $a_1 = 1 + \frac{1}{a_0} = 2$, $a_2 = 1 + \frac{1}{a_1} = \frac{3}{2} = 1.5$, $a_3 = 1 + \frac{1}{a_2} = \frac{5}{3} \approx 1.667$, $a_4 = 1 + \frac{1}{a_3} = \frac{8}{5} = 1.6$, $a_5 = 1 + \frac{1}{a_4} = \frac{13}{8} = 1.625$.

c. From the list above, the values of the sequence alternately decrease and increase, so we would expect that the limit is somewhere between 1.6 and 1.625.

d. Assume that the limit is equal to L. Then from $a_{n+1} = 1 + \frac{1}{a_n}$, we have $\lim_{n\to\infty} a_{n+1} = 1 + \frac{1}{\lim_{n\to\infty} a_n}$, so $L = 1 + \frac{1}{L}$, and thus $L^2 - L - 1 = 0$. Therefore, $L = \frac{1\pm\sqrt{5}}{2}$, and because L is clearly positive, it must be equal to $\frac{1+\sqrt{5}}{2} \approx 1.618$.

e. Here $a_0 = a$ and $a_{n+1} = a + \frac{b}{a_n}$. Assuming that $\lim_{n\to\infty} a_n = L$ we have $L = a + \frac{b}{L}$, so $L^2 = aL + b$, and thus $L^2 - aL - b = 0$. Therefore, $L = \frac{a\pm\sqrt{a^2+4b}}{2}$, and because $L > 0$ we have $L = \frac{a+\sqrt{a^2+4b}}{2}$.

8.2.100

a. With $p = 0.5$ we have for $a_{n+1} = a_n^p$:

n	1	2	3	4	5	6	7
a_n	0.707	0.841	0.971	0.958	0.979	0.989	0.995

Experimenting with recurrence (1) one sees that for $0 < p \le 1$ the sequence converges to 1, while for $p > 1$ the sequence diverges to ∞.

b. With $p = 1.2$ and $a_n = p^{a_{n-1}}$ we obtain

n	1	2	3	4	5	6	7	8	9	10
a_n	1.2	1.2446	1.2547	1.2570	1.2577	1.2577	1.2577	1.2577	1.2577	1.2577

With recurrence (2), in addition to converging for $p < 1$ it also converges for values of p less than approximately 1.444. Here is a table of approximate values for different values of p:

p	1.1	1.2	1.3	1.4	1.44	1.444	1.445
$\lim_{n\to\infty} a_n$	1.1118	1.25776	1.471	1.887	2.39385	2.587	Diverges

It appears that the upper limit of convergence is about 1.444.

8.2.101

a. $f_0 = f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5, f_5 = 8, f_6 = 13, f_7 = 21, f_8 = 34, f_9 = 55, f_{10} = 89$.

b. The sequence is clearly not bounded.

c. $\frac{f_{10}}{f_9} \approx 1.61818$

d. We use induction. Note that $\frac{1}{\sqrt{5}}\left(\varphi + \frac{1}{\varphi}\right) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\left(\frac{1+2\sqrt{5}+5+4}{2(1+\sqrt{5})}\right) = 1 = f_1$. Also note that $\frac{1}{\sqrt{5}}\left(\varphi^2 - \frac{1}{\varphi^2}\right) = \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2} - \frac{2}{3+\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\left(\frac{9+6\sqrt{5}+5-4}{2(3+\sqrt{5})}\right) = 1 = f_2$. Now note that

$$\begin{aligned} f_{n-1} + f_{n-2} &= \frac{1}{\sqrt{5}}(\varphi^{n-1} - (-1)^{n-1}\varphi^{1-n} + \varphi^{n-2} - (-1)^{n-2}\varphi^{2-n}) \\ &= \frac{1}{\sqrt{5}}((\varphi^{n-1} + \varphi^{n-2}) - (-1)^n(\varphi^{2-n} - \varphi^{1-n})). \end{aligned}$$

Now, note that $\varphi - 1 = \frac{1}{\varphi}$, so that

$$\varphi^{n-1} + \varphi^{n-2} = \varphi^{n-1}\left(1 + \frac{1}{\varphi}\right) = \varphi^{n-1} \cdot \varphi = \varphi^n$$

and

$$\varphi^{2-n} - \varphi^{1-n} = \varphi^{-n}(\varphi^2 - \varphi) = \varphi^{-n}(\varphi(\varphi - 1)) = \varphi^{-n}.$$

Making these substitutions, we get

$$f_n = f_{n-1} + f_{n-2} = \frac{1}{\sqrt{5}}(\varphi^n - (-1)^n\varphi^{-n})$$

8.2.102

a. We show that the arithmetic mean of any two positive numbers exceeds their geometric mean. Let a, $b > 0$; then $\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a - 2\sqrt{ab} + b) = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 \geq 0$. Because in addition $a_0 > b_0$, we have $a_n > b_n$ for all n.

b. To see that $\{a_n\}$ is decreasing, note that

$$a_{n+1} = \frac{a_n + b_n}{2} < \frac{a_n + a_n}{2} = a_n.$$

Similarly,

$$b_{n+1} = \sqrt{a_n b_n} > \sqrt{b_n b_n} = b_n,$$

so that $\{b_n\}$ is increasing.

c. $\{a_n\}$ is monotone and nonincreasing by part (b), and bounded below by part (a) (it is bounded below by any of the b_n), so it converges by the monotone convergence theorem. Similarly, $\{b_n\}$ is monotone and nondecreasing by part (b) and bounded above by part (a), so it too converges.

d.

$$a_{n+1} - b_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n} = \frac{1}{2}(a_n - 2\sqrt{a_n b_n} + b_n) < \frac{1}{2}(a_n - 2\sqrt{b_n^2} + b_n) = \frac{1}{2}(a_n - b_n).$$

Thus the difference between a_{n+1} and b_{n+1} is less than half the difference between a_n and b_n, so that difference goes to zero and the two limits are the same.

e. The AGM of 12 and 20 is approximately 15.745; Gauss' constant is $\frac{1}{\mathrm{AGM}(1,\sqrt{2})} \approx 0.8346$.

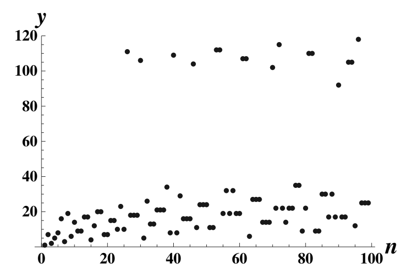
8.2.103

a.

$$\begin{array}{rl} 2: & 1 \\ 3: & 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1 \\ 4: & 2,\ 1 \\ 5: & 16,\ 8,\ 4,\ 2,\ 1 \\ 6: & 3,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1 \\ 7: & 22,\ 11,\ 34,\ 17,\ 52,\ 26,\ 13,\ 40,\ 20,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1 \\ 8: & 4,\ 2,\ 1 \\ 9: & 28,\ 14,\ 7,\ 22,\ 11,\ 34,\ 17,\ 52,\ 26,\ 13,\ 40,\ 20,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1 \\ 10: & 5,\ 16,\ 8,\ 4,\ 2,\ 1 \end{array}$$

b. From the above, $H_2 = 1, H_3 = 7$, and $H_4 = 2$.

c. This plot is for $1 \le n \le 100$. Like hailstones, the numbers in the sequence a_n rise and fall but eventually crash to the earth. The conjecture appears to be true.



8.2.104 $\{a_n\} \ll \{b_n\}$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. But $\lim_{n\to\infty} \frac{ca_n}{db_n} = \frac{c}{d} \lim_{n\to\infty} \frac{a_n}{b_n} = 0$, so that $\{ca_n\} \ll \{db_n\}$.

8.2.105

a. Note that $a_2 = \sqrt{3a_1} = \sqrt{3\sqrt{3}} > \sqrt{3} = a_1$. Now assume that $\sqrt{3} = a_1 < a_2 < \dots a_{k-1} < a_k$. Then

$$a_{k+1} = \sqrt{3a_k} > \sqrt{3a_{k-1}} = a_k.$$

Thus $\{a_n\}$ is increasing.

b. Clearly because $a_1 = \sqrt{3} > 0$ and $\{a_n\}$ is increasing, the sequence is bounded below by $\sqrt{3} > 0$. Further, $a_1 = \sqrt{3} < 3$; assume that $a_k < 3$. Then $a_{k+1} = \sqrt{3a_k} < \sqrt{3 \cdot 3} = 3$, so that $a_{k+1} < 3$. So by induction, $\{a_k\}$ is bounded above by 3.

c. Because $\{a_n\}$ is bounded and monotonically increasing, $\lim_{n\to\infty} a_n$ exists by Theorem 8.5.

d. Because the limit exists, we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{3a_n} = \sqrt{3} \lim_{n\to\infty} \sqrt{a_n} = \sqrt{3}\sqrt{\lim_{n\to\infty} a_n}.$$

Let $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$; then $L = \sqrt{3}\sqrt{L}$, so that $L = 3$.

8.2.106 By Theorem 8.6,

$$\lim_{n\to\infty} \frac{2\ln n}{\sqrt{n}} = 2 \lim_{n\to\infty} \frac{\ln n}{n^{1/2}} = 0,$$

so that $\sqrt{n}$ has the larger growth rate. Using computational software, we see that $\sqrt{74} \approx 8.60233 < 2\ln 74 \approx 8.60813$, while $\sqrt{75} \approx 8.66025 > 2\ln 75 \approx 8.63493$.

8.2.107 By Theorem 8.6,

$$\lim_{n\to\infty} \frac{n^5}{e^{n/2}} = 2^5 \lim_{n\to\infty} \frac{(n/2)^5}{e^{n/2}} = 0,$$

so that $e^{n/2}$ has the larger growth rate. Using computational software we see that $e^{35/2} \approx 3.982 \times 10^7 < 35^5 \approx 5.252 \times 10^7$, while $e^{36/2} \approx 6.566 \times 10^7 > 36^5 \approx 6.047 \times 10^7$.

8.2.108 By Theorem 8.6, $\ln n^{10} \ll n^{1.001}$, so that $n^{1.001}$ has the larger growth rate. Using computational software we see that $35^{1.001} \approx 35.1247 < \ln 35^{10} \approx 35.5535$ while $36^{1.001} \approx 36.1292 > \ln 36^{10} \approx 35.8352$.

8.2.109 Experiment with a few widely separated values of n:

n	$n!$	$n^{0.7n}$
1	1	1
10	3.63×10^6	10^7
100	9.33×10^{157}	10^{140}
1000	4.02×10^{2567}	10^{2100}

It appears that $n^{0.7n}$ starts out larger, but is overtaken by the factorial somewhere between $n = 10$ and $n = 100$, and that the gap grows wider as n increases. Looking between $n = 10$ and $n = 100$ revels that for $n = 18$, we have $n! \approx 6.402 \times 10^{15} < n^{0.7n} \approx 6.553 \times 10^{15}$ while for $n = 19$ we have $n! \approx 1.216 \times 10^{17} > n^{0.7n} \approx 1.017 \times 10^{17}$.

8.2.110 By Theorem 8.6,

$$\lim_{n\to\infty} \frac{n^9 \ln^3 n}{n^{10}} = \lim_{n\to\infty} \frac{\ln^3 n}{n} = 0,$$

so that n^{10} has a larger growth rate. Using computational software we see that $93^{10} \approx 4.840 \times 10^{19} < 93^9 \ln^3 93 \approx 4.846 \times 10^{19}$ while $94^{10} \approx 5.386 \times 10^{19} > 94^9 \ln^3 94 \approx 5.374 \times 10^{19}$.

8.2.111 First note that for $a = 1$ we already know that $\{n^n\}$ grows fast than $\{n!\}$. So if $a > 1$, then $n^{an} \geq n^n$, so that $\{n^{an}\}$ grows faster than $\{n!\}$ for $a > 1$ as well. To settle the case $a < 1$, recall Stirling's formula which states that for large values of n,

$$n! \sim \sqrt{2\pi n} n^n e^{-n}.$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} \frac{n!}{n^{an}} &= \lim_{n\to\infty} \frac{\sqrt{2\pi n} n^n e^{-n}}{n^{an}} \\
&= \sqrt{2\pi} \lim_{n\to\infty} n^{\frac{1}{2}+(1-a)n} e^{-n} \\
&\geq \sqrt{2\pi} \lim_{n\to\infty} n^{(1-a)n} e^{-n} \\
&= \sqrt{2\pi} \lim_{n\to\infty} e^{(1-a)n\ln n} e^{-n} \\
&= \sqrt{2\pi} \lim_{n\to\infty} e^{((1-a)\ln n - 1)n}.
\end{aligned}$$

If $a < 1$ then $(1-a)\ln n - 1 > 0$ for large values of n because $1 - a > 0$, so that this limit is infinite. Hence $\{n!\}$ grows faster than $\{n^{an}\}$ exactly when $a < 1$.

8.3 Infinite Series

8.3.1 A geometric series is a series in which the ratio of successive terms in the underlying sequence is a constant. Thus a geometric series has the form $\sum ar^k$ where r is the constant. One example is $3 + 6 + 12 + 24 + 48 + \cdots$ in which $a = 3$ and $r = 2$.

8.3.2 A geometric sum is the sum of a finite number of terms which have a constant ratio; a geometric series is the sum of an infinite number of such terms.

8.3.3 The ratio is the common ratio between successive terms in the sum.

8.3.4 Yes, because there are only a finite number of terms.

8.3.5 No. For example, the geometric series with $a_n = 3 \cdot 2^n$ does not have a finite sum.

8.3.6 The series converges if and only if $|r| < 1$.

8.3.7 $S = 1 \cdot \frac{1-3^9}{1-3} = \frac{19682}{2} = 9841.$

8.3.8 $S = 1 \cdot \frac{1-(1/4)^{11}}{1-(1/4)} = \frac{4^{11}-1}{3 \cdot 4^{10}} = \frac{4194303}{3 \cdot 1048576} = \frac{1398101}{1048576} \approx 1.333.$

8.3.9 $S = 1 \cdot \frac{1-(4/25)^{21}}{1-4/25} = \frac{25^{21}-4^{21}}{25^{21}-4 \cdot 25^{20}} \approx 1.1905.$

8.3.10 $S = 16 \cdot \frac{1-2^9}{1-2} = 511 \cdot 16 = 8176.$

8.3.11 $S = 1 \cdot \frac{1-(-3/4)^{10}}{1+3/4} = \frac{4^{10}-3^{10}}{4^{10}+3 \cdot 4^9} = \frac{141361}{262144} \approx 0.5392.$

8.3.12 $S = (-2.5) \cdot \frac{1-(-2.5)^5}{1+2.5} = -70.46875.$

8.3.13 $S = 1 \cdot \frac{1-\pi^7}{1-\pi} = \frac{\pi^7-1}{\pi-1} \approx 1409.84.$

8.3.14 $S = \frac{4}{7} \cdot \frac{1-(4/7)^{10}}{3/7} = \frac{375235564}{282475249} \approx 1.328.$

8.3.15 $S = 1 \cdot \frac{1-(-1)^{21}}{2} = 1.$

8.3.16 $\frac{65}{27}$.

8.3.17 $\frac{1093}{2916}$.

8.3.18 $\frac{1}{5}\left(\frac{1-(3/5)^6}{1-3/5}\right) = \frac{7448}{15625}$.

8.3.19 $\frac{1}{1-1/4} = \frac{4}{3}$.

8.3.20 $\frac{1}{1-3/5} = \frac{5}{2}$.

8.3.21 $\frac{1}{1-0.9} = 10$.

8.3.22 $\frac{1}{1-2/7} = \frac{7}{5}$.

8.3.23 Divergent, because $r > 1$.

8.3.24 $\frac{1}{1-1/\pi} = \frac{\pi}{\pi-1}$.

8.3.25 $\frac{e^{-2}}{1-e^{-2}} = \frac{1}{e^2-1}$.

8.3.26 $\frac{5/4}{1-1/2} = \frac{5}{2}$.

8.3.27 $\frac{2^{-3}}{1-2^{-3}} = \frac{1}{7}$.

8.3.28 $\dfrac{3 \cdot 4^3/7^3}{1-4/7} = \dfrac{64}{49}$.

8.3.29 $\dfrac{1/625}{1-1/5} = \dfrac{1}{500}$.

8.3.30 Note that this is the same as $\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^k$. Then $S = \dfrac{1}{1-3/4} = 4$.

8.3.31 $\dfrac{1}{1-e/\pi} = \dfrac{\pi}{\pi - e}$. (Note that $e < \pi$, so $r < 1$ for this series.)

8.3.32 $\dfrac{1/16}{1-3/4} = \dfrac{1}{4}$.

8.3.33 $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k 5^{3-k} = 5^3\sum_{k=0}^{\infty}\left(\frac{1}{20}\right)^k = 5^3 \cdot \frac{1}{1-1/20} = \frac{5^3 \cdot 20}{19} = \frac{2500}{19}$.

8.3.34 $\dfrac{3^6/8^6}{1-(3/8)^3} = \dfrac{729}{248320}$

8.3.35 $\dfrac{1}{1+9/10} = \dfrac{10}{19}$.

8.3.36 $-\dfrac{2/3}{1+2/3} = -\dfrac{2}{5}$.

8.3.37 $3 \cdot \dfrac{1}{1+1/\pi} = \dfrac{3\pi}{\pi+1}$.

8.3.38 $\displaystyle\sum_{k=1}^{\infty}\left(-\frac{1}{e}\right)^k = -\frac{1/e}{1+1/e} = -\frac{1}{e+1}$.

8.3.39 $\dfrac{0.15^2}{1.15} = \dfrac{9}{460} \approx 0.0196$.

8.3.40 $-\dfrac{3/8^3}{1+1/8^3} = -\dfrac{1}{171}$.

8.3.41

a. $0.\overline{3} = 0.333\ldots = \sum_{k=1}^{\infty} 3(0.1)^k$.

b. The limit of the sequence of partial sums is $1/3$.

8.3.42

a. $0.\overline{6} = 0.666\ldots = \sum_{k=1}^{\infty} 6(0.1)^k$.

b. The limit of the sequence of partial sums is $2/3$.

8.3.43

a. $0.\overline{1} = 0.111\ldots = \sum_{k=1}^{\infty}(0.1)^k$.

b. The limit of the sequence of partial sums is $1/9$.

8.3.44

a. $0.\overline{5} = 0.555\ldots = \sum_{k=1}^{\infty} 5(0.1)^k$.

b. The limit of the sequence of partial sums is $5/9$.

8.3.45

a. $0.\overline{09} = 0.0909\ldots = \sum_{k=1}^{\infty} 9(0.01)^k$.

b. The limit of the sequence of partial sums is $1/11$.

8.3.46

a. $0.\overline{27} = 0.272727\ldots = \sum_{k=1}^{\infty} 27(0.01)^k$.

b. The limit of the sequence of partial sums is $3/11$.

8.3.47

a. $0.\overline{037} = 0.037037037\ldots = \sum_{k=1}^{\infty} 37(0.001)^k$.

b. The limit of the sequence of partial sums is $37/999 = 1/27$.

8.3.48

a. $0.\overline{027} = 0.027027027\ldots = \sum_{k=1}^{\infty} 27(0.001)^k$

b. The limit of the sequence of partial sums is $27/999 = 1/37$.

8.3.49 $0.\overline{12} = 0.121212\ldots = \displaystyle\sum_{k=0}^{\infty} .12 \cdot 10^{-2k} = \frac{.12}{1-1/100} = \frac{12}{99} = \frac{4}{33}$.

8.3.50 $1.\overline{25} = 1.252525\ldots = 1 + \displaystyle\sum_{k=0}^{\infty} .25 \cdot 10^{-2k} = 1 + \frac{.25}{1-1/100} = 1 + \frac{25}{99} = \frac{124}{99}$.

8.3.51 $0.\overline{456} = 0.456456456\ldots = \displaystyle\sum_{k=0}^{\infty} .456 \cdot 10^{-3k} = \frac{.456}{1-1/1000} = \frac{456}{999} = \frac{152}{333}$.

8.3.52 $1.00\overline{39} = 1.00393939\ldots = 1 + \displaystyle\sum_{k=0}^{\infty} .0039 \cdot 10^{-2k} = 1 + \frac{.0039}{1-1/100} = 1 + \frac{.39}{99} = 1 + \frac{39}{9900} = \frac{9939}{9900} = \frac{3313}{3300}$.

8.3.53 $0.00\overline{952} = 0.00952952\ldots = \displaystyle\sum_{k=0}^{\infty} .00952 \cdot 10^{-3k} = \frac{.00952}{1-1/1000} = \frac{9.52}{999} = \frac{952}{99900} = \frac{238}{24975}$.

8.3.54 $5.12\overline{83} = 5.12838383\ldots = 5.12 + \displaystyle\sum_{k=0}^{\infty} .0083 \cdot 10^{-2k} = 5.12 + \frac{.0083}{1-1/100} = \frac{512}{100} + \frac{.83}{99} = \frac{128}{25} + \frac{83}{9900} = \frac{50771}{9900}$.

8.3.55 The second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{1}{1+1} - \frac{1}{n+2} = \frac{n}{2n+4}$, and $\lim_{n\to\infty} \frac{n}{2n+4} = \frac{1}{2}$.

8.3.56 The second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{1}{1+2} - \frac{1}{n+3} = \frac{n}{3n+6}$, and $\lim_{n\to\infty} \frac{n}{3n+9} = \frac{1}{3}$.

8.3.57 $\dfrac{1}{(k+6)(k+7)} = \dfrac{1}{k+6} - \dfrac{1}{k+7}$, so the series given is the same as $\sum_{k=1}^{\infty}\left(\frac{1}{k+6} - \frac{1}{k+7}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{1}{1+6} - \frac{1}{n+7}$. Thus $\lim_{n\to\infty} S_n = \frac{1}{7}$.

8.3.58 $\dfrac{1}{(3k+1)(3k+4)} = \dfrac{1}{3}\left(\dfrac{1}{3k+1} - \dfrac{1}{3k+4}\right)$, so the series given can be written $\displaystyle\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{1}{3k+1} - \frac{1}{3k+4}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term (because $3(k+1)+1 = 3k+4$), so we are left with $S_n = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{3n+4}\right) = \frac{n+1}{3n+4}$ and $\lim_{n\to\infty} \frac{n+1}{3n+4} = \frac{1}{3}$.

8.3.59 Note that $\frac{4}{(4k-3)(4k+1)} = \frac{1}{4k-3} - \frac{1}{4k+1}$. Thus the given series is the same as $\displaystyle\sum_{k=3}^{\infty}\left(\frac{1}{4k-3} - \frac{1}{4k+1}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term (because $4(k+1)-3 = 4k+1$), so we have $S_n = \frac{1}{9} - \frac{1}{4n+1}$, and thus $\lim_{n\to\infty} S_n = \dfrac{1}{9}$.

8.3.60 Note that $\frac{2}{(2k-1)(2k+1)} = \frac{1}{2k-1} - \frac{1}{2k+1}$. Thus the given series is the same as $\displaystyle\sum_{k=3}^{\infty}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term (because $2(k+1)-1 = 2k+1$), so we have $S_n = \frac{1}{5} - \frac{1}{2n+1}$. Thus, $\lim_{n\to\infty} S_n = \dfrac{1}{5}$.

8.3.61 $\ln\left(\dfrac{k+1}{k}\right) = \ln(k+1) - \ln k$, so the series given is the same as $\sum_{k=1}^{\infty}(\ln(k+1) - \ln k)$, in which the first part of each term cancels with the second part of the next term, so we have $S_n = \ln(n+1) - \ln 1 = \ln(n+1)$, and thus the series diverges.

8.3.62 Note that $S_n = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{n+1} - \sqrt{n})$. The second part of each term cancels with the first part of the previous term. Thus, $S_n = \sqrt{n+1} - 1$. and because $\lim_{n\to\infty} \sqrt{n+1} - 1 = \infty$, the series diverges.

8.3.63 $\dfrac{1}{(k+p)(k+p+1)} = \dfrac{1}{k+p} - \dfrac{1}{k+p+1}$, so that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+p+1)} = \sum_{k=1}^{\infty}\left(\frac{1}{k+p} - \frac{1}{k+p+1}\right)$ and this series telescopes to give $S_n = \frac{1}{p+1} - \frac{1}{n+p+1} = \frac{n}{n(p+1)+(p+1)^2}$ so that $\lim_{n\to\infty} S_n = \frac{1}{p+1}$.

8.3.64 $\dfrac{1}{(ak+1)(ak+a+1)} = \dfrac{1}{a}\left(\dfrac{1}{ak+1} - \dfrac{1}{ak+a+1}\right)$, so that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(ak+1)(ak+a+1)} =$ $\displaystyle\frac{1}{a}\sum_{k=1}^{\infty}\left(\frac{1}{ak+1} - \frac{1}{ak+a+1}\right)$. This series telescopes - the second term of each summand cancels with the first term of the succeeding summage – so that $S_n = \frac{1}{a}\left(\frac{1}{a+1} - \frac{1}{an+a+1}\right)$, and thus the limit of the sequence is $\frac{1}{a(a+1)}$.

8.3.65 Let $a_n = \dfrac{1}{\sqrt{n+1}} - \dfrac{1}{\sqrt{n+3}}$. Then the second term of a_n cancels with the first term of a_{n+2}, so the series telescopes and $S_n = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{n-1+3}} - \frac{1}{\sqrt{n+3}}$ and thus the sum of the series is the limit of S_n, which is $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}}$.

8.3.66 The first term of the k^{th} summand is $\sin(\frac{(k+1)\pi}{2k+1})$; the second term of the $(k+1)^{\text{st}}$ summand is $-\sin(\frac{(k+1)\pi}{2(k+1)-1})$; these two are equal except for sign, so they cancel. Thus $S_n = -\sin 0 + \sin(\frac{(n+1)\pi}{2n+1}) = \sin(\frac{(n+1)\pi}{2n+1})$. Because $\frac{(n+1)\pi}{2n+1}$ has limit $\pi/2$ as $n \to \infty$, and because the sine function is continuous, it follows that $\lim_{n\to\infty} S_n$ is $\sin(\frac{\pi}{2}) = 1$.

8.3.67 $16k^2 + 8k - 3 = (4k+3)(4k-1)$, so $\frac{1}{16k^2+8k-3} = \frac{1}{(4k+3)(4k-1)} = \frac{1}{4}\left(\frac{1}{4k-1} - \frac{1}{4k+3}\right)$. Thus the series given is equal to $\displaystyle\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{1}{4k-1} - \frac{1}{4k+3}\right)$. This series telescopes, so $S_n = \frac{1}{4}\left(-1 - \frac{1}{4n+3}\right)$, so the sum of the series is equal to $\lim_{n\to\infty} S_n = -\frac{1}{4}$.

8.3.68 This series clearly telescopes to give $S_n = -\tan^{-1}(1) + \tan^{-1}(n) = \tan^{-1}(n) - \frac{\pi}{4}$. Then because $\lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2}$, the sum of the series is equal to $\lim_{n\to\infty} S_n = \frac{\pi}{4}$.

8.3.69

a. True. $\left(\dfrac{\pi}{e}\right)^{-k} = \left(\dfrac{e}{\pi}\right)^{k}$; because $e < \pi$, this is a geometric series with ratio less than 1.

b. True. If $\displaystyle\sum_{k=12}^{\infty} a^k = L$, then $\displaystyle\sum_{k=0}^{\infty} a^k = \left(\sum_{k=0}^{11} a^k\right) + L$.

c. False. For example, let $0 < a < 1$ and $b > 1$.

d. True. Suppose $a > \frac{1}{2}$. Then we want $a = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. Solving for r gives $r = 1 - \frac{1}{a}$. Because $a > 0$ we have $r < 1$; because $a > \frac{1}{2}$ we have $r > 1 - \frac{1}{1/2} = -1$. Thus $|r| < 1$ so that $\sum_{k=0}^{\infty} r^k$ converges, and it converges to a.

e. True. Suppose $a > -\frac{1}{2}$. Then we want $a = \sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$. Solving for r gives $r = \frac{a}{a+1}$. For $a \geq 0$, clearly $0 \leq r < 1$ so that $\sum_{k=1}^{\infty} r^k$ converges to a. For $-\frac{1}{2} < a < 0$, clearly $r < 0$, but $|a| < |a+1|$, so that $|r| < 1$. Thus in this case $\sum_{k=1}^{\infty} r^k$ also converges to a.

8.3.70 We have

$$S_n = \left(\sin^{-1} 1 - \sin^{-1}\frac{1}{2}\right) + \left(\sin^{-1}\frac{1}{2} - \sin^{-1}\frac{1}{3}\right) + \cdots + \left(\sin^{-1}\frac{1}{n} - \sin^{-1}\frac{1}{n+1}\right).$$

Note that the first part of each term cancels the second part of the previous term, so the nth partial sum telescopes to be $\sin^{-1} 1 - \sin^{-1} \frac{1}{n+1}$. Because $\sin^{-1} 1 = \frac{\pi}{2}$ and $\lim_{n\to\infty} \sin^{-1} \dfrac{1}{n+1} = \sin^{-1} 0 = 0$, we have $\lim_{n\to\infty} S_n = \dfrac{\pi}{2}$.

8.3.71 This can be written as $\dfrac{1}{3}\displaystyle\sum_{k=1}^{\infty}\left(-\frac{2}{3}\right)^k$. This is a geometric series with ratio $r = -\frac{2}{3}$ so the sum is $\frac{1}{3} \cdot \frac{-2/3}{1-(-2/3)} = \frac{1}{3} \cdot \left(-\frac{2}{5}\right) = -\frac{2}{15}$.

8.3.72 This can be written as $\dfrac{1}{e}\displaystyle\sum_{k=1}^{\infty}\left(\frac{\pi}{e}\right)^k$. This is a geometric series with $r = \frac{\pi}{e} > 1$, so the series diverges.

8.3.73 Note that

$$\frac{\ln((k+1)k^{-1})}{(\ln k)\ln(k+1)} = \frac{\ln(k+1)}{(\ln k)\ln(k+1)} - \frac{\ln k}{(\ln k)\ln(k+1)} = \frac{1}{\ln k} - \frac{1}{\ln(k+1)}.$$

In the partial sum S_n, the first part of each term cancels the second part of the preceding term, so we have $S_n = \frac{1}{\ln 2} - \frac{1}{\ln(n+1)}$. Thus we have $\lim_{n\to\infty} S_n = \dfrac{1}{\ln 2}$.

8.3.74

a. Because the first part of each term cancels the second part of the previous term, the nth partial sum telescopes to be $S_n = \frac{1}{2} - \frac{1}{2^{n+1}}$. Thus, the sum of the series is $\lim_{n\to\infty} S_n = \dfrac{1}{2}$.

b. Note that $\frac{1}{2^k} - \frac{1}{2^{k+1}} = \frac{2^{k+1}-2^k}{2^k 2^{k+1}} = \frac{1}{2^{k+1}}$. Thus, the original series can be written as $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2^{k+1}}$ which is geometric with $r = 1/2$ and $a = 1/4$, so the sum is $\frac{1/4}{1-1/2} = \frac{1}{2}$.

8.3.75

a. Because the first part of each term cancels the second part of the previous term, the nth partial sum telescopes to be $S_n = \frac{4}{3} - \frac{4}{3^{n+1}}$. Thus, the sum of the series is $\lim_{n\to\infty} S_n = \dfrac{4}{3}$.

b. Note that $\frac{4}{3^k} - \frac{4}{3^{k+1}} = \frac{4\cdot 3^{k+1}-4\cdot 3^k}{3^k 3^{k+1}} = \frac{8}{3^{k+1}}$. Thus, the original series can be written as $\displaystyle\sum_{k=1}^{\infty}\frac{8}{3^{k+1}}$ which is geometric with $r = 1/3$ and $a = 8/9$, so the sum is $\frac{8/9}{1-1/3} = \frac{8}{9} \cdot \frac{3}{2} = \frac{4}{3}$.

8.3.76 It will take Achilles 1/5 hour to cover the first mile. At this time, the tortoise has gone 1/5 mile more, and it will take Achilles 1/25 hour to reach this new point. At that time, the tortoise has gone another 1/25 of a mile, and it will take Achilles 1/125 hour to reach this point. Adding the times up, we have

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots = \frac{1/5}{1-1/5} = \frac{1}{4},$$

so it will take Achilles 1/4 of an hour (15 minutes) to catch the tortoise.

8.3.77 At the n^{th} stage, there are 2^{n-1} triangles of area $A_n = \frac{1}{8}A_{n-1} = \frac{1}{8^{n-1}}A_1$, so the total area of the triangles formed at the n^{th} stage is $\dfrac{2^{n-1}}{8^{n-1}}A_1 = \left(\dfrac{1}{4}\right)^{n-1} A_1$. Thus the total area under the parabola is

$$\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1} A_1 = A_1 \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1} = A_1 \frac{1}{1-1/4} = \frac{4}{3}A_1.$$

8.3.78

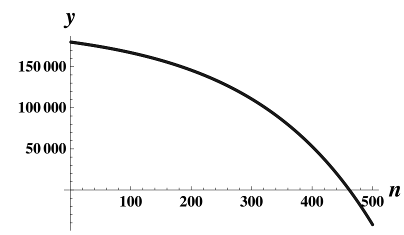
a. Note that $\frac{3^k}{(3^{k+1}-1)(3^k-1)} = \frac{1}{2}\cdot\left(\frac{1}{3^k-1} - \frac{1}{3^{k+1}-1}\right)$. Then

$$\sum_{k=1}^{\infty} \frac{3^k}{(3^{k+1}-1)(3^k-1)} = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{3^k-1} - \frac{1}{3^{k+1}-1}\right).$$

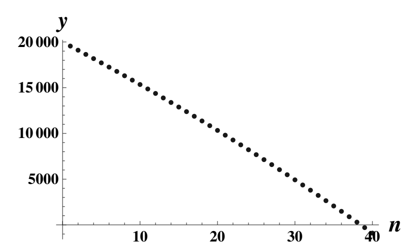
This series telescopes to give $S_n = \frac{1}{2}\left(\frac{1}{3-1} - \frac{1}{3^{n+1}-1}\right)$, so that the sum of the series is $\lim_{n\to\infty} S_n = \frac{1}{4}$.

b. We mimic the above computations. First, $\frac{a^k}{(a^{k+1}-1)(a^k-1)} = \frac{1}{a-1}\cdot\left(\frac{1}{a^k-1} - \frac{1}{a^{k+1}-1}\right)$, so we see that we cannot have $a = 1$, because the fraction would then be undefined. Continuing, we obtain $S_n = \frac{1}{a-1}\left(\frac{1}{a-1} - \frac{1}{a^{n+1}-1}\right)$. Now, $\lim_{n\to\infty} \frac{1}{a^{n+1}-1}$ converges if and only if the denominator grows without bound; this happens if and only if $|a| > 1$. Thus, the original series converges for $|a| > 1$, when it converges to $\frac{1}{(a-1)^2}$. Note that this is valid even for a negative.

8.3.79 It appears that the loan is paid off after about 470 months. Let B_n be the loan balance after n months. Then $B_0 = 180000$ and $B_n = 1.005 \cdot B_{n-1} - 1000$. Then $B_n = 1.005 \cdot B_{n-1} - 1000 = 1.005(1.005 \cdot B_{n-2} - 1000) - 1000 = (1.005)^2 \cdot B_{n-2} - 1000(1 + 1.005) = (1.005)^2 \cdot (1.005 \cdot B_{n-3} - 1000) - 1000(1+1.005) = (1.005)^3 \cdot B_{n-3} - 1000(1 + 1.005 + (1.005)^2) = \cdots = (1.005)^n B_0 - 1000(1+1.005+(1.005)^2+\cdots+(1.005)^{n-1}) = (1.005)^n \cdot 180000 - 1000\left(\frac{(1.005)^n-1}{1.005-1}\right)$. Solving this equation for $B_n = 0$ gives $n \approx 461.667$ months, so the loan is paid off after 462 months.



8.3.80 It appears that the loan is paid off after about 38 months. Let B_n be the loan balance after n months. Then $B_0 = 20000$ and $B_n = 1.0075 \cdot B_{n-1} - 60$. Then $B_n = 1.0075 \cdot B_{n-1} - 600 = 1.0075(1.0075 \cdot B_{n-2} - 600) - 600 = (1.0075)^2 \cdot B_{n-2} - 600(1 + 1.0075) = (1.0075)^2(1.0075 \cdot B_{n-3} - 600) - 600(1 + 1.0075) = (1.0075)^3 \cdot B_{n-3} - 600(1 + 1.0075 + (1.0075)^2) = \cdots = (1.0075)^n B_0 - 600(1 + 1.0075 + (1.0075)^2 + \cdots + (1.0075)^{n-1}) = (1.0075)^n \cdot 20000 - 600\left(\frac{(1.0075)^n-1}{1.0075-1}\right)$. Solving this equation for $B_n = 0$ gives $n \approx 38.501$ months, so the loan is paid off after 39 months.



8.3.81 $F_n = (1.015)F_{n-1} - 120 = (1.015)((1.015)F_{n-2} - 120) - 120 = (1.015)((1.015)((1.015)F_{n-3} - 120) - 120) - 120 = \cdots = (1.015)^n(4000) - 120(1 + (1.015) + (1.015)^2 + \cdots + (1.015)^{n-1})$. This is equal to

$$(1.015)^n(4000) - 120\left(\frac{(1.015)^n - 1}{1.015 - 1}\right) = (-4000)(1.015)^n + 8000.$$

The long term population of the fish is 0.

8.3.82 Let A_n be the amount of antibiotic in your blood after n 6-hour periods. Then $A_0 = 200, A_n = 0.5A_{n-1} + 200$. We have $A_n = .5A_{n-1} + 200 = .5(.5A_{n-2} + 200) + 200 = .5(.5(.5A_{n-3} + 200) + 200) + 200 = \cdots = .5^n(200) + 200(1 + .5 + .5^2 + \cdots + .5^{n-1})$. This is equal to

$$.5^n(200) + 200\left(\frac{.5^n - 1}{.5 - 1}\right) = (.5^n)(200 - 400) + 400 = (-200)(.5^n) + 400.$$

The limit of this expression as $n \to \infty$ is 400, so the steady-state amount of antibiotic in your blood is 400 mg.

8.3.83 Under the one-child policy, each couple will have one child. Under the one-son policy, we compute the expected number of children as follows: with probability 1/2 the first child will be a son; with probability $(1/2)^2$, the first child will be a daughter and the second child will be a son; in general, with probability $(1/2)^n$, the first $n-1$ children will be girls and the n^{th} a boy. Thus the expected number of children is the sum $\displaystyle\sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i$. To evaluate this series, use the following "trick": Let $f(x) = \displaystyle\sum_{i=1}^{\infty} ix^i$. Then $f(x) + \displaystyle\sum_{i=1}^{\infty} x^i = \sum_{i=1}^{\infty}(i+1)x^i$. Now, let

$$g(x) = \sum_{i=1}^{\infty} x^{i+1} = -1 - x + \sum_{i=0}^{\infty} x^i = -1 - x + \frac{1}{1-x}$$

and

$$g'(x) = f(x) + \sum_{i=1}^{\infty} x^i = f(x) - 1 + \sum_{i=0}^{\infty} x^i = f(x) - 1 + \frac{1}{1-x}.$$

Evaluate $g'(x) = -1 - \frac{1}{(1-x)^2}$; then

$$f(x) = 1 - \frac{1}{1-x} - 1 - \frac{1}{(1-x)^2} = \frac{-1 + x + 1}{(1-x)^2} = \frac{x}{(1-x)^2}$$

Finally, evaluate at $x = \frac{1}{2}$ to get $f\left(\frac{1}{2}\right) = \sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i = \frac{1/2}{(1-1/2)^2} = 2$. There will thus be twice as many children under the one-son policy as under the one-child policy.

8.3.84 Let L_n be the amount of light transmitted through the window the n^{th} time the beam hits the second pane. Then the amount of light that was available before the beam went through the pane was $\frac{L_n}{1-p}$, so $\frac{pL_n}{1-p}$ is reflected back to the first pane, and $\frac{p^2L_n}{1-p}$ is then reflected back to the second pane. Of that, a fraction equal to $1-p$ is transmitted through the window. Thus

$$L_{n+1} = (1-p)\frac{p^2L_n}{1-p} = p^2L_n.$$

The amount of light transmitted through the window the first time is $(1-p)^2$. Thus the total amount is

$$\sum_{i=0}^{\infty} p^{2n}(1-p)^2 = \frac{(1-p)^2}{1-p^2} = \frac{1-p}{1+p}.$$

8.3.85 Ignoring the initial drop for the moment, the height after the n^{th} bounce is $10p^n$, so the total time spent in that bounce is $2 \cdot \sqrt{2 \cdot 10p^n/g}$ seconds. The total time before the ball comes to rest (now including the time for the initial drop) is then $\sqrt{20/g} + \sum_{i=1}^{\infty} 2 \cdot \sqrt{2 \cdot 10p^n/g} = \sqrt{\frac{20}{g}} + 2\sqrt{\frac{20}{g}} \sum_{i=1}^{\infty}(\sqrt{p})^n = \sqrt{\frac{20}{g}} + 2\sqrt{\frac{20}{g}}\frac{\sqrt{p}}{1-\sqrt{p}} = \sqrt{\frac{20}{g}}\left(1 + \frac{2\sqrt{p}}{1-\sqrt{p}}\right) = \sqrt{\frac{20}{g}}\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right)$ seconds.

8.3.86

a. The fraction of available wealth spent each month is $1-p$, so the amount spent in the n^{th} month is $W(1-p)^n$. The total amount spent is then $\sum_{n=1}^{\infty} W(1-p)^n = \frac{W(1-p)}{1-(1-p)} = W\left(\frac{1-p}{p}\right)$ dollars.

b. As $p \to 1$, the total amount spent approaches 0. This makes sense, because in the limit, if everyone saves all of the money, none will be spent. As $p \to 0$, the total amount spent gets larger and larger. This also makes sense, because almost all of the available money is being respent each month.

8.3.87

a. I_{n+1} is obtained by I_n by dividing each edge into three equal parts, removing the middle part, and adding two parts equal to it. Thus 3 equal parts turn into 4, so $L_{n+1} = \frac{4}{3}L_n$. This is a geometric sequence with a ratio greater than 1, so the n^{th} term grows without bound.

b. As the result of part (a), I_n has $3 \cdot 4^n$ sides of length $\frac{1}{3^n}$; each of those sides turns into an added triangle in I_{n+1} of side length 3^{-n-1}. Thus the added area in I_{n+1} consists of $3 \cdot 4^n$ equilateral triangles with side 3^{-n-1}. The area of an equilateral triangle with side x is $\dfrac{x^2\sqrt{3}}{4}$. Thus $A_{n+1} = A_n + 3 \cdot 4^n \cdot \frac{3^{-2n-2}\sqrt{3}}{4} = A_n + \frac{\sqrt{3}}{12} \cdot \left(\frac{4}{9}\right)^n$, and $A_0 = \frac{\sqrt{3}}{4}$. Thus $A_{n+1} = A_0 + \sum_{i=0}^{n} \frac{\sqrt{3}}{12} \cdot \left(\frac{4}{9}\right)^i$, so that

$$A_\infty = A_0 + \frac{\sqrt{3}}{12}\sum_{i=0}^{\infty}\left(\frac{4}{9}\right)^i = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\frac{1}{1-4/9} = \frac{\sqrt{3}}{4}\left(1+\frac{3}{5}\right) = \frac{2}{5}\sqrt{3}.$$

8.3.88

a. $5\sum_{i=1}^{\infty} 10^{-k} = 5\sum_{i=1}^{\infty}\left(\frac{1}{10}\right)^k = 5\left(\frac{1/10}{9/10}\right) = \frac{5}{9}$.

b. $54\sum_{i=1}^{\infty} 10^{-2k} = 54\sum_{i=1}^{\infty}\left(\frac{1}{100}\right)^k = 54\left(\frac{1/100}{99/100}\right) = \frac{54}{99}$.

c. Suppose $x = 0.n_1n_2\ldots n_pn_1n_2\ldots$. Then we can write this decimal as $n_1n_2\ldots n_p\sum_{i=1}^{\infty}10^{-ip} = n_1n_2\ldots n_p\sum_{i=1}^{\infty}\left(\frac{1}{10^p}\right)^i = n_1n_2\ldots n_p\frac{1/10^p}{(10^p-1)/10^p} = \frac{n_1n_2\ldots n_p}{999\ldots 9}$, where here $n_1n_2\ldots n_p$ does not mean multiplication but rather the digits in a decimal number, and where there are p 9's in the denominator.

d. According to part (c), $0.12345678912345678912\ldots = \frac{123456789}{999999999}$

e. Again using part (c), $0.\bar{9} = \frac{9}{9} = 1$.

8.3.89 $|S - S_n| = \left|\sum_{i=n}^{\infty} r^k\right| = \left|\frac{r^n}{1-r}\right|$ because the latter sum is simply a geometric series with first term r^n and ratio r.

8.3.90

a. Solve $\frac{0.6^n}{0.4} < 10^{-6}$ for n to get $n = 29$.

b. Solve $\frac{0.15^n}{0.85} < 10^{-6}$ for n to get $n = 8$.

8.3.91

a. Solve $\left|\frac{(-0.8)^n}{1.8}\right| = \frac{0.8^n}{1.8} < 10^{-6}$ for n to get $n = 60$.

b. Solve $\frac{0.2^n}{0.8} < 10^{-6}$ for n to get $n = 9$.

8.3.92

a. Solve $\frac{0.72^n}{0.28} < 10^{-6}$ for n to get $n = 46$.

b. Solve $\left|\frac{(-0.25)^n}{1.25}\right| = \frac{0.25^n}{1.25} < 10^{-6}$ for n to get $n = 10$.

8.3.93

a. Solve $\frac{1/\pi^n}{1-1/\pi} < 10^{-6}$ for n to get $n = 13$.

b. Solve $\frac{1/e^n}{1-1/e} < 10^{-6}$ for n to get $n = 15$.

8.3.94

a. $f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$; because f is represented by a geometric series, $f(x)$ exists only for $|x| < 1$. Then $f(0) = 1$, $f(0.2) = \frac{1}{0.8} = 1.25$, $f(0.5) = \frac{1}{1-0.5} = 2$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

8.3.95

a. $f(x) = \sum_{k=0}^{\infty}(-1)^k x^k = \frac{1}{1+x}$; because f is a geometric series, $f(x)$ exists only when the ratio, $-x$, is such that $|-x| = |x| < 1$. Then $f(0) = 1$, $f(0.2) = \frac{1}{1.2} = \frac{5}{6}$, $f(0.5) = \frac{1}{1+.05} = \frac{2}{3}$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

8.3.96

a. $f(x) = \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2}$. f is a geometric series, so $f(x)$ is defined only when the ratio, x^2, is less than 1, which means $|x| < 1$. Then $f(0) = 1$, $f(0.2) = \frac{1}{1-.04} = \frac{25}{24}$, $f(0.5) = \frac{1}{1-0.25} = \frac{4}{3}$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

8.3.97 $f(x)$ is a geometric series with ratio $\frac{1}{1+x}$; thus $f(x)$ converges when $\left|\frac{1}{1+x}\right| < 1$. For $x > -1$, $\left|\dfrac{1}{1+x}\right| = \dfrac{1}{1+x}$ and $\dfrac{1}{1+x} < 1$ when $1 < 1 + x$, $x > 0$. For $x < -1$, $\left|\dfrac{1}{1+x}\right| = \dfrac{1}{-1-x}$, and this is less than 1 when $1 < -1 - x$, i.e. $x < -2$. So $f(x)$ converges for $x > 0$ and for $x < -2$. When $f(x)$ converges, its value is $\frac{1}{1-\frac{1}{1+x}} = \frac{1+x}{x}$, so $f(x) = 3$ when $1 + x = 3x$, $x = \frac{1}{2}$.

8.3.98

a. Clearly for $k < n$, h_k is a leg of a right triangle whose hypotenuse is r_k and whose other leg is formed where the vertical line (in the picture) meets a diameter of the next smaller sphere; thus the other leg of the triangle is r_{k+1}. The Pythagorean theorem then implies that $h_k^2 = r_k^2 - r_{k+1}^2$.

b. The height is $H_n = \sum_{i=1}^{n} h_i = r_n + \sum_{i=1}^{n-1} \sqrt{r_i^2 - r_{i+1}^2}$ by part (a).

c. From part (b), because $r_i = a^{i-1}$,

$$\begin{aligned}
H_n = r_n + \sum_{i=1}^{n-1} \sqrt{r_i^2 - r_{i+1}^2} &= a^{n-1} + \sum_{i=1}^{n-1} \sqrt{a^{2i-2} - a^{2i}} \\
&= a^{n-1} + \sum_{i=1}^{n-1} a^{i-1}\sqrt{1-a^2} = a^{n-1} + \sqrt{1-a^2}\sum_{i=1}^{n-1} a^{i-1} \\
&= a^{n-1} + \sqrt{1-a^2}\left(\frac{1-a^{n-1}}{1-a}\right)
\end{aligned}$$

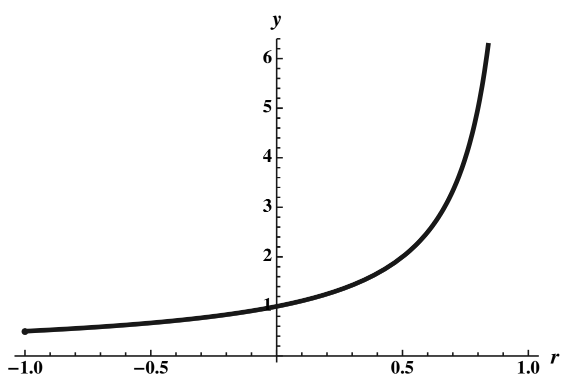
d. $\lim_{n\to\infty} H_n = \lim_{n\to\infty} a^{n-1} + \sqrt{1-a^2} \lim_{n\to\infty} \frac{1-a^{n-1}}{1-a} = 0 + \sqrt{1-a^2}\left(\frac{1}{1-a}\right) = \sqrt{\frac{1-a^2}{(1-a)(1-a)}} = \sqrt{\frac{1+a}{1-a}}$.

8.3.99

a. Using Theorem 8.7 in each case except for $r = 0$ gives

r	$f(r)$
−0.9	0.526
−0.7	0.588
−0.5	0.667
−0.2	0.833
0	1
0.2	1.250
0.5	2
0.7	3.333
0.9	10

b. A plot of f is



c. For $-1 < r < 1$ we have $f(r) = \frac{1}{1-r}$, so that

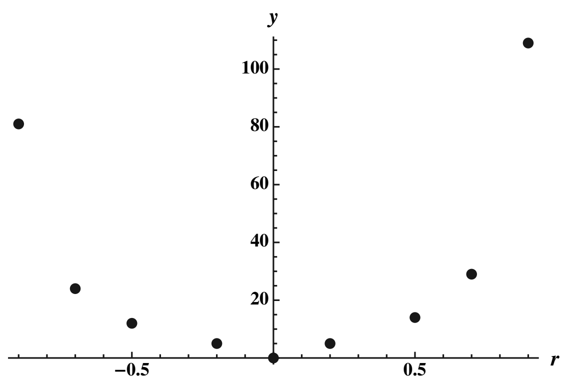
$$\lim_{r\to -1^+} f(r) = \lim_{r\to -1^+} \frac{1}{1-r} = \frac{1}{2}, \qquad \lim_{r\to 1^-} f(r) = \lim_{r\to 1^-} \frac{1}{1-r} = \infty.$$

8.3.100

a. In each case (except for $r = 0$ where $N(r)$ is clearly 0), compute $|S - S_n|$ for various values of n gives the following results:

r	$N(r)$	$\lvert S - S_{N(r)-1}\rvert$	$\lvert S - S_{N(r)}\rvert$
-0.9	81	1.0×10^{-4}	9.3×10^{-5}
-0.7	24	1.1×10^{-4}	7.9×10^{-5}
-0.5	12	1.6×10^{-4}	8.1×10^{-5}
-0.2	5	2.7×10^{-4}	5.3×10^{-5}
0	0	—	0
0.2	5	4.0×10^{-4}	8.0×10^{-5}
0.5	14	1.2×10^{-4}	6.1×10^{-5}
0.7	29	1.1×10^{-4}	7.5×10^{-5}
0.9	109	1.0×10^{-4}	9.3×10^{-5}

b. A plot of r versus $N(r)$ for these values of r is



c. The rate of convergence is faster for r closer to 0, since $N(r)$ is smaller. The reason for this is that r^k gets smaller faster as k increases when $|r|$ is closer to zero than when it is closer to 1.

8.4 The Divergence and Integral Tests

8.4.1 If the sequence of terms has limit 1, then the corresponding series diverges. It is necessary (but not sufficient) that the sequence of terms has limit 0 in order for the corresponding series to be convergent.

8.4.2 No. For example, the harmonic serkes $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges although $\frac{1}{k} \to 0$ as $k \to \infty$.

8.4.3 Yes. Either the series and the integral both converge, or both diverge, if the terms are positive and decreasing.

8.4.4 It converges for $p > 1$, and diverges for all other values of p.

8.4.5 For the same values of p as in the previous problem – it converges for $p > 1$, and diverges for all other values of p.

8.4.6 Let S_n be the partial sums. Then $S_{n+1} - S_n = a_{n+1} > 0$ because $a_{n+1} > 0$. Thus the sequence of partial sums is increasing.

8.4.7 The remainder of an infinite series is the error in approximating a convergent infinite series by a finite number of terms.

8.4.8 Yes. Suppose $\sum a_k$ converges to S, and let the sequence of partial sums be $\{S_n\}$. Then for any $\epsilon > 0$ there is some N such that for any $n > N$, $|S - S_n| < \epsilon$. But $|S - S_n|$ is simply the remainder R_n when the series is approximated to n terms. Thus $R_n \to 0$ as $n \to \infty$.

8.4.9 $a_k = \frac{k}{2k+1}$ and $\lim_{k\to\infty} a_k = \frac{1}{2}$, so the series diverges.

8.4.10 $a_k = \frac{k}{k^2+1}$ and $\lim_{k\to\infty} a_k = 0$, so the divergence test is inconclusive.

8.4.11 $a_k = \frac{k}{\ln k}$ and $\lim_{k\to\infty} a_k = \infty$, so the series diverges.

8.4.12 $a_k = \frac{k^2}{2^k}$ and $\lim_{k\to\infty} a_k = 0$, so the divergence test is inconclusive.

8.4.13 $a_k = \frac{1}{1000+k}$ and $\lim_{k\to\infty} a_k = 0$, so the divergence test is inconclusive.

8.4.14 $a_k = \frac{k^3}{k^3+1}$ and $\lim_{k\to\infty} a_k = 1$, so the series diverges.

8.4.15 $a_k = \frac{\sqrt{k}}{\ln^{10} k}$ and $\lim_{k\to\infty} a_k = \infty$, so the series diverges.

8.4.16 $a_k = \frac{\sqrt{k^2+1}}{k}$ and $\lim_{k\to\infty} a_k = 1$, so the series diverges.

8.4.17 $a_k = k^{1/k}$. In order to compute $\lim_{k\to\infty} a_k$, we let $y_k = \ln a_k = \frac{\ln k}{k}$. By Theorem 9.6, (or by L'Hôpital's rule), $\lim_{k\to\infty} y_k = 0$, so $\lim_{k\to\infty} a_k = e^0 = 1$. The given series thus diverges.

8.4.18 By Theorem 9.6 $k^3 \ll k!$, so $\lim_{k\to\infty} \frac{k^3}{k!} = 0$. The divergence test is inconclusive.

8.4.19 Clearly $\frac{1}{e^x} = e^{-x}$ is continuous, positive, and decreasing for $x \geq 2$ (in fact, for all x), so the integral test applies. Because

$$\int_2^\infty e^{-x}\,dx = \lim_{c\to\infty} \int_2^c e^{-x}\,dx = \lim_{c\to\infty} (-e^{-x})\Big|_2^c = \lim_{c\to\infty} (e^{-2} - e^{-c}) = e^{-2},$$

the Integral Test tells us that the original series converges as well.

8.4.20 Let $f(x) = \frac{x}{\sqrt{x^2+4}}$. $f(x)$ is continuous for $x \geq 1$. Note that $f'(x) = \frac{4}{(\sqrt{x^2+4})^3} > 0$. Thus f is increasing, and the conditions of the Integral Test aren't satisfied. The given series diverges by the Divergence Test.

8.4.21 Let $f(x) = x \cdot e^{-2x^2}$. This function is continuous for $x \geq 1$. Its derivative is $e^{-2x^2}(1 - 4x^2) < 0$ for $x \geq 1$, so $f(x)$ is decreasing. Because $\int_1^\infty x \cdot e^{-2x^2}\,dx = \frac{1}{4e^2}$, the series converges.

8.4.22 Let $f(x) = \frac{1}{\sqrt[3]{x+10}}$. $f(x)$ is obviously continuous and decreasing for $x \geq 1$. Because $\int_1^\infty \frac{1}{\sqrt[3]{x+10}}\,dx = \infty$, the series diverges.

8.4.23 Let $f(x) = \frac{1}{\sqrt{x+8}}$. $f(x)$ is obviously continuous and decreasing for $x \geq 1$. Because $\int_1^\infty \frac{1}{\sqrt{x+8}}\,dx = \infty$, the series diverges.

8.4.24 Let $f(x) = \frac{1}{x(\ln x)^2}$. $f(x)$ is continuous and decreasing for $x \geq 2$. Because $\int_2^\infty f(x)\,dx = \frac{1}{\ln 2}$ the series converges.

8.4.25 Let $f(x) = \frac{x}{e^x}$. $f(x)$ is clearly continuous for $x > 1$, and its derivative, $f'(x) = \frac{e^x - xe^x}{e^{2x}} = (1-x)\frac{e^x}{e^{2x}}$, is negative for $x > 1$ so that $f(x)$ is decreasing. Because $\int_1^\infty f(x)\,dx = 2e^{-1}$, the series converges.

8.4.26 Let $f(x) = \frac{1}{x \cdot \ln x \cdot \ln \ln x}$. $f(x)$ is continuous and decreasing for $x > 3$, and $\int_3^\infty \frac{1}{x \cdot \ln x \cdot \ln \ln x}\,dx = \infty$. The given series therefore diverges.

8.4.27 The integral test does not apply, because the sequence of terms is not decreasing.

8.4.28 $f(x) = \frac{x}{(x^2+1)^3}$ is decreasing and continuous, and $\int_1^\infty \frac{x}{(x^2+1)^3}\,dx = \frac{1}{16}$. Thus, the given series converges.

8.4.29 This is a p-series with $p = 10$, so this series converges.

8.4.30 $\sum_{k=2}^\infty \frac{k^e}{k^\pi} = \sum_{k=2}^\infty \frac{1}{k^{\pi-e}}$. Note that $\pi - e \approx 3.1416 - 2.71828 < 1$, so this series diverges.

8.4.31 $\sum_{k=3}^\infty \frac{1}{(k-2)^4} = \sum_{k=1}^\infty \frac{1}{k^4}$, which is a p-series with $p = 4$, thus convergent.

8.4.32 $\sum_{k=1}^\infty 2k^{-3/2} = 2\sum_{k=1}^\infty \frac{1}{k^{3/2}}$ is a p-series with $p = 3/2$, thus convergent.

8.4.33 $\sum_{k=1}^\infty \frac{1}{\sqrt[3]{k}} = \sum_{k=1}^\infty \frac{1}{k^{1/3}}$ is a p-series with $p = 1/3$, thus divergent.

8.4.34 $\sum_{k=1}^\infty \frac{1}{\sqrt[3]{27k^2}} = \frac{1}{3}\sum_{k=1}^\infty \frac{1}{k^{2/3}}$ is a p-series with $p = 2/3$, thus divergent.

8.4.35

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^6}\,dx = \frac{1}{5n^5}$.

b. We solve $\frac{1}{5n^5} < 10^{-3}$ to get $n = 3$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^6}\,dx = S_n + \frac{1}{5(n+1)^5}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^6}\,dx = S_n + \frac{1}{5n^5}$.

d. $S_{10} \approx 1.017341512$, so $L_{10} \approx 1.017341512 + \frac{1}{5\cdot 11^5} \approx 1.017342754$, and $U_{10} \approx 1.017341512 + \frac{1}{5\cdot 10^5} \approx 1.017343512$.

8.4.36

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^8}\,dx = \frac{1}{7n^7}$.

b. We solve $\frac{1}{7n^7} < 10^{-3}$ to obtain $n = 3$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^8}\,dx = S_n + \frac{1}{7(n+1)^7}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^8}\,dx = S_n + \frac{1}{7n^7}$.

d. $S_{10} \approx 1.004077346$, so $L_{10} \approx 1.004077346 + \frac{1}{7\cdot 11^7} \approx 1.004077353$, and $U_{10} \approx 1.004077346 + \frac{1}{7\cdot 10^7} \approx 1.004077360$.

8.4.37

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{3^x}\,dx = \frac{1}{3^n \ln 3}$.

b. We solve $\frac{1}{3^n \ln 3} < 10^{-3}$ to obtain $n = 7$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{3^x}\,dx = S_n + \frac{1}{3^{n+1}\ln 3}$, and $U_n = S_n + \int_n^\infty \frac{1}{3^x}\,dx = S_n + \frac{1}{3^n \ln 3}$.

d. $S_{10} \approx 0.4999915325$, so $L_{10} \approx 0.4999915325 + \frac{1}{3^{11}\ln 3} \approx 0.4999966708$, and $U_{10} \approx 0.4999915325 + \frac{1}{3^{10}\ln 3} \approx 0.5000069475$.

8.4.38

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x\ln^2 x}\,dx = \frac{1}{\ln n}$.

b. We solve $\frac{1}{\ln n} < 10^{-3}$ to get $n = e^{1000} \approx 10^{434}$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x\ln^2 x}\,dx = S_n + \frac{1}{\ln(n+1)}$, and $U_n = S_n + \int_n^\infty \frac{1}{x\ln^2 x}\,dx = S_n + \frac{1}{\ln n}$.

d. $S_{11} = \sum_{k=2}^{11} \frac{1}{k\ln^2 k} \approx 1.700396385$, so $L_{11} \approx 1.700396385 + \frac{1}{\ln 12} \approx 2.102825989$, and $U_{11} \approx 1.700396385 + \frac{1}{\ln 11} \approx 2.117428776$.

8.4.39

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^{3/2}}\,dx = 2n^{-1/2}$.

b. We solve $2n^{-1/2} < 10^{-3}$ to get $n > 4 \times 10^6$, so let $n = 4 \times 10^6 + 1$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^{3/2}}\,dx = S_n + 2(n+1)^{-1/2}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^{3/2}}\,dx = S_n + 2n^{-1/2}$.

d. $S_{10} = \sum_{k=1}^{10} \frac{1}{k^{3/2}} \approx 1.995336493$, so $L_{10} \approx 1.995336493 + 2 \cdot 11^{-1/2} \approx 2.598359182$, and $U_{10} \approx 1.995336493 + 2 \cdot 10^{-1/2} \approx 2.627792025$.

8.4.40

a. The remainder R_n is bounded by $\int_n^\infty e^{-x}\,dx = e^{-n}$.

b. We solve $e^{-n} < 10^{-3}$ to get $n = 7$.

c. $L_n = S_n + \int_{n+1}^\infty e^{-x}\,dx = S_n + e^{-(n+1)}$, and $U_n = S_n + \int_n^\infty e^{-x}\,dx = S_n + e^{-n}$.

d. $S_{10} = \sum_{k=1}^{10} e^{-k} \approx 0.5819502852$, so $L_{10} \approx 0.5819502852 + e^{-11} \approx 0.5819669869$, and $U_{10} \approx 0.5819502852 + e^{-10} \approx 0.5819956851$.

8.4.41

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$.

b. We solve $\frac{1}{2n^2} < 10^{-3}$ to get $n = 23$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^3}\,dx = S_n + \frac{1}{2(n+1)^2}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^3}\,dx = S_n + \frac{1}{2n^2}$.

d. $S_{10} \approx 1.197531986$, so $L_{10} \approx 1.197531986 + \frac{1}{2 \cdot 11^2} \approx 1.201664217$, and $U_{10} \approx 1.197531986 + \frac{1}{2 \cdot 10^2} \approx 1.202531986$.

8.4.42

a. The remainder R_n is bounded by $\int_n^\infty xe^{-x^2}\,dx = \frac{1}{2e^{n^2}}$.

b. We solve $\frac{1}{2e^{n^2}} < 10^{-3}$ to get $n = 3$.

c. $L_n = S_n + \int_{n+1}^\infty xe^{-x^2}\,dx = S_n + \frac{1}{2e^{(n+1)^2}}$, and $U_n = S_n + \int_n^\infty xe^{-x^2}\,dx = S_n + \frac{1}{2e^{n^2}}$.

d. $S_{10} \approx 0.4048813986$, so $L_{10} \approx 0.4048813986 + \frac{1}{2e^{11^2}} \approx 0.4048813986$, and $U_{10} \approx 0.4048813986 + \frac{1}{2e^{10^2}} \approx 0.4048813986$.

8.4.43 This is a geometric series with $a = \frac{1}{3}$ and $r = \frac{1}{12}$, so $\sum_{k=1}^\infty \frac{4}{12^k} = \frac{1/3}{1-1/12} = \frac{1/3}{11/12} = \frac{4}{11}$.

8.4.44 This is a geometric series with $a = 3/e^2$ and $r = 1/e$, so $\sum_{k=2}^\infty 3e^{-k} = \frac{3/e^2}{1-(1/e)} = \frac{3/e^2}{(e-1)/e} = \frac{3}{e(e-1)}$.

8.4.45 $$\sum_{k=0}^\infty \left(3\left(\frac{2}{5}\right)^k - 2\left(\frac{5}{7}\right)^k\right) = 3\sum_{k=0}^\infty \left(\frac{2}{5}\right)^k - 2\sum_{k=0}^\infty \left(\frac{5}{7}\right)^k = 3\left(\frac{1}{3/5}\right) - 2\left(\frac{1}{2/7}\right) = 5 - 7 = -2.$$

8.4.46 $$\sum_{k=1}^\infty \left(2\left(\frac{3}{5}\right)^k + 3\left(\frac{4}{9}\right)^k\right) = 2\sum_{k=1}^\infty \left(\frac{3}{5}\right)^k + 3\sum_{k=1}^\infty \left(\frac{4}{9}\right)^k = 2\left(\frac{3/5}{2/5}\right) + 3\left(\frac{4/9}{5/9}\right) = 3 + \frac{12}{5} = \frac{27}{5}.$$

8.4.47 $$\sum_{k=1}^\infty \left(\frac{1}{3}\left(\frac{5}{6}\right)^k + \frac{3}{5}\left(\frac{7}{9}\right)^k\right) = \frac{1}{3}\sum_{k=1}^\infty \left(\frac{5}{6}\right)^k + \frac{3}{5}\sum_{k=1}^\infty \left(\frac{7}{9}\right)^k = \frac{1}{3}\left(\frac{5/6}{1/6}\right) + \frac{3}{5}\left(\frac{7/9}{2/9}\right) = \frac{5}{3} + \frac{21}{10} = \frac{113}{30}.$$

8.4.48 $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{2}(0.2)^k+\frac{3}{2}(0.8)^k\right)=\frac{1}{2}\sum_{k=0}^{\infty}(0.2)^k+\frac{3}{2}\sum_{k=0}^{\infty}(0.8)^k=\frac{1}{2}\left(\frac{1}{0.8}\right)+\frac{3}{2}\left(\frac{1}{0.2}\right)=\frac{5}{8}+\frac{15}{2}=\frac{65}{8}.$

8.4.49 $\displaystyle\sum_{k=1}^{\infty}\left(\left(\frac{1}{6}\right)^k+\left(\frac{1}{3}\right)^{k-1}\right)=\sum_{k=1}^{\infty}\left(\frac{1}{6}\right)^k+\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^{k-1}=\frac{1/6}{5/6}+\frac{1}{2/3}=\frac{17}{10}.$

8.4.50 $\displaystyle\sum_{k=0}^{\infty}\frac{2-3^k}{6^k}=\sum_{k=0}^{\infty}\left(\frac{2}{6^k}-\frac{3^k}{6^k}\right)=2\sum_{k=0}^{\infty}\left(\frac{1}{6}\right)^k-\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k=2\left(\frac{1}{5/6}\right)-\frac{1}{1/2}=\frac{2}{5}.$

8.4.51

a. True. The two series differ by a finite amount ($\sum_{k=1}^{9}a_k$), so if one converges, so does the other.

b. True. The same argument applies as in part (a).

c. False. If $\sum a_k$ converges, then $a_k\to 0$ as $k\to\infty$, so that $a_k+0.0001\to 0.0001$ as $k\to\infty$, so that $\sum(a_k+0.0001)$ cannot converge.

d. False. Suppose $p=-1.0001$. Then $\sum p^k$ diverges but $p+0.001=-0.9991$ so that $\sum(p+.0001)^k$ converges.

e. False. Let $p=1.0005$; then $-p+.001=-(p-.001)=-.9995$, so that $\sum k^{-p}$ converges (p-series) but $\sum k^{-p+.001}$ diverges.

f. False. Let $a_k=\frac{1}{k}$, the harmonic series.

8.4.52 Diverges by the Divergence Test because $\displaystyle\lim_{k\to\infty}a_k=\lim_{k\to\infty}\sqrt{\frac{k+1}{k}}=1\neq 0.$

8.4.53 Converges by the Integral Test because $\displaystyle\int_1^{\infty}\frac{1}{(3x+1)(3x+4)}\,dx=\int_1^{\infty}\frac{1}{3(3x+1)}-\frac{1}{3(3x+4)}\,dx=\lim_{b\to\infty}\int_1^b\left(\frac{1}{3(3x+1)}-\frac{1}{3(3x+4)}\right)dx=\lim_{b\to\infty}\frac{1}{9}\left(\ln\left(\frac{3x+1}{3x+4}\right)\right)\Bigg|_1^b=\lim_{b\to\infty}=-\frac{1}{9}\cdot\ln(4/7)\approx 0.06217<\infty.$

Alternatively, this is a telescoping series with nth partial sum equal to $S_n=\frac{1}{3}\left(\frac{1}{4}-\frac{1}{3n+4}\right)$ which converges to $\frac{1}{12}$.

8.4.54 Converges by the Integral Test because $\displaystyle\int_0^{\infty}\frac{10}{x^2+9}\,dx=\frac{10}{3}\lim_{b\to\infty}\left(\tan^{-1}(x/3)\Big|_0^b\right)=\frac{10}{3}\frac{\pi}{2}\approx 5.236<\infty.$

8.4.55 Diverges by the Divergence Test because $\displaystyle\lim_{k\to\infty}a_k=\lim_{k\to\infty}\frac{k}{\sqrt{k^2+1}}=1\neq 0.$

8.4.56 Converges because it is the sum of two geometric series. In fact, $\sum_{k=1}^{\infty}\frac{2^k+3^k}{4^k}=\sum_{k=1}^{\infty}(2/4)^k+\sum_{k=1}^{\infty}(3/4)^k=\frac{1/2}{1-(1/2)}+\frac{3/4}{1-(3/4)}=1+3=4.$

8.4.57 Converges by the Integral Test because $\displaystyle\int_2^{\infty}\frac{4}{x\ln^2 x}\,dx=\lim_{b\to\infty}\left(\frac{-4}{\ln x}\Bigg|_2^b\right)=\frac{4}{\ln 2}<\infty.$

8.4.58

a. In order for the series to converge, the integral $\int_2^{\infty}\frac{1}{x(\ln x)^p}\,dx$ must exist. But

$$\int\frac{1}{x(\ln x)^p}\,dx=\frac{1}{1-p}(\ln x)^{1-p},$$

so in order for this improper integral to exist, we must have that $1-p<0$ or $p>1$.

b. The series converges faster for $p = 3$ because the terms of the series get smaller faster.

8.4.59

a. Note that $\int \frac{1}{x \ln x (\ln \ln x)^p}\, dx = \frac{1}{1-p}(\ln \ln x)^{1-p}$, and thus the improper integral with bounds n and ∞ exists only if $p > 1$ because $\ln \ln x > 0$ for $x > e$. So this series converges for $p > 1$.

b. For large values of z, clearly $\sqrt{z} > \ln z$, so that $z > (\ln z)^2$. Write $z = \ln x$; then for large x, $\ln x > (\ln \ln x)^2$; multiplying both sides by $x \ln x$ we have that $x \ln^2 x > x \ln x (\ln \ln x)^2$, so that the first series converges faster because the terms get smaller faster.

8.4.60

a. $\sum \frac{1}{k^{2.5}}$.

b. $\sum \frac{1}{k^{0.75}}$.

c. $\sum \frac{1}{k^{3/2}}$.

8.4.61 Let $S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$. Then this looks like a left Riemann sum for the function $y = \frac{1}{\sqrt{x}}$ on $[1, n+1]$. Because each rectangle lies above the curve itself, we see that S_n is bounded below by the integral of $\frac{1}{\sqrt{x}}$ on $[1, n+1]$. Now,

$$\int_1^{n+1} \frac{1}{\sqrt{x}}\, dx = \int_1^{n+1} x^{-1/2}\, dx = 2\sqrt{x}\Big|_1^{n+1} = 2\sqrt{n+1} - 2.$$

This integral diverges as $n \to \infty$, so the series does as well by the bound above.

8.4.62 $\sum_{k=1}^{\infty}(a_k \pm b_k) = \lim_{n\to\infty} \sum_{k=1}^{n}(a_k \pm b_k) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \pm \lim_{n\to\infty} \sum_{k=1}^{n} b_k = A \pm B$.

8.4.63 $\sum_{k=1}^{\infty} ca_k = \lim\limits_{n\to\infty} \sum_{k=1}^{n} ca_k = \lim\limits_{n\to\infty} c\sum_{k=1}^{n} a_k = c \lim\limits_{n\to\infty} \sum_{k=1}^{n} a_k$, so that one sum diverges if and only if the other one does.

8.4.64 $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges by the Integral Test, because $\int_2^{\infty} \frac{1}{x \ln x} = \lim_{b\to\infty}\left(\ln \ln x\big|_2^b\right) = \infty$.

8.4.65 To approximate the sequence for $\zeta(m)$, note that the remainder R_n after n terms is bounded by

$$\int_n^{\infty} \frac{1}{x^m}\, dx = \frac{1}{m-1} n^{1-m}.$$

For $m = 3$, if we wish to approximate the value to within 10^{-3}, we must solve $\dfrac{1}{2}n^{-2} < 10^{-3}$, so that $n = 23$, and $\displaystyle\sum_{k=1}^{23} \frac{1}{k^3} \approx 1.201151926$. The true value is ≈ 1.202056903.

For $m = 5$, if we wish to approximate the value to within 10^{-3}, we must solve $\dfrac{1}{4}n^{-4} < 10^{-3}$, so that $n = 4$, and $\displaystyle\sum_{k=1}^{4} \frac{1}{k^5} \approx 1.036341789$. The true value is ≈ 1.036927755.

8.4.66

a. Starting with $\cot^2 x < \dfrac{1}{x^2} < 1 + \cot^2 x$, substitute $k\theta$ for x:

$$\begin{aligned}
\cot^2(k\theta) &< \frac{1}{k^2\theta^2} < 1 + \cot^2(k\theta),\\
\sum_{k=1}^{n}\cot^2(k\theta) &< \sum_{k=1}^{n}\frac{1}{k^2\theta^2} < \sum_{k=1}^{n}(1+\cot^2(k\theta)),\\
\sum_{k=1}^{n}\cot^2(k\theta) &< \frac{1}{\theta^2}\sum_{k=1}^{n}\frac{1}{k^2} < n + \sum_{k=1}^{n}\cot^2(k\theta).
\end{aligned}$$

Note that the identity is valid because we are only summing for k up to n, so that $k\theta < \frac{\pi}{2}$.

b. Substitute $\dfrac{n(2n-1)}{3}$ for the sum, using the identity:

$$\begin{aligned}
\frac{n(2n-1)}{3} &< \frac{1}{\theta^2}\sum_{k=1}^{n}\frac{1}{k^2} < n + \frac{n(2n-1)}{3},\\
\theta^2\frac{n(2n-1)}{3} &< \sum_{k=1}^{n}\frac{1}{k^2} < \theta^2\frac{n(2n+2)}{3},\\
\frac{n(2n-1)\pi^2}{3(2n+1)^2} &< \sum_{k=1}^{n}\frac{1}{k^2} < \frac{n(2n+2)\pi^2}{3(2n+1)^2}.
\end{aligned}$$

c. By the Squeeze Theorem, if the expressions on either end have equal limits as $n \to \infty$, the expression in the middle does as well, and its limit is the same. The expression on the left is

$$\pi^2\frac{2n^2-n}{12n^2+12n+3} = \pi^2\frac{2-n^{-1}}{12+12n^{-1}+3n^{-2}},$$

which has a limit of $\dfrac{\pi^2}{6}$ as $n \to \infty$. The expression on the right is

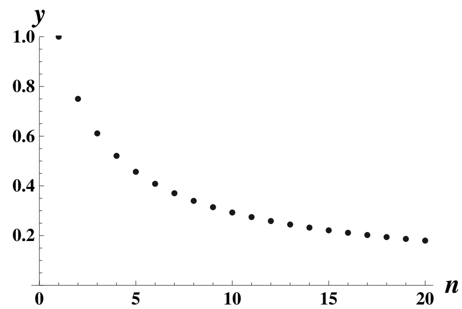
$$\pi^2\frac{2n^2+2n}{12n^2+12n+3} = \pi^2\frac{2+2n^{-1}}{12+12n^{-1}+3n^{-3}},$$

which has the same limit. Thus $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2} = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$.

8.4.67 $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{k=1}^{\infty}\frac{1}{(2k)^2} + \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$, splitting the series into even and odd terms. But $\sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}$. Thus $\frac{\pi^2}{6} = \frac{1}{4}\frac{\pi^2}{6} + \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$, so that the sum in question is $\frac{3\pi^2}{24} = \frac{\pi^2}{8}$.

8.4.68

a. $\{F_n\}$ is a decreasing sequence because each term in F_n is smaller than the corresponding term in F_{n-1} and thus the sum of terms in F_n is smaller than the sum of terms in F_{n-1}.

b. 

c. It appears that $\lim_{n\to\infty} F_n = 0$.

8.4.69

a. $x_1 = \sum_{k=2}^{2} \frac{1}{k} = \frac{1}{2}$, $x_2 = \sum_{k=3}^{4} \frac{1}{k} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$, $x_3 = \sum_{k=4}^{6} \frac{1}{k} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{37}{60}$.

b. x_n has n terms. Each term is bounded below by $\frac{1}{2n}$ and bounded above by $\frac{1}{n+1}$. Thus $x_n \geq n \cdot \frac{1}{2n} = \frac{1}{2}$, and $x_n \leq n \cdot \frac{1}{n+1} < n \cdot \frac{1}{n} = 1$.

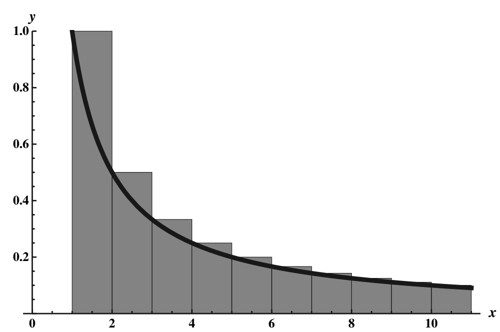
c. The right Riemann sum for $\int_1^2 \frac{dx}{x}$ using n subintervals has n rectangles of width $\frac{1}{n}$; the right edges of those rectangles are at $1 + \frac{i}{n} = \frac{n+i}{n}$ for $i = 1, 2, \ldots, n$. The height of such a rectangle is the value of $\frac{1}{x}$ at the right endpoint, which is $\frac{n}{n+i}$. Thus the area of the rectangle is $\frac{1}{n} \cdot \frac{n}{n+i} = \frac{1}{n+i}$. Adding up over all the rectangles gives x_n.

d. The limit $\lim_{n\to\infty} x_n$ is the limit of the right Riemann sum as the width of the rectangles approaches zero. This is precisely $\int_1^2 \frac{dx}{x} = \ln x \Big|_1^2 = \ln 2$.

8.4.70

a. The first diagram is a left Riemann sum for $f(x) = \frac{1}{x}$ on the interval $[1, 11]$ (we assume $n = 10$ for purposes of drawing a graph). The area under the curve is $\int_1^{n+1} \frac{1}{x}\, dx = \ln(n+1)$, and the sum of the areas of the rectangles is obviously $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Thus

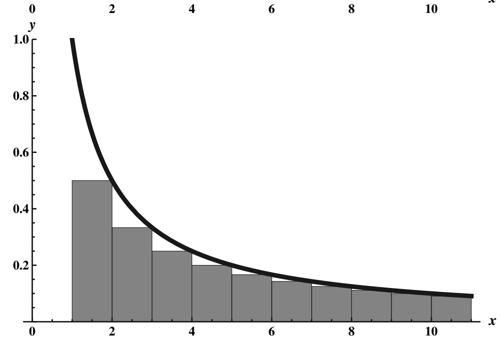
$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$



The second diagram is a right Riemann sum for the same function on the same interval. Considering only $[1, n]$, we see that, comparing the area under the curve and the sum of the areas of the rectangles, that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n.$$

Adding 1 to both sides gives the desired inequality.



b. According to part (a), $\ln(n+1) < S_n$ for $n = 1, 2, 3, \ldots,$, so that $E_n = S_n - \ln(n+1) > 0$.

c. Using the second figure above and assuming $n = 9$, the final rectangle corresponds to $\frac{1}{n+1}$, and the area under the curve between $n+1$ and $n+2$ is clearly $\ln(n+2) - \ln(n+1)$.

d. $E_{n+1} - E_n = S_{n+1} - \ln(n+2) - (S_n - \ln(n+1)) = \frac{1}{n+1} - (\ln(n+2) - \ln(n+1))$. But this is positive because of the bound established in part (c).

e. Using part (a), $E_n = S_n - \ln(n+1) < 1 + \ln n - \ln(n+1) < 1$.

f. E_n is a monotone (increasing) sequence that is bounded, so it has a limit.

g. The first ten values (E_1 through E_{10}) are

$$.3068528194, .401387711, .447038972, .473895421, .491573864,$$
$$.504089851, .513415601, .520632566, .526383161, .531072981.$$

$E_{1000} \approx 0.576716082$.

h. For $S_n > 10$ we need $10 - 0.5772 = 9.4228 > \ln(n+1)$. Solving for n gives $n \approx 12366.16$, so $n = 12367$.

8.4.71

a. Note that the center of gravity of any stack of dominoes is the average of the locations of their centers. Define the midpoint of the zeroth (top) domino to be $x = 0$, and stack additional dominoes down and to its right (to increasingly positive x-coordinates). Let $m(n)$ be the x-coordinate of the midpoint of the n^{th} domino. Then in order for the stack not to fall over, the left edge of the n^{th} domino must be placed directly under the center of gravity of dominos 0 through $n-1$, which is $\frac{1}{n}\sum_{i=0}^{n-1} m(i)$, so that $m(n) = 1 + \frac{1}{n}\sum_{i=0}^{n-1} m(i)$. We claim that in fact $m(n) = \sum_{k=1}^{n} \frac{1}{k}$. Use induction. This is certainly true for $n = 1$. Note first that $m(0) = 0$, so we can start the sum at 1 rather than at 0. Now, $m(n) = 1 + \frac{1}{n}\sum_{i=1}^{n-1} m(i) = 1 + \frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=1}^{i} \frac{1}{j}$. Now, 1 appears $n-1$ times in the double sum, 2 appears $n-2$ times, and so forth, so we can rewrite this sum as $m(n) = 1 + \frac{1}{n}\sum_{i=1}^{n-1} \frac{n-i}{i} = 1 + \frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{n}{i} - 1\right) = 1 + \frac{1}{n}\left(n\sum_{i=1}^{n-1}\frac{1}{i} - (n-1)\right) = \sum_{i=1}^{n-1}\frac{1}{i} + 1 - \frac{n-1}{n} = \sum_{i=1}^{n}\frac{1}{i}$, and we are done by induction (noting that the statement is clearly true for $n = 0$, $n = 1$). Thus the maximum overhang is $\sum_{k=2}^{n}\frac{1}{k}$.

b. For an infinite number of dominos, because the overhang is the harmonic series, the distance is potentially infinite.

8.4.72

a. The circumference of the kth layer is $2\pi \cdot \frac{1}{k}$, so its area is $2\pi \cdot \frac{1}{k}$ and thus the total vertical surface area $\sum_{k=1}^{\infty} 2\pi \cdot \frac{1}{k} = 2\pi\sum_{k=1}^{\infty}\frac{1}{k} = \infty$. The horizontal surface area, however, is π, since looking at the cake from above, the horizontal surface covers the circle of radius 1, which has area $\pi \cdot 1^2 = \pi$.

b. The volume of a cylinder of radius r and height h is $\pi r^2 h$, so the volume of the kth layer is $\pi \cdot \frac{1}{k^2} \cdot 1 = \frac{\pi}{k^2}$. Thus the volume of the cake is

$$\sum_{k=1}^{\infty}\frac{\pi}{k^2} = \pi\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^3}{6} \approx 5.168.$$

c. This cake has infinite surface area, yet it has finite volume!

8.4.73

a. Dividing both sides of the recurrence equation by f_n gives $\frac{f_{n+1}}{f_n} = 1 + \frac{f_{n-1}}{f_n}$. Let the limit of the ratio of successive terms be L. Taking the limit of the previous equation gives $L = 1 + \frac{1}{L}$. Thus $L^2 = L + 1$, so $L^2 - L - 1 = 0$. The quadratic formula gives $L = \frac{1 \pm \sqrt{1 - 4\cdot(-1)}}{2}$, but we know that all the terms are positive, so we must have $L = \frac{1+\sqrt{5}}{2} = \phi \approx 1.618$.

b. Write the recurrence in the form $f_{n-1} = f_{n+1} - f_n$ and divide both sides by f_{n+1}. Then we have $\frac{f_{n-1}}{f_{n+1}} = 1 - \frac{f_n}{f_{n+1}}$. Taking the limit gives $1 - \frac{1}{\phi}$ on the right-hand side.

c. Consider the harmonic series with the given groupings, and compare it with the sum of $\frac{f_{k-1}}{f_{k+1}}$ as shown. The first three terms match exactly. The sum of the next two are $\frac{1}{4}+\frac{1}{5} > \frac{1}{5}+\frac{1}{5}=\frac{2}{5}$. The sum of the next three are $\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$. The sum of the next five are $\frac{1}{9}+\cdots+\frac{1}{13} > 5\cdot\frac{1}{13}=\frac{5}{13}$. Thus the harmonic series is bounded below by the series $\sum_{k=1}^{\infty}\frac{f_{k-1}}{f_{k+1}}$.

d. The result above implies that the harmonic series diverges, because the series $\sum_{k=1}^{\infty}\frac{f_{k-1}}{f_{k+1}}$ diverges, since its general term has limit $1-\frac{1}{\phi}\neq 0$.

8.5 The Ratio, Root, and Comparison Tests

8.5.1 Given a series $\sum a_k$ of positive terms, compute $\lim_{k\to\infty}\frac{a_{k+1}}{a_k}$ and call it r. If $0\le r<1$, the given series converges. If $r>1$ (including $r=\infty$), the given series diverges. If $r=1$, the test is inconclusive.

8.5.2 Given a series $\sum a_k$ of positive terms, compute $\lim_{k\to\infty}\sqrt[k]{a_k}$ and call it r. If $0\le r<1$, the given series converges. If $r>1$ (including $r=\infty$), the given series diverges. If $r=1$, the test is inconclusive.

8.5.3 Given a series of positive terms $\sum a_k$ that you suspect converges, find a series $\sum b_k$ that you know converges, for which $\lim_{k\to\infty}\frac{a_k}{b_k}=L$ where $L\ge 0$ is a finite number. If you are successful, you will have shown that the series $\sum a_k$ converges.

Given a series of positive terms $\sum a_k$ that you suspect diverges, find a series $\sum b_k$ that you know diverges, for which $\lim_{k\to\infty}\frac{a_k}{b_k}=L$ where $L>0$ (including the case $L=\infty$). If you are successful, you will have shown that $\sum a_k$ diverges.

8.5.4 The Divergence Test.

8.5.5 The Ratio Test.

8.5.6 The Comparison Test or the Limit Comparison Test.

8.5.7 The difference between successive partial sums is a term in the sequence. Because the terms are positive, differences between successive partial sums are as well, so the sequence of partial sums is increasing.

8.5.8 No. They all determine convergence or divergence by approximating or bounding the series by some other series known to converge or diverge; thus, the actual value of the series cannot be determined.

8.5.9 The ratio between successive terms is $\frac{a_{k+1}}{a_k}=\frac{1}{(k+1)!}\cdot\frac{(k)!}{1}=\frac{1}{k+1}$, which goes to zero as $k\to\infty$, so the given series converges by the Ratio Test.

8.5.10 The ratio between successive terms is $\frac{a_{k+1}}{a_k}=\frac{2^{k+1}}{(k+1)!}\cdot\frac{(k)!}{2^k}=\frac{2}{k+1}$; the limit of this ratio is zero, so the given series converges by the Ratio Test.

8.5.11 The ratio between successive terms is $\frac{a_{k+1}}{a_k}=\frac{(k+1)^2}{4^(k+1)}\cdot\frac{4^k}{(k)^2}=\frac{1}{4}\left(\frac{k+1}{k}\right)^2$. The limit is $1/4$ as $k\to\infty$, so the given series converges by the Ratio Test.

8.5.12 The ratio between successive terms is

$$\frac{a_{k+1}}{a_k}=\frac{(k+1)^{(k+1)}}{2^{(k+1)}}\cdot\frac{2^k}{k^k}=\frac{k+1}{2}\left(\frac{k+1}{k}\right)^k.$$

Note that $\lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k=e$, but $\lim_{k\to\infty}\frac{k+1}{2}=\infty$, so the given series diverges by the Ratio Test.

8.5.13 The ratio between successive terms is $\frac{a_{k+1}}{a_k}=\frac{(k+1)e^{-(k+1)}}{(k)e^{-(k)}}=\frac{k+1}{(k)e}$. The limit of this ratio as $k\to\infty$ is $1/e<1$, so the given series converges by the Ratio Test.

8.5.14 The ratio between successive terms is $\frac{a_{k+1}}{a_k} = \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \left(\frac{k+1}{k}\right)^k$. This has limit e as $k \to \infty$, so the limit of the ratio of successive terms is $e > 1$, so the given series diverges by the Ratio Test.

8.5.15 The ratio between successive terms is $\frac{2^{k+1}}{(k+1)^{99}} \cdot \frac{(k)^{99}}{2^k} = 2\left(\frac{k}{k+1}\right)^{99}$; the limit as $k \to \infty$ is 2, so the given series diverges by the Ratio Test.

8.5.16 The ratio between successive terms is $\frac{(k+1)^6}{(k+1)!} \cdot \frac{(k)!}{(k)^6} = \frac{1}{k+1}\left(\frac{k+1}{k}\right)^6$; the limit as $k \to \infty$ is zero, so the given series converges by the Ratio Test.

8.5.17 The ratio between successive terms is $\frac{((k+1)!)^2}{(2(k+1))!} \cdot \frac{(2k)!}{((k)!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)}$; the limit as $k \to \infty$ is $1/4$, so the given series converges by the Ratio Test.

8.5.18 Note that this series is $\sum_{k=1}^{\infty} \frac{2^k}{k^4}$. The ratio between successive terms is $\frac{2^{k+1}k^4}{2^k(k+1)^4} = 2\left(\frac{k}{k+1}\right)^4 \to 2$ as $k \to \infty$. So the given series diverges by the ratio test.

8.5.19 The kth root of the kth term is $\frac{10k^3+3}{9k^3+k+1}$. The limit of this as $k \to \infty$ is $\frac{10}{9} > 1$, so the given series diverges by the Root Test.

8.5.20 The kth root of the kth term is $\frac{2k}{k+1}$. The limit of this as $k \to \infty$ is $2 > 1$, so the given series diverges by the Root Test.

8.5.21 The kth root of the kth term is $\frac{k^{2/k}}{2}$. The limit of this as $k \to \infty$ is $\frac{1}{2} < 1$, so the given series converges by the Root Test.

8.5.22 The kth root of the kth term is $\left(1 + \frac{3}{k}\right)^k$. The limit of this as $k \to \infty$ is $= e^3 > 1$, so the given series diverges by the Root Test.

8.5.23 The kth root of the kth term is $\left(\frac{k}{k+1}\right)^{2k}$. The limit of this as $k \to \infty$ is $e^{-2} < 1$, so the given series converges by the Root Test.

8.5.24 The kth root of the kth term is $\frac{1}{\ln(k+1)}$. The limit of this as $k \to \infty$ is 0, so the given series converges by the Root Test.

8.5.25 The kth root of the kth term is$\left(\frac{1}{k^k}\right)$. The limit of this as $k \to \infty$ is 0, so the given series converges by the Root Test.

8.5.26 The kth root of the kth term is $\frac{k^{1/k}}{e}$. The limit of this as $k \to \infty$ is $\frac{1}{e} < 1$, so the given series converges by the Root Test.

8.5.27 $\frac{1}{k^2+4} < \frac{1}{k^2}$, and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$ converges as well, by the Comparison Test.

8.5.28 Use the Limit Comparison Test with $\left\{\frac{1}{k^2}\right\}$. The ratio of the terms of the two series is $\frac{k^4+k^3-k^2}{k^4+4k^2-3}$ which has limit 1 as $k \to \infty$. Because the comparison series converges, the given series does as well.

8.5.29 Use the Limit Comparison Test with $\left\{\frac{1}{k}\right\}$. The ratio of the terms of the two series is $\frac{k^3-k}{k^3+4}$ which has limit 1 as $k \to \infty$. Because the comparison series diverges, the given series does as well.

8.5.30 Use the Limit Comparison Test with $\left\{\frac{1}{k}\right\}$. The ratio of the terms of the two series is $\frac{0.0001k}{k+4}$ which has limit 0.0001 as $k \to \infty$. Because the comparison series diverges, the given series does as well.

8.5.31 For all k, $\frac{1}{k^{3/2}+1} < \frac{1}{k^{3/2}}$. The series whose terms are $\frac{1}{k^{3/2}}$ is a p-series which converges, so the given series converges as well by the Comparison Test.

8.5.32 Use the Limit Comparison Test with $\{1/k\}$. The ratio of the terms of the two series is $k\sqrt{\frac{k}{k^3+1}} = \sqrt{\frac{k^3}{k^3+1}}$, which has limit 1 as $k \to \infty$. Because the comparison series diverges, the given series does as well.

8.5.33 $\sin(1/k) > 0$ for $k \geq 1$, so we can apply the Comparison Test with $1/k^2$. $\sin(1/k) < 1$, so $\frac{\sin(1/k)}{k^2} < \frac{1}{k^2}$. Because the comparison series converges, the given series converges as well.

8.5.34 Use the Limit Comparison Test with $\{1/3^k\}$. The ratio of the terms of the two series is $\frac{3^k}{3^k - 2^k} = \frac{1}{1 - \left(\frac{2^k}{3^k}\right)}$, which has limit 1 as $k \to \infty$. Because the comparison series converges, the given series does as well.

8.5.35 Use the Limit Comparison Test with $\{1/k\}$. The ratio of the terms of the two series is $\frac{k}{2k - \sqrt{k}} = \frac{1}{2 - 1/\sqrt{k}}$, which has limit $1/2$ as $k \to \infty$. Because the comparison series diverges, the given series does as well.

8.5.36 $\frac{1}{k\sqrt{k+2}} < \frac{1}{k\sqrt{k}} = \frac{1}{k^{3/2}}$. Because the series whose terms are $\frac{1}{k^{3/2}}$ is a $p-$series with $p > 1$, it converges. Because the comparison series converges, the given series converges as well.

8.5.37 Use the Limit Comparison Test with $\frac{k^{2/3}}{k^{3/2}}$. The ratio of corresponding terms of the two series is $\frac{\sqrt[3]{k^2+1}}{\sqrt{k^3+1}} \cdot \frac{k^{3/2}}{k^{2/3}} = \frac{\sqrt[3]{k^2+1}}{\sqrt[3]{k^2}} \cdot \frac{\sqrt{k^3}}{\sqrt{k^3+1}}$, which has limit 1 as $k \to \infty$. The comparison series is the series whose terms are $k^{2/3-3/2} = k^{-5/6}$, which is a p-series with $p < 1$, so it, and the given series, both diverge.

8.5.38 For all k, $\frac{1}{(k \ln k)^2} < \frac{1}{k^2}$. Because the series whose terms are $\frac{1}{k^2}$ converges, the given series converges as well.

8.5.39

a. False. For example, let $\{a_k\}$ be all zeros, and $\{b_k\}$ be all 1's.

b. True. This is a result of the Comparison Test.

c. True. Both of these statements follow from the Comparison Test.

d. True. The limit of the ratio is always 1 in the case, so the test is inconclusive.

8.5.40 Use the Divergence Test: $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \left(1 - \frac{1}{k}\right)^k = \frac{1}{e} \neq 0$, so the given series diverges.

8.5.41 Use the Divergence Test: $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \left(1 + \frac{2}{k}\right)^k = e^2 \neq 0$, so the given series diverges.

8.5.42 Use the Root Test: The kth root of the kth term is $\frac{k^2}{2k^2+1}$. The limit of this as $k \to \infty$ is $\frac{1}{2} < 1$, so the given series converges by the Root Test.

8.5.43 Use the Ratio Test: the ratio of successive terms is $\frac{(k+1)^{100}}{(k+2)!} \cdot \frac{(k+1)!}{k^{100}} = \left(\frac{k+1}{k}\right)^{100} \cdot \frac{1}{k+2}$. This has limit $1^{100} \cdot 0 = 0$ as $k \to \infty$, so the given series converges by the Ratio Test.

8.5.44 Use the Comparison Test. Note that $\sin^2 k \leq 1$ for all k, so $\frac{\sin^2 k}{k^2} \leq \frac{1}{k^2}$ for all k. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so does the given series.

8.5.45 Use the Root Test. The kth root of the kth term is $(k^{1/k} - 1)^2$, which has limit 0 as $k \to \infty$, so the given series converges by the Root Test.

8.5.46 Use the Limit Comparison Test with the series whose kth term is $\left(\frac{2}{e}\right)^k$. Note that $\lim_{k\to\infty} \frac{2^k}{e^k - 1} \cdot \frac{e^k}{2^k} = \lim_{k\to\infty} \frac{e^k}{e^k - 1} = 1$. The given series thus converges because $\sum_{k=1}^{\infty} \left(\frac{2}{e}\right)^k$ converges (because it is a geometric series with $r = \frac{2}{e} < 1$). Note that it is also possible to show convergence with the Ratio Test.

8.5.47 Use the Divergence Test: $\lim_{k\to\infty} \frac{k^2+2k+1}{3k^2+1} = \frac{1}{3} \neq 0$, so the given series diverges.

8.5.48 Use the Limit Comparison Test with the series whose kth term is $\frac{1}{5^k}$. Note that $\lim_{k\to\infty} \frac{1}{5^k-1} \cdot \frac{5^k}{1} = 1$, and the series $\sum_{k=1}^{\infty} \frac{1}{5^k}$ converges because it is a geometric series with $r = \frac{1}{5}$. Thus, the given series also converges.

8.5.49 Use the Limit Comparison Test with the harmonic series. Note that $\lim_{k\to\infty} \frac{\frac{1}{\ln k}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{\ln k} = \infty$, and because the harmonic series diverges, the given series does as well.

8.5.50 Use the Limit Comparison Test with the series whose kth term is $\frac{1}{5^k}$. Note that $\lim_{k\to\infty} \frac{1}{5^k-3^k} \cdot \frac{5^k}{1} = \lim_{k\to\infty} \frac{1}{1-(3/5)^k} = 1$, and the series $\sum_{k=3}^{\infty} \frac{1}{5^k}$ converges because it is a geometric series with $r = \frac{1}{5}$. Thus, the given series also converges.

8.5.51 Use the Limit Comparison Test with the series whose kth term is $\frac{1}{k^{3/2}}$. Note that $\lim_{k\to\infty} \frac{1}{\sqrt{k^3-k+1}} \cdot \frac{\sqrt{k^3}}{1} = \lim_{k\to\infty} \sqrt{\frac{k^3}{k^3-k+1}} = \sqrt{1} = 1$, and the series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges because it is a p-series with $p = \frac{3}{2}$. Thus, the given series also converges.

8.5.52 Use the Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{((k+1)!)^3}{(3k+3)!} \cdot \frac{(3k)!}{(k!)^3} = \frac{(k+1)^3}{(3k+1)(3k+2)(3k+3)}$, which has limit $1/27$ as $k \to \infty$. Thus the given series converges.

8.5.53 Use the Comparison Test. Each term $\frac{1}{k} + 2^{-k} > \frac{1}{k}$. Because the harmonic series diverges, so does this series.

8.5.54 Use the Comparison Test with $\{5/k\}$. Note that $\frac{5\ln k}{k} > \frac{5}{k}$ for $k > 1$. Because the series whose terms are $5/k$ diverges, the given series diverges as well.

8.5.55 Use the Ratio Test. $\frac{a_{k+1}}{a_k} = \frac{2^{k+1}(k+1)!}{(k+1)^{k+1}} \cdot \frac{(k)^k}{2^k(k)!} = 2\left(\frac{k}{k+1}\right)^k$, which has limit $\frac{2}{e}$ as $k \to \infty$, so the given series converges.

8.5.56 Use the Root Test. $\lim\limits_{k\to\infty} \left(1 - \frac{1}{k}\right)^k = e^{-1} < 1$, so the given series converges.

8.5.57 Use the Limit Comparison Test with $\{1/k^3\}$. The ratio of corresponding terms is $\frac{k^{11}}{k^{11}+3}$, which has limit 1 as $k \to \infty$. Because the comparison series converges, so does the given series.

8.5.58 Use the Root Test. $\lim\limits_{k\to\infty} \frac{1}{1+p} = \frac{1}{1+p} < 1$ because $p > 0$, so the given series converges.

8.5.59 This is a p-series with exponent greater than 1, so it converges.

8.5.60 Use the Comparison Test: $\frac{1}{k^2 \ln k} < \frac{1}{k^2}$. Because the series whose terms are $\frac{1}{k^2}$ is a convergent $p-$series, the given series converges as well.

8.5.61 $\ln\left(\frac{k+2}{k+1}\right) = \ln(k+2) - \ln(k+1)$, so this series telescopes. We get $\sum_{k=1}^{n} \ln\left(\frac{k+2}{k+1}\right) = \ln(n+2) - \ln 2$. Because $\lim_{n\to\infty} \ln(n+2) - \ln 2 = \infty$, the sequence of partial sums diverges, so the given series is divergent.

8.5.62 Use the Divergence Test. Note that $\lim_{k\to\infty} k^{-1/k} = \lim_{k\to\infty} \frac{1}{\sqrt[k]{k}} = 1 \neq 0$, so the given series diverges.

8.5.63 For $k > 7$, $\ln k > 2$ so note that $\frac{1}{k^{\ln k}} < \frac{1}{k^2}$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the given series converges as well.

8.5.64 Use the Limit Comparison Test with $\{1/k^2\}$. Note that $\frac{\sin^2(1/k)}{1/k^2} = \left(\frac{\sin(1/k)}{1/k}\right)^2$. Because $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, the limit of this expression is $1^2 = 1$ as $k \to \infty$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the given series does as well.

8.5.65 Use the Limit Comparison Test with the harmonic series. $\frac{\tan(1/k)}{1/k}$ has limit 1 as $k \to \infty$ because $\lim\limits_{x\to 0} \frac{\tan x}{x} = 1$. Thus the original series diverges.

8.5.66 Use the Root Test. $\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{100} \cdot \frac{1}{k} = 0$, so the given series converges.

8.5.67 Note that $\dfrac{1}{(2k+1)\cdot(2k+3)} = \dfrac{1}{2}\left(\dfrac{1}{2k+1} - \dfrac{1}{2k+3}\right)$. Thus this series telescopes.

$$\sum_{k=0}^{n} \frac{1}{(2k+1)(2k+3)} = \frac{1}{2}\sum_{k=0}^{n}\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right) = \frac{1}{2}\left(-\frac{1}{2n+3} + 1\right),$$

so the given series converges to $1/2$, because that is the limit of the sequence of partial sums.

8.5.68 This series is $\sum_{k=1}^{\infty} \frac{k-1}{k^2} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k^2}\right)$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, if the original series also converged, we would have that $\sum_{k=1}^{\infty} \frac{1}{k}$ converged, which is false. Thus the original series diverges.

8.5.69 This series is $\sum_{k=1}^{\infty} \frac{k^2}{k!}$. By the Ratio Test, $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(k+1)!} \cdot \frac{k!}{k^2} = \frac{1}{k+1}\left(\frac{k+1}{k}\right)^2$, which has limit 0 as $k \to \infty$, so the given series converges.

8.5.70 For any p, if k is sufficently large then $k^{1/p} > \ln k$ because powers grow faster than logs, so that $k > (\ln k)^p$ and thus $1/k < 1/(\ln k)^p$. Because $\sum 1/k$ diverges, we see that the original series diverges for all p.

8.5.71 For $p \le 1$ and $k > e$, $\frac{\ln k}{k^p} > \frac{1}{k^p}$. The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges, so the given series diverges. For $p > 1$, let $q < p - 1$; then for sufficiently large k, $\ln k < k^q$, so that by the Comparison Test, $\frac{\ln k}{k^p} < \frac{k^q}{k^p} = \frac{1}{k^{p-q}}$. But $p - q > 1$, so that $\sum_{k=1}^{\infty} \frac{1}{k^{p-q}}$ is a convergent p-series. Thus the original series is convergent precisely when $p > 1$.

8.5.72 For $p \ne 1$,

$$\int_2^{\infty} \frac{dx}{x\ln x(\ln\ln x)^p} = \lim_{b\to\infty}\left(\left.\frac{(\ln\ln x)^{1-p}}{1-p}\right|_2^b\right).$$

This improper integral converges if and only $p > 1$. If $p = 1$, we have

$$\int_2^{\infty} \frac{dx}{x(\ln x)\ln\ln x} = \lim_{b\to\infty} \ln\ln\ln x\Big|_2^b = \infty.$$

Thus the original series converges for $p > 1$.

8.5.73 For $p \le 1$, $\frac{(\ln k)^p}{k^p} > \frac{1}{k^p}$ for $k \ge 3$, and $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges for $p \le 1$, so the original series diverges. For $p > 1$, let $q < p - 1$; then for sufficiently large k, $(\ln k)^p < k^q$. Note that $\frac{(\ln k)^p}{k^p} < \frac{k^q}{k^p} = \frac{1}{k^{p-q}}$. But $p - q > 1$, so $\sum_{k=1}^{\infty} \frac{1}{k^{p-q}}$ converges, so the given series converges. Thus, the given series converges exactly for $p > 1$.

8.5.74 Using the Ratio Test, $\frac{a_{k+1}}{a_k} = \frac{(k+1)!p^{k+1}}{(k+2)^{k+1}} \cdot \frac{(k+1)^k}{(k)!p^k} = \frac{(k+1)p(k+1)^k}{(k+2)^{k+1}} = p\left(\frac{k+1}{k+2}\right)^{k+1} = p \cdot \left(\frac{1}{1+\frac{1}{k+1}}\right)^{k+1}$, which has limit pe^{-1}. The series converges if the ratio limit is less than 1, so if $p < e$. If $p > e$, the given series diverges by the Ratio Test. If $p = e$, the given series diverges by the Divergence Test.

8.5.75 Use the Ratio Test:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)p^{k+1}}{k+2} \cdot \frac{k+1}{kp^k} = p,$$

so the given series converges for $p < 1$ and diverges for $p > 1$. For $p = 1$ the given series diverges by limit comparison with the harmonic series.

8.5.76 $\ln\left(\frac{k}{k+1}\right)^p = p(\ln(k) - \ln(k+1))$, so

$$\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)^p = p\sum_{k=1}^{\infty}(\ln(k) - \ln(k+1))$$

which telescopes, and the n^{th} partial sum is $-p\ln(n+1)$, and $\lim_{n\to\infty} -p\ln(n+1)$ is not a finite number for any value of p other than 0. The given series diverges for all values of p other than $p = 0$.

8.5.77 $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(1-\frac{p}{k}\right)^k = e^{-p} \neq 0$, so this sequence diverges for all p by the Divergence Test.

8.5.78 Use the Limit Comparison Test: $\lim_{k\to\infty} \frac{a_k^2}{a_k} = \lim_{k\to\infty} a_k = 0$, because $\sum a_k$ converges. By the Limit Comparison Test, the series $\sum a_k^2$ must converge as well.

8.5.79 These tests apply only for series with positive terms, so assume $r > 0$. Clearly the series do not converge for $r = 1$, so we assume $r \neq 1$ in what follows. Using the Integral Test, $\sum r^k$ converges if and only if $\int_1^\infty r^x\,dx$ converges. This improper integral has value $\lim_{b\to\infty} \left.\frac{r^x}{\ln r}\right|_1^b$, which converges only when $\lim_{b\to\infty} r^b$ exists, which occurs only for $r < 1$. Using the Ratio Test, $\frac{a_{k+1}}{a_k} = \frac{r^{k+1}}{r^k} = r$, so by the Ratio Test, the series converges if and only if $r < 1$. Using the Root Test, $\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{r^k} = \lim_{k\to\infty} r = r$, so again we have convergence if and only if $r < 1$. By the Divergence Test, we know that a geometric series diverges if $|r| \geq 1$.

8.5.80

a. Use the Limit Comparison Test with the divergent harmonic series. Note that $\lim_{k\to\infty} \frac{\sin(1/k)}{1/k} = 1$, because $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Because the comparison series diverges, the given series does as well.

b. We use the Limit Comparison Test with the convergent series $\sum \frac{1}{k^2}$. Note that $\lim_{k\to\infty} \frac{(1/k)\sin(1/k)}{1/k^2} = \lim_{k\to\infty} \frac{\sin(1/k)}{1/k} = 1$, so the given series converges.

8.5.81 To prove case (2), assume $L = 0$ and that $\sum b_k$ converges. Because $L = 0$, for every $\varepsilon > 0$, there is some N such that for all $n > N$, $|\frac{a_k}{b_k}| < \varepsilon$. Take $\varepsilon = 1$; this then says that there is some N such that for all $n > N$, $0 < a_k < b_k$. By the Comparison Test, because $\sum b_k$ converges, so does $\sum a_k$. To prove case (3), because $L = \infty$, then $\lim_{k\to\infty} \frac{b_k}{a_k} = 0$, so by the argument above, we have $0 < b_k < a_k$ for sufficient large k. But $\sum b_k$ diverges, so by the Comparison Test, $\sum a_k$ does as well.

8.5.82 The series clearly converges for $x = 0$. For $x \neq 0$, we have $\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} = \frac{x}{k+1}$. This has limit 0 as $k \to \infty$ for any value of x, so the series converges for all $x \geq 0$.

8.5.83 The series clearly converges for $x = 0$. For $x \neq 0$, we have $\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{x^k} = x$. This has limit x as $k \to \infty$, so the series converges for $x < 1$. It clearly does not converge for $x = 1$. So the series converges for $x \in [0, 1)$.

8.5.84 The series clearly converges for $x = 0$. For $x \neq 0$, we have $\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{k+1} \cdot \frac{k}{x^k} = x \cdot \frac{k}{k+1}$, which has limit x as $k \to \infty$. Thus this series converges for $x < 1$; additionally, for $x = 1$ (where the Ratio Test is inconclusive), the series is the harmonic series which diverges. So the series converges for $x \in [0, 1)$.

8.5.85 The series clearly converges for $x = 0$. For $x \neq 0$, we have $\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{(k+1)^2} \cdot \frac{k^2}{x^k} = x\left(\frac{k}{k+1}\right)^2$, which has limit x as $k \to \infty$. Thus the series converges for $x < 1$. When $x = 1$, the series is $\frac{1}{k^2}$, which converges. Thus the original series converges for $0 \leq x \leq 1$.

8.5.86 The series clearly converges for $x = 0$. For $x \neq 0$, we have $\frac{a_{k+1}}{a_k} = \frac{x^{2k+2}}{(k+1)^2} \cdot \frac{k^2}{x^{2k}} = x^2\left(\frac{k}{k+1}\right)^2$, which has limit x^2 as $k \to \infty$, so the series converges for $x < 1$. When $x = 1$, the series is $\frac{1}{k^2}$, which converges. Thus this series converges for $0 \leq x \leq 1$.

8.5.87 The series clearly converges for $x = 0$. For $x \neq 0$, we have $\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{2^{k+1}} \cdot \frac{2^k}{x^k} = \frac{x}{2}$, which has limit $x/2$ as $k \to \infty$. Thus the series converges for $0 \leq x < 2$. For $x = 2$, it is obviously divergent.

8.5.88

a. Let P_n be the n^{th} partial product of the a_k: $P_n = \prod_{k=1}^n a_k$. Then $\sum_{k=1}^n \ln a_k = \ln \prod_{k=1}^n a_k = \ln P_n$. If $\sum \ln a_k$ is a convergent series, then $\sum_{k=1}^\infty \ln a_k = \lim_{n\to\infty} \ln P_n = L < \infty$. But then $e^L = \lim_{n\to\infty} e^{\ln P_n} = \lim_{n\to\infty} P_n$, so that the infinite product converges.

b.

n	2	3	4	5	6	7	8
P_n	3/4	2/3	5/8	3/5	7/12	4/7	9/16

It appears that $P_n = \frac{n+1}{2n}$, so that $\lim_{n\to\infty} P_n = \frac{1}{2}$.

c. Because $\lim_{n\to\infty} \prod_{k=2}^n \left(1 - \frac{1}{k^2}\right) = \frac{1}{2}$, taking logs and using part (a) we see that $\lim_{n\to\infty} \sum_{k=1}^n \ln\left(1 - \frac{1}{k^2}\right) = \ln \frac{1}{2} = -\ln 2$.

8.5.89

a. $\ln \prod_{k=0}^\infty e^{1/2^k} = \sum_{k=0}^\infty \frac{1}{2^k} = 2$, so that the original product converges to e^2.

b. $\ln \prod_{k=2}^\infty \left(1 - \frac{1}{k}\right) = \ln \prod_{k=2}^\infty \frac{k-1}{k} = \sum_{k=2}^\infty \ln \frac{k-1}{k} = \sum_{k=2}^\infty (\ln(k-1) - \ln(k))$. This series telescopes to give $S_n = -\ln(n)$, so the original series has limit $\lim_{n\to\infty} P_n = \lim_{n\to\infty} e^{-\ln(n)} = 0$.

8.5.90 The sum on the left is simply the left Riemann sum over n equal intervals between 0 and 1 for $f(x) = x^p$. The limit of the sum is thus $\int_0^1 x^p\,dx = \frac{1}{p+1} x^{p+1}\Big|_0^1 = \frac{1}{p+1}$, because p is positive.

8.5.91

a. Use the Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{1 \cdot 3 \cdot 5 \cdots (2k+1)}{p^{k+1}(k+1)!} \cdot \frac{p^k (k)!}{1 \cdot 3 \cdot 5 \cdots (2k-1)} = \frac{(2k+1)}{(k+1)p}$$

and this expression has limit $\frac{2}{p}$ as $k \to \infty$. Thus the series converges for $p > 2$.

b. Following the hint, when $p = 2$ we have $\displaystyle\sum_{k=1}^\infty \frac{(2k)!}{2^k k!(2 \cdot 4 \cdot 6 \cdots 2k)} = \sum_{k=1}^\infty \frac{(2k)!}{(2^k)^2 (k!)^2}$. Using Stirling's formula, the numerator is asymptotic to $2\sqrt{\pi}\sqrt{k}(2k)^{2k} e^{-2k} = 2\sqrt{\pi}\sqrt{k}(2^k)^2 (k^k)^2 e^{-2k}$ while the denominator is asymptotic to $(2^k)^2 2\pi k (k^k)^2 e^{-2k}$, so the quotient is asymptotic to $\frac{1}{\sqrt{\pi}\sqrt{k}}$. Thus the original series diverges for $p = 2$ by the Limit Comparison Test with the divergent p-series $\sum_{k=1}^\infty \frac{1}{k^{1/2}}$.

8.6 Alternating Series

8.6.1 Because $S_{n+1} - S_n = (-1)^n a_{n+1}$ alternates signs.

8.6.2 Check that the terms of the series are nonincreasing in magnitude after some finite number of terms, and that $\lim_{k\to\infty} a_k = 0$.

8.6.3 We have

$$S = S_{2n+1} + (a_{2n} - a_{2n+1}) + (a_{2n+2} - a_{2n+3}) + \cdots$$

and each term of the form $a_{2k} - a_{2k+1} > 0$, so that $S_{2n+1} < S$. Also

$$S = S_{2n} + (-a_{2n+1} + a_{2n+2}) + (-a_{2n+3} + a_{2n+4}) + \cdots$$

and each term of the form $-a_{2k+1} + a_{2k+2} < 0$, so that $S < S_{2n}$. Thus the sum of the series is trapped between the odd partial sums and the even partial sums.

8.6.4 The difference between L and S_n is bounded in magnitude by a_{n+1}.

8.6.5 The remainder is less than the first neglected term because

$$S - S_n = (-1)^{n+1}(a_{n+1} + (-a_{n+2} + a_{n+3}) + \cdots)$$

so that the sum of the series *after* the first disregarded term has the opposite sign from the first disregarded term.

8.6.6 The alternating harmonic series $\sum(-1)^k \frac{1}{k}$ converges, but not absolutely.

8.6.7 No. If the terms are positive, then the absolute value of each term is the term itself, so convergence and absolute convergence would mean the same thing in this context.

8.6.8 The idea of the proof is to note that $0 \le |a_k| + a_k \le 2|a_k|$ and apply the Comparison Test to conclude that if $\sum |a_k|$ converges, then so does $\sum 2|a_k|$, and thus so must $\sum(|a_k| + a_k)$, and then conclude that $\sum a_k$ must converge as well.

8.6.9 Yes. For example, $\sum \frac{(-1)^k}{k^3}$ converges absolutely and thus not conditionally (see the definition).

8.6.10 The alternating harmonic series $\sum(-1)^k \frac{1}{k}$ converges conditionally, but not absolutely.

8.6.11 The terms of the series decrease in magnitude, and $\lim_{k\to\infty} \frac{1}{2k+1} = 0$, so the given series converges.

8.6.12 The terms of the series decrease in magnitude, and $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$, so the given series converges.

8.6.13 $\lim_{k\to\infty} \frac{k}{3k+2} = \frac{1}{3} \ne 0$, so the given series diverges.

8.6.14 $\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e \ne 0$, so the given series diverges.

8.6.15 The terms of the series decrease in magnitude, and $\lim\limits_{k\to\infty} \frac{1}{k^3} = 0$, so the given series converges.

8.6.16 The terms of the series decrease in magnitude, and $\lim\limits_{k\to\infty} \frac{1}{k^2+10} = 0$, so the given series converges.

8.6.17 The terms of the series decrease in magnitude, and $\lim\limits_{k\to\infty} \frac{k^2}{k^3+1} = \lim\limits_{k\to\infty} \frac{1/k}{1+1/k^3} = 0$, so the given series converges.

8.6.18 The terms of the series eventually decrease in magnitude, because if $f(x) = \frac{\ln x}{x^2}$, then $f'(x) = \frac{x(1-2\ln x)}{x^4} = \frac{1-2\ln x}{x^3}$, which is negative for large enough x. Further, $\lim\limits_{k\to\infty} \frac{\ln k}{k^2} = \lim\limits_{k\to\infty} \frac{1/k}{2k} = \lim\limits_{k\to\infty} \frac{1}{2k^2} = 0$. Thus the given series converges.

8.6.19 $\lim\limits_{k\to\infty} \frac{k^2-1}{k^2+3} = 1$, so the terms of the series do not tend to zero and thus the given series diverges.

8.6.20 $\sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k = \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{5}\right)^k$. $(1/5)^k$ is decreasing, and tends to zero as $k \to \infty$, so the given series converges.

8.6.21 $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)=1$, so the given series diverges.

8.6.22 Note that $\cos(\pi k)=(-1)^k$, and so the given series is alternating. Because $\lim_{k\to\infty}\frac{1}{k^2}=0$ and $\frac{1}{k^2}$ is decreasing, the given series is convergent.

8.6.23 The derivative of $f(k)=\frac{k^{10}+2k^5+1}{k(k^{10}+1)}$ is $f'(k)=\frac{-(k^{20}+2k^{10}+12k^{15}-8k^5+1)}{k^2(k^{10}+1)^2}$. The numerator is negative for large enough values of k, and the denominator is always positive, so the derivative is negative for large enough k. Also, $\lim_{k\to\infty}\frac{k^{10}+2k^5+1}{k(k^{10}+1)}=\lim_{k\to\infty}\frac{1+2k^{-5}+k^{-10}}{k+k^{-9}}=0$. Thus the given series converges.

8.6.24 Clearly $\frac{1}{k\ln^2 k}$ is nonincreasing, and $\lim_{k\to\infty}\frac{1}{k\ln^2 k}=0$, so the given series converges.

8.6.25 $\lim_{k\to\infty}k^{1/k}=1$ (for example, take logs and apply L'Hôpital's rule), so the given series diverges by the Divergence Test.

8.6.26 $a_{k+1}<a_k$ because $\frac{a_{k+1}}{a_k}=\frac{(k+1)!}{(k+1)^{k+1}}\cdot\frac{k^k}{k!}=\left(\frac{k}{k+1}\right)^k<1$. Additionally, $\frac{k!}{k^k}\to 0$ as $k\to\infty$, so the given series converges.

8.6.27 $\frac{1}{\sqrt{k^2+4}}$ is decreasing and tends to zero as $k\to\infty$, so the given series converges.

8.6.28 $\lim_{k\to\infty}k\sin(1/k)=\lim_{k\to\infty}\frac{\sin(1/k)}{1/k}=1$, so the given series diverges.

8.6.29 We want $\frac{1}{n+1}<10^{-4}$, or $n+1>10^4$, so $n=10^4$.

8.6.30 The series starts with $k=0$, so we want $\frac{1}{n!}<10^{-4}$, or $n!>10^4=10000$. This happens for $n=8$.

8.6.31 The series starts with $k=0$, so we want $\frac{1}{2n+1}<10^{-4}$, or $2n+1>10^4$, $n=5000$.

8.6.32 We want $\frac{1}{(n+1)^2}<10^{-4}$, or $(n+1)^2>10^4$, so $n=100$.

8.6.33 We want $\frac{1}{(n+1)^4}<10^{-4}$, or $(n+1)^4>10^4$, so $n=10$.

8.6.34 The series starts with $k=0$, so we want $\frac{1}{(2n+1)^3}<10^{-4}$, or $2n+1>10^{4/3}$, so $n=11$.

8.6.35 The series starts with $k=0$, so we want $\frac{1}{3n+1}<10^{-4}$, or $3n+1>10^4$, $n=3334$.

8.6.36 We want $\frac{1}{(n+1)^6}<10^{-4}$, or $(n+1)^6>10^4=10000$, so $n=4$.

8.6.37 The series starts with $k=0$, so we want $\frac{1}{4^n}\left(\frac{2}{4n+1}+\frac{2}{4n+2}+\frac{1}{4n+3}\right)<10^{-4}$, or $\frac{4^n(4n+1)(4n+2)(4n+3)}{4(20n^2+21n+5)}>10000$, which occurs first for $n=6$.

8.6.38 The series starts with $k=0$, so we want $\frac{1}{3n+2}<10^{-4}$, so $3n+2>10000$, $n=3333$.

8.6.39 To figure out how many terms we need to sum, we must find n such that $\frac{1}{(n+1)^5}<10^{-3}$, so that $(n+1)^5>1000$; this occurs first for $n=3$. Thus $\frac{-1}{1}+\frac{1}{2^5}-\frac{1}{3^5}\approx -0.973$.

8.6.40 To figure out how many terms we need to sum, we must find n such that $\frac{1}{(2(n+1)+1)^3}<10^{-3}$, or $(2n+3)^3>10^3$, so $2n+3>10$ and $n=4$. Thus the approximation is $\sum_{k=1}^{4}\frac{(-1)^n}{(2n+1)^3}\approx -0.306$.

8.6.41 To figure out how many terms we need to sum, we must find n so that $\frac{n+1}{(n+1)^2+1}<10^{-3}$, so that $\frac{(n+1)^2+1}{n+1}=n+1+\frac{1}{n+1}>1000$. This occurs first for $n=999$. We have $\sum_{k=1}^{999}\frac{(-1)^k k}{k^2+1}\approx -0.269$.

8.6.42 To figure out how many terms we need to sum, we must find n such that $\frac{n+1}{(n+1)^4+1} < 10^{-3}$, so that $\frac{(n+1)^4+1}{n+1} = (n+1)^3 + \frac{1}{n+1} > 1000$, which occurs for $n = 9$. We have $\sum_{k=1}^{9} \frac{(-1)^k k}{k^4+1} \approx -0.409$.

8.6.43 To figure how many terms we need to sum, we must find n such that $\frac{1}{(n+1)^{n+1}} < 10^{-3}$, or $(n+1)^{n+1} > 1000$, so $n = 4$ ($5^5 = 3125$). Thus the approximation is $\sum_{k=1}^{4} \frac{(-1)^n}{n^n} \approx -.783$.

8.6.44 To figure how many terms we need to sum, we must find n such that $\frac{1}{(2(n+1)+1)!} < 10^{-3}$, or $(2n+3)! > 1000$, so $2n+3 \geq 7$ and $n = 2$. The approximation is $\sum_{k=1}^{2} \frac{(-1)^{n+1}}{(2n+1)!} \approx 0.158$

8.6.45 The series of absolute values is a p-series with $p = 2/3$, so it diverges. The given alternating series does converge, though, by the Alternating Series Test. Thus, the given series is conditionally convergent.

8.6.46 The series of absolute values is a p-series with $p = 1/2$, so it diverges. The given alternating series does converge, though, by the Alternating Series Test. Thus, the given series is conditionally convergent.

8.6.47 The series of absolute values is a p-series with $p = 3/2$, so it converges absolutely.

8.6.48 The series of absolute values is $\sum \frac{1}{3^k}$, which converges, so the series converges absolutely.

8.6.49 The series of absolute values is $\sum \frac{|\cos(k)|}{k^3}$, which converges by the Comparison Test because $\frac{|\cos(k)|}{k^3} \leq \frac{1}{k^3}$. Thus the series converges absolutely.

8.6.50 The series of absolute values is $\sum \frac{k^2}{\sqrt{k^6+1}}$. The limit comparison test with $\frac{1}{k}$ gives $\lim_{k\to\infty} \frac{k^3}{\sqrt{k^6+1}} = \lim_{k\to\infty} \sqrt{\frac{k^6}{k^6+1}} = 1$. Because the comparison series diverges, so does the series of absolute values. The original series converges conditionally, however, because the terms are nonincreasing and $\lim_{k\to\infty} \frac{k^2}{\sqrt{k^6+1}} = \lim_{k\to\infty} \sqrt{\frac{k^4}{k^6+1}} = 0$.

8.6.51 The absolute value of the kth term of this series has limit $\pi/2$ as $k \to \infty$, so the given series is divergent by the Divergence Test.

8.6.52 The series of absolute values is a geometric series with $r = \frac{1}{e}$ and $|r| < 1$, so the given series converges absolutely

8.6.53 The series of absolute values is $\sum \frac{k}{2k+1}$, but $\lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2}$, so by the Divergence Test, this series diverges. The original series does not converge conditionally, either, because $\lim_{k\to\infty} a_k = \frac{1}{2} \neq 0$.

8.6.54 The series of absolute values is $\sum \frac{1}{\ln k}$, which diverges, so the series does not converge absolutely. However, because $\lim_{k\to\infty} \frac{1}{\ln k} \to 0$ and the terms are nonincreasing, the series does converge conditionally.

8.6.55 The series of absolute values is $\sum \frac{\tan^{-1}(k)}{k^3}$, which converges by the Comparison Test because $\frac{\tan^{-1}(k)}{k^3} < \frac{\pi}{2}\frac{1}{k^3}$, and $\sum \frac{\pi}{2}\frac{1}{k^3}$ converges because it is a constant multiple of a convergent $p-$series. So the original series converges absolutely.

8.6.56 The series of absolute values is $\sum \frac{e^k}{(k+1)!}$. Using the ratio test, $\frac{a_{k+1}}{a_k} = \frac{e^{k+1}}{(k+2)!} \cdot \frac{(k+1)!}{e^k} = \frac{e}{k+2}$, which tends to zero as $k \to \infty$, so the original series converges absolutely.

8.6.57

a. False. For example, consider the alternating harmonic series.

b. True. This is part of Theorem 8.21.

c. True. This statement is simply saying that a convergent series converges.

d. True. This is part of Theorem 8.21.

e. False. Let $a_k = \frac{1}{k}$.

f. True. Use the Comparison Test: $\lim_{k\to\infty} \frac{a_k^2}{a_k} = \lim_{k\to\infty} a_k = 0$ because $\sum a_k$ converges, so $\sum a_k^2$ and $\sum a_k$ converge or diverge together. Because the latter converges, so does the former.

g. True, by definition. If $\sum |a_k|$ converged, the original series would converge absolutely, not conditionally.

8.6.58 Neither condition is satisfied. $\frac{a_{k+1}}{a_k} = \frac{(k+1)(2k+1)}{(2k+3)k} = \frac{2k^2+3k+1}{2k^2+3k} > 1$, and $\lim_{k\to\infty} a_k = \frac{1}{2}$.

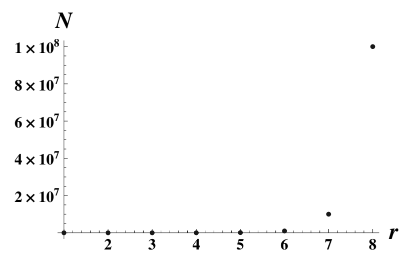
8.6.59 $\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = 2 \cdot \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}$, and thus $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{6} - \frac{1}{2} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{12}$.

8.6.60 $\sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} = 2\sum_{k=1}^{\infty} \frac{1}{(2k)^4} = 2 \cdot \frac{1}{16} \sum_{k=1}^{\infty} \frac{1}{k^4}$, and thus $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} = \frac{\pi^4}{90} - \frac{1}{8} \cdot \frac{\pi^4}{90} = \frac{7\pi^4}{720}$.

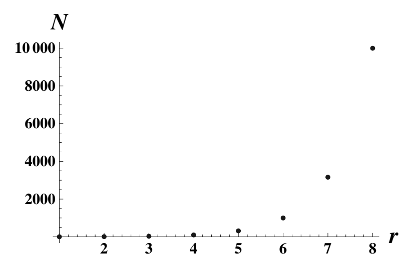
8.6.61 Write $r = -s$; then $0 < s < 1$ and $\sum r^k = \sum (-1)^k s^k$. Because $|s| < 1$, the terms s^k are nonincreasing and tend to zero, so by the Alternating Series Test, the series $\sum (-1)^k s^k = \sum r^k$ converges.

8.6.62

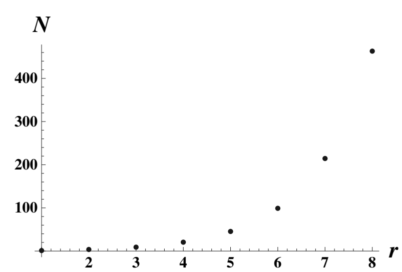
a. As p gets larger, fewer terms are needed to achieve a particular level of accuracy; this means that for larger p, the series converge faster.



$p = 1$

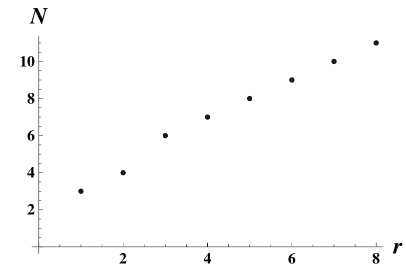


$p = 2$



$p = 3$

b. This graph shows that $\sum \frac{1}{k!}$ converges much faster than any of the powers of k.



8.6.63 Let $S = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$. Then

$$\begin{array}{rcccccccccc} S &=& \left(1-\frac{1}{2}\right) &+& \left(\frac{1}{3}-\frac{1}{4}\right) &+& \left(\frac{1}{5}-\frac{1}{6}\right) &+& \left(\frac{1}{7}-\frac{1}{8}\right) &+& \dots \\ \frac{1}{2}S &=& \frac{1}{2} &-& \frac{1}{4} &+& \frac{1}{6} &-& \frac{1}{8} &+& \dots \end{array}$$

Add these two series together to get

$$\frac{3}{2}S = \frac{3}{2}\ln 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \cdots$$

To see that the results are as desired, consider a collection of four terms:

$$\begin{array}{ccccc} \cdots + & \left(\frac{1}{4k+1} - \frac{1}{4k+2}\right) & + & \left(\frac{1}{4k+3} - \frac{1}{4k+4}\right) & + \quad \dots \\ \dots & + \quad \frac{1}{4k+2} & - & \frac{1}{4k+4} & + \quad \cdots \end{array}$$

Adding these results in the desired sign pattern. This repeats for each group of four elements.

8.6.64

a. Note that we can write

$$S_n = -\frac{a_1}{2} + \frac{1}{2}\left(\sum_{k=1}^{n-1}(-1)^k(a_i - a_{i+1})\right) + \frac{(-1)^n a_n}{2},$$

so that

$$S_n + \frac{(-1)^{n+1}a_{n+1}}{2} = -\frac{a_1}{2} + \frac{1}{2}\left(\sum_{k=1}^{n}(-1)^k d_i\right)$$

where $d_i = a_i - a_{i+1}$. Now consider the expression on the right-hand side of this last equation as the nth partial sum of a series which converges to S. Because the d_i's are decreasing and positive, the error made by stopping the sum after n terms is less than the absolute value of the first omitted term, which would be $\frac{1}{2}|d_{n+1}| = \frac{1}{2}|a_{n+1} - a_{n+2}|$. The method in the text for approximating the error simply takes the absolute value of the first unused term as an approximation of $|S - S_n|$. Here, S_n is modified by adding half the next term. Because the terms are decreasing in magnitude, this should be a better approximation to S than just S_n itself; the right side shows that this intuition is correct, because $\frac{1}{2}|a_{n+1} - a_{n+2}|$ is at most a_{n+1} and is generally less than that (because generally $a_{n+2} < a_{n+1}$).

b. i. Using the method from the text, we need n such that $\frac{1}{n+1} < 10^{-6}$, i.e. $n > 10^6 - 1$. Using the modified method from this problem, we want $\frac{1}{2}|a_{n+1} - a_{n+2}| < 10^{-6}$, so

$$\frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2(n+1)(n+2)} < 10^{-6}$$

This is true when $10^6 < 2(n+1)(n+2)$, which requires $n > 705.6$, so $n \geq 706$.

ii. Using the method from the book, we need n such that $k \ln k > 10^6$, which means $k \geq 87848$. Using the method of this problem, we want

$$\frac{1}{2}\left|\left(\frac{1}{k\ln k} - \frac{1}{(k+1)\ln(k+1)}\right)\right| = \left|\frac{(k+1)\ln(k+1) - k\ln k}{2k(k+1)\ln k\ln(k+1)}\right| < 10^{-6},$$

so that $|2k(k+1)\ln k\ln(k+1)| > |10^6(k\ln k - (k+1)\ln(k+1))|$, which means $k \geq 319$.

iii. Using the method from the book, we need k such that $\sqrt{k} > 10^6$, so $k > 10^{12}$. Using the method of this problem, we want

$$\frac{1}{2}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = \frac{\sqrt{k+1} - \sqrt{k}}{2\sqrt{k(k+1)}} < 10^{-6}$$

which means that $k > 3968.002$ so that $k \geq 3969$.

8.6.65 Both series diverge, so comparisons of their values are not meaningful.

8.6.66

a. The first ten terms are

$$(2-1) + \left(1 - \frac{1}{2}\right) + \left(\frac{2}{3} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{2}{5} - \frac{1}{5}\right)$$

Suppose that $k = 2i$ is even (and so $k-1 = 2i-1$ is odd). Then the sum of the $(k-1)$st term and the kth term is $\frac{4}{k} - \frac{2}{k} = \frac{2}{k} = \frac{1}{i}$. Then the sum of the first $2n$ terms of the given series is $\sum_{i=1}^{n} \frac{1}{i}$.

b. Note that $\lim_{k\to\infty} \frac{4}{k+1} = \lim_{k\to\infty} \frac{2}{k} = 0$. Thus given $\epsilon > 0$ there exists N_1 so that for $k > N_1$, we have $\frac{4}{k+1} < \epsilon$. Also, there exist N_2 so that for $k > N_2$, $\frac{2}{k} < \epsilon$. Let N be the larger of N_1 or N_2. Then for $k > N$, we have $a_k < \epsilon$, as desired.

c. The series can be seen to diverge because the even partial sums have limit ∞. This does not contradict the alternating series test because the terms a_k are not nonincreasing.

Chapter Eight Review

1

a. False. Let $a_n = 1 - \frac{1}{n}$. This sequence has limit 1.

b. False. The terms of a sequence tending to zero is necessary but not sufficient for convergence of the series.

c. True. This is the definition of convergence of a series.

d. False. If a series converges absolutely, the definition says that it does not converge conditionally.

e. True. It has limit 1 as $n \to \infty$.

f. False. The subsequence of the even terms has limit 1 and the subsequence of odd terms has limit -1, so the sequence does not have a limit.

g. False. It diverges by the Divergence Test because $\lim_{k\to\infty} \frac{k^2}{k^2+1} = 1 \neq 0$.

h. True. The given series converges by the Limit Comparison Test with the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, and thus its sequence of partial sums converges.

2 $\lim_{n\to\infty} \dfrac{n^2+4}{\sqrt{4n^4+1}} = \lim_{n\to\infty} \dfrac{1+4n^{-2}}{\sqrt{4+n^{-4}}} = \dfrac{1}{2}.$

3 $\lim_{n\to\infty} \dfrac{8^n}{n!} = 0$ because exponentials grow more slowly than factorials.

4 After taking logs, we want to compute

$$\lim_{n\to\infty} 2n\ln(1+3/n) = \lim_{n\to\infty} \frac{\ln(1+3/n)}{1/(2n)}.$$

By L'Hôpital's rule, this is $\lim_{n\to\infty} \frac{6n}{n+3}$ (after some algebraic manipulations), which is 6. Thus the original limit is e^6.

5 Take logs and compute $\lim_{n\to\infty}(1/n)\ln n = \lim_{n\to\infty}(\ln n)/n = \lim_{n\to\infty}\frac{1}{n} = 0$ by L'Hôpital's rule. Thus the original limit is $e^0 = 1$.

6 $\lim_{n\to\infty}(n-\sqrt{n^2-1}) = \lim_{n\to\infty} \frac{n-\sqrt{n^2-1}}{1} \cdot \frac{n+\sqrt{n^2-1}}{n+\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{1}{n+\sqrt{n^2+1}} = 0.$

7 Take logs, and then evaluate $\lim_{n\to\infty} \frac{1}{\ln n}\ln(1/n) = \lim_{n\to\infty}(-1) = -1$, so the original limit is e^{-1}.

8 This series oscillates among the values $\pm 1/2, \pm\sqrt{3}/2, \pm 1$, and 0, so it has no limit.

9 $a_n = (-1/0.9)^n = (-10/9)^n$. The terms grow without bound so the sequence does not converge.

10 $\lim_{n\to\infty} \tan^{-1} n = \lim_{x\to\infty} \tan^{-1} x = \dfrac{\pi}{2}.$

11

a. $S_1 = \frac{1}{3}$, $S_2 = \frac{11}{24}$, $S_3 = \frac{21}{40}$, $S_4 = \frac{17}{30}$.

b. $S_n = \dfrac{1}{2}\left(\dfrac{1}{1} + \dfrac{1}{2} - \dfrac{1}{n+1} - \dfrac{1}{n+2}\right)$, because the series telescopes.

c. From part (b), $\lim_{n\to\infty} S_n = \frac{3}{4}$, which is the sum of the series.

12 This is a geometric series with ratio 9/10, so the sum is $\frac{9/10}{1-9/10} = 9$.

13 $\sum_{k=1}^{\infty} 3(1.001)^k = 3\sum_{k=1}^{\infty}(1.001)^k$. This is a geometric series with ratio greater than 1, so it diverges.

14 This is a geometric series with ratio $-1/5$, so the sum is $\frac{1}{1+1/5} = \frac{5}{6}$

15 $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, so the series telescopes, and $S_n = 1 - \frac{1}{n+1}$. Thus $\lim_{n\to\infty} S_n = 1$, which is the value of the series.

16 This series clearly telescopes, and $S_n = \frac{1}{\sqrt{n}} - 1$, so $\lim_{n\to\infty} S_n = -1$.

17 This series telescopes. $S_n = 3 - \frac{3}{3n+1}$, so that $\lim_{n\to\infty} S_n = 3$, which is the value of the series.

18 $\sum_{k=1}^{\infty} 4^{-3k} = \sum_{k=1}^{\infty}(1/64)^k$. This is a geometric series with ratio 1/64, so its sum is $\frac{1/64}{1-1/64} = \frac{1}{63}$.

19 $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^{k+2}} = \frac{1}{9}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{9}\cdot\frac{2/3}{1-2/3} = \frac{2}{9}.$

20 This is the difference of two convergent geometric series (because both have ratios less than 1). Thus the sum of the series is equal to

$$\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k - \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{k+1} = \frac{1}{1-1/3} - \frac{2/3}{1-2/3} = \frac{3}{2} - 2 = -\frac{1}{2}.$$

21

a. It appears that the series converges, because the sequence of partial sums appears to converge to 1.5.

b. The convergence is uncertain.

c. This series clearly appears to diverge, because the partial sums seem to be growing without bound.

22 This is p-series with $p = 3/2 > 1$, so this series is convergent.

23 The series can be written $\sum \frac{1}{k^{2/3}}$, which is a p-series with $p = 2/3 < 1$, so this series diverges.

24 $a_k = \frac{2k^2+1}{\sqrt{k^3+2}} = \sqrt{\frac{4k^4+4k^2+1}{k^3+2}}$, so the sequence of terms diverges. By the Divergence Test, the given series diverges as well.

25 This is a geometric series with ratio $2/e < 1$, so the series converges.

26 Note that $\frac{1}{a_k} = \left(\left(1+\frac{3}{k}\right)^k\right)^2$, so $\lim_{k\to\infty} \frac{1}{a_k} = \lim_{k\to\infty} \left(\left(1+\frac{3}{k}\right)^k\right)^2 = (e^3)^2$, so $\lim_{k\to\infty} a_k = \frac{1}{e^6} \neq 0$, so the given series diverges by the Divergence Test.

27 Applying the Ratio Test:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{2^{k+1}(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{2^k k!} = \lim_{k\to\infty} 2\left(\frac{k}{k+1}\right)^k = \frac{2}{e} < 1,$$

so the given series converges.

28 Use the Limit Comparison Test with $\frac{1}{k}$:

$$\frac{1}{\sqrt{k^2+k}} \Big/ \frac{1}{k} = \frac{k}{\sqrt{k^2+k}} = \sqrt{\frac{k^2}{k^2+k}},$$

which has limit 1 as $k \to \infty$. Because $\sum 1/k$ diverges, the original series does as well.

29 Use the Comparison Test: $\frac{3}{2+e^k} < \frac{3}{e^k}$, but $\sum \frac{3}{e^k}$ converges because it is a geometric series with ratio $\frac{1}{e} < 1$. Thus the original series converges as well.

30 $\lim_{k\to\infty} a_k = \lim_{k\to\infty} k\sin(1/k) = \lim_{k\to\infty} \frac{\sin(1/k)}{1/k} = 1$, so the given series diverges by the Divergence Test.

31 $a_k = \frac{k^{1/k}}{k^3} = \frac{1}{k^{3-1/k}}$. For $k \geq 2$, then, $a_k < \frac{1}{k^2}$. Because $\sum \frac{1}{k^2}$ converges, the given series also converges, by the Comparison Test.

32 Use the Comparison Test: $\frac{1}{1+\ln k} > \frac{1}{k}$ for $k > 1$. Because $\sum \frac{1}{k}$ diverges, the given series does as well.

33 Use the Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{(k+1)^5}{e^{k+1}} \cdot \frac{e^k}{k^5} = \frac{1}{e} \cdot \left(\frac{k+1}{k}\right)^5$, which has limit $1/e < 1$ as $k \to \infty$. Thus the given series converges.

34 For $k > 5$, we have $k^2 - 10 > (k-1)^2$, so that $a_k = \frac{2}{k^2-10} < \frac{2}{(k-1)^2}$. Because $\sum \frac{2}{(k-1)^2}$ converges, the original series does as well.

35 Use the Comparison Test. Because $\lim_{k\to\infty} \dfrac{\ln k}{k^{1/2}} = 0$, we have that for sufficiently large k, $\ln k < k^{1/2}$, so that $a_k = \frac{2\ln k}{k^2} < \frac{2k^{1/2}}{k^2} = \frac{2}{k^{3/2}}$. Now $\sum \frac{2}{k^{3/2}}$ is convergent, because it is a p-series with $p = 3/2 > 1$. Thus the original series is convergent.

36 By the Ratio Test: $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{k+1}{e^{k+1}} \cdot \frac{e^k}{k} = \lim_{k\to\infty} \frac{1}{e} \cdot \frac{k+1}{k} = \frac{1}{e} < 1$. Thus the given series converges.

37 Use the Ratio Test. The ratio of successive terms is $\frac{2\cdot 4^{k+1}}{(2k+3)!} \cdot \frac{(2k+1)!}{2\cdot 4^k} = \frac{4}{(2k+3)(2k+2)}$. This has limit 0 as $k \to \infty$, so the given series converges.

38 Use the Ratio Test. The ratio of successive term is $\frac{9^{k+1}}{(2k+2)!} \cdot \frac{(2k)!}{9^k} = \frac{9}{(2k+2)(2k+1)}$. This has limit 0 as $k \to \infty$, so the given series converges.

39 Use the Limit Comparison Test with the harmonic series. Note that $\lim_{k\to\infty} \dfrac{\coth k}{k} \cdot \dfrac{k}{1} = \lim_{k\to\infty} \coth k = 1$. Because the harmonic series diverges, the given series does as well.

40 Use the Limit Comparison Test with the convergent geometric series whose kth term is $\frac{1}{e^k}$. We have $\lim_{k\to\infty} \frac{1}{\sinh k} \cdot \frac{e^k}{1} = \lim_{k\to\infty} \frac{2e^k}{e^k - e^{-k}} = 2\lim_{k\to\infty} \frac{1}{1-e^{-2k}} = 2$. The given series is therefore convergent.

41 Use the Divergence Test. $\lim_{k\to\infty} \tanh k = \lim_{k\to\infty} \frac{e^k + e^{-k}}{e^k - e^{-k}} = 1 \neq 0$, so the given series diverges.

42 Use the Limit Comparison Test with the convergent geometric series whose kth term is $\frac{1}{e^k}$. We have $\lim_{k\to\infty} \frac{1}{\cosh k} \cdot \frac{e^k}{1} = \lim_{k\to\infty} \frac{2e^k}{e^k + e^{-k}} = 2\lim_{k\to\infty} \frac{1}{1+e^{-2k}} = 2$. The given series is therefore convergent.

43 $|a_k| = \frac{1}{k^2-1}$. Use the Limit Comparison Test with the convergent series $\sum \frac{1}{k^2}$. Because $\lim_{k\to\infty} \frac{\frac{1}{k^2-1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1$, the given series converges absolutely.

44 This series does not converge, because $\lim_{k\to\infty} |a_k| = \lim_{k\to\infty} \frac{k^2+4}{2k^2+1} = \frac{1}{2}$.

45 Use the Ratio Test on the absolute values of the sequence of terms: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k+1}{e^{k+1}} \cdot \frac{e^k}{k} = \lim_{k\to\infty} \frac{1}{e} \cdot \frac{k+1}{k} = \frac{1}{e} < 1$. Thus, the original series is absolutely convergent.

46 Using the Limit Comparison Test with the harmonic series, we consider $\lim_{k\to\infty} a_k/(1/k) = \lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}}$ $= \lim_{k\to\infty} \sqrt{\frac{k^2}{k^2+1}} = 1$; because the comparison series diverges, so does the original series. Thus the series is not absolutely convergent. However, the terms are clearly decreasing to zero, so it is conditionally convergent.

47 Use the Ratio Test on the absolute values of the sequence of terms: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{10}{k+1} = 0$, so the series converges absolutely.

48 $\sum \frac{1}{k\ln k}$ does not converge because $\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \ln(\ln x)\Big|_2^\infty = \infty$, so the improper integral diverges. Thus the given series does not converge absolutely. However, it does converge conditionally because the terms are decreasing and approach zero.

49 Because $k^2 \ll 2^k$, $\lim_{k\to\infty} \frac{-2\cdot(-2)^k}{k^2} \neq 0$. The given series thus diverges by the Divergence Test.

50 The series of absolute values converges, by the Limit Comparison Test with the convergent geometric series whose kth term is $\frac{1}{e^k}$. This follows because $\lim_{k\to\infty} \frac{1}{e^k+e^{-k}} \cdot \frac{e^k}{1} = \lim_{k\to\infty} \frac{1}{1+e^{-2k}} = 1$.

51

a. For $|x| < 1$, $\lim_{k\to\infty} x^k = 0$, so this limit is zero.

b. This is a geometric series with ratio $-4/5$, so the sum is $\frac{1}{1+4/5} = \frac{5}{9}$.

52

a. $\lim_{k\to\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{k\to\infty} \frac{1}{k(k+1)} = 0$.

b. This series telescopes, and $S_n = 1 - \frac{1}{n+1}$, so $\lim_{n\to\infty} S_n = 1$, which is the sum of the series.

53 Consider the constant sequence with $a_k = 1$ for all k. The sequence $\{a_k\}$ converges to 1, but the corresponding series $\sum a_k$ diverges by the divergence test.

54 This is not possible. If the series $\sum_{k=1}^{\infty} a_k$ converges, then we must have $\lim_{k\to\infty} a_k = 0$.

55

a. This sequence converges because $\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = \frac{1}{1+0} = 1$.

b. Because the sequence of terms has limit 1 (which means its limit isn't zero) this series diverges by the divergence test.

56 No. The geometric sequence converges for $-1 < r \le 1$, while the geometric series converges for $-1 < r < 1$. So the geometric sequence converges for $r = 1$ but the geometric series does not.

57 Because the series converges, we must have $\lim_{k\to\infty} a_k = 0$. Because it converges to 8, the partial sums converge to 8, so that $\lim_{k\to\infty} S_k = 8$.

58 R_n is given by

$$R_n \le \int_n^{\infty} \frac{1}{x^5}\,dx = \lim_{b\to\infty} \left(-\frac{1}{4x^4}\bigg|_n^b \right) = \frac{1}{4n^4}.$$

Thus to approximate the sum to within 10^{-4}, we need $\frac{1}{4n^4} < 10^{-4}$, so $4n^4 > 10^4$ and $n = 8$.

59 The series converges absolutely for $p > 1$, conditionally for $0 < p \le 1$ in which case $\{k^{-p}\}$ is decreasing to zero.

60 By the Integral Test, the series converges if and only if the following integral converges:

$$\int_2^{\infty} \frac{1}{x\ln^p(x)}\,dx = \lim_{b\to\infty} \left(\frac{1}{1-p} \ln^{(1-p)}(x)\bigg|_2^b \right) = \lim_{b\to\infty} \frac{1}{1-p} \ln^{(1-p)}(b) - \left(\frac{1}{1-p}\right) \cdot \ln^{(1-p)}(2).$$

This limit exists only if $1 - p < 0$, i.e. $p > 1$. Note that the above calculation is for the case $p \ne 1$. In the case $p = 1$, the integral also diverges.

61 The sum is 0.2500000000 to ten decimal places. The maximum error is

$$\int_{20}^{\infty} \frac{1}{5^x}\,dx = \lim_{b\to\infty} \left(-\frac{1}{5^x \ln 5}\bigg|_{20}^b \right) = \frac{1}{5^{20}\ln 5} \approx 6.5 \times 10^{-15}.$$

62 The sum is 1.037. The maximum error is

$$\int_{20}^{\infty} \frac{1}{x^5}\,dx = \lim_{b\to\infty} \left(-\frac{1}{4x^4}\bigg|_{20}^b \right) = \frac{1}{4 \cdot 20^4} \approx 1.6 \times 10^{-6}.$$

63 The maximum error is a_{n+1}, so we want $a_{n+1} = \frac{1}{(k+1)^4} < 10^{-8}$, or $(k+1)^4 > 10^8$, so $k = 100$.

64

a. $\sum_{k=0}^{\infty} e^{kx} = \sum_{k=0}^{\infty} (e^x)^k = \frac{1}{1-e^x} = 2$, so $1 - e^x = 1/2$. Thus $e^x = 1/2$ and $x = -\ln(2)$.

b. $\sum_{k=0}^{\infty} (3x)^k = \frac{1}{1-3x} = 4$, so that $1 - 3x = \frac{1}{4}$, $x = \frac{1}{4}$.

c. The x's cancel, so the equation reads $\sum_{k=0}^{\infty} \left(\frac{1}{k-1/2} - \frac{1}{k+1/2}\right) = 6$. The series telescopes, so that the left side, up to n, is

$$\sum_{k=0}^{n} \left(\frac{1}{k-1/2} - \frac{1}{k+1/2}\right) = \frac{1}{-1/2} - \frac{1}{n+1/2} = -2 - \frac{1}{n+1/2}$$

and in the limit the equation then reads $-2 = 6$, so that there is no solution.

65

a. Let T_n be the amount of additional tunnel dug during week n. Then $T_0 = 100$ and $T_n = .95 \cdot T_{n-1} = (.95)^n T_0 = 100(0.95)^n$, so the total distance dug in N weeks is

$$S_N = 100 \sum_{k=0}^{N-1} (0.95)^k = 100 \left(\frac{1-(0.95)^N}{1-0.95}\right) = 2000(1 - 0.95^N).$$

Then $S_{10} \approx 802.5$ meters and $S_{20} \approx 1283.03$ meters.

b. The longest possible tunnel is $S_\infty = 100 \sum_{k=0}^{\infty} (0.95)^k = \frac{100}{1-.95} = 2000$ meters.

66 Let t_n be the time required to dig meters $(n-1) \cdot 100$ through $n \cdot 100$, so that $t_1 = 1$ week. Then $t_n = 1.1 \cdot t_{n-1} = (1.1)^{n-1} t_1 = (1.1)^{n-1}$ weeks. The time required to dig 1500 meters is then

$$\sum_{k=1}^{15} t_k = \sum_{k=1}^{15} (1.1)^{k-1} \approx 31.77 \text{ weeks.}$$

So it is not possible.

67

a. The area of a circle of radius r is πr^2. For $r = 2^{1-n}$, this is $2^{2-2n}\pi$. There are 2^{n-1} circles on the n^{th} page, so the total area of circles on the n^{th} page is $2^{n-1} \cdot \pi 2^{2-2n} = 2^{1-n}\pi$.

b. The sum of the areas on all pages is $\sum_{k=1}^{\infty} 2^{1-k}\pi = 2\pi \sum_{k=1}^{\infty} 2^{-k} = 2\pi \cdot \frac{1/2}{1/2} = 2\pi$.

68 $x_0 = 1$, $x_1 \approx 1.540302$, $x_2 \approx 1.57079$, $x_3 \approx 1.570796327$, which is $\frac{\pi}{2}$ to nine decimal places. Thus $p = 2$.

69

a. $B_n = 1.0025 B_{n-1} + 100$ and $B_0 = 100$.

b. $B_n = 100 \cdot 1.0025^n + 100 \cdot \frac{1-1.0025^n}{1-1.0025} = 100 \cdot 1.0025^n - 40000(1 - 1.0025^n) = 40000(1.0025^{n+1} - 1)$.

70

a. $a_n = \displaystyle\int_0^1 x^n \, dx = \frac{1}{n+1} x^{n+1} \Big|_0^1 = \frac{1}{n+1}$, so $\displaystyle\lim_{n\to\infty} a_n = 0$.

b. $b_n = \displaystyle\int_1^n \frac{1}{x^p} \, dx = \frac{1}{1-p} x^{1-p} \Big|_1^n = \frac{1}{1-p}(n^{1-p} - 1)$. Because $p > 1$, $n^{1-p} \to 0$ as $n \to \infty$, so that $\displaystyle\lim_{n\to\infty} b_n = \frac{1}{p-1}$.

71

a. $T_1 = \frac{\sqrt{3}}{16}$ and $T_2 = \frac{7\sqrt{3}}{64}$.

b. At stage n, 3^{n-1} triangles of side length $1/2^n$ are removed. Each of those triangles has an area of $\dfrac{\sqrt{3}}{4 \cdot 4^n} = \dfrac{\sqrt{3}}{4^{n+1}}$, so a total of

$$3^{n-1} \cdot \frac{\sqrt{3}}{4^{n+1}} = \frac{\sqrt{3}}{16} \cdot \left(\frac{3}{4}\right)^{n-1}$$

is removed at each stage. Thus

$$T_n = \frac{\sqrt{3}}{16} \sum_{k=1}^{n} \left(\frac{3}{4}\right)^{k-1} = \frac{\sqrt{3}}{16} \sum_{k=0}^{n-1} \left(\frac{3}{4}\right)^{k} = \frac{\sqrt{3}}{4} \left(1 - \left(\frac{3}{4}\right)^{n}\right).$$

c. $\lim_{n \to \infty} T_n = \frac{\sqrt{3}}{4}$ because $\left(\frac{3}{4}\right)^n \to 0$ as $n \to \infty$.

d. The area of the triangle was originally $\frac{\sqrt{3}}{4}$, so none of the original area is left.

72 Because the given sequence is non-decreasing and bounded above by 1, it must have a limit. A reasonable conjecture is that the limit is 1.

Chapter 9

Power Series

9.1 Approximating Functions With Polynomials

9.1.1 Let the polynomial be $p(x)$. Then $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$.

9.1.2 It generally increases, because the more derivatives of f are taken into consideration, the better "fit" the polynomial will provide to f.

9.1.3 The approximations are $p_0(0.1) = 1$, $p_1(0.1) = 1 + \frac{0.1}{2} = 1.05$, and $p_2(0.1) = 1 + \frac{0.1}{2} - \frac{.01}{8} = 1.04875$.

9.1.4 The first three terms: $f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$.

9.1.5 The remainder is the difference between the value of the Taylor polynomial at a point and the true value of the function at that point, $R_n(x) = f(x) - p_n(x)$.

9.1.6 This is explained in Theorem 9.2. The idea is that the error when using an nth order Taylor polynomial centered at a is $|R_n(x)| \le M \cdot \frac{|x-a|^{n+1}}{(n+1)!}$ where M is an upper bound for the $(n+1)$st derivative of f for values between a and x.

9.1.7

a. Note that $f(1) = 8$, and $f'(x) = 12\sqrt{x}$, so $f'(1) = 12$. Thus, $p_1(x) = 8 + 12(x-1)$.

b. $f''(x) = 6/\sqrt{x}$, so $f''(1) = 6$. Thus $p_2(x) = 8 + 12(x-2) + 3(x-1)^2$.

c. $p_1(1.1) = 12 \cdot 0.1 + 8 = 9.2$. $p_2(1.1) = 3(.1)^2 + 12 \cdot 0.1 + 8 = 9.23$.

9.1.8

a. Note that $f(1) = 1$, and that $f'(x) = -1/x^2$, so $f'(1) = -1$. Thus, $p_1(x) = 1 - (x-1) = -x + 2$.

b. $f''(x) = 2/x^3$, so $f''(1) = 2$. Thus, $p_2(x) = 2 - x + (x-1)^2$.

c. $p_1(1.05) = 0.95$. $p_2(1.05) = (0.05)^2 - 0.05 + 2 = .953$.

9.1.9

a. $f'(x) = -e^{-x}$, so $p_1(x) = f(0) + f'(0)x = 1 - x$.

b. $f''(x) = e^{-x}$, so $p_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 - x + \frac{1}{2}x^2$.

c. $p_1(0.2) = 0.8$, and $p_2(0.2) = 1 - 0.2 + \frac{1}{2}(0.04) = 0.82$.

9.1.10

a. $f'(x) = \frac{1}{2}x^{-1/2}$, so $p_1(x) = f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4)$.

b. $f''(x) = -\frac{1}{4}x^{-3/2}$, so $p_2(x) = f(4) + f'(4)(x-4) + \frac{1}{2}f''(4)(x-4)^2 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$.

c. $p_1(3.9) = 2 + \frac{1}{4}(-0.1) = 2 - 0.025 = 1.975$, and $p_2(3.9) = 2 - 0.025 - \frac{1}{64}(0.001) = 1.975$.

9.1.11

a. $f'(x) = -\frac{1}{(x+1)^2}$, so $p_1(x) = f(0) + f'(0)x = 1 - x$.

b. $f''(x) = \frac{2}{(x+1)^3}$, so $p_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 - x + x^2$.

c. $p_1(0.05) = 0.95$, and $p_2(0.05) = 1 - 0.05 + 0.0025 = 0.953$.

9.1.12

a. $f'(x) = -\sin x$, so $p_1(x) = \cos(\pi/4) - \sin(\pi/4)(x - \pi/4) = \frac{\sqrt{2}}{2}(1 - (x - \pi/4))$.

b. $f''(x) = -\cos x$, so

$$\begin{aligned} p_2(x) &= \cos(\pi/4) - \sin(\pi/4)(x-\pi/4) - \frac{1}{2}\cos(\pi/4)(x-\pi/4)^2 \\ &= \frac{\sqrt{2}}{2}\left(1 - (x-\pi/4) - \frac{1}{2}(x-\pi/4)^2\right). \end{aligned}$$

c. $p_1(0.24\pi) \approx 0.729$, $p_2(0.24\pi) \approx 0.729$.

9.1.13

a. $f'(x) = (1/3)x^{-2/3}$, so $p_1(x) = f(8) + f'(8)(x-8) = 2 + \frac{1}{12}(x-8)$.

b. $f''(x) = (-2/9)x^{-5/3}$, so $p_2(x) = f(8) + f'(8)(x-8) + \frac{1}{2}f''(8)(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$.

c. $p_1(7.5) \approx 1.958$, $p_2(7.5) \approx 1.957$.

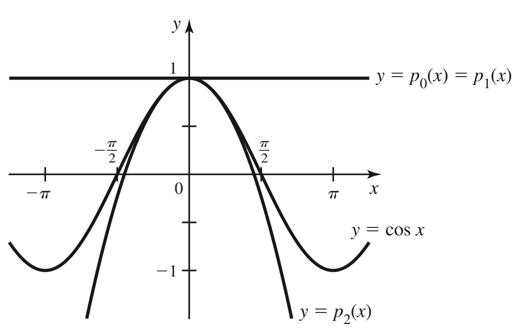
9.1.14

a. $f'(x) = \frac{1}{1+x^2}$, so $p_1(x) = f(0) + f'(0)x = x$.

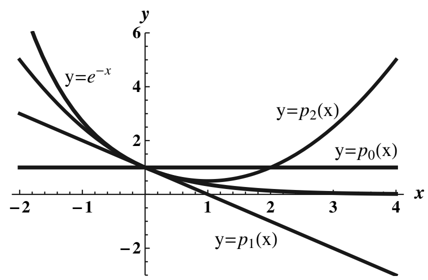
b. $f''(x) = -\frac{2x}{(1+x^2)^2}$, so $p_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = x$.

c. $p_1(0.1) = p_2(0.1) = 0.1$.

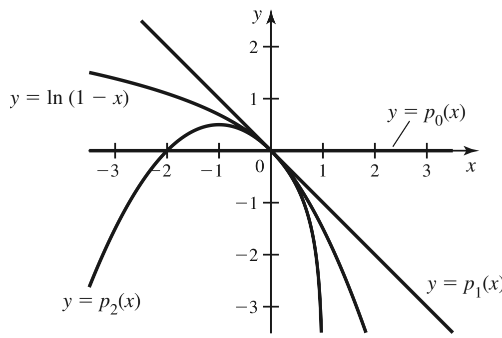
9.1.15 $f(0) = 1$, $f'(0) = -\sin 0 = 0$, $f''(0) = -\cos 0 = -1$, so that $p_0(x) = 1$, $p_1(x) = 1$, $p_2(x) = 1 - \frac{1}{2}x^2$.



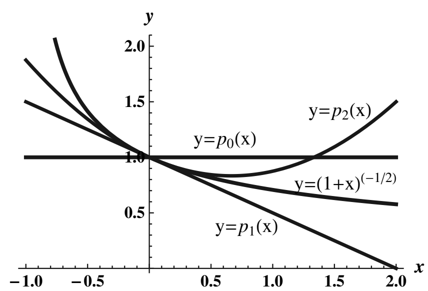
9.1.16 $f(0) = 1$, $f'(0) = -e^0 = -1$, $f''(0) = e^0 = 1$, so that $p_0(x) = 1$, $p_1(x) = 1 - x$, $p_2(x) = 1 - x + \frac{x^2}{2}$.



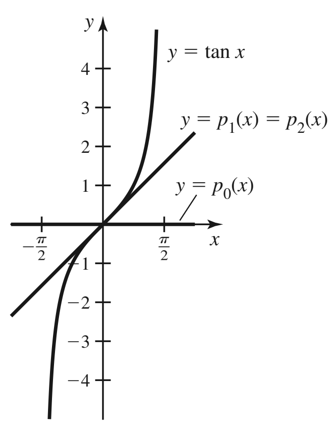
9.1.17 $f(0) = 0$, $f'(0) = -\frac{1}{1-0} = -1$,$f''(0) = -\frac{1}{(1-0)^2} = -1$, so that $p_0(x) = 0$, $p_1(x) = -x$, $p_2(x) = -x - \frac{1}{2}x^2$.



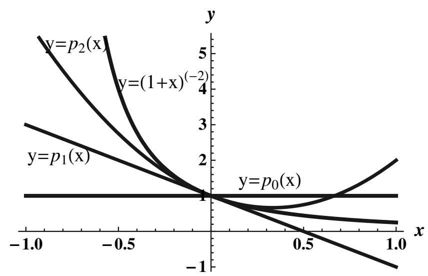
9.1.18 $f(0) = 1$, $f'(0) = (-1/2)(0+1)^{-3/2} = -1/2$, $f''(0) = (3/4)(0+1)^{-5/2} = 3/4$, so that $p_0(x) = 1$, $p_1(x) = 1 - \frac{x}{2}$, $p_2(x) = 1 - \frac{x}{2} + \frac{3}{8}x^2$.



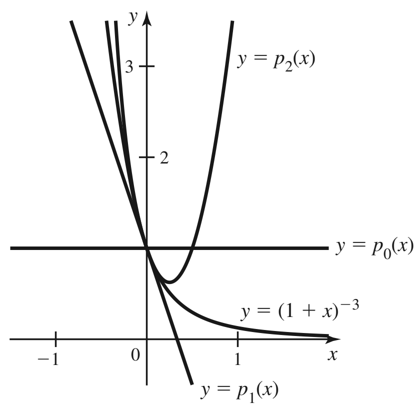
9.1.19 $f(0) = 0$. $f'(x) = \sec^2 x$, $f''(x) = 2\tan x \sec^2 x$, so that $f'(0) = 1$, $f''(0) = 0$. Thus $p_0(x) = 0$, $p_1(x) = x$, $p_2(x) = x$.



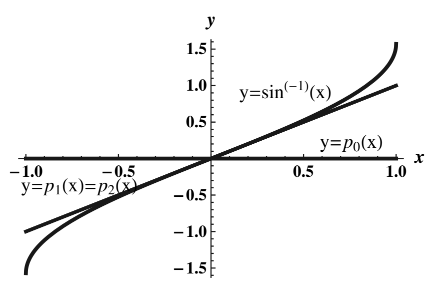
9.1.20 $f(0) = 1$, $f'(0) = (-2)(1+0)^{-3} = -2$, $f''(0) = 6(1+0)^{-4} = 6$. Thus $p_0(x) = 1$, $p_1(x) = 1 - 2x$, $p_2(x) = 1 - 2x + 3x^2$.



9.1.21 $f(0) = 1$, $f'(0) = -3(1+0)^{-4} = -3$, $f''(0) = 12(1+0)^{-5} = 12$, so that $p_0(x) = 1$, $p_1(x) = 1 - 3x$, $p_2(x) = 1 - 3x + 6x^2$.



9.1.22 $f(0) = 0$, $f'(x) = \frac{1}{\sqrt{1-x^2}}$, $f''(x) = \frac{x}{(1-x^2)^{3/2}}$, so that $f'(0) = 1$, $f''(0) = 0$. Thus $p_0(x) = 0$, $p_1(x) = x$, $p_2(x) = x$.



9.1.23

a. $p_2(0.05) \approx 1.025$.

b. The absolute error is $\sqrt{1.05} - p_2(0.05) \approx 7.68 \times 10^{-6}$.

9.1.24

a. $p_2(0.1) \approx 1.032$.

b. The absolute error is $1.1^{1/3} - p_2(0.1) \approx 5.8 \times 10^{-5}$.

9.1.25

a. $p_2(0.08) \approx 0.962$.

b. The absolute error is $p_2(0.08) - \frac{1}{\sqrt{1.08}} \approx 1.5 \times 10^{-4}$.

9.1.26

a. $p_2(0.06) = 0.058$.

b. The absolute error is $\ln 1.06 - p_2(0.06) \approx 6.9 \times 10^{-5}$.

9.1.27

a. $p_2(0.15) \approx 0.861$.

b. The absolute error is $p_2(0.15) - e^{-0.15} \approx 5.4 \times 10^{-4}$.

9.1.28

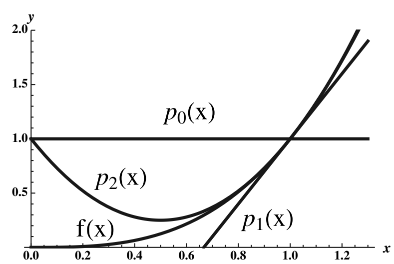
a. $p_2(0.12) \approx 0.726$.

b. The absolute error is $p_2(0.12) = \frac{1}{1.12^3} \approx 1.5 \times 10^{-2}$.

9.1.29

a. Note that $f(1) = 1$, $f'(1) = 3$, and $f''(1) = 6$. Thus, $p_0(x) = 1$, $p_1(x) = 1 + 3(x-1)$, and $p_2(x) = 1 + 3(x-1) + 3(x-1)^2$.

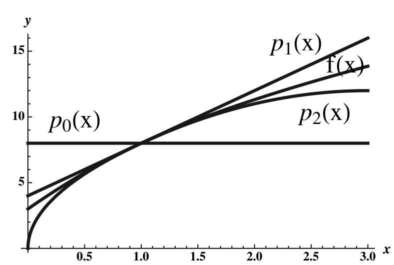
b.



9.1.30

a. Note that $f(1) = 8$, $f'(1) = \frac{4}{\sqrt{1}} = 4$, and $f''(1) = \frac{-2}{(1)^{3/2}} = -2$ Thus, $p_0(x) = 8$, $p_1(x) = 8 + 4(x-1)$, $p_2(x) = 8 + 4(x-1) - (x-1)^2$.

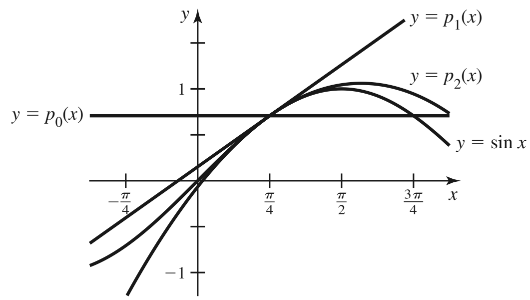
b.



9.1.31

a. $p_0(x) = \frac{\sqrt{2}}{2}$, $p_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4})$, $p_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x - \frac{\pi}{4})^2$.

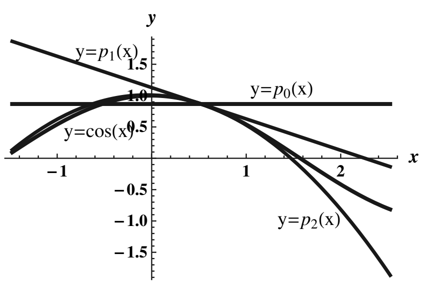
b.



9.1.32

a. $p_0(x) = \frac{\sqrt{3}}{2}$, $p_1(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right)$, $p_2(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2$.

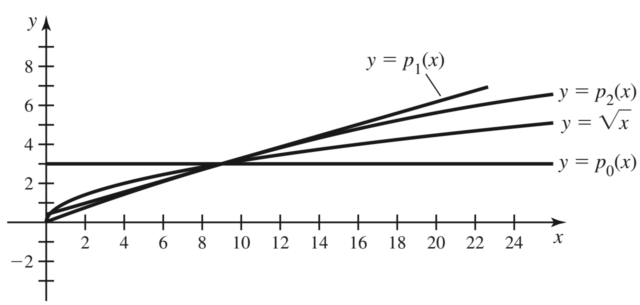
b.



9.1.33

a. $p_0(x) = 3$, $p_1(x) = 3 + \frac{1}{6}(x-9)$, $p_2(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2$.

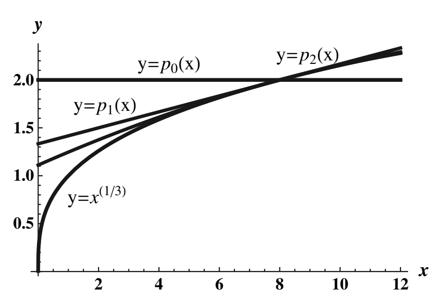
b.



9.1.34

a. $p_0(x) = 2$, $p_1(x) = 2 + \frac{1}{12}(x-8)$, $p_2(x) = x + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$.

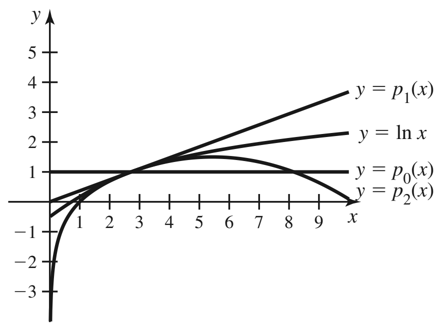
b.



9.1.35

a. $p_0(x) = 1$, $p_1(x) = 1 + \frac{1}{e}(x-e)$, $p_2(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2$.

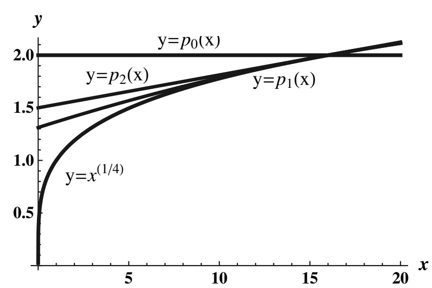
b.



9.1.36

a. $p_0(x) = 2$, $p_1(x) = 2 + \frac{1}{32}(x-16)$, $p_2(x) = 2 + \frac{1}{32}(x-16) - \frac{3}{4096}(x-16)^2$.

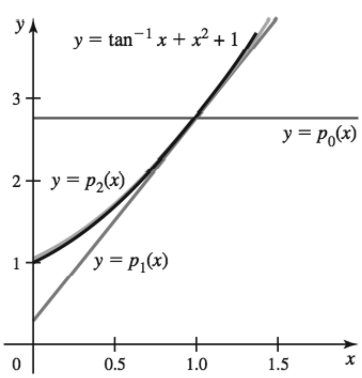
b.



9.1.37

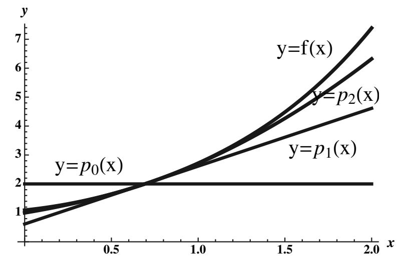
a. $f(1) = \frac{\pi}{4} + 2$, $f'(1) = \frac{1}{2} + 2 = \frac{5}{2}$. $f''(1) = -\frac{1}{2} + 2 = \frac{3}{2}$. $p_0(x) = 2 + \frac{\pi}{4}$, $p_1(x) = 2 + \frac{\pi}{4} + \frac{5}{2}(x-1)$, $p_2(x) = 2 + \frac{\pi}{4} + \frac{5}{2}(x-1) + \frac{3}{4}(x-1)^2$.

b.



9.1.38

a. $f(\ln 2) = 2$, $f'(\ln 2) = 2$, $f''(\ln 2) = 2$. So $p_0(x) = 2$, $p_1(x) = 2 + 2(x - \ln 2)$, $p_2(x) = 2 + 2(x - \ln 2) + (x - \ln 2)^2$.

b. 

9.1.39

a. Ue the Taylor polynomial centered at 0 with $f(x) = e^x$. We have $p_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$. $p_3(0.12) \approx 1.127$.

b. $|f(0.12) - p_3(0.12)| \approx 8.9 \times 10^{-6}$.

9.1.40

a. Use the Taylor polynomial centered at 0 with $f(x) = \cos(x)$. We have $p_3(x) = 1 - \frac{1}{2}x^2$. $p_3(-0.2) = 0.98$.

b. $|f(0.12) - p_3(0.12)| \approx 6.7 \times 10^{-5}$.

9.1.41

a. Use the Taylor polynomial centered at 0 with $f(x) = \tan(x)$. We have $p_3(x) = x + \frac{1}{3}x^3$. $p_3(-0.1) \approx -0.100$.

b. $|p_3(-0.1) - f(-0.1)| \approx 1.3 \times 10^{-6}$.

9.1.42

a. Use the Taylor polynomial centered at 0 with $f(x) = \ln(1 + x)$. We have $p_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$. $p_3(0.05) \approx 0.0488$.

b. $|p_3(0.05) - f(0.05)| \approx 1.5 \times 10^{-6}$.

9.1.43

a. Use the Taylor polynomial centered at 0 with $f(x) = \sqrt{1 + x}$. We have $p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$. $p_3(0.06) \approx 1.030$.

b. $|f(0.06) - p_3(0.06)| \approx 4.9 \times 10^{-7}$.

9.1.44

a. Use the Taylor polynomial centered at 81 with $f(x) = \sqrt[4]{x}$. We have $p_3(x) = 3 + \frac{1}{108}(x-81) - \frac{1}{23328}(x-81)^2 + \frac{7}{22674816}(x - 81)^3$. $p_3(79) \approx 2.981$.

b. $|p_3(79) - f(79)| \approx 4.3 \times 10^{-8}$.

9.1.45

a. Use the Taylor polynomial centered at 100 with $f(x) = \sqrt{x}$. We have $p_3(x) = 10 + \frac{1}{20}(x - 100) - \frac{1}{8000}(x - 100)^2 + \frac{1}{1600000}(x - 100)^3$. $p_3(101) \approx 10.050$.

b. $|p_3(101) - f(101)| \approx 3.9 \times 10^{-9}$.

9.1.46

a. Use the Taylor polynomial centered at 125 with $f(x) = \sqrt[3]{x}$. We have $p_3(x) = 5 + \frac{1}{75}(x - 125) - \frac{1}{28125}(x - 125)^2 + \frac{1}{6328125}(x - 125)^3$. $p_3(125) \approx 5.013$.

b. $|p_3(126) - f(126)| \approx 8.4 \times 10^{-10}$.

9.1.47

a. Use the Taylor polynomial centered at 0 with $f(x) = \sinh(x)$. Note that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$ and $f'''(0) = 1$. Then we have $p_3(x) = x + x^3/6$, so $\sinh(.5) \approx (.5)^3/6 + .5 \approx 0.521$.

b. $|p_3(.5) - \sinh(.5)| \approx 2.6 \times 10^{-4}$.

9.1.48

a. Use the Taylor polynomial centered at 0 with $f(x) = \tanh(x)$, Note that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -2$. Then we have $p_3(x) = -x^3/3 + x$, so $\tanh(.5) \approx -(.5)^2/3 + .5 \approx 0.449$.

b. $|p_3(x) - \tanh(.5)| \approx 3.8 \times 10^{-3}$.

9.1.49 With $f(x) = \sin x$ we have $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for c between 0 and x.

9.1.50 With $f(x) = \cos 2x$ we have $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for c between 0 and x.

9.1.51 With $f(x) = e^{-x}$ we have $f^{(n+1)}(x) = (-1)^{n+1}e^{-x}$, so that $R_n(x) = \dfrac{(-1)^{n+1}e^{-c}}{(n+1)!}x^{n+1}$ for c between 0 and x.

9.1.52 With $f(x) = \cos x$ we have $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}\left(x - \dfrac{\pi}{2}\right)^{n+1}$ for c between $\frac{\pi}{2}$ and x.

9.1.53 With $f(x) = \sin x$ we have $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}\left(x - \dfrac{\pi}{2}\right)^{n+1}$ for c between $\frac{\pi}{2}$ and x.

9.1.54 With $f(x) = \frac{1}{1-x}$ we have $f^{(n+1)}(x) = (-1)^{n+1}\frac{1}{(1-x)^{n+2}}$ so that $R_n(x) = \dfrac{(-1)^{n+1}}{(1-c)^{n+2}}\left(x^{n+1}\right)$ for c between 0 and x.

9.1.55 $f(x) = \sin x$, so $f^{(5)}(x) = \cos x$. Because $\cos x$ is bounded in magnitude by 1, the remainder is bounded by $|R_4(x)| \le \frac{0.3^5}{5!} \approx 2.0 \times 10^{-5}$.

9.1.56 $f(x) = \cos x$, so $f^{(4)}(x) = \cos x$. Because $\cos x$ is bounded in magnitude by 1, the remainder is bounded by $|R_3(x)| \le \frac{0.45^4}{4!} \approx 1.7 \times 10^{-3}$.

9.1.57 $f(x) = e^x$, so $f^{(5)}(x) = e^x$. Because $e^{0.25}$ is bounded by 2, $|R_4(x)| \le 2 \cdot \frac{0.25^5}{5!} \approx 1.63 \times 10^{-5}$.

9.1.58 $f(x) = \tan x$, so $f^{(3)}(x) = 2\sec^2 x(\sec^2 x + 2\tan^2 x)$. Now, since both $\tan x$ and $\sec x$ are increasing on $[0, \pi/2]$, and $0.3 < \frac{\pi}{6} \approx 0.524$, we can get an upper bound on $f^{(3)}(x)$ on $[0, 0.3]$ by evaluating at $\frac{\pi}{6}$; this gives $f^{(3)}(x) < \frac{16}{3}$ on $[0, 0.3]$. Thus $|R_2(x)| \le \frac{16}{3} \cdot \frac{0.3^3}{3!} = 2.4 \times 10^{-2}$.

9.1.59 $f(x) = e^{-x}$, so $f^{(5)}(x) = -e^{-x}$. Because $f^{(5)}$ achieves its maximum magnitude in the range at $x = 0$, which has absolute value 1, $|R_4(x)| \le 1 \cdot \frac{0.5^5}{5!} \approx 2.6 \times 10^{-4}$.

9.1.60 $f(x) = \ln(1 + x)$, so $f^{(4)}(x) = -\frac{6}{(x+1)^4}$. On $[0, 0.4]$, the maximum magnitude is 6, so $|R_3(x)| \le 6 \cdot \frac{0.4^4}{4!} = 6.4 \times 10^{-3}$.

9.1.61 Here $n = 3$ or 4, so use $n = 4$, and $M = 1$ because $f^{(5)}(x) = \cos x$, so that $R_4(x) \le \frac{(\pi/4)^5}{5!} \approx 2.49 \times 10^{-3}$.

9.1.62 $n = 2$ or 3, so use $n = 3$, and $M = 1$ because $f^{(4)}(x) = \cos x$, so that $|R_3(x)| \le \frac{(\pi/4)^4}{4!} \approx 1.6 \times 10^{-2}$.

9.1.63 $n = 2$ and $M = e^{1/2} < 2$, so $|R_2(x)| \le 2 \cdot \frac{(1/2)^3}{3!} \approx 4.2 \times 10^{-2}$.

9.1.64 $n = 1$ or 2, so use 2, and $f^{(3)}(x) = 2\sec^2 x(\sec^2 x + 2\tan^2 x)$. On $[\frac{-\pi}{6}, \frac{\pi}{6}]$ this achieves its maximum value at $\pm\frac{\pi}{6}$; that value is $\frac{16}{3}$. Thus $|R_2(x)| \le \frac{16}{3} \cdot \frac{(\pi/6)^3}{3!} \approx 1.28 \times 10^{-1}$.

9.1.65 $n = 2$; $f^{(3)}(x) = \frac{2}{(1+x)^3}$, which achieves its maximum at $x = -0.2$: $|f^{(3)}(x)| = \frac{2}{0.8^3} < 4$. Then $|R_2(x)| \le 4 \cdot \frac{0.2^3}{3!} \approx 5.4 \times 10^{-3}$.

9.1.66 $n = 1$, $f''(x) = -\frac{1}{4}(1+x)^{-3/2}$, which achieves its maximum magnitude at $x = -0.1$, where it is less than 1/3. Thus $R_1(x) \le \frac{1}{3} \cdot \frac{0.1^2}{2!} \approx 1.7 \times 10^{-3}$.

9.1.67 Use the Taylor series for e^x at $x = 0$. The derivatives of e^x are e^x. On $[-0.5, 0]$, the maximum magnitude of any derivative is thus 1 at $x = 0$, so $|R_n(-0.5)| \le \frac{0.5^{n+1}}{(n+1)!}$, so for $R_n(-0.5) < 10^{-3}$ we need $n = 4$.

9.1.68 Use the Taylor series at $x = 0$ for $\sin x$. The magnitude of any derivative of $\sin x$ is bounded by 1, so $|R_n(0.2)| \le \frac{0.2^{n+1}}{(n+1)!}$, so for $R_n(0.2) < 10^{-3}$ we need $n = 3$.

9.1.69 Use the Taylor series for $\cos x$ at $x = 0$. The magnitude of any derivative of $\cos x$ is bounded by 1, so $|R_n(-0.25)| \le \frac{0.25^{n+1}}{(n+1)!}$, so for $|R_n(-0.25)| < 10^{-3}$ we need $n = 3$.

9.1.70 Use the Taylor series for $f(x) = \ln(1+x)$ at $x = 0$. Then $|f^{(n+1)}(x)| = \frac{n!}{(1+x)^{n+1}}$, which for $x \in [-0.15, 0]$ achieves its maximum at $x = -.15$. This maximum is less than $(1.2)^{n+1} \cdot n!$. Thus $|R_n(-0.15)| \le (1.2)^{n+1} \cdot n! \cdot \frac{.18^{n+1}}{(n+1)!} = \frac{1.2\cdot(0.15)^{n+1}}{n}$, so for $|R_n(-0.15)| < 10^{-3}$ we need $n = 3$.

9.1.71 Use the Taylor series for $f(x) = \sqrt{x}$ at $x = 1$. Then $|f^{(n+1)}(x)| = \frac{1\cdot3\cdot\cdots\cdot(2n-1)}{2^{n+1}}x^{-(2n+1)/2}$, which achieves its maximum on $[1, 1.06]$ at $x = 1$. Then

$$|R_n(1.06)| \le \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2^{n+1}} \cdot \frac{(1.06-1)^{n+1}}{(n+1)!},$$

and for $|R_n(0.06)| < 10^{-3}$ we need $n = 1$.

9.1.72 Use the Taylor series for $f(x) = \sqrt{1/(1-x)}$ at $x = 0$. Then $|f^{(n+1)}(x)| = \dfrac{1 \cdot 3 \cdot \cdots \cdot (2n+1)}{2^{n+1}}(1-x)^{(-3-2n)/2}$, which achieves its maximum on $[0, 0.15]$ at $x = 0.15$. Thus

$$\begin{aligned} |R_n(0.15)| &\le \frac{1 \cdot 3 \cdot \cdots \cdot (2n+1)}{2^{n+1}} \cdot \left(\frac{1}{1-0.15}\right)^{(2n+3)/2} \cdot \frac{0.15^{n+1}}{(n+1)!} \\ &= \frac{1 \cdot 3 \cdot \cdots \cdot (2n+1)}{2^{n+1}(n+1)!} \cdot \left(\frac{0.15^{n+1}}{0.85^{(2n+3)/2}}\right), \end{aligned}$$

and for $|R_n(0.15)| < 10^{-3}$ we need $n = 3$.

9.1.73

a. False. If $f(x) = e^{-2x}$, then $f^{(n)}(x) = (-1)^n 2^n e^{-2x}$, so that $f^{(n)}(0) \ne 0$ and all powers of x are present in the Taylor series.

b. True. The constant term of the Taylor series is $f(0) = 1$. Higher-order terms all involve derivatives of $f(x) = x^5 - 1$ evaluated at $x = 0$; clearly for $n < 5$, $f^{(n)}(0) = 0$, and for $n > 5$, the derivative itself vanishes. Only for $n = 5$, where $f^{(5)}(x) = 5!$, is the derivative nonzero, so the coefficient of x^5 in the Taylor series is $f^{(5)}(0)/5! = 1$ and the Taylor polynomial of order 10 is in fact $x^5 - 1$. Note that this statement is true of any polynomial of degree at most 10.

c. True. The odd derivatives of $\sqrt{1+x^2}$ vanish at $x=0$, while the even ones do not.

d. True. Clearly the second-order Taylor polynomial for f at a has degree at most 2. However, the coefficient of $(x-a)^2$ is $\frac{1}{2}f''(a)$, which is zero because f has an inflection point at a.

9.1.74 Let $p(x) = \sum_{k=0}^{n} c_k(x-a)^k$ be the n^{th} polynomial for $f(x)$ at a. Because $f(a) = p(a)$, it follows that $c_0 = f(0)$. Now, the k^{th} derivative of $p(x)$, $1 \le k \le n$, is $p^{(k)}(x) = k!c_k +$ terms involving $(x-a)^i$, $i > 0$, so that $f^{(k)}(a) = p^{(k)}(a) = k! \cdot c_k$ so that $c_k = \frac{f^{(k)}(a)}{k!}$.

9.1.75

a. This matches (C) because for $f(x) = (1+2x)^{1/2}$, $f''(x) = -(1+2x)^{-3/2}$ so $\frac{f''(0)}{2!} = -\frac{1}{2}$.

b. This matches (E) because for $f(x) = (1+2x)^{-1/2}$, $f''(x) = 3(1+2x)^{-5/2}$, so $\frac{f''(0)}{2!} = \frac{3}{2}$.

c. This matches (A) because $f^{(n)}(x) = 2^n e^{2x}$, so that $f^{(n)}(0) = 2^n$, which is (A)'s pattern.

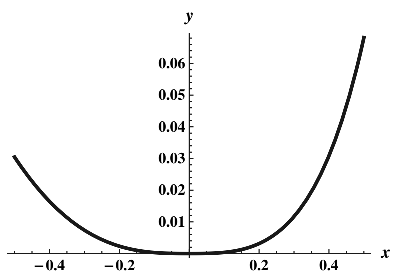
d. This matches (D) because $f''(x) = 8(1+2x)^{-3}$ and $f''(0) = 8$, so that $f''(0)/2! = 4$

e. This matches (B) because $f'(x) = -6(1+2x)^{-4}$ so that $f'(0) = -6$.

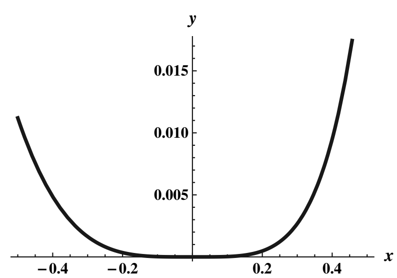
f. This matches (F) because $f^{(n)}(x) = (-2)^n e^{-2x}$, so $f^{(n)}(0) = (-2)^n$, which is (F)'s pattern.

9.1.76

a.



$|\ln(1-x) - p_2(x)|$



$|\ln(1-x) - p_3(x)|$

b. The error seems to be largest at $x = \frac{1}{2}$ and smallest at $x = 0$.

c. The error bound found in Example 7 for $|\ln(1-x) - p_3(x)|$ was 0.25. The actual error seems much less than that, about 0.02.

9.1.77

a. $p_2(0.1) = 0.1$. The maximum error in the approximation is $1 \cdot \frac{0.1^3}{3!} \approx 1.67 \times 10^{-4}$.

b. $p_2(0.2) = 0.2$. The maximum error in the approximation is $1 \cdot \frac{0.2^3}{3!} \approx 1.33 \times 10^{-3}$.

9.1.78

a. $p_1(0.1) = 0.1$. $f''(x) = 2\tan x(1+\tan^2 x)$. Because $\tan(0.1) < 0.2$, $|f''(c)| \le 2(.2)(1+.2^2) = 0.416$. Thus the maximum error is $\frac{0.416}{2!} \cdot 0.1^2 \approx 2.1 \times 10^{-3}$.

b. $p_1(0.2) = 0.2$. The maximum error is $\frac{0.416}{2} \cdot 0.2^2 \approx 8.3 \times 10^{-3}$.

9.1.79

a. $p_3(0.1) = 1 - .01/2 = 0.995$. The maximum error is $1 \cdot \frac{0.1^4}{4!} \approx 4.2 \times 10^{-6}$.

b. $p_3(0.2) = 1 - .04/2 = 0.98$. The maximum error is $1 \cdot \frac{0.2^4}{4!} \approx 6.7 \times 10^{-5}$.

9.1.80

a. $p_2(0.1) = 0.1$ (we can take $n = 2$ because the coefficient of x^2 in $p_2(x)$ is 0). $f^{(3)}(x) = \frac{6x^2-2}{(x^2+1)^3}$ has a maximum magnitude value of 2, the maximum error is $2 \cdot \frac{0.1^3}{3!} \approx 3.3 \times 10^{-4}$.

b. $p_2(0.2) = 0.2$. The maximum error is $2 \cdot \frac{0.2^3}{3!} \approx 2.7 \times 10^{-3}$.

9.1.81

a. $p_1(0.1) = 1.05$. Because $|f''(x)| = \frac{1}{4}(1+x)^{-3/2}$ has a maximum value of $1/4$ at $x = 0$, the maximum error is $\frac{1}{4} \cdot \frac{0.1^2}{2} \approx 1.3 \times 10^{-3}$.

b. $p_1(0.2) = 1.1$. The maximum error is $\frac{1}{4} \cdot \frac{0.2^2}{2} = 5 \times 10^{-3}$.

9.1.82

a. $p_2(0.1) = 0.1 - 0.01/2 = 0.095$. Because $|f^{(3)}(x)| = \frac{2}{(x+1)^3}$ achieves a maximum of 2 at $x = 0$, the maximum error is $2 \cdot \frac{0.1^3}{3!} \approx 3.3 \times 10^{-4}$.

b. $p_2(0.2) = 0.2 - 0.04/2 = 0.18$. The maximum error is $2 \cdot \frac{0.2^3}{3!} \approx 2.7 \times 10^{-3}$.

9.1.83

a. $p_1(0.1) = 1.1$. Because $f''(x) = e^x$ is less than 2 on $[0, 0.1]$, the maximum error is less than $2 \cdot \frac{0.1^2}{2!} = 10^{-2}$.

b. $p_1(0.2) = 1.2$. The maximum error is less than $2 \cdot \frac{0.2^2}{2!} = .04 = 4 \times 10^{-2}$.

9.1.84

a. $p_1(0.1) = 0.1$. Because $f''(x) = \frac{x}{(1-x^2)^{3/2}}$ is less than 1 on $[0, 0.2]$, the maximum error is $1 \cdot \frac{0.1^3}{3!} \approx 1.7 \times 10^{-4}$.

b. $p_1(0.2) = 0.2$. The maximum error is $1 \cdot \frac{0.2^3}{3!} \approx 1.3 \times 10^{-3}$.

9.1.85

a.

	$\lvert\sec x - p_2(x)\rvert$	$\lvert\sec x - p_4(x)\rvert$
-0.2	3.4×10^{-4}	5.5×10^{-6}
-0.1	2.1×10^{-5}	8.5×10^{-8}
0.0	0	0
0.1	2.1×10^{-5}	8.5×10^{-8}
0.2	3.4×10^{-4}	5.5×10^{-6}

b. The errors are equal for positive and negative x. This makes sense, because $\sec(-x) = \sec x$ and $p_n(-x) = p_n(x)$ for $n = 2, 4$. The errors appear to get larger as x gets farther from zero.

9.1.86

a.

	$\lvert\cos x - p_2(x)\rvert$	$\lvert\cos x - p_4(x)\rvert$
-0.2	6.66×10^{-5}	8.88×10^{-8}
-0.1	4.17×10^{-6}	1.39×10^{-9}
0.0	0	0
0.1	4.17×10^{-6}	1.39×10^{-9}
0.2	6.66×10^{-5}	8.88×10^{-8}

b. The errors are equal for positive and negative x. This makes sense, because $\cos(-x) = \cos x$ and $p_n(-x) = p_n(x)$ for $n = 2, 4$. The errors appear to get larger as x gets farther from zero.

9.1.87

a.

	$\lvert e^{-x} - p_1(x)\rvert$	$\lvert e^{-x} - p_2(x)\rvert$
-0.2	2.14×10^{-2}	1.40×10^{-3}
-0.1	5.17×10^{-3}	1.71×10^{-4}
0.0	0	0
0.1	4.84×10^{-3}	1.63×10^{-4}
0.2	1.87×10^{-2}	1.27×10^{-3}

b. The errors are different for positive and negative displacements from zero, and appear to get larger as x gets farther from zero.

9.1.88

a.

	$\lvert f(x) - p_1(x)\rvert$	$\lvert f(x) - p_2(x)\rvert$
-0.2	2.31×10^{-2}	3.14×10^{-4}
-0.1	5.36×10^{-3}	3.61×10^{-4}
0.0	0	0
0.1	4.69×10^{-3}	3.10×10^{-4}
0.2	1.77×10^{-2}	2.32×10^{-3}

b. The errors are different for positive and negative displacements from zero, and appear to get larger as x gets farther from zero.

9.1.89

a.

	$\lvert \tan x - p_1(x)\rvert$	$\lvert \tan x - p_3(x)\rvert$
-0.2	2.71×10^{-3}	4.34×10^{-5}
-0.1	3.35×10^{-4}	1.34×10^{-6}
0.0	0	0
0.1	3.35×10^{-4}	1.34×10^{-6}
0.2	2.71×10^{-3}	4.34×10^{-5}

b. The errors are equal for positive and negative x. This makes sense, because $\tan(-x) = -\tan x$ and $p_n(-x) = -p_n(x)$ for $n = 1, 3$. The errors appear to get larger as x gets farther from zero.

9.1.90 The true value of $\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}} \approx 0.966$. The 6^{th}-order Taylor polynomial for $\cos x$ centered at $x = 0$ is

$$p_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}.$$

Evaluating the polynomials at $x = \pi/12$ produces the following table:

n	$p_n\left(\frac{\pi}{12}\right)$	$\lvert p_n\left(\frac{\pi}{12}\right) - \cos\frac{\pi}{12}\rvert$
1	1.0000000000	3.41×10^{-2}
2	0.9657305403	1.95×10^{-4}
3	0.9657305403	1.95×10^{-4}
4	0.9659262729	4.47×10^{-7}
5	0.9659262729	4.47×10^{-7}
6	0.9659258257	5.47×10^{-10}

The 6^{th}-order Taylor polynomial for $\cos x$ centered at $x = \pi/6$ is

$$p_6(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{48}\left(x - \frac{\pi}{6}\right)^4 - \frac{1}{240}\left(x - \frac{\pi}{6}\right)^5 - \frac{\sqrt{3}}{1440}\left(x - \frac{\pi}{6}\right)^6.$$

Evaluating the polynomials at $x = \pi/12$ produces the following table:

n	$p_n\left(\frac{\pi}{12}\right)$	$\lvert p_n\left(\frac{\pi}{12}\right) - \cos\frac{\pi}{12}\rvert$
1	0.9969250977	3.10×10^{-2}
2	0.9672468750	1.32×10^{-3}
3	0.9657515877	1.74×10^{-4}
4	0.9659210972	4.73×10^{-6}
5	0.9659262214	3.95×10^{-7}
6	0.9659258342	7.88×10^{-9}

Comparing the tables shows that using the polynomial centered at $x = 0$ is more accurate when n is even while using the polynomial centered at $x = \pi/6$ is more accurate when n is odd. To see why, consider the remainder. Let $f(x) = \cos x$. By Theorem 9.2, the magnitude of the remainder when approximating $f(\pi/12)$ by the polynomial p_n centered at 0 is:

$$\left|R_n\left(\frac{\pi}{12}\right)\right| = \frac{|f^{(n+1)}(c)|}{(n+1)!}\left(\frac{\pi}{12}\right)^{n+1}$$

for some c with $0 < c < \frac{\pi}{12}$, while the magnitude of the remainder when approximating $f(\pi/12)$ by the polynomial p_n centered at $\pi/6$ is:

$$\left|R_n\left(\frac{\pi}{12}\right)\right| = \frac{|f^{(n+1)}(c)|}{(n+1)!}\left(\frac{\pi}{12}\right)^{n+1}$$

for some c with $\frac{\pi}{12} < c < \frac{\pi}{6}$. When n is odd, $|f^{(n+1)}(c)| = |\cos c|$. Because $\cos x$ is a positive and decreasing function over $[0, \pi/6]$, the magnitude of the remainder in using the polynomial centered at $\pi/6$ will be less than the remainder in using the polynomial centered at 0, and the former polynomial will be more accurate. When n is even, $|f^{(n+1)}(c)| = |\sin c|$. Because $\sin x$ is a positive and increasing function over $[0, \pi/6]$, the remainder in using the polynomial centered at 0 will be less than the remainder in using the polynomial centered at $\pi/6$, and the former polynomial will be more accurate.

9.1.91 The true value of $e^{0.35} \approx 1.419067549$. The 6^{th}-order Taylor polynomial for e^x centered at $x = 0$ is

$$p_6(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}.$$

Evaluating the polynomials at $x = 0.35$ produces the following table:

n	$p_n(0.35)$	$\lvert p_n(0.35) - e^{0.35}\rvert$
1	1.350000000	6.91×10^{-2}
2	1.411250000	7.82×10^{-3}
3	1.418395833	6.72×10^{-4}
4	1.419021094	4.65×10^{-5}
5	1.419064862	2.69×10^{-6}
6	1.419067415	1.33×10^{-7}

The 6^{th}-order Taylor polynomial for e^x centered at $x = \ln 2$ is

$$p_6(x) = 2 + 2(x-\ln 2) + (x-\ln 2)^2 + \frac{1}{3}(x-\ln 2)^3 + \frac{1}{12}(x-\ln 2)^4 + \frac{1}{60}(x-\ln 2)^5 + \frac{1}{360}(x-\ln 2)^6.$$

Evaluating the polynomials at $x = 0.35$ produces the following table:

n	$p_n(0.35)$	$\lvert p_n(0.35) - e^{0.35}\rvert$
1	1.313705639	1.05×10^{-1}
2	1.431455626	1.24×10^{-2}
3	1.417987101	1.08×10^{-3}
4	1.419142523	7.50×10^{-5}
5	1.419063227	4.32×10^{-6}
6	1.419067762	2.13×10^{-7}

Comparing the tables shows that using the polynomial centered at $x = 0$ is more accurate for all n. To see why, consider the remainder. Let $f(x) = e^x$. By Theorem 9.2, the magnitude of the remainder when approximating $f(0.35)$ by the polynomial p_n centered at 0 is:

$$|R_n(0.35)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}(0.35)^{n+1} = \frac{e^c}{(n+1)!}(0.35)^{n+1}$$

for some c with $0 < c < 0.35$ while the magnitude of the remainder when approximating $f(0.35)$ by the polynomial p_n centered at $\ln 2$ is:

$$|R_n(0.35)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|0.35 - \ln 2|^{n+1} = \frac{e^c}{(n+1)!}(\ln 2 - 0.35)^{n+1}$$

for some c with $0.35 < c < \ln 2$. Because $\ln 2 - 0.35 \approx 0.35$, the relative size of the magnitudes of the remainders is determined by e^c in each remainder. Because e^x is an increasing function, the remainder in using the polynomial centered at 0 will be less than the remainder in using the polynomial centered at $\ln 2$, and the former polynomial will be more accurate.

9.1.92

a. Let x be a point in the interval on which the derivatives of f are assumed continuous. Then f' is continuous on $[a, x]$, and the Fundamental Theorem of Calculus implies that because f is an antiderivative of f', then $\int_a^x f'(t)\,dt = f(x) - f(a)$, or $f(x) = f(a) + \int_a^x f'(t)\,dt$.

b. Using integration by parts with $u = f'(t)$ and $dv = dt$, note that we may choose any antiderivative of dv; we choose $t - x = -(x - t)$. Then

$$\begin{aligned} f(x) &= f(a) - f'(t)(x-t)\Big|_{t=a}^{x} + \int_a^x (x-t)f''(t)\,dt \\ &= f(a) - f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt. \end{aligned}$$

c. Integrate by parts again, using $u = f''(t)$, $dv = (x-t)\,dt$, so that $v = -\frac{(x-t)^2}{2}$:

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt \\ &= f(a) + f'(a)(x-a) - \frac{(x-t)^2}{2}f''(t)\Big|_a^x + \frac{1}{2}\int_a^x (x-t)^2 f'''(t)\,dt \\ &= f(a) + f'(a)(x-a) + \frac{f''(t)}{2}(x-a)^2 + \frac{1}{2}\int_a^x (x-t)^2 f'''(t)\,dt. \end{aligned}$$

It is clear that continuing this process will give the desired result, because successive integral of $x - t$ give $-\frac{1}{k!}(x-t)^k$.

d. **Lemma:** Let g and h be continuous functions on the interval $[a, b]$ with $g(t) \geq 0$. Then there is a number c in $[a, b]$ with

$$\int_a^b h(t)g(t)\,dt = h(c)\int_a^b g(t)\,dt.$$

Proof: We note first that if $g(t) = 0$ for all t in $[a, b]$, then the result is clearly true. We can thus assume that there is some t in $[a, b]$ for which $g(t) > 0$. Because g is continuous, there must be an interval about this t on which g is strictly positive, so we may assume that

$$\int_a^b g(t)\,dt > 0.$$

Because h is continuous on $[a, b]$, the Extreme Value Theorem shows that h has an absolute minimum value m and an absolute maximum value M on the interval $[a, b]$. Thus

$$m \leq h(t) \leq M$$

for all t in $[a, b]$, so

$$m\int_a^b g(t)\,dt \leq \int_a^b h(t)g(t)\,dt \leq M\int_a^b g(t)\,dt.$$

Because $\int_a^b g(t)\,dt > 0$, we have

$$m \leq \frac{\int_a^b h(t)g(t)\,dt}{\int_a^b g(t)\,dt} \leq M.$$

Now there are points in $[a, b]$ at which $h(t)$ equals m and M, so the Intermediate Value Theorem shows that there is a point c in $[a, b]$ at which

$$h(c) = \frac{\int_a^b h(t)g(t)\,dt}{\int_a^b g(t)\,dt}$$

or

$$\int_a^b h(t)g(t)\,dt = h(c)\int_a^b g(t)\,dt.$$

Applying the lemma with $h(t) = \frac{f^{(n+1)}(t)}{n!}$, $g(t) = (x-t)^n$, we see that $R_n(x) = \frac{f^{(n+1)}(c)}{n!}\int_a^x (x-t)^n\,dt = \frac{f^{(n+1)}(c)}{n!} \cdot \frac{1}{n+1}(x-a)^{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some $c \in [a, b]$.

9.1.93

a. The slope of the tangent line to $f(x)$ at $x = a$ is by definition $f'(a)$; by the point-slope form for the equation of a line, we have $y - f(a) = f'(a)(x - a)$, or $y = f(a) + f'(a)(x - a)$.

b. The Taylor polynomial centered at a is $p_1(x) = f(a) + f'(a)(x - a)$, which is the tangent line at a.

9.1.94

a. $p_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$, so that $p_2'(x) = f'(a) + f''(a)(x - a)$ and $p_2''(x) = f''(a)$. If f has a local maximum at a, then $f'(a) = 0, f''(a) \leq 0$, but then $p_2'(a) = 0$ and $p_2''(a) \leq 0$ by the above, so that $p_2(x)$ also has a local maximum at a.

b. Similarly, if f has a local minimum at a, then $f'(a) = 0, f''(a) \geq 0$, but then $p_2'(a) = 0$ and $p_2''(a) \geq 0$ by the above, so that $p_2(x)$ also has a local minimum at a.

c. Recall that f has an inflection point at a if the second derivative of f changes sign at a. But $p_2''(x)$ is a constant, so p_2 does not have an inflection point at a (or anywhere else).

d. No. For example, let $f(x) = x^3$. Then $p_2(x) = 0$, so that the second-order Taylor polynomial has a local maximum at $x = 0$, but $f(x)$ does not. It also has a local minimum at $x = 0$, but $f(x)$ does not.

9.1.95

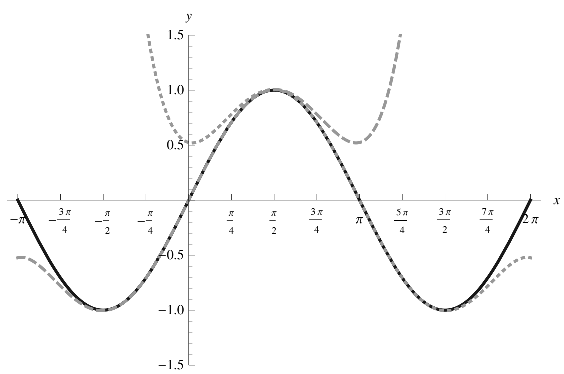
a. We have

$$\begin{aligned} f(0) &= f^{(4)}(0) = \sin 0 = 0 & f(\pi) &= f^{(4)}(\pi) = \sin \pi = 0 \\ f'(0) &= f^{(5)}(0) = \cos 0 = 1 & f'(\pi) &= f^{(5)}(0) = \cos \pi = -1 \\ f''(0) &= -\sin 0 = 0 & f''(\pi) &= -\sin \pi = 0 \\ f'''(0) &= -\cos 0 = -1 & f'''(\pi) &= -\cos \pi = 1. \end{aligned}$$

Thus

$$\begin{aligned} p_5(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\ q_5(x) &= -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5. \end{aligned}$$

b. A plot of the three functions, with $\sin x$ the black solid line, $p_5(x)$ the dashed line, and $q_5(x)$ the dotted line is below.



$p_5(x)$ and $\sin x$ are almost indistinguishable on $[-\pi/2, \pi/2]$, after which $p_5(x)$ diverges pretty quickly from $\sin x$. $q_5(x)$ is reasonably close to $\sin x$ over the entire range, but the two are almost indistinguishable on $[\pi/2, 3\pi/2]$. $p_5(x)$ is a better approximation than $q_5(x)$ on about $[-\pi, \pi/2)$, while $q_5(x)$ is better on about $(\pi/2, 2\pi]$.

c. Evaluating the errors gives

x	$\lvert\sin x - p_5(x)\rvert$	$\lvert\sin x - q_5(x)\rvert$
$\frac{\pi}{4}$	3.6×10^{-5}	7.4×10^{-2}
$\frac{\pi}{2}$	4.5×10^{-3}	4.5×10^{-3}
$\frac{3\pi}{4}$	7.4×10^{-2}	3.6×10^{-5}
$\frac{5\pi}{4}$	2.3	3.6×10^{-5}
$\frac{7\pi}{4}$	20.4	7.4×10^{-2}

d. $p_5(x)$ is a better approximation than $q_5(x)$ only at $x = \frac{\pi}{4}$, in accordance with part (b). The two are equal at $x = \frac{\pi}{2}$, after which $q_5(x)$ is a substantially better approximation than $p_5(x)$.

9.1.96

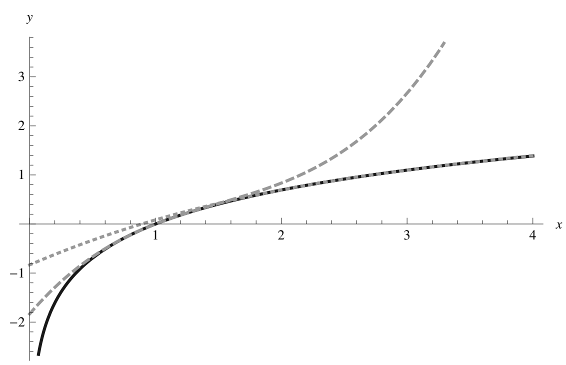
a. We have

$$f(1) = \ln 1 = 0 \qquad f(e) = \ln e = 1$$
$$f'(1) = 1 \qquad f'(e) = \frac{1}{3}$$
$$f''(1) = -1 \qquad f''(e) = -\frac{1}{e^2}$$
$$f'''(1) = 2 \qquad f'''(e) = \frac{2}{e^3}.$$

Thus

$$p_3(x) = (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$
$$q_3(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3.$$

b. A plot of the three functions, with $\ln x$ the black solid line, $p_3(x)$ the dashed line, and $q_3(x)$ the dotted line is below.



c. Evaluating the errors gives

x	$\lvert\ln x - p_3(x)\rvert$	$\lvert\ln x - q_3(x)\rvert$
0.5	2.6×10^{-2}	3.6×10^{-1}
1.0	0	8.4×10^{-2}
1.5	1.1×10^{-2}	1.6×10^{-2}
2.0	1.4×10^{-1}	1.5×10^{-3}
2.5	5.8×10^{-1}	1.1×10^{-5}
3.0	1.6	2.7×10^{-5}
3.5	3.3	1.4×10^{-3}

d. $p_3(x)$ is a better approximation than $q_3(x)$ for $x = 0.5$, 1.0, and 1.5, and $q_3(x)$ is a better approximation for the other points. To see why this is true, note that on $[0.5, 4]$ that $f^{(4)}(x) = -\frac{6}{x^4}$ is bounded in magnitude by $\frac{6}{0.5^4} = 96$, so that (using P_3 for the error term for p_3 and Q_3 as the error term for q_3)

$$P_3(x) \le 96 \cdot \frac{|x-1|^4}{4!} = 4\,|x-1|^4\,, \qquad Q_3(x) \le 96 \cdot \frac{|x-e|^4}{4!} = 4\,|x-e|^4\,.$$

Thus the relative sizes of $P_3(x)$ and $Q_3(x)$ are governed by the distance of x from 1 and e. Looking at the different possibilities for x reveals why the results in part (c) hold.

9.1.97

a. We have

$$f(36) = \sqrt{36} = 6 \qquad f(49) = \sqrt{49} = 7$$
$$f'(36) = \frac{1}{2} \cdot \frac{1}{\sqrt{36}} = \frac{1}{12} \qquad f'(49) = \frac{1}{2} \cdot \frac{1}{\sqrt{49}} = \frac{1}{14}.$$

Thus

$$p_1(x) = 6 + \frac{1}{12}(x-36) \qquad q_1(x) = 7 + \frac{1}{14}(x-49).$$

b. Evaluating the errors gives

x	$\lvert\sqrt{x} - p_1(x)\rvert$	$\lvert\sqrt{x} - q_1(x)\rvert$
37	5.7×10^{-4}	6.0×10^{-2}
39	5.0×10^{-3}	4.1×10^{-2}
41	1.4×10^{-2}	2.5×10^{-2}
43	2.6×10^{-2}	1.4×10^{-2}
45	4.2×10^{-2}	6.1×10^{-3}
47	6.1×10^{-2}	1.5×10^{-3}

c. $p_1(x)$ is a better approximation than $q_1(x)$ for $x \le 41$, and $q_1(x)$ is a better approximation for $x \ge 43$. To see why this is true, note that $f''(x) = -\frac{1}{4}x^{-3/2}$, so that on $[36, 49]$ it is bounded in magnitude by $\frac{1}{4} \cdot 36^{-3/2} = \frac{1}{864}$. . Thus (using P_1 for the error term for p_1 and Q_1 for the error term for q_1)

$$P_1(x) \le \frac{1}{864} \cdot \frac{|x-36|^2}{2!} = \frac{1}{1728}(x-36)^2, \qquad Q_1(x) \le \frac{1}{864} \cdot \frac{|x-49|^2}{2!} = \frac{1}{1728}(x-49)^2.$$

It follows that the relative sizes of $P_1(x)$ and $Q_1(x)$ are governed by the distance of x from 36 and 49. Looking at the different possibilities for x reveals why the results in part (b) hold.

9.1.98

a. The quadratic Taylor polynomial for $\sin x$ centered at $\frac{\pi}{2}$ is

$$\begin{aligned} p_2(x) &= \sin\frac{\pi}{2} + \cos\frac{\pi}{2} \cdot \left(x - \frac{\pi}{2}\right) - \frac{1}{2}\sin\frac{\pi}{2} \cdot \left(x - \frac{\pi}{2}\right)^2 \\ &= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 \\ &= -\frac{1}{2}x^2 + \frac{\pi}{2}x + 1 - \frac{\pi^2}{8}. \end{aligned}$$

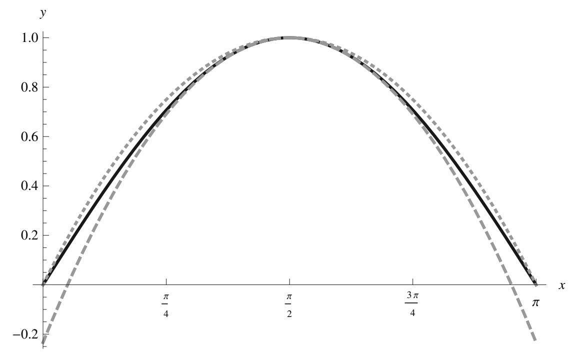
b. Let $q(x) = ax^2 + bx + c$. Because $q(0) = \sin 0 = 0$, we must have $c = 0$, so that $q(x) = ax^2 + bx$. Then the other two conditions give us a pair of linear equation in a and b:

$$\begin{aligned} \frac{\pi^2}{4}a + \frac{\pi}{2}b &= 1 \\ \pi^2 a + \pi b &= 0 \end{aligned}$$

where the first equation comes from the fact that $q(\pi/2) = \sin(\pi/2) = 1$ and the second from the fact that $q(\pi) = \sin\pi = 0$. Solving the linear system of equations gives $b = \frac{4}{\pi}$ and $a = -\frac{4}{\pi^2}$, so that

$$q(x) = -\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x.$$

c. A plot of the three function, with $\sin x$ the black solid line, $p_2(x)$ the dashed line, and $q(x)$ the dotted line is below.



d. Evaluating the errors gives

x	$\lvert\sin x - p_2(x)\rvert$	$\lvert\sin x - q(x)\rvert$
$\frac{\pi}{4}$	1.6×10^{-2}	4.3×10^{-2}
$\frac{\pi}{2}$	0	0
$\frac{3\pi}{4}$	1.6×10^{-2}	4.3×10^{-2}
π	2.3×10^{-1}	0

e. q is a better approximation than p at $x = \pi$, and the two are equal at $x = \frac{\pi}{2}$. At the other two points, however, $p_2(x)$ is a better approximation than $q(x)$. Clearly $q(x)$ will be exact at $x = 0$, $x = \frac{\pi}{2}$, and $x = \pi$, because it was chosen that way. Also clearly $p_2(x)$ will be exact at $x = \frac{\pi}{2}$ since it is the Taylor polynomial centered at $\frac{\pi}{2}$. The fact that $p_2(x)$ is a better approximation than $q(x)$ at the two intermediate points is a result of the way the polynomials were constructed: the goal of $p_2(x)$ was to be as good an approximation as possible near $x = \frac{\pi}{2}$, while the goal of $q(x)$ was to match $\sin x$ at three given points. Overall, it appears that $q(x)$ does a better job over the full range (the total area between $q(x)$ and $\sin x$ is certainly smaller than the total area between $p_2(x)$ and $\sin x$).

9.2 Properties of Power Series

9.2.1 $c_0 + c_1x + c_2x^2 + c_3x^3$.

9.2.2 $c_0 + c_1(x-3) + c_2(x-3)^2 + c_3(x-3)^3$.

9.2.3 Generally the Ratio Test or Root Test is used.

9.2.4 Theorem 9.3 says that on the interior of the interval of convergence, a power series centered at a converges absolutely, and that the interval of convergence is symmetric about a. So it makes sense to try to find this interval using the Ratio Test, and check the endpoints individually.

9.2.5 The radius of convergence does not change, but the interval of convergence may change at the endpoints.

9.2.6 $2R$, because for $|x| < 2R$ we have $|x/2| < R$ so that $\sum c_k(x/2)^k$ converges.

9.2.7 $|x| < \frac{1}{4}$.

9.2.8 $(-1)^k c_k x^k = c_k(-x)^k$, so the two series have the same radius of convergence, because $|-x| = |x|$.

9.2.9 Using the Root Test: $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} |2x| = |2x|$. So the radius of convergence is $\frac{1}{2}$. At $x = 1/2$ the series is $\sum 1$ which diverges, and at $x = -1/2$ the series is $\sum(-1)^k$ which also diverges. So the interval of convergence is $(-1/2, 1/2)$.

9.2.10 Using the Ratio Test: $\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty} \left|\frac{(2x)^{k+1}}{(k+1)!} \cdot \frac{k!}{(2x)^k}\right| = \lim\limits_{k\to\infty} \left|\frac{2x}{k+1}\right| = 0$. So the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

9.2.11 Using the Root Test, $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} = \lim\limits_{k\to\infty} \frac{|x-1|}{k^{1/k}} = |x-1|$. So the radius of convergence is 1. At $x = 2$, we have the harmonic series (which diverges) and at $x = 0$ we have the alternating harmonic series (which converges). Thus the interval of convergence is $[0, 2)$.

9.2.12 Using the Ratio Test: $\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty} \left|\frac{(x-1)^{k+1}}{(k+1)!} \cdot \frac{k!}{(x-1)^k}\right| = \lim\limits_{k\to\infty} \left|\frac{x-1}{k+1}\right| = 0$. Thus the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

9.2.13 Using the Ratio Test: $\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty} \left|\frac{(k+1)^{k+1}x^{k+1}}{k^k x^k}\right| = \lim\limits_{k\to\infty} (k+1)\left(\frac{k+1}{k}\right)^k |x| = \infty$ (for $x \neq 0$) because $\lim\limits_{k\to\infty} \left(\frac{k+1}{k}\right)^k = e$. Thus, the radius of convergence is 0, the series only converges at $x = 0$.

9.2.14 Using the Ratio Test: $\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty} \left|\frac{(k+1)!(x-10)^{k+1}}{k!(x-10)^k}\right| = \lim\limits_{k\to\infty} (k+1)|x-10| = \infty$ (for $x \neq 10$). Thus, the radius of convergence is 0, the series only converges at $x = 10$.

9.2.15 Using the Root Test: $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} = \lim\limits_{k\to\infty} \sin(1/k)|x| = \sin(0)|x| = 0$. Thus, the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

9.2.16 Using the Root Test: $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} = \lim\limits_{k\to\infty} \frac{2|x-3|}{k^{1/k}} = 2|x-3|$. Thus, the radius of convergence is $1/2$. When $x = 7/2$, we have the harmonic series (which diverges), and when $x = 5/2$, we have the alternating harmonic series which converges. The interval of convergence is thus $[5/2, 7/2)$.

9.2.17 Using the Root Test: $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} = \lim\limits_{k\to\infty} \frac{|x|}{3} = \frac{|x|}{3}$, so the radius of convergence is 3. At -3, the series is $\sum(-1)^k$, which diverges. At 3, the series is $\sum 1$, which diverges. So the interval of convergence is $(-3, 3)$.

9.2.18 Using the Root Test: $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} = \lim\limits_{k\to\infty} \frac{|x|}{5} = \frac{|x|}{5}$, so the radius of convergence is 5. At 5, we obtain $\sum(-1)^k$ which diverges. At -5, we have $\sum 1$, which also diverges. So the interval of convergence is $(-5, 5)$.

9.2.19 Using the Root Test: $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} = \lim\limits_{k\to\infty} \frac{|x|}{k} = 0$, so the radius of convergence is infinite and the interval of convergence is $(-\infty, \infty)$.

9.2.20 Using the Ratio Test: $\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty} \left|\left(\frac{(k+1)(x-4)^{k+1}}{2^{k+1}} \cdot \frac{2^k}{k(x-4)^k}\right)\right| = \lim\limits_{k\to\infty} \left(\frac{k+1}{k} \cdot \frac{|x-4|}{2}\right) = \frac{|x-4|}{2}$, so that the radius of convergence is 2. The interval is $(2, 6)$, because at the left endpoint, the series becomes $\sum k$ (which diverges) and at the right endpoint, it becomes $\sum(-1)^k k$ (which diverges).

9.2.21 Using the Ratio Test: $\lim\limits_{k\to\infty} \left|\frac{(k+1)^2 x^{2k+2}}{(k+1)!} \cdot \frac{k!}{k^2 x^{2k}}\right| = \lim\limits_{k\to\infty} \frac{k+1}{k^2} x^2 = 0$, so the radius of convergence is infinite, and the interval of convergence is $(-\infty, \infty)$.

9.2.22 Using the Root Test: $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} = \lim\limits_{k\to\infty} k^{1/k}|x-1| = |x-1|$. The radius of convergence is therefore 1. At both $x = 2$ and $x = 0$ the series diverges by the Divergence Test. The interval of convergence is therefore $(0, 2)$.

9.2.23 Using the Ratio Test: $\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{2k+3}}{3^k} \cdot \frac{3^{k-1}}{x^{2k+1}}\right| = \frac{x^2}{3}$ so that the radius of convergence is $\sqrt{3}$. At $x = \sqrt{3}$, the series is $\sum 3\sqrt{3}$, which diverges. At $x = -\sqrt{3}$, the series is $\sum(-3\sqrt{3})$, which also diverges, so the interval of convergence is $(-\sqrt{3}, \sqrt{3})$.

9.2.24 $\sum\left(\frac{-x}{10}\right)^{2k} = \sum\left(\frac{x^2}{100}\right)^k$. Using the Root Test: $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\frac{x^2}{100} = \frac{x^2}{100}$, so that the radius of convergence is 10. At $x = \pm 10$, the series is then $\sum 1$, which diverges, so the interval of convergence is $(-10, 10)$.

9.2.25 Using the Root Test: $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\frac{(|x-1|)k}{k+1} = |x-1|$, so the series converges when $|x-1| < 1$, so for $0 < x < 2$. The radius of convergence is 1. At $x = 2$, the series diverges by the Divergence Test. At $x = 0$, the series diverges as well by the Divergence Test. Thus the interval of convergence is $(0, 2)$.

9.2.26 Using the Ratio Test:

$$\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|} = \left|\frac{(-2)^{k+1}(x+3)^{k+1}}{3^{k+2}} \cdot \frac{3^{k+1}}{(-2)^k(x+3)^k}\right| = \frac{2}{3}|x+3|.$$

Thus the series converges when $\frac{2}{3}|x+3| < 1$, or $-\frac{9}{2} < x < -\frac{3}{2}$. At $x = -\frac{9}{2}$, the series diverges by the Divergence Test. At $x = -\frac{3}{2}$, the series diverges by the Divergence Test. Thus the interval of convergence is $\left(-\frac{9}{2}, -\frac{3}{2}\right)$.

9.2.27 Using the Ratio Test: $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)^{20}x^{k+1}}{(2k+3)!} \cdot \frac{(2k+1)!}{x^k k^{20}}\right| = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^{20}\frac{|x|}{(2k+2)(2k+3)} = 0$, so the radius of convergence is infinite, and the interval of convergence is $(-\infty, \infty)$.

9.2.28 Using the Root Test: $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\frac{|x^3|}{27} = \frac{|x^3|}{27}$, so the radius of convergence is 3. The series is divergent by the Divergence Test for $x = \pm 3$, so the interval of convergence is $(-3, 3)$.

9.2.29 $f(3x) = \frac{1}{1-3x} = \sum_{k=0}^{\infty} 3^k x^k$, which converges for $|x| < 1/3$, and diverges at the endpoints.

9.2.30 $g(x) = \frac{x^3}{1-x} = \sum_{k=0}^{\infty} x^{k+3}$, which converges for $|x| < 1$ and is divergent at the endpoints.

9.2.31 $h(x) = \frac{2x^3}{1-x} = \sum_{k=0}^{\infty} 2x^{k+3}$, which converges for $|x| < 1$ and is divergent at the endpoints.

9.2.32 $f(x^3) = \frac{1}{1-x^3} = \sum_{k=0}^{\infty} x^{3k}$. By the Root Test, $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \left|x^3\right|$, so this series also converges for $|x| < 1$. It is divergent at the endpoints.

9.2.33 $p(x) = \frac{4x^{12}}{1-x} = \sum_{k=0}^{\infty} 4x^{k+12} = 4\sum_{k=0}^{\infty} x^{k+12}$, which converges for $|x| < 1$. It is divergent at the endpoints.

9.2.34 $f(-4x) = \frac{1}{1+4x} = \sum_{k=0}^{\infty}(-4x)^k = \sum_{k=0}^{\infty}(-1)^k 4^k x^k$, which converges for $|x| < 1/4$ and is divergent at the endpoints.

9.2.35 $f(3x) = \ln(1-3x) = -\sum_{k=1}^{\infty}\frac{(3x)^k}{k} = -\sum_{k=1}^{\infty}\frac{3^k}{k}x^k$. Using the Ratio Test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{3k}{k+1}|x| = 3|x|,$$

so the radius of convergence is $1/3$. The series diverges at $1/3$ (harmonic series), and converges at $-1/3$ (alternating harmonic series).

9.2.36 $g(x) = x^3\ln(1-x) = -\sum_{k=1}^{\infty}\frac{x^{k+3}}{k}$. Using the Ratio Test: $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{k}{k+1}|x| = |x|$, so the radius of convergence is 1. The series diverges at 1 and converges at -1.

9.2.37 $h(x) = x\ln(1-x) = -\sum_{k=1}^{\infty}\frac{x^{k+1}}{k}$. Using the Ratio Test: $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{k}{k+1}|x| = |x|$, so the radius of convergence is 1, and the series diverges at 1 (harmonic series) but converges at -1 (alternating harmonic series).

9.2.38 $f(x^3) = \ln(1-x^3) = -\sum_{k=1}^{\infty} \frac{x^{3k}}{k}$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k}{k+1}\left|x^3\right| = \left|x^3\right|$, so the radius of convergence is 1. The series diverges at 1 (harmonic series) but converges at -1 (alternating harmonic series).

9.2.39 $p(x) = 2x^6\ln(1-x) = -2\sum_{k=1}^{\infty} \frac{x^{k+6}}{k}$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k}{k+1}|x| = |x|$, so the radius of convergence is 1. The series diverges at 1 (harmonic series) but converges at -1 (alternating harmonic series).

9.2.40 $f(-4x) = \ln(1+4x) = -\sum_{k=1}^{\infty} \frac{(-4x)^k}{k}$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k}{k+1}4|x| = 4|x|$, so the radius of convergence is 1/4. The series converges at 1/4 (alternating harmonic series) but diverges at $-1/4$ (harmonic series).

9.2.41 The power series for $f(x)$ is $\sum_{k=0}^{\infty}(2x)k$, convergent for $-1 < 2x < 1$, so for $-1/2 < x < 1/2$. The power series for $g(x) = f'(x)$ is $\sum_{k=1}^{\infty} k(2x)^{k-1}\cdot 2 = 2\sum_{k=1}^{\infty} k(2x)^{k-1}$, also convergent on $|x| < 1/2$.

9.2.42 The power series for $f(x)$ is $\sum_{k=0}^{\infty} x^k$, convergent for $-1 < x < 1$, so the power series for $g(x) = \frac{1}{2}f''(x)$ is $\frac{1}{2}\sum_{k=2}^{\infty} k(k-1)x^{k-2} = \frac{1}{2}\sum_{k=0}^{\infty}(k+1)(k+2)x^k$, also convergent on $|x| < 1$.

9.2.43 The power series for $f(x)$ is $\sum_{k=0}^{\infty} x^k$, convergent for $-1 < x < 1$, so the power series for $g(x) = \frac{1}{6}f'''(x)$ is $\frac{1}{6}\sum_{k=3}^{\infty} k(k-1)(k-2)x^{k-3} = \frac{1}{6}\sum_{k=0}^{\infty}(k+1)(k+2)(k+3)x^k$, also convergent on $|x| < 1$.

9.2.44 The power series for $f(x)$ is $\sum_{k=0}^{\infty}(-1)^k x^{2k}$, convergent on $|x| < 1$. Because $g(x) = -\frac{1}{2}f'(x)$, the power series for g is $-\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k 2kx^{2k-1} = \sum_{k=1}^{\infty}(-1)^{k+1}kx^{2k-1}$, also convergent on $|x| < 1$.

9.2.45 The power series for $\frac{1}{1-3x}$ is $\sum_{k=0}^{\infty}(3x)^k$, convergent on $|x| < 1/3$. Because $g(x) = \ln(1-3x) = -3\int \frac{1}{1-3x}\,dx$ and because $g(0) = 0$, the power series for $g(x)$ is $-3\sum_{k=0}^{\infty} 3^k \frac{1}{k+1}x^{k+1} = -\sum_{k=1}^{\infty} \frac{3^k}{k}x^k$, also convergent on $[-1/3, 1/3)$.

9.2.46 The power series for $\frac{x}{1+x^2}$ is $x\sum_{k=0}^{\infty}(-1)^k x^{2k} = \sum_{k=0}^{\infty}(-1)^k x^{2k+1}$, convergent on $|x| < 1$. Because $g(x) = 2\int f(x)\,dx$, and because $g(0) = 0$, the power series for $g(x)$ is $2\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+2}x^{2k+2} = \sum_{k=0}^{\infty}(-1)^k\frac{1}{k+1}x^{2k+2}$. This can be written as $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}x^{2k}$, which is convergent on $[-1, 1]$.

9.2.47 Start with $g(x) = \frac{1}{1+x}$. The power series for $g(x)$ is $\sum_{k=0}^{\infty}(-1)^k x^k$. Because $f(x) = g(x^2)$, its power series is $\sum_{k=0}^{\infty}(-1)^k x^{2k}$. The radius of convergence is still 1, and the series is divergent at both endpoints. The interval of convergence is $(-1, 1)$.

9.2.48 Start with $g(x) = \frac{1}{1-x}$. The power series for $g(x)$ is $\sum_{k=0}^{\infty} x^k$. Because $f(x) = g(x^4)$, its power series is $\sum_{k=0}^{\infty} x^{4k}$. The radius of convergence is still 1, and the series is divergent at both endpoints. The interval of convergence is $(-1, 1)$.

9.2.49 Note that $f(x) = \frac{3}{3+x} = \frac{1}{1+(1/3)x}$. Let $g(x) = \frac{1}{1+x}$. The power series for $g(x)$ is $\sum_{k=0}^{\infty}(-1)^k x^k$, so the power series for $f(x) = g((1/3)x)$ is $\sum_{k=0}^{\infty}(-1)^k 3^{-k}x^k = \sum_{k=0}^{\infty}\left(\frac{-x}{3}\right)^k$. Using the Ratio Test: $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{3^{-(k+1)}x^{k+1}}{3^{-k}x^k}\right| = \frac{|x|}{3}$, so the radius of convergence is 3. The series diverges at both endpoints. The interval of convergence is $(-3, 3)$.

9.2.50 Note that $f(x) = \frac{1}{2}\ln(1-x^2)$. The power series for $g(x) = \ln(1-x)$ is $-\sum_{k=1}^{\infty}\frac{1}{k}x^k$, so the power series for $f(x) = \frac{1}{2}g(x^2)$ is $\frac{-1}{2}\sum_{k=1}^{\infty}\frac{1}{k}x^{2k}$. The radius of convergence is still 1. The series diverges at both 1 and -1, its interval of convergence is $(-1, 1)$.

9.2.51 Note that $f(x) = \ln\sqrt{4-x^2} = \frac{1}{2}\ln(4-x^2) = \frac{1}{2}\left(\ln 4 + \ln\left(1-\frac{x^2}{4}\right)\right) = \ln 2 + \frac{1}{2}\ln\left(1-\frac{x^2}{4}\right)$. Now, the power series for $g(x) = \ln(1-x)$ is $-\sum_{k=1}^{\infty}\frac{1}{k}x^k$, so the power series for $f(x)$ is $\ln 2 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\frac{x^{2k}}{4^k} = \ln 2 - \sum_{k=1}^{\infty}\frac{x^{2k}}{k2^{2k+1}}$. Now, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{2k+2}}{(k+1)2^{2k+3}}\cdot\frac{k2^{2k+1}}{x^{2k}}\right| = \lim_{k\to\infty}\frac{k}{4(k+1)}x^2 = \frac{x^2}{4}$, so that the radius of convergence is 2. The series diverges at both endpoints, so its interval of convergence is $(-2, 2)$.

9.2.52 By Example 5, the Taylor series for $g(x) = \tan^{-1} x$ is $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$, so that $f(x) = g((2x)^2)$ has Taylor series $\sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{4k+2}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 4^{2k+1}}{2k+1} x^{4k+2}$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{4^{2k+3} x^{4k+6}}{2k+3} \cdot \frac{2k+1}{4^{2k+1} x^{4k+2}}\right| = \lim_{k\to\infty} \frac{16(2k+1)}{2k+3} x^4 = 16x^4$, so that the radius of convergence is $1/2$. The interval of convergence is $(-1/2, 1/2)$.

9.2.53

a. True. This power series is centered at $x = 3$, so its interval of convergence will be symmetric about 3.

b. True. Use the Root Test.

c. True. Substitute x^2 for x in the series.

d. True. Because the power series is zero on the interval, all its derivatives are as well, which implies (differentiating the power series) that all the c_k are zero.

9.2.54 Using the Root Test: $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k |x| = ex$. Thus, the radius of convergence is $\frac{1}{e}$.

9.2.55 Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)!\, x^{k+1}}{(k+1)^{k+1}} \cdot \frac{k^k}{k!\, x^k}\right| = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k |x| = \frac{1}{e}|x|$. The radius of convergence is therefore e.

9.2.56 $1 + \sum_{k=1}^{\infty} \frac{1}{2k} x^k$

9.2.57 $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} x^k$

9.2.58 $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(k+1)^2}$

9.2.59 $\sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{k!}$

9.2.60 The power series for $f(ax)$ is $\sum c_k (ax)^k$. Then $\sum c_k (ax)^k$ converges if and only if $|ax| < R$ (because $\sum c_k x^k$ converges for $|x| < R$), which happens if and only if $|x| < \frac{R}{|a|}$.

9.2.61 The power series for $f(x-a)$ is $\sum c_k (x-a)^k$. Then $\sum c_k (x-a)^k$ converges if and only if $|x - a| < R$, which happens if and only if $a - R < x < a + R$, so the radius of convergence is the same.

9.2.62 Let's first consider where this series converges. By the Root Test, $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} (x^2+1)^2 = (x^2+1)^2$, which is always greater than 1 for $x \neq 0$. This series also diverges when $x = 0$, because there we have the divergent series $\sum 1$. Because this series diverges everywhere, it doesn't represent any function, except perhaps the empty function.

9.2.63 This is a geometric series with ratio $\sqrt{x} - 2$, so its sum is $\frac{1}{1-(\sqrt{x}-2)} = \frac{1}{3-\sqrt{x}}$. Again using the Root Test, $\lim_{k\to\infty} \sqrt[k]{|a_k|} = |\sqrt{x} - 2|$, so the interval of convergence is given by $|\sqrt{x} - 2| < 1$, so $1 < \sqrt{x} < 3$ and $1 < x < 9$. The series diverges at both endpoints.

9.2.64 This series is $\frac{1}{4} \sum_{k=1}^{\infty} \frac{x^{2k}}{k}$. Because $\sum_{k=1}^{\infty} \frac{x^k}{k}$ is the power series for $-\ln(1-x)$, the power series given is $-\frac{1}{4}\ln(1-x^2)$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \left|\lim_{k\to\infty} \frac{x^{2k+2}}{4k+4} \cdot \frac{4k}{x^{2k}}\right| = \lim_{k\to\infty} \frac{k}{k+1} x^2 = x^2$, so the radius of convergence is 1. The series diverges at both endpoints (it is a multiple of the harmonic series). The interval of convergence is $(-1, 1)$.

9.2.65 This is a geometric series with ratio e^{-x}, so its sum is $\frac{1}{1-e^{-x}}$. By the Root Test, $\lim_{k\to\infty} \sqrt[k]{|a_k|} = e^{-x}$, so the power series converges for $x > 0$.

9.2.66 This is a geometric series with ratio $\frac{x-2}{9}$, so its sum is $\frac{(x-2)/9}{1-(x-2)/9} = \frac{x-2}{9-(x-2)} = \frac{x-2}{11-x}$. Using the Root Test: $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \left|\frac{x-2}{9}\right| = \left|\frac{x-2}{9}\right|$, so the series converges for $|x-2| < 9$, or $-7 < x < 11$. It diverges at both endpoints.

9.2.67 This is a geometric series with ratio $(x^2-1)/3$, so its sum is $\frac{1}{1-\frac{x^2-1}{3}} = \frac{3}{3-(x^2-1)} = \frac{3}{4-x^2}$. Using the Root Test, the series converges for $\left|x^2-1\right| < 3$, so that $-2 < x^2 < 4$ or $-2 < x < 2$. It diverges at both endpoints.

9.2.68 Replacing x by $x-1$ gives $\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k}$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(x-1)^{k+1}}{k+1} \cdot \frac{k}{(x-1)^k}\right| = \lim_{k\to\infty} \frac{k}{k+1}|x-1| = |x-1|$, so that the series converges for $|x-1| < 1$. Checking the endpoints, the interval of convergence is $(0, 2]$.

9.2.69 The power series for e^x is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. Substitute $-x$ for x to get $e^{-x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$. The series converges for all x.

9.2.70 Substitute $2x$ for x in the power series for e^x to get $e^{2x} = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k$. The series converges for all x.

9.2.71 Substitute $-3x$ for x in the power series for e^x to get $e^{-3x} = \sum_{k=0}^{\infty} \frac{(-3x)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{3^k}{k!} x^k$. The series converges for all x.

9.2.72 Multiply the power series for e^x by x^2 to get $x^2 e^x = \sum_{k=0}^{\infty} \frac{x^{k+2}}{k!}$, which converges for all x.

9.2.73 The power series for $x^m f(x)$ is $\sum c_k x^{k+m}$. The radius of convergence of this power series is determined by the limit

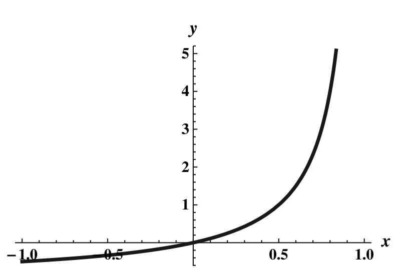
$$\lim_{k\to\infty} \left|\frac{c_{k+1} x^{k+1+m}}{c_k x^{k+m}}\right| = \lim_{k\to\infty} \left|\frac{c_{k+1} x^{k+1}}{c_k x^k}\right|,$$

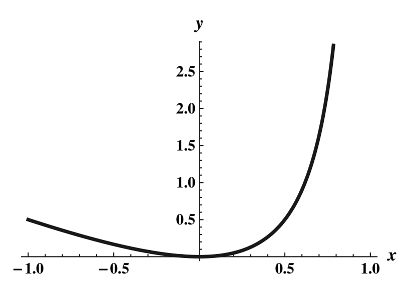
and the right-hand side is the limit used to determine the radius of convergence for the power series for $f(x)$. Thus the two have the same radius of convergence.

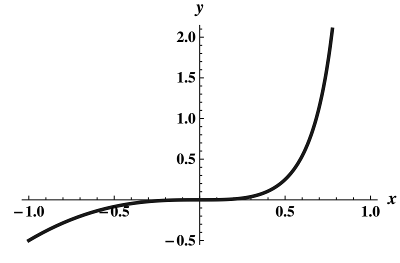
9.2.74

a. $R_n = f(x) - S_n(x) = \sum_{k=n}^{\infty} x^k$. This is a geometric series with ratio x. Its sum is then $R_n = \frac{x^n}{1-x}$ as desired.

b. $R_n(x)$ increases without bound as x approaches 1, and its absolute value smallest at $x = 0$ (where it is zero). In general, for $x > 0$, $R_n(x) < R_{n-1}(x)$, so the approximations get better the more terms of the series are included.

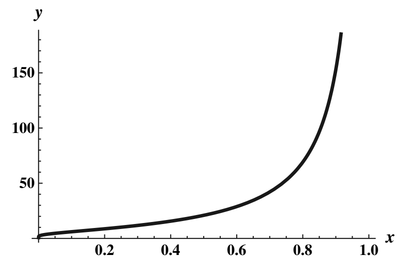






c. To minimize $|R_n(x)|$, set its derivative to zero. Assuming $n > 1$, we have $R_n'(x) = \frac{n(1-x)x^{n-1}+x^n}{(1-x)^2}$, which is zero for $x = 0$. There is a minimum at this critical point.

d. The following is a plot that shows, for each $x \in (0,1)$, the n required so that $R_n(x) < 10^{-6}$. The closer x gets to 1, the more terms are required in order for the estimate given by the power series to be accurate. The number of terms increases rapidly as $x \to 1$.



9.2.75

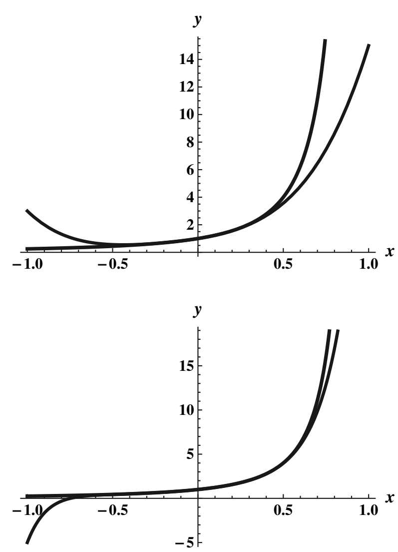
a. $f(x)g(x) = c_0d_0 + (c_0d_1 + c_1d_0)x + (c_0d_2 + c_1d_1 + c_2d_0)x^2 + \ldots$

b. The coefficient of x^n in $f(x)g(x)$ is $\sum_{i=0}^{n} c_i d_{n-i}$.

9.2.76 The function $\frac{1}{\sqrt{1-x^2}}$ is the derivative of the inverse sine function, and $\sin^{-1}(0) = 0$, so the power series for $\sin^{-1} x$ is the integral of the given power series, or $x + \frac{1}{6}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \ldots$. This can also be written $x + \sum_{k=1}^{\infty} \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k\cdot(2k+1)}x^{2k+1}$.

9.2.77

a. For both graphs, the difference between the true value and the estimate is greatest at the two ends of the range; the difference at 0.9 is greater than that at -0.9.



b. The difference between $f(x)$ and $S_n(x)$ is greatest for $x = 0.9$; at that point, $f(x) = \frac{1}{(1-0.9)^2} = 100$, so we want to find n such that $S_n(x)$ is within 0.01 of 100. We find that $S_{111} \approx 99.98991435$ and $S_{112} \approx 99.99084790$, so $n = 112$.

9.3 Taylor Series

9.3.1 The nth Taylor Polynomial is the nth sum of the corresponding Taylor Series.

9.3.2 In order to have a Taylor series centered at a, a function f must have derivatives of all orders on some interval containing a.

9.3.3 The n^{th} coefficient is $\frac{f^{(n)}(a)}{n!}$.

9.3.4 The interval of convergence is found in the same manner that it is found for a more general power series.

9.3.5 Substitute x^2 for x in the Taylor series. By theorems proved in the previous section about power series, the interval of convergence does not change except perhaps at the endpoints of the interval.

9.3.6 The Taylor series terminates if $f^{(n)}(0) = 0$ for $n > N$ for some N. For $(1+x)^p$, this occurs if and only if p is an integer ≥ 0.

9.3.7 It means that the limit of the remainder term is zero.

9.3.8 The Maclaurin series is $e^{2x} = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!}$. This is determined by substituting $2x$ for x in the Maclaurin series for e^x.

9.3.9

a. Note that $f(0) = 1$, $f'(0) = -1$, $f''(0) = 1$, and $f'''(0) = -1$. So the Maclaurin series is $1 - x + x^2/2 - x^3/6 + \cdots$.

b. $\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$.

c. The series converges on $(-\infty, \infty)$, as can be seen from the Ratio Test.

9.3.10

a. Note that $f(0) = 1$, $f'(0) = 0$, $f''(0) = -4$, $f'''(0) = 0$, $f^{(4)}(0) = 16$, ... Thus the Maclaurin series is $1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \cdots$.

b. $\sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k}}{(2k)!}$

c. The series converges on $(-\infty, \infty)$, as can be seen from the Ratio Test.

9.3.11

a. Because the series for $\frac{1}{1+x}$ is $1 - x + x^2 - x^3 + \cdots$, the series for $\frac{1}{1+x^2}$ is $1 - x^2 + x^4 - x^6 + \cdots$.

b. $\sum_{k=0}^{\infty} (-1)^k x^{2k}$.

c. The absolute value of the ratio of consecutive terms is x^2, so by the Ratio Test, the radius of convergence is 1. The series diverges at the endpoints by the Divergence Test, so the interval of convergence is $(-1, 1)$.

9.3.12

a. Note that $f(0) = 0$, $f'(0) = 4$, $f''(0) = -16$, $f'''(0) = 128$, and $f''''(0) = -1526$. Thus, the series is given by $4x - \frac{16x^2}{2} + \frac{128x^3}{6} - \frac{1536x^4}{24} + \cdots$.

b. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k-1)!(4x)^k}{k!} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(4x)^k}{k}$.

c. The absolute value of the ratio of consecutive terms is $\frac{4|x|k}{k+1}$, which has limit $4|x|$ as $k \to \infty$, so the interval of convergence is $(-1/4, 1/4]$. Note that for $x = 1/4$ we have the alternating harmonic series, while for $x = -1/4$ we have negative 1 times the harmonic series, which diverges.

9.3.13

a. Note that $f(0) = 1$, and that $f^{(n)}(0) = 2^n$. Thus, the series is given by $1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{(2x)^k}{k!}$.

c. The absolute value of the ratio of consecutive terms is $\frac{2|x|}{n}$, which has limit 0 as $n \to \infty$. So by the Ratio Test, the interval of convergence is $(-\infty, \infty)$.

9.3.14

a. Substitute $2x$ for x in the Taylor series for $(1+x)^{-1}$, to obtain the series $1 - 2x + 4x^2 - 8x^3 + \cdots$.

b. $\sum_{k=0}^{\infty} (-1)^k (2x)^k$.

c. The Root Test shows that the series converges absolutely for $|2x| < 1$, or $|x| < 1/2$. The interval of convergence is $(-1/2, 1/2)$, because the series at both endpoints diverge by the Divergence Test.

9.3.15

a. By integrating the Taylor series for $\frac{1}{1+x^2}$ (which is the derivative of $\tan^{-1}(x)$), we obtain the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. Then by replacing x by $x/2$ we have $\frac{x}{2} - \frac{x^3}{3 \cdot 2^3} + \frac{x^5}{5 \cdot 2^5} - \frac{x^7}{7 \cdot 2^7} + \cdots$.

b. $\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1) \cdot 2^{2k+1}} x^{2k+1}$.

c. By the Ratio Test (the ratio of consecutive terms has limit $\frac{x^2}{4}$), the radius of convergence is $|x| < 2$. Also, at the endpoints we have convergence by the Alternating Series Test, so the interval of convergence is $[-2, 2]$.

9.3.16

a. Substitute $3x$ for x in the Taylor series for $\sin x$, to obtain the series $3x - \frac{9x^3}{2} + \frac{81x^5}{40} - \frac{243x^7}{560} + \cdots$.

b. $\sum_{k=0}^{\infty} (-1)^k \frac{3^{2k+1}}{(2k+1)!} x^{2k+1}$.

c. The ratio of successive terms is $\frac{9}{2n(2n+1)} x^2$, which has limit zero as $n \to \infty$, so the interval of convergence is $(-\infty, \infty)$.

9.3.17

a. Note that $f(0) = 1$, $f'(0) = \ln 3$, $f''(0) = \ln^2 3$, $f''(0) = \ln^3 3$. So the first four terms of the desired series are $1 + (\ln 3)x + \frac{\ln^2 3}{2} x^2 + \frac{\ln^3 3}{6} x^3 + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{(\ln^k 3) x^k}{k!}$.

c. The ratio of successive terms is $\frac{(\ln^{k+1} 3) x^{k+1}}{(k+1)!} \cdot \frac{k!}{(\ln^k 3) x^k} = \frac{\ln 3}{k+1} x$, and the limit as $k \to \infty$ of this quantity is 0, so the interval of convergence is $(-\infty, \infty)$.

9.3.18

a. Note that $f(0) = 0$, $f'(0) = \frac{1}{\ln 3}$, $f''(0) = -\frac{1}{\ln 3}$, $f'''(0) = \frac{2}{ln 3}$, $f''''(0) = -\frac{6}{\ln 3}$. So the first terms of the desired series are $0 + \frac{x}{\ln 3} - \frac{x^2}{2 \ln 3} + \frac{x^3}{3 \ln 3} - \frac{x^4}{4 \ln 3} + \cdots$.

b. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k \ln 3}$.

c. The absolute value of the ratio of successive terms is $\left| \frac{x^{k+1}}{(k+1) \ln 3} \cdot \frac{k \ln 3}{x^k} \right| = \frac{k}{k+1} |x|$, which has limit $|x|$ as $k \to \infty$. Thus the radius of convergence is 1. At $x = -1$ we have a multiple of the harmonic series (which diverges) and at $x = 1$ we have a multiple of the alternating harmonic series (which converges) so the interval of convergence is $(-1, 1]$.

9.3.19

a. Note that $f(0) = 1$, $f'(0) = 0$, $f''(0) = 9$, $f'''(0) = 0$, etc. The first terms of the series are $1 + 9x^2/2 + 81x^4/4! + 3^6x^6/6! + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{(3x)^{2k}}{(2k)!}$.

c. The absolute value of the ratio of successive terms is $\left|\frac{(3x)^{2k+2}}{(2k+2)!} \cdot \frac{(2k)!}{(3x)^{2k}}\right| = \frac{1}{(2k+2)(2k+1)} \cdot 9x^2$, which has limit 0 as $x \to \infty$. The interval of convergence is therefore $(-\infty, \infty)$.

9.3.20

a. Note that $f(0) = 0$, $f'(0) = 2$, $f''(0) = 0$, $f'''(0) = 8$, etc. The first terms of the series are $2x + 8x^3/6 + 32x^5/5! + 128x^7/7! + \cdots$, or $2x + \frac{4x^3}{3} + \frac{4x^5}{15} + \frac{8x^7}{315} + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{2^{2k+1}x^{2k+1}}{(2k+1)!}$.

c. The absolute value of the ratio of successive terms is $\left|\frac{2^{2k+3}x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{2^{2k+1}x^{2k+1}}\right| = \frac{4}{(2k+3)(2k+2)}x^2$, which has limit 0 as $x \to \infty$. The interval of convergence is therefore $(-\infty, \infty)$.

9.3.21

a. Note that $f(\pi/2) = 1$, $f'(\pi/2) = \cos(\pi/2) = 0$, $f''(\pi/2) = -\sin(\pi/2) = -1$, $f'''(\pi/2) = -\cos(\pi/2) = 0$, and so on. Thus the series is given by $1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x - \frac{\pi}{2}\right)^6 + \cdots$.

b. $\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k}$.

9.3.22

a. Note that $f(\pi) = -1$, $f'(\pi) = -\sin\pi = 0$, $f''(\pi) = -\cos\pi = 1$, $f'''(\pi) = -\sin\pi = 0$, and so on. Thus the series is given by $-1 + \frac{1}{2}(x - \pi)^2 - \frac{1}{24}(x - \pi)^4 + \frac{1}{720}(x - \pi)^6 + \cdots$.

b. $\sum_{k=0}^{\infty}(-1)^{k+1} \frac{1}{(2k)!}(x - \pi)^{2k}$.

9.3.23

a. Note that $f^{(k)}(1) = (-1)^k \frac{k!}{1^{k+1}} = (-1)^k \cdot k!$. Thus the series is given by $1-(x-1)+(x-1)^2-(x-1)^3+\cdots$.

b. $\sum_{k=0}^{\infty}(-1)^k(x - 1)^k$.

9.3.24

a. Note that $f^{(k)}(2) = (-1)^k \frac{k!}{2^{k+1}}$. Thus the series is given by $\frac{1}{2} - \frac{x-2}{4} + \frac{1}{8}(x - 2)^2 - \frac{1}{16}(x - 2)^3 + \frac{1}{32}(x - 2)^4 + \cdots$.

b. $\sum_{k=0}^{\infty}(-1)^k \frac{1}{2^{k+1}}(x - 2)^k$.

9.3.25

a. Note that $f^{(k)}(3) = (-1)^{k-1}\frac{(k-1)!}{3^k}$. Thus the series is given by $\ln(3)+\frac{x-3}{3}-\frac{1}{18}(x-3)^2+\frac{1}{81}(x-3)^3+\cdots$.

b. $\ln 3 + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k \cdot 3^k}(x - 3)^k$.

9.3.26

a. Note that $f^{(k)}(\ln 2) = 2$. Thus the series is given by $2 + 2(x - \ln(2)) + (x - \ln(2))^2 + \frac{1}{3}(x - \ln(2))^3 + \frac{1}{12}(x - \ln(2))^4 + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{2}{k!}(x - \ln(2))^k$.

9.3.27

a. Note that $f(1) = 2$, $f'(1) = 2\ln 2$, $f''(1) = 2\ln^2 2$, $f'''(1) = 2\ln^3 2$. The first terms of the series are $2 + (2\ln 2)(x-1) + (\ln^2 2)(x-1)^2 + \frac{(\ln^3 2)(x-1)^3}{3} + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{2(x-1)^k \ln^k 2}{k!}$.

9.3.28

a. Note that $f(2) = 100$, $f'(2) = 100\ln 10$, $f''(2) = 100\ln^2 10$, $f'''(2) = 100\ln^3 10$. The first terms of the series are $100 + 100(\ln 10)(x-2) + 50(\ln^2 10)(x-2)^2 + \frac{50}{3}(\ln^3 10)(x-2)^3 + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{100(x-2)^k \ln^k 10}{k!}$.

9.3.29 Because the Taylor series for $\ln(1+x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, the first four terms of the Taylor series for $\ln(1+x^2)$ are $x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$, obtained by substituting x^2 for x.

9.3.30 Because the Taylor series for $\sin x$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, the first four terms of the Taylor series for $\sin x^2$ are $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$, obtained by substituting x^2 for x.

9.3.31 Because the Taylor series for $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, the first four terms of the Taylor series for $\frac{1}{1-2x}$ are $1 + 2x + 4x^2 + 8x^3 + \cdots$ obtained by substituting $2x$ for x.

9.3.32 Because the Taylor series for $\ln(1+x)$ is $x - x^2/2 + x^3/3 - x^4/4 + \cdots$, the first four terms of the Taylor series for $2x - 2x^2 + 8x^3/3 - 4x^4 + \cdots$ obtained by substituting $2x$ for x.

9.3.33 The Taylor series for $e^x - 1$ is the Taylor series for e^x, less the constant term of 1, so it is $x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$. Thus, the first four terms of the Taylor series for $\frac{e^x-1}{x}$ are $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$, obtained by dividing the terms of the first series by x.

9.3.34 Because the Taylor series for $\cos x$ is $1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, the first four terms of the Taylor series for $\cos x^3$ are $1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots$, obtained by substituting x^3 for x.

9.3.35 Because the Taylor series for $(1+x)^{-1}$ is $1 - x + x^2 - x^3 + \cdots$, if we substitute x^4 for x, we obtain $1 - x^4 + x^8 - x^{12} + \cdots$.

9.3.36 The Taylor series for $\tan^{-1} x$ is $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots$. Thus, the Taylor series for $\tan^{-1} x^2$ is $x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} - \cdots$ and, multiplying by x, the Taylor series for $x\tan^{-1} x^2$ is $x^3 - \frac{x^7}{3} + \frac{x^{11}}{5} - \frac{x^{15}}{7} - \cdots$.

9.3.37 The Taylor series for $\sinh x$ is $x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots$. Thus, the Taylor series for $\sinh x^2$ is $x^2 + \frac{x^6}{6} + \frac{x^{10}}{120} + \frac{x^{14}}{5040} + \cdots$ obtained by substituting x^2 for x.

9.3.38 The Taylor series for $\cosh x$ is $1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots$. Thus, the Taylor series for $\cosh 3x$ is $1 + \frac{9x^2}{2} + \frac{81x^4}{24} + \frac{729x^6}{720} + \cdots$, obtained by substituting $3x$ for x.

9.3.39

a. The binomial coefficients are $\binom{-2}{0} = 1$, $\binom{-2}{1} = \frac{-2}{1!} = -2$, $\binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3$, $\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4$.

Thus the first four terms of the series are $1 - 2x + 3x^2 - 4x^3 + \cdots$.

b. $1 - 2 \cdot 0.1 + 3 \cdot 0.01 - 4 \cdot 0.001 = 0.826$

9.3.40

a. The binomial coefficients are $\binom{1/2}{0} = 1$, $\binom{1/2}{1} = \frac{1/2}{1!} = \frac{1}{2}$, $\binom{1/2}{2} = \frac{(1/2)(-1/2)}{2!} = -\frac{1}{8}$, $\binom{1/2}{3} = \frac{(1/2)(-1/2)(-3/2)}{3!} = \frac{1}{16}$, so the first four terms of the series are $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$.

b. $1 + \frac{1}{2} \cdot .06 - \frac{1}{8} \cdot .06^2 + \frac{1}{16} \cdot .06^3 \approx 1.030$

9.3.41

a. The binomial coefficients are $\binom{1/4}{0} = 1$, $\binom{1/4}{1} = \frac{1/4}{1} = \frac{1}{4}$, $\binom{1/4}{2} = \frac{(1/4)(-3/4)}{2!} = -\frac{3}{32}$, $\binom{1/4}{3} = \frac{(1/4)(-3/4)(-7/4)}{3!} = \frac{7}{128}$, so the first four terms of the series are $1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 + \cdots$.

b. Substitute $x = 0.12$ to get approximately 1.029.

9.3.42

a. The binomial coefficients are $\binom{-3}{0} = 1$, $\binom{-3}{1} = -3$, $\binom{-3}{2} = \frac{(-3)(-4)}{2!} = 6$, $\binom{-3}{3} = \frac{(-3)(-4)(-5)}{3!} = -10$, so the first four terms of the series are $1 - 3x + 6x^2 - 10x^3 + \cdots$.

b. Substitute $x = 0.1$ to get 0.750.

9.3.43

a. The binomial coefficients are $\binom{-2/3}{0} = 1$, $\binom{-2/3}{1} = -\frac{2}{3}$, $\binom{-2/3}{2} = \frac{(-2/3)(-5/3)}{2!} = \frac{5}{9}$, $\binom{-2/3}{3} = \frac{(-2/3)(-5/3)(-8/3)}{3!} = -\frac{40}{81}$, so the first four terms of the series are $1 - \frac{2}{3}x + \frac{5}{9}x^2 - \frac{40}{81}x^3 + \cdots$.

b. Substitute $x = 0.18$ to get 0.89512.

9.3.44

a. The binomial coefficients are $\binom{2/3}{0} = 1$, $\binom{2/3}{1} = \frac{2}{3}$, $\binom{2/3}{2} = \frac{(2/3)(-1/3)}{2!} = -\frac{1}{9}$, $\binom{2/3}{3} = \frac{(2/3)(-1/3)(-4/3)}{3!} = \frac{4}{81}$, so the first four terms of the series are $1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 + \cdots$

b. Substitute $x = 0.02$ to get ≈ 1.013289284.

9.3.45 $\sqrt{1+x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \cdots$. By the Ratio Test, the radius of convergence is 1. At the endpoints, the series obtained are convergent by the Alternating Series Test. Thus, the interval of convergence is $[-1, 1]$.

9.3.46 $\sqrt{4+x} = 2\sqrt{1+x/4} = 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \cdots$. The interval of convergence is $(-4, 4]$.

9.3.47 $\sqrt{9-9x} = 3\sqrt{1-x} = 3 - \frac{3}{2}x - \frac{3}{8}x^2 - \frac{3}{16}x^3 - \cdots$. The interval of convergence is $[-1, 1]$.

9.3.48 $\sqrt{1-4x} = 1 - 2x - 2x^2 - 4x^3 - \cdots$, obtained by substituting $-4x$ for x in the original series. The interval of convergence of $[-1/4, 1/4)$.

9.3.49 $\sqrt{a^2+x^2} = a\sqrt{1+\frac{x^2}{a^2}} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \cdots$. The series converges when $\frac{x^2}{a^2}$ is less than 1 in magnitude, so the radius of convergence is a. The series given by the endpoints is convergent by the Alternating Series Test, so the interval of convergence is $[-a, a]$.

9.3.50 $\sqrt{4-16x^2} = 2\sqrt{1-(2x)^2} = 2 - 4x^2 - 4x^4 - 8x^6 - \cdots$. Because $2x$ was substituted for x to produce this series, this series converges when $-1 < 2x < 1$, or $-\frac{1}{2} < x < \frac{1}{2}$. Because only even powers of x appear in the series, the series at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ are identical, and are convergent. Thus the interval of convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

9.3.51 $(1+4x)^{-2} = 1 - 2(4x) + 3(4x)^2 - 4(4x)^3 + \cdots = 1 - 8x + 48x^2 - 256x^3 + \cdots$.

9.3.52 $\frac{1}{(1-4x)^2} = (1-4x)^{-2} = 1 - 2(-4x) + 3(-4x)^2 - 4(-4x)^3 + \cdots = 1 + 8x + 48x^2 + 256x^3$.

9.3.53 $\frac{1}{(4+x^2)^2} = (4+x^2)^{-2} = \frac{1}{16}(1+(x^2/4))^{-2} = \frac{1}{16}\left(1 - 2\cdot\frac{x^2}{4} + 3\cdot\frac{x^4}{16} - 4\cdot\frac{x^6}{64} + \cdots\right) = \frac{1}{16} - \frac{1}{32}x^2 + \frac{3}{256}x^4 - \frac{1}{256}x^6 + \cdots$

9.3.54 Note that $x^2 - 4x + 5 = 1 + (x-2)^2$, so $(1+(x-2)^2)^{-2} = 1 - 2(x-2)^2 + 3(x-2)^4 - 4(x-2)^6 + \cdots$.

9.3.55 $(3+4x)^{-2} = \frac{1}{9}\left(1+\frac{4x}{3}\right)^{-2} = \frac{1}{9} - \frac{2}{9}\left(\frac{4x}{3}\right) + \frac{3}{9}\left(\frac{4x}{3}\right)^2 - \frac{4}{9}\left(\frac{4x}{3}\right)^3 + \cdots$.

9.3.56 $(1+4x^2)^{-2} = (1+(2x)^2)^{-2} = 1 - 2(2x)^2 + 3(2x)^4 - 4(2x)^6 + \cdots = 1 - 8x^2 + 48x^4 - 256x^6 + \cdots$.

9.3.57 The interval of convergence for the Taylor series for $f(x) = \sin x$ is $(-\infty, \infty)$. The remainder is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c. Because $f^{(n+1)}(x)$ is $\pm\sin x$ or $\pm\cos x$, we have

$$\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{1}{(n+1)!}\left|x^{n+1}\right| = 0$$

for any x.

9.3.58 The interval of convergence for the Taylor series for $f(x) = \cos 2x$ is $(-\infty, \infty)$. The remainder is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c. The nth derivative of $\cos 2x$ is 2^n times either $\pm\sin x$ or $\pm\cos x$, so that $f^{(n+1)}$ is bounded by 2^{n+1} in magnitude. Thus $\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}\left|x^{n+1}\right| = \lim_{n\to\infty} \frac{(2|x|)^{n+1}}{(n+1)!} = 0$ for any x.

9.3.59 The interval of convergence for the Taylor series for e^{-x} is $(-\infty, \infty)$. The remainder is $R_n(x) = \frac{(-1)^{n+1}e^{-c}}{(n+1)!}x^{n+1}$ for some c. Thus $\lim_{n\to\infty} |R_n(x)| = 0$ for any x.

9.3.60 The interval of convergence for the Taylor series for $f(x) = \cos x$ is $(-\infty, \infty)$. The remainder is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-\pi/2)^{n+1}$ for some c. Because $f^{n+1}(x)$ is $\pm\cos x$ or $\pm\sin x$, we have

$$\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{1}{(n+1)!}\left|(x-\pi/2)^{n+1}\right| = 0$$

for any x.

9.3.61

a. False. Not all of its derivatives are defined at zero - in fact, none of them are.

b. True. The derivatives of $\csc x$ involve positive powers of $\csc x$ and $\cot x$, both of which are defined at $\pi/2$, so that $\csc x$ has continuous derivatives at $\pi/2$.

c. False. For example, the Taylor series for $f(x^2)$ doesn't converge at $x = 1.9$, because the Taylor series for $f(x)$ doesn't converge at $1.9^2 = 3.61$.

d. False. The Taylor series centered at 1 involves derivatives of f evaluated at 1, not at 0.

e. True. The follows because the Taylor series must itself be an even function.

9.3.62

a. The relevant Taylor series are: $\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots$, and $2\sin x = 2x - \frac{1}{3}x^3 + \frac{1}{60}x^5 - \cdots$. Thus, the first four terms of the resulting series are $\cos 2x + 2\sin x = 1 + 2x - 2x^2 - \frac{1}{3}x^3 + \frac{2}{3}x^4 + \cdots$.

b. Because each series converges (absolutely) on $(-\infty, \infty)$, so does their sum. The radius of convergence is ∞.

9.3.63

a. The relevant Taylor series are: $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\cdots$ and $e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\frac{x^6}{6!}+\cdots$. Thus the first four terms of the resulting series are $\frac{1}{2}\left(e^x+e^{-x}\right) = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots$.

b. Because each series converges (absolutely) on $(-\infty,\infty)$, so does their sum. The radius of convergence is ∞.

9.3.64

a. The first four terms of the Taylor series for $\sin x$ are $x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}$, so the first four terms for $\frac{\sin x}{x}$ are $1-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}$.

b. The radius of convergence is the same as that for $\sin x$, namely ∞.

9.3.65

a. Use the binomial theorem. The binomial coefficients are $\binom{-2/3}{0}=1$, $\binom{-2/3}{1}=-\frac{2}{3}$, $\binom{-2/3}{2}=\frac{(-2/3)(-5/3)}{2!}=\frac{5}{9}$, $\binom{-2/3}{3}=\frac{(-2/3)(-5/3)(-8/3)}{3!}=-\frac{40}{81}$ and then, substituting x^2 for x, we obtain $1-\frac{2}{3}x^2+\frac{5}{9}x^4-\frac{40}{81}x^6+\cdots$.

b. From Theorem 9.6 the radius of convergence is determined from $\left|x^2\right|<1$, so it is 1.

9.3.66

a. The first four terms of $\cos x$ are $1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}$, so the first four terms of $\cos x^2$ are $1-\frac{x^4}{2}+\frac{x^8}{24}-\frac{x^{12}}{720}$, and thus the first four terms of $x^2\cos x^2$ are $x^2-\frac{x^6}{2}+\frac{x^{10}}{24}-\frac{x^{14}}{720}$.

b. The radius of convergence is ∞.

9.3.67

a. From the binomial formula, the Taylor series for $(1-x)^p$ is $\sum\binom{p}{k}(-1)^k x^k$, so the Taylor series for $(1-x^2)^p$ is $\sum\binom{p}{k}(-1)^k x^{2k}$. Here $p=1/2$, and the binomial coefficients are $\binom{1/2}{0}=1$, $\binom{1/2}{1}=\frac{1/2}{1!}=\frac{1}{2}$, $\binom{1/2}{2}=\frac{(1/2)(-1/2)}{2!}=-\frac{1}{8}$, $\binom{1/2}{3}=\frac{(1/2)(-1/2)(-3/2)}{3!}=\frac{1}{16}$ so that $(1-x^2)^{1/2}=1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6+\cdots$.

b. From Theorem 9.6 the radius of convergence is determined from $\left|x^2\right|<1$, so it is 1.

9.3.68

a. Because $b^x=e^{x\ln b}$, the Taylor series is $1+x\ln b+\frac{1}{2!}(x\ln b)^2+\frac{1}{3!}(x\ln b)^3+\cdots$

b. Because the series for e^x converges on $(-\infty,\infty)$, the radius of convergence for the series in part a is ∞.

9.3.69

a. $f(x)=(1+x^2)^{-2}$; using the binomial series and substituting x^2 for x we obtain $1-2x^2+3x^4-4x^6+\cdots$.

b. From Theorem 9.6 the radius of convergence is determined from $\left|x^2\right|<1$, so it is 1.

9.3.70 Because $f(36)=6$, and $f'(x)=\frac{1}{2}x^{-1/2}$, $f'(36)=\frac{1}{12}$, $f''(x)=-\frac{1}{4}x^{-3/2}$, $f''(36)=-\frac{1}{864}$, $f'''(x)=\frac{3}{8}x^{-5/2}$, and $f'''(36)=\frac{3}{62208}$, the first four terms of the Taylor series are $6+\frac{1}{12}(x-36)-\frac{1}{864\cdot 2!}(x-36)^2+\frac{3}{62208\cdot 3!}(x-36)^3$. Evaluating at $x=39$ we get 6.245008681.

9.3.71 Because $f(64)=4$, and $f'(x)=\frac{1}{3}x^{-2/3}$, $f'(64)=\frac{1}{48}$, $f''(x)=-\frac{2}{9}x^{-5/3}$, $f''(64)=-\frac{1}{4608}$, $f'''(x)=\frac{10}{27}x^{-8/3}$, and $f'''(64)=\frac{10}{1769472}=\frac{5}{884736}$, the first four terms of the Taylor series are $4+\frac{1}{48}(x-64)-\frac{1}{4608\cdot 2!}(x-64)^2+\frac{5}{884736\cdot 3!}(x-64)^3$. Evaluating at $x=60$, we get 3.914870274.

9.3.72 Because $f(4) = \frac{1}{2}$, and $f'(x) = -\frac{1}{2}x^{-3/2}$, $f'(4) = -\frac{1}{16}$, $f''(x) = \frac{3}{4}x^{-5/2}$, $f''(4) = \frac{3}{128}$, $f'''(x) = -\frac{15}{8}x^{-7/2}$, and $f'''(4) = -\frac{15}{1024}$, the first four terms of the Taylor series are $\frac{1}{2} - \frac{1}{16}(x-4) + \frac{3}{128\cdot 2!}(x-4)^2 - \frac{15}{1024\cdot 3!}(x-4)^3$. Evaluating at $x = 3$, we get 0.5766601563.

9.3.73 Because $f(16) = 2$, and $f'(x) = \frac{1}{4}x^{-3/4}$, $f'(16) = \frac{1}{32}$, $f''(x) = -\frac{3}{16}x^{-7/4}$, $f''(16) = -\frac{3}{2048}$, $f'''(x) = \frac{21}{64}x^{-11/4}$, and $f'''(16) = \frac{21}{131072}$, the first four terms of the Taylor series are $2 + \frac{1}{32}(x-16) - \frac{3}{2048\cdot 2!}(x-16)^2 + \frac{21}{131072\cdot 3!}(x-16)^3$. Evaluating at $x = 13$, we get 1.898937225.

9.3.74 Evaluate the binomial coefficient $\binom{-1}{k} = \frac{(-1)(-2)\cdots(-1-k+1)}{k!} = (-1)^k$, so that the binomial expansion for $(1+x)^{-1}$ is $\sum_{k=0}^{\infty}(-1)^k x^k$. Substituting $-x$ for x, we obtain $(1-x)^{-1} = \sum_{k=0}^{\infty}(-1)^k(-x)^k = \sum_{k=0}^{\infty} x^k$.

9.3.75 Evaluate the binomial coefficient $\binom{1/2}{k} = \frac{(1/2)(-1/2)(-3/2)\cdots(1/2-k+1)}{k!} = \frac{(1/2)(-1/2)\cdots((3-2k)/2)}{k!} = (-1)^{k-1}2^{-k}\frac{1\cdot 3\cdots(2k-3)}{k!} = (-1)^{k-1}2^{-k}\frac{(2k-2)!}{2^{k-1}\cdot(k-1)!\cdot k!} = (-1)^{k-1}2^{1-2k}\cdot\frac{1}{k}\binom{2k-2}{k-1}$. This is the coefficient of x^k in the Taylor series for $\sqrt{1+x}$. Substituting $4x$ for x, the Taylor series becomes $\sum_{k=0}^{\infty}(-1)^{k-1}2^{1-2k}\cdot\frac{1}{k}\binom{2k-2}{k-1}(4x)^k = \sum_{k=0}^{\infty}(-1)^{k-1}\frac{2}{k}\binom{2k-2}{k-1}x^k$. If we can show that k divides $\binom{2k-2}{k-1}$, we will be done, for then the coefficient of x^k will be an integer. But $\binom{2k-2}{k-1} - \binom{2k-2}{k-2} = \frac{(2k-2)!}{(k-1)!(k-1)!} - \frac{(2k-2)!}{(k-2)!k!} = \frac{(2k-2)!}{(k-1)!(k-1)!} - \frac{(2k-2)!(k-1)}{(k-1)!(k-1)!k} = \frac{k(2k-2)!-(k-1)(2k-2)!}{k(k-1)!(k-1)!} = \frac{1}{k}\frac{(2k-2)!}{(k-1)!(k-1)!} = \frac{1}{k}\binom{2k-2}{k-1}$ and thus we have shown that k divides $\binom{2k-2}{k-1}$.

9.3.76 The two Taylor series are:

$$8 + \frac{1}{16}(x-64) - \frac{1}{4096}(x-64)^2 + \frac{1}{524288}(x-64)^3 - \frac{5}{268435456}(x-64)^4 + \cdots$$
$$9 + \frac{1}{18}(x-81) - \frac{1}{5832}(x-81)^2 + \frac{1}{944784}(x-81)^3 - \frac{5}{612220032}(x-81)^4 + \cdots.$$

Evaluating these Taylor series at $n = 2, 3, 4$ (after the quadratic, cubic, and quartic terms) we obtain the errors:

n	64	81
2	9.064×10^{-4}	-8.297×10^{-4}
3	-7.019×10^{-5}	-5.813×10^{-5}
4	6.106×10^{-6}	-4.550×10^{-6}

The errors using the Taylor series centered at 81 are consistently smaller.

9.3.77

a. The Maclaurin series for $\sin x$ is $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$. Squaring the first four terms yields

$$\begin{aligned}&\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7\right)^2\\ &= x^2 - \frac{2}{3!}x^4 + \left(\frac{2}{5!} + \frac{1}{3!3!}\right)x^6 + \left(-2\cdot\frac{1}{7!} - 2\cdot\frac{1}{3!5!}\right)x^8\\ &= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8.\end{aligned}$$

b. The Maclaurin series for $\cos x$ is $1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$. Substituting $2x$ for x in the Maclaurin series for $\cos x$ and then computing $(1-\cos 2x)/2$, we obtain

$$\begin{aligned}&(1 - (1-\frac{1}{2}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6) + \frac{1}{8!}(2x)^8)/2\\ &= (2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 - \frac{2}{315}x^8)/2\\ &= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8,\end{aligned}$$

and the two are the same.

c. If $f(x) = \sin^2 x$, then $f(0) = 0$, $f'(x) = \sin 2x$, so $f'(0) = 0$. $f''(x) = 2\cos 2x$, so $f''(x) = 2$, $f'''(x) = -4\sin 2x$, so $f'''(0) = 0$. Note that from this point $f^{(n)}(0) = 0$ if n is odd and $f^{(n)}(0) = \pm 2^{n-1}$ if n is even, with the signs alternating for every other even n. Thus, the series for $\sin^2 x$ is

$$2x^2/2 - 8x^4/4! + 32x^6/6! - 128x^8/8! + \cdots = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8 + \cdots .$$

9.3.78

a. The Maclaurin series for $\cos x$ is $1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$. Squaring the first four terms yields

$$\begin{aligned}
&(1-\frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6)^2 \\
&\quad = 1 - (\frac{1}{2} + \frac{1}{2})x^2 + (\frac{1}{4!} + \frac{1}{4!} + \frac{1}{4})x^4 + (-\frac{1}{6!} - \frac{1}{6!} - \frac{1}{2\cdot 4!} - \frac{1}{2\cdot 4!})x^6 \\
&\quad = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6.
\end{aligned}$$

b. Substituting $2x$ for x in the Maclaurin series for $\cos x$ and then computing $(1+\cos 2x)/2$, we obtain

$$\begin{aligned}
&(1 + 1-\frac{1}{2}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6)/2 \\
&\quad = (2 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6)/2 \\
&\quad = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6,
\end{aligned}$$

and the two are the same.

c. If $f(x) = \cos^2 x$, then $f(0) = 1$. Also, $f'(x) = -2\cos x \sin x = -\sin 2x$. So $f'(0) = 0$. $f''(x) = -2\cos 2x$, so $f''(0) = -2$. $f'''(x) = 8\sin 2x$, so $f'''(0) = 0$. Note that from this point on, $f^{(n)}(0) = 0$ if n is odd, and $f^{(n)}(0) = \pm 2^{n-1}$ if n is even, with the signs alternating for every other even n. Thus, the series for $\cos^2 x$ is

$$1 - 2x^2/2 + 8x^4/4! - 32x^6/6! + \cdots = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots .$$

9.3.79 There are many solutions. For example, first find a series that has $(-1,1)$ as an interval of convergence, say $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$. Then the series $\frac{1}{1-x/2} = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$ has $(-2,2)$ as its interval of convergence. Now shift the series up so that it is centered at 4. We have $\sum_{k=0}^{\infty} \left(\frac{x-4}{2}\right)^k$, which has interval of convergence $(2,6)$.

9.3.80 $-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}x^5$.

9.3.81 $\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}x^5$.

9.3.82

a. The Maclaurin series in question are

$$\begin{aligned}
\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \\
e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots ,
\end{aligned}$$

so substituting the series for $\sin x$ for x in the series for e^x (and considering only those terms that will give us an exponent at most 3), we obtain $e^{\sin x} = 1 + (x - \frac{1}{3!}x^3) + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = 1 + x + \frac{1}{2}x^2 + \cdots$.

b. The Maclaurin series in question are

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$
$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots,$$

so substituting the series for $\tan x$ for x in the series for e^x (and considering only those terms that will give us an exponent at most 3), we obtain $e^{\tan x} = 1 + (x + \frac{1}{3}x^3) + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = 1 + x + \frac{1}{2}x^2 + \cdots$.

c. The Maclaurin series in question are

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$
$$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots,$$

so substituting the series for $\sin x$ for x in the series for $\sqrt{1+x^2}$ (and considering only those terms that will give us an exponent at most 4), we obtain $\sqrt{1+\sin^2 x} = 1 + \frac{1}{2}(x - \frac{1}{3!}x^3)^2 - \frac{1}{8}x^4 + \cdots = 1 + \frac{1}{2}x^2 - \frac{7}{24}x^4 + \cdots$.

9.3.83 Use the Taylor series for $\cos x$ centered at $\pi/4$: $\frac{\sqrt{2}}{2}(1 - (x-\pi/4) - \frac{1}{2}(x-\pi/4)^2 + \frac{1}{6}(x-\pi/4)^3 + \cdots)$. The remainder after n terms (because the derivatives of $\cos x$ are bounded by 1 in magnitude) is $|R_n(x)| \le \frac{1}{(n+1)!} \cdot \left(\frac{\pi}{4} - \frac{2\pi}{9}\right)^{n+1}$.

Solving for $|R_n(x)| < 10^{-4}$, we obtain $n = 3$. Evaluating the first four terms (through $n = 3$) of the series we get 0.7660427050. The true value is ≈ 0.7660444431.

9.3.84 Use the Taylor series for $\sin x$ centered at π: $-(x-\pi) + \frac{1}{6}(x-\pi)^3 - \frac{1}{120}(x-\pi)^5 + \cdots$. The remainder after n terms (because the derivatives of $\sin x$ are bounded by 1 in magnitude) is $|R_n(x)| \le \frac{1}{(n+1)!} \cdot (\pi - 0.98\pi)^{n+1}$.

Solving for $|R_n(x)| < 10^{-4}$, we obtain $n = 2$. Evaluating the first term of the series gives 0.06283185307. The true value is ≈ 0.06279051953.

9.3.85 Use the Taylor series for $f(x) = x^{1/3}$ centered at 64: $4 + \frac{1}{48}(x-64) - \frac{1}{9216}(x-64)^2 + \cdots$. Because we wish to evaluate this series at $x = 83$, $|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}(83-64)^{n+1}$. We compute that $|f^{(n+1)}(c)| = \frac{2\cdot5\cdots(3n-1)}{3^{n+1}c^{(3n+2)/3}}$, which is maximized at $c = 64$. Thus

$$|R_n(x)| \le \frac{2\cdot5\cdots(3n-1)}{3^{n+1}64^{(3n+2)/3}(n+1)!}19^{n+1}$$

Solving for $|R_n(x)| < 10^{-4}$, we obtain $n = 5$. Evaluating the terms of the series through $n = 5$ gives 4.362122553. The true value is ≈ 4.362070671.

9.3.86 Use the Taylor series for $f(x) = x^{-1/4}$ centered at 16: $\frac{1}{2} - \frac{1}{128}(x-16) + \frac{5}{16384}(x-16)^2 + \cdots$. Because we wish to evaluate this series at $x = 17$, $|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}(17-16)^{n+1}$. We compute that $|f^{(n+1)}(c)| = \frac{1\cdot5\cdots(4n+1)}{4^{n+1}c^{(4n+5)/4}}$ which is maximized at $c = 16$. Thus

$$|R_n(x)| \le \frac{1\cdot5\cdots(4n+1)}{4^{n+1}16^{(4n+5)/4}(n+1)!}1^{n+1}$$

Solving for $|R_n(x)| < 10^{-4}$, we obtain $n = 2$. Evaluating the terms of the series through $n = 2$ gives 0.4924926758. The true value is ≈ 0.4924790605.

9.3.87

a. Use the Taylor series for $(125+x)^{1/3}$ centered at $x=0$. Using the first four terms and evaluating at $x=3$ gives a result (5.03968) accurate to within 10^{-4}.

b. Use the Taylor series for $x^{1/3}$ centered at $x=125$. Note that this gives the identical Taylor series except that the exponential terms are $(x-125)^n$ rather than x^n. Thus we need terms up through $(x-125)^3$, just as before, evaluated at $x=128$, and we obtain the identical result.

c. Because the two Taylor series are the same except for the shifting, the results are equivalent.

9.3.88 Suppose that f is differentiable.
Consider the remainder after the zeroth term of the Taylor series. Taylor's Theorem says that

$$R_0(x)=\frac{f'(c)}{1!}(x-a)^1 \quad \text{for some } c \text{ between } x \text{ and } a,$$

but $f(x)=f(a)+R_0(x)$, which gives $f(x)=f(a)+f'(c)(x-a)$. Rearranging, we obtain $f'(c)=\frac{f(x)-f(a)}{x-a}$ for some c between x and a, which is the conclusion of the Mean Value Theorem.

9.3.89 Consider the remainder after the first term of the Taylor series. Taylor's Theorem indicates that $R_1(x)=\frac{f''(c)}{2}(x-a)^2$ for some c between x and a, so that $f(x)=f(a)+f'(a)(x-a)+\frac{f''(c)}{2}(x-a)^2$. But $f'(a)=0$, so that for every x in an interval containing a, there is a c between x and a such that $f(x)=f(a)+\frac{f''(c)}{2}(x-a)^2$.

a. If $f''(x)>0$ on the interval containing a, then for every x in that interval, we have $f(x)=f(a)+\frac{f''(c)}{2}(x-a)^2$ for some c between x and a. But $f''(c)>0$ and $(x-a)^2>0$, so that $f(x)>f(a)$ and a is a local minimum.

b. If $f''(x)<0$ on the interval containing a, then for every x in that interval, we have $f(x)=f(a)+\frac{f''(c)}{2}(x-a)^2$ for some c between x and a. But $f''(c)<0$ and $(x-a)^2>0$, so that $f(x)<f(a)$ and a is a local maximum.

9.3.90

a. To show that $f'(0)=0$, we compute the limits of the left and right difference quotients and show that they are both zero:

$$\lim_{x\to 0^+}\frac{e^{-1/x^2}-0}{x}=\lim_{x\to 0^+}\frac{e^{-1/x^2}}{x} \qquad \text{and} \qquad \lim_{x\to 0^-}\frac{e^{-1/x^2}-0}{x}=\lim_{x\to 0^-}\frac{e^{-1/x^2}}{x}.$$

For the limit from the right, use the substitution $x=\frac{1}{\sqrt{y}}$; then $y=x^2$ and the limit becomes

$$\lim_{y\to\infty}e^{-y}\sqrt{y}=\lim_{y\to\infty}\frac{\sqrt{y}}{e^y}=0,$$

because exponentials dominate power functions. Similarly, for the limit from the left, use the substitution $x=-\frac{1}{\sqrt{y}}$; then again $y=x^2$ and the limit becomes

$$\lim_{y\to\infty}(-e^{-y}\sqrt{y})=-\lim_{y\to\infty}\frac{\sqrt{y}}{e^y}=0.$$

Since the left and right limits are both zero, it follows that f is differentiable at $x=0$, and its derivative is zero.

b. Because $f^{(k)}(0)=0$, the Taylor series centered at 0 has only one term:$f(x)=f(0)=0$, so the Taylor series is zero.

c. It does not converge to $f(x)$ because $f(x)\neq 0$ for all $x\neq 0$.

9.4 Working with Taylor Series

9.4.1 Replace f and g by their Taylor series centered at a, and evaluate the limit.

9.4.2 Integrate the Taylor series for $f(x)$ centered at a, and evaluate it at the endpoints.

9.4.3 Substitute -0.6 for x in the Taylor series for e^x centered at 0. Note that this series is an alternating series, so the error can easily be estimated by looking at the magnitude of the first neglected term.

9.4.4 Take the Taylor series for $\sin^{-1}(x)$ centered at 0 and evaluate it at $x = 1$, then multiply the result by 2.

9.4.5 The series is $f'(x) = \sum_{k=1}^{\infty} kc_k x^{k-1}$, which converges for $|x| < b$.

9.4.6 It must have derivatives of all orders on some interval containing a.

9.4.7 Because $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$, we have $\frac{e^x-1}{x} = 1 + x/2! + \cdots$, so $\lim_{x\to 0} \frac{e^x-1}{x} = 1$.

9.4.8 Because $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, we have $\frac{\tan^{-1} x - x}{x^3} = \frac{-1}{3} + \frac{x^2}{5} - \cdots$.
So $\lim_{x\to 0} \frac{\tan^{-1} x - x}{3} = \frac{-1}{3}$.

9.4.9 Because $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots$, we have $\frac{-x-\ln(1-x)}{x^2} = \frac{1}{2} + \frac{x}{3} + \frac{x^2}{4} + \cdots$, so $\lim_{x\to 0} \frac{-x-\ln(1-x)}{x^2} = \frac{1}{2}$.

9.4.10 Because $\sin 2x = 2x - \frac{4x^3}{3} + \frac{4x^5}{15} + \cdots$, we have $\frac{\sin 2x}{x} = 2 - \frac{4x^2}{3} + \frac{4x^4}{15} + \cdots$, so $\lim_{x\to 0} \frac{\sin 2x}{x} = 2$.

9.4.11 We compute that

$$\begin{aligned} \frac{e^x - e^{-x}}{x} &= \frac{1}{x}\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)\right) \\ &= \frac{1}{x}\left(2x + \frac{x^3}{3} + \cdots\right) = 2 + \frac{x^2}{3} + \cdots \end{aligned}$$

so the limit of $\dfrac{e^x - e^{-x}}{x}$ as $x \to 0$ is 2.

9.4.12 Because $-e^x = -1 - x - x^2/2 - x^3/6 + \cdots$, we have $\frac{1+x-e^x}{4x^2} = -\frac{1}{8} - \frac{x}{24} + \cdots$, so $\lim_{x\to 0} \frac{1+x-e^x}{4x^2} = -\frac{1}{8}$.

9.4.13 We compute that

$$\begin{aligned} \frac{2\cos 2x - 2 + 4x^2}{2x^4} &= \frac{1}{2x^4}\left(2(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \frac{(2x)^6}{720} + \cdots) - 2 + 4x^2\right) \\ &= \frac{1}{2x^4}\left(\frac{(2x)^4}{12} - \frac{(2x)^6}{360} + \cdots\right) = \frac{2}{3} - \frac{4x^2}{45} + \cdots \end{aligned}$$

so the limit of $\dfrac{2\cos 2x - 2 + 4x^2}{2x^4}$ as $x \to 0$ is $\dfrac{2}{3}$.

9.4.14 We substitute $t = \dfrac{1}{x}$ and find $\displaystyle\lim_{t\to 0} \frac{\sin t}{t}$. We compute that

$$\frac{\sin t}{t} = \frac{1}{t}\left(t - \frac{t^3}{6} + \cdots\right) = 1 - \frac{t^2}{6} + \cdots$$

so the limit of $x \sin\left(\dfrac{1}{x}\right)$ as $x \to \infty$ is 1.

9.4.15 We have $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$, so that

$$\frac{\ln(1+x) - x + x^2/2}{x^3} = \frac{x^3/3 - x^4/4 + \cdots}{x^3} = \frac{1}{3} - \frac{x}{4} + \cdots$$

so that $\lim_{x\to 0} \frac{\ln(1+x) - x + x^2/2}{x^3} = \frac{1}{3}$.

9.4.16 The Taylor series for $\ln(x-3)$ centered at $x = 4$ is

$$(x-4) - \frac{1}{2}(x-4)^2 + \cdots .$$

We compute that

$$\begin{aligned}\frac{x^2-16}{\ln(x-3)} &= \frac{x^2-16}{(x-4) - \frac{1}{2}(x-4)^2 + \cdots} = \frac{(x-4)(x+4)}{(x-4) - \frac{1}{2}(x-4)^2 + \cdots} \\ &= \frac{x+4}{1 - \frac{1}{2}(x-4) + \cdots}\end{aligned}$$

so the limit of $\frac{x^2-16}{\ln(x-3)}$ as $x \to 4$ is 8.

9.4.17 We compute that

$$\begin{aligned}\frac{3\tan^{-1} x - 3x + x^3}{x^5} &= \frac{1}{x^5}\left(3\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) - 3x + x^3\right) \\ &= \frac{1}{x^5}\left(\frac{3x^5}{5} - \frac{3x^7}{7} + \cdots\right) = \frac{3}{5} - \frac{3x^2}{7} + \cdots\end{aligned}$$

so the limit of $\frac{3\tan^{-1} x - 3x + x^3}{x^5}$ as $x \to 0$ is $\frac{3}{5}$.

9.4.18 The Taylor series for $\sqrt{1+x}$ centered at 0 is

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots .$$

We compute that

$$\begin{aligned}\frac{\sqrt{1+x} - 1 - (x/2)}{4x^2} &= \frac{1}{4x^2}\left(\left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots\right) - 1 - \frac{x}{2}\right) \\ &= \frac{1}{4x^2}\left(-\frac{x^2}{8} + \frac{x^3}{16} + \cdots\right) = -\frac{1}{32} + \frac{x}{64} + \cdots\end{aligned}$$

so the limit of $\frac{\sqrt{1+x} - 1 - (x/2)}{4x^2}$ as $x \to 0$ is $-\frac{1}{32}$.

9.4.19 The Taylor series for $\sin 2x$ centered at 0 is

$$\sin 2x = 2x - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \frac{1}{7!}(2x)^7 + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots .$$

Thus

$$\begin{aligned}\frac{12x - 8x^3 - 6\sin 2x}{x^5} &= \frac{12 - 8x^3 - (12x - 8x^3 + \frac{8}{5}x^5 - \frac{16}{105}x^7 + \cdots)}{x^5} \\ &= -\frac{8}{5} + \frac{16}{105}x^2 - \cdots ,\end{aligned}$$

so $\lim_{x\to 0} \frac{12x - 8x^3 - 6\sin 2x}{x^5} = -\frac{8}{5}$.

9.4.20 The Taylor series for $\ln x$ centered at 1 is

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots .$$

We compute that

$$\frac{x-1}{\ln x} = \frac{x-1}{(x-1) - \frac{1}{2}(x-1)^2 + \cdots} = \frac{1}{1 - \frac{1}{2}(x-1) + \cdots}$$

so the limit of $\dfrac{x-1}{\ln x}$ as $x \to 1$ is 1.

9.4.21 The Taylor series for $\ln(x-1)$ centered at 2 is

$$\ln(x-1) = (x-2) - \frac{1}{2}(x-2)^2 + \cdots .$$

We compute that

$$\frac{x-2}{\ln(x-1)} = \frac{x-2}{(x-2) - \frac{1}{2}(x-2)^2 + \cdots} = \frac{1}{1 - \frac{1}{2}(x-2) + \cdots}$$

so the limit of $\dfrac{x-2}{\ln(x-1)}$ as $x \to 2$ is 1.

9.4.22 Because $e^{1/x} = 1 + (1/x) + 1/(2x^2) + \cdots$, we have

$$x(e^{1/x} - 1) = 1 + 1/(2x) + \cdots .$$

Thus, $\lim_{x\to\infty} x(e^{1/x} - 1) = 1$.

9.4.23 Computing Taylor series centers at 0 gives

$$e^{-2x} = 1 - 2x + \frac{1}{2!}(-2x)^2 + \frac{1}{3!}(-2x)^3 + \cdots = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \cdots$$

$$e^{-x/2} = 1 - \frac{x}{2} + \frac{1}{2!}\left(-\frac{x}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x}{2}\right)^3 + \cdots = 1 - \frac{x}{2} + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \cdots .$$

Thus

$$\begin{aligned}\frac{e^{-2x} - 4e^{-x/2} + 3}{2x^2} &= \frac{1 - 2x + 2x^2 - \frac{4}{3}x^3 + \cdots - (4 - 2x + \frac{1}{2}x^2 - \frac{1}{12}x^3 + \cdots) + 3}{2x^2} \\ &= \frac{\frac{3}{2}x^2 - \frac{5}{4}x^3 + \cdots}{2x^2} \\ &= \frac{3}{4} - \frac{5}{8}x + \cdots\end{aligned}$$

so $\displaystyle\lim_{x\to 0} \frac{e^{-2x} - 4e^{-x/2} + 3}{2x^2} = \frac{3}{4}$.

9.4.24 The Taylor series for $(1-2x)^{-1/2}$ centered at 0 is

$$(1-2x)^{-1/2} = 1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \cdots .$$

We compute that

$$\begin{aligned}\frac{(1-2x)^{-1/2} - e^x}{8x^2} &= \frac{1}{8x^2}\left(\left(1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \cdots\right) - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\right) \\ &= \frac{1}{8x^2}\left(x^2 + \frac{7x^3}{3} + \cdots\right) = \frac{1}{8} + \frac{7x}{24} + \cdots\end{aligned}$$

so the limit of $\dfrac{(1-2x)^{-1/2} - e^x}{8x^2}$ as $x \to 0$ is $\dfrac{1}{8}$.

9.4.25

a. $f'(x) = \frac{d}{dx}(\sum_{k=0}^{\infty} \frac{x^k}{k!}) = \sum_{k=1}^{\infty} k \frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = f(x)$.

b. $f'(x) = e^x$ as well.

c. The series converges on $(-\infty, \infty)$.

9.4.26

a. $f'(x) = \frac{d}{dx}(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}) = \sum_{k=1}^{\infty}(-1)^k (2k) \frac{x^{2k-1}}{(2k)!} = \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k-1}}{(2k-1)!} = -\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$.

b. $f'(x) = -\sin x$.

c. The series converges on $(-\infty, \infty)$, because the series for $\cos x$ does.

9.4.27

a. $f'(x) = \frac{d}{dx}(\ln(1+x)) = \frac{d}{dx}(\sum_{k=1}^{\infty}(-1)^{k+1} \frac{1}{k} x^k) = \sum_{k=1}^{\infty}(-1)^{k+1} x^{k-1} = \sum_{k=0}^{\infty}(-1)^k x^k$.

b. This is the power series for $\frac{1}{1+x}$.

c. The Taylor series for $\ln(1+x)$ converges on $(-1, 1)$, as does the Taylor series for $\frac{1}{1+x}$.

9.4.28

a. $f'(x) = \frac{d}{dx}(\sin x^2) = \frac{d}{dx}(\sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{(2k+1)!}) = \sum_{k=0}^{\infty}(-1)^k \cdot 2(2k+1) \frac{x^{4k+1}}{(2k+1)!} = 2\sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+1}}{(2k)!} = 2x\sum_{k=0}^{\infty}(-1)^k \frac{x^{4k}}{(2k)!}$.

b. This is the power series for $2x \cos x^2$.

c. Because the Taylor series for $\sin x^2$ converges everywhere, the Taylor series for $2x \cos x^2$ does as well.

9.4.29

a.

$$f'(x) = \frac{d}{dx}(e^{-2x}) = \frac{d}{dx}(\sum_{k=0}^{\infty} \frac{(-2x)^k}{k!}) = \frac{d}{dx}(\sum_{k=0}^{\infty}(-2)^k \frac{x^k}{k!}) = -2\sum_{k=1}^{\infty}(-2)^{k-1} \frac{x^{k-1}}{(k-1)!} = -2\sum_{k=0}^{\infty} \frac{(-2x)^k}{k!}.$$

b. This is the Taylor series for $-2e^{-2x}$.

c. Because the Taylor series for e^{-2x} converges on $(-\infty, \infty)$, so does this one.

9.4.30

a. We have

$$f'(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k\right) = \frac{d}{dx}\left(1 + \sum_{k=1}^{\infty} x^k\right) = \sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty}(k+1)x^k.$$

b. From the formula for $(1+x)^p$ in Table 9.5, we see that the Taylor series for $\frac{1}{(1-x)^2}$ is

$$\sum_{k=0}^{\infty} \frac{(-2)(-3)\cdots(-2-k+1)}{k!}(-x)^k = \sum_{k=0}^{\infty}(-1)^k(-1)^k \frac{(k+1)!}{k!} x^k = \sum_{k=0}^{\infty}(k+1)x^k,$$

so that $f'(x)$ is simply $\frac{1}{(1-x)^2}$ as expected.

c. Since the Taylor series for $\frac{1}{1-x}$ converges on $(-1,1)$, so does the series for $\frac{1}{(1-x)^2}$. Checking the endpoints, we see that the series diverges at both endpoints by the Divergence test, so that the interval of convergence for $f'(x)$ is also $(-1,1)$.

9.4.31

a. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$, so $\frac{d}{dx} \tan^{-1} x^2 = 1 - x^2 + x^4 - x^6 + \cdots$.

b. This is the series for $\frac{1}{1+x^2}$.

c. Because the series for $\tan^{-1} x$ has a radius of convergence of 1, this series does too. Checking the endpoints shows that the interval of convergence is $(-1,1)$.

9.4.32

a. $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots$, so $\frac{d}{dx}[-\ln(1-x)] = 1 + x + x^2 + x^3 + \cdots$.

b. This is the series for $\frac{1}{1-x}$.

c. The interval of convergence for $\frac{1}{1-x}$ is $(-1,1)$.

9.4.33

a. Because $y(0) = 2$, we have $0 = y'(0) - y(0) = y'(0) - 2$ so that $y'(0) = 2$. Differentiating the equation gives $y''(0) = y'(0)$, so that $y''(0) = 2$. Successive derivatives also have the value 2 at 0, so the Taylor series is $2\sum_{k=0}^{\infty} \frac{t^k}{k!}$.

b. $2\sum_{k=0}^{\infty} \frac{t^k}{k!} = 2e^t$.

9.4.34

a. Because $y(0) = 0$, we see that $y'(0) = 8$. Differentiating the equation gives $y''(0) + 4y'(0) = 0$, so $y''(0) + 4 \cdot 8 = 0$, $y''(0) = -4 \cdot 8$. Continuing, $y'''(0) + 4 \cdot (-4 \cdot 8) = 0$, so $y'''(0) = 4 \cdot 4 \cdot 8$, and in general $y^{(k)}(0) = (-1)^{k+1} 2 \cdot 4^k$ for $k \geq 1$, so the Taylor series is $2\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(4t)^k}{k!}$.

b. $2\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(4t)^k}{k!} = 2(1 - e^{-4t})$.

9.4.35

a. $y(0) = 2$, so that $y'(0) = 16$. Differentiating, $y''(t) - 3y'(t) = 0$, so that $y''(0) = 48$, and in general $y^{(k)}(0) = 3y^{(k-1)}(0) = 3^{k-1} \cdot 16$. Thus the power series is $2 + \frac{16}{3}\sum_{k=1}^{\infty} \frac{(3t)^k}{k!} = 2 + \sum_{k=1}^{\infty} \frac{3^{k-1} 16}{k!} t^k$.

b. $2 + \frac{16}{3}\sum_{k=1}^{\infty} \frac{(3t)^k}{k!} = 2 + \frac{16}{3}(e^{3t} - 1) = \frac{16}{3}e^{3t} - \frac{10}{3}$.

9.4.36

a. $y(0) = 2$, so $y'(0) = 12 + 9 = 21$. Differentiating, $y^{(n)}(0) = 6y^{(n-1)}(0)$ for $n > 1$, so that $y^{(n)}(0) = 6^{n-1} \cdot 21$ for $n \geq 1$. Thus the power series is $2 + \sum_{k=1}^{\infty} 21 \cdot 6^{k-1} \frac{t^k}{k!} = 2 + \frac{7}{2}\sum_{k=1}^{\infty} \frac{(6t)^k}{k!}$.

b. $2 + \frac{7}{2}\sum_{k=1}^{\infty} \frac{(6t)^k}{k!} = 2 + \frac{7}{2}(e^{6t} - 1) = \frac{7}{2}e^{6t} - \frac{3}{2}$.

9.4.37 The Taylor series for e^{-x^2} is $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$. Thus, the desired integral is $\int_0^{0.25} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}\,dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)k!}\Big|_0^{0.25} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)k!4^{2k+1}}$. Because this is an alternating series, to approximate it to within 10^{-4}, we must find n such that $a_{n+1} < 10^{-4}$, or $\frac{1}{(2n+3)(n+1)! \cdot 4^{2n+3}} < 10^{-4}$. This occurs for $n = 1$, so $\sum_{k=0}^{1} (-1)^k \frac{1}{(2k+1) \cdot k! \cdot 4^{2k+1}} = \frac{1}{4} - \frac{1}{192} \approx 0.245$.

9.4.38 The Taylor series for $\sin x^2$ is $\sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{(2k+1)!}$. Thus the desired integral is

$$\int_0^{0.2} \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{(2k+1)!}\, dx = \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+3}}{(4k+3)(2k+1)!}\bigg|_0^{0.2} = \sum_{k=0}^{\infty}(-1)^k \frac{0.2^{4k+3}}{(4k+3)(2k+1)!}.$$

Because this is an alternating series, to approximate it to within 10^{-4}, we must find n such that $a_{n+1} < 10^{-4}$, or $\frac{0.2^{4n+7}}{(4n+7)(2n+3)!} < 10^{-4}$. This occurs first for $n = 0$, so we obtain $\frac{0.2^3}{3 \cdot 1!} \approx 2.67 \times 10^{-3}$.

9.4.39 The Taylor series for $\cos 2x^2$ is $\sum_{k=0}^{\infty}(-1)^k \frac{(2x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k \frac{4^k x^{4k}}{(2k)!}$. Note that $\cos x$ is an even function, so we compute the integral from 0 to 0.35 and double it:

$$2\int_0^{0.35} \sum_{k=0}^{\infty}(-1)^k \frac{4^k x^{4k}}{(2k)!}\, dx = 2\left(\sum_{k=0}^{\infty}(-1)^k \frac{4^k x^{4k+1}}{(4k+1)(2k)!}\right)\bigg|_0^{0.35} = 2\left(\sum_{k=0}^{\infty}(-1)^k \frac{4^k (0.35)^{4k+1}}{(4k+1)(2k)!}\right).$$

Because this is an alternating series, to approximate it to within $\frac{1}{2} \cdot 10^{-4}$, we must find n such that $a_{n+1} < \frac{1}{2} \cdot 10^{-4}$, or $\frac{4^{n+1}(0.35)^{4n+5}}{(4n+3)(2n+2)!} < \frac{1}{2} \cdot 10^{-4}$. This occurs first for $n = 1$, and we have $2\left(.35 - \frac{4 \cdot (0.35)^5}{5 \cdot 2!}\right) \approx 0.696$.

9.4.40 The Taylor series for $(1 + x^4)^{1/2}$ is $\sum_{k=0}^{\infty} \binom{1/2}{k} x^{4k}$, so the desired integral is

$$\int_0^{0.2} \sum_{k=0}^{\infty} \binom{1/2}{k} x^{4k}\, dx = \sum_{k=0}^{\infty} \frac{1}{4k+1}\binom{1/2}{k} x^{4k+1}\bigg|_0^{0.2} = \sum_{k=0}^{\infty} \frac{1}{4k+1}\binom{1/2}{k}(0.2)^{4k+1}.$$

This is an alternating series because the binomial coefficients alternate in sign, so to approximate it to within 10^{-4}, we must find n such that $a_{n+1} < 10^{-4}$, or $\left|\frac{1}{4n+5}\binom{1/2}{n+1}(0.2)^{4n+5}\right| < 10^{-4}$. This happens first for $n = 0$, so the approximation is $\binom{1/2}{0} \cdot 0.2 = 0.2$.

9.4.41 $\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - \cdots$, so $\int \tan^{-1} x\, dx = \int (x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - \cdots)\, dx = C + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \frac{x^8}{56} + \cdots$. Thus, $\int_0^{0.35} \tan^{-1} x\, dx = \frac{(0.35)^2}{2} - \frac{(0.35)^4}{12} + \frac{(0.35)^6}{30} - \frac{(0.35)^8}{56} + \cdots$. Note that this series is alternating, and $\frac{(0.35)^6}{30} < 10^{-4}$, so we add the first two terms to approximate the integral to the desired accuracy. Calculating gives approximately 0.060.

9.4.42 $\ln(1 + x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$, so $\int \ln(1 + x^2)\, dx = \int (x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots)\, dx = C + \frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{21} - \frac{x^9}{36} + \frac{x^{11}}{55} + \cdots$. Thus, $\int_0^{0.4} \ln(1 + x^2)\, dx = \frac{(0.4)^3}{3} - \frac{(0.4)^5}{10} + \frac{(0.4)^7}{21} - \frac{(0.4)^9}{36} + \cdots$. Because $\frac{(0.4)^7}{21} < 10^{-4}$, we add the first two terms to approximate the integral to the desired accuracy. Calculating gives approximately 0.020.

9.4.43 The Taylor series for $(1+x^6)^{-1/2}$ is $\sum_{k=0}^{\infty} \binom{-1/2}{k} x^{6k}$, so the desired integral is $\int_0^{0.5} \sum_{k=0}^{\infty} \binom{-1/2}{k} x^{6k}\, dx$ $= \sum_{k=0}^{\infty} \frac{1}{6k+1}\binom{-1/2}{k} x^{6k+1}\bigg|_0^{0.5} = \sum_{k=0}^{\infty} \frac{1}{6k+1}\binom{-1/2}{k}(0.5)^{6k+1}$. This is an alternating series because the binomial coefficients alternate in sign, so to approximate it to within 10^{-4}, we must find n such that $a_{n+1} < 10^{-4}$, or $\left|\frac{1}{6n+7}\binom{-1/2}{n+1}(0.5)^{6n+7}\right| < 10^{-4}$. This occurs first for $n = 1$, so we have $\binom{-1/2}{0}0.5 + \frac{1}{7}\binom{-1/2}{1}(0.5)^7 \approx 0.499$.

9.4.44 The Taylor series for $\frac{\ln(1+t)}{t}$ centered at 0 is $\sum_{k=0}^{\infty}(-1)^k \frac{t^k}{k+1}$. The desired integral is thus $\int_0^{0.2} \sum_{k=0}^{\infty}(-1)^k \frac{t^k}{k+1}\, dt = \sum_{k=0}^{\infty}(-1)^k \frac{t^{k+1}}{(k+1)^2}\bigg|_0^{0.2} = \sum_{k=0}^{\infty}(-1)^k \frac{(0.2)^{k+1}}{(k+1)^2}$. This is an alternating series, so to approximate it to within 10^{-4}, we must find n such that $a_{n+1} < 10^{-4}$, or $\frac{(0.2)^{n+2}}{(n+2)^2} < 10^{-4}$. This occurs first for $n = 3$, so we have $\sum_{k=0}^{3}(-1)^k \frac{(0.2)^{k+1}}{(k+1)^2} \approx 0.191$.

9.4.45 Use the Taylor series for e^x at 0: $1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}$.

9.4.46 Use the Taylor series for e^x at 0: $1+\frac{1/2}{1!}+\frac{(1/2)^2}{2!}+\frac{(1/2)^3}{3!}=1+\frac{1}{2}+\frac{1}{8}+\frac{1}{8\cdot 3!}$.

9.4.47 Use the Taylor series for $\cos x$ at 0: $1-\frac{2^2}{2!}+\frac{2^4}{4!}-\frac{2^6}{6!}$

9.4.48 Use the Taylor series for $\sin x$ at 0: $1-\frac{1^3}{3!}+\frac{1^5}{5!}-\frac{1^7}{7!}=1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}$.

9.4.49 Use the Taylor series for $\ln(1+x)$ evaluated at $x=1/2$: $\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{8}-\frac{1}{4}\cdot\frac{1}{16}$.

9.4.50 Use the Taylor series for $\tan^{-1}x$ evaluated at 1/2: $\frac{1}{2}-\frac{1}{3}\cdot\frac{1}{8}+\frac{1}{5}\cdot\frac{1}{32}-\frac{1}{7}\cdot\frac{1}{128}$.

9.4.51 The Taylor series for f centered at 0 is $\frac{-1+\sum_{k=0}^{\infty}\frac{x^k}{k!}}{x}=\frac{\sum_{k=1}^{\infty}\frac{x^k}{k!}}{x}=\sum_{k=1}^{\infty}\frac{x^{k-1}}{k!}=\sum_{k=0}^{\infty}\frac{x^k}{(k+1)!}$. Evaluating both sides at $x=1$, we have $e-1=\sum_{k=0}^{\infty}\frac{1}{(k+1)!}$.

9.4.52 The Taylor series for f centered at 0 is $\frac{-1+\sum_{k=0}^{\infty}\frac{x^k}{k!}}{x}=\frac{\sum_{k=1}^{\infty}\frac{x^k}{k!}}{x}=\sum_{k=1}^{\infty}\frac{x^{k-1}}{k!}=\sum_{k=0}^{\infty}\frac{x^k}{(k+1)!}$. Differentiating, the Taylor series for $f'(x)$ is $f'(x)=\frac{(x-1)e^x+1}{x^2}=\sum_{k=1}^{\infty}\frac{kx^{k-1}}{(k+1)!}$. Evaluating both sides at 2 gives $\frac{e^2+1}{4}=\sum_{k=1}^{\infty}\frac{k\cdot 2^{k-1}}{(k+1)!}$.

9.4.53 The Maclaurin series for $\ln(1+x)$ is $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. By the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\left|\frac{x^{k+1}k}{x^k(k+1)}\right|=|x|$, so the radius of convergence is 1. The series diverges at -1 and converges at 1, so the interval of convergence is $(-1,1]$. Evaluating at 1 gives $\ln 2=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$.

9.4.54 The Taylor series for $\ln(1+x)$ at 0 is $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. By the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\left|\frac{x^{k+1}k}{x^k(k+1)}\right|=|x|$, so the radius of convergence is 1. The series diverges at -1 and converges at 1, so the interval of convergence is $(-1,1]$. Evaluate both sides at $-1/2$ to get $f(\frac{-1}{2})=\ln(1/2)=-\ln 2=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(-1/2)^k}{k}=-\sum_{k=1}^{\infty}\frac{1}{k\cdot 2^k}$, so that $\ln 2=\sum_{k=1}^{\infty}\frac{1}{k\cdot 2^k}$.

9.4.55 $\sum_{k=0}^{\infty}\frac{x^k}{2^k}=\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k=\frac{1}{1-\frac{x}{2}}=\frac{2}{2-x}$.

9.4.56 $\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{3^k}=\sum_{k=0}^{\infty}\left(\frac{-x}{3}\right)^k=\frac{1}{1+\frac{x}{3}}=\frac{3}{3+x}$.

9.4.57 $\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{4^k}=\sum_{k=0}^{\infty}\left(\frac{-x^2}{4}\right)^k=\frac{1}{1+\frac{x^2}{4}}=\frac{4}{4+x^2}$.

9.4.58 $\sum_{k=0}^{\infty}2^kx^{2k+1}=x\sum_{k=0}^{\infty}(2x^2)^k=\frac{x}{1-2x^2}$.

9.4.59 $\ln(1+x)=-\sum_{k=1}^{\infty}(-1)^k\frac{x^k}{k}$, so $\ln(1-x)=-\sum_{k=1}^{\infty}\frac{x^k}{k}$, and finally $-\ln(1-x)=\sum_{k=1}^{\infty}\frac{x^k}{k}$.

9.4.60 $\sum_{k=0}^{\infty}\frac{(-1)^kx^{k+1}}{4^k}=-4\sum_{k=0}^{\infty}\left(\frac{-x}{4}\right)^{k+1}=-4(-1+\sum_{k=0}^{\infty}\left(\frac{-x}{4}\right)^k)=4-\frac{4}{1+\frac{x}{4}}=4-\frac{16}{4+x}=\frac{4x}{4+x}$

9.4.61

$$\begin{aligned}\sum_{k=1}^{\infty}(-1)^k\frac{kx^{k+1}}{3^k} &= \sum_{k=1}^{\infty}(-1)^k\frac{k}{3^k}x^{k+1}=\sum_{k=1}^{\infty}k\left(-\frac{1}{3}\right)^kx^{k+1}\\ &= x^2\sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^kkx^{k-1}=x^2\sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^k\frac{d}{dx}(x^k)\\ &= x^2\frac{d}{dx}\left(\sum_{k=1}^{\infty}\left(-\frac{x}{3}\right)^k\right)=x^2\frac{d}{dx}\left(\frac{1}{1+\frac{x}{3}}\right)=-\frac{3x^2}{(x+3)^2}.\end{aligned}$$

9.4.62 By Exercise 53, $\sum_{k=1}^{\infty}\frac{x^k}{k}=-\ln(1-x)$, so $\sum_{k=1}^{\infty}\frac{x^{2k}}{k}=\sum_{k=1}^{\infty}\frac{(x^2)^k}{k}=-\ln(1-x^2)$.

9.4.63 $\sum_{k=2}^{\infty} \frac{k(k-1)x^k}{3^k} = x^2 \sum_{k=2}^{\infty} \frac{k(k-1)x^{k-2}}{3^k} = x^2 \frac{d^2}{dx^2}\left(\sum_{k=2}^{\infty} \frac{x^k}{3^k}\right)$
$= x^2 \frac{d^2}{dx^2}\left(\sum_{k=2}^{\infty} \left(\frac{x}{3}\right)^k\right) = x^2 \frac{d^2}{dx^2}\left(\frac{x^2}{9} \cdot \frac{1}{1-\frac{x}{3}}\right) = x^2 \frac{d^2}{dx^2}\left(\frac{x^2}{9-3x}\right) = x^2 \frac{-6}{(x-3)^3} = \frac{-6x^2}{(x-3)^3}$.

9.4.64 $\sum_{k=2}^{\infty} \frac{x^k}{k(k-1)} = \sum_{k=2}^{\infty} \frac{x^k}{k-1} - \sum_{k=2}^{\infty} \frac{x^k}{k} = x\sum_{k=1}^{\infty} \frac{x^k}{k} - \sum_{k=1}^{\infty} \frac{x^k}{k} + x, = -x\ln(1-x) + \ln(1-x) + x = x + (1-x)\ln(1-x)$.

9.4.65

a. False. This is because $\frac{1}{1-x}$ is not continuous at 1, which is in the interval of integration.

b. False. The Ratio Test shows that the radius of convergence for the Taylor series for $\tan^{-1} x$ centered at 0 is 1.

c. True. $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$. Substitute $x = \ln 2$.

9.4.66 The Taylor series for e^{ax} centered at 0 is

$$e^{ax} = 1 + ax + \frac{(ax)^2}{2} + \frac{(ax)^3}{6} + \cdots .$$

We compute that

$$\begin{aligned} \frac{e^{ax}-1}{x} &= \frac{1}{x}\left(\left(1 + ax + \frac{(ax)^2}{2} + \frac{(ax)^3}{6} + \cdots\right) - 1\right) \\ &= \frac{1}{x}\left(ax + \frac{(ax)^2}{2} + \frac{(ax)^3}{6} + \cdots\right) = a + \frac{a^2x}{2} + \frac{a^3x^2}{6} + \cdots \end{aligned}$$

so the limit of $\dfrac{e^{ax}-1}{x}$ as $x \to 0$ is a.

9.4.67 The Taylor series for $\sin x$ centered at 0 is

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots .$$

We compute that

$$\begin{aligned} \frac{\sin ax}{\sin bx} &= \frac{ax - \frac{(ax)^3}{6} + \frac{(ax)^5}{120} - \cdots}{bx - \frac{(bx)^3}{6} + \frac{(bx)^5}{120} - \cdots} \\ &= \frac{a - \frac{a^3x^2}{6} + \frac{a^5x^4}{120} - \cdots}{b - \frac{b^3x^2}{6} + \frac{b^5x^4}{120} - \cdots} \end{aligned}$$

so the limit of $\dfrac{\sin ax}{\sin bx}$ as $x \to 0$ is $\dfrac{a}{b}$.

9.4.68 The Taylor series for $\sin ax$ centered at 0 is

$$\sin ax = ax - \frac{(ax)^3}{6} + \frac{(ax)^5}{120} - \cdots$$

and the Taylor series for $\tan^{-1} ax$ centered at 0 is

$$\tan^{-1} ax = ax - \frac{(ax)^3}{3} + \frac{(ax)^5}{5} - \cdots .$$

We compute that

$$\begin{aligned} \frac{\sin ax - \tan^{-1} ax}{bx^3} &= \frac{1}{bx^3}\left(\left(ax - \frac{(ax)^3}{6} + \frac{(ax)^5}{120} - \cdots\right) - \left(ax - \frac{(ax)^3}{3} + \frac{(ax)^5}{5} - \cdots\right)\right) \\ &= \frac{1}{bx^3}\left(\frac{(ax)^3}{6} - \frac{23(ax)^5}{120} + \cdots\right) = \frac{a^3}{6b} - \frac{23a^5}{120b}x^2 + \cdots \end{aligned}$$

so the limit of $\dfrac{\sin ax - \tan^{-1} ax}{bx^3}$ as $x \to 0$ is $\dfrac{a^3}{6b}$.

9.4.69 Compute instead the limit of the log of this expression, $\lim_{x\to 0}\frac{\ln(\sin x/x)}{x^2}$. If the Taylor expansion of $\ln(\sin x/x)$ is $\sum_{k=0}^{\infty} c_k x^k$, then $\lim_{x\to 0}\frac{\ln(\sin x/x)}{x^2} = \lim_{x\to 0}\sum_{k=0}^{\infty} c_k x^{k-2} = \lim_{x\to 0} c_0x^{-2} + c_1x^{-1} + c_2$, because the higher-order terms have positive powers of x and thus approach zero as x does. So compute the terms of the Taylor series of $\ln\left(\frac{\sin x}{x}\right)$ up through the quadratic term. The relevant Taylor series are: $\frac{\sin x}{x} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \cdots$, $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$ and we substitute the Taylor series for $\frac{\sin x}{x} - 1$ for x in the Taylor series for $\ln(1+x)$. Because the lowest power of x in the first Taylor series is 2, it follows that only the linear term in the series for $\ln(1+x)$ will give any powers of x that are at most quadratic. The only term that results is $-\frac{1}{6}x^2$. Thus $c_0 = c_1 = 0$ in the above, and $c_2 = -\frac{1}{6}$, so that $\lim_{x\to 0}\frac{\ln(\sin x/x)}{x^2} = -\frac{1}{6}$ and thus $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$.

9.4.70 We can find the Taylor series for $\ln(x+\sqrt{1+x^2})$ by substituting into $\ln(1+t)$ the Taylor series for $x + \sqrt{x^2+1} - 1$. The Taylor series in question are: $x + \sqrt{x^2+1} - 1 = x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \ldots$, $\ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5 - \frac{1}{6}t^6 + \frac{1}{7}t^7 - \ldots$. Substituting the former into the latter and simplifying (not a simple task!), we obtain $\ln(x+\sqrt{x^2+1}) = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \ldots$. Using the second definition, start with the Taylor series for $(1+t^2)^{-1/2}$, which is $1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \ldots$, and integrate it: $\int_0^x \left(1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \ldots\right) dt = \left(t - \frac{1}{6}t^3 + \frac{3}{40}t^5 - \frac{5}{112}t^7 + \ldots\right)\Big|_0^x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \ldots$.

9.4.71 The Taylor series we need are $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots$, $e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \ldots$. We are looking for powers of x^3 and x^4 that occur when the first series is substituted for t in the second series. Clearly there will be no odd powers of x, because $\cos x$ has only even powers. Thus the coefficient of x^3 is zero, so that $f^{(3)}(0) = 0$. The coefficient of x^4 comes from the expansion of $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ in each term of e^t. Higher powers of x clearly cannot contribute to the coefficient of x^4. Thus consider $\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)^k$. The term $-\frac{1}{2}x^2$ generates $\binom{k}{2}$ terms of value $\frac{1}{4}x^4$ for $k \ge 2$, while the other term generates k terms of value $\frac{1}{24}x^4$ for $k \ge 1$. These terms all have to be divided by the $k!$ appearing in the series for e^t. So the total coefficient of x^4 is $\frac{1}{24}\sum_{k=1}^{\infty}\frac{k}{k!} + \frac{1}{4}\sum_{k=2}^{\infty}\binom{k}{2}\frac{1}{k!}, = \frac{1}{24}\sum_{k=1}^{\infty}\frac{1}{(k-1)!} + \frac{1}{4}\sum_{k=2}^{\infty}\frac{1}{2\cdot(k-2)!}, = \frac{1}{24}\sum_{k=0}^{\infty}\frac{1}{k!} + \frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{k!}, = \frac{1}{24}e + \frac{1}{8}e = \frac{e}{6}$ Thus $f^{(4)}(0) = \frac{e}{6}\cdot 4! = 4e$.

9.4.72 The Taylor series for $(1+x)^{-1/3}$ is $(1+x)^{-1/3} = 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \frac{35}{243}x^4 - \ldots$, so we want the coefficients of x^3 and x^4 in $(x^2+1)\left(1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \frac{35}{243}x^4\right)$. The coefficient of x^3 is $-\frac{1}{3} - \frac{14}{81} = -\frac{41}{81}$, and the coefficient of x^4 is $\frac{2}{9} + \frac{35}{243} = \frac{89}{243}$. Thus $f^{(3)}(0) = 6\cdot\frac{-41}{81} = \frac{-82}{27}$, and $f^{(4)}(0) = 24\cdot\frac{89}{243} = \frac{712}{81}$.

9.4.73 The Taylor series for $\sin t^2$ is $\sin t^2 = t^2 - \frac{1}{3!}t^6 + \frac{1}{5!}t^{10} - \ldots$, so that $\int_0^x \sin t^2\,dt = \frac{1}{3}t^3 - \frac{1}{7\cdot 3!}t^7 + \ldots\Big|_0^x = \frac{1}{3}x^3 - \frac{1}{7\cdot 3!}x^7 + \ldots$. Thus $f^{(3)}(0) = \frac{3!}{3} = 2$ and $f^{(4)}(0) = 0$.

9.4.74 $\frac{1}{1+t^4} = 1 - t^4 + t^8 + \ldots$, so that $\int_0^x \frac{1}{1+t^4}dt = t - \frac{1}{5}t^5 + \frac{1}{9}t^9 + \ldots\Big|_0^x = x - \frac{1}{5}x^5 + \ldots$ so that both $f^{(3)}(0)$ and $f^{(4)}(0)$ are zero.

9.4.75 Consider the series $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$. Differentiating both sides gives $\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{x}\sum_{k=0}^{\infty} kx^k$ so that $\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^k$. Evaluate both sides at $x = 1/2$ to see that the sum of the series is $\frac{1/2}{(1-1/2)^2} = 2$. Thus the expected number of tosses is 2.

9.4.76

a. $\sum_{k=0}^{\infty}\frac{1}{6}\left(\frac{5}{6}\right)^{2k} = \frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{25}{36}\right)^k = \frac{1}{6}\cdot\frac{1}{1-25/36} = \frac{6}{11}$.

b. Consider the series $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$. Differentiating both sides gives $\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}$ Evaluating at $x = 5/6$ and multiplying the result by $1/6$, we get $\frac{1}{6}\cdot\frac{1}{(1-5/6)^2} = 6$.

9.4.77

a. We look first for a Taylor series for $(1-k^2\sin^2\theta)^{-1/2}$. Because $(1-k^2x^2)^{-1/2} = (1-(kx)^2)^{-1/2} = \sum_{i=0}^{\infty}\binom{-1/2}{i}(kx)^{2i}$, and $\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \dots$, substituting the second series into the first gives $\frac{1}{\sqrt{1-k^2\sin^2\theta}} = 1 + \frac{1}{2}k^2\theta^2 + \left(-\frac{1}{6}k^2 + \frac{3}{8}k^4\right)\theta^4 + \left(\frac{1}{45}k^2 - \frac{1}{4}k^4 + \frac{5}{16}k^6\right)\theta^6 +$ $\left(\frac{-1}{630}k^2 + \frac{3}{40}k^4 - \frac{5}{16}k^6 + \frac{35}{128}k^8\right)\theta^8 + \dots$.
Integrating with respect to θ and evaluating at $\pi/2$ (the value of the antiderivative is 0 at 0) gives $\frac{1}{2}\pi + \frac{1}{48}k^2\pi^3 + \frac{1}{160}\left(-\frac{1}{6}k^2 + \frac{3}{8}k^4\right)\pi^5 + \frac{1}{896}\left(\frac{1}{45}k^2 - \frac{1}{4}k^4 + \frac{5}{16}k^6\right)\pi^7 + \frac{1}{4608}\left(-\frac{1}{630}k^2 + \frac{3}{40}k^4 - \frac{5}{16}k^6 + \frac{35}{128}k^8\right)\pi^9$. Evaluating these terms for $k = 0.1$ gives $F(0.1) \approx 1.574749680$. (The true value is approximately 1.574745562.)

b. The terms above, with coefficients of k^n converted to decimal approximations, is $1.5707 + .3918\cdot k^2 + .3597\cdot k^4 - .9682\cdot k^6 + 1.7689\cdot k^8$. The coefficients are all less than 2 and do not appear to be increasing very much if at all, so if we want the result to be accurate to within 10^{-3} we should probably take n such that $k^n < \frac{1}{2}\times 10^{-3} = .0005$, so $n = 4$ for this value of k.

c. By the above analysis, we would need a larger n because $0.2^n > 0.1^n$ for a given value of n.

9.4.78

a. $\frac{\sin t}{t} = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$.

b. $\int_0^x \frac{\sin t}{t}\,dt = \sum_{k=0}^{\infty}\int_0^x(-1)^k\frac{t^{2k}}{(2k+1)!}\,dt = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)(2k+1)!}$.

c. This is an alternating series, so we want n such that $a_{n+1} < 10^{-3}$, or $\frac{0.5^{2n+3}}{(2n+3)(2n+3)!} < 10^{-3}$ (resp. $\frac{1^{2n+3}}{(2n+3)(2n+3)!} < 10^{-3}$), which gives $n = 1$ (resp. $n = 2$). Thus $\mathrm{Si}(0.5) \approx \frac{0.5}{1} - \frac{0.5^3}{3\cdot 3!} \approx 0.4930555556$, $\mathrm{Si}(1.0) \approx 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} \approx 0.9461111111$.

9.4.79

a. By the Fundamental Theorem, $S'(x) = \sin x^2$, $C'(x) = \cos x^2$.

b. The relevant Taylor series are $\sin t^2 = t^2 - \frac{1}{3!}t^6 + \frac{1}{5!}t^{10} - \frac{1}{7!}t^{14} + \dots$, and $\cos t^2 = 1 - \frac{1}{2!}t^4 + \frac{1}{4!}t^8 - \frac{1}{6!}t^{12} + \dots$. Integrating, we have $S(x) = \frac{1}{3}x^3 - \frac{1}{7\cdot 3!}x^7 + \frac{1}{11\cdot 5!}x^{11} - \frac{1}{15\cdot 7!}x^{15} + \dots$, and $C(x) = x - \frac{1}{5\cdot 2!}x^5 + \frac{1}{9\cdot 4!}x^9 - \frac{1}{13\cdot 6!}x^{13} + \dots$.

c. $S(0.05) \approx \frac{1}{3}(0.05)^3 - \frac{1}{42}(0.05)^7 + \frac{1}{1320}(0.05)^{11} - \frac{1}{75600}(0.05)^{15} \approx 4.166664807\times 10^{-5}$. $C(-0.25) \approx (-0.25) - \frac{1}{10}(-0.25)^5 + \frac{1}{216}(-0.25)^9 - \frac{1}{9360}(-0.25)^{13} \approx -.2499023616$.

d. The series is alternating. Because $a_{n+1} = \frac{1}{(4n+7)(2n+3)!}(0.05)^{4n+7}$, and this is less than 10^{-4} for $n = 0$, only one term is required.

e. The series is alternating. Because $a_{n+1} = \frac{1}{(4n+5)(2n+2)!}(0.25)^{4n+5}$, and this is less than 10^{-6} for $n = 1$, two terms are required.

9.4.80

a. $\frac{d}{dx}\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}(e^{-x^2})$.

b. $e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots = \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{k!}$, so that the Maclaurin series for the error function is $\mathrm{erf(x)} = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \dots\right)$.

c. $\mathrm{erf}(0.15) \approx \frac{2}{\sqrt{\pi}}\left(0.15 - \frac{0.15^3}{3} + \frac{0.15^5}{10} - \frac{0.15^7}{42}\right) \approx 0.1679959712$.

$\mathrm{erf}(-0.09) \approx \frac{2}{\sqrt{\pi}}\left(-0.09 + \frac{0.09^3}{3} - \frac{0.09^5}{10} + \frac{0.09^7}{42}\right) \approx -.1012805939$.

d. The first omitted term in each case is $\frac{x^9}{9\cdot 5!} = \frac{x^9}{1080}$. For $x = 0.15$, this is $\approx 3.56 \times 10^{-11}$. For $x = -0.09$, this is (in absolute value) $\approx 3.59 \times 10^{-13}$.

9.4.81

a. $J_0(x) = 1 - \frac{1}{4}x^2 + \frac{1}{16\cdot 2!^2}x^4 - \frac{1}{2^6\cdot 3!^2}x^6 + \dots$.

b. Using the Ratio Test: $\left|\frac{a_{k+1}}{a_k}\right| = \frac{x^{2k+2}}{2^{2k+2}((k+1)!)^2} \cdot \frac{2^{2k}(k!)^2}{x^{2k}} = \frac{x^2}{4(k+1)^2}$, which has limit 0 as $k \to \infty$ for any x. Thus the radius of convergence is infinite and the interval of convergence is $(-\infty, \infty)$.

c. Starting only with terms up through x^8, we have $J_0(x) = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \frac{1}{147456}x^8 + \dots$, $J_0'(x) = -\frac{1}{2}x + \frac{1}{16}x^3 - \frac{1}{384}x^5 + \frac{1}{18432}x^7 + \dots$, $J_0''(x) = -\frac{1}{2} + \frac{3}{16}x^2 - \frac{5}{384}x^4 + \frac{7}{18432}x^6 + \dots$ so that $x^2J_0(x) = x^2 - \frac{1}{4}x^4 + \frac{1}{64}x^6 - \frac{1}{2304}x^8 + \frac{1}{147456}x^{10} + \dots$, $xJ_0'(x) = -\frac{1}{2}x^2 + \frac{1}{16}x^4 - \frac{1}{384}x^6 + \frac{1}{18432}x^8 + \dots$, $x^2J_0''(x) = -\frac{1}{2}x^2 + \frac{3}{16}x^4 - \frac{5}{384}x^6 + \frac{7}{18432}x^8 + \dots$, and $x^2J_0''(x) + xJ_0'(x) + x^2J_0(x) = 0$.

9.4.82 $\sec x = \frac{1}{\cos x} = \frac{1}{1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots} = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \dots$

9.4.83

a. The power series for $\cos x$ has only even powers of x, so that the power series has the same value evaluated at $-x$ as it does at x.

b. The power series for $\sin x$ has only odd powers of x, so that evaluating it at $-x$ gives the opposite of its value at x.

9.4.84 Long division gives $\csc x = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \cdots$, so that $\csc x \approx \frac{1}{x} + \frac{1}{6}x$ as $x \to 0^+$.

9.4.85

a. Because $f(a) = g(a) = 0$, we use the Taylor series for $f(x)$ and $g(x)$ centered at a to compute that

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots}{g(a) + g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \cdots} \\ &= \lim_{x\to a}\frac{f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots}{g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \cdots} \\ &= \lim_{x\to a}\frac{f'(a) + \frac{1}{2}f''(a)(x-a) + \cdots}{g'(a) + \frac{1}{2}g''(a)(x-a) + \cdots} = \frac{f'(a)}{g'(a)}.\end{aligned}$$

Because $f'(x)$ and $g'(x)$ are assumed to be continuous at a and $g'(a) \neq 0$,

$$\frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

and we have that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

which is one form of L'Hôpital's Rule.

b. Because $f(a) = g(a) = f'(a) = g'(a) = 0$, we use the Taylor series for $f(x)$ and $g(x)$ centered at a to compute that

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \cdots}{g(a) + g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \frac{1}{6}g'''(a)(x-a)^3 + \cdots} \\ &= \lim_{x\to a}\frac{\frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \cdots}{\frac{1}{2}g''(a)(x-a)^2 + \frac{1}{6}g'''(a)(x-a)^3 + \cdots} \\ &= \lim_{x\to a}\frac{\frac{1}{2}f''(a) + \frac{1}{6}f'''(a)(x-a) + \cdots}{\frac{1}{2}g''(a) + \frac{1}{6}g'''(a)(x-a) + \cdots} = \frac{f''(a)}{g''(a)}.\end{aligned}$$

Because $f''(x)$ and $g''(x)$ are assumed to be continuous at a and $g''(a) \neq 0$,

$$\frac{f''(a)}{g''(a)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

and we have that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

which is consistent with two applications of L'Hôpital's Rule.

9.4.86

a. Clearly $x = \sin s$ because BE, of length x, is the side opposite the angle measured by s in a right triangle with unit length hypotenuse.

b. In the formula $\frac{1}{2}r^2\theta$ for the formula for the area of a circular sector, we have $r = 1$, and $\theta = s$, so that the area is in fact $\frac{s}{2}$. But the area can also be expressed as an integral as follows: the area of the sector is the area under the circle between P and F (i.e. the area of the region $PAEF$), minus the area of the right triangle PEF. The area of the right triangle is $\frac{1}{2}x\sqrt{1-x^2}$ by the Pythagorean theorem and the formula for the area of a triangle. Equating these two formulae for the area of the sector, we have $\frac{s}{2} = \int_0^x \sqrt{1-t^2}\,dt - \frac{1}{2}x\sqrt{1-x^2}$, so $s = 2\int_0^x \sqrt{1-t^2}\,dt - x\sqrt{1-x^2}$.

c. The Taylor series for $\sqrt{1-t^2}$ is $1 - \frac{1}{2}t^2 - \frac{1}{8}t^4 - \frac{1}{16}t^6 - \frac{5}{128}t^8 - \ldots$. Integrating and evaluating at x we have $s = \sin^{-1} x = 2\left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 - \frac{5}{1152}x^9\right) - x\left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8\right) + \cdots = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots$.

d. Suppose $x = \sin s = a_0 + a_1 s + a_2 s^2 + \ldots$. Then $x = \sin(\sin^{-1}(x)) = a_0 + a_1(x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots) + a_2((x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots)^2 + \ldots$. Equating coefficients yields $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = \frac{-1}{6}$, and so on.

Chapter Nine Review

1

a. True. The approximations tend to get better as n increases in size, and also when the value being approximated is closer to the center of the series. Because 2.1 is closer to 2 than 2.2 is, and because $3 > 2$, we should have $|p_3(2.1) - f(2.1)| < |p_2(2.2) - f(2.2)|$.

b. False. The interval of convergence may or may not include the endpoints.

c. True. The interval of convergence is an interval centered at 0, and the endpoints may or may not be included.

d. True. Because $f(x)$ is a polynomial, all its derivatives vanish after a certain point (in this case, $f^{(12)}(x)$ is the last nonzero derivative).

2 $p_3(x) = 2x - \frac{(2x)^3}{3!}$.

3 $p_2(x) = 1$.

4 $p_2(x) = 1 - x + \frac{x^2}{2}$.

5 $p_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$.

6 $p_2(x) = \frac{\sqrt{2}}{2}\left(1 - (x - \pi/4) - \frac{1}{2}(x - \pi/4)^2\right)$.

7 $p_2(x) = x - 1 - \frac{1}{2}(x-1)^2$.

8 $p_4(x) = 8x^3/3! + 2x = 4x^3/3 + 2x$.

9 $p_3(x) = \frac{5}{4} + \frac{3(x-\ln 2)}{4} + \frac{5(x-\ln 2)^2}{8} + \frac{(x-\ln 2)^3}{8}$.

10

a. $p_0(x) = p_1(x) = 1$, and $p_2(x) = 1 - \frac{x^2}{2}$.

b.

n	$p_n(-0.08)$	$\lvert p_n(-0.08) - \cos(-0.08)\rvert$
0	1	3.2×10^{-3}
1	1	3.2×10^{-3}
2	0.997	1.7×10^{-6}

11

a. $p_0(x) = 1$, $p_1(x) = 1 + x$, and $p_2(x) = 1 + x + \frac{x^2}{2}$.

b.

n	$p_n(-0.08)$	$\lvert p_n(-0.08) - e^{-0.08}\rvert$
0	1	7.7×10^{-2}
1	0.92	3.1×10^{-3}
2	0.923	8.4×10^{-5}

12

a. $p_0(x) = 1$, $p_1(x) = 1 + \frac{1}{2}x$, and $p_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$.

b.

n	$p_n(0.08)$	$\lvert p_n(0.08) - \sqrt{1+0.08}\rvert$
0	1	3.9×10^{-2}
1	1.04	7.7×10^{-4}
2	1.039	3.0×10^{-5}

13

a. $p_0(x) = \frac{\sqrt{2}}{2}$, $p_1(x) = \frac{\sqrt{2}}{2}(1 + (x - \pi/4))$, and $p_2(x) = \frac{\sqrt{2}}{2}\left(1 + (x - \pi/4) - \frac{1}{2}(x - \pi/4)^2\right)$.

b.

n	$p_n(\pi/5)$	$\lvert p_n(\pi/5) - \sin(\pi/5)\rvert$
0	0.707	1.2×10^{-1}
1	0.596	8.2×10^{-3}
2	0.587	4.7×10^{-4}

14 The bound is $|R_n(x)| \le M\frac{|x|^{n+1}}{(n+1)!}$, where M is a bound for $|e^x|$ (because e^x is its own derivative) on $[-1, 1]$. Thus take $M = 3$ so that $|R_3(x)| \le \frac{3x^4}{4!} = \frac{x^4}{8}$. But $|x| < 1$, so this is at most $\frac{1}{8}$.

15 The derivatives of $\sin x$ are bounded in magnitude by 1, so $|R_n(x)| \le M\frac{|x|^{n+1}}{(n+1)!} \le \frac{|x|^{n+1}}{(n+1)!}$. But $|x| < \pi$, so $|R_3(x)| \le \frac{\pi^4}{24}$.

16 The third derivative of $\ln(1-x)$ is $\frac{-2}{(x-1)^3}$, which is bounded in magnitude by 16 on $|x| < 1/2$ (at $x = 1/2$). Thus $|R_3(x)| \le 16\frac{|x|^4}{4!} \le 16\frac{1}{2^4 4!} = \frac{1}{4!}$.

17 Using the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)^2x^{k+1}}{(k+1)!}\cdot\frac{k!}{k^2x^k}\right| = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^2\frac{|x|}{k+1} = 0$, so the interval of convergence is $(-\infty, \infty)$.

18 Using the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{4k+4}}{(k+1)^2}\cdot\frac{k^2}{x^{4k}}\right| = \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^2 x^4 = x^4$, so that the radius of convergence is 1. Because $\sum\frac{1}{k^2}$ converges, the given power series converges at both endpoints, so its interval of convergence is $[-1, 1]$.

19 Using the Ratio Test, $\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\left|\frac{(x+1)^{2k+2}}{(k+1)!}\cdot\frac{k!}{(x+1)^{2k}}\right| = \lim_{k\to\infty}\frac{1}{k+1}(x+1)^2 = 0$, so the interval of convergence is $(-\infty, \infty)$.

20 Using the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(x-1)^{k+1}}{(k+1)5^{k+1}}\cdot\frac{k5^k}{(x-1)^k}\right| = \lim_{k\to\infty}\frac{k}{5k+5}|x-1| = \frac{1}{5}(|x-1|)$, so the series converges when $|1/5(x-1)| < 1$, or $-5 < x-1 < 5$, so that $-4 < x < 6$. At $x = -4$, the series is the alternating harmonic series. At $x = 6$, it is the harmonic series, so the interval of convergence is $[-4, 6)$.

21 By the Root Test, $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\left(\frac{|x|}{9}\right)^3 = \frac{|x^3|}{729}$, so the series converges for $|x| < 9$. The series given by letting $x = \pm 9$ are both divergent by the Divergence Test. Thus, $(-9, 9)$ is the interval of convergence.

22 By the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(x+2)^{k+1}}{\sqrt{k+1}}\cdot\frac{\sqrt{k}}{(x+2)^k}\right| = \lim_{k\to\infty}\sqrt{\frac{k}{k+1}}(|x+2|) = |x+2|$, so that the series converges for $|x+2| < 1$, so $-3 < x < -1$. At $x = -3$, we have a series which converges by the Alternating Series Test. At $x = -1$, we have the divergent $p-$series with $p = 1/2$. Thus, $[-3, -1)$ is the interval of convergence.

23 By the Ratio Test, $\lim_{k\to\infty}\left|\frac{(x+2)^{k+1}}{2^{k+1}\ln(k+1)}\cdot\frac{2^k\ln k}{(x+2)^k}\right| = \lim_{k\to\infty}\frac{\ln k}{2\ln(k+1)}|x+2| = \frac{|x+2|}{2}$. The radius of convergence is thus 2, and a check of the endpoints gives the divergent series $\sum\frac{1}{\ln k}$ at $x = 0$ and the convergent alternating series $\sum\frac{(-1)^k}{\ln k}$ at $x = -4$. The interval of convergence is therefore $[-4, 0)$.

24 By the Ratio Test, $\lim_{k\to\infty}\left|\frac{x^{2k+3}}{2k+3}\cdot\frac{2k+1}{x^{2k+1}}\right| = x^2$. The radius of convergence is thus 1. At each endpoint we have a divergent series, so the interval of convergence is $(-1, 1)$.

25 The Maclaurin series for $f(x)$ is $\sum_{k=0}^{\infty} x^{2k}$. By the Root Test, this converges for $\left|x^2\right| < 1$, so $-1 < x < 1$. It diverges at both endpoints, so the interval of convergence is $(-1, 1)$.

26 The Maclaurin series for $f(x)$ is determined by replacing x by $(-x)^3$ in the power series for $\frac{1}{1-x}$, so it is $\sum_{k=0}^{\infty}(-1)^k x^{3k}$. The radius of convergence is still 1. The series diverges at both endpoints, so the interval of convergence is $(-1, 1)$.

27 The Maclaurin series for $f(x)$ is $\sum_{k=0}^{\infty}(-5x)^k = \sum_{k=0}^{\infty}(-5)^k x^k$. By the Root Test, this has radius of convergence 1/5. Checking the endpoints, we obtain an interval of convergence of $(-1/5, 1/5)$.

28 Replace x by $-x$ in the original power series, and multiply the result by $10x$, to get the Maclaurin series for $f(x)$, which is $\sum_{k=0}^{\infty}(-1)^k 10x^{k+1}$. By the Ratio Test, the radius of convergence is 1. Checking the endpoints, we obtain an interval of convergence of $(-1, 1)$.

29 Note that $\frac{1}{1-10x} = \sum_{k=0}^{\infty}(10x)^k$, so $\frac{1}{10}\cdot\frac{1}{1-10x} = \frac{1}{10}\sum_{k=0}^{\infty}(10x)^k$. Taking the derivative of $\frac{1}{10}\cdot\frac{1}{1-10x}$ gives $f(x)$. Thus, the Maclaurin series for $f(x)$ is $\frac{1}{10}\sum_{k=1}^{\infty}10k(10x)^{k-1} = \sum_{k=1}^{\infty}k(10x)^{k-1}$. Using the Ratio Test, we see that the radius of convergence is $1/10$, and checking endpoints we obtain an interval of convergence of $(-1/10, 1/10)$.

30 Integrating $\frac{1}{1-x}$ and then replacing x by $4x$ gives $-f(x)$, so the series for $f(x)$ is $-\sum_{k=0}^{\infty}\frac{1}{k+1}(4x)^{k+1}$. The Ratio Test shows that the series has a radius of convergence of $1/4$; checking the endpoints, we obtain an interval of convergence of $[-1/4, 1/4)$.

31 The first three terms are $1 + 3x + \frac{9x^2}{2}$. The series is $\sum_{k=0}^{\infty}\frac{(3x)^k}{k!}$.

32 The first three terms are $1-(x-1)+(x-1)^2$. The series is $\sum_{k=0}^{\infty}(-1)^k(x-1)^k$.

33 The first three terms are $-(x-\pi/2)+\frac{1}{6}(x-\pi/2)^3 - \frac{1}{120}(x-\pi/2)^5$. The series is

$$\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{(2k+1)!}\left(x-\frac{\pi}{2}\right)^{2k+1}.$$

34 The first three terms for $\frac{1}{1+x}$ are $1-x+x^2$, so the first three terms of $x^2\cdot\frac{1}{1+x}$ are $x^2-x^3+x^4$. The series is $\sum_{k=0}^{\infty}(-1)^k x^{k+2}$.

35 The first three terms are $4x - \frac{1}{3}(4x)^3 + \frac{1}{5}(4x)^5$. The series is $\sum_{k=0}^{\infty}(-1)^k\frac{(4x)^{2k+1}}{2k+1}$.

36 The nth derivative of $f(x) = \sin(2x)$ is $\pm 2^n$ times either $\sin 2x$ or $\cos 2x$. Evaluated at $-\frac{\pi}{2}$, the even derivatives are therefore zero, and the $(2n+1)^{\text{st}}$ derivative is $(-1)^{n+1}2^{2n+1}$. The Taylor series for $\sin 2x$ around $x = -\frac{\pi}{2}$ is thus $-2\left(x+\frac{\pi}{2}\right) + \frac{2^3}{3!}\left(x+\frac{\pi}{2}\right)^3 - \frac{2^5}{5!}\left(x+\frac{\pi}{2}\right)^5 + \cdots$, and the general series is $\sum_{k=0}^{\infty}(-1)^{k+1}\frac{2^{2k+1}}{(2k+1)!}\left(x+\frac{\pi}{2}\right)^{2k+1}$.

37 The nth derivative of $\cosh 3x$ at $x=0$ is 0 if n is odd and is 3^n if n is even. The first 3 terms of the series are thus $1+\frac{9x^2}{2!}+\frac{81x^4}{4!}$. The whole series can be written as $\sum_{k=0}^{\infty}\frac{(3x)^{2k}}{(2k)!}$.

38 $f(0) = \frac{1}{4}$, $f'(x) = \frac{-2x}{(x^2+4)^2}$, so $f'(0) = 0$. $f''(x) = \frac{6x^2-8}{(x^2+4)^3}$, so $f''(0) = -\frac{1}{8}$. $f'''(0) = 0$, and $f''''(0) = \frac{3}{8}$. The first three terms are $\frac{1}{4} - \frac{x^2}{16} + \frac{x^4}{64}$. The series is given by $\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{4^{k+1}}$.

39 $f(x) = \binom{1/3}{0} + \binom{1/3}{1}x + \binom{1/3}{2}x^2 + \cdots = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \cdots$.

40 $f(x) = \binom{-1/2}{0} + \binom{-1/2}{1}x + \binom{-1/2}{2}x^2 + \cdots = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$.

41 $f(x) = \binom{-3}{0} + \binom{-3}{1}\frac{x}{2} + \binom{-3}{2}\frac{x^2}{4} + \cdots = 1 - \frac{3}{2}x + \frac{3}{2}x^2 + \cdots$.

42 $f(x) = \binom{-5}{0} + \binom{-5}{1}(2x) + \binom{-5}{2}(2x)^2 + \cdots = 1 - 10x + 60x^2 + \cdots$.

43 $R_n(x) = \frac{(-1)^{n+1}e^{-c}}{(n+1)!}x^{n+1}$ for some c between 0 and x, and $\lim_{n\to\infty}|R_n(x)| \le e^{-|x|}\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0$, because $n!$ grows faster than $|x|^n$ as $n\to\infty$ for all x.

44 $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c between 0 and x. Because all derivatives of $\sin x$ are bounded in magnitude by 1, we have $\lim_{n\to\infty}|R_n(x)| \le \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0$ because $n!$ grows faster than $|x|^n$ as $n\to\infty$ for all x.

45 $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c in $(-1/2, 1/2)$. Now, $\left|f^{(n+1)}(c)\right| = \frac{n!}{(1+c)^{n+1}}$, so $\lim_{n\to\infty}|R_n(x)| \le \lim_{n\to\infty}(2|x|)^{n+1}\cdot\frac{1}{n+1} \le \lim_{n\to\infty}1^{n+1}\frac{1}{n+1} = 0$.

46 $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c in $(-1/2, 1/2)$. Now the $(n+1)^{\text{st}}$ derivative of $(\sqrt{1+x})$ is $\pm\frac{1\cdot3\cdot5\cdots(2n-1)}{2^{n+1}(1+x)^{(2n+1)/2}}$, so for c in $(-1/2, 1/2)$, this is bounded in magnitude by $\frac{1\cdot3\cdot5\cdots(2n-1)}{2^{n+1}(1/2)^{(2n+1)/2}} = \frac{1\cdot3\cdot5\cdots(2n-1)}{2^{1/2}}$, and thus

$$\begin{aligned}
\lim_{n\to\infty}|R_n(x)| &= \lim_{n\to\infty}\left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \\
&\le \lim_{n\to\infty}\frac{1\cdot3\cdot5\cdot(2n-1)}{\sqrt{2}}\cdot\frac{1}{2^{n+1}\cdot(n+1)!} \\
&= \lim_{n\to\infty}\frac{1\cdot3\cdot5\cdots(2n-1)}{\sqrt{2}}\cdot\frac{1}{2\cdot4\cdot6\cdots(2n+2)} \\
&= \lim_{n\to\infty}\left(\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\cdot\frac{1}{2n+2}\right) = 0.
\end{aligned}$$

for x in $(-1/2, 1/2)$.

47 The Taylor series for $\cos x$ centered at 0 is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots.$$

We compute that

$$\begin{aligned}
\frac{x^2/2 - 1 + \cos x}{x^4} &= \frac{1}{x^4}\left(x^2/2 - 1 + \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)\right) \\
&= \frac{1}{x^4}\left(\frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) = \frac{1}{24} - \frac{x^2}{720} + \cdots
\end{aligned}$$

so the limit of $\frac{x^2/2 - 1 + \cos x}{x^4}$ as $x \to 0$ is $\frac{1}{24}$.

48 The Taylor series for $\sin x$ centered at 0 is

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

and the Taylor series for $\tan^{-1} x$ centered at 0 is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

We compute that

$$\begin{aligned}
&\frac{2\sin x - \tan^{-1} x - x}{2x^5} \\
&= \frac{1}{2x^5}\left(2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) - x\right) \\
&= \frac{1}{2x^5}\left(\frac{11x^5}{60} + \frac{359x^7}{2520} - \cdots\right) = -\frac{11}{120} + \frac{359x^2}{5040} - \cdots
\end{aligned}$$

so the limit of $\frac{2\sin x - \tan^{-1} x - x}{2x^5}$ as $x \to 0$ is $-\frac{11}{120}$.

49 The Taylor series for $\ln(x-3)$ centered at 4 is

$$\ln(x-3) = (x-4) - \frac{1}{2}(x-4)^2 + \frac{1}{3}(x-4)^3 - \cdots.$$

We compute that

$$\begin{aligned}\frac{\ln(x-3)}{x^2-16} &= \frac{1}{(x-4)(x+4)}\left((x-4)-\frac{1}{2}(x-4)^2+\frac{1}{3}(x-4)^3-\cdots\right)\\ &= \frac{1}{(x-4)(x+4)}\left((x-4)\left(1-\frac{1}{2}(x-4)+\frac{1}{3}(x-4)^2-\cdots\right)\right)\\ &= \frac{1}{x+4}\left(1-\frac{1}{2}(x-4)+\frac{1}{3}(x-4)^2-\cdots\right)\end{aligned}$$

so the limit of $\dfrac{\ln(x-3)}{x^2-16}$ as $x \to 4$ is $\dfrac{1}{8}$.

50 The Taylor series for $\sqrt{1+2x}$ centered at 0 is

$$\sqrt{1+2x} = 1 + x - \frac{x^2}{2} + \frac{x^3}{2} - \cdots .$$

We compute that

$$\begin{aligned}\frac{\sqrt{1+2x}-1-x}{x^2} &= \frac{1}{x^2}\left(\left(1+x-\frac{x^2}{2}+\frac{x^3}{2}-\cdots\right)-1-x\right)\\ &= \frac{1}{x^2}\left(-\frac{x^2}{2}+\frac{x^3}{2}-\cdots\right) = -\frac{1}{2}+\frac{x}{2}-\cdots\end{aligned}$$

so the limit of $\dfrac{\sqrt{1+2x}-1-x}{x^2}$ as $x \to 0$ is $-\dfrac{1}{2}$.

51 The Taylor series for $\sec x$ centered at 0 is

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$$

and the Taylor series for $\cos x$ centered at 0 is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots .$$

We compute that

$$\begin{aligned}&\frac{\sec x - \cos x - x^2}{x^4}\\ &= \frac{1}{x^4}\left(\left(1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}+\cdots\right)-\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots\right)-x^2\right)\\ &= \frac{1}{x^4}\left(\frac{x^4}{6}+\frac{31x^6}{360}+\cdots\right) = \frac{1}{6}+\frac{31x^2}{360}+\cdots\end{aligned}$$

so the limit of $\dfrac{\sec x - \cos x - x^2}{x^4}$ as $x \to 0$ is $\dfrac{1}{6}$.

52 The Taylor series for $(1+x)^{-2}$ centered at 0 is

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

and the Taylor series for $\sqrt[3]{1-6x}$ centered at 0 is

$$\sqrt[3]{1-6x} = 1 - 2x - 4x^2 - \frac{40x^3}{3} - \cdots .$$

We compute that

$$\begin{aligned}&\frac{(1+x)^{-2}-\sqrt[3]{1-6x}}{2x^2}\\ &= \frac{1}{2x^2}\left(\left(1-2x+3x^2-4x^3+\cdots\right)-\left(1-2x-4x^2-\frac{40x^3}{3}-\cdots\right)\right)\\ &= \frac{1}{2x^2}\left(7x^2+\frac{28x^3}{3}+\cdots\right)=\frac{7}{2}+\frac{14x}{3}+\cdots\end{aligned}$$

so the limit of $\dfrac{(1+x)^{-2}-\sqrt[3]{1-6x}}{2x^2}$ as $x\to 0$ is $\dfrac{7}{2}$.

53 We have $e^{-x^2}=1-x^2+\frac{x^4}{2}-\frac{x^6}{6}+\frac{x^8}{24}-\cdots$, so $\int e^{-x^2}\,dx=\int(1-x^2+\frac{x^4}{2}-\frac{x^6}{6}+\frac{x^8}{24}-\cdots)\,dx=C+x-\frac{x^3}{3}+\frac{x^5}{10}-\frac{x^7}{42}+\cdots$. Thus, $\int_0^{1/2}e^{-x^2}\,dx=(0.5)-\frac{(0.5)^3}{3}+\frac{(0.5)^5}{10}-\frac{(0.5)^7}{42}+\cdots$. Because $(0.5)^7/42<.001$, we can calculate the approximation using the first three numbers shown, arriving at approximately 0.461.

54 $\tan^{-1}x=x-x^3/3+x^5/5-x^7/7+x^9/9-\cdots$, so $\int\tan^{-1}(x)\,dx=\int(x-x^3/3+x^5/5-x^7/7+x^9/9-\cdots)\,dx=C+\frac{x^2}{2}-\frac{x^4}{12}+\frac{x^6}{30}-\frac{x^8}{56}+\cdots$. Thus, $\int_0^{0.5}\tan^{-1}x\,dx=\frac{(0.5)^2}{2}-\frac{(0.5)^4}{12}+\frac{(0.5)^6}{30}-\frac{(0.5)^8}{56}+\cdots$. Note that this series is alternating, and $\frac{(0.5)^6}{30}<.001$, so we add the first two terms showing to approximate the integral to the desired accuracy. Calculating gives approximately 0.120.

55 $x\cos x=x-\frac{x^3}{2}+\frac{x^5}{24}-\frac{x^7}{720}+\cdots$, so $\int x\cos x\,dx=\int(x-\frac{x^3}{2}+\frac{x^5}{24}-\frac{x^7}{720}+\cdots)\,dx=C+\frac{x^2}{2}-\frac{x^4}{8}+\frac{x^6}{144}-\frac{x^8}{5760}+\frac{x^{10}}{403200}-\cdots$. Thus $\int_0^1 x\cos x\,dx=\frac{1}{2}-\frac{1}{8}+\frac{1}{144}-\frac{1}{5760}+\cdots$. Because $\frac{1}{5760}<.001$, we add the first three terms to approximate to the desired accuracy. Calculating gives $\int_0^1 x\cos x\,dx\approx 0.382$.

56 $x^2\tan^{-1}x=x^3-x^5/3+x^7/5-x^9/7+x^{11}/9+\cdots$, so $\int x^2\tan^{-1}(x)\,dx=\int(x^3-x^5/3+x^7/5-x^9/7+x^{11}/9-\cdots)\,dx=C+\frac{x^4}{4}-\frac{x^6}{18}+\frac{x^8}{40}-\frac{x^{10}}{70}+\cdots$. Thus, $\int_0^{0.5}x^2\tan^{-1}x\,dx=\frac{(0.5)^4}{4}-\frac{(0.5)^6}{18}+\frac{(0.5)^8}{40}-\frac{(0.5)^{10}}{70}+\cdots$. Note that this series is alternating, and $\frac{(0.5)^6}{18}<.001$, so we use the first term showing to approximate the integral to the desired accuracy. Calculating gives approximately 0.015.

57 The series for $f(x)=\sqrt{x}$ centered at $a=121$ is $11+\frac{x-121}{22}-\frac{(x-121)^2}{10648}+\frac{(x-121)^3}{2576816}+\cdots$. Letting $x=119$ gives $\sqrt{119}\approx 11-\frac{1}{11}-\frac{1}{2\cdot 11^3}-\frac{1}{2\cdot 11^5}$.

58 Because 20 degrees corresponds to $\frac{\pi}{9}$ radians, we consider the series for $\sin x$ centered at 0. We have $\sin x\approx x-x^3/3!+x^5/5!-x^7/7!+\cdots$, so $\sin\pi/9\approx\frac{\pi}{9}-\frac{(\pi/9)^3}{3!}+\frac{(\pi/9)^5}{5!}-\frac{(\pi/9)^7}{7!}$.

59 $\tan^{-1}x=x-x^3/3+x^5/5-x^7/7+x^9/9+\cdots$, so $\tan^{-1}(-1/3)\approx\frac{-1}{3}+\frac{1}{3\cdot 3^3}-\frac{1}{5\cdot 3^5}+\frac{1}{7\cdot 3^7}$.

60 $\sinh x=x+\frac{x^3}{6}+\frac{x^5}{120}+\frac{x^7}{5040}+\cdots$, so $\sinh(-1)\approx(-1)+\frac{(-1)^3}{6}+\frac{(-1)^5}{120}+\frac{(-1)^7}{5040}$.

61 Because $y(0)=4$, we have $y'(0)-16+12=0$, so $y'(0)=4$. Differentiating the equation $n-1$ times and evaluating at 0 we obtain $y^{(n)}(0)=4y^{(n-1)}(0)$, so that $y^{(n)}(0)=4^n$. The Taylor series for $y(x)$ is thus $y(x)=4+4x+\frac{4^2x^2}{2!}+\frac{4^3x^3}{3!}+\cdots$, or $y(x)=3+e^{4x}$.

62 We begin with $e^{-102x^2}=1-102x^2+\frac{102^2x^4}{2!}+\cdots$. For $n=2$, we have $11.4\int_0^{0.14}(1-102x^2)\,dx=11.4(x-34x^3)|_0^{0.14}=0.5324256$. For $n=3$, $11.4\int_0^{0.14}(1-102x^2+5202x^4)\,dx=11.4(x-34x^3+1040.4x^5)|_0^{0.14}\approx 1.170314983$. Clearly the second estimate is too high, because the true probability cannot exceed 1. The true value is approximately 0.9547855902.

63

a. The Taylor series for $\ln(1+x)$ is $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. Evaluating at $x=1$ gives $\ln 2=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$.

b. The Taylor series for $\ln(1-x)$ is $-\sum_{k=1}^{\infty} \frac{x^k}{k}$. Evaluating at $x = 1/2$ gives $\ln(1/2) = -\sum_{k=1}^{\infty} \frac{1}{k2^k}$, so that $\ln 2 = \sum_{k=1}^{\infty} \frac{1}{k2^k}$.

c. $f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$. Using the two Taylor series above we have $f(x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} - \left(-\sum_{k=1}^{\infty} \frac{x^k}{k}\right) = \sum_{k=1}^{\infty}(1+(-1)^{k+1})\frac{x^k}{k} = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$.

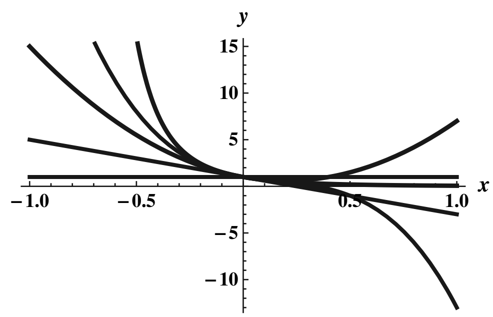
d. Because $\frac{1+x}{1-x} = 2$ when $x = \frac{1}{3}$, the resulting infinite series for $\ln 2$ is $2\sum_{k=0}^{\infty} \frac{1}{3^{2k+1}(2k+1)}$.

e. The first four terms of each series are: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx 0.5833333333$, $\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} \approx 0.6822916667$, $\frac{2}{3} + \frac{2}{81} + \frac{2}{1215} + \frac{2}{15309} \approx 0.6931347573$ The true value is $\ln 2 \approx 0.6931471806$. The third series converges the fastest, because it has 3^{k+1} in the denominator as opposed to 2^k, so its terms get small faster.

64

a. $p_3(x) = 1 - 4x + 10x^2 - 20x^3$.

b.



c. The constant polynomial looks like $f(x)$ only at 0. The linear polynomial looks like $f(x)$ on about $(-.1, .1)$. The quadratic approximation looks like $f(x)$ on about $(-.1, .1)$ as well, and the cubic approximation looks like $f(x)$ on about $(-.2, .2)$.

Chapter 10

Parametric and Polar Curves

10.1 Parametric Equations

10.1.1 Given an input value of t, the point $(x(t), y(t))$ can be plotted in the xy-plane, generating a curve.

10.1.2 $x = 6\cos t$ and $y = 6\sin t$ for $0 \le t \le 2\pi$ generates the circle, because $x^2 + y^2 = 36\cos^2 t + 36\sin^2 t = 36$. Similarly, $x = 6\sin t$ and $y = 6\cos t$ for $0 \le t \le 2\pi$ generates the same curve.

10.1.3 Let $x = R\cos(\pi t/5)$ and $y = -R\sin(\pi t/5)$. Note that as t ranges from 0 to 10, $\pi t/5$ ranges from 0 to 2π. Because $x^2 + y^2 = R^2$, this curve represents a circle of radius R. Note also that for $t = 0$ the initial point is $(R, 0)$, and for small values of t the plotted points are in the third quadrant — so the curve is being traced with clockwise orientation.

10.1.4 Let $x = t$ and $y = -2t + 5$ for $t \in (-\infty, \infty)$.

10.1.5 Let $x = t$ and $y = t^2$ for $t \in (-\infty, \infty)$.

10.1.6 The former represents the part of the parabola $y = x^2$ lying in the first quadrant. The latter represents the part of that same parabola lying in the second quadrant.

10.1.7 Solving the first equation for t gives $t = \frac{1-x}{2}$. Substitute that value for t in the second equation to get $y = 3\left(\frac{1-x}{2}\right)^2$; simplifying gives $y = \frac{3}{4}x^2 - \frac{3}{2}x + \frac{3}{4}$.

10.1.8 With $t = 0$ the corresponding point on the curve is $(-2\sin 0, 2\cos 0) = (0, 2)$. As t increases from 0, the x-coordinate becomes negative, while the y coordinate decreases. Thus the curve is generated counterclockwise, running successively through $(0, 2)$, then $(-2, 0)$ for $t = \frac{\pi}{2}$, then $(0, -2)$ for $t = \pi$, then $(2, 0)$ for $t = \frac{3\pi}{2}$, and finally back to $(0, 2)$ for $t = 2\pi$. Then it repeats.

10.1.9 The slope of the tangent line is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, so at $t = a$ the slope is given by $\frac{g'(a)}{f'(a)}$, $f'(a) \ne 0$.

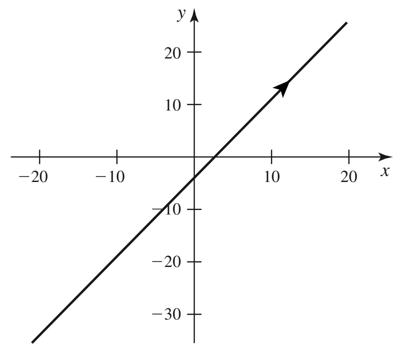
10.1.10 There is a horizontal tangent line at $t = a$ where $g'(a) = 0$, provided $f'(a) \ne 0$, so these points can be found by solving $g'(t) = 0$ and checking that any solution $t = a$ satisfies $f'(a) \ne 0$.

10.1.11

a.

t	x	y
-10	-20	-34
-5	-10	-19
0	0	-4
5	10	11
10	20	26

b.



c. Solving $x = 2t$ for t yields $t = x/2$, so $y = 3t - 4 = 3x/2 - 4$.

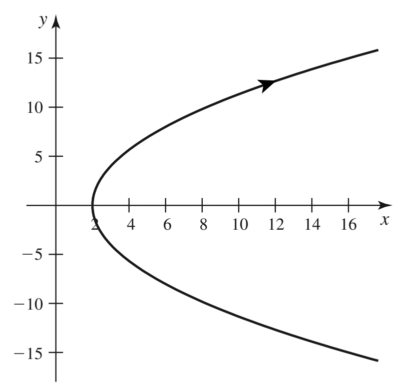
d. The curve is the line segment from $(-20, -34)$ to $(20, 26)$.

10.1.12

a.

t	x	y
-4	18	-16
-2	6	-8
0	2	0
2	6	8
4	18	16

b.



c. Solving $y = 4t$ for t yields $t = y/4$, so $x = t^2 + 2 = y^2/16 + 2$.

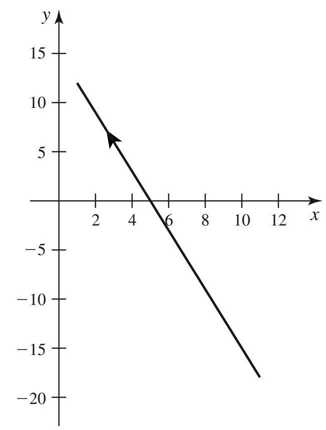
d. The curve is part of the parabola $x = y^2/16 + 2$ from $(18, -16)$ to $(18, 16)$.

10.1.13

a.

t	x	y
-5	11	-18
-3	9	-12
0	6	-3
3	3	6
5	1	12

b.



c. Solving $x = -t + 6$ for t yields $t = 6 - x$, so $y = 3t - 3 = 18 - 3x - 3 = 15 - 3x$.

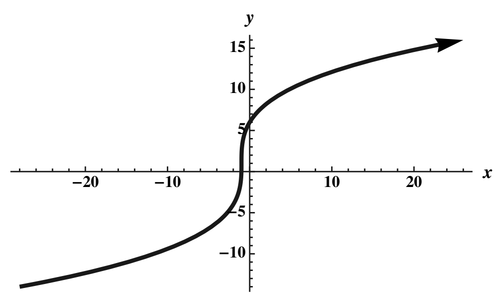
d. The curve is the line segment from $(11, -18)$ to $(1, 12)$.

10.1.14

a.

t	x	y
-3	-28	-14
-2	-9	-9
-1	-2	-4
0	-1	1
1	0	6
2	7	11
3	26	16

b.



c. Because $t = \sqrt[3]{x+1}$, we have $y = 5\sqrt[3]{x+1} + 1$.

d. The curve is a shifted and scaled version of the cube root function.

10.1.15

a. Solving $x = \sqrt{t} + 4$ for t yields $t = (x-4)^2$. Thus, $y = 3\sqrt{t} = 3(x-4)$, where x ranges from 4 to 8. Note that all $t \geq 0$, $x > 0$, and $y > 0$.

b. The curve is the line segment from $(4, 0)$ to $(8, 12)$.

10.1.16

a. Solving $y = t+2$ for t yields $t = y-2$. Thus, $x = (t+1)^2 = (y-2+1)^2 = (y-1)^2$, where $-8 \leq y \leq 12$.

b. The curve is the part of the parabola $x = (y-1)^2$ from $(81, -8)$ to $(121, 12)$.

10.1.17

a. Because $\cos^2 t + \sin^2 t = 1$, we have $x^2 + y = 1$, so $y = 1 - x^2$, $-1 \leq x \leq 1$.

b. This is a parabola opening downward with a vertex at $(0, 1)$, and starting at $(1, 0)$ and ending at $(-1, 0)$.

10.1.18

a. Note that $(1 - \sin^2 s) - \cos^2 s = 0$, so $x - y^2 = 0$, so $x = y^2$, $-1 \le y \le 1$.

b. This is a parabola opening to the right with a vertex at $(0, 0)$, starting at $(1, -1)$ and ending at $(1, 1)$.

10.1.19

a. Solving $x = r - 1$ for r yields $r = x + 1$. Thus, $y = r^3 = (x + 1)^3$, where $-5 \le x \le 3$.

b. The curve is the part of the standard cubic curve, shifted one unit to the left, from $(-5, -64)$ to $(3, 64)$.

10.1.20

a. Solving $x = e^{2t}$ for t yields $t = \ln(\sqrt{x})$. Thus, $y = e^t + 1 = \sqrt{x} + 1$, where $1 \le x \le e^{50}$.

b. The curve is the part of the standard square root function, shifted one unit vertically, from the point $(1, 2)$ to $(e^{50}, e^{25} + 1)$.

10.1.21 Note that $x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9$, so this represents an arc of the circle of radius 3 centered at the origin from $(-3, 0)$ to $(3, 0)$ traversed counterclockwise.

10.1.22 Note that $x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9$, so this represents an arc of the circle of radius 3 centered at the origin from $(3, 0)$ to $(0, 3)$ traversed counterclockwise.

10.1.23 Note that $x^2 + (y - 1)^2 = \cos^2 t + \sin^2 t = 1$, so we have a circle of radius 1 centered at $(0, 1)$, traversed counterclockwise starting at $(1, 1)$.

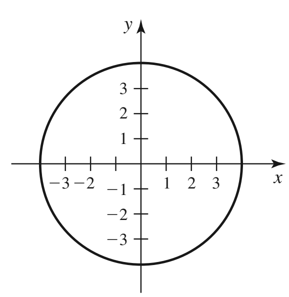
10.1.24 Note that $(x + 3)^2 + (y - 5)^2 = 4$. This is a circle of radius 2 centered at $(-3, 5)$ and traversed clockwise starting at $(-3, 7)$.

10.1.25 Note that $x^2 + y^2 = 49\cos^2 2t + 49\sin^2 2t = 49$, so this represents an arc of the circle of radius 7 centered at the origin from $(-7, 0)$ to $(-7, 0)$ traversed counterclockwise. (So the whole circle is represented.)

10.1.26 Note that $(x - 1)^2 + (y - 2)^2 = 9\sin^2 4\pi t + 9\cos^2 4\pi t = 9$, so this represents the circle of radius 3 centered at $(1, 2)$ from $(1, 5)$ to $(1, 5)$ traversed counterclockwise.

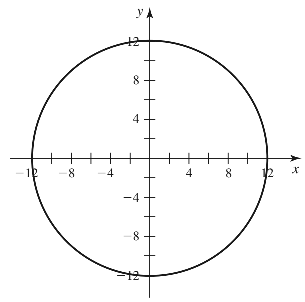
10.1.27

Let $x = 4\cos t$ and $y = 4\sin t$ for $0 \le t \le 2\pi$.
Then $x^2 + y^2 = 16\cos^2 t + 16\sin^2 t = 16$.



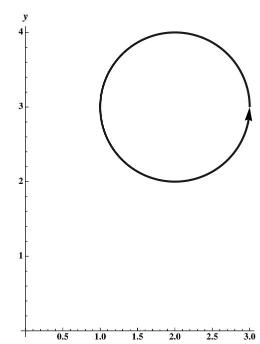
10.1.28

Let $x = 12\sin t$ and $y = 12\cos t$ for $0 \leq t \leq 2\pi$. Then $x^2 + y^2 = 144\cos^2 t + 144\sin^2 t = 144$, and for $t = 0$ the value of (x, y) is $(0, 12)$.



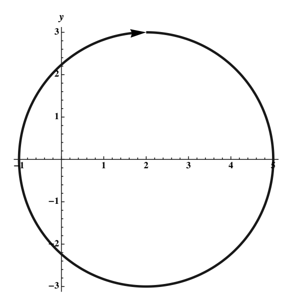
10.1.29

Let $x = \cos t + 2$ and $y = \sin t + 3$ for $0 \leq t \leq 2\pi$. Then $(x-2)^2 + (y-3)^2 = 1$, which is a circle with the desired center and radius and orientation.



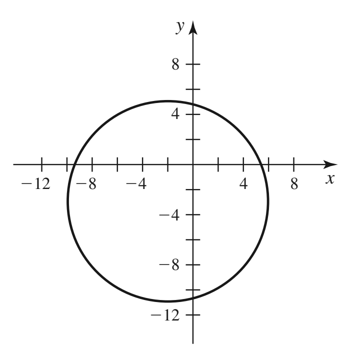
10.1.30

Let $x = 3\sin t + 2$ and $y = 3\cos t$ for $0 \leq t \leq 2\pi$. Then $(x-2)^2 + y^2 = 9$, which is a circle with the desired center and radius and orientation.



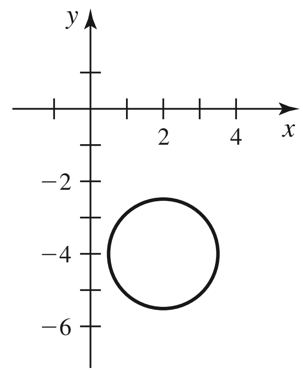
10.1.31

Let $x = -2 + 8\sin t$ and $y = -3 + 8\cos t$ for $0 \leq t \leq 2\pi$. Then $(x+2)^2 + (y+3)^2 = 64\sin^2 t + 64\cos^2 t = 64$.



10.1.32

Let $x = 2-(3/2)\cos t$ and $y = -4+(3/2)\sin t$ for $\pi \le t \le 3\pi$. Then $(x-2)^2+(y+4)^2 = \frac{9}{4}$. Note that for $t = \pi$, we have $x = 7/2$ and $y = -4$.



10.1.33 Let t be time in minutes, so $0 \le t \le 1.5$ Let $x = 400\cos(4\pi/3)t$ and $y = 400\sin(4\pi/3)t$. Then because $x^2+y^2 = 400^2$, the path is a circle of radius 400. Note that the values of x and y are the same at $t = 0$ and $t = 1.5$, and that the circle is traversed counterclockwise.

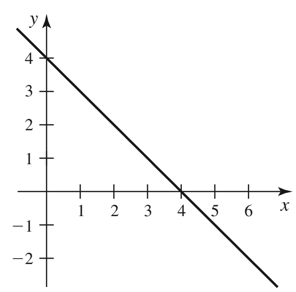
10.1.34 Let t be time in seconds, so $0 \le t \le 60$ Let $x = 15\sin(\pi/30)t$ and $y = 15\cos(\pi/30)t$. Then because $x^2+y^2 = 15^2$, the path is a circle of radius 15. Note that the values of x and y are the same at $t = 0$ and $t = 60$, and that the circle is traversed clockwise.

10.1.35 Let t be time in seconds, so $0 \le t \le 24$ Let $x = 50\cos(\pi/12)t$ and $y = 50\sin(\pi/12)t$. Then because $x^2+y^2 = 50^2$, the path is a circle of radius 50. Note that the values of x and y are the same at $t = 0$ and $t = 24$, and that the circle is traversed counterclockwise.

10.1.36 Let t be time in minutes, so $0 \le t \le 3$. Because the low point is the origin, the circle we seek has its center at $(0, 20)$ and a radius of 20. Let $x = -20\sin(2\pi/3)t$ and $y = 20 - 20\cos(2\pi/3)t$. Then because $x^2+(y-20)^2 = 20^2$, the path is a circle of radius 20. Note that the values of x and y are the same for $t = 0$ and $t = 3$.

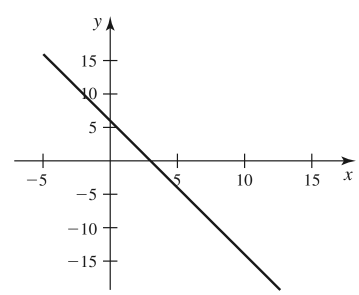
10.1.37

Because $t = x-3$, we have $y = 1-(x-3) = 4-x$, so the line has slope -1. When $t = 0$, we have the point $(3, 1)$.



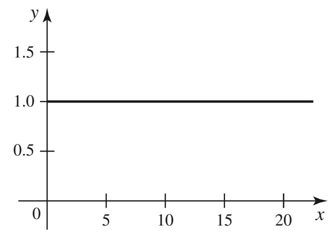
10.1.38

Because $t = \frac{4-x}{3}$, we have $y = -2+6(\frac{4-x}{3}) = 6-2x$, so the line has slope -2. When $t = 0$, we have the point $(4, -2)$.



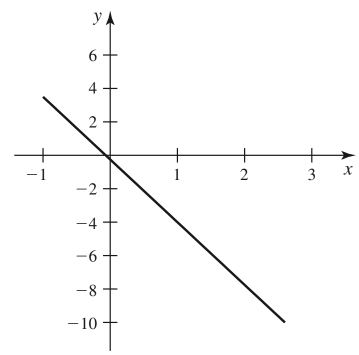
10.1.39

Because $y = 1$, this is a horizontal line with slope 0. When $t = 0$, we have the point $(8, 1)$.



10.1.40

Because $t = \frac{3}{2}(x - 1)$, we have $y = -\frac{1}{4} - \frac{15}{4}x$, so the line has slope $-\frac{15}{4}$. When $t = 0$, we have the point $(1, -4)$.



10.1.41 Let $x = x_0 + at$ and $y = y_0 + bt$. Letting $(x_0, y_0) = (0, 0)$, and then finding a and b so that the curve is at the point Q when $t = 1$ yields $x = 2t$, $y = 8t$ for $0 \le t \le 1$.

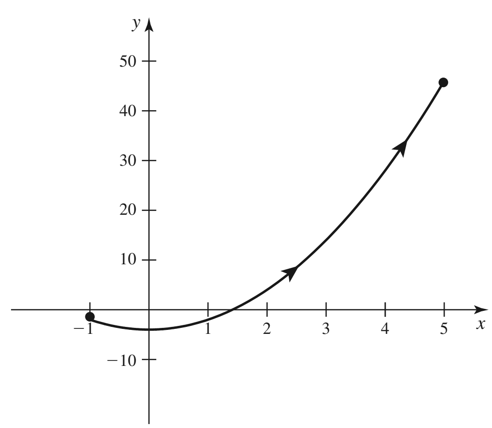
10.1.42 Let $x = x_0 + at$ and $y = y_0 + bt$. Letting $(x_0, y_0) = (1, 3)$, and then finding a and b so that the curve is at the point Q when $t = 1$ yields $x = 1 - 3t$, $y = 3 + 3t$ for $0 \le t \le 1$.

10.1.43 Let $x = x_0 + at$ and $y = y_0 + bt$. Letting $(x_0, y_0) = (-1, -3)$, and then finding a and b so that the curve is at the point Q when $t = 1$ yields $x = -1 + 7t$, $y = -3 - 13t$ for $0 \le t \le 1$.

10.1.44 Let $x = x_0 + at$ and $y = y_0 + bt$, and parametrize from $t = 0$ to $t = 1$. Because the point is at $(8, 2)$ when $t = 0$, we have $x_0 = 8$ and $y_0 = 2$. At $t = 1$, the point is at $(-2, -3)$, so that $-2 = 8 + a$ and $-3 = 2 + b$. Thus $a = -10$ and $b = -5$, and our equations are $x = 8 - 10t$ and $y = 2 - 5t$ for $0 \le t \le 1$.

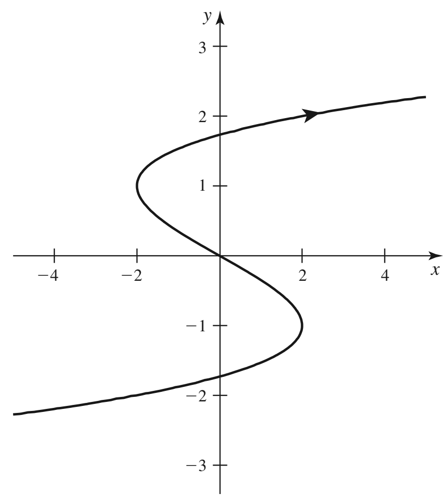
10.1.45

Let $x = t$ and $y = 2t^2 - 4$, $-1 \le t \le 5$.



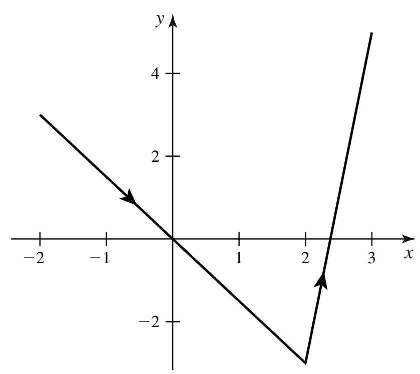
10.1.46

Let $x = t^3 - 3t$ and $y = t$, $-\infty < t < \infty$.



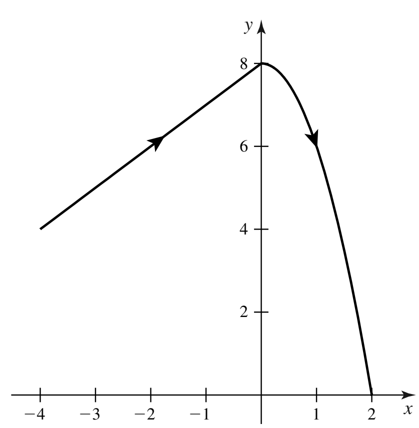
10.1.47

Let $x = -2 + 4t$ and $y = 3 - 6t$, $0 \le t \le 1$, and $x = t + 1$, $y = 8t - 11$ for $1 \le t \le 2$.

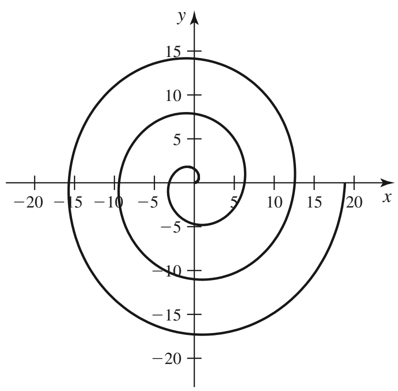


10.1.48

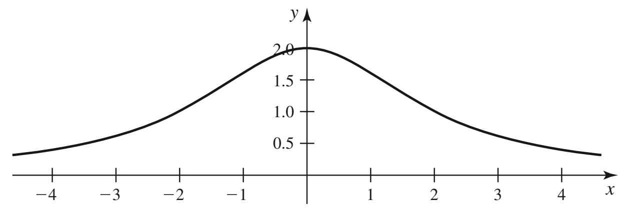
Let $x = -4 + 4t$ and $y = 4 + 4t$, $0 \le t \le 1$, and $x = t - 1$, $y = 8 - 2(t - 1)^2$ for $1 \le t \le 3$.



10.1.49



10.1.50



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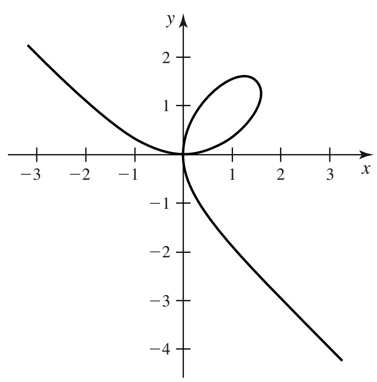
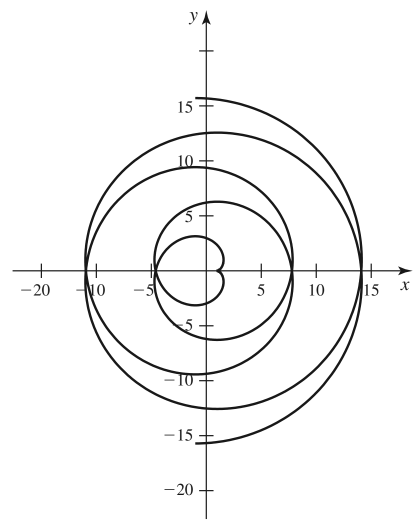


10.1.52



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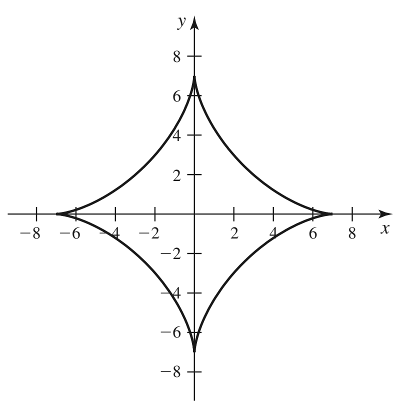
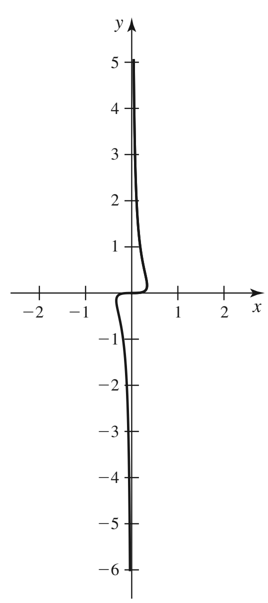


10.1.54



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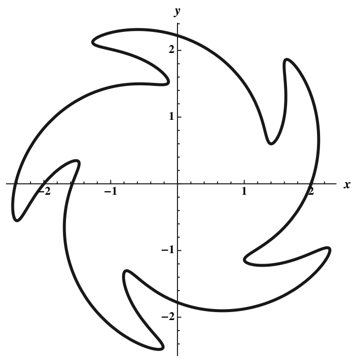
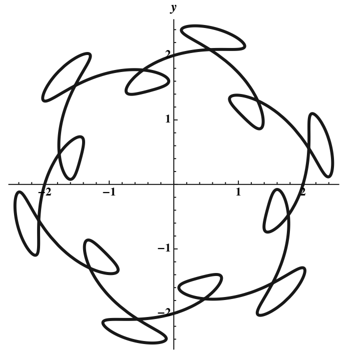
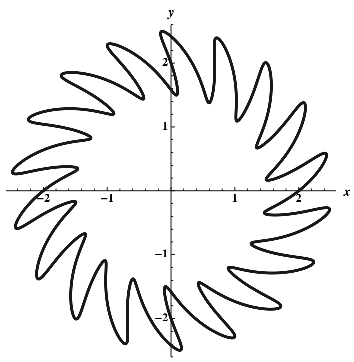


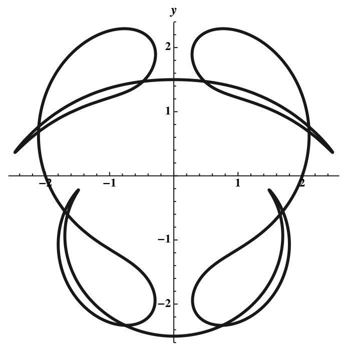
10.1.56



10.1.57

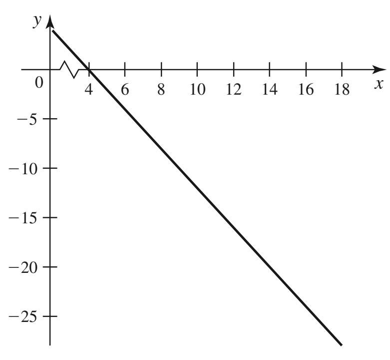


10.1.58



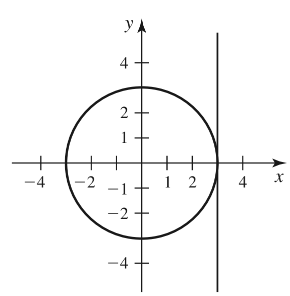
10.1.59

a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{8}{4} = -2$ for all t. Because the curve is a line, the tangent line to the curve at the given point is the line itself.

b. 

10.1.60

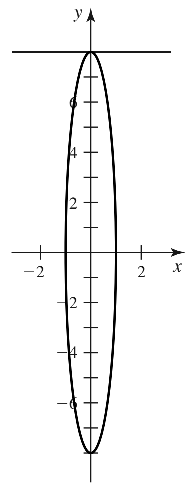
a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{3\sin t}{3\cos t} = -\tan t$. At the given value of t, the value of $\frac{dy}{dx}$ doesn't exist, and the tangent line is the vertical line $x = 3$.

b. 

10.1.61

a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{8\cos t}{-\sin t} = -8\cot t$. At the given value of t, the value of $\frac{dy}{dx}$ is $-8\cot\pi/2 = 0$. The tangent line at the point $(0, 8)$ is thus the horizontal line $y = 8$.

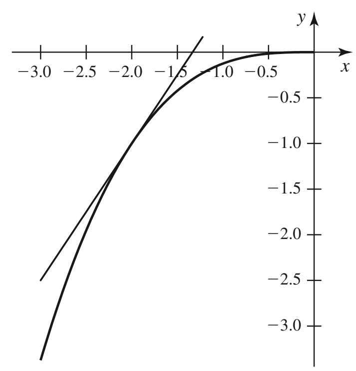
b.



10.1.62

a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2}$. At the given value of t, the value of $\frac{dy}{dx}$ is $\frac{3}{2}$, and the tangent line is $y = \frac{3}{2}x + 2$, tangent at the point $(-2, -1)$.

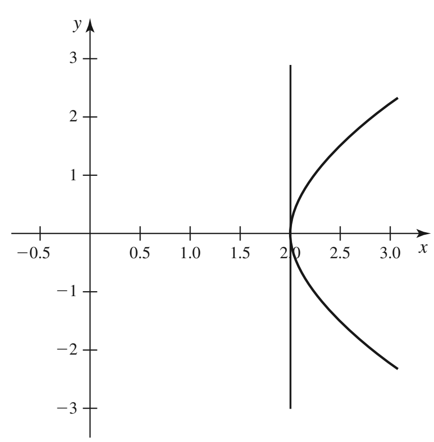
b.



10.1.63

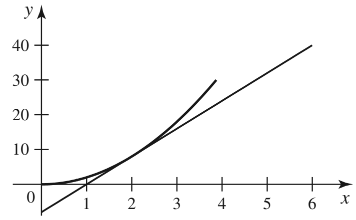
a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}} = \frac{t^2+1}{t^2-1}$. At the given value of t, the derivative doesn't exist, and the tangent line is the vertical line $x = 2$, tangent at the point $(2, 0)$.

b.



10.1.64

a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{1/(2\sqrt{t})} = 4\sqrt{t}$. At the given value of t, the value of $\frac{dy}{dx}$ is 8. The equation of the tangent line is $y = 8x - 8$, tangent at the point $(2, 8)$.

b. 

10.1.65

a. False. This generates a circle in the counterclockwise direction.

b. True. Note that when t is increased by one, the value of $2\pi t$ is increased by 2π, which is the period of both the sine and the cosine functions.

c. False. This generates only the portion of the parabola in the first quadrant, omitting the portion in the second quadrant.

d. True. They describe the portion of the unit circle in the 4th and 1st quadrants.

e. True. This ellipse has vertical tangents at $t = 0$ and $t = \pi$.

10.1.66 The point corresponding to $t = \pi/4$ is $(\sqrt{2}/2, \sqrt{2}/2)$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t}$. At $t = \pi/4$, we have a slope of -1. The equation of the tangent line is thus $y - \sqrt{2}/2 = -1(x - \sqrt{2}/2)$, or $y = -x + \sqrt{2}$.

10.1.67 The point corresponding to $t = 2$ is $(3, 10)$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2+1}{2t}$, so the slope at $t = 2$ is $\frac{13}{4}$. The equation of the tangent line is therefore $y - 10 = \frac{13}{4}(x - 3)$, or $y = \frac{13}{4}x + \frac{1}{4}$.

10.1.68 The point corresponding to $t = 0$ is $(1, 0)$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{(t+1)e^t}$, so the slope at $t = 0$ is 1. The equation of the tangent line is thus $y = x - 1$.

10.1.69 The point corresponding to $t = \pi/4$ is $\left(\frac{4\sqrt{2}+\pi\sqrt{2}}{8}, \frac{4\sqrt{2}-\pi\sqrt{2}}{8}\right)$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t-(\cos t-t\sin t)}{-\sin t+(\sin t+t\cos t)} = \tan t$. At $t = \pi/4$, we have a slope of 1. The equation of the tangent line is thus $y - \frac{4\sqrt{2}-\pi\sqrt{2}}{8} = 1\left(x - \frac{4\sqrt{2}+\pi\sqrt{2}}{8}\right)$, or $y = x - \frac{\pi\sqrt{2}}{4}$.

10.1.70 Let $x = -t$ and $y = t^2 + 1$, for $0 \le t < \infty$.

10.1.71 Let $x = 1 + 2t$ and $y = 1 + 4t$, for $-\infty < t < \infty$. Note that $y = 2(1 + 2t) - 1$, so $y = 2x - 1$.

10.1.72 Let $x = -2 - 6\cos t$ and $y = 2 - 6\sin t$, for $0 \le t \le \pi$. Then $(x + 2)^2 + (y - 2)^2 = 36$, so the curve represented is part of the circle of radius 6 centered at $(-2, 2)$. Note also that as t runs from 0 to π, the portion of the circle traversed is the lower portion, from $(-8, 2)$ to $(4, 2)$.

10.1.73 Let $x = t^2$ and $y = t$, for $0 \le t < \infty$. Note that $x = t^2 = y^2$, and that the starting point is $(0, 0)$.

10.1.74

a. This corresponds to graph (D). Note that $t = 0$ corresponds to the point $(-2, 0)$ and as $t \to \infty$, both $x \to \infty$ and $y \to \infty$.

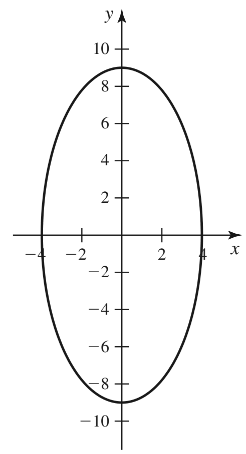
b. This corresponds to graph (B). Note that $-1 \le x \le 1$ and $-1 \le y \le 1$ for all values of t.

c. This corresponds to graph (A). Note that as $t \to -\infty$, we have $x \to -\infty$ and $y \to -\infty$.

d. This corresponds to graph (C). Note that $-3 \le x \le 3$ and $-3 \le y \le 3$ for all values of t.

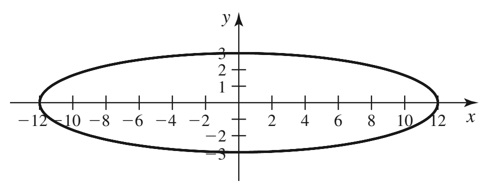
10.1.75

The entire curve is traversed for $0 \leq t \leq 2\pi$.



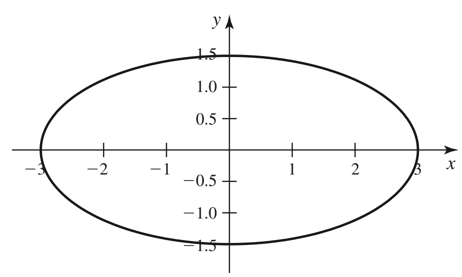
10.1.76

The entire curve is traversed for $0 \leq t \leq \pi$.



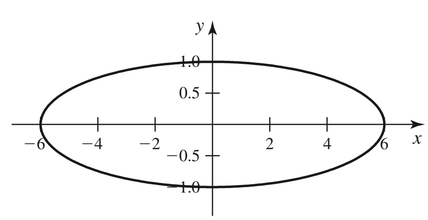
10.1.77

Let $x = 3\cos t$ and $y = \frac{3}{2}\sin t$ for $0 \leq t \leq 2\pi$. Then the major axis on the x-axis has length $2 \cdot 3 = 6$ and the minor axis on the y-axis has length $2 \cdot \frac{3}{2} = 3$. Note that $\left(\frac{x}{3}\right)^2 + \left(\frac{2y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$.



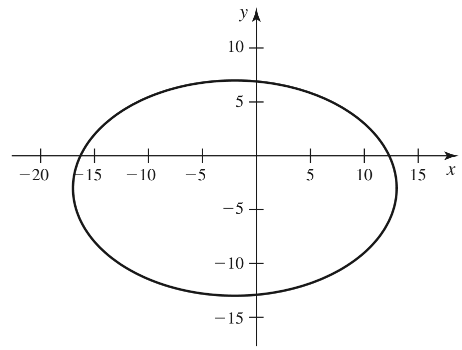
10.1.78

Let $x = 6\cos t$ and $y = -\sin t$ for $0 \leq t \leq 2\pi$. Then the major axis on the x-axis has length $2 \cdot 6 = 12$ and the minor axis on the y-axis has length $2 \cdot 1 = 2$. Note that $\left(\frac{x}{6}\right)^2 + (y)^2 = \cos^2 t + \sin^2 t = 1$.



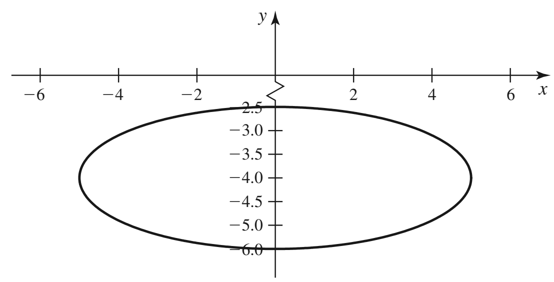
10.1.79

Let $x = 15\cos t - 2$ and $y = 10\sin t - 3$ for $0 \le t \le 2\pi$. Note that $\left(\frac{x+2}{15}\right)^2 + \left(\frac{y+3}{10}\right)^2 = \cos^2 t + \sin^2 t = 1$.
Then the major axis has length 30 and the minor axis has length 20.



10.1.80

Let $x = 5\cos t$ and $y = -\frac{3}{2}\sin t - 4$ for $0 \le t \le 2\pi$. Note that $\left(\frac{x}{5}\right)^2 + \left(\frac{y+4}{3/2}\right)^2 = \cos^2 t + \sin^2 t = 1$.
Then the major axis has length 10 and the minor axis has length 3.



10.1.81

a. For $(1+s, 2s) = (1+2t, 3t)$, we must have$1+s = 1+2t$ and $2s = 3t$, so that $s = 2t$ and $2s = 3t$. The only solution to this pair of equations is $s = t = 0$, so these two lines intersect when $s = t = 0$, at the point $(1, 0)$.

b. For $(2+5s, 1+s) = (4+10t, 3+2t)$, we must have $2+5s = 4+10t$ and $1+s = 3+2t$, so that $s = 2+2t$ and $s = 2+2t$. This pair of equations has no solutions, so the lines are parallel.

c. For $(1+3s, 4+2s) = (4?3t, 6+4t)$, we must have $1+3s = 4?3t$ and $4+2s = 6+4t$, so that $s = 1?t$ and $s = 1+2t$. The only solution to this pair of equations is $s = 1$ and $t = 0$, so these two lines intersect for these values of s and t, at the point $(4, 6)$.

10.1.82 All three represent portions of the parabola $x = 2\cdot(y-4)^2$ where x is between 0 and 32. However, the curve in part **b** only represents the portion of the parabola where $y \ge 4$, because for that curve, $y = 4+t^2 \ge 4$.

10.1.83 Note that $x^2 + y^2 = 4\sin^2 8t + 4\cos^2 8t = 4$, so the curve is the circle $x^2 + y^2 = 4$.

10.1.84 Note that $4x^2 + y^2 = 4\sin^2 8t + 4\cos^2 8t = 4$, so the curve is the ellipse $4x^2 + y^2 = 4$.

10.1.85 Note that because $t = x$, we have $y = \sqrt{4-t^2} = \sqrt{4-x^2}$.

10.1.86 Note that $x^2 = t+1$, so $y = \frac{1}{t+1} = \frac{1}{x^2}$.

10.1.87 Because $\sec^2 t - 1 = \tan^2 t$, we have $y = x^2$.

10.1.88 Note that $(\sqrt[n]{x/a})^2 + (\sqrt[n]{y/b})^2 = \sin^2 t + \cos^2 t$, so $(\sqrt[n]{x/a})^2 + (\sqrt[n]{y/b})^2 = 1$.

10.1.89 $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\cos t}{-4\sin t} = -\cot t$. We seek t so that $\cot t = -1/2$, so $t = \cot^{-1}(-1/2)$. The corresponding points on the curve are $(-\frac{4\sqrt{5}}{5}, \frac{8\sqrt{5}}{5})$ and $(\frac{4\sqrt{5}}{5}, -\frac{8\sqrt{5}}{5})$.

10.1.90 $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{8\cos t}{-2\sin t} = -4\cot t$. We seek t so that $\cot t = 1/4$, so $t = \cot^{-1}(1/4)$. The corresponding points on the curve are $(\frac{2\sqrt{17}}{17}, \frac{32\sqrt{17}}{17})$ and $(-\frac{2\sqrt{17}}{17}, -\frac{32\sqrt{17}}{17})$.

10.1.91 $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+(1/t^2)}{1-(1/t^2)} = \frac{t^2+1}{t^2-1}$. We seek t so that $\frac{t^2+1}{t^2-1} = 1$, which never occurs. Thus, there are no points on this curve with slope 1.

10.1.92 $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{4}{(1/2\sqrt{t})} = -8\sqrt{t}$ for $t \neq 0$. Note that this isn't 0 for t on the interval $(0, \infty)$, but it is the case that $\lim_{t\to 0^+} \frac{dy}{dx} = 0$, so there is a flat tangent line at the point $(2, 2)$, as long as the point is approached from the right.

10.1.93 Note that in equation B, the parameter is scaled by a factor of 3. Thus, the curves are the same when the corresponding interval for t is scaled by a factor of $1/3$, so for $a = 0$ and $b = \frac{2\pi}{3}$. In fact, the same curve will be generated for $a = p$, $b = p + 2\pi/3$ where p is any real number.

10.1.94 Note that equation B can be obtained from A by replacing t by $t^{1/3}$. Thus, the curves are the same when $a = (-2)^3 = -8$ and $b = 2^3 = 8$.

10.1.95

a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{2\cos 2t}$. This is zero when $\cos t = 0$ but $\cos 2t \neq 0$, which occurs for $t = \pi/2$ and $t = 3\pi/2$. The corresponding points on the graph are $(0, 2)$ and $(0, -2)$.

b. Using the derivative obtained above, we seek points where $\cos 2t = 0$ but $\cos t \neq 0$. This occurs for $t = \pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$. The corresponding points on the curve are $(1, \sqrt{2})$, $(-1, \sqrt{2})$, $(-1, -\sqrt{2})$, and $(1, -\sqrt{2})$.

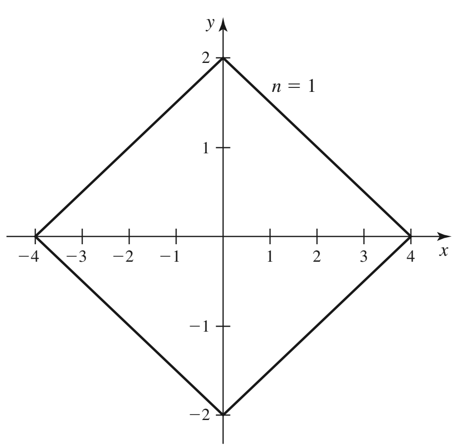
10.1.96

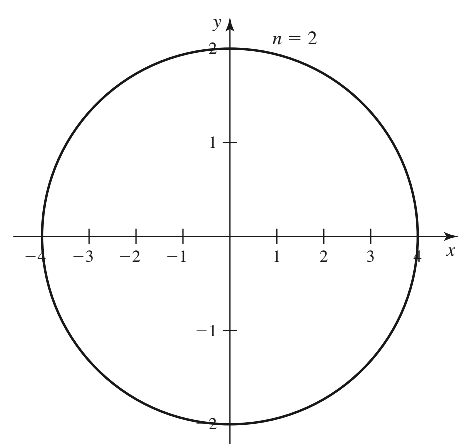
a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos 3t}{4\cos 4t}$. This is zero when $\cos 3t = 0$ but $\cos 4t \neq 0$, which occurs for $t = \pi/6$, $\pi/2$, $5\pi/6$, $7\pi/6$, $3\pi/2$, and $t = 11\pi/6$. The corresponding points on the graph are the four points $(\pm\frac{\sqrt{3}}{2}, \pm 1)$ and the two points $(0, \pm 1)$.

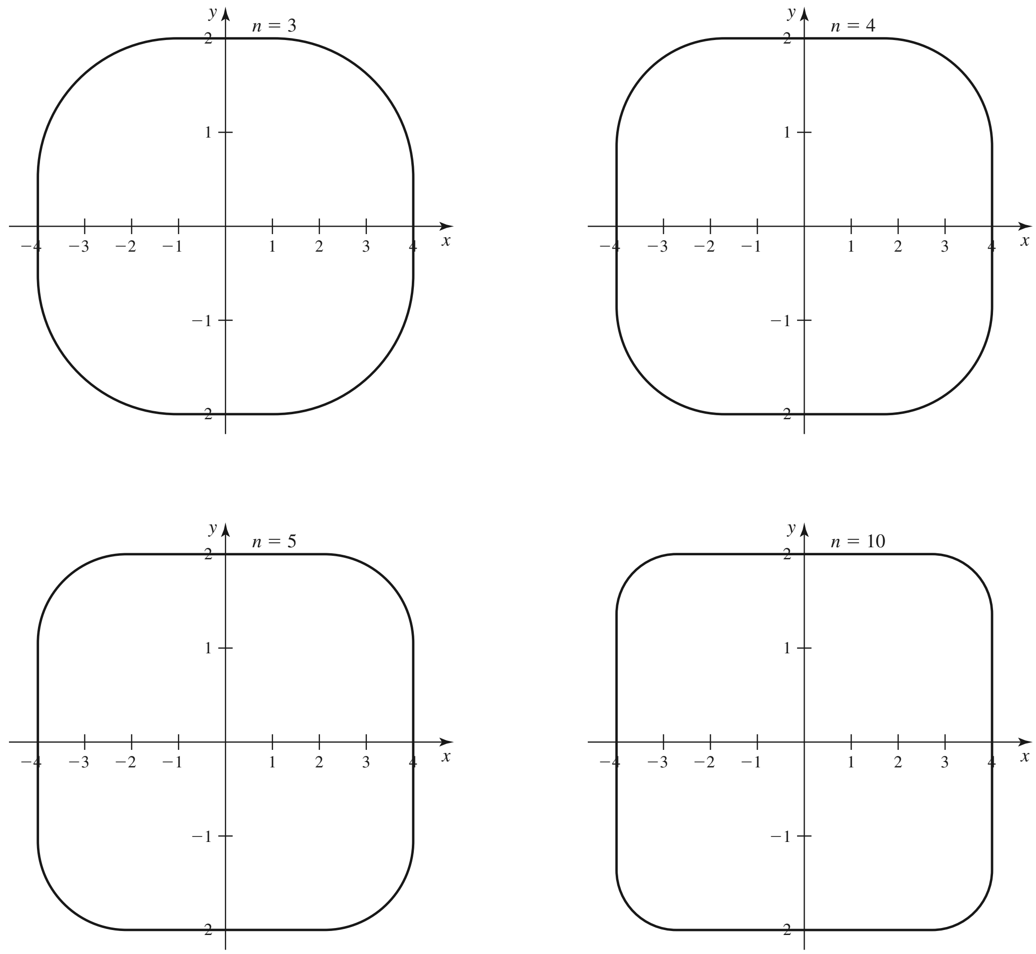
b. Using the derivative obtained above, we seek points where $\cos 4t = 0$ but $\cos 3t \neq 0$. This occurs for $t = \frac{(2n+1)\pi}{8}$, $n = 0, 1, \ldots, 7$. The corresponding points on the curve are the four points $(\pm 1, \pm\sin(\pi/8))$ and $(\pm 1, \pm\sin(3\pi/8))$.

10.1.97

a. Let $\text{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$ Let $x = a \cdot \text{sgn}(\cos t)\,|\cos(t)|^{2/n}$ and $y = b \cdot \text{sgn}(\sin(t))\,|\sin(t)|^{2/n}$.

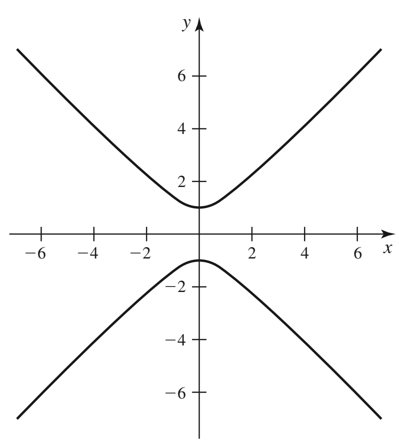
b. 



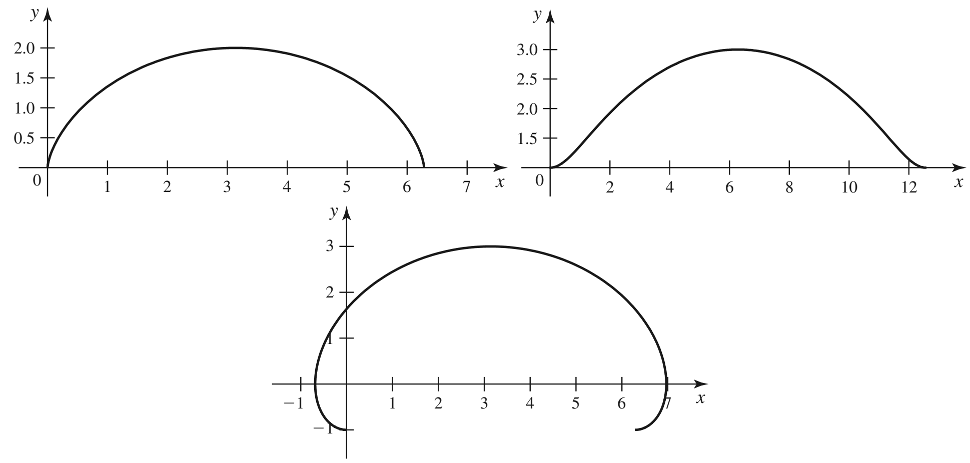


c. As n increases from near 0 to near 1, the curves change from star-shaped to a rectangular shape with corners at $(\pm a, 0)$ and $(0, \pm b)$. As n increases from 1 on, the curves become more rectangular with corners at $(\pm a, \pm b)$.

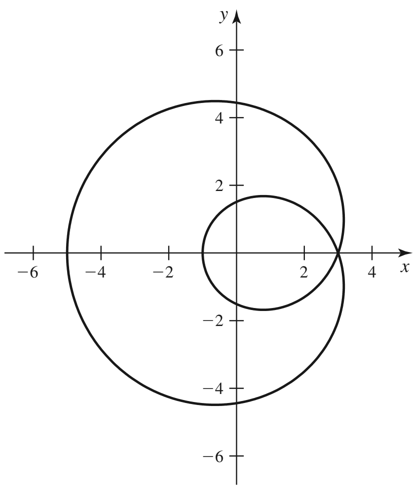
10.1.98



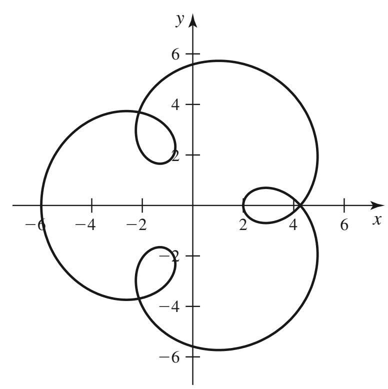
10.1.99 The first graphic shown is for $a = 1$ and $b = 1$. The second is for $a = 2$, $b = 1$, and the third is for $a = 1$, $b = 2$.



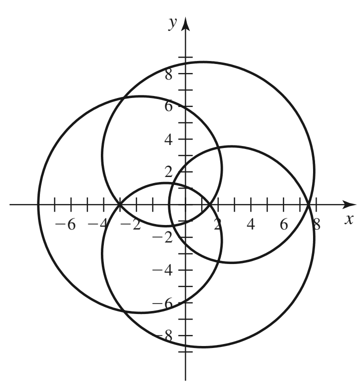
10.1.100



$a = 1,\ b = 1,\ c = 3.$

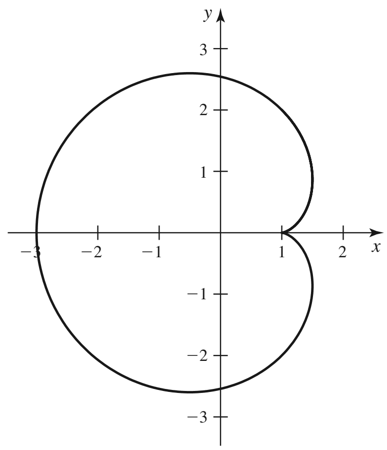


$a = 3,\ b = 1,\ c = 2.$

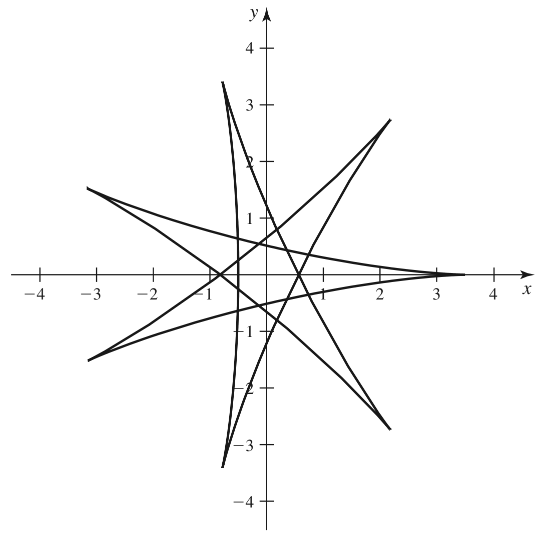


$a = 3,\ b = 1,\ c = 5.$

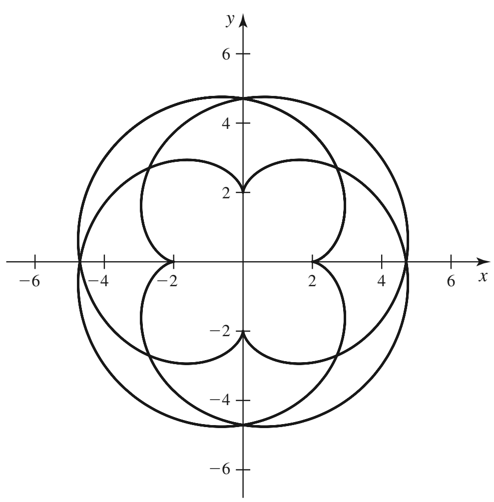
10.1.101



$a = 1,\ b = 2.$



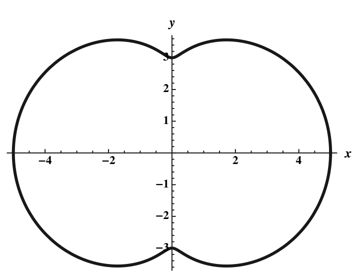
$a = 3.5,\ b = 2.$



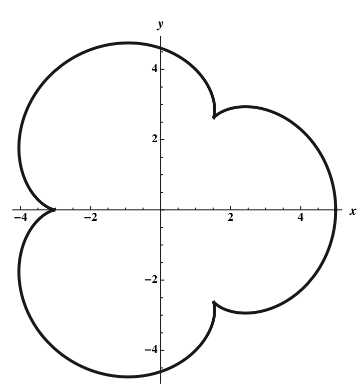
$a = 2,\ b = 3.5.$

Note that for $a < b$, we have cusps pointing inward, while for $a > b$, the cusps point outward.

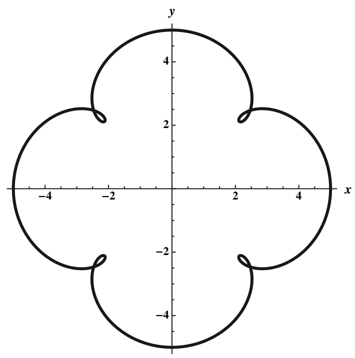
10.1.102



a.



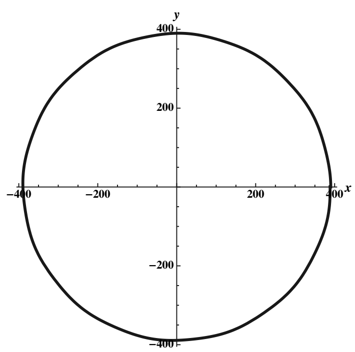
b.

c. 

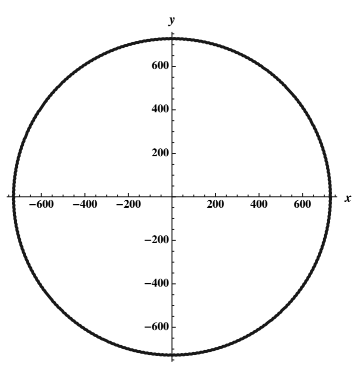
For a fixed a, there appear to be loops when $n > a$.

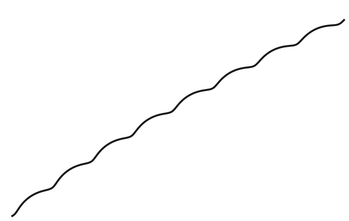
10.1.103

a.

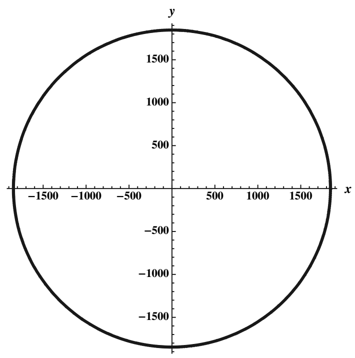


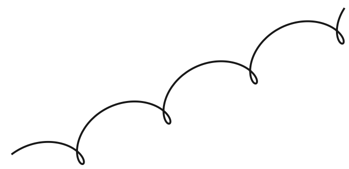
b.



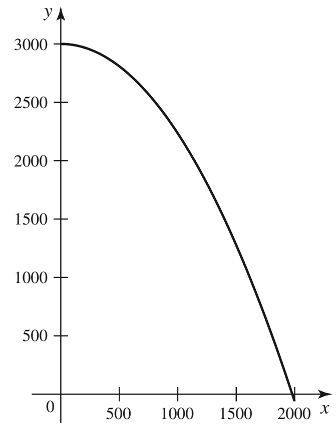


c.





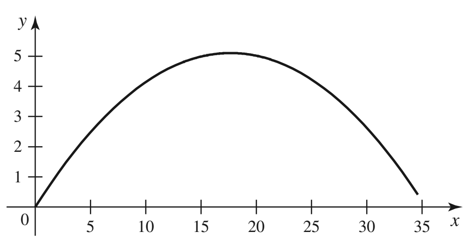
10.1.104 The packages lands when $y = 0$, so we seek a solution to $0 = -4.9t^2 + 3000$. So $t = \sqrt{\frac{3000}{4.9}} \approx 24.744$ seconds, at which point $x \approx 80 \cdot 24.744 \approx 1979.487$ meters.



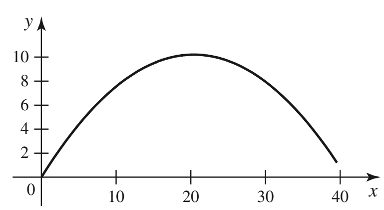
10.1.105 The package lands when $y = 0$, so when $-4.9t^2 + 4000 = 0$ for $t > 0$. This occurs when $t = \sqrt{\frac{4000}{4.9}} \approx 28.571$ seconds. At that time, $x \approx 100 \cdot 28.57 = 2857$ meters.

10.1.106

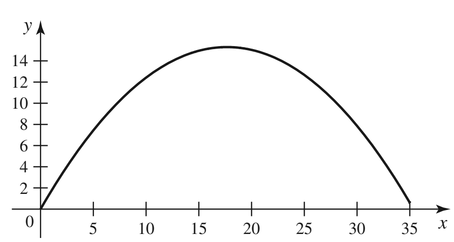
a.



$\theta = \pi/6$.



$\theta = \pi/4$.



$\theta = \pi/3$.

b. The maximum appears to be reached when $\theta = \pi/4$.

10.1.107 Let $x = 1 + \cos^2 t - \sin^2 t$ and $y = t$, for $-\infty < t < \infty$. Note that because $1 - \sin^2 t = \cos^2 t$, we have $x = 2\cos^2 t$, $y = t$.

10.1.108 Note that $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, so $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx} \cdot \frac{dx}{dt}\right) = \frac{dy}{dx}\frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{d}{dt}\left(\frac{dy}{dx}\right)$.
Also $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \cdot \frac{dx}{dt}$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Thus,

$$\frac{d^2y}{dt^2} = \frac{dy}{dx} \cdot \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{d^2y}{dx^2} \cdot \frac{dx}{dt} = \frac{dy/dt}{dx/dt} \cdot \frac{d^2x}{dt^2} + \frac{d^2y}{dx^2} \cdot \left(\frac{dx}{dt}\right)^2.$$

Solving for $\frac{d^2y}{dx^2}$ yields $y'' = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t))^3} = \frac{f'(t)g''(t) - f''(t)g'(t)}{(f'(t))^3}$.

10.1.109 Suppose that $a^2 + c^2 = b^2 + d^2$, and that $ab + cd = 0$. Note that

$$x^2 + y^2 = a^2\cos^2 t + 2ab\sin t\cos t + b^2\sin^2 t + c^2\cos^2 t + 2cd\sin t\cos t + d^2\sin^2 t,$$

which can be rewritten as

$$(a^2 + c^2)\cos^2 t + (b^2 + d^2)\sin^2 t + (2ab + 2cd)\sin t\cos t.$$

Because $b^2 + d^2 = a^2 + c^2$ and because $2ab + 2cd = 0$, we can write this as

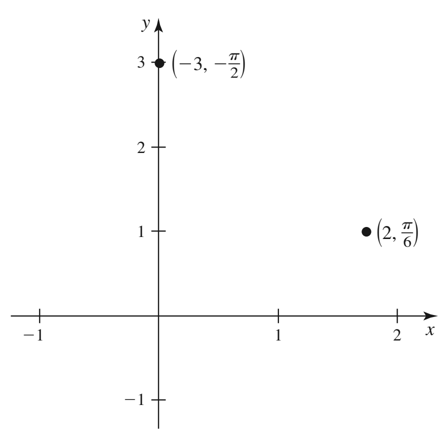
$$(a^2 + c^2)(\cos^2 t + \sin^2 t) = R^2,$$

so we have the circle $x^2 + y^2 = R^2$, as desired.

10.1.110 Note that if we let $x = t^{\frac{1}{t-1}}$ and $y = t^{\frac{t}{t-1}}$, $1 < t < \infty$, then we can see that $x\ln y = y\ln x$. Let L_1 represent this curve, and let L_2 be the curve with the same parametric equations but for $0 < t < 1$, and let L_3 be the line $y = x$. The region where $y^x > x^y$ is the region below L_1 and above L_3, and below L_3 but above L_2.

10.2 Polar Coordinates

10.2.1 The coordinates $(2, \pi/6)$, $(2, -11\pi/6)$, and $(-2, 7\pi/6)$ all give rise to the same point. Also, the coordinates $(-3, -\pi/2)$, $(3, \pi/2)$ and $(-3, 3\pi/2)$ give rise to the same point.



10.2.2 For a point with polar coordinates (r, θ), we have the Cartesian coordinates $x = r\cos\theta$ and $y = r\sin\theta$.

10.2.3 If a point has Cartesian coordinates (x, y) then $r^2 = x^2 + y^2$ and $\tan\theta = y/x$ for $x \neq 0$. If $x = 0$, then $\theta = \pi/2$ and $r = y$.

10.2.4 A cicle of radius $|a|$ centered at the origin has polar equation $r = |a|$.

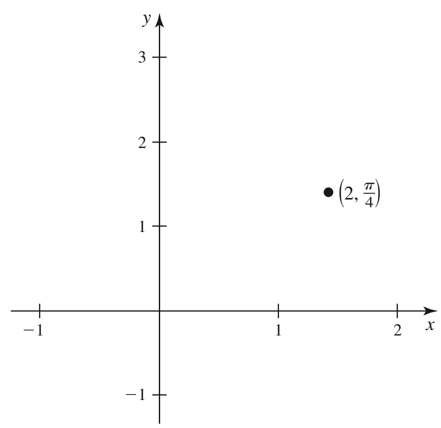
10.2.5 Because $x = r\cos\theta$, we have that the vertical line $x = 5$ has polar equation $r = 5\sec\theta$.

10.2.6 Because $y = r\sin\theta$, the horizontal line $y = 5$ has polar equation $r = 5\csc\theta$.

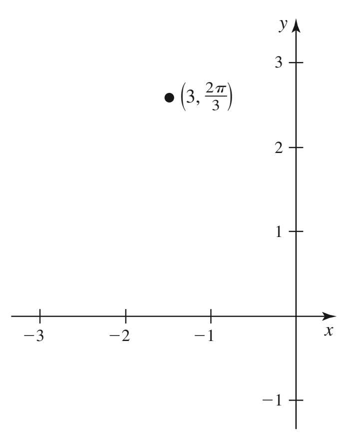
10.2.7 x-axis symmetry occurs if (r, θ) on the graph implies $(r, -\theta)$ is on the graph. y-axis symmetry occurs if (r, θ) on the graph implies $(r, \pi - \theta) = (-r, -\theta)$ is on the graph. Symmetry about the origin occurs if (r, θ) on the graph implies $(-r, \theta) = (r, \theta + \pi)$ is on the graph.

10.2.8 Graph $r = f(\theta)$ as if r and θ were Cartesian coordinates with θ on the horizontal axis and r on the vertical axis. Choose an interval in θ on which the entire polar curve is produced. Then use this graph as a guide to sketch the points (r, θ) on the final polar curve.

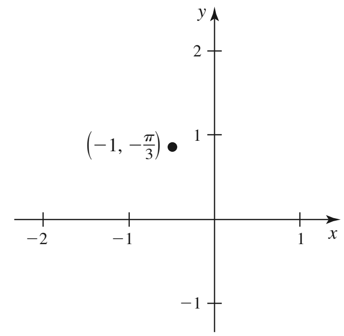
10.2.9 The coordinates $(2, \pi/4)$, $(-2, 5\pi/4)$, and $(2, 9\pi/4)$ represent the same point.



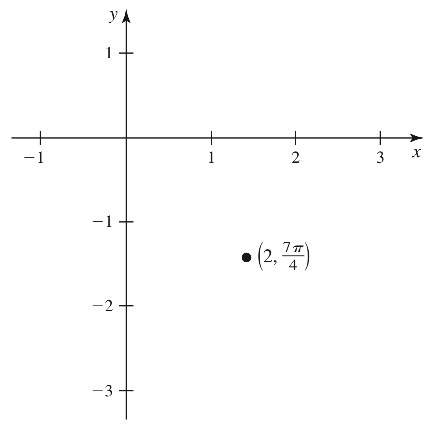
10.2.10 The coordinates $(3, 2\pi/3)$, $(-3, 5\pi/3)$ and $(3, 8\pi/3)$ represent the same point.



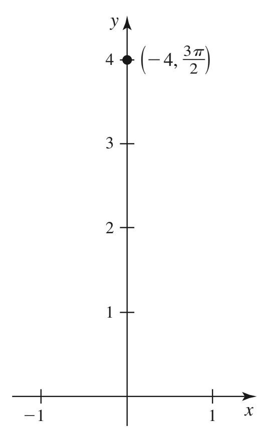
10.2.11 The coordinates $(-1, -\pi/3)$, $(1, 2\pi/3)$ and $(1, -4\pi/3)$ represent the same point.



10.2.12 The coordinates $(2, 7\pi/4)$, $(-2, 3\pi/4)$ and $(2, -\pi/4)$ represent the same point.



10.2.13 The coordinates $(-4, 3\pi/2)$, $(4, \pi/2)$ and $(-4, -\pi/2)$ represent the same point.



10.2.14 $A = (4, \pi/6) = (-4, 7\pi/6)$. $B = (3, \pi/4) = (-3, 5\pi/4)$. $C = (2, \pi/3) = (-2, 4\pi/3)$. $D = (4, \pi/2) = (-4, 3\pi/2)$. $E = (2, 4\pi/3) = (-2, \pi/3)$. $F = (4, -\pi/3) = (-4, 2\pi/3)$.

10.2.15 $x = 3\cos(\pi/4) = \frac{3\sqrt{2}}{2}$. $y = 3\sin(\pi/4) = \frac{3\sqrt{2}}{2}$.

10.2.16 $x = \cos(2\pi/3) = -1/2$. $y = \sin(2\pi/3) = \frac{\sqrt{3}}{2}$.

10.2.17 $x = \cos(-\pi/3) = \frac{1}{2}$. $y = \sin(-\pi/3) = -\frac{\sqrt{3}}{2}$.

10.2.18 $x = 2\cos(7\pi/4) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$. $y = 2\sin(7\pi/4) = -\sqrt{2}$.

10.2.19 $x = -4\cos(3\pi/4) = 2\sqrt{2}$. $y = -4\sin(3\pi/4) = -2\sqrt{2}$.

10.2.20 $x = 4\cos(5\pi) = -4$. $y = 4\sin(5\pi) = 0$.

10.2.21 $r^2 = x^2 + y^2 = 4 + 4 = 8$, so $r = \sqrt{8}$. $\tan\theta = 1$, so $\theta = \pi/4$, so $(2\sqrt{2}, \pi/4)$ is one representation of this point, and $(-2\sqrt{2}, -3\pi/4)$ is another.

10.2.22 $r^2 = x^2 + y^2 = 1 + 0$, so $r = \pm 1$. $\tan\theta = 0$, so $\theta = 0, \pi$. $(-1, 0)$ is one representation of this point, and $(1, \pi)$ is another.

10.2.23 $r^2 = x^2 + y^2 = 1 + 3 = 4$, so $r = \pm 2$. $\tan\theta = \sqrt{3}$, so $\theta = \pi/3$, $4\pi/3$. $(2, \pi/3)$ is one representation of this point, and $(-2, -2\pi/3)$ is another.

10.2.24 $r^2 = 81$, so $r = \pm 9$. $\tan\theta = 0$, so $\theta = 0, \pi$. One representation of the given point is $(9, \pi)$, and $(-9, 0)$ is another.

10.2.25 $r^2 = 64$, so $r = \pm 8$. $\tan\theta = -\sqrt{3}$, so $\theta = -\pi/3$, $2\pi/3$. One representation of the given point is $(8, 2\pi/3)$, and $(-8, -\pi/3)$ is another.

10.2.26 $r^2 = 16 + 48 = 64$, so $r = \pm 8$. $\tan\theta = \sqrt{3}$. One representation of the given point is $(8.\pi/3)$, and another is $(-8, 4\pi/3)$.

10.2.27 $x = r\cos\theta = -4$, so this is the vertical line $x = -4$ through $(-4, 0)$.

10.2.28 $y = r\sin\theta = \cot\theta\csc\theta\sin\theta = \cot\theta = \frac{x}{y}$. Thus, $y^2 = x$. This curve is a parabola with vertex at $(0, 0)$ which opens to the right.

10.2.29 Because $x^2 + y^2 = r^2 = 4$, this is a circle of radius 2 centered at the origin.

10.2.30 Because $y = r\sin\theta = 3\csc\theta\sin\theta = 3$, this is the horizontal line $y = 3$.

10.2.31 Note that $x^2 + y^2 = r^2 = 4\sin^2\theta + 8\sin\theta\cos\theta + 4\cos^2\theta = 4 + 8\sin\theta\cos\theta$. Also note that $x = r\cos\theta = 2\sin\theta\cos\theta + 2\cos^2\theta$ and $y = r\sin\theta = 2\sin^2\theta + 2\sin\theta\cos\theta$. Thus, $2x + 2y = 4 + 8\sin\theta\cos\theta$. If we combine these, we see that $x^2 + y^2 - (2x + 2y) = 0$. Thus $(x^2 - 2x + 1) + (y^2 - 2y + 1) = 2$, so we have the circle $(x - 1)^2 + (y - 1)^2 = 2$. This is a circle of radius $\sqrt{2}$ centered at $(1, 1)$.

10.2.32 We have $r\sin\theta = \pm r\cos\theta$, so $y = \pm x$. These are lines through the origin with slopes ± 1.

10.2.33 $r\cos\theta = \sin 2\theta = 2\sin\theta\cos\theta$. Note that if $\cos\theta = 0$, then r can be any real number, and the equation is satisfied. For $\cos\theta \neq 0$, we have $x = r\cos\theta = 2\sin\theta\cos\theta$, so $r = 2\sin\theta$, and thus $y = r\sin\theta = 2\sin^2\theta$. Thus $x^2 + y^2 - 2y = 4\sin^2\theta\cos^2\theta + 4\sin^2\theta\sin^2\theta - 4\sin^2\theta = 4\sin^2\theta(\sin^2\theta + \cos^2\theta) - 4\sin^2\theta = 4\sin^2\theta - 4\sin^2\theta = 0$. Note also that $x^2 + y^2 - 2y = 0$ is equivalent to $x^2 + (y - 1)^2 = 1$, so we have a circle of radius one centered at $(0, 1)$, as well as the line $x = 0$ which is the y-axis.

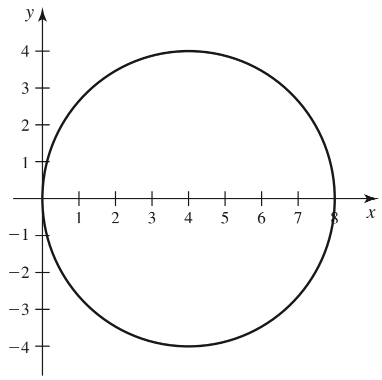
10.2.34 $r = \sin\theta\sec^2\theta$, so $x = r\cos\theta = \tan\theta = \frac{y}{x}$, so $y = x^2$, the standard parabola.

10.2.35 $r = 8\sin\theta$, so $r^2 = 8r\sin\theta$, so $x^2 + y^2 = 8y$. This can be written $x^2 + (y - 4)^2 = 16$, which represents a circle of radius 4 centered at $(0, 4)$.

10.2.36 The given equation implies that $2r\cos\theta + 3r\sin\theta = 1$, so $2x + 3y = 1$. This is a line with slope $-\frac{2}{3}$ and y-intercept $\frac{1}{3}$.

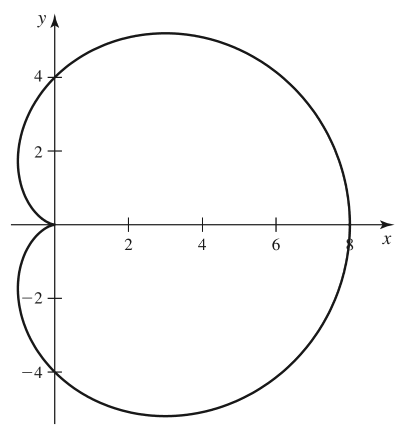
10.2.37

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
r	8	$4\sqrt{3}$	$4\sqrt{2}$	4	0	-4	$-4\sqrt{2}$	$-4\sqrt{3}$	-8

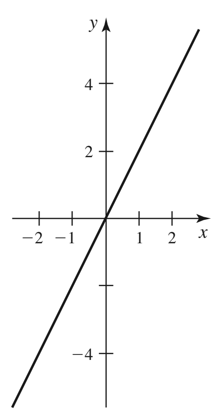


10.2.38

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	8	$4+2\sqrt{2}$	4	$4-2\sqrt{2}$	0	$4-2\sqrt{2}$	4	$4+2\sqrt{2}$	8

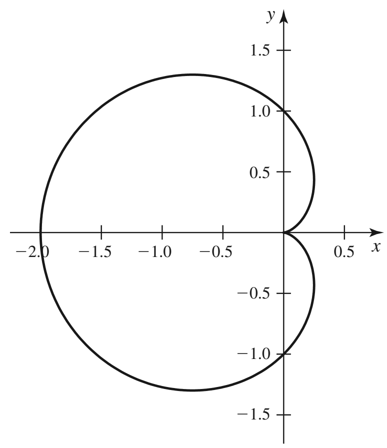


10.2.39 $r(\sin\theta - 2\cos\theta) = 0$ when $r = 0$ or when $\tan\theta = 2$, so the curve is a straight line through the origin of slope 2.

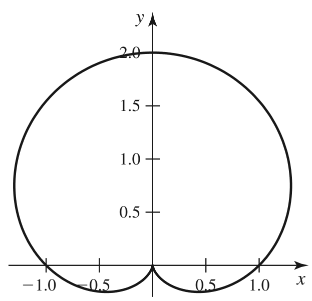


10.2.40

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	0	$\frac{\sqrt{2}-1}{\sqrt{2}}$	1	$\frac{\sqrt{2}+1}{\sqrt{2}}$	2	$\frac{\sqrt{2}+1}{\sqrt{2}}$	1	$\frac{\sqrt{2}-1}{\sqrt{2}}$	0



10.2.41



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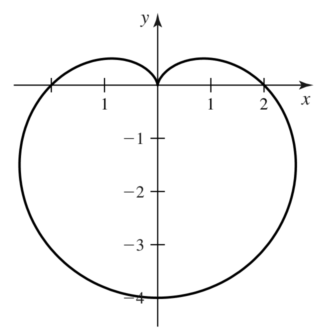

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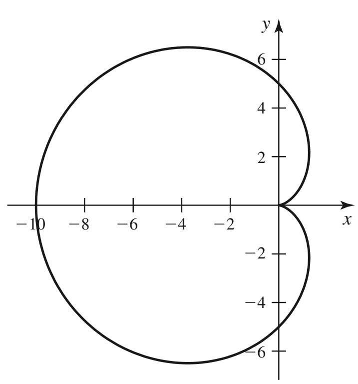

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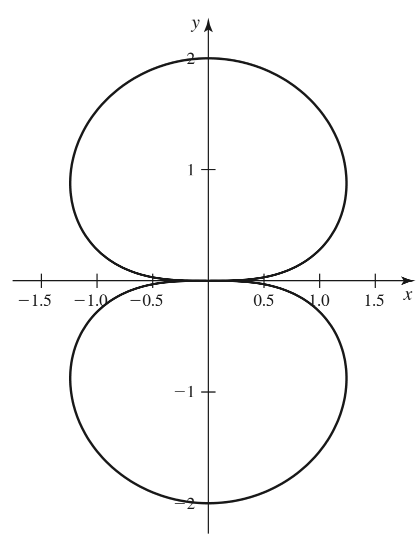
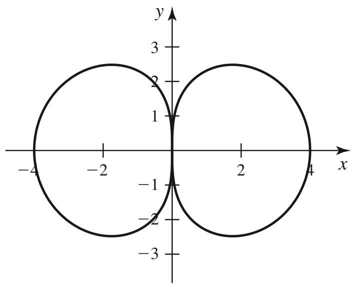


10.2.45



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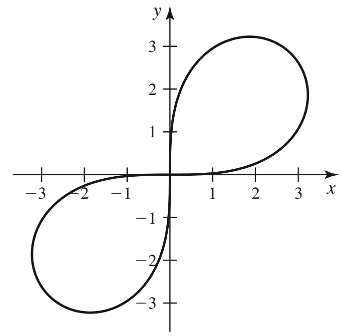

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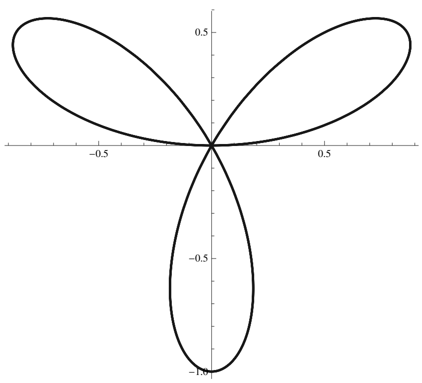
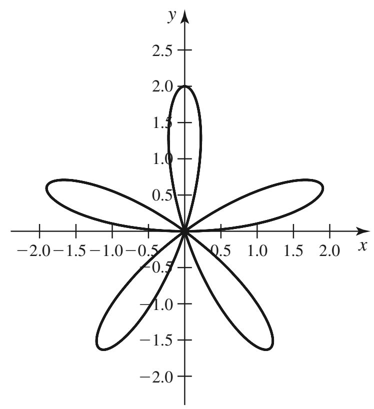
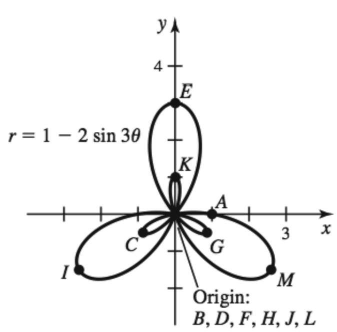


10.2.48

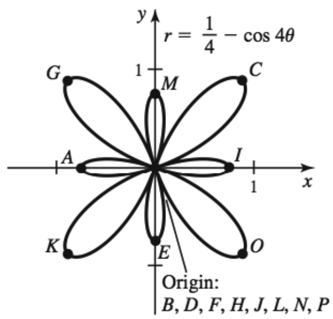


10.2.49 Points B, D, F, H, J and L have y-coordinate 0, so the graph is at the pole for each of these points. Points E, I, and M have maximal radius, so these correspond to the points at the tips of the outer loops. The points C, G and K correspond to the tips of the smaller loops. Point A corresponds to the polar point $(1, 0)$.



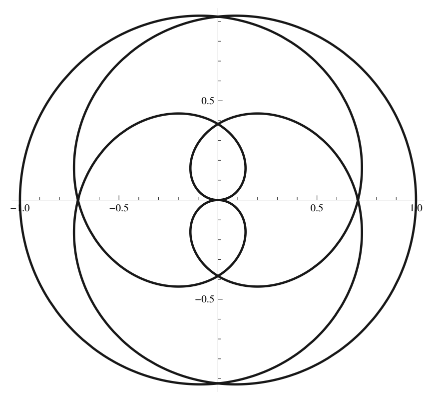
10.2.50 Points B, D, H and J have y-coordinate 0, so the graph is at the pole for each of these points. Points A and K lie where the graph intersects the negative x-axis. C and I are at the top of the two large loops, while F is is where the graph intersects the positive x-axis. E and G are the extreme points of the large wide loop.

10.2.51 Points B, D, F, H, J, L, N and P are at the origin. C, G, K and O are on the ends of the long loops, while A, E, I and M are at the ends of the smaller loops.

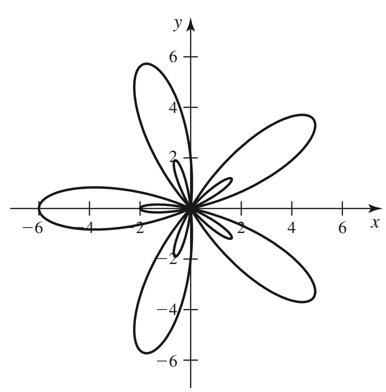


10.2.52 Points C, E, G and I are at the origin. B and D are at the ends of the two bigger loops, F and H are at ends of the two smaller loops. A and J are the points where the graph intersects the positive x-axis.

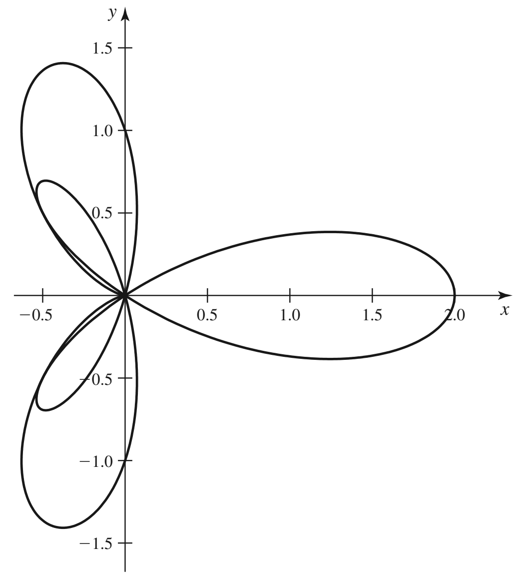
10.2.53 The interval $[0, 8\pi]$ generates the entire graph.



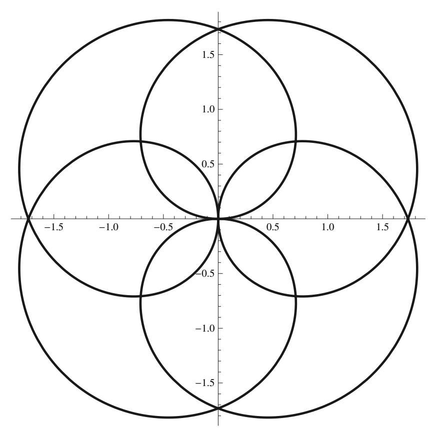
10.2.54 The interval $[0, 2\pi]$ generates the entire graph.



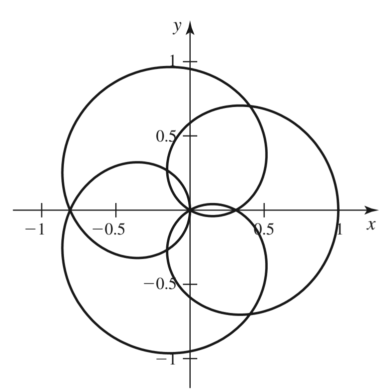
10.2.55 The interval $[0, 2\pi]$ generates the entire graph.



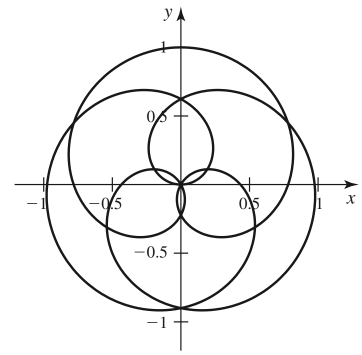
10.2.56 The interval $[0, 6\pi]$ generates the entire graph.



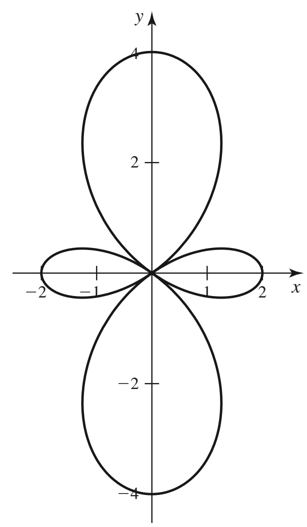
10.2.57 The interval $[0, 5\pi]$ generates the entire graph.



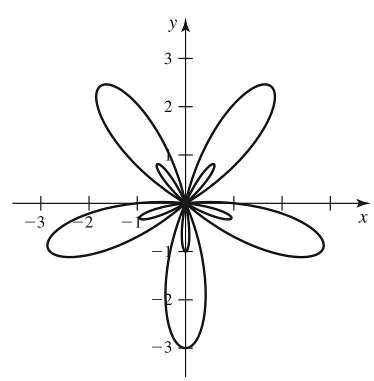
10.2.58 The interval $[0, 7\pi]$ generates the entire graph.



10.2.59 The interval $[0, 2\pi]$ generates the entire graph.



10.2.60 The interval $[0, 2\pi]$ generates the entire graph.



10.2.61

a. True. Note that $r^2 = 8$ and $\tan\theta = -1$.

b. True. Their intersection point (in Cartesian coordinates) is $(4, -2)$.

c. False. They intersect at the polar coordinates $(2, \pi/4)$ and $(2, 5\pi/4)$.

d. True. Note that for $\theta = \frac{3\pi}{2}$ we have $r = -3$. But the polar point $\left(-3, \frac{3\pi}{2}\right)$ is the same as the polar point $\left(3, \frac{\pi}{2}\right)$.

e. True. The first is the line $x = 2$ because $x = r\cos\theta = 2\sec\theta\cos\theta = 2$, and the second is $y = 3$ because $y = r\sin\theta = 3\csc\theta\sin\theta = 3$.

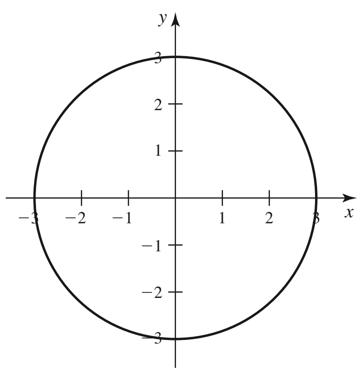
10.2.62 We have $y = r\sin\theta = 3$, so $r = \frac{3}{\sin\theta} = 3\csc\theta$.

10.2.63 We have $r\sin\theta = r^2\cos^2\theta$, so $r = \frac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta$.

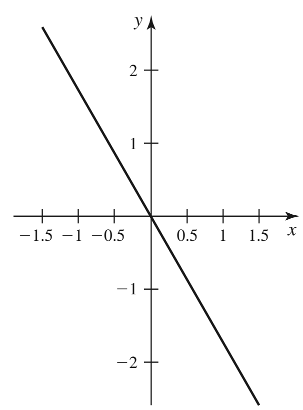
10.2.64 We have $(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$, so $r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = 1$, and thus $2r\cos\theta = r^2$. Thus $r = 2\cos\theta$.

10.2.65 We have $r\sin\theta = \frac{1}{r\cos\theta}$, so $r^2 = \sec\theta\csc\theta$.

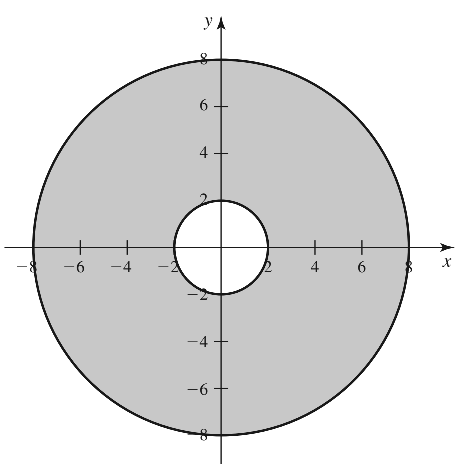
10.2.66



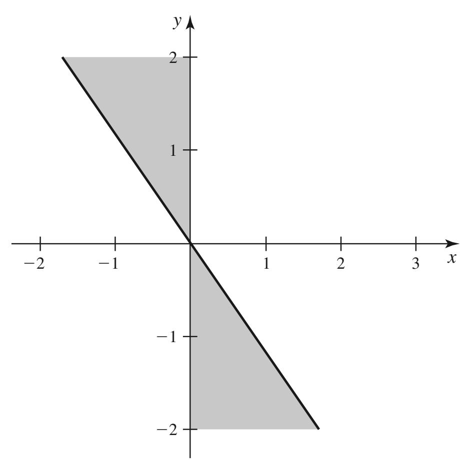
10.2.67



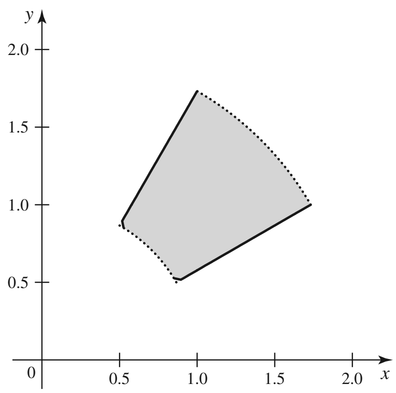
10.2.68



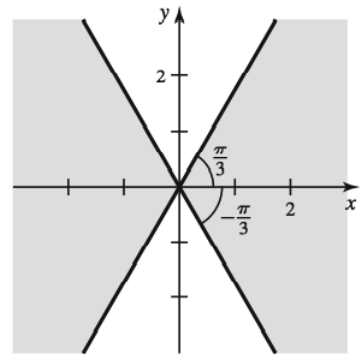
10.2.69



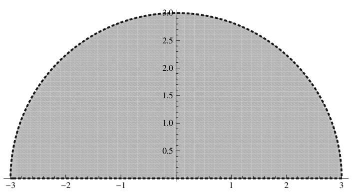
10.2.70



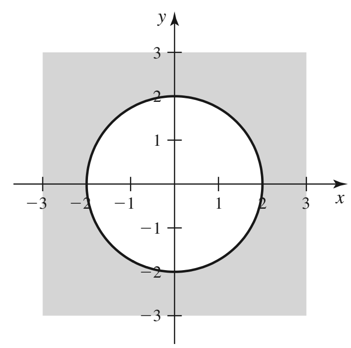
10.2.71



10.2.72



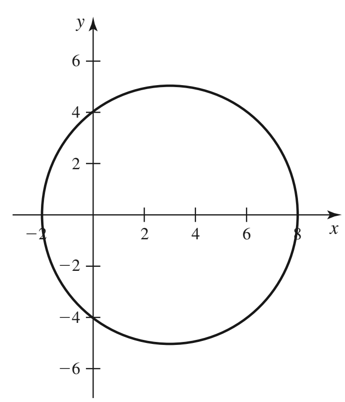
10.2.73



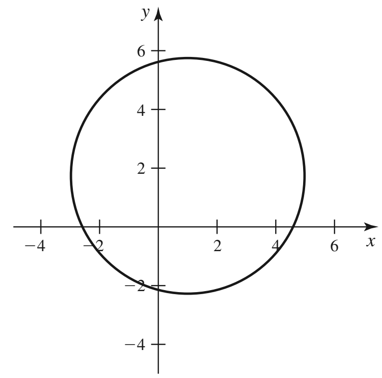
10.2.74 Let $x = r\cos\theta$ and $y = r\sin\theta$, so that $r^2 = x^2 + y^2$. Then the given equation can be written $(x^2 + y^2) - 2ax - 2by + (a^2 + b^2) = R^2$, which in turn can be written as $(x - a)^2 + (y - b)^2 = R^2$, which is the equation of a circle of radius R centered at (a, b).

10.2.75 Consider the circle with center $C(r_0, \theta_0)$, and let A be the origin and $B(r, \theta)$ be a point on the circle not collinear with A and C. Note that the length of side BC is R, and that the angle CAB has measure $\theta - \theta_0$. Applying the law of cosines to triangle CAB yields the equation $R^2 = r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)$, which is equivalent to the given equation.

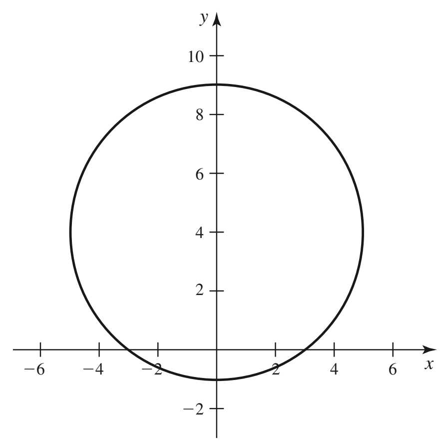
10.2.76 In relation to number 66, we have $2a = 6$, so $a = 3$ and $b = 0$. So $R^2 - a^2 - b^2 = R^2 - 9 = 16$, and thus $R^2 = 25$. Thus we have a circle centered at $(3, 0)$ with radius 5.



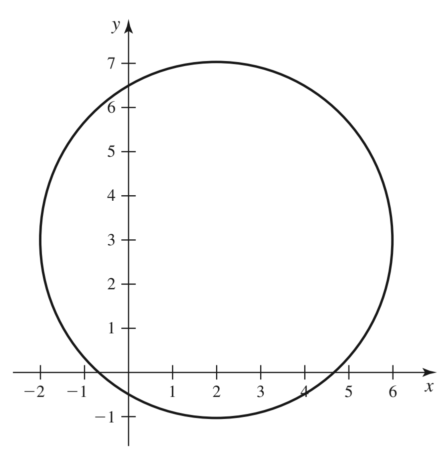
10.2.77 In relation to number 75, we have $r_0 = 2$ and $\theta_0 = \pi/3$, and $R^2 - 4 = 12$, so $R^2 = 16$. Thus this is a circle with polar center $(2, \pi/3)$ and radius 4.



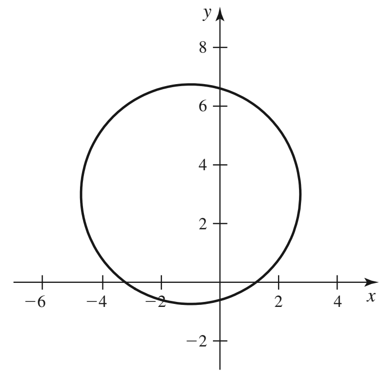
10.2.78 In relation to number 75, we have $r_0 = 4$ and $\theta_0 = \pi/2$, and $R^2 - 16 = 9$, so $R^2 = 25$. Thus this is a circle with polar center $(4, \pi/2)$ and radius 5.



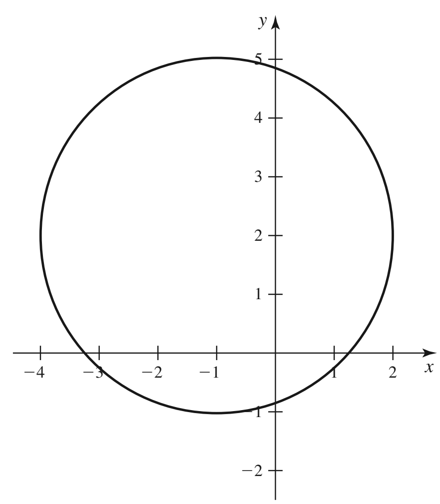
10.2.79 In relation to number 66, we have $a = 2$ and $b = 3$. So $R^2 - a^2 - b^2 = R^2 - 13 = 3$, and thus $R^2 = 16$. Thus we have a circle centered at $(2, 3)$ with radius 4.



10.2.80 In relation to number 66, we have $a = -1$ and $b = 3$. So $R^2 - a^2 - b^2 = R^2 - 10 = 4$, and thus $R^2 = 14$. Thus we have a circle centered at $(-1, 3)$ with radius $\sqrt{14}$.



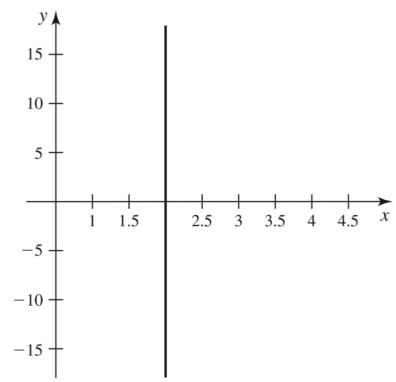
10.2.81 In relation to number 66, we have $a = -1$ and $b = 2$. So $R^2 - a^2 - b^2 = R^2 - 5 = 4$, and thus $R^2 = 9$. Thus we have a circle centered at $(-1, 2)$ with radius 3.



10.2.82 The radius of a circle inscribed in a triangle with side lengths a, b, and c is $\frac{2A}{a+b+c}$ where A is the area of the triangle. So for the bigger circle, $R = r_0 = \frac{2}{2+2\sqrt{2}} = \frac{1}{1+\sqrt{2}}$. For each of the smaller circles, we have $R = \frac{1}{2+\sqrt{2}}$. The area inside the three circles is thus $2\pi \cdot \frac{1}{(2+\sqrt{2})^2} + \pi \cdot \frac{1}{(1+\sqrt{2})^2} \approx 1.078$. Because the area of the square is 2, there is more area inside the circles than outside the circles but inside the square. Using problem 75, the equation of the largest circle is $r^2 - 2r\left(\frac{1}{1+\sqrt{2}}\right)\cos(\theta - \pi/2) = 0$. The smaller circle in the 3rd quadrant has center with polar radius $r_0 = \frac{\sqrt{2}}{2} - \frac{1}{2+\sqrt{2}} = \sqrt{2} - 1$, so its equation is $r^2 - 2r\left(\sqrt{2} - 1\right)\cos(\theta - 5\pi/4) = R^2 - r_0^2 = \sqrt{2} - 3/2$, and the other circle has equation $r^2 - 2r\left(\sqrt{2} - 1\right)\cos(\theta + \pi/4) = \sqrt{2} - 3/2$.

10.2.83

a. On all three intervals, the graph is the same vertical line, oriented upward.



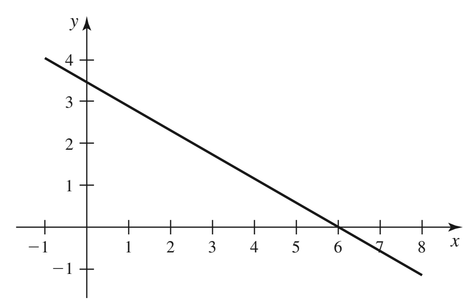
b. For $\theta \neq \frac{2m+1}{2}\pi$ where m is an integer, we have $\cos\theta \neq 0$, so the equation is equivalent to $x = r\cos\theta = 2$. So the graph is a vertical line.

10.2.84

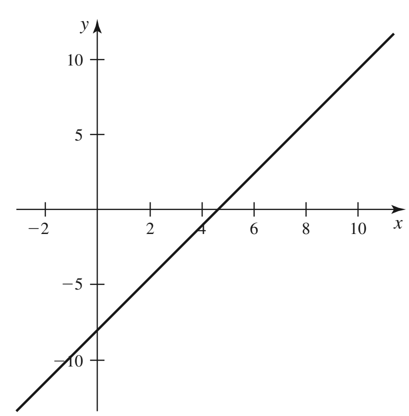
a. Given $y = mx + b$, let $x = r\cos\theta$ and $y = r\sin\theta$. Then $r\sin\theta = m(r\cos\theta) + b$, so $r\sin\theta - mr\cos\theta = b$, and thus $r(\sin\theta - m\cos\theta) = b$, and $r = \frac{b}{\sin\theta - m\cos\theta}$, provided $\sin\theta - m\cos\theta \neq 0$.

b. Using the right triangle shown, we see that $\frac{r_0}{r} = \cos(\theta_0 - \theta)$, so $r_0 = r\cos(\theta_0 - \theta)$.

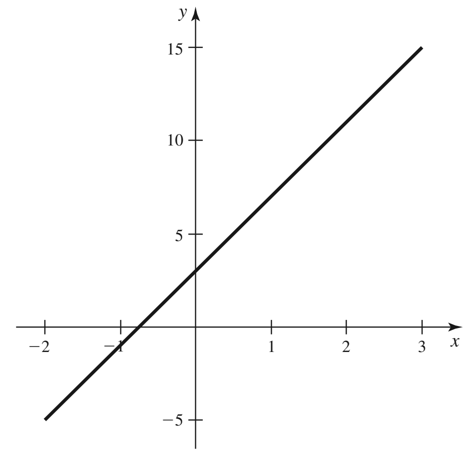
10.2.85 Using problem 84b, this is the line with $r_0 = 3$ and $\theta_0 = \frac{\pi}{3}$. So it is the line through the polar point $(3, \pi/3)$ in the direction of angle $\pi/3 + \pi/2 = 5\pi/6$. The Cartesian equation is $y = -\frac{x}{\sqrt{3}} + 2\sqrt{3}$.



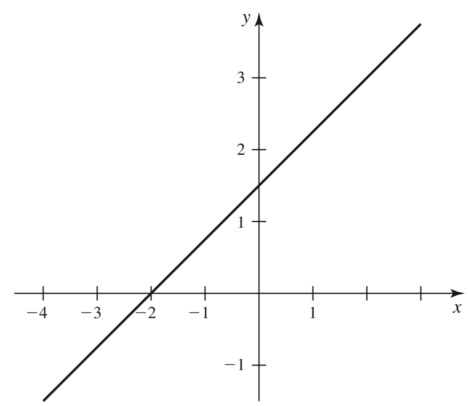
10.2.86 Using problem 84b, this is the line with $r_0 = 4$ and $\theta_0 = -\frac{\pi}{6}$. So it is the line through the polar point $(4, -\pi/6)$ in the direction of angle $-\pi/6 + \pi/2 = \pi/3$. The Cartesian equation is $y = \sqrt{3}x - 8$.



10.2.87 Using problem 84a, this is the line with $b = 3$ and $m = 4$, so $y = 4x + 3$.



10.2.88 Using problem 84a, this is the line with $b = 3/2$ and $m = 3/4$, so $y = \frac{3}{4}x + \frac{3}{2}$.



10.2.89

a. This matches (A), because we have $|a| = 1 = |b|$, and the graph is a cardioid.

b. This matches (C). This has an inner loop because $|a| = 1 < 2 = |b|$. Note that $r = 1$ when $\theta = 0$, so it can't be (D).

c. This matches (B). This has $|a| = 2 > 1 = |b|$, so it has an oval-like shape.

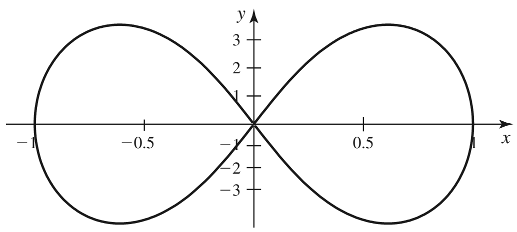
d. This matches (D). This has an inner loop because $|a| = 1 < 2 = |b|$. Note that $r = -1$ when $\theta = 0$, so this can't be (C).

e. This matches (E). Note that there is an inner loop because $|a| = 1 < 2 = |b|$, and that $r = 3$ when $\theta = \pi/2$.

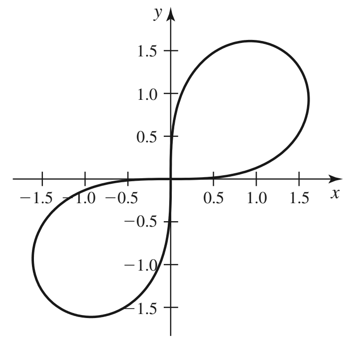
f. This matches (F).

10.2.90 As $b \to \infty$, the inner loop approaches the outer loop, so that for large b the graph appears to be a single circle with diameter b. Thus, there is no limiting curve as $b \to \infty$.

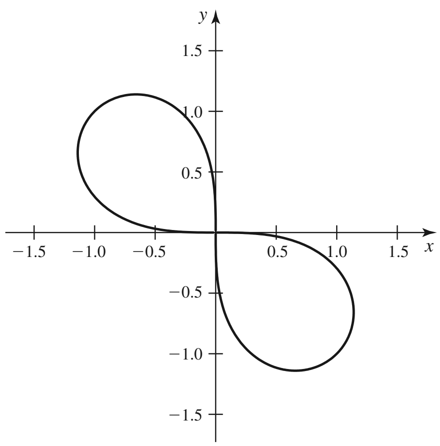
10.2.91



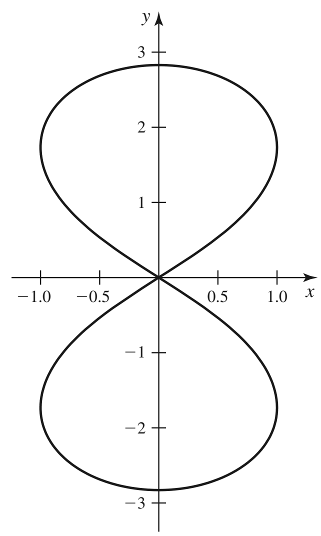
10.2.92



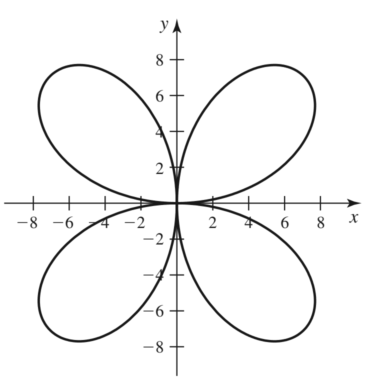
10.2.93



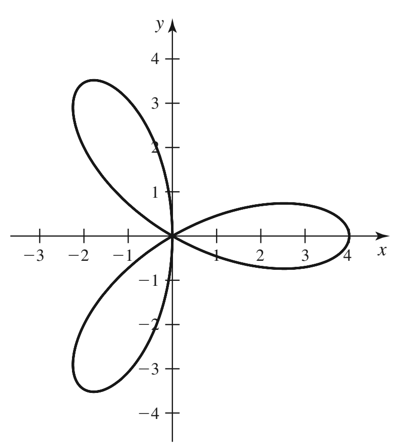
10.2.94



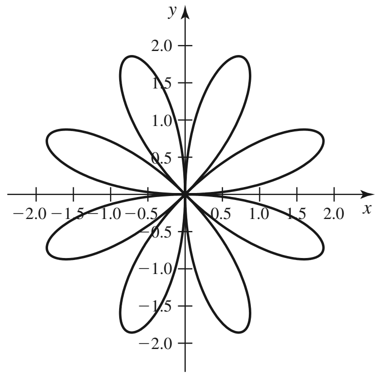
10.2.95



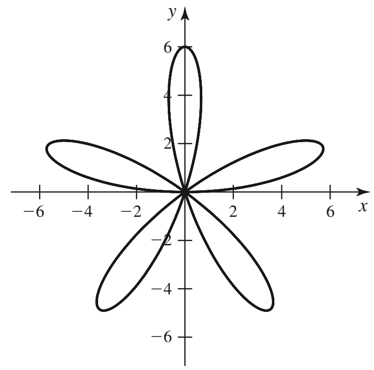
10.2.96



10.2.97

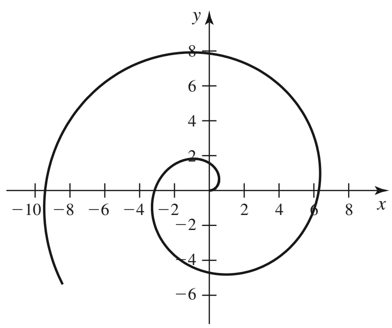


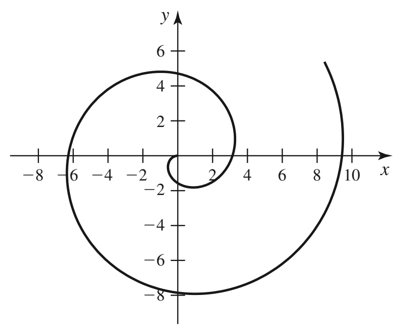
10.2.98



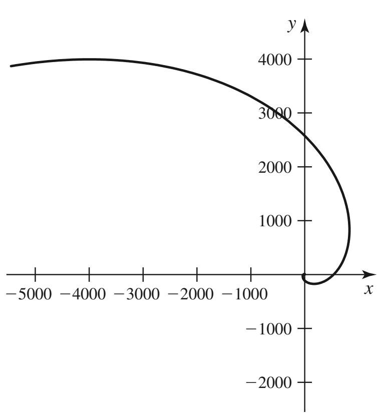
10.2.99 Note that $a \sin m\theta = 0$ for $\theta = \frac{k\pi}{m}$, $k = 1, 2, \ldots, 2m$. Thus the graph is back at the pole $r = 0$ for each of these values, and each of these gives rise to a distinct petal of the rose if m is odd. If m is even, then by symmetry, each petal for $k = 1, 2, \ldots \frac{m}{2}$ is equivalent to one for $k = \frac{m}{2} + 1, \frac{m}{2} + 2, \ldots, m$. (Note that this follows because the sine function is odd.) A similar result holds for the rose $r = a \cos \theta$.

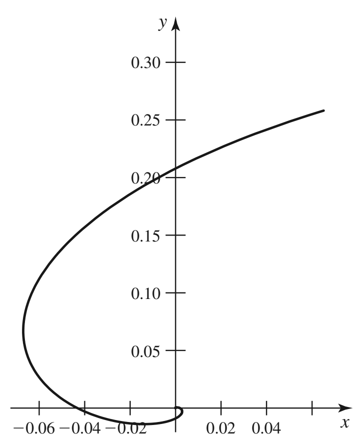
10.2.100 The spirals wind outward counterclockwise.



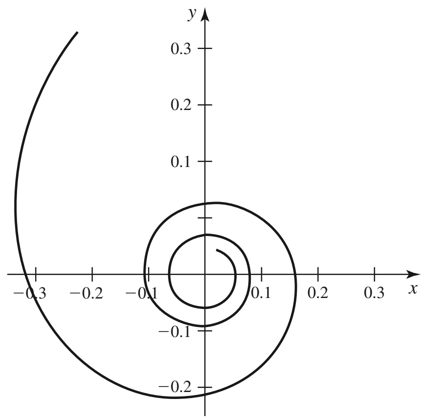


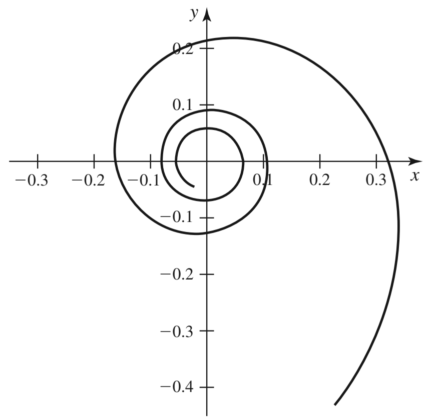
10.2.101 For $a = 1$, the spiral winds outward counterclockwise. For $a = -1$, the spiral winds inward counterclockwise.





10.2.102 The spirals wind inward counterclockwise for $a = 1$ and outward clockwise for $a = -1$.





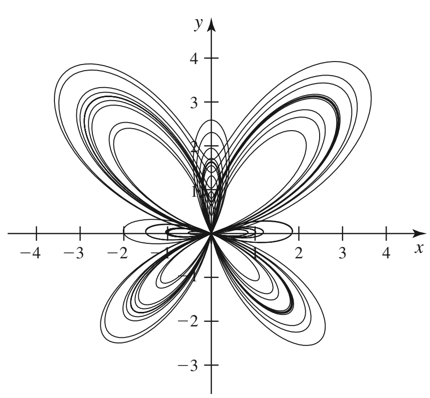
10.2.103 Suppose $2\cos\theta = 1 + \cos\theta$. Then $\cos\theta = 1$, so this occurs for $\theta = 0$ and $\theta = 2\pi$. At those values, $r = 2$, so the curves intersect at the polar point $(2, 0)$. The curves also intersect when $r = 0$, which occurs for $\theta = \pi/2$ and $\theta = 3\pi/2$ for the first curve and $\theta = \pi$ for the second.

10.2.104 Suppose $4\cos\theta = 1 + 2\cos\theta + \cos^2\theta$. Then $(\cos\theta - 1)^2 = 0$, so $\theta = 0$. At that value, $r = 2$, so the curves intersect at the polar point $(2, 0)$. The curves also intersect when $r = 0$, which occurs for the first curve at $\pi/2$ and $3\pi/2$, and for the second curve at π. Also, the curves intersect when $4\cos\theta = -1 - 2\cos\theta - \cos^2\theta$, which occurs for $\cos^2\theta + 6\cos\theta + 1 = 0$, or (using the quadratic formula) $\theta = \cos^{-1}(-3 + 2\sqrt{2}) \approx 1.743$. This leads to the polar intersection points at approximately $(0.828, \pm 1.743)$.

10.2.105 Suppose $1 - \sin\theta = 1 + \cos\theta$, or $\tan\theta = -1$. Then $\theta = 3\pi/4$ or $\theta = 7\pi/4$. So the curves intersect at the polar points $(1 + \sqrt{2}/2, 7\pi/4)$ and $(1 - \sqrt{2}/2, 3\pi/4)$. They also intersect at the pole $(0, 0)$, which occurs for the first curve at $\pi/2$ and for the second curve at π.

10.2.106 Suppose $\cos 2\theta = \sin 2\theta \geq 0$. Then $2\theta = \frac{\pi}{4}$, so $\theta = \frac{\pi}{8}, \frac{9\pi}{8}$. The curves intersect at $\pi/8$ and $9\pi/8$ where both $\cos 2\theta$ and $\sin 2\theta$ have value $\frac{\sqrt{2}}{2}$. The curves also intersect at the pole, which occurs for the first curve at $\pi/4$, and $3\pi/4$, $5\pi/4$ and $7\pi/4$, and for the second curve at 0, $\pi/2$, π, and $3\pi/2$. Thus the intersection points are $(0, 0)$, and approximately $(0.841, 0.393)$ and $(-0.841, 0.393)$.

10.2.107



a.

b. It adds multiple layers of the same type of curve as $\sin^5\theta/12$ oscillates between -1 and 1 for $0 \leq \theta \leq 24\pi$.

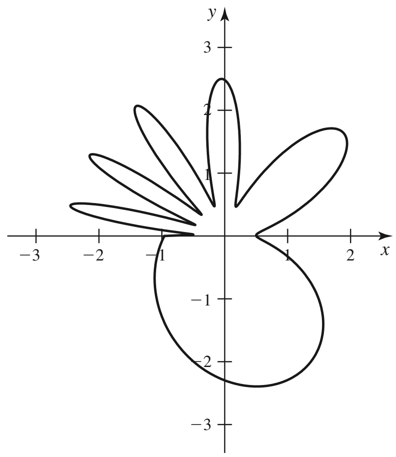
10.2.108

a. $f(0) = \cos(1) - 1.5$, and $f(2\pi) = \cos(((1 + 12\pi)^{1/2\pi})^{2\pi}) - 1.5 = \cos(1 + 12\pi) - 1.5 = \cos(1) - 1.5 = f(0)$. The points correspond to the polar points $(-0.960, 0)$.

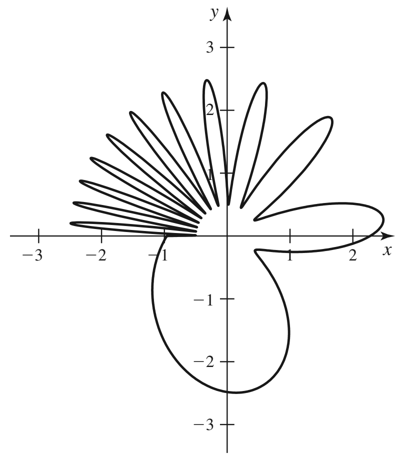
b. No. The curve for $-\pi \le \theta \le 0$ has nowhere where the absolute value of the radius is equal to 1, whereas the curve for $\pi \le \theta \le 2\pi$ has numerous places where this is true, because a^x has a much bigger range on $[0, \pi]$ than on $[-\pi, 0]$.

c. Because $((1+2k\pi)^{1/2\pi})^0 = 1$ and $((1+2k\pi)^{1/2\pi})^{2\pi} = 1 + 2k\pi$, we have that $f(0) = \cos(1) - b = \cos(1 + 2k\pi) - b = \cos(1) - b = f(2\pi)$.

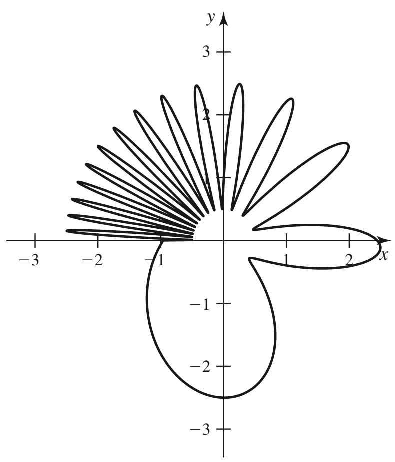
d.



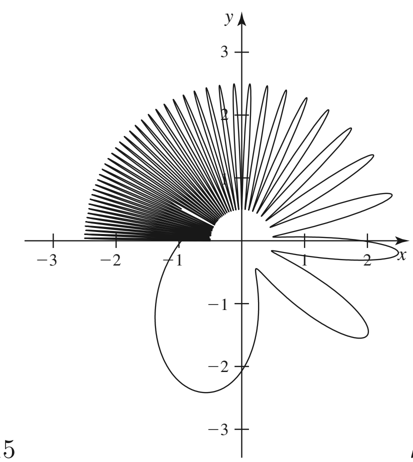
$k = 6$



$k = 12$



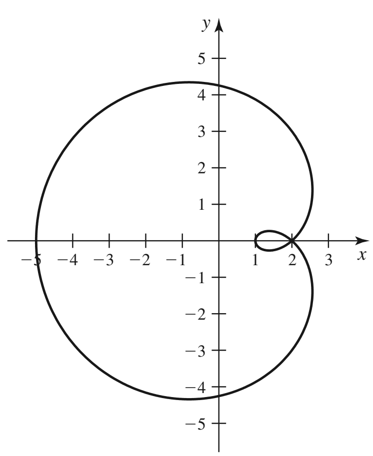
$k = 15$



$k = 45$

10.2.109

a.



b. $r = 3 - 4\cos \pi t$ is a limaçon, and $x - 2 = r \cos \pi t$ and $y = r \sin \pi t$ is a circle, and the composition of a limaçon and a circle is a limaçon.

10.2.110

a. The region is given by $\{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$.

b. The inflow is given by $\{(r,\theta) : 1 \le r \le 2,\ \theta = 0\}$. The outflow is given by $\{(r,\theta) : 1 \le r \le 2,\ \theta = \pi\}$.

c. The tangential velocity at $(1.5, \pi/4)$ is $v(1.5) = 10 \cdot 1.5 = 15$ meters per second. At $(1.2, 3\pi/4)$ it is $v(1.2) = 10 \cdot 1.2 = 12$ meters per second, so it is greater at 1.5.

d. The velocity is greater at $r = 1.3$, because $\frac{20}{1.3} > \frac{20}{1.8}$.

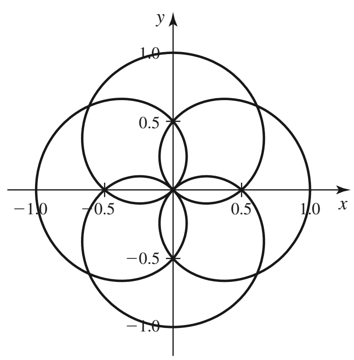
e. $\int_1^2 10r\,dr = 5r^2\Big|_1^2 = 15$, while $\int_1^2 \frac{20}{r}\,dr = 20\ln r|_1^2 \approx 13.86$, so the flow is greater in part c).

10.2.111 With $r = a\cos\theta + b\sin\theta$, we have $r^2 = ar\cos\theta + br\sin\theta$, or $x^2 + y^2 = ax + by$, so $\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2+b^2}{4}$. Thus, the center is $(a/2, b/2)$ and $r = \frac{\sqrt{a^2+b^2}}{2}$.

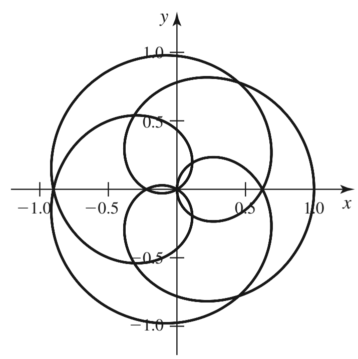
10.2.112 Note that $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, so $r^2 = a^2(\cos^2\theta - \sin^2\theta)$, so $r^4 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$, so $(x^2+y^2)^2 = a^2(x^2 - y^2)$.

10.2.113 Because $\sin(\theta/2) = \sin(\pi - \theta/2) = \sin((2\pi - \theta)/2)$, we have that the graph is symmetric with respect to the x-axis.

10.2.114



a.



b.

c. If n is even, then the whole curve is generated for $0 \le \theta \le 2m\pi$. If n is odd, then the whole curve is generated for $0 \le \theta \le m\pi$.

10.3 Calculus in Polar Coordinates

10.3.1 Because $x = r\cos\theta$ and $y = r\sin\theta$, we have $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin(\theta)$.

10.3.2 We need $\frac{dy}{dx}$, which can be computed using the formula $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$, which will then need to be evaluated at $\theta = \theta_0$.

10.3.3 Because slope is given relative to the horizontal and vertical coordinates, it is given by $\frac{dy}{dx}$, not by $\frac{dr}{d\theta}$.

10.3.4 This would be given by $\frac{1}{2}\int_\alpha^\beta (f(\theta)^2 - g(\theta)^2)\,d\theta$.

10.3.5 $\frac{dy}{dx} = \frac{-\cos\theta\sin\theta + (1-\sin\theta)\cos\theta}{-\cos^2\theta - (1-\sin\theta)\sin\theta}$. At $(1/2, \pi/6)$, we have $\frac{dy}{dx} = \frac{0}{-1} = 0$. The given curve intersects the origin $r = 0$ for $\theta = \pi/2$. At this point, $\frac{dy}{dx}$ does not exist, and the tangent line is vertical. (It is the line $\theta = \pi/2$.)

10.3.6 $\frac{dy}{dx} = \frac{-4\sin^2\theta+4\cos^2\theta}{-8\cos\theta\sin\theta}$. At $(2, \pi/3)$ we have $\frac{dy}{dx} = \frac{-2}{-2\sqrt{3}} = \frac{\sqrt{3}}{3}$. The given curve intersects the origin $r = 0$ for $\theta = \pi/2$ and $\theta = 3\pi/2$. At these points, the derivative does not exist, and the tangent line is vertical, so $\theta = \pi/2$ is the tangent line.

10.3.7 $\frac{dy}{dx} = \frac{16\cos\theta\sin\theta}{-8\sin^2\theta+8\cos^2\theta}$. At $(4, 5\pi/6)$ we have $\frac{dy}{dx} = \frac{-4\sqrt{3}}{4} = -\sqrt{3}$. The given curve intersects the origin $r = 0$ for $\theta = 0$ and $\theta = \pi$. At these points, the derivative is 0, and the tangent line is horizontal, so $\theta = 0$ is the tangent line.

10.3.8 $\frac{dy}{dx} = \frac{\cos\theta\sin\theta+(4+\sin\theta)\cos\theta}{\cos^2\theta-(4+\sin\theta)\sin\theta}$. At $(4, 0)$ we have $\frac{dy}{dx} = \frac{4}{1} = 4$. At $(3, 3\pi/2)$ we have $\frac{dy}{dx} = \frac{0}{3} = 0$. The given curve does not intersect the origin, because $r \geq 3$ for all θ.

10.3.9 $\frac{dy}{dx} = \frac{-3\sin^2\theta+(6+3\cos\theta)\cos\theta}{-3\cos\theta\sin\theta-(6+3\cos\theta)\sin\theta}$. At both $(3, \pi)$ and $(9, 0)$, this doesn't exist. The given curve does not intersect the origin, because $r \geq 3$ for all θ.

10.3.10 $\frac{dy}{dx} = \frac{6\cos(3\theta)\sin\theta+2\sin(3\theta)\cos\theta}{6\cos(3\theta)\cos\theta-2\sin(3\theta)\sin\theta}$. The tips of the leaves occur at $\theta = \pi/6$, $\pi/2$ and $5\pi/6$. At $\pi/6$, we have $\frac{dy}{dx} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$. At $\pi/2$ we have $\frac{dy}{dx} = \frac{0}{2} = 0$. At $5\pi/6$ we have $\frac{dy}{dx} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$. The graph intersects the origin for $\theta = 0$, $\theta = \pi/3$, $\theta = 2\pi/3$ and $\theta = \pi$, and these are the corresponding equations of the tangent lines. (Note that the lines $\theta = 0$ and $\theta = \pi$ are the same.)

10.3.11 $\frac{dy}{dx} = \frac{-8\sin(2\theta)\sin\theta+4\cos(2\theta)\cos\theta}{-8\sin(2\theta)\cos\theta-4\cos(2\theta)\sin\theta}$. The tips of the leaves occur at $\theta = 0$, $\pi/2$, π and $3\pi/2$. At 0 and at π, we have that $\frac{dy}{dx}$ doesn't exist. At $\pi/2$ and $3\pi/2$ we have $\frac{dy}{dx} = 0$. The graph intersects the origin for $\theta = \pi/4$, $\theta = 3\pi/4$, $\theta = 5\pi/4$ and $\theta = 7\pi/4$, and thus the two distinct tangent lines are $\theta = \pi/4$ and $\theta = 3\pi/4$.

10.3.12 $\frac{dy}{dx} = \frac{0+3(\sqrt{2}/2)}{0-3(\sqrt{2}/2)} = -1$. The curve is at the origin when $\sin 2\theta = -\frac{1}{2}$, which occurs when $2\theta = 7\pi/6, 11\pi/6, 19\pi/6$, and $23\pi/6$, or $\theta = 7\pi/12, 11\pi/12, 19\pi/12$, and $23\pi/12$.

10.3.13 The curve hits the origin at $\pm\pi/4$, where the tangent lines are given by $\theta = \pi/4$ and $\theta = -\pi/4$. The slopes of those lines are given by $\tan(\pi/4) = 1$ and $\tan(-\pi/4) = -1$.

10.3.14 $\frac{dy}{dx} = \frac{2\sin\theta+2\theta\cos\theta}{2\cos\theta-2\theta\sin\theta}$. At $(\pi/2, \pi/4)$ this is $\frac{\sqrt{2}+\pi\sqrt{2}/4}{\sqrt{2}-\pi\sqrt{2}/4} \approx 8.32$. The graph intersects the origin at $\theta = 0$, where there is a horizontal tangent.

10.3.15 Note that the curve is at the origin at $\pi/2$, so there is vertical tangent at $(0, \pi/2)$. Also, $\frac{dy}{dx} = \frac{-4\sin^2\theta+4\cos^2\theta}{-8\sin\theta\cos\theta} = \frac{1-2\sin^2\theta}{\sin(2\theta)}$. Thus, there are horizontal tangents at $\pi/4$ and $3\pi/4$ (at the polar points $(2\sqrt{2}, \pi/4)$ and $(-2\sqrt{2}, 3\pi/4)$). There is also a vertical tangent where $\theta = 0$, at the point $(4, 0)$.

10.3.16 Note that the curve is at the origin at $3\pi/2$, so there is vertical tangent at $(0, 3\pi/2)$. Also, $\frac{dy}{dx} = \frac{2\cos\theta\sin\theta+(2+2\sin\theta)\cos\theta}{2\cos^2\theta-(2+2\sin\theta)\sin\theta} = \frac{\cos\theta(2+4\sin\theta)}{(2-4\sin^2\theta)-2\sin\theta}$. Thus, the are horizontal tangents where this expression is 0 at $\pi/2$ and $7\pi/6$ and $(11\pi/6)$ (at the polar points $(4, \pi/2)$ and $(1, 7\pi/6)$ and $(1, 11\pi/6)$). There are also vertical tangents where the denominator is 0 and the numerator isn't, which occurs at the point $(3, \pi/6)$ and at $(3, 5\pi/6)$.

10.3.17 Using the double angle identities somewhat liberally:

$$\frac{dy}{dx} = \frac{2\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta}{2\cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta} = \frac{\sin\theta(\cos 2\theta + \cos^2\theta)}{\cos\theta(\cos(2\theta) - \sin^2\theta)} = \frac{\sin\theta(3\cos^2\theta - 1)}{\cos\theta(1 - 3\sin^2\theta)} = \frac{\sin\theta(3\cos^2\theta - 1)}{\cos\theta(3\cos^2\theta - 2)}.$$

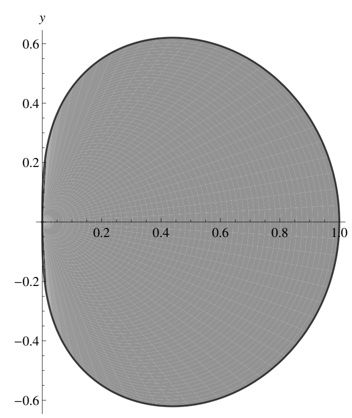
The numerator is 0 for $\theta = 0$ and for $\theta = \pm\cos^{-1}(\pm\sqrt{3}/3)$, so there are horizontal tangents at the corresponding points $(0, 0)$, $(0.943, 0.955)$, $(-0.943, 2.186)$, $(0.943, 4.097)$, and $(-0.943, 5.328)$. The denominator is 0 for $\theta = \pi/2$ and $3\pi/2$, and for $\theta = \pm\cos^{-1}(\pm\sqrt{6}/3)$, so there are vertical tangents at $(0, 0)$, $(0.943, 0.615)$, $(-0.943, 2.526)$, $(0.943, 3.757)$, and $(-0.943, 5.668)$.

10.3.18 The curve intersects the origin at $\theta = 7\pi/6$ and $\theta = 11\pi/6$, so those don't give rise to vertical or horizontal tangents. We have $\frac{dy}{dx} = \frac{6\cos\theta\sin\theta+(3+6\sin\theta)\cos\theta}{6\cos\theta\cos\theta-(3+6\sin\theta)\sin\theta} = \frac{\cos\theta(1+4\sin\theta)}{(2-\sin\theta-4\sin^2\theta)}$. Thus there are horizontal tangents for $\theta = \pi/2$ and $3\pi/2$, at the corresponding points $(9, \pi/2)$ and $(-3, 3\pi/2)$, and at the points where $\sin(\theta) = -1/4$, which are $(3/2, 3.394)$ and $(3/2, 6.031)$. There are vertical tangents where the denominator is 0, which occurs for $\theta = \sin^{-1}\left(-\frac{1}{8} \pm \frac{\sqrt{33}}{8}\right)$, so the corresponding points are $(-2.06, 5.28)$, and $(6.56, .634)$.

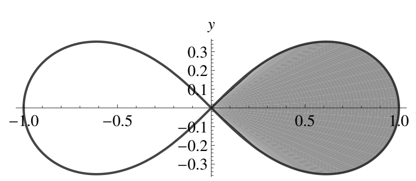
10.3.19 The curve intersects the origin at $\theta = \pi/2$, and there is a vertical tangent at $(0, \pi/2)$. $\frac{dy}{dx} = \frac{-\cos\theta\sin\theta+(1-\sin\theta)\cos\theta}{-\cos^2\theta-(1-\sin\theta)\sin\theta} = \frac{\cos\theta(1-2\sin\theta)}{\sin^2\theta-\cos^2\theta-\sin\theta} = \frac{\cos\theta(1-2\sin\theta)}{2\sin^2\theta-\sin\theta-1}$. There are horizontal tangents when $\sin\theta = 1/2$, which occurs for $\theta = \pi/6, 5\pi/6$, and when $\cos\theta = 0$ (but not $\sin\theta = 1$) which occurs at $\theta = 3\pi/2$. So the horizontal tangents are at $(1/2, \pi/6)$, $((1/2, 5\pi/6)$, and $(2, 3\pi/2)$. There are vertical tangents when $2\sin^2\theta - \sin\theta - 1 = (2\sin\theta + 1)(\sin\theta - 1) = 0$, or $\theta = 7\pi/6$ and $\theta = 11\pi/6$. The vertical tangents are thus at $(3/2, 7\pi/6)$, $(3/2, 11\pi/6)$, and $(0, \pi/2)$, as well as the aforementioned $(0, \pi/2)$.

10.3.20 Note that this curve is actually the vertical line $x = 1$, so it has no horizontal tangents, and a vertical tangent at every θ, so at $(\sec\theta, \theta)$ for every θ.

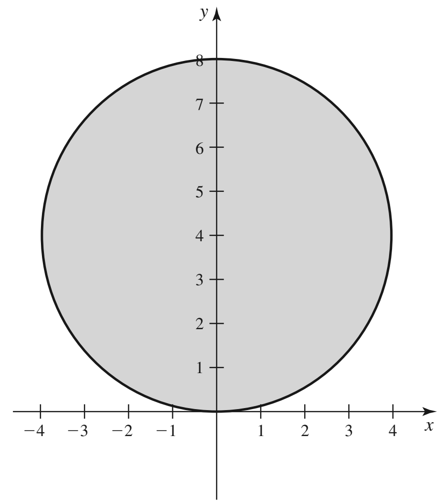
10.3.21 $A = 2 \cdot \frac{1}{2}\int_0^{\pi/2} \cos\theta\, d\theta = \sin\theta\Big|_0^{\pi/2} = 1.$



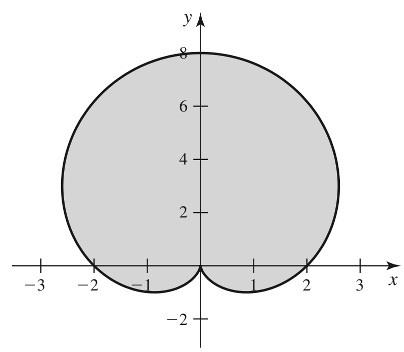
10.3.22 $A = 2 \cdot \frac{1}{2}\int_0^{\pi/4} \cos 2\theta\, d\theta = \frac{1}{2}\left(\sin 2\theta\right)\Big|_0^{\pi/4} = \frac{1}{2}.$



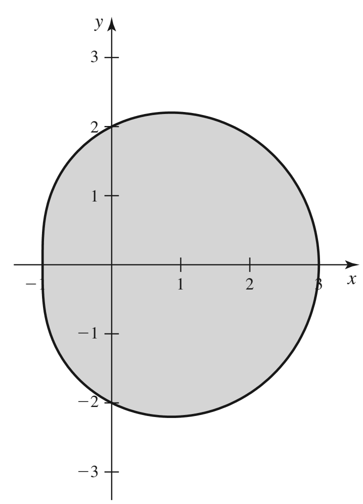
10.3.23 $A = \frac{1}{2}\int_0^{\pi}(8\sin\theta)^2\,d\theta = 32\int_0^{\pi}\sin^2\theta\,d\theta = 32\int_0^{\pi}\frac{1-\cos 2\theta}{2}\,d\theta = 32\left(\frac{1}{2}\theta - \frac{\sin\theta\cos\theta}{2}\right)\Big|_0^{\pi} = 16\pi.$



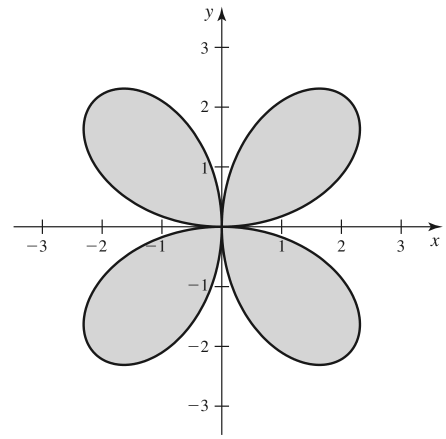
10.3.24
$$A = \frac{1}{2}\int_0^{2\pi}(4+4\sin\theta)^2\,d\theta$$
$$= 8\int_0^{2\pi}(1+2\sin\theta+\sin^2\theta)\,d\theta$$
$$= 8\left(\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{\sin\theta\cos\theta}{2}\right)\Big|_0^{2\pi} = 24\pi.$$



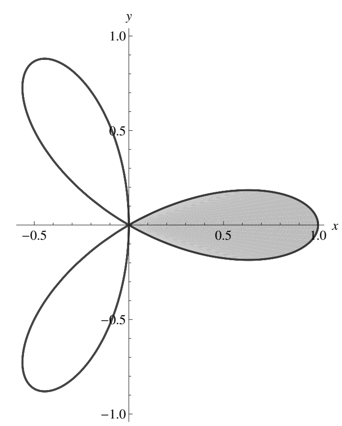
10.3.25 Using symmetry, we have $\frac{1}{2}\cdot 2\int_0^{\pi}(2+\cos\theta)^2\,d\theta = \int_0^{\pi}(4+4\cos\theta+\cos^2\theta)\,d\theta = \left(4\theta + 4\sin\theta + \frac{1}{2}\theta + \frac{\sin\theta\cos\theta}{2}\right)\Big|_0^{\pi} = \frac{9\pi}{2}.$



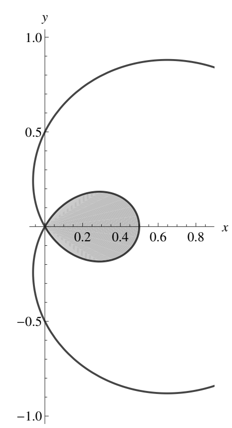
10.3.26 Because there are 4 symmetric leaves, we compute the area of 1/2 of one of the leaves, and then multiply by 8 to get the total area. We have $\frac{1}{2}\int_0^{\pi/4} 9\sin^2(2\theta)\,d\theta = \frac{9}{2}\int_0^{\pi/4}\sin^2(2\theta)\,d\theta = \frac{9}{4}\left(\theta - \frac{\sin 2\theta\cos 2\theta}{2}\right)\Big|_0^{\pi/4} = \frac{9\pi}{16}$. So the total area is $8\cdot\frac{9\pi}{16} = \frac{9\pi}{2}$.



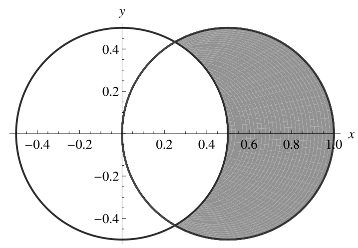
10.3.27 $2\cdot\frac{1}{2}\int_0^{\pi/6}\cos^2 3\theta\,d\theta = \int_0^{\pi/6}\frac{1+\cos 6\theta}{2}\,d\theta = \left(\theta/2 + \frac{\sin 6\theta}{6}\right)\Big|_0^{\pi/6} = \frac{\pi}{12}$.



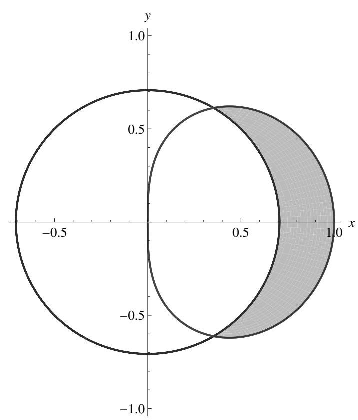
10.3.28 $2\cdot\frac{1}{2}\int_0^{\pi/3}(\cos\theta - 1/2)^2\,d\theta = \int_0^{\pi/3}(\cos^2\theta - \cos\theta + 1/4)\,d\theta = \int_0^{\pi/3}(\cos(2\theta)/2 - \cos\theta + 3/4)\,d\theta = (\sin(2\theta)/4 - \sin\theta + 3\theta/4)\Big|_0^{\pi/3} = \sqrt{3}/8 - \sqrt{3}/2 + \pi/4 = \pi/4 - \frac{3\sqrt{3}}{8}$.



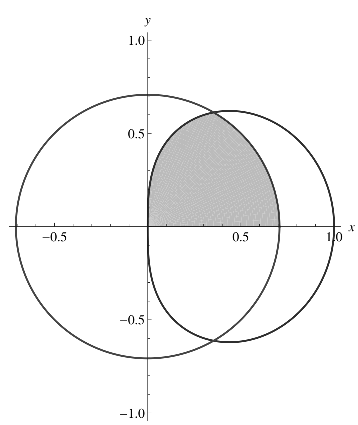
10.3.29 The area is given by $2 \cdot \frac{1}{2}\int_0^{\pi/3}(\cos^2\theta - (1/2)^2)\,d\theta = \int_0^{\pi/3}(\cos(2\theta)/2 + \frac{1}{4})\,d\theta = (\sin(2\theta)/4 + \theta/4)\Big|_0^{\pi/3} = \frac{\sqrt{3}}{8} + \frac{\pi}{12}$.



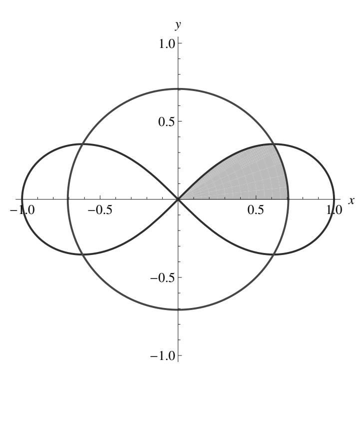
10.3.30 We have already computed the area inside $\sqrt{\cos\theta}$ to be 1. Now we must take away the portion of the circle with radius $1/\sqrt{2}$ between $\theta = -\pi/3$ and $\theta = \pi/3$. This is 1/3 of a circle, so the area being removed is $(1/3)\pi(1/2) = \pi/6$. We must also remove the area of the regions inside $\sqrt{\cos\theta}$ between $-\pi/2$ and $-\pi/3$ and $\pi/3$ and $\pi/2$. These have area $2 \cdot \frac{1}{2}\int_{\pi/3}^{\pi/2}\cos\theta\,d\theta = (\sin\theta)\Big|_{\pi/3}^{\pi/2} = 1 - \sqrt{3}/2$. So the area is $1 - (\pi/6 + (1 - \sqrt{3}/2)) = \sqrt{3}/2 - \pi/6$.



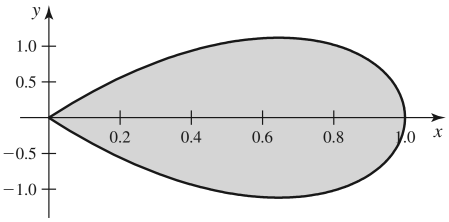
10.3.31 The region inside the circle between 0 and $\pi/3$ is 1/6 the area of a circle of radius $1/\sqrt{2}$ so it has area $(1/6)\pi(1/2) = \pi/12$. The rest of the area is represented by $\frac{1}{2}\int_{\pi/3}^{\pi}\cos\theta\,d\theta = \frac{1}{2}(\sin\theta)\Big|_{\pi/3}^{\pi} = \frac{1}{2}\left(1 - \sqrt{3}/2\right)$. The total area is therefore $\frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}$.



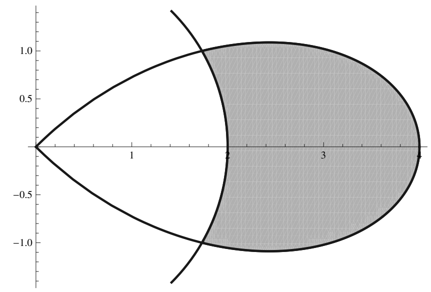
10.3.32 The region inside the circle between 0 and $\pi/6$ is 1/12 the area of a circle of radius $1/\sqrt{2}$ so it has area $(1/12)\pi(1/2) = \pi/24$. The rest of the area is represented by $\frac{1}{2}\int_{\pi/6}^{\pi/4}\cos 2\theta\,d\theta = \frac{1}{4}(\sin 2\theta)\Big|_{\pi/6}^{\pi/4} = \frac{1}{4}\left(1 - \sqrt{3}/2\right)$. The total area is thus $\frac{\pi}{24} + \frac{1}{4} - \frac{\sqrt{3}}{8}$.



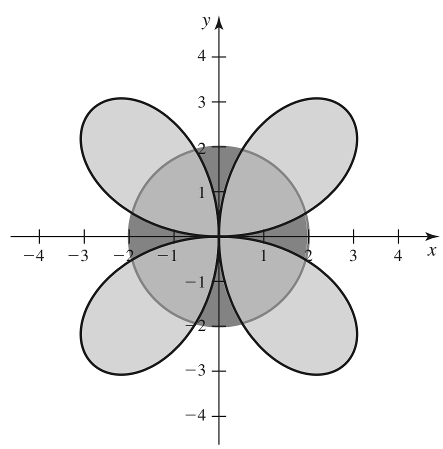
10.3.33 Using symmetry, we compute the area of 1/2 of one leaf, and then double it. We have $A = \frac{1}{2}\int_0^{\pi/10}\cos^2(5\theta)\,d\theta = \frac{1}{10}\int_0^{\pi/2}\cos^2 u\,du = \frac{1}{10}\left(\frac{1}{2}u + \frac{\cos u \sin u}{2}\right)\Big|_0^{\pi/2} = \frac{\pi}{40}$. So the area of one leaf is $2\cdot\frac{\pi}{40} = \frac{\pi}{20}$.



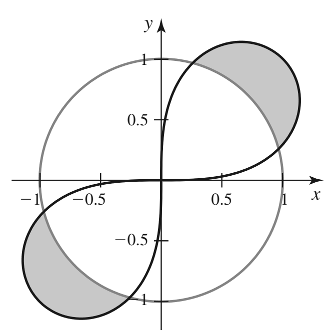
10.3.34 The curves intersect where $4\cos 2\theta = 2$, or $\theta = \pi/6$. By symmetry, we can compute the area of 1/2 of the tip of one leaf, and then multiply by 8. The area of 1/2 of the tip of one leaf is given by $\frac{1}{2}\int_0^{\pi/6}(4\cos(2\theta)^2 - 4)\,d\theta = \int_0^{\pi/6}(8\cos^2(2\theta) - 2)\,d\theta = (4\theta + \sin(4\theta) - 2\theta)\Big|_0^{\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$. Thus the total area desired is $8\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) = \frac{8\pi}{3} + 4\sqrt{3}$.



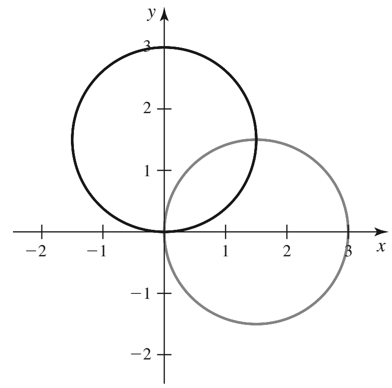
10.3.35 Note that the area inside one leaf of the rose but outside the circle is given by $\frac{1}{2}\int_{\pi/12}^{5\pi/12}(16\sin^2(2\theta) - 4)\,d\theta = (2\theta - \sin(4\theta))\Big|_{\pi/12}^{5\pi/12} = \sqrt{3} + \frac{2\pi}{3}$. Also, the area inside one leaf of the rose is $\frac{1}{2}\int_0^{\pi/2} 16\sin^2(2\theta)\,d\theta = (4\theta - \sin(4\theta))\Big|_0^{\pi/2} = 2\pi$. Thus the area inside one leaf of the rose and inside the circle must be $2\pi - (\sqrt{3} + \frac{2\pi}{3}) = \frac{4\pi}{3} - \sqrt{3}$, and the total area inside the rose and inside the circle must be $4(\frac{4\pi}{3} - \sqrt{3}) = \frac{16\pi}{3} - 4\sqrt{3}$.



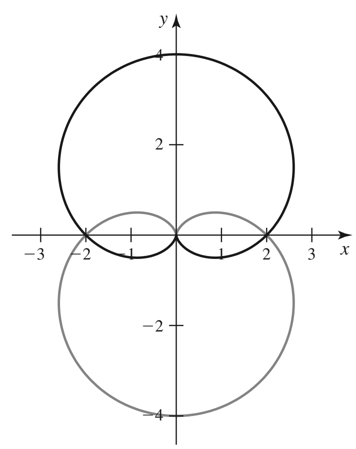
10.3.36 The curves intersect for $2\sin(2\theta) = 1$, which occurs in the first quadrant at $\theta = \pi/12$ and $\theta = 5\pi/12$. So one half of the total desired area is given by $\frac{1}{2}\int_{\pi/12}^{5\pi/12}(2\sin(2\theta) - 1)\,d\theta = \frac{1}{2}(-\cos(2\theta) - \theta)\Big|_{\pi/12}^{5\pi/12} = -\frac{1}{2}\left(-\sqrt{3} + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$. So the total desired area is $\sqrt{3} - \frac{\pi}{3}$.



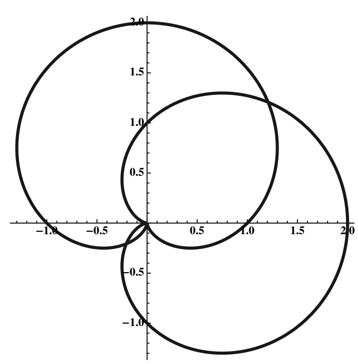
10.3.37 These curves intersect when $\sin\theta = \cos\theta$, which occurs at $\theta = \pi/4$ and $\theta = 5\pi/4$, and when $r = 0$ which occurs for $\theta = 0$ and $\theta = \pi$ for the first curve and $\theta = \pi/2$ and $\theta = 3\pi/2$ for the second curve. Only two of these intersection points are unique: the origin and the point $(3\sqrt{2}/2, \pi/4) = (-3\sqrt{2}/2, 5\pi/4)$.



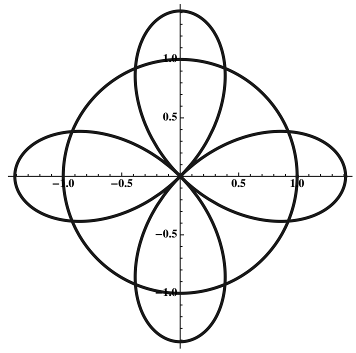
10.3.38 The curves intersect where $2 + 2\sin\theta = 2 - 2\sin\theta$, which occurs when $\sin\theta = 0$. The curves also intersect at the origin, which occurs for the first curve at $\theta = 3\pi/2$ and for the second curve at $\pi/2$. The only points of intersection are the origin, $(2, 0)$ and $(2, \pi)$.



10.3.39 The curves intersect when $\sin\theta = \cos\theta$, which occurs for $\theta = \pi/4$ and $\theta = 5\pi/4$. The corresponding points are $\left(\frac{2+\sqrt{2}}{2}, \frac{\pi}{4}\right)$ and $\left(\frac{2-\sqrt{2}}{2}, \frac{5\pi}{4}\right)$. They also intersect at the pole: the first curve is at the pole at $(0, \pi)$ and the other at $(0, 3\pi/2)$.



10.3.40 These curves intersect when $\cos(2\theta) = \frac{\sqrt{2}}{2}$, which occurs when $2\theta = \pi/4, 7\pi/4, \ldots$, so for $\theta = \pi/8, 7\pi/8, \ldots$. The intersection points are thus $(1, \pi/8)$, $(1, 7\pi/8)$, $(1, 9\pi/8)$, $(1, 15\pi/8)$, $(1, 17\pi/8)$, $(1, 23\pi/8)$, $(1, 25\pi/8)$, and $((1, 31\pi/8)$.



10.3.41 By symmetry, we need to compute the area inside $r = 3\sin\theta$ between 0 and $\pi/4$ and then double that result. We have $2 \cdot \frac{1}{2}\int_0^{\pi/4} 9\sin^2\theta\, d\theta = \frac{9}{2}\int_0^{\pi/4}(1-\cos(2\theta))\, d\theta = \frac{9}{2}\left(\theta - (1/2)\sin(2\theta)\right)\Big|_0^{\pi/4} = \frac{9}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{9}{8}(\pi - 2)$.

10.3.42 By symmetry, we need to compute the area of the region inside $r = 2 - 2\sin\theta$ between 0 and $\pi/2$ and then quadruple it. We have $4 \cdot \frac{1}{2}\int_0^{\pi/2}(2 - 2\sin\theta)^2\, d\theta = 8\int_0^{\pi/2}(1 - 2\sin\theta + \sin^2\theta)\, d\theta = 8\int_0^{\pi/2}((3/2) - 2\sin\theta - (1/2)\cos 2\theta)\, d\theta = (12\theta + 16\cos\theta - 2\sin(2\theta))\Big|_0^{\pi/2} = 6\pi + 0 - 0 - (0 + 16 - 0) = 6\pi - 16$.

10.3.43 By symmetry, we can compute the area between $\pi/4$ and $5\pi/4$ inside $r = 1+\cos\theta$ and then double it. This will include both the bigger and smaller enclosed regions. We have $2 \cdot \frac{1}{2}\int_{\pi/4}^{5\pi/4}(1+\cos\theta)^2\, d\theta = \int_{\pi/4}^{5\pi/4}(1 + 2\cos\theta + (1/2)(1+\cos(2\theta)))\, d\theta = \int_{\pi/4}^{5\pi/4}((3/2) + 2\cos\theta + (1/2)(\cos 2\theta))\, d\theta = (3\theta/2 + 2\sin\theta + (1/4)\sin 2\theta)\Big|_{\pi/4}^{5\pi/4}$ $= \left(\frac{15\pi}{8} - \sqrt{2} + \frac{1}{4}\right) - \left(\frac{3\pi}{8} + \sqrt{2} + \frac{1}{4}\right) = \frac{3\pi}{2} - 2\sqrt{2}$.

10.3.44 By symmetry, we can compute the area between 0 and $\pi/8$ within the circle $r = 1$ and add it to the area between $\pi/8$ and $\pi/4$ within the curve $\sqrt{2}\cos 2\theta$ and then multiply this by 8. The area within the circle between 0 and $\pi/8$ is (1/16)th the area of the circle, so this area is $\frac{\pi}{16}$. The area between $\pi/8$ and $\pi/4$ within $\sqrt{2}\cos 2\theta$ is given by $\frac{1}{2}\int_{\pi/8}^{\pi/4} 2\cos^2 2\theta\, d\theta = \frac{1}{2}\int_{\pi/8}^{\pi/4}(1 + \cos 4\theta)\, d\theta = \frac{1}{2}\left(\theta + (1/4)\sin 4\theta\right)\Big|_{\pi/8}^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4} + 0 - \left(\frac{\pi}{8} + \frac{1}{4}\right)\right) = \frac{\pi}{16} - \frac{1}{8}$. Adding this to the previously computed area gives that 1/8 of the total area is $\frac{\pi}{16} - \frac{1}{8} + \frac{\pi}{16} = \frac{\pi}{8} - \frac{1}{8}$. Thus the total area we are seeking is $\pi - 1$.

10.3.45

a. False. The area is given by $\frac{1}{2}\int_\alpha^\beta f(\theta)^2\, d\theta$.

b. False. The slope is given by $\frac{dy}{dx}$, which can be computed using the formula

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

10.3.46 The polar point $(-1, 3\pi/2)$ is equivalent to the polar point $(1, \pi/2)$ which does satisfy the equation.

10.3.47 The circles intersect for $\theta = \pi/6$ and $\theta = 5\pi/6$.

The area inside $r = 2\sin\theta$ but outside of $r = 1$ would be given by $\frac{1}{2}\int_{\pi/6}^{5\pi/6}(4\sin^2\theta - 1)\, d\theta = \frac{1}{2}\left(x - \sin(2x)\right)\Big|_{\pi/6}^{5\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$. The total area of $r = 2\sin\theta$ is π. Thus, the area inside both circles is $\pi - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.

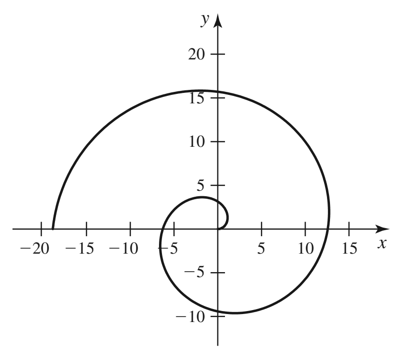
10.3.48 The inner loop is traced from $\theta = 2\pi/3$ to $\theta = 4\pi/3$. So the area is given by $\frac{1}{2}\int_{2\pi/3}^{4\pi/3}(2+4\cos\theta)^2\,d\theta = \int_{2\pi/3}^{4\pi/3}(2+8\cos\theta+8\cos^2\theta)\,d\theta = \left(2\theta+8\sin\theta+4\theta+2\sin(2\theta)\right)\Big|_{2\pi/3}^{4\pi/3} = 4\pi - 6\sqrt{3}$.

10.3.49 The inner loop is traced out between $\theta = \pi/6$ and $\theta = 5\pi/6$, so its area is given by $\frac{1}{2}\int_{\pi/6}^{5\pi/6}(3 - 6\sin\theta)^2\,d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6}(9 - 36\sin\theta + 36\sin^2\theta)\,d\theta = \frac{3}{2}\left(3\theta + 12\cos\theta + 6\theta - 3\sin(2\theta)\right)\Big|_{\pi/6}^{5\pi/6} = 9\pi - \frac{27\sqrt{3}}{2}$.

We can determine the area inside the outer loop by using symmetry and doubling the area of the region traced out between $5\pi/6$ and $3\pi/2$. Thus the area inside the outer region is $2\cdot\frac{1}{2}\int_{5\pi/6}^{3\pi/2}(3-6\sin\theta)^2\,d\theta = 3\left(3\theta + 12\cos\theta + 6\theta - 3\sin(2\theta)\right)\Big|_{5\pi/6}^{3\pi/2} = 18\pi + \frac{27\sqrt{3}}{2}$. So the area outside the inner loop and inside the outer loop is $18\pi + \frac{27\sqrt{3}}{2} - \left(9\pi - \frac{27\sqrt{3}}{2}\right) = 9\pi + 27\sqrt{3}$.

10.3.50 The curves intersect at $\theta = \pi/3$, and using symmetry, the area we seek is $2\cdot\frac{1}{2}\int_0^{\pi/3}(1+\cos\theta)^2\,d\theta + 2\cdot\frac{1}{2}\int_{\pi/3}^{\pi/2}(3\cos\theta)^2\,d\theta = \int_0^{\pi/3}(1+2\cos\theta+\cos^2\theta)\,d\theta + \int_{\pi/3}^{\pi/2}9\cos^2\theta\,d\theta = \left(\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{\pi/3} + \left(\frac{9\theta}{2} + \frac{9\sin 2\theta}{4}\right)\Big|_{\pi/3}^{\pi/2} = \frac{\pi}{2} + 2\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8} + \frac{9\pi}{4} - \left(\frac{3\pi}{2} + \frac{9\sqrt{3}}{8}\right) = \frac{5\pi}{4}$.

10.3.51 The first horizontal tangent line is at the origin. The next is at approximately $(4.0576, 2.0288)$, and the third at approximately $(9.8262, 4.9131)$. The first vertical tangent line is at approximately $(1.7206, 0.8603)$, the next is at about $(6.8512, 3.4256)$, and the next at approximately $(12.8746, 6.4373)$.



10.3.52

a. The area of one half of one leaf is $\frac{1}{2}\int_0^{\pi/(4m)}\cos^2(2m\theta)\,d\theta = \left(\frac{\theta}{4} + \frac{\sin(4m\theta)}{16m}\right)\Big|_0^{\pi/(4m)} = \frac{\pi}{16m}$. So the area of all $8m$ half-leaves is $\frac{\pi}{2}$.

b. The area of one half of one leaf is $\frac{1}{2}\int_0^{\pi/(4m+2)}\cos^2((2m+1)\theta)\,d\theta = \left(\frac{\theta}{4} + \frac{\sin(2(2m+1)\theta)}{4(4m+2)}\right)\Big|_0^{\pi/(4m+2)} = \frac{\pi}{8\cdot(2m+1)}$. So the area of all $2(2m+1)$ half-leaves is $\frac{\pi}{4}$.

10.3.53

a. $A_n = \frac{1}{2}\int_{(2n-2)\pi}^{(2n-1)\pi} e^{-2\theta}\,d\theta - \frac{1}{2}\int_{2n\pi}^{(2n+1)\pi} e^{-2\theta}\,d\theta = -\frac{1}{4}e^{-(4n-2)\pi} + \frac{1}{4}e^{-(4n-4)\pi} + \frac{1}{4}e^{-(4n+2)\pi} - \frac{1}{4}e^{-4n\pi}$.

b. Each term tends to 0 as $n \to \infty$ so $\lim_{n\to\infty} A_n = 0$.

c. $\dfrac{A_{n+1}}{A_n} = \dfrac{e^{-(4n+2)\pi} + e^{-(4n)\pi} + e^{-(4n+6)\pi} - e^{-(4n+4)\pi}}{e^{-(4n-2)\pi} + e^{-(4n-4)\pi} + e^{-(4n+2)\pi} - e^{-4n\pi}} = e^{-4\pi}$, so $\lim_{n\to\infty}\frac{A_{n+1}}{A_n} = e^{-4\pi}$.

10.3.54 The area of one half of one leaf is $\frac{1}{2}\int_0^{\pi/6} 4\cdot\cos^2(3\theta)\,d\theta = \left(\theta + \frac{\sin(6\theta)}{6}\right)\Big|_0^{\pi/6} = \frac{\pi}{6}$. So the area of all 6 half-leaves is π.

10.3.55 One half of the area is given by $\frac{1}{2}\int_0^{\pi/2} 6\sin 2\theta\,d\theta = -\frac{3}{2}\cos 2\theta\Big|_0^{\pi/2} = 3$, so the total area is 6.

10.3.56 By symmetry, we can compute the area between $\theta = 5\pi/6$ and $\theta = 3\pi/2$ and double it. Thus, the total area we seek is given by $\int_{5\pi/6}^{3\pi/2}(2-4\sin\theta)^2\,d\theta = \int_{5\pi/6}^{3\pi/2}(4-16\sin\theta+16\sin^2\theta)\,d\theta =$
$(4\theta + 16\cos\theta + 8\theta - 4\sin(2\theta))\Big|_{5\pi/6}^{3\pi/2} = 6\sqrt{3}+8\pi$.

10.3.57 The area is given by

$$\frac{1}{2}\int_0^{2\pi}(4-2\cos\theta)^2\,d\theta = \int_0^{2\pi}(8-8\cos\theta+2\cos^2\theta)\,d\theta = \left(8\theta - 8\sin\theta + \theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} = 18\pi.$$

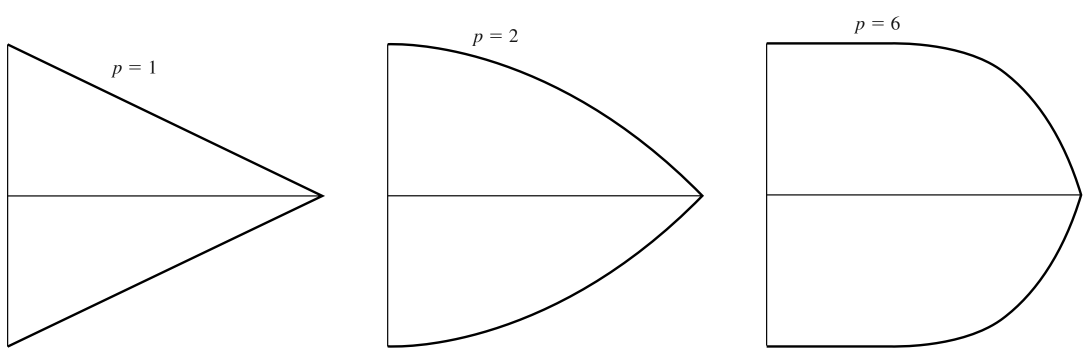
10.3.58

a. Because V and R are constants, the function is a parabola which opens downward with vertex at $(0, V)$, so the velocity is maximal when $r = 0$.

b. The average velocity is $\frac{1}{\pi R^2}\cdot 2\pi\cdot\int_0^R V\cdot\left(1-\frac{r^2}{R^2}\right)r\,dr = \frac{2\pi}{\pi R^2}V\left(-\frac{r^4}{4R^2}+\frac{r^2}{2}\right)\Big|_0^R = \frac{2}{R^2}\cdot\frac{VR^2}{4} = \frac{V}{2}$.

c. The average velocity is $\frac{1}{\pi R^2}\cdot 2\pi\cdot\int_0^R V\cdot\left(1-\frac{r^2}{R^2}\right)^{1/p} r\,dr = \frac{2V}{R^2}\cdot\frac{1}{R^{2/p}}\int_0^R (R^2-r^2)^{1/p}\cdot r\,dr = \frac{2V}{R^2}\cdot \frac{1}{R^{1/p}}\left(-\frac{p\left(R^2-r^2\right)^{\frac{1}{p}+1}}{2p+2}\right)\Big|_0^R = \frac{2V}{R^2}\cdot\frac{1}{R^{2/p}}\cdot\frac{p(R^2)^{(p+1)/p}}{2p+2} = \frac{2pV}{2p+2}$.

c. $\lim_{p\to\infty} V_{\text{avg}} = V\cdot\lim_{p\to\infty}\frac{2p}{2p+2} = V$.



10.3.59 Suppose that the goat is tethered at the origin, and that the center of the corral is $(1,\pi)$. The circle that the goat can graze is $r = a$, and the corral is given by $r = -2\cos\theta$. The intersection occurs for $\theta = \cos^{-1}(-a/2)$.

The area grazed by the goat is twice the area of the sector of the circle $r = a$ between $\cos^{-1}(-a/2)$ and π, plus twice the area of the circle $r = -2\cos\theta$ between $\pi/2$ and $\cos^{-1}(-a/2)$. Thus we need to compute $A = \int_{\cos^{-1}(-a/2)}^{\pi} a^2\,d\theta + \int_{\pi/2}^{\cos^{-1}(-a/2)} 4\cos^2\theta\,d\theta = a^2\pi - a^2\cos^{-1}(-a/2) + (2\cos\theta\sin\theta + 2\theta)\Big|_{\pi/2}^{\cos^{-1}(-a/2)} =$
$a^2(\pi - \cos^{-1}(-a/2)) - \pi - \frac{1}{2}a\sqrt{4-a^2} + 2\cos^{-1}(-a/2)$. Note that $\pi - \cos^{-1}(-a/2) = \cos^{-1}(a/2)$, so this can be written as $(a^2-2)\cos^{-1}(a/2) + \pi - \frac{1}{2}a\sqrt{4-a^2}$. Note that for $a = 0$ this is 0, and for $a = 2$, this is π, as desired.

10.3.60 Imagine that the boundary of the concrete slab is the fence from the previous problem Then the area the goat could graze in the previous problem becomes the area it can't graze in this problem. If the slab weren't there, the goat could graze a region of area πa^2. Thus, the goat can graze a region of area $\pi a^2 - \left((a^2-2)\cos^{-1}(a/2) + \pi - \frac{1}{2}a\sqrt{4-a^2}\right) = \pi(a^2-1) + \frac{1}{2}a\sqrt{4-a^2} + (2-a^2)\cos^{-1}(a/2)$. If $a = 0$, this quantity is 0, while if $a = 2$, this quantity is 3π.

10.3.61 Again, suppose that the goat is tethered at the origin, and that the center of the corral is $(1, \pi)$. The equation of the corral fence is given by $r = -2\cos\theta$. Note that to the right of the vertical line $\theta = \pi/2$, the goat can graze a half-circle of area $\pi a^2/2$. Also, there is a region in the 2nd quadrant and one in the 3rd quadrant of equal size that can also be grazed. Let this region have area A, so that the total area grazed will then be $\frac{\pi a^2}{2} + 2A$.

Imagine that the goat is walking "west" from the polar point $(a, \pi/2)$, and is keeping the rope taut until his whole rope is along the fence in the third quadrant. Let ϕ be the central angle angle from the origin to the polar point $(1, \pi)$ to the point on the fence that the goat's rope is touching as he makes this walk. When the goat is at $(a, \pi/2)$, we have $\phi = 0$. When the goat is all the way to the fence, we have $\phi = a$. Then length of the rope not along the fence is $a - \phi$. Thus, the value of A is $\frac{1}{2}\int_0^a (a-\phi)^2\,d\phi = \frac{1}{2}\left(a^2\phi - a\phi^2 + \frac{\phi^3}{3}\right)\Big|_0^a = \frac{a^3}{6}$.

Thus, the goat can graze a region of area $\frac{\pi a^2}{2} + \frac{a^3}{3}$.

10.3.62

a. The slope of the line tangent to $r = f(\theta)$ at P is $\left.\frac{dy}{dx}\right|_P$. Also, the slope of a line intersecting the x-axis at an angle α is $\tan\alpha$. (Note that in the picture, $\tan(\pi-\alpha) = -\tan(\alpha) = \frac{\text{rise}}{-\text{run}} =$ $-$slope of the tangent line.)

b. Draw a vertical line through P and let Q be the point where this line intersects the x-axis. Then in triangle OPQ we see $\tan\theta = \frac{y}{x}$.

c. Note that

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\tan\theta + \frac{f(\theta)}{f'(\theta)}}{1 - \frac{f(\theta)}{f'(\theta)}\tan\theta} = \tan\alpha.$$

Because $\alpha = \phi + \theta$ and $\tan(\phi+\theta) = \frac{\tan\theta + \tan\phi}{1 - \tan\phi\tan\theta}$, we see that $\tan\phi = \frac{f(\theta)}{f'(\theta)}$.

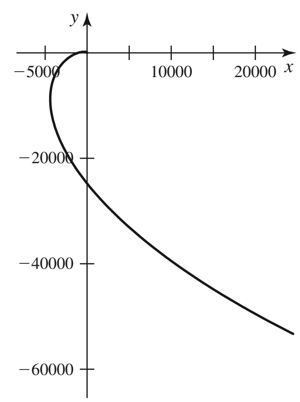
d. l is parallel to the x-axis when $\frac{dy}{dx} = 0$, or when $f'(\theta)\sin\theta + f(\theta)\cos\theta = 0$, hence if $\tan\theta = -\frac{f(\theta)}{f'(\theta)}$.

e. l is parallel to the y-axis when $\frac{dx}{dy} = 0$, which occurs when $f'(\theta)\cos\theta - f(\theta)\sin\theta = 0$, hence if $\tan\theta = \frac{f(\theta)}{f'(\theta)}$.

10.3.63

a. If $\cot\phi = \frac{f'(\theta)}{f(\theta)}$ is constant for all θ, then $\phi = \cot^{-1}\left(\frac{f'(\theta)}{f(\theta)}\right)$ is constant. Then $\frac{d}{d\theta}\ln(f(\theta)) = \frac{1}{f(\theta)} \cdot f'(\theta) =$ $\cot\phi$ is constant.

b. If $f(\theta) = Ce^{k\theta}$, then $\cot\phi = \frac{f'(\theta)}{f(\theta)} = \frac{kCe^{k\theta}}{Ce^{k\theta}} = k$.



c.

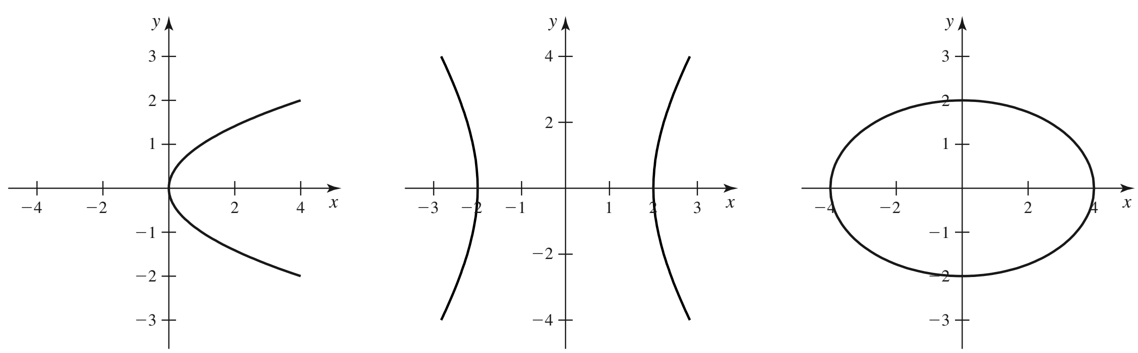
10.4 Conic Sections

10.4.1 A parabola is the set of points in the plane which are equidistant from a given fixed point and a given fixed line.

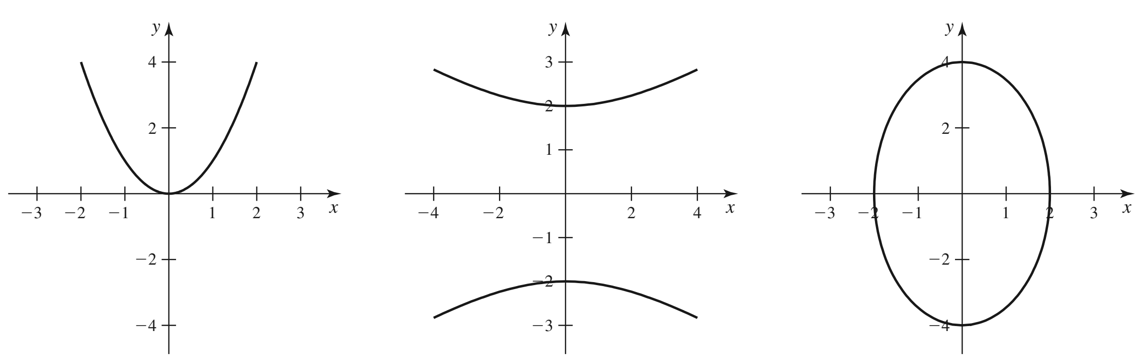
10.4.2 An ellipse is the set of points in the plane with the property that the sum of the distances from the point to two given fixed points is a given constant.

10.4.3 A hyperbola is the set of points in the plane with the property that the difference of the distances from the point to two given fixed points is a given constant.

10.4.4



10.4.5



10.4.6 $x^2 = 4py$, where $p < 0$.

10.4.7 $\left(\dfrac{x}{a}\right)^2 + \dfrac{y^2}{a^2 - c^2} = 1$.

10.4.8 $\left(\dfrac{y}{a}\right)^2 - \dfrac{x^2}{c^2 - a^2} = 1$.

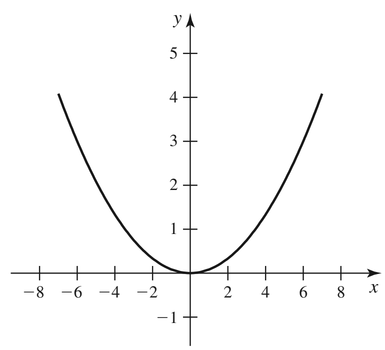
10.4.9 The foci for both are $(\pm ae, 0)$.

10.4.10 By theorem 11.4, this is given by $r = \frac{ed}{1+e\cos\theta}$, $-\pi < \theta < \pi$.

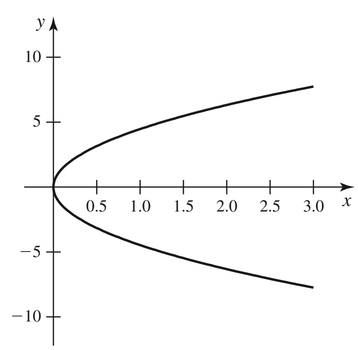
10.4.11 The asymptotes are $y = -\frac{b}{a} \cdot x$ and $y = \frac{b}{a} \cdot x$.

10.4.12 If $e = 1$, the conic section is a parabola. If $e > 1$, it is a hyperbola. If $0 < e < 1$, it is an ellipse.

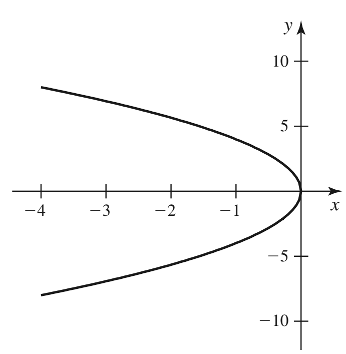
10.4.13 Directrix: $y = -3$. Focus: $(0, 3)$.



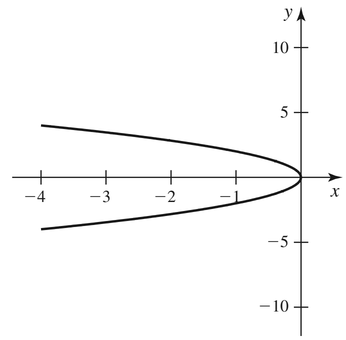
10.4.14 Directrix: $x = -5$. Focus: $(5, 0)$.



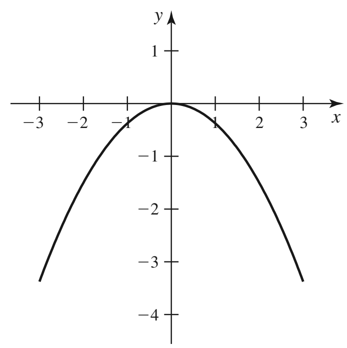
10.4.15 Directrix: $x = 4$. Focus: $(-4, 0)$.



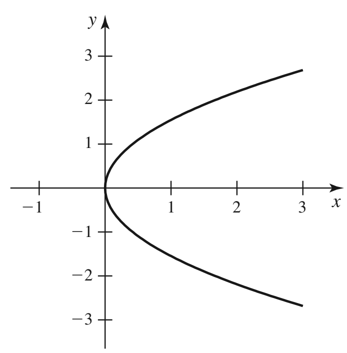
10.4.16 Directrix: $x = 1$. Focus: $(-1, 0)$.



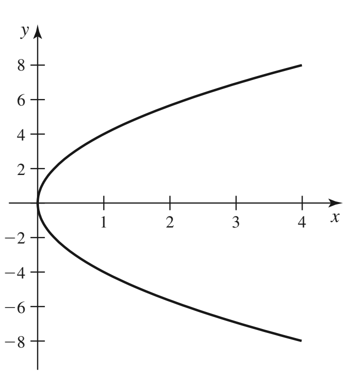
10.4.17 Directrix: $y = \frac{2}{3}$. Focus: $(0, -\frac{2}{3})$.



10.4.18 Directrix: $x = -\frac{3}{5}$. Focus: $(\frac{3}{5}, 0)$.

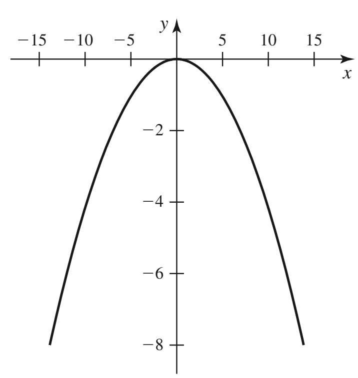


10.4.19



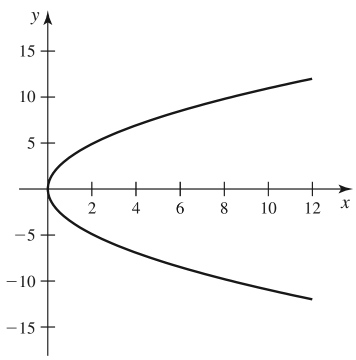
$y^2 = 16x$.

10.4.20



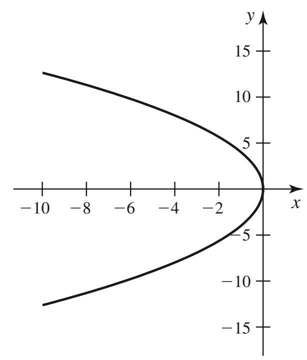
$x^2 = -24y$.

10.4.21



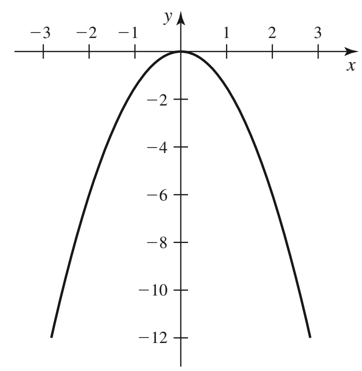
$y^2 = 12x$.

10.4.22

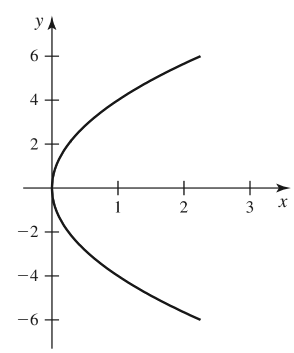


$y^2 = -16x$.

10.4.23 $x^2 = 4py$ and $4 = 4p(-6)$, so $p = -\frac{1}{6}$ and $x^2 = -\frac{2}{3} \cdot y$.



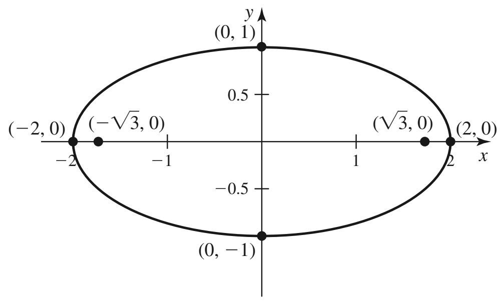
10.4.24 $y^2 = 4px$ and $(-4)^2 = 4p(1)$, so $p = 4$ and $y^2 = 16x$.



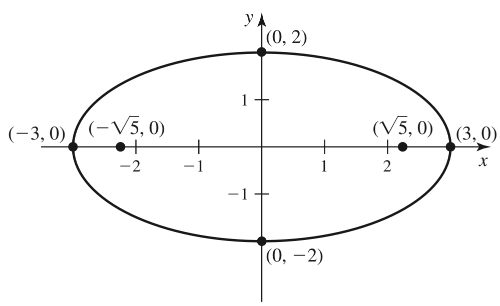
10.4.25 Because the vertex is $(-1, 0)$ and the parabola is symmetric about the x-axis, we have $y^2 = 4p(x+1)$ and because the directrix is one unit left of the vertex, we obtain $p = 1$ and $y^2 = 4(x + 1)$.

10.4.26 Because the vertex is $(0, 4)$ and the parabola is symmetric about the y-axis, we have $x^2 = 4p(y-2)$ and because the directrix is 2 units above the vertex, we obtain $p = -2$ and $x^2 = -8(y - 2)$.

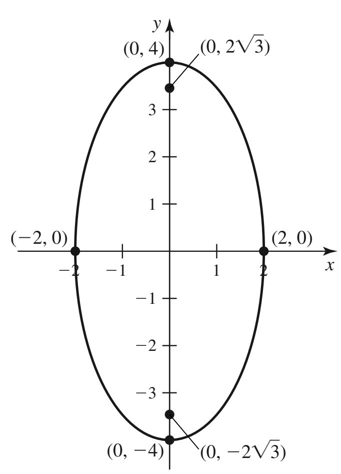
10.4.27 Vertices are $(\pm 2, 0)$, and the foci are $(\pm\sqrt{3}, 0)$. The major axis has length 4 and the minor axis has length 2.



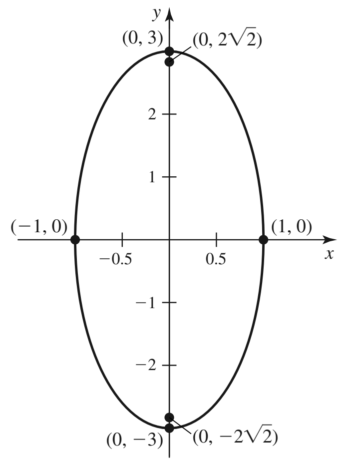
10.4.28 Vertices are $(\pm 3, 0)$, and the foci are $(\pm\sqrt{5}, 0)$. The major axis has length 6 and the minor axis has length 4.



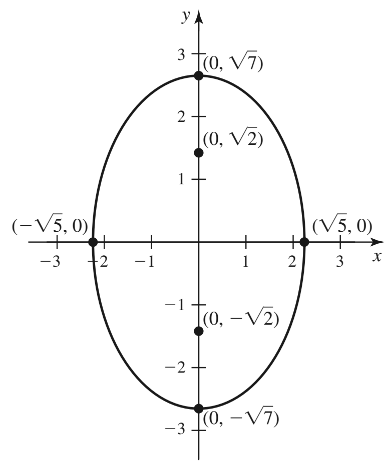
10.4.29 Vertices are $(0, \pm 4)$, and the foci are $(0, \pm 2\sqrt{3})$. The major axis has length 8 and the minor axis has length 4.



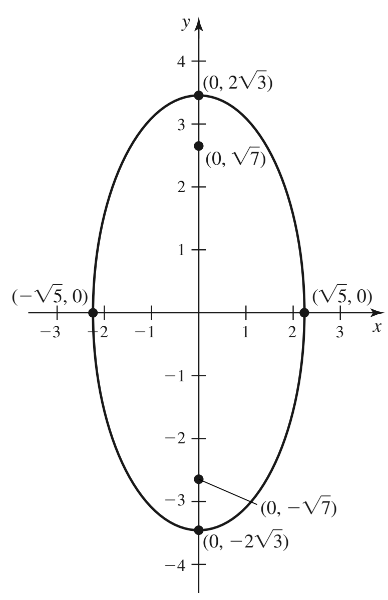
10.4.30 Vertices are $(0, \pm 3)$, and the foci are $(0, \pm 2\sqrt{2})$. The major axis has length 6 and the minor axis has length 2.



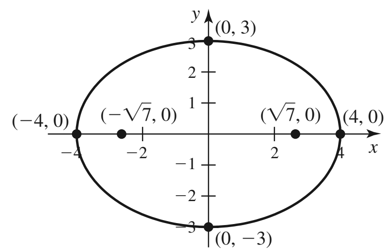
10.4.31 Vertices are $(0, \pm\sqrt{7})$, and the foci are $(0, \pm\sqrt{2})$. The major axis has length $2\sqrt{7}$ and the minor axis has length $2\sqrt{5}$.



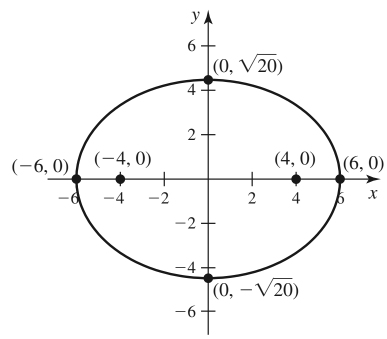
10.4.32 Vertices are $(0, \pm 2\sqrt{3})$, and the foci are $(0, \pm\sqrt{7})$. The major axis has length $4\sqrt{3}$ and the minor axis has length $2\sqrt{5}$.



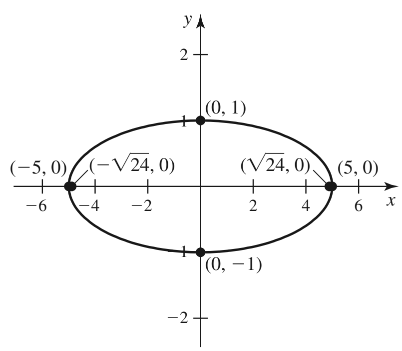
10.4.33 $a = 4$, and $b = 3$, so the equation is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.



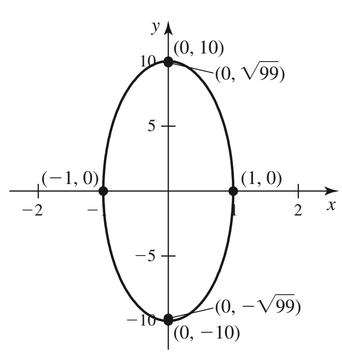
10.4.34 $a = 6$, and $a^2 = b^2 + c^2$ where $c = 4$, so $b^2 = 36 - 16 = 20$, and the equation is $\frac{x^2}{36} + \frac{y^2}{20} = 1$.



10.4.35 $a = 5$, and the equation is of the form $\frac{x^2}{25} + \frac{y^2}{b^2} = 1$. Because $(4, \frac{3}{5})$ is on the curve, we have $\frac{16}{25} + \frac{9}{25b^2} = 1$, so $b = 1$. The equation is $\frac{x^2}{25} + y^2 = 1$.



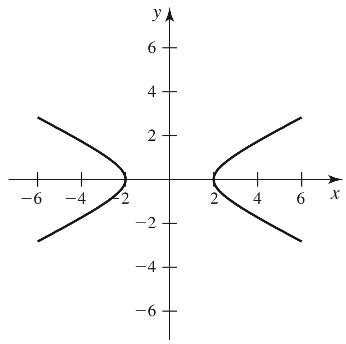
10.4.36 $a = 10$, and the equation is of the form $\frac{y^2}{100} + \frac{x^2}{b^2} = 1$, and because $(\sqrt{3}/2, 5)$ is on the curve, we have $\frac{1}{4} + \frac{3}{4b^2} = 1$, so $b = 1$. The equation is $x^2 + \frac{y^2}{100} = 1$.



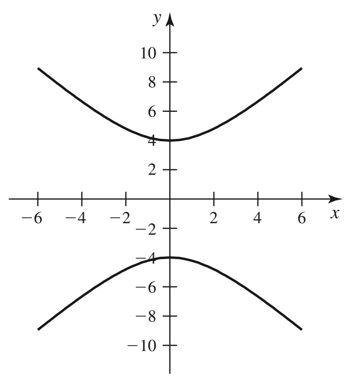
10.4.37 $a = 3$ and $b = 2$, so the equation is $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

10.4.38 $a = 10$ and $b = 8$, so $\frac{x^2}{100} + \frac{y^2}{64} = 1$.

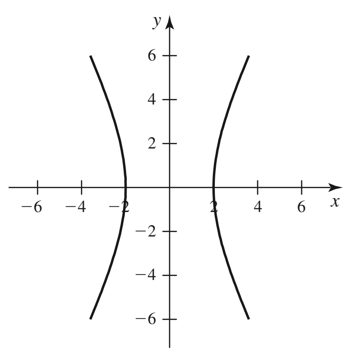
10.4.39 The vertices are $(\pm 2, 0)$, and the foci are $(\pm\sqrt{5}, 0)$. The asymptotes are $y = \frac{\pm 1}{2} \cdot x$.



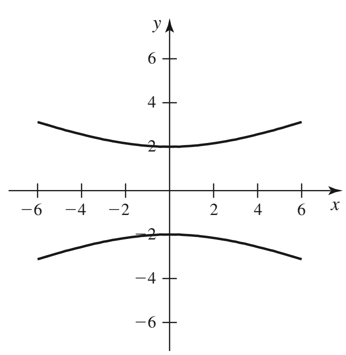
10.4.40 The vertices are $(0, \pm 4)$, and the foci are $(0, \pm 5)$. The asymptotes are $y = \frac{\pm 4}{3} \cdot x$.



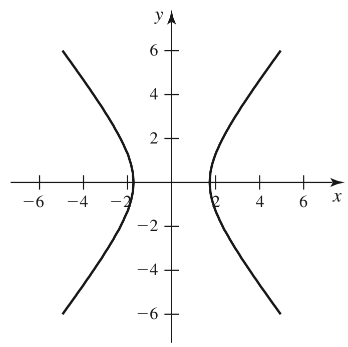
10.4.41 The vertices are $(\pm 2, 0)$, and the foci are $(\pm 2\sqrt{5}, 0)$. The asymptotes are $y = \pm 2x$.



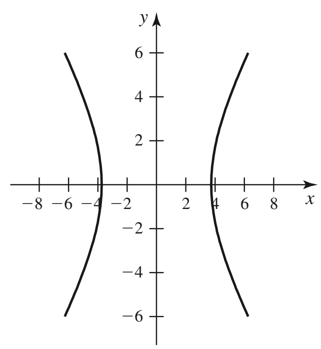
10.4.42 The vertices are $(0, \pm 2)$, and the foci are $(0, \pm 29)$. The asymptotes are $y = \frac{\pm 2}{5} \cdot x$.



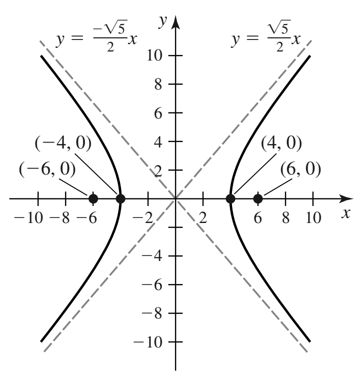
10.4.43 The vertices are $(\pm\sqrt{3}, 0)$, and the foci are $(\pm 2\sqrt{2}, 0)$. The asymptotes are $y = \pm\sqrt{\frac{5}{3}} \cdot x$.



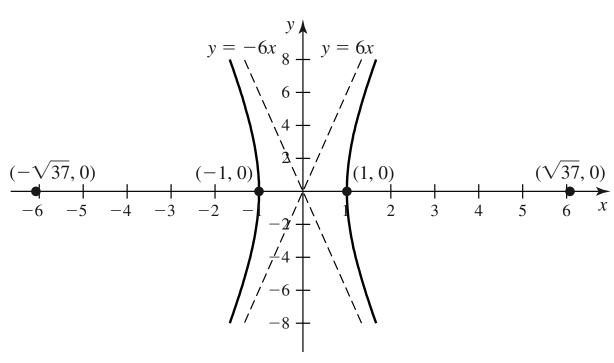
10.4.44 The vertices are $(\pm\sqrt{14}, 0)$, and the foci are $(\pm\sqrt{34}, 0)$. The asymptotes are $y = \pm\sqrt{\frac{10}{7}} \cdot x$.



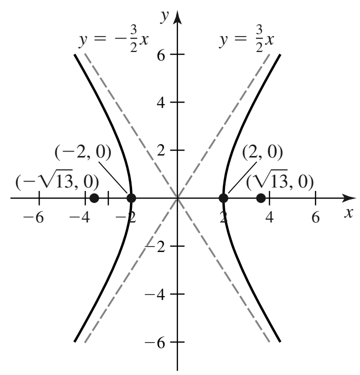
10.4.45 We have $a = 4$ and $c = 6$, so $b^2 = c^2 - a^2 = 20$, so the equation is $\frac{x^2}{16} - \frac{y^2}{20} = 1$.



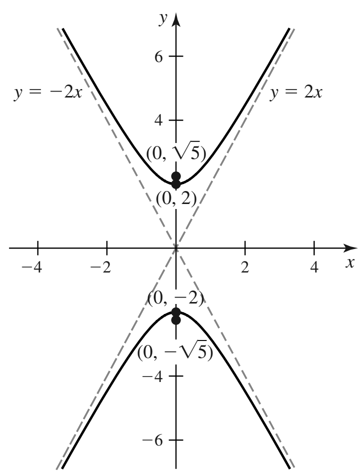
10.4.46 We have $a = 1$, so the equation is of the form $x^2 - \frac{y^2}{b^2} = 1$. Because $(5/3, 8)$ is on the curve, we have $\frac{25}{9} - \frac{64}{b^2} = 1$, so $b = 6$. The equation is $x^2 - \frac{y^2}{36} = 1$.



10.4.47 We have $a = 2$, and because the asymptoes are $y = \frac{\pm bx}{a}$, we have that $b = 3$, so the equation is $\frac{x^2}{4} - \frac{y^2}{9} = 1$.



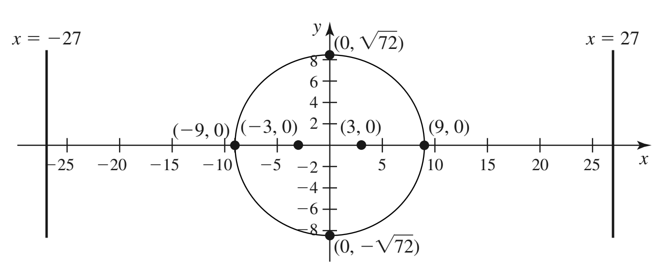
10.4.48 We have $a = 2$ and because the asymptotes are $y = \frac{\pm a}{b} \cdot x$, we have $b = 1$, and the equation is $\frac{y^2}{4} - x^2 = 1$.



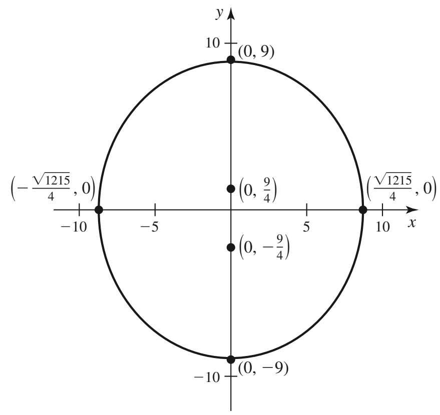
10.4.49 We have $a = 4$ and $c = 5$, so $b^2 = 25 - 16 = 9$, so $b = 3$ and the equation is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

10.4.50 We have $a = 6$ and $c = 10$, and $b^2 = 100 - 36 = 64$, so $b = 8$, and the equation is $\frac{y^2}{36} - \frac{x^2}{64} = 1$.

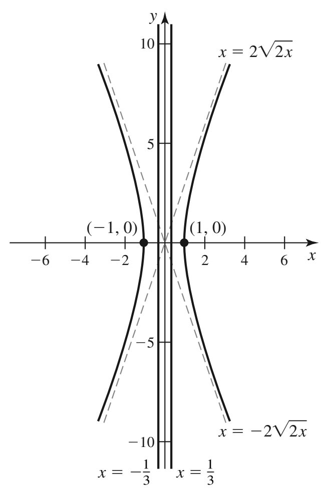
10.4.51 We have $a = 9$ and $e = \frac{1}{3}$, so $c = ae = 3$, and $b^2 = a^2 - c^2 = 72$, so the equation is $\frac{x^2}{81} + \frac{y^2}{72} = 1$.



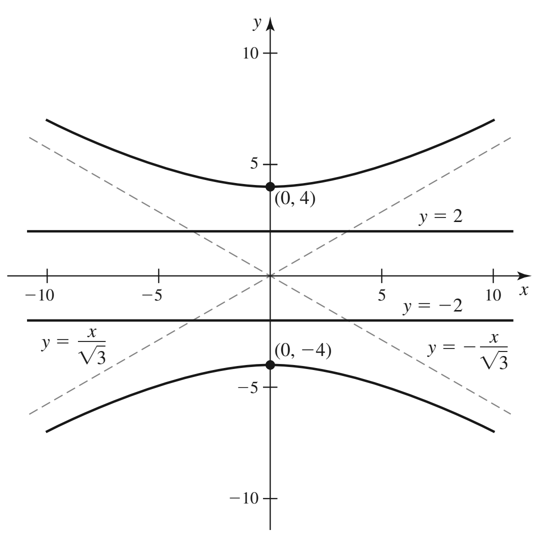
10.4.52 We have $a = 9$ and $e = \frac{1}{4}$, so $c = ae = \frac{9}{4}$, and $b^2 = a^2 - c^2 = 81 - \frac{81}{16} = \frac{1215}{16}$. Thus the equation is $\frac{16x^2}{1215} + \frac{y^2}{81} = 1$.



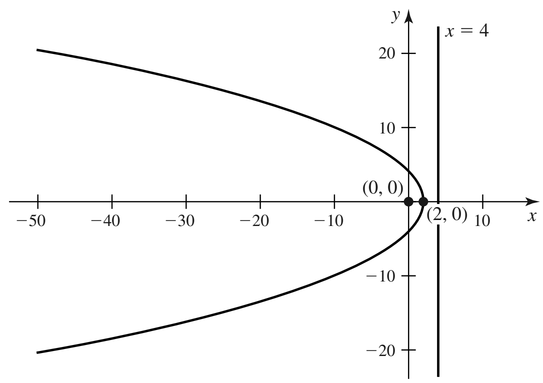
10.4.53 We have $a = 1$ and $e = 3$, so $c = ae = 3$ and $b^2 = c^2 - a^2 = 9 - 1 = 8$. Thus, the equation is $x^2 - \frac{y^2}{8} = 1$.



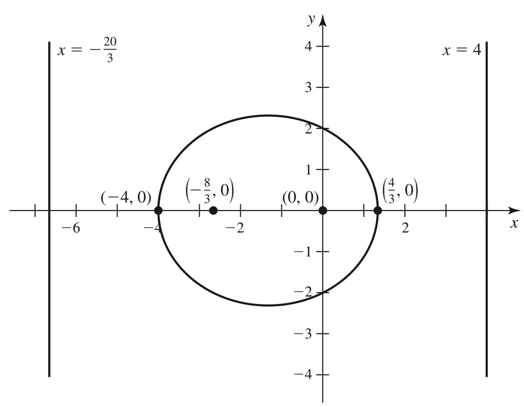
10.4.54 We have $a = 4$ and $e = 2$, so $c = ae = 8$ and $b^2 = c^2 - a^2 = 64 - 16 = 48$. Thus, the equation is $\frac{y^2}{16} - \frac{x^2}{48} = 1$.



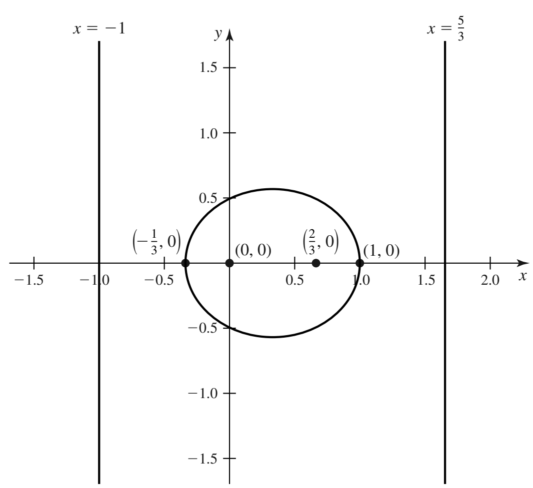
10.4.55 The vertex is $(2, 0)$. The focus is $(0, 0)$, and the directrix is the line $x = 4$.



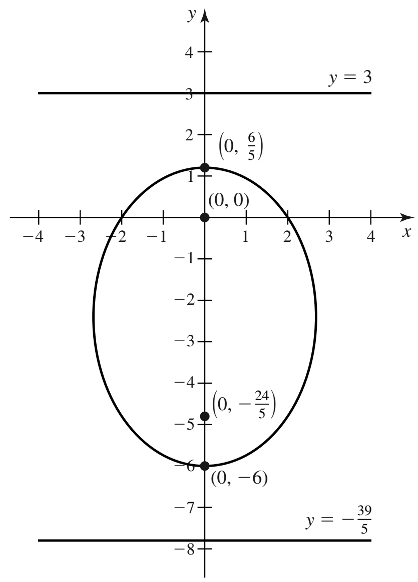
10.4.56 The vertices are $(4/3, 0)$ and $(-4, 0)$. The center is $(-4/3, 0)$. The foci are $(0, 0)$ and $(-8/3, 0)$.



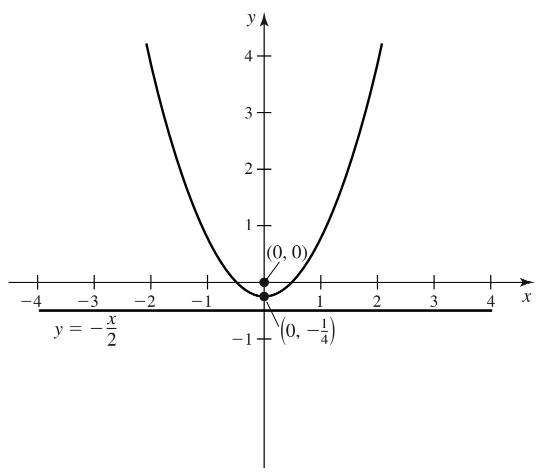
10.4.57 The vertices are $(1, 0)$ and $(-1/3, 0)$. The center is $(1/3, 0)$. The directrices are $x = -1$ and $x = 5/3$.



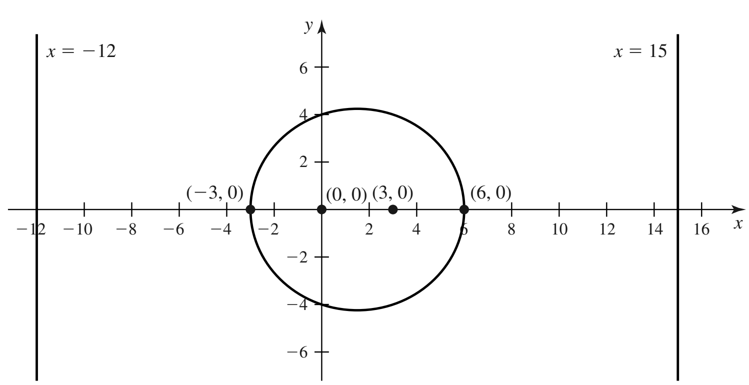
10.4.58 The vertices are $(0, 6/5)$ and $(0, -6)$. The center is $(0, -12/5)$. The foci are $(0, 0)$ and $(-24/5, 0)$.



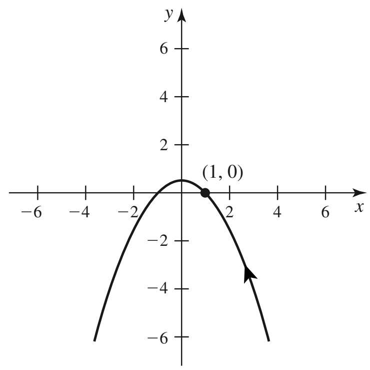
10.4.59 The vertex is $(0, -1/4)$, and the focus is $(0, 0)$. The directrix is the line $y = -\frac{1}{2}$.



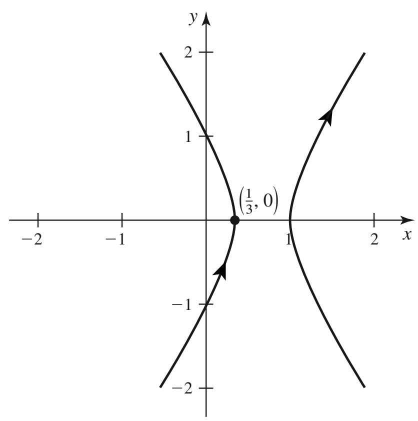
10.4.60 The vertices are $(6, 0)$ and $(-3, 0)$. The center is $(3/2, 0)$. The foci are $(0, 0)$ and $(3, 0)$.



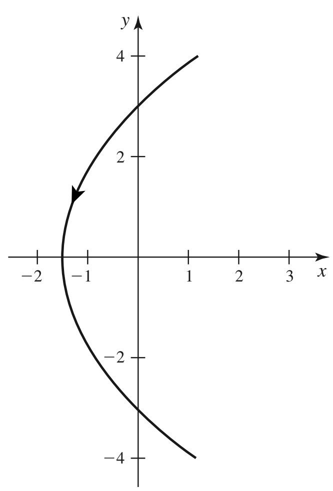
10.4.61 The parabola starts at $(1, 0)$ and goes through quadrants I, II, and III for $\theta \in [0, 3\pi/2]$. It then approaches $(1, 0)$ by traveling through quadrant IV for $\theta \in (3\pi/2, 2\pi)$.



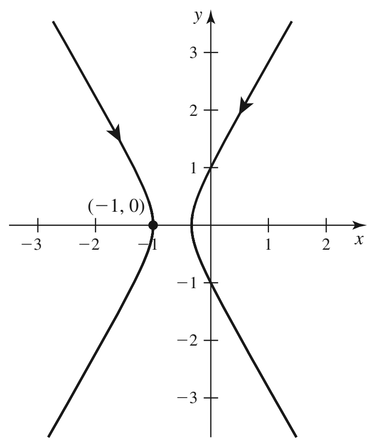
10.4.62 Note that the value of r for $\theta = 0$ is $1/3$. As θ proceeds to $\pi/2$, the curve is traced in the first quadrant and approaches the polar point $(1, \pi/2)$ From $\pi/2$ to π, the curve approaches the asymptote, and then appears along the asymptote in the fourth quadrant and heads toward the polar point $(-1, \pi)$. From π to 2π, the curve approaches the asymptote in the first quadrant, and then reappears in the third quadrant along the asymptote, and heads toward the point $(1/3, 2\pi)$.



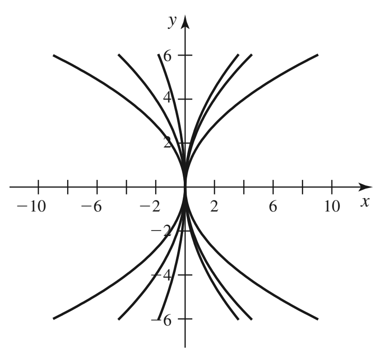
10.4.63 The parabola begins in the first quadrant and passes through the points $(0, 3)$ and then $(-3/2, 0)$ and $(0 - 3)$ as θ ranges from 0 to 2π.



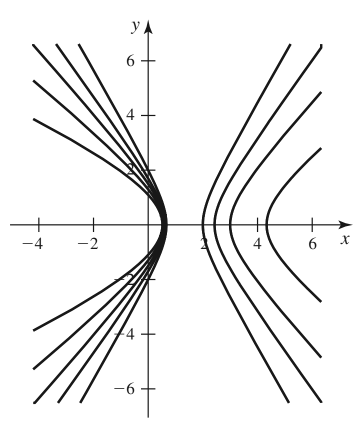
10.4.64 As θ ranges from 0 to $\pi/3$, the branch of the hyperbola in quadrant III starts at the point $(-1,0)$, and approaches the asymptote (note that $r \to \infty$ as $\theta \to \pi/3^-$.) As θ takes on the values from $\pi/3$ to $\pi/2$, the portion of the parabola in quadrant I appears and heads toward the point $(0,1)$. For θ ranging from $\pi/2$ to $3\pi/2$, the curve ranges from $(0,1)$ to $(-1/3,0)$ to $(0,-1)$. From $\theta = 3\pi/2$ to $\theta = 5\pi/3$, the curve approaches the asymptote in quadrant IV. From $5\pi/3$ to 2π, the curve reappears along the asymptote in quadrant II, and approaches the point $(-1,0)$.



10.4.65 For negative p, the parabola opens to the left and for positive p it opens to the right. As p increases to 0, the parabola opens wider and as p decreases (for $p > 0$), it gets narrower.



10.4.66 As e gets larger, the vertices move closer to each other.



10.4.67

a. True. Note that if $x = 0$, the equation becomes $-y^2 = 9$, which has no solution.

b. True. The slopes of the tangent lines range continuously from $-\infty$ to 0 to ∞ and then back through 0 to $-\infty$ again.

c. True. Given c and d, one can compute a, b, and e. See the summary after Theorem 10.3.

d. True. The vertex is exactly halfway between the focus and the directrix.

10.4.68 Using implicit differentiation, we have $2yy' = 8$, and at the point $(8, -8)$, we have $y' = -\frac{1}{2}$. So $y - (-8) = -\frac{1}{2}(x - 8)$, or $y = -\frac{1}{2}x - 4$ is the equation of the tangent line.

10.4.69 Differentiating gives $2x = -6y'$, so at $(-6, -6)$ we obtain $-12 = -6y'$, so $y' = 2$. Thus $y - (-6) = 2(x - (-6))$, or $y = 2x + 6$ is the equation of the tangent line.

10.4.70 We have

$$\frac{dy}{dx} = \frac{-\frac{\cos\theta\sin\theta}{(1+\sin\theta)^2} + \frac{\cos\theta}{1+\sin\theta}}{-\frac{\cos^2\theta}{(1+\sin\theta)^2} - \frac{\sin\theta}{1+\sin\theta}} = \frac{\sin\theta - 1}{\cos\theta}.$$

At $\theta = \frac{\pi}{6}$ we have $y' = \frac{(1/2)-1}{\sqrt{3}/2} = -\frac{\sqrt{3}}{3}$. The equation of the tangent line is therefore $y - \frac{1}{3} = -\frac{\sqrt{3}}{3}(x - \frac{\sqrt{3}}{3})$, or $y = -\frac{\sqrt{3}}{3}x + \frac{2}{3}$.

10.4.71 Differentiating implicitly, we have $2yy' - \frac{x}{32} = 0$, so at $(6, -5/4)$ we have $-\frac{5}{2}y' - \frac{3}{16} = 0$, so $y' = -\frac{3}{40}$. The equation of the tangent line is $y + \frac{5}{4} = -\frac{3}{40}(x - 6)$, or $y = -\frac{3x}{40} - \frac{4}{5}$.

10.4.72 We have an ellipse with focus at the origin and directrix $x = 2$. Because $(2/3, 0)$ is a vertex, $e = \frac{|PF|}{|PL|} = \frac{2/3}{4/3} = \frac{1}{2}$ and $r(\theta) = \frac{\frac{1}{2}\cdot 2}{1+\frac{1}{2}\cos\theta} = \frac{2}{2+\cos\theta}$.

10.4.73 We have a hyperbola with focal point at the origin and directrix $y = -2$. Furthermore $P = (0, -4/3)$ is a vertex. Thus, $e = \frac{|PF|}{|PL|} = \frac{4/3}{2/3} = 2$, and $r(\theta) = \frac{2(2)}{1-2\sin\theta} = \frac{4}{1-2\sin\theta}$.

10.4.74

a. $e = \frac{|PF|}{|PL|}$, so $r = |PF| = e\,|PL| = e\,|-d - r\cos\theta|$, or $r = e(d+r\cos\theta)$. Solving for r yields $r = \frac{ed}{1-e\cos\theta}$.

b. $r = |PF| = e\,|PL|$, or $r = e(d - r\sin\theta)$. Solving for r yields $r = \frac{ed}{1+e\sin\theta}$.

c. $r = |PF| = e\,|PL|$, or $r = e(-d - r\sin\theta)$. So $r = e(d + r\sin\theta)$, and solving for r yields $r = \frac{ed}{1-e\sin\theta}$.

10.4.75 The points on the intersection of the two circles are a distance of $2a + r$ from F_1 and a distance of r from F_2. So for P an intersection point, we have $|PF_1| - |PF_2| = 2a$ for all r, and the set of all such points form a hyperbola with foci F_1 and F_2.

10.4.76

a. Making use of the substitution $\frac{x}{\sqrt{2}} = \sin t$, we have

$$\begin{aligned} A &= \int_0^1 \left(\sqrt{1 - \frac{x^2}{2}} - \frac{x^2}{\sqrt{2}}\right) dx = \int_0^{\pi/4} \sqrt{1 - \sin^2 t}\sqrt{2}\cos t\, dt - \frac{1}{\sqrt{2}}\int_0^1 x^2\, dx \\ &= \sqrt{2}\cdot\frac{1}{2}\left(\cos x\sin x + x\right)\Big|_0^{\pi/4} - \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{2}(\sin(\pi/4)\cos(\pi/4) + \frac{\pi}{4}) - \frac{1}{3\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\cdot(\frac{1}{2} + \frac{\pi}{4}) - \frac{\sqrt{2}}{6} = \frac{\sqrt{2}}{12} + \frac{\sqrt{2}\pi}{8}. \end{aligned}$$

b. About the x-axis, we obtain $\pi\int_0^1\left(1-\frac{x^2}{2}-\frac{x^4}{2}\right)dx=\pi\left(x-\frac{x^3}{6}-\frac{x^5}{10}\right)\Big|_0^1=\frac{11\pi}{15}\approx 2.304.$

About the y-axis, we obtain

$$2\pi\int_0^1\left(x\sqrt{1-\frac{x^2}{2}}-\frac{x^2}{\sqrt{2}}\right)dx=2\pi\int_{1/2}^1 u^{1/2}\,du-\frac{1}{\sqrt{2}}\int_0^1 x^3\,dx$$
$$=2\pi\left(\frac{2}{3}u^{3/2}\right)\Big|_{1/2}^1-\left(\frac{1}{4\sqrt{2}}x^4\right)\Big|_0^1=2\pi\left(\frac{2}{3}-\frac{2}{6\sqrt{2}}-\frac{1}{4\sqrt{2}}\right)\approx 1.597.$$

So the volume about the x-axis is greater.

10.4.77 Using implicit differentiation, we have $\frac{2x}{a^2}+\frac{2yy'}{b^2}=0$, or $y'=-\frac{b^2x}{a^2y}$. If (x_0,y_0) is the point of tangency, then $-\frac{b^2x_0}{a^2y_0}=\frac{y-y_0}{x-x_0}$, so $\frac{x_0(x-x_0)}{a^2}=-\frac{y_0(y-y_0)}{b^2}$, so $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$.

10.4.78 Using implicit differentiation, we have $\frac{2x}{a^2}-\frac{2yy'}{b^2}=0$, so $y'=\frac{b^2x}{a^2y}$. At (x_0,y_0), we have $y'=\frac{b^2x_0}{a^2y_0}$, so the tangent line is given by $y-y_0=\frac{b^2x_0}{a^2y_0}(x-x_0)$, or $\frac{y(y-y_0)}{b^2}=\frac{x_0(x-x_0)}{a^2}$, or $\frac{y_0y}{b^2}-\frac{x_0x}{a^2}=\frac{y_0^2}{b^2}-\frac{x_0^2}{a^2}=-1$, so $\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$.

10.4.79 $V_x=\pi\int_{-a}^a\left(b^2-\frac{b^2x^2}{a^2}\right)dx=\pi b^2\int_{-a}^a\left(1-\frac{x^2}{a^2}\right)dx=\pi b^2\left(x-\frac{x^3}{3a^2}\right)\Big|_{-a}^a=\frac{4\pi b^2a}{3}$.

$V_y=\pi\int_{-b}^b\left(a^2-\frac{a^2y^2}{b^2}\right)dy=\pi a^2\int_{-b}^b\left(1-\frac{y^2}{b^2}\right)dy=\pi a^2\left(y-\frac{y^3}{3b^2}\right)\Big|_{-b}^b=\frac{4\pi a^2b}{3}$.

These are different if $a\neq b$. In the case $a=b$, both volumes give $\frac{4\pi a^3}{3}$, the volume of a sphere.

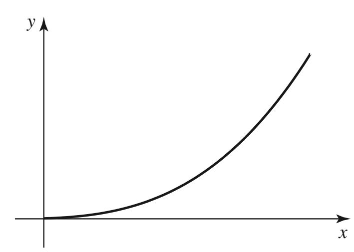
10.4.80

a. The focus is at $c=\sqrt{a^2+b^2}$. We have $A=2b\int_a^c\sqrt{\frac{x^2}{a^2}-1}\,dx=\frac{2b}{a}\int_a^c\sqrt{x^2-a^2}\,dx$. Using either the substitution $x=a\sec\theta$ (or a table of integrals), we have

$$A=\frac{2b}{a}\cdot\left(\frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\ln\left(2\left(\sqrt{x^2-a^2}+x\right)\right)\right)\Big|_a^c$$

so

$$A=ab\ln(a)-ab\ln(\sqrt{a^2+b^2}+b)+\frac{\sqrt{a^2+b^2}}{a}\cdot b^2.$$

b. 

10.4.81

a. $V_x=\pi\int_a^c\left(\sqrt{\frac{b^2x^2}{a^2}-b^2}\right)^2dx=\pi\int_a^c\left(\frac{b^2x^2}{a^2}-b^2\right)dx=\pi b^2\left(\frac{x^3}{3a^2}-x\right)\Big|_a^c=\pi b^2\left(\frac{c^3}{3a^2}-c-\frac{a}{3}+a\right)=\frac{\pi b^2}{3a^2}(c^3-3ca^2+2a^3)=\frac{\pi b^2}{3a^2}(a-c)^2(2a+c)$.

b. $V_y=2\cdot 2\pi\int_a^c a^2b\sqrt{\frac{x^2}{a^2}-1}\,dx=2\pi\int_0^{b^2/a^2}a^2b\sqrt{u}\,du=2\pi a^2b\left(\frac{2}{3}u^{3/2}\right)\Big|_0^{b^2/a^2}=2\pi a^2b\frac{2b^3}{3a^3}=\frac{4\pi b^4}{3a}$.

10.4.82 $V_R=2\pi\int_0^{\sqrt{h/a}}x(h-ax^2)\,dx=2\pi\int_0^{\sqrt{h/a}}(xh-ax^3)\,dx=2\pi\left(\frac{hx^2}{2}-\frac{ax^4}{4}\right)\Big|_0^{\sqrt{h/a}}=2\pi\left(\frac{h^2}{2a}-\frac{h^2}{4a}\right)=\frac{\pi h^2}{2a}$. The cone has height h and radius $\sqrt{h/a}$, so $V_c=\frac{1}{3}\pi\left(\sqrt{\frac{h}{a}}\right)^2\cdot h=\frac{\pi h^2}{3a}$, and $V_R=\frac{3}{2}\cdot\frac{\pi h^2}{3a}=\frac{\pi h^2}{2a}$.

10.4.83

a. The slope of a line making an angle θ with the horizontal is $\tan\theta$. The slope of the tangent line at (x_0, y_0) is $y' = \frac{x}{2p}$, so $y' = \frac{x_0}{2p}$, so $\tan\theta = \frac{x_0}{2p}$.

b. The distance from $(0, y_0)$ to $(0, p)$ is $p - y_0$, and $\tan\phi = \frac{\text{opposite}}{\text{adjacent}} = \frac{p-y_0}{x_0}$.

c. Because l is perpendicular to $y = y_0$, we have $\alpha + \theta = \pi/2$, or $\alpha = \frac{\pi}{2} - \theta$, so $\tan\alpha = \cot\theta = \frac{2p}{x_0}$.

d. $\tan\beta = \tan(\theta+\phi) = \frac{\frac{x_0}{wp} + \frac{p-y_0}{x_0}}{1 - \frac{p-y_0}{2p}} = \frac{x_0^2+2p^2-2py_0}{x_0(p+y_0)}$. Now because $x_0^2 = 4py_0$, we obtain $\tan\beta = \frac{4py_0+2p^2-2py_0}{x_0(p+y_0)}$ $= \frac{2p(p+y_0)}{x_0(p+y_0)} = \frac{2p}{x_0}$.

e. Because α and β are acute, we have that $\tan\alpha = \tan\beta$, so $\alpha = \beta$.

10.4.84 We have a vertex at $(0, 0)$ and the parabola passes through $(640, 152)$, so $152 = a(640)^2$, so $a = \frac{19}{51200} \approx 0.000371$. Thus, $y = \frac{19}{51200}x^2$, and the guy wire has length $L = \frac{19}{51200}(500^2) = \frac{11875}{128} \approx 92.77$ meters.

10.4.85 Assume the two fixed points are at $(c, 0)$ and $(-c, 0)$. Let P be the point $(0, b)$, and note that P is equidistant from the two given points, so we must have $b^2 + c^2 = a^2$ by the Pythagorean theorem. Now let $Q = (u, 0)$ be on the ellipse for $u > c$. Then $u - c + (c + u) = 2a$, so $u = a$. Now let $R = (x, y)$ be an arbitrary point on the ellipse (assume $x > 0$ and $y > 0$ – the other cases are similar.) Using the triangles formed between the foci, R, and the projection of R onto the x-axis, we have $\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(c-x)^2 + y^2}$. Squaring both sides gives $(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(c-x)^2 + y^2} + (c - x)^2 + y^2$. Isolating the root gives $\sqrt{(c-x)^2 + y^2} = \frac{1}{4a}\left((c - x)^2 + y^2 - (c + x)^2 - y^2 + 4a^2\right)$, so $\sqrt{(c-x)^2 + y^2} = a - \frac{c}{a}x$. Squaring again yields $(c-x)^2+y^2 = a^2 - 2xc + \frac{c^2}{a^2}x^2$, so $c^2 - 2cx + x^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2$, or $x^2\left(1 - \frac{c^2}{a^2}\right) + y^2 = a^2 - c^2$. Thus $\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$, which can be written $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, because $b^2 = a^2 - c^2$.

10.4.86 The intersection points of the branches with the x-axis are at $(-a, 0)$ and $(a, 0)$ because the distances to $(c, 0)$ and $(-c, 0)$ are $c+a$ and $c-a$, so the difference is $\pm 2a$. Consider one point on the right branch (the left branch will follow by a similar argument.) Let the distance from $(-c, 0)$ to (x, y) be u and the distance from $(c, 0)$ to (x, y) be v. Then $u = \sqrt{(c+x)^2 + y^2}$ and $v = \sqrt{(c-x)^2 + y^2}$, and because $u-v = 2a$, we have $\sqrt{(c+x)^2 + y^2} = 2a + \sqrt{(c-x)^2 + y^2}$. Squaring gives $(c+x)^2+y^2 = 4a^2+4a\sqrt{(c-x)^2 + y^2}+(c-x)^2+y^2$. Isolating the root gives $\sqrt{(c-x)^2 + y^2} = \frac{1}{4a}\left((c + x)^2 + y^2 - (c - x)^2 - y^2 - 4a^2\right)$, so $\sqrt{(c-x)^2 + y^2} = -a + \frac{c}{a}x$.

Squaring again yields $(c - x)^2 + y^2 = a^2 - 2xc + \frac{c^2}{a^2}x^2$, so $c^2 - 2cx + x^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2$, or $x^2\left(1 - \frac{c^2}{a^2}\right) + y^2 = a^2 - c^2$. Thus $\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$, which can be written $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = c^2 - a^2$.

10.4.87 Let the parabola be symmetric about the y-axis with vertex at the origin. Let the circle have radius r and be centered at $(r + a, 0)$, and let the line be $y = -a$. The distance form the point $P(x, y)$ to the line is $u = y + a$. The distance from the point P to the circle is $v = \sqrt{x^2 + (r + a - y)^2} - r$. Setting $u = v$ yields $y + a = \sqrt{x^2 + (r + a - y)^2} - r$, so $y + r + a = \sqrt{x^2 + (r + a - y)^2}$, and squaring gives $y^2 + 2(r + a)y + (r + a)^2 = x^2 + (r + a - y)^2$, so $y^2 + 2(r + a)y + (r + a)^2 = x^2 + (r + a)^2 - 2(r + a)y + y^2$, and thus $4(r + a)y = x^2$, so $y = \frac{1}{4(r+a)}x^2$, the equation of a parabola.

10.4.88 With focus at the origin, the cartesian equation of an ellipse with the second focus at $(-2c, 0)$ and major axis length $2a$, minor axis length $2b$ is $\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1$. Using $c = ae$ and polar coordinates yields $\frac{(r\cos\theta+ae)^2}{a^2} + \frac{r^2\sin^2\theta}{a^2(1-e^2)} = 1$. Thus, $(1 - e^2)(r^2\cos^2\theta + 2aer\cos\theta + a^2e^2) + r^2\sin^2\theta = a^2(1 - e^2)$, so $r^2 - e^2r^2\cos^2\theta + 2ae(1 - e^2)r\cos\theta + (1 - e^2)a^2e^2 = a^2(1 - e^2)$. Gathering like terms gives $(1 - e^2\cos^2\theta)r^2 + 2ae(1 - e^2)\cos\theta \cdot r - a^2(1 - e^2)^2 = 0$. Using the quadratic formula, we have

$$r = \frac{-2ae(1 - e^2)\cos\theta + \sqrt{4a^2e^2(1 - e^2)^2\cos^2\theta + 4a^2(1 - e^2)^2(1 - e^2\cos^2\theta)}}{2(1 - e^2\cos^2\theta)}.$$

This can be written as

$$r = \frac{-2ae(1-e^2)\cos\theta + 2a(1-e^2)}{2(1-e^2\cos^2\theta)} = \frac{a(1-e^2)(-e\cos\theta+1)}{(1-e\cos\theta)(1+e\cos\theta)} = \frac{a(1-e^2)}{1+e\cos\theta}.$$

10.4.89 Let the hyperbolas be centered at the origin with equations $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{B^2} - \frac{x^2}{A^2} = 1$ and eccentricities $e = \frac{c}{a}$ and $E = \frac{C}{B}$, respectively. Because the hyperbolas share a set of asymptotes $A = ra$ and $B = rb$ fro some $r > 0$, and

$$\begin{aligned} C^2 &= A^2 + B^2 = (ra)^2 + (rb)^2 \\ &= r^2(a^2+b^2) = r^2c^2. \end{aligned}$$

Then we have

$$\begin{aligned} e^{-2} + E^{-2} &= \left(\frac{c}{a}\right)^{-2} + \left(\frac{C}{B}\right)^{-2} = \frac{a^2}{c^2} + \frac{B^2}{C^2} \\ &= \frac{a^2}{c^2} + \frac{r^2b^2}{r^2c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1. \end{aligned}$$

10.4.90 The focal chord of slope $m \neq 0$ has equation $y = m(x-p)$. Because $y^2 = 4px$, the focal chord and the parabola intersect for $(mx-mp)^2 = 4px$, which occurs (via the quadratic formula) at $x = \frac{(m^2+2\pm2\sqrt{m^2+1})p}{m^2}$. The corresponding y-values are $\frac{(m^2+2\pm2\sqrt{m^2+1})p}{m} - mp$. Now $y' = \frac{2p}{y}$, so $y' = \frac{m}{1\pm\sqrt{m^2+1}}$ at the two points. The product of these two values of y' is -1, so the two lines are perpendicular. If we call the intersection points found above (x_0, y_0) and (x_1, y_1), then the two lines intersect for

$$\frac{m}{1+\sqrt{m^2+1}}(x-x_0) + y_0 = \frac{m}{1-\sqrt{m^2+1}}(x-x_1) + y_1,$$

which when solved for x gives $x = -p$, so the two lines meet on the directrix.

Note that in the case of a vertical chord, we have $(x_0, y_0) = (p, 2p)$ and $(x_1, y_1) = (p, -2p)$, and thus the slopes of the tangent lines are 1 and -1, so their product is still -1 and thus they are perpendicular. Then the tangent lines meet when $1(x-p) + 2p = -1(x-p) - 2p$, which occurs when $x = -p$, so they still meet on the directrix.

10.4.91 The latus rectum L intersects the parabola at $x = p$, $y = \pm 2p$. The distance between any point $P(x, y)$ on the parabola to the left of L and L is $p - x$. The distance from F to P is $\sqrt{(x-p)^2 + y^2} = \sqrt{x^2 - 2px + p^2 + 4px} = \sqrt{x^2 + 2px + p^2} = x + p$ (because both x and p are positive.) Thus $D + |FP| = p - x + x + p = 2p$.

10.4.92 Because the latus rectum intersects the parabola at $(p, 2p)$ and $(p, -2p)$, its length is $4\,|p|$.

10.4.93 Let P be a point on the intersection of the latus rectum and the ellipse. The length of the latus rectum is twice the distance from P to the focus. Let l be the length from P to the focus, and let L be the distance from P to the other focal point. Then $l + L = 2a$, so $L^2 = 4c^2 + l^2$, and thus $(2a-l)^2 = 4c^2 + l^2$, and solving for l yields $l = a - \frac{c^2}{a}$. Because $c^2 = a^2 - b^2$, this can be written as $l = a - \frac{a^2-b^2}{a} = a - (a - \frac{b^2}{a}) = \frac{b^2}{a}$. The length of the latus rectum is therefore $\frac{2b^2}{a}$. Now because $e = \frac{c}{a}$, we have $\sqrt{1-e^2} = \sqrt{1 - \frac{a^2-b^2}{a^2}} = \sqrt{\frac{b^2}{a^2}} = \frac{b}{a}$. The length of the latus rectum can thus also be written as $2b \cdot \frac{b}{a} = 2b\sqrt{1-e^2}$.

10.4.94 Let P be a point on the intersection of the latus rectum and the hyperbola. The length of the latus rectum is twice the distance from P to the focus. Let l be the length from P to the focus, and let L be the distance from P to the other focal point. Then $L - l = 2a$, so $L^2 = 4c^2 + l^2$, and thus $(2a+l)^2 = 4c^2 + l^2$, and solving for l yields $l = \frac{c^2}{a} - a$. Because $c^2 = a^2 + b^2$, this can be written as $l = \frac{a^2+b^2}{a} - a = \frac{b^2}{a}$. The length of the latus rectum is therefore $\frac{2b^2}{a}$. Now because $e = \frac{c}{a}$, we have $\sqrt{e^2-1} = \sqrt{\frac{a^2+b^2}{a^2} - 1} = \sqrt{\frac{b^2}{a^2}} = \frac{b}{a}$. The length of the latus rectum can thus also be written as $2b \cdot \frac{b}{a} = 2b\sqrt{e^2-1}$.

10.4.95 Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$ and let the equation of the hyperbola be $\frac{x^2}{r^2}-\frac{y^2}{c^2-r^2}=1$. Let (x_0,y_0) be a point of intersection. By evaluating both equations at the point of intersection and subtracting, we obtain the result

$$\frac{x_0^2}{a^2}-\frac{x_0^2}{r^2}+\frac{y_0^2}{a^2-c^2}+\frac{y_0^2}{c^2-r^2}=0,$$

which can be written

$$\frac{r_0^2x_0^2-a^2x_0^2}{a^2r^2}+\frac{(c^2-r^2)y_0^2+(a^2-c^2)y_0^2}{(a^2-c^2)(c^2-r^2)}=0.$$

This equation can be rewritten in the form $\frac{x_0^2}{y_0^2}=\frac{a^2r^2}{(a^2-c^2)(c^2-r^2)}$, which we will use later.

Now implicitly differentiating the equation for the ellipse yields $\frac{2x}{a^2}+\frac{2yy'}{a^2-c^2}=0$, and thus the slope of the tangent line to the ellipse at (x_0,y_0) is $y_e'=-\frac{x_0}{y_o}\cdot\frac{a^2-c^2}{a^2}$. Differentiating the equation of the hyperbola gives $\frac{2x}{r^2}-\frac{2yy'}{c^2-r^2}=0$, so the slope of the tangent line to the hyperbola at the point of intersection is $y_h'=\frac{x_0}{y_0}\cdot\frac{c^2-r^2}{r^2}$.

Now consider the product

$$-1\cdot y_e'\cdot y_h'=\frac{x_0^2}{y_0^2}\cdot\frac{(a^2-c^2)(c^2-r^2)}{a^2r^2}.$$

By the result of the first paragraph, this is equal to 1, and thus the two curves are perpendicular at the point of intersection.

10.4.96 The vertical distance at x_0 is given by $d(x_0)=\frac{bx_0}{a}-\sqrt{\frac{x_0^2b^2}{a^2}-a^2}=\frac{b}{a}\left(x_0-\sqrt{x_0^2-\frac{a^4}{b^2}}\right)$. We have

$$\lim_{x_0\to\infty}d(x_0)=\frac{b}{a}\lim_{x_0\to\infty}\left(x_0-\sqrt{x_0^2-\frac{a^4}{b^2}}\right)=\frac{b}{a}\lim_{x_0\to\infty}\left(\frac{x_0^2-\left(x_0^2-\frac{a^4}{b^2}\right)}{x_0+\sqrt{x_0^2-\frac{a^4}{b^2}}}\right)=0.$$

10.4.97

a. The curve and the line intersect when $x^2-m^2(x^2-4x+4)-1=0$, which occurs for $\frac{2m^2\pm\sqrt{1+3m^2}}{m^2-1}$, assuming $m\neq\pm1$. So there are two solutions in this case – but if $-1<m<1$, one of the solutions is negative (the intersection lies on the other branch of the hyperbola.) If $m^2=1$, then the equation becomes $4x-5=0$, and there is only the solution $x=\frac{5}{4}$. So there are two intersection points on the right branch exactly for $|m|>1$. We have $v(m)=\frac{2m^2+\sqrt{1+3m^2}}{m^2-1}$ and $u(m)=\frac{2m^2-\sqrt{1+3m^2}}{m^2-1}$.

b. $\lim_{m\to1^+}u(m)=\lim_{m\to1^+}u(m)\cdot\frac{2m^2+\sqrt{1+3m^2}}{2m^2+\sqrt{1+3m^2}}=\lim_{m\to1^+}\frac{4m^4-3m^2-1}{(m^2-1)(2m^2+\sqrt{1+3m^2})}=$
$\lim_{m\to1^+}\frac{(m^2-1)(4m^2+1)}{(m^2-1)(2m^2+\sqrt{1+3m^2})}=\frac{5}{4}$.

$\lim_{m\to1^+}v(m)=\lim_{m\to1^+}v(m)\cdot\frac{2m^2-\sqrt{1+3m^2}}{2m^2-\sqrt{1+3m^2}}=\lim_{m\to1^+}\frac{4m^4-3m^2-1}{(m^2-1)(2m^2-\sqrt{1+3m^2})}=$
$\lim_{m\to1^+}\frac{(m^2-1)(4m^2+1)}{(m^2-1)(2m^2-\sqrt{1+3m^2})}=\lim_{m\to1^+}\frac{(4m^2+1)}{(2m^2-\sqrt{1+3m^2})}=\infty$.

c. $$\lim_{m\to\infty}u(m)=\lim_{m\to\infty}\frac{2-\sqrt{\frac{1}{m^4}+\frac{3}{m^2}}}{1-\frac{1}{m^2}}=2.$$

$$\lim_{m\to\infty}v(m)=\lim_{m\to\infty}\frac{2+\sqrt{\frac{1}{m^4}+\frac{3}{m^2}}}{1-\frac{1}{m^2}}=2.$$

d. The expression $\lim_{m\to\infty}A(m)$ represents the area of the region bounded by the hyperbola and the line $x=2$. It is given by $2\int_1^2\sqrt{x^2-1}\,dx=2\left(\frac{x}{2}\sqrt{x^2-1}-\frac{1}{2}\ln(x+\sqrt{x^2-1})\right)\Big|_1^2=2\sqrt{3}-\ln(2+\sqrt{3})$.

10.4.98

a. The area of the anvil is $A = 4\int_0^p \sqrt{1+y^2}\,dy = 4\int_0^{\tan^{-1}(p)} \sec^3 t\,dt = 2p\sqrt{1+p^2} + 2\ln(\sqrt{1+p^2}+p)$, where this last integral can be evaluated using the techniques of chapter 7 (or a table of integrals.)

The area of R is equal to the area of S when $2 = p\sqrt{1+p^2} + \ln(\sqrt{1+p^2}+p)$. Using a CAS, the result is $p \approx 0.8927$.

b. For R to have twice the area of S, we need $4 = p\sqrt{1+p^2} + \ln(\sqrt{1+p^2}+p)$, which occurs for $p \approx 1.5279$.

10.4.99

a. With $x^2 = a^2\cos^2 t + 2ab\sin t\cos t + b^2\sin^2 t$, $y^2 = c^2\cos^2 t + 2cd\sin t\cos t + d^2\sin^2 t$, and $xy = ac\cos^2 t + (ad+bc)\sin t\cos t + bd\sin^2 t$, we have $Ax^2+Bxy+Cy^2 = (Aa^2+Bac+Cc^2)\cos^2 t + (2Aab + B(ad+bc)+2Ccd)\sin t\cos t + (Ab^2+Bbd+Cd^2)\sin^2 t = K$. Thus we have an equation of the desired form as long as there exist A, B, C, and K so that $A(a^2-b^2)+B(ac-bd)+C(c^2-d^2) = 0$ and $2Aab + B(ad+bc) + 2Ccd = 0$. This turns out to be the case when $ad - bc \neq 0$. Note that the value of K is $Aa^2 + Bac + Cc^2$.

b. Suppose that $ad - bc \neq 0$, but $ac + bd = 0$. Then $\frac{b}{a} = -\frac{c}{d}$, and $\tan^{-1}(b/a) = \tan^{-1}(-c/d)$.

Note that $x = \sqrt{a^2+b^2}\cos(t + \tan^{-1}(-b/a))$, $y = \sqrt{c^2+d^2}\sin(t+\tan^{-1}(c/d))$. This can be seen by applying the trigonometric identities for the sum of two angles. Then $\frac{x^2}{a^2+b^2} + \frac{y^2}{c^2+d^2} = \cos^2(t + \tan^1(b/a)) + \sin^2(\tan^{-1}(-c/d)) = 1$.

c. Using the work in part b), we see that the equation is $\frac{x^2}{a^2+b^2} + \frac{y^2}{c^2+d^2} = 1$, or $x^2+y^2 = r^2$, where $r^2 = a^2+b^2 = c^2+d^2$.

Chapter Ten Review

1

a. False. For example, $x = r\cos t$, $y = r\sin t$ for $0 \le t \le 2\pi$ and $x = r\sin t$, $y = r\cos t$ for $0 \le t \le 2\pi$ generate the same circle.

b. False. Because $e^t > 0$ for all t, this only describes the portion of that line where $x > 0$.

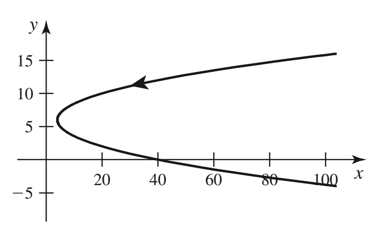
c. True. They both describe the point whose cartesian coordinates are $(3\cos(-3\pi/4), 3\sin(-3\pi/4)) = (-3\cos(\pi/4), -3\sin(\pi/4)) = (-3/\sqrt{2}, -3/\sqrt{2})$.

d. False. The given integral counts the inner loop twice.

e. True. This follows because the equation $0 - x^2/4 = 1$ has no real solutions.

f. True. Note that the given equation can be written as $(x-1)^2 + 4y^2 = 4$, or $\frac{(x-1)^2}{4} + y^2 = 1$.

2



a.

b. $x = t^2 + 4 = (6-y)^2 + 4$.

c. The curve is a parabola which opens in the positive x-direction, with vertex at $(4, 6)$.

d. $\frac{dy}{dx} = -\frac{1}{2t}$. At the point $(5,5)$ we have $t = 1$, so $\frac{dy}{dx} = -\frac{1}{2}$.

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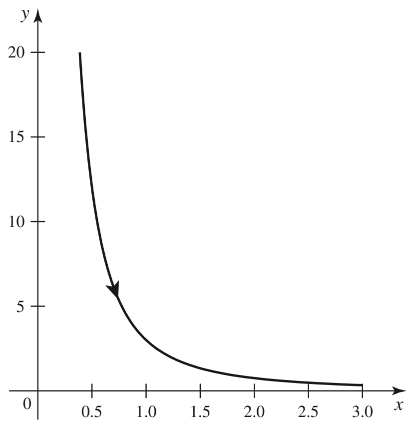
3

a. 

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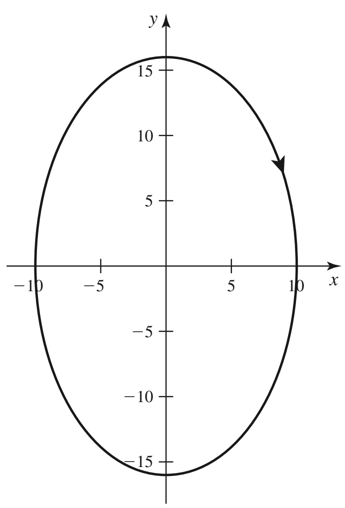
b. $y = 3(e^t)^{-2} = \frac{3}{x^2}$.

c. The curve represents the portion of $\frac{3}{x^2}$ for $x > 0$.

d. $\frac{dy}{dx} = -\frac{6}{x^3}$, so at $(1, 3)$ we have $\frac{dy}{dx} = -6$.

4

a. 

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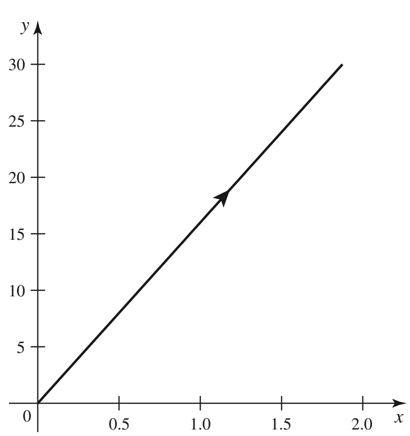
b. $\left(\frac{x}{10}\right)^2 + \left(\frac{y}{16}\right)^2 = \sin^2 2t + \cos^2 2t = 1$.

c. The curve represents an ellipse traced clockwise.

d. $\frac{dy}{dx} = -\frac{32 \sin 2t}{20 \cos 2t}$, and at $t = \pi/6$ this is equal to $-\frac{16\sqrt{3}}{10}$.

5

a. 

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b. Because $\ln t^2 = 2 \ln t$ for $t > 0$, we have $y = 16x$ for $0 \le x \le 2$.

c. The curve represents a line segment from $(0, 0)$ to $(2, 32)$.

d. $\frac{dy}{dx} = 16$ for all value of x.

6 As derived in the last problem in section 10.4, this describes a circle provided $ad-bc \neq 0$, but $ac+bd=0$, and $a^2+b^2=c^2+d^2$. In this case, the circle has radius $r=\sqrt{a^2+b^2}$.

7 Note that $\left(\frac{x}{4}\right)^2+\left(\frac{y}{3}\right)^2=1$. This represents an ellipse generated counterclockwise.

8 Note that $\left(\frac{x+1}{4}\right)^2+\left(\frac{y-2}{4}\right)^2=1$, so $(x+1)^2+(y-2)^2=16$. This is a circle of radius 4 centered at $(-1,2)$ generated counterclockwise.

9 Note that $(x+3)^2+(y-6)^2=1$. This is the right half of a circle of radius 1 centered at $(-3,6)$. It is generated clockwise.

10 If we let $r=1+\cos t$, then $x=r\cos t$ and $y=r\sin t$. The curve $r=1+\cos t$ is a cardioid.

11 $x=3\sin t$, $y=3\cos t$, $0\le t\le 2\pi$.

12 $x=3\cos t$, $y=2\sin t$, $0\le t\le \pi$.

13 $x=3\cos t$, $y=2\sin t$, $-\pi/2\le t\le \pi/2$.

14 $x=t$, $y=4t+11$, $-\infty\le t\le\infty$.

15 From P to Q, we use $(x(t),y(t))=tQ+(1-t)P=(t,t)+(t-1,0)=(2t-1,t)$. So $x(t)=2t-1$, $y(t)=t$, $0\le t\le 1$.

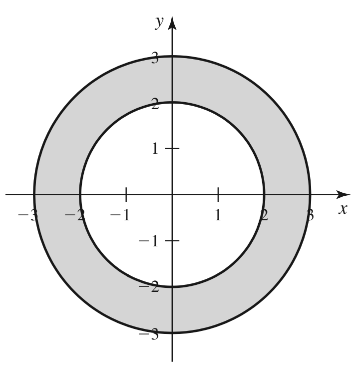
From Q to P, we use $(x(t),y(t))=tP+(1-t)Q=(-t,0)+(1-t,1-t)=(1-2t,1-t)$, for $0\le t\le 1$. Thus $x(t)=1-2t$, $y(t)=1-t$, $0\le t\le 1$.

16 $x=t$, $y=t^3+2t$, $0\le t\le 2$.

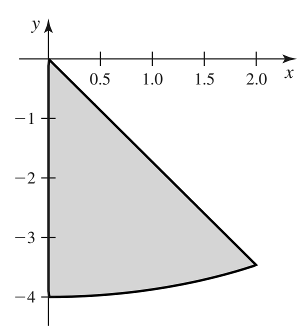
17 $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\sin t}{1-\cos t}$. At $t=\pi/6$, the slope of the tangent line is $\frac{1}{2-\sqrt{3}}=2+\sqrt{3}$. So the equation of the tangent line is $y-(1-\sqrt{3}/2)=(2+\sqrt{3})(x-(\pi/6-1/2))$, or $y=(2+\sqrt{3})x+(2-\frac{\pi}{3}-\frac{\pi\sqrt{3}}{6})$.

At $t=2\pi/3$, the slope of the tangent line is $\frac{\sqrt{3}}{3}$, so the equation of the tangent line is $y-\frac{3}{2}=\frac{\sqrt{3}}{3}(x-(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}))$, or $y=\frac{x}{\sqrt{3}}+2-\frac{2\pi}{3\sqrt{3}}$.

18



19



20

a. This matches (F). Note that there are 8 solutions to the equation $3\sin 4\theta=3$ for $0\le t\le 2\pi$, corresponding to the tips of the petals.

b. This matches (D). Note that for every value of θ for $-\pi/2<\theta<\pi/2$, there are two symmetric values for r.

c. This matches (B). Note that this limaçon has its largest value for r at $\theta=3\pi/2$.

d. This matches (E). Note that this limaçon has its largest value for r at $\theta = 0$.

e. This matches (C). Note that there are 3 unique solutions to $r = 3\cos\theta = 3$ for $0 \le \theta \le \pi$ that correspond to the tips of the petals. Note that the curve is generated for $0 \le \theta \le \pi$.

f. This matches (A). Note that $r \to 0$ as $\theta \to \infty$, and $r \to \infty$ as $\theta \to -\infty$.

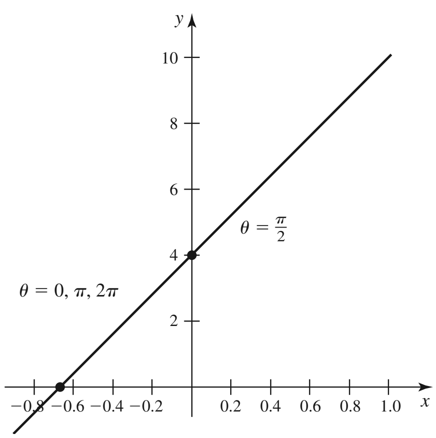
21 Liz should choose the cardioid, which is $r = 1 - \sin\theta$.

22 Jake should send $r^2 = \cos 2\theta$.

23 Letting $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$, we have $x^2 + y^2 + 2y - 6x = 0$, which can be written as $x^2 - 6x + 9 + y^2 + 2y + 1 = 10$, or $(x-3)^2 + (y+1)^2 = 10$, so this is a circle of radius $\sqrt{10}$ centered at $(3, -1)$.

24

a. We can write the equation as $r\sin\theta - 6r\cos\theta = 4$, or $y - 6x = 4$. This is a straight line with slope 6 and y-intercept 4.

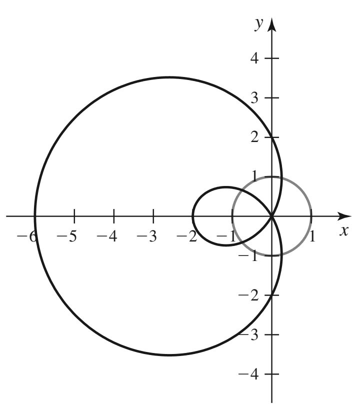
b. 

c. Note that $\sin\theta - 6\cos\theta = 0$ for $\theta = \tan^{-1}(6)$. The whole curve can be generated for $\tan^{-1}(6) - \pi < \theta < \tan^{-1}(6) + \pi$.

25 If $x = r\cos\theta$ and $y = r\sin\theta$, then $(r\cos\theta - 4)^2 + r^2\sin^2\theta = 16$, so $r^2\cos^2\theta - 8r\cos\theta + 16 + r^2\sin^2\theta = 16$, so $r^2 = 8r\cos\theta$, and thus $r = 8\cos\theta$. The complete circle can be described by $-\pi/2 \le \theta \le \pi/2$.

26 We have $r\cos\theta = r^2\sin^2\theta$, so $r = \cot\theta\csc\theta$. The whole parabola is described by $0 < \theta < \pi$.

27

a. 

There are 4 intersection points.

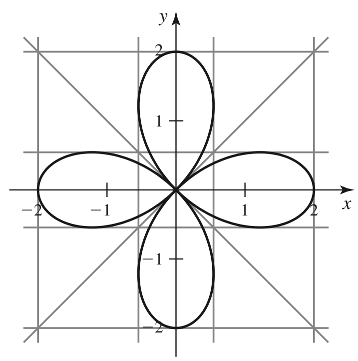
b. Note that $2 - 4\cos\theta = 1$ for $\theta = \cos^{-1}(1/4) \approx 1.32$, and $2 - 4\cos\theta = -1$ for $\theta = \cos^{-1}(3/4) \approx .73$. The points of intersection (in polar form) are approximately $(1, 1.32)$, $(1, 2\pi - 1.32) \approx (1, 4.96)$, $(-1, .73)$, and $(-1, 2\pi - .73) \approx (-1, 5.56)$.

28

a. $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-4\sin(2\theta)\sin\theta + 2\cos(2\theta)\cos\theta}{-4\sin(2\theta)\cos\theta - 2\cos(2\theta)\sin\theta}$. This is 0 when $-4\sin(2\theta)\sin\theta + 2\cos(2\theta)\cos\theta = -8\sin^2\theta\cos\theta + 2\cos^3\theta - 2\sin^2\theta\cos\theta = 0$, which occurs for $\cos\theta = 0$, and for $2\cos^2\theta - 10\sin^2\theta = 0$, or $\tan^2\theta = \frac{1}{5}$. So there are 6 places with horizontal tangent lines: at $\theta = \pm\pi/2$, $\theta = \pm\tan^{-1}(\sqrt{1/5})$, and $\theta = \pi \pm \tan^{-1}(\sqrt{1/5})$.

Vertical tangent lines occur when $-4\sin(2\theta)\cos\theta - 2\cos(2\theta)\sin\theta = -8\sin\theta\cos^2\theta - 2\cos^2\theta\sin\theta + 2\sin^3\theta = 0$. Thus occurs when $\sin\theta = 0$, and when $-8\cos^2\theta - 2\cos^2\theta + 2\sin^2\theta = 0$, which can be written as $\tan^2\theta = 5$. So the vertical tangent lines occur at $\theta = 0$, $\theta = \pi$ and $\theta = \pm\tan^{-1}(\sqrt{5})$ and $\theta = \pi \pm \tan^{-1}(\sqrt{5})$.

b. The curve is at the origin for $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$. At these values, $\frac{dy}{dx} = \pm 1$, so the tangent lines have the equation $y = x$ or $y = -x$.

c. 

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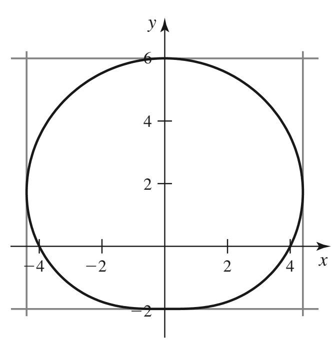
29

a. $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos\theta\sin\theta + (4 + 2\sin\theta)\cos\theta}{2\cos\theta\cos\theta - (4 + 2\sin\theta)\sin\theta} = \frac{4\cos\theta + 4\sin\theta\cos\theta}{2\cos^2\theta - 2\sin^2\theta - 4\sin\theta}$.

This is 0 when $\cos\theta = 0$, and when $4\sin\theta = -4$, so the only solutions are $\theta = \pi/2, 3\pi/2$.

The denominator is 0 when $2 - 4\sin^2\theta - 4\sin\theta = 0$ which occurs (using the quadratic formula) for $\sin\theta = -\frac{1}{2} + \frac{\sqrt{3}}{2}$, so there are vertical tangent lines at $\theta = \sin^{-1}(-\frac{1}{2} + \frac{\sqrt{3}}{2})$ and $\theta = \pi - \sin^{-1}(-\frac{1}{2} + \frac{\sqrt{3}}{2})$.

b. The curve is never at the origin.

c. 

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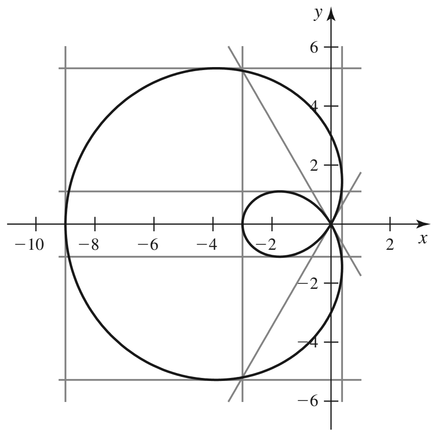
30

a. $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{6\sin\theta \cdot \sin\theta + (3 - 6\cos\theta)\cos\theta}{6\sin\theta\cos\theta - (3 - 6\cos\theta)\sin\theta} = \frac{6 - 12\cos^2\theta + 3\cos\theta}{12\sin\theta\cos\theta - 3\sin\theta}$.

This is 0 when $\cos^2\theta - \frac{1}{4}\cos\theta - \frac{1}{2} = 0$, which (by the quadratic formula) occurs where $\cos\theta = \frac{1}{8} \pm \frac{\sqrt{33}}{8}$, so for $\theta \approx .568$, 2.206, 4.078, and 5.715.

The denominator is 0 when $\sin\theta = 0$ and when $12\cos\theta - 3 = 0$, or $\theta = \pm\cos^{-1}(1/4)$.

b. The curve is at the origin for $\theta = \pm\pi/3$, and because $\tan\pi/3 = \sqrt{3}$, the tangent lines have the equations $y = \pm\sqrt{3}x$.

c. 

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31

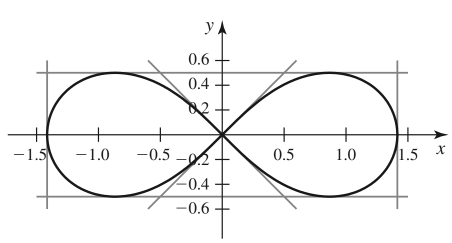
a. Note that the whole curve is generated for $-\pi/4 \le \theta \le \pi/4$, so we restrict ourselves to that domain. Write the equations as $r = \sqrt{2\cos 2\theta}$. Then

$$\frac{dy}{d\theta} = \sqrt{2\cos 2\theta}\cos\theta - \sin\theta\frac{2\sin 2\theta}{\sqrt{2\cos 2\theta}} = \frac{\cos\theta}{\sqrt{2\cos 2\theta}}\left(2\cos 2\theta - 4\sin^2\theta\right) = \frac{\cos\theta}{\sqrt{2\cos 2\theta}}\left(2 - 8\sin^2\theta\right).$$

Also, $\frac{dx}{d\theta} = -\sqrt{2\cos 2\theta}\sin\theta + \cos\theta\frac{2\sin 2\theta}{\sqrt{2\cos 2\theta}} = \frac{\sin\theta}{\sqrt{2\cos 2\theta}}\left(-4\cos^2\theta - 2(\cos 2\theta)\right) = \frac{(2-8\cos^2\theta)\sin\theta}{\sqrt{2\cos(2\theta)}}$. Thus $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \cot\theta\left(\frac{1-4\sin^2\theta}{1-4\cos^2\theta}\right)$.

This expression is 0 on the given domain only for $\sin^2\theta = \frac{1}{4}$, so there are horizontal tangent lines at $\theta = \pm\frac{\pi}{6}$. There are vertical tangent lines on the given domain only for $\theta = 0$ In cartesian coordinates, the lines are $x = \pm\sqrt{2}$.

b. The curve is at the origin for $\theta = \pm\pi/4$, and because $\tan\pi/4 = 1$, the tangent lines have the equations $y = \pm x$.

c. 

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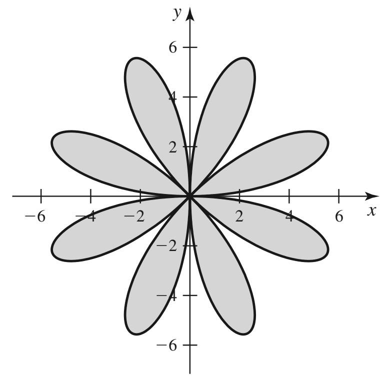
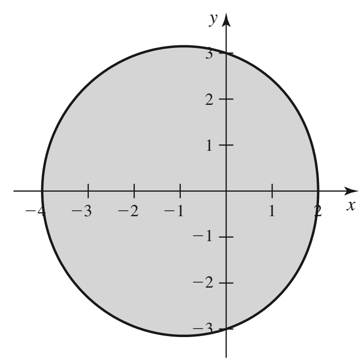
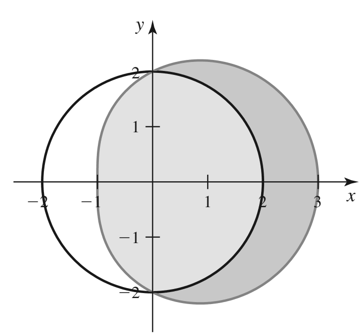
32 One leaf is traced for $0 \le \theta \le \pi/4$, so $A = 8 \cdot \frac{1}{2}\int_0^{\pi/4}(3\sin(4\theta))^2\,d\theta = 36\int_0^{\pi/4}\sin^2 4\theta\,d\theta = 36\left(-\frac{1}{8}\sin(4\theta)\cos(4\theta) + \frac{\theta}{2}\right)\Big|_0^{\pi/4} = 36\left(\frac{\pi}{8}\right) = \frac{9\pi}{2}$.



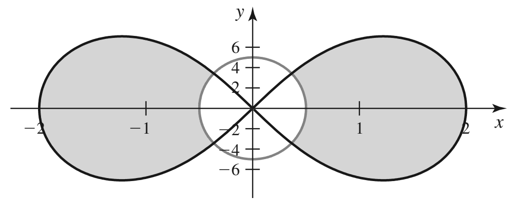
33 The area is given by $A = \frac{1}{2}\int_0^{2\pi}(3 - \cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(9 - 6\cos\theta + \cos^2\theta)\,d\theta = \frac{1}{2}\left(9\theta - 6\sin\theta + \frac{1}{2}(\cos\theta\sin\theta + \theta)\right)\Big|_0^{2\pi} = \frac{1}{2}(18\pi + \pi) = \frac{19\pi}{2}$.



34 The curves intersect at $\theta = \pm\pi/2$. By symmetry, the area is twice the area outside the circle and inside the limaçon between 0 and $\pi/2$. We have $A = 2 \cdot \frac{1}{2}\int_0^{\pi/2}((2 + \cos\theta)^2 - 2^2)\,d\theta = \int_0^{\pi/2}(4\cos\theta + \cos^2\theta)\,d\theta = \left(4\sin\theta + \frac{1}{2}(\cos\theta\sin\theta + \theta)\right)\Big|_0^{\pi/2} = 4 + \frac{\pi}{4}$.



35 The curves intersect at $\theta = \pm\frac{1}{2}\cos^{-1}(1/16)$. By symmetry the total desired area is $A = 4 \cdot \frac{1}{2}\int_0^{\cos^{-1}(1/16)/2}(4\cos 2\theta - \frac{1}{4})\,d\theta = 2\left(2\sin 2\theta - \frac{\theta}{4}\right)\Big|_0^{\cos^{-1}(1/16)/2} = \frac{1}{4}\sqrt{255} - \frac{\cos^{-1}(1/16)}{4}$.



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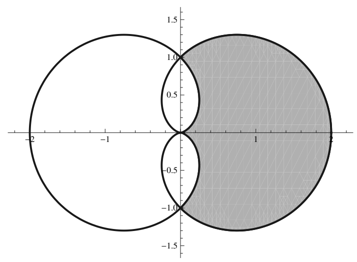
36 By symmetry, we can compute the area within the curve $r = 1 - \cos\theta$ for $0 \le \theta \le \pi/2$ and then quadruple it. We have

$$4 \cdot \frac{1}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta = 2\int_0^{\pi/2}(1 - 2\cos\theta + (1/2) + (1/2)\cos 2\theta)\,d\theta$$

$$= (3\theta - 4\sin\theta + (1/2)\sin 2\theta)\Big|_0^{\pi/2} = \frac{3\pi}{2} - 4 + 0 - (0 - 0 + 0) = \frac{3\pi}{2} - 4.$$



37 Note that we can compute the area within $1 + \cos\theta$ between 0 and $\pi/2$ and then subtract 1/4 of the area from the previous problem, and then double this difference. If we compute $\frac{1}{2}\int_0^{\pi/2}(1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}(1 + 2\cos\theta + \cos^2\theta)\,d\theta = \int_0^{\pi/2}(3/4 + \cos\theta + (1/4)\cos 2\theta)\,d\theta = (3\theta/4 + \sin\theta + (1/8)\sin 2\theta)\Big|_0^{\pi/2} = 3\pi/8 + 1 + 0 - (0 + 0 + 0) = 3\pi/8 + 1$.

If we subtract 1/4 of the previous result, we have $3\pi/8 + 1 - (3\pi/8 - 1) = 2$. Doubling this gives a final result of 4.

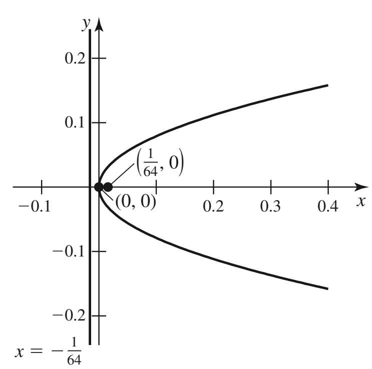
38

a. This represents a parabola.

b. We can write $y^2 = \frac{1}{16}x = 4 \cdot \frac{1}{64}x$, so $(p, 0) = (\frac{1}{64}, 0)$ is the focus, and the directrix is $x = -\frac{1}{64}$. The vertex is $(0, 0)$.

c. $e = 1$, because that is the case for all parabolas.

d.



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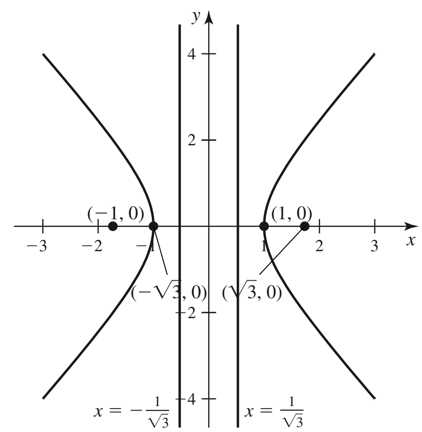
39

a. This represents a hyperbola with $a = 1$ and $b = \sqrt{2}$.

b. The vertices are $(\pm 1, 0)$, the foci are $(\pm c, 0)$ where $c^2 = a^2 + b^2 = 3$, so they are $(\pm\sqrt{3}, 0)$. The directrices are $x = \frac{\pm a^2}{c} = \frac{\pm 1}{\sqrt{3}}$.

c. The eccentricity is $e = \frac{c}{a} = \sqrt{3}$.

d. 

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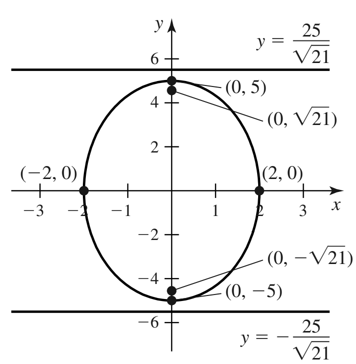
40

a. This represents an ellipse with $a = 5$ and $b = 2$.

b. The vertices are $(0, \pm 5)$. The foci are $(0, \pm c)$ where $c^2 = a^2 - b^2 = 25 - 4 = 21$, so they are $(0, \pm\sqrt{21})$. The directrices are $y = \frac{\pm a^2}{c} = \frac{\pm 25}{\sqrt{21}}$.

c. The eccentricity is $e = \frac{c}{a} = \frac{\sqrt{21}}{5}$.

d. 

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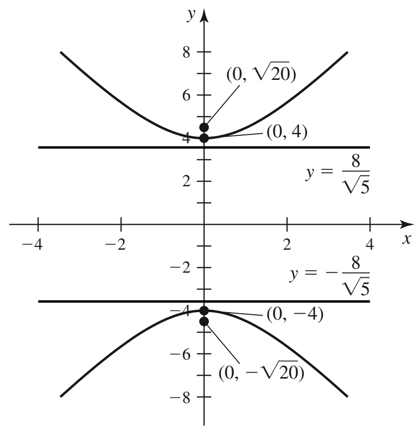
41

a. This can be written as $\frac{y^2}{16} - \frac{x^2}{4} = 1$. It is a hyperbola with $a = 4$ and $b = 2$.

b. The vertices are $(0, \pm 4)$. The foci are $(0, \pm c)$ where $c^2 = a^2 + b^2 = 16 + 4 = 20$, so they are $(0, \pm\sqrt{20})$. The directrices are $y = \frac{\pm a^2}{c} = \frac{\pm 16}{\sqrt{20}} = \frac{\pm 8}{\sqrt{5}}$.

c. The eccentricity is $e = \frac{c}{a} = \frac{\sqrt{20}}{4} = \frac{\sqrt{5}}{2}$.

d. 

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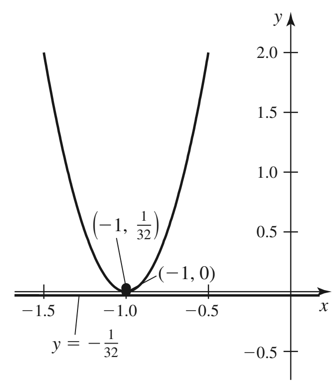
42

a. This can be written in the form $y = 8(x+1)^2$, so it is a parabola opening upward.

b. The vertex is $(-1, 0)$, and because $\frac{1}{8}y = (x+1)^2$, we have $p = \frac{1}{32}$ and the focus is $(-1, \frac{1}{32})$. The directrix is $y = -\frac{1}{32}$.

c. The eccentricity is $e = 1$, as it is for all parabolas.

d. 

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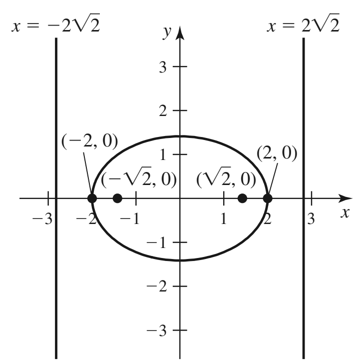
43

a. This can be written as $\frac{x^2}{4} + \frac{y^2}{2} = 1$, so it is an ellipse with $a = 2$ and $b = \sqrt{2}$.

b. The vertices are $(\pm 2, 0)$. The foci are $(\pm c, 0)$ where $c^2 = a^2 - b^2 = 4 - 2 = 2$, so they are $(\pm\sqrt{2}, 0)$. The directrices are $x = \frac{\pm a^2}{c} = \frac{\pm 4}{\sqrt{2}} = \pm 2\sqrt{2}$.

c. The eccentricity is $e = \frac{c}{a} = \frac{\sqrt{2}}{2}$.

d. 

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44

a. This matches graph (E).

b. This matches graph (D).

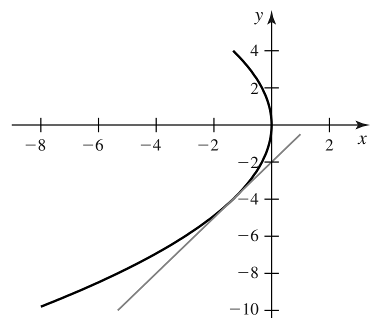
c. This matches graph (B).

d. This matches graph (F).

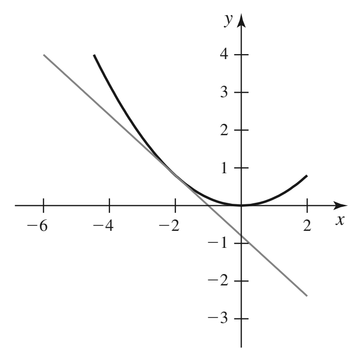
e. This matches graph (C).

f. This matches graph (A).

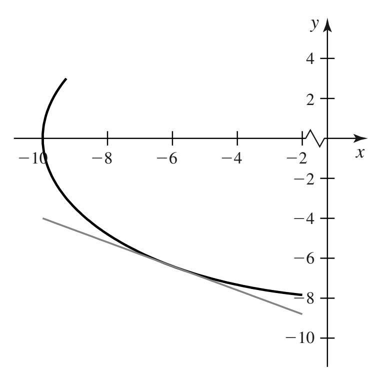
45 $2y\frac{dy}{dx} = -12$, so at the point in question, $\frac{dy}{dx} = 3/2$. So the equation of the tangent line is $y + 4 = \frac{3}{2}\left(x + \frac{4}{3}\right)$, or $y = \frac{3}{2}x - 2$.



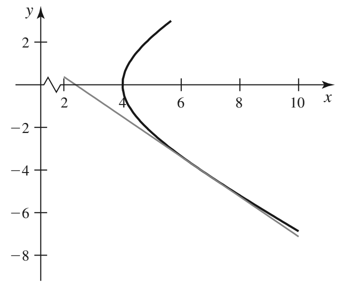
46 $2x = 5\frac{dy}{dx}$, so at the given point, we have $\frac{dy}{dx} = -\frac{4}{5}$. So the equation of the tangent line is $y - \frac{4}{5} = -\frac{4}{5}(x+2)$, or $y = -\frac{4}{5}x - \frac{4}{5}$.



47 $\frac{x}{50} + \frac{y}{32} \cdot \frac{dy}{dx} = 0$, so at the given point, $\frac{dy}{dx} = -\frac{6}{10} = -\frac{3}{5}$. So the equation of the tangent line is $y + \frac{32}{5} = -\frac{3}{5}(x+6)$, or $y = -\frac{3}{5}x - 10$.



48 $\frac{x}{8} - \frac{2y}{9} \cdot \frac{dy}{dx} = 0$, so at the given point, $\frac{dy}{dx} = -\frac{15}{16}$. The equation of the tangent line is therefore $y + 4 = -\frac{15}{16}\left(x - \frac{20}{3}\right)$, or $y = -\frac{15}{16}x + \frac{9}{4}$.



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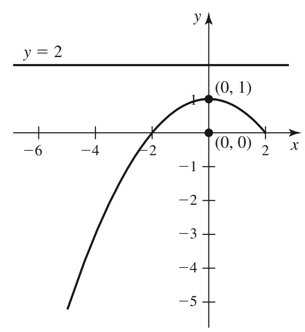
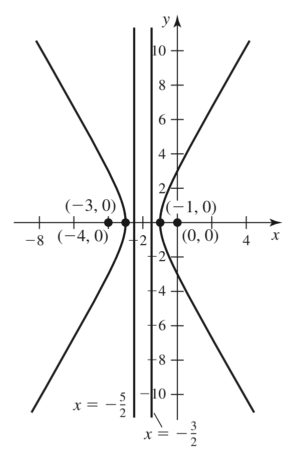
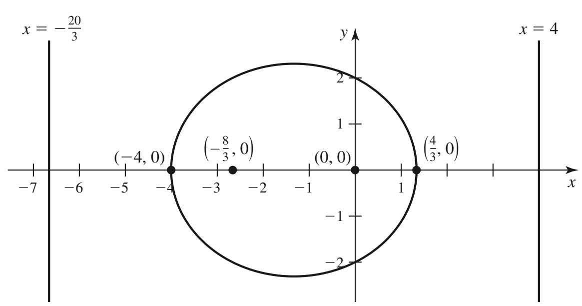
49 The eccentricity is 1, and the directrix is $y = 2$. The vertex is $(0, 1)$ and the focus is $(0, 0)$.



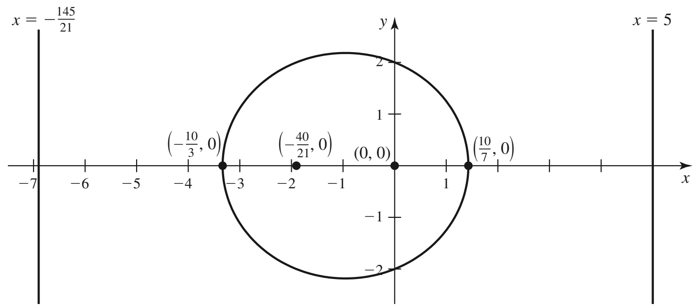
50 The eccentricity is 2, and the directrices are $x = -\frac{3}{2}$ and $x = -\frac{5}{2}$. The vertices are $(-1, 0)$ and $(-3, 0)$ and the foci are $(0, 0)$ and $(-4, 0)$.



51 The eccentricity is $\frac{1}{2}$, and the directrices are $x = 4$ and $x = -\frac{20}{3}$. The vertices are $(\frac{4}{3}, 0)$ and $(-4, 0)$ and the foci are $(0, 0)$ and $(-\frac{8}{3}, 0)$.



52 The eccentricity is $\frac{2}{5}$. The vertices are $(10/7, 0)$ and $(-10/3, 0)$, so the center is $(-20/21, 0)$. The foci are $(0, 0)$ and $(-40/21, 0)$. The directrices are $x = -\frac{145}{21}$ and $x = 5$.

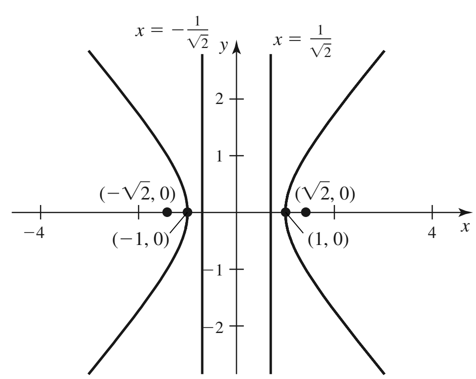


53

a. Recall that $\cos 2\theta = \cos^2\theta - \sin^2\theta$, so $r^2\cos(2\theta) = 1$ becomes $r^2(\cos^2\theta - \sin^2\theta) = x^2 - y^2 = 1$. The curve is a hyperbola.

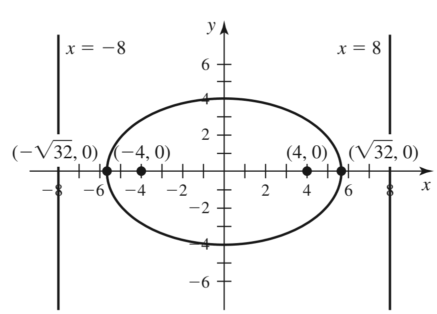
b. With $a = b = 1$, we have $c^2 = 2$, so the vertices are $(\pm 1, 0)$ and the foci are $(\pm\sqrt{2}, 0)$. The directrices are $x = \pm\frac{a^2}{c} = \pm\frac{1}{\sqrt{2}}$. The eccentricity is $e = \frac{c}{a} = \sqrt{2}$.

c. It does not have the form as in Theorem 11.4 because it does not have a focus at the origin.

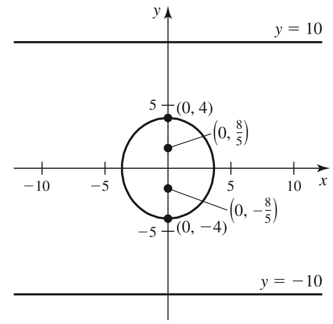


.

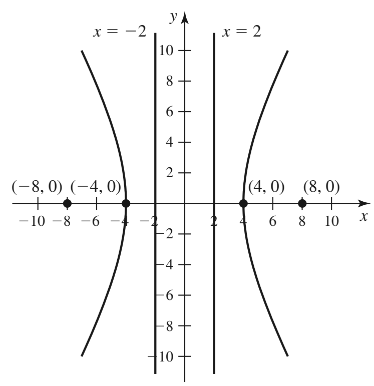
54 Because the center is halfway between the foci, it is $(0, 0)$. We must have $c = 4$ and because $\frac{a^2}{c} = d = 8$, we have $a^2 = 32$. So $b^2 = a^2 - c^2 = 32 - 16 = 16$. The ellipse has equation $\frac{x^2}{32} + \frac{y^2}{16} = 1$. The eccentricity is $\frac{c}{a} = \frac{4}{4\sqrt{2}} = \frac{\sqrt{2}}{2}$.



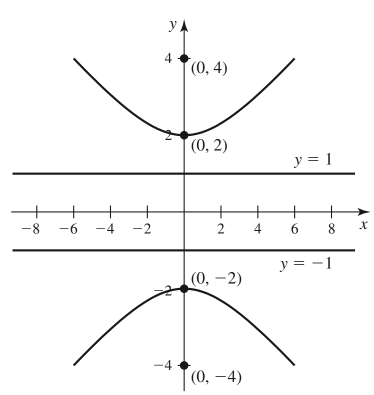
55 Because the center is halfway between the vertices, it is $(0,0)$. We must have $a = 4$ and because $\frac{a^2}{c} = d = 10$, we have $c = \frac{8}{5}$. So $b^2 = a^2 - c^2 = 16 - \frac{64}{25} = \frac{336}{25}$. The ellipse has equation $\frac{25x^2}{336} + \frac{y^2}{16} = 1$. The eccentricity is $\frac{c}{a} = \frac{8/5}{4} = \frac{2}{5}$.



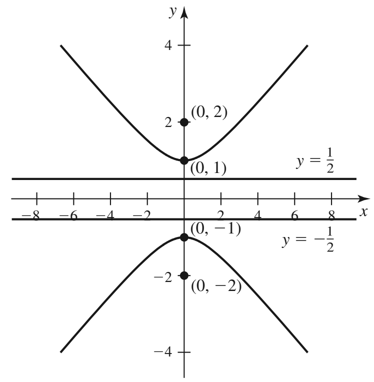
56 Because the center is halfway between the vertices, it is $(0,0)$. We must have $a = 4$ and because $\frac{a^2}{c} = d = 2$, we have $c = 8$. So $b^2 = 64 - 16 = 48$. The hyperbola has equation $\frac{x^2}{16} - \frac{y^2}{48} = 1$. The eccentricity is $\frac{c}{a} = \frac{8}{4} = 2$.



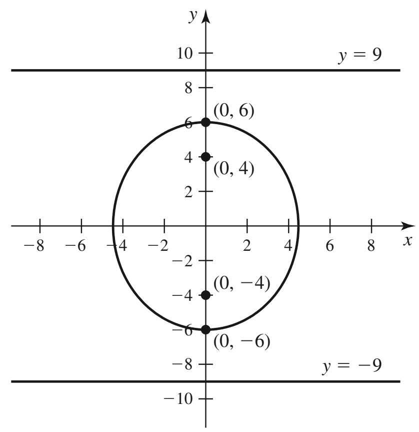
57 Because the center is halfway between the vertices, it is $(0,0)$. We must have $a = 2$ and because $\frac{a^2}{c} = d = 1$, we have $c = 4$. So $b^2 = 16-4 = 12$. The hyperbola has equation $\frac{y^2}{4} - \frac{x^2}{12} = 1$. The eccentricity is $\frac{c}{a} = \frac{4}{2} = 2$.



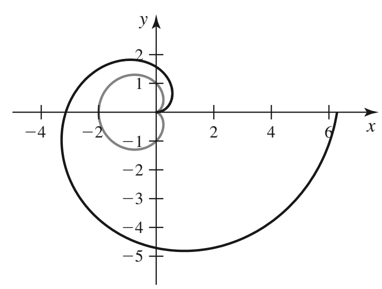
58 We have $c = 2$, $e = \frac{c}{a} = 2$, so $a = 1$. Also, $b^2 = c^2 - a^2 = 3$, so the equation is $\frac{y^2}{1} - \frac{x^2}{3} = 1$. We have $d = \frac{a^2}{c} = \frac{1}{2}$. The vertices are $(0, \pm 1)$ and the directrices are $y = \pm\frac{1}{2}$.



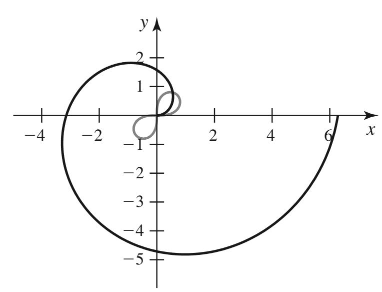
59 We have $a = 6$, $c = 4$ and $e = \frac{c}{a} = \frac{4}{6} = \frac{2}{3}$. Also, $b^2 = a^2 - c^2 = 36 - 16 = 20$, and the equation is $\frac{y^2}{36} + \frac{x^2}{20} = 1$. The vertices are $(\pm 2\sqrt{5}, 0)$. The directrices are $y = \pm\frac{a^2}{c} = \pm 9$.



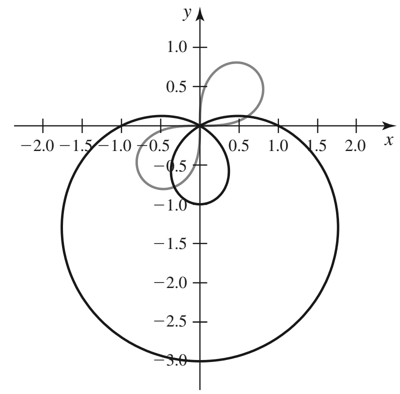
60 $1 - \cos\theta = \theta$, so $\theta = 0$ is a solution. Note that if $f(\theta) = 1 - \cos\theta - \theta$, then $f'(\theta) = \sin\theta - 1 \le 0$ for all θ. Thus this function is non-decreasing and the only solution is the one already found.



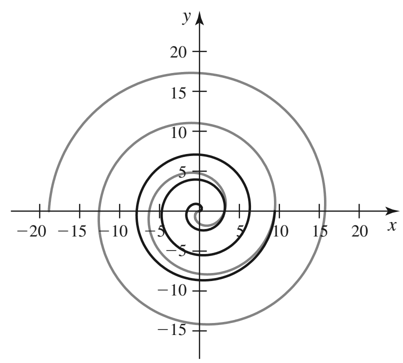
61 $\sin 2\theta = \theta^2$ when $\theta = 0$. Graphing the functions reveals a root near $\theta = 1$. A CAS reveals the intersection point to be $\theta \approx .9669$. In polar coordinates, the intersection points are $(0, 0)$ and $(.9669, .9669)$.



62 $\sin 2\theta = 2\sin\theta\cos\theta$, and $(1-2\sin\theta)^2 = 1-4\sin\theta+4\sin^2\theta$. The equation $1-4\sin\theta+4\sin^2\theta = 2\sin\theta\cos\theta$ does not lend itself to an analytic solution, however. A graphing utility shows three points of intersection, and a CAS reveals the origin as an intersection point, as well as the approximate polar intersection points $(.6148, .1938)$ and $(-.8445, 1.1738)$.



63 The curves intersect for $\theta = 0$. Note also that when $\theta = k\pi$ for k an odd integer, the curve $r = -\theta$ is at the polar point $(-k\pi, k\pi) = (k\pi, 0)$. And for $\theta = 2k\pi$, the curve $r = \frac{\theta}{2}$ is at the point $(k\pi, 0)$. So the curves intersect at these points.



64 Note that $a = ed$ and $b = a\sqrt{1-e^2}$, so the ellipse given by

$$r = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

has the same area as the original ellipse, but is centered at the origin. We compute the area of this ellipse instead. Using symmetry, we have

$$A = 4\cdot\frac{1}{2}\int_0^{\pi/2}\frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\,d\theta = 2\int_0^{\pi/2}\frac{a^2b^2\sec^2\theta}{a^2\tan^2\theta + b^2}\,d\theta = 2a^2\int_0^{\pi/2}\frac{\sec^2\theta}{\frac{a^2}{b^2}\tan^2\theta + 1}\,d\theta.$$

Let $u = \frac{a}{b}\tan\theta$ so that $du = \frac{a}{b}\sec^2\theta\,d\theta$. Then we have $A = 2ab\int_0^\infty \frac{1}{1+u^2}\,du = 2ab\cdot\lim_{z\to\infty}\tan^{-1} z = \frac{2ab\pi}{2} = \pi ab$.

65 By symmetry, we can focus on the region in the first quadrant. That area is given by $A = xy$ where $y = \sqrt{b^2 - \frac{b^2}{a^2}x^2}$. So

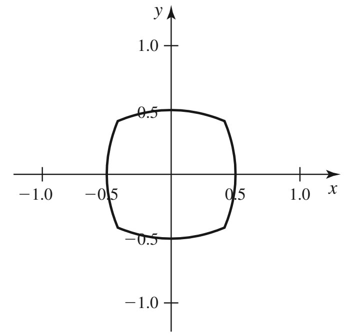
$$A(x) = x\sqrt{b^2 - \frac{b^2}{a^2}x^2},$$

so

$$A'(x) = \sqrt{b^2 - \frac{b^2}{a^2}x^2} - \frac{b^2x^2}{a^2\sqrt{b^2 - \frac{b^2}{a^2}x^2}}.$$

Setting the derivative equal to 0 and clearing denominators yields $\left(b^2 - \frac{b^2}{a^2}x^2\right)a^2 - b^2x^2 = 0$, and solving for x gives the critical point $x = \frac{\sqrt{2}}{2}a$. Because this is the only critical point and it clearly does not give a minimum (because $A(0) = A(a) = 0$), it must yield a maximum. The whole rectangle has dimensions $\sqrt{2}a \times \sqrt{2}b$, and area $2ab$.

66 We focus on the first quadrant and then use symmetry for the rest. Consider points within the triangle with vertices $(0,0)$, $(a,0)$ and (a,a). Any point (x,y) within this triangle is closer to the line $x=a$ than any other side of the square, so the distance from this point to the square is $a-x$. The distance from (x,y) to the origin is $\sqrt{x^2+y^2}$. So we have $x^2+y^2=x^2-2ax+a^2$, or the parabola $y^2=-2ax+a^2$.
In the triangle with vertices $(0,0)$, $(0,a)$ and (a,a), the point (x,y) is closer to the line $y=a$, so the distance from that point to the line is $a-y$. Therefore, the points we are seeking lie along a curve where $x^2+y^2=y^2-2ay+a^2$, or the parabola $x^2=-2ay+a^2$. Note that the two curves mentioned intersect intersect along the line $y=x$ at the point $(a(\sqrt{2}-1), a(\sqrt{2}-1))$. The curve desired is thus the union of portions of 4 parabolas.



67 The area of the ellipse in the first quadrant is $\frac{\pi ab}{4}$, so we are seeking θ_0 so that

$$\frac{\pi ab}{8} = \frac{1}{2}\int_0^{\theta_0} \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta = \frac{a^2}{2}\int_0^{\theta_0} \frac{\sec^2\theta}{\frac{a^2}{b^2}\tan^2\theta+1}\,d\theta.$$

Let $u=\frac{a}{b}\tan\theta$ so that $du=\frac{a}{b}\sec^2\theta\,d\theta$. Then we have $\frac{\pi ab}{8}=\frac{ab}{2}\int_0^{\frac{a}{b}\tan\theta_0}\frac{1}{1+u^2}\,du=\frac{ab}{2}\tan^{-1}(\frac{a}{b}\tan(\theta_0))$. Note that this equation is satisfied when $\tan(\theta_0)=\frac{b}{a}$, because then the expression on the right-hand side of that equation is $\frac{ab}{2}\cdot\frac{\pi}{4}=\frac{\pi ab}{8}$. So the desired value of m is $\tan(\theta_0)=\frac{b}{a}$.

68

a. The curves are tangent when there is only one point of intersection in the first quadrant. This occurs when $x^2-p^2x^4=1$ has only one solution. This quadratic-type equation $-p^2(x^2)^2+(x^2)-1=0$ has solution $x^2=\frac{1\pm\sqrt{1-4p^2}}{2p^2}$, and the discriminant $1-4p^2$ is 0 for $p=1/2$.

b. The two curves intersect for $x^2=\frac{1\pm 0}{2p^2}=2$, so for $x=\sqrt{2}$. The corresponding value for y is $\frac{1}{2}(\sqrt{2})^2=1$.

c. Using the same line of reasoning, we seek the value of p so that $\frac{x^2}{a^2}-\frac{p^2x^4}{b^2}=1$, which yields the quadratic-type equation $p^2a^2(x^2)^2-b^2(x^2)+a^2b^2=0$. The discriminant is 0 when $b^4-4p^2a^4b^2=0$, which occurs when $p=\frac{b}{2a^2}$. The point of intersection is $x=\sqrt{2}a$. The corresponding value of y is $px^2=b$.

69 Note that $Q=(a\cos\theta, a\sin\theta)$ and $R=(b\cos\theta, b\sin\theta)$, where θ is the angle formed by l and the x-axis. Then $P=(a\sin\theta, b\cos\theta)$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, because it satisfies that equation.

70 The focal point is at the origin, the directrix is $x=-\frac{3}{2}$, so $d=\frac{3}{2}$, and $r=\frac{ed}{1-e\cos\theta}$ where $e=\frac{c}{a}$. Because c is the distance from the center to the focal point, we have $c=2$, and because a is the distance from the center to a vertex, we have $a=1$. Thus $e=2$ and $r=\frac{3}{1-2\cos\theta}$.

71 The focal point is at the origin, the directrix is $y=-d$, so we have an equation of the form $r=\frac{ed}{1-e\sin\theta}$. Because c is the distance from the center to the focal point, we have $c=3/8$, and because a is the distance from the center to a vertex, we have $a=9/8$. Then we have $e=\frac{c}{a}=\frac{3/8}{9/8}=\frac{1}{3}$, and $d=\frac{a^2}{c}-\frac{3}{8}=3$. Thus $r=\frac{1}{1-\frac{1}{3}\sin\theta}=\frac{3}{3-\sin\theta}$.

Chapter 11

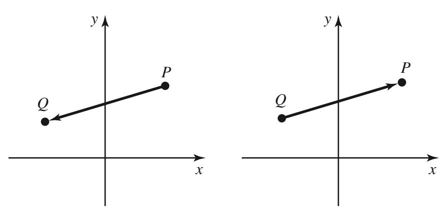
Vectors and Vector-Valued Functions

11.1 Vectors in the Plane

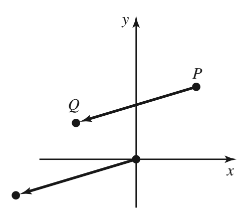
11.1.1 The coordinates of a point determine its location, but a given point has no width or breadth, so it has no size or direction. A nonzero vector has size (magnitude) and direction, but it has no location in the sense that it can be translated to a different initial point and be considered the same vector.

11.1.2 A position vector is one with its tail at the origin.

11.1.3



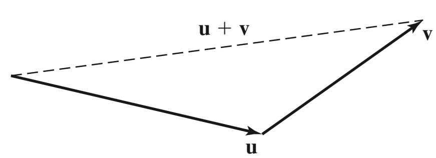
11.1.4



11.1.5 Two vectors are equal if they have the same magnitude and direction. Given a position vector, any translation of that vector to a different initial point yields an equivalent vector. Because there are infinitely many such translations which don't change the given vector's direction or magnitude, there are infinitely many vectors equivalent to the given one.

11.1.6

To find the sum $\mathbf{u} + \mathbf{v}$, translate $\mathbf{v}$ so that its tail is at the head of $\mathbf{u}$. The sum of the two vectors is the one whose tail is the tail of $\mathbf{u}$ and whose head is the head of $\mathbf{v}$.



11.1.7 If $c > 0$ is given, the scalar multiple $c\mathbf{v}$ of the vector $\mathbf{v}$ is obtained by scaling the magnitude of $\mathbf{v}$ by a factor of c, and keeping the direction the same. If $c < 0$, then the head and tail of $\mathbf{v}$ are interchanged, and then the vector's magnitude is scaled by a factor of $|c|$.

11.1.8 If $P(x_0, y_0)$ and $Q(x_1, y_1)$ are given, then the vector $\overrightarrow{PQ}$ is given by $\langle x_1 - x_0, y_1 - y_0 \rangle$.

11.1.9 $\mathbf{u} + \mathbf{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$.

11.1.10 $c\mathbf{v} = c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle$.

11.1.11 $|\mathbf{v}| = |\langle v_1, v_2 \rangle| = \sqrt{v_1^2 + v_2^2}$.

11.1.12 $\mathbf{v} = \langle v_1, v_2 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$.

11.1.13 If $P(p_1, p_2)$ and $Q(q_1, q_2)$ are given, then $\left|\overrightarrow{PQ}\right| = |\langle q_1 - p_1, q_2 - p_2 \rangle| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$.

11.1.14 Given a vector $\mathbf{v} = \langle v_1, v_2 \rangle$, the vectors $\frac{1}{\sqrt{v_1^2+v_2^2}} \cdot \mathbf{v} = \frac{\mathbf{v}}{|\mathbf{v}|}$ and $-\frac{1}{\sqrt{v_1^2+v_2^2}} \cdot \mathbf{v} = -\frac{\mathbf{v}}{|\mathbf{v}|}$ are unit vectors parallel to $\mathbf{v}$.

11.1.15 The vector $10 \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = 10 \cdot \frac{1}{\sqrt{9+4}} \cdot \langle 3, -2 \rangle = \langle \frac{30}{\sqrt{13}}, -\frac{20}{\sqrt{13}} \rangle$ has the desired properties.

11.1.16 The unit vector in the desired direction is given by $\langle \sqrt{2}/2, -\sqrt{2}/2 \rangle$, so the desired vector is $100 \cdot \langle \sqrt{2}/2, -\sqrt{2}/2 \rangle = \langle 50\sqrt{2}, -50\sqrt{2} \rangle$.

11.1.17 The vectors in choices a, c, and e are all equal to $\overrightarrow{CE}$.

11.1.18 The vectors in choices b, c, and e are equal to $\overrightarrow{BK}$.

11.1.19

a. $3\mathbf{v}$ b. $2\mathbf{u}$ c. $-3\mathbf{u}$ d. $-2\mathbf{u}$ e. $\mathbf{v}$

11.1.20

a. $2\mathbf{v}$ b. $-2\mathbf{v}$ c. $3\mathbf{u}$ d. $-5\mathbf{v}$ e. $-3\mathbf{u}$

11.1.21

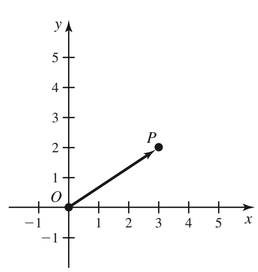
a. $3\mathbf{u} + 3\mathbf{v}$ b. $\mathbf{u} + 2\mathbf{v}$ c. $2\mathbf{u} + 5\mathbf{v}$ d. $-2\mathbf{u} + 3\mathbf{v}$ e. $3\mathbf{u} + 2\mathbf{v}$
f. $-3\mathbf{u} - 2\mathbf{v}$ g. $-2\mathbf{u} - 4\mathbf{v}$ h. $\mathbf{u} - 4\mathbf{v}$ i. $-\mathbf{u} - 6\mathbf{v}$

11.1.22

a. $\mathbf{u} + 3\mathbf{v}$ b. $\mathbf{u} + 3\mathbf{v}$ c. $2\mathbf{u} + 2\mathbf{v}$ d. $2\mathbf{u} - 3\mathbf{v}$ e. $-\mathbf{u} - 2\mathbf{v}$
f. $4\mathbf{u} + 4\mathbf{v}$ g. $-\mathbf{u} + 2\mathbf{v}$ h. $-\mathbf{u} + 2\mathbf{v}$ i. $2\mathbf{u} - 4\mathbf{v}$

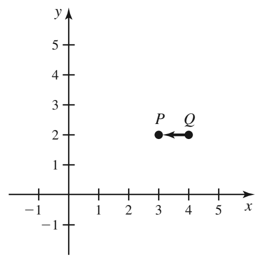
11.1.23

a. $\overrightarrow{OP}$

i. 

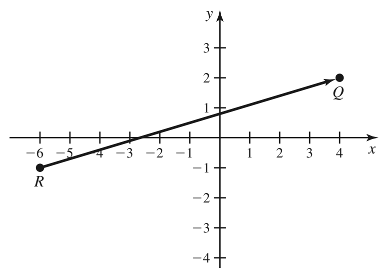
ii. $|3\mathbf{i} + 2\mathbf{j}| = \sqrt{13}$.

b. $\overrightarrow{QP}$

i. 

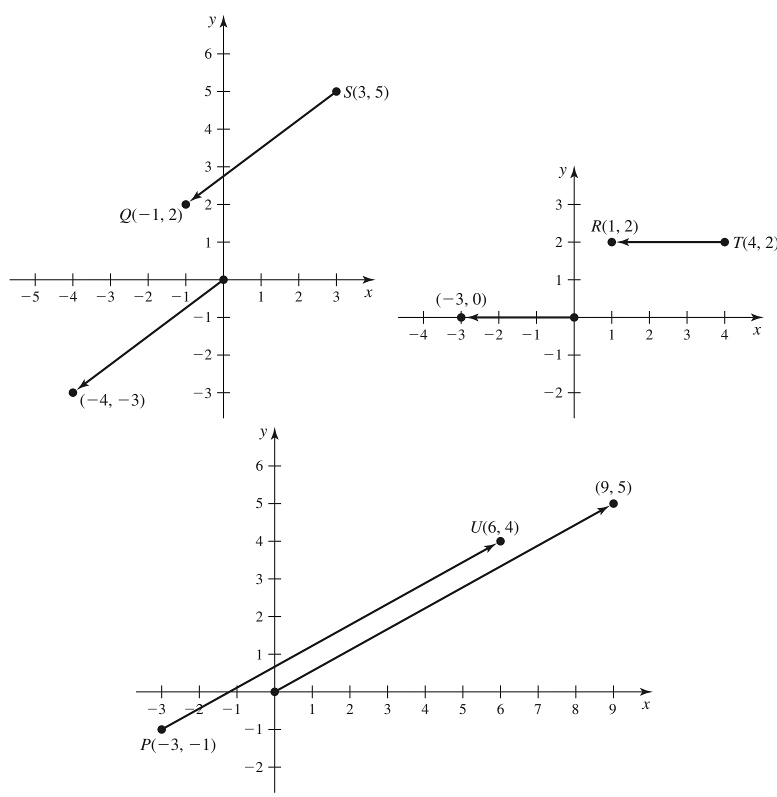
ii. $|-\mathbf{i} + 0 \cdot \mathbf{j}| = 1.$

c. $\overrightarrow{RQ}$

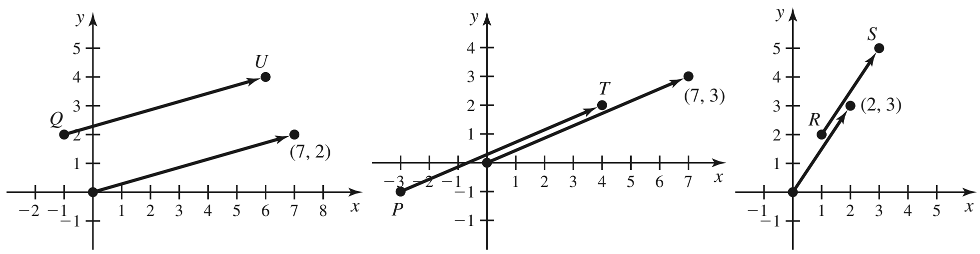
i. 

ii. $|10\mathbf{i} + 3\mathbf{j}| = \sqrt{109}.$

11.1.24 $\overrightarrow{PU} = \langle 9, 5 \rangle$, $\overrightarrow{TR} = \langle -3, 0 \rangle$, $\overrightarrow{SQ} = \langle -4, -3 \rangle$.



11.1.25 $\overrightarrow{QU} = \langle 7, 2 \rangle$, $\overrightarrow{PT} = \langle 7, 3 \rangle$, $\overrightarrow{RS} = \langle 2, 3 \rangle$.



11.1.26 $\overrightarrow{PQ} = \langle 2, 3\rangle$, $\overrightarrow{RS} = \langle 2, 3\rangle$, and $\overrightarrow{TU} = \langle 2, 2\rangle$, so $\overrightarrow{PQ} = \overrightarrow{RS}$.

11.1.27 $\overrightarrow{QT} = \langle 5, 0\rangle$, while $\overrightarrow{SU} = \langle 3, -1\rangle$.

11.1.28 $\mathbf{u} + \mathbf{v} = \langle 4, -2\rangle + \langle -4, 6\rangle = \langle 0, 4\rangle$.

11.1.29 $\mathbf{w} - \mathbf{u} = \langle 0, 8\rangle - \langle 4, -2\rangle = \langle -4, 10\rangle$.

11.1.30 $2\mathbf{u} + 3\mathbf{v} = 2\langle 4, -2\rangle + 3\langle -4, 6\rangle = \langle -4, 14\rangle$.

11.1.31 $\mathbf{w} - 3\mathbf{v} = \langle 0, 8\rangle - 3\langle -4, 6\rangle = \langle 12, -10\rangle$.

11.1.32 $10\mathbf{u} - 3\mathbf{v} + \mathbf{w} = 10\langle 4, -2\rangle - 3\langle -4, 6\rangle + \langle 0, 8\rangle = \langle 52, -30\rangle$.

11.1.33 $8\mathbf{w} + \mathbf{v} - 6\mathbf{u} = 8\langle 0, 8\rangle + \langle -4, 6\rangle - 6\langle 4, -2\rangle = \langle -28, 82\rangle$.

11.1.34 $|\mathbf{u} + \mathbf{v}| = |\langle 4, -3\rangle| = \sqrt{16 + 9} = 5$.

11.1.35 $|-2\mathbf{v}| = |\langle -2, -2\rangle| = \sqrt{4 + 4} = 2\sqrt{2}$.

11.1.36 $|\mathbf{u} + \mathbf{v} + \mathbf{w}| = |\langle 3, -4\rangle + \langle 1, 1\rangle + \langle -1, 0\rangle| = |\langle 3, -3\rangle| = 3\sqrt{2}$.

11.1.37 $|2\mathbf{u} + 3\mathbf{v} - 4\mathbf{w}| = |2\langle 3, -4\rangle + 3\langle 1, 1\rangle - 4\langle -1, 0\rangle| = |\langle 13, -5\rangle| = \sqrt{194}$.

11.1.38 Two vectors parallel to $\mathbf{u}$ with magnitude four times that of $\mathbf{u}$ are the vectors $\pm 4\mathbf{u} = \pm 4\langle 3, -4\rangle = \pm\langle 12, -16\rangle$. So they are $\langle 12, -16\rangle$ and $\langle -12, 16\rangle$.

11.1.39 The vectors we seek are $\pm 3\mathbf{v}$, so they are $\langle 3, 3\rangle$ and $\langle -3, -3\rangle$.

11.1.40 $|2\mathbf{u}| = |\langle 6, -8\rangle| = \sqrt{36 + 84} = 10$. $|7\mathbf{v}| = |\langle 7, 7\rangle| = \sqrt{98} = 7\sqrt{2}$. Because $7\sqrt{2} < 10$, $2\mathbf{u}$ has larger magnitude.

11.1.41 $|\mathbf{u} - \mathbf{v}| = |\langle 3, -4\rangle - \langle 1, 1\rangle| = |\langle 2, -5\rangle| = \sqrt{29}$. $|\mathbf{w} - \mathbf{u}| = |\langle -1, 0\rangle - \langle 3, -4\rangle| = |\langle -4, 4\rangle| = \sqrt{32} = 4\sqrt{2}$. $\mathbf{w} - \mathbf{u}$ has the greater magnitude.

11.1.42 $\overrightarrow{PQ} = \langle 3, -4\rangle - \langle -4, 1\rangle = \langle 7, -5\rangle = 7\mathbf{i} - 5\mathbf{j}$.

11.1.43 $\overrightarrow{QR} = \langle 2, 6\rangle - \langle 3, -4\rangle = \langle -1, 10\rangle = -\mathbf{i} + 10\mathbf{j}$.

11.1.44 $\mathbf{u} = \frac{\overrightarrow{QR}}{|\overrightarrow{QR}|} = \frac{\langle -1, 10\rangle}{\sqrt{101}} = -\frac{1}{\sqrt{101}}\mathbf{i} + \frac{10}{\sqrt{101}}\mathbf{j}$.

11.1.45 $\mathbf{u} = \frac{\overrightarrow{PR}}{|\overrightarrow{PR}|} = \frac{\langle 2,6\rangle - \langle -4,1\rangle}{|\overrightarrow{PR}|} = \frac{\langle 6,5\rangle}{\sqrt{36+25}} = \langle 6/\sqrt{61}, 5/\sqrt{61}\rangle$ is one unit vector, while the other is $-\mathbf{u} = \langle -6/\sqrt{61}, -5/\sqrt{61}\rangle$

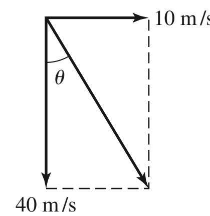
11.1.46 Using the result of the previous problem, the two vectors are $4\langle 6/\sqrt{61}, 5/\sqrt{61}\rangle = \langle 24/\sqrt{61}, 20/\sqrt{61}\rangle$ and $-4\langle 6/\sqrt{61}, 5/\sqrt{61}\rangle = \langle -24/\sqrt{61}, -20/\sqrt{61}\rangle$.

11.1.47 $\overrightarrow{QP} = \langle -4, 1\rangle - \langle 3, -4\rangle = \langle -7, 5\rangle$. A unit vector parallel to $\overrightarrow{QP}$ is $\frac{1}{\sqrt{74}}\langle -7, 5\rangle$. So the desired vectors are $\frac{4}{\sqrt{74}}\langle -7, 5\rangle$ and $-\frac{4}{\sqrt{74}}\langle -7, 5\rangle$.

11.1.48 Let $\mathbf{w} = \langle 0, -10\rangle$ represent the water, and let $\mathbf{b} = \langle 20, 0\rangle$ represent the boat relative to the shore. Then the vector $\mathbf{v}$ which represents the boat relative to the water is given by $\mathbf{v} + \mathbf{w} = \mathbf{b}$, so $\mathbf{v} = \mathbf{b} - \mathbf{w} = \langle 20, 10\rangle$. Note that $|\langle 20, 10\rangle| = 10\sqrt{5} \approx 22.4$ miles per hour represents the speed of the boat. The direction is the direction of the vector $\langle 2, 1\rangle$ which is $\tan^{-1}(1/2) \cdot \frac{180}{\pi} \approx 26.6$ degrees north of east.

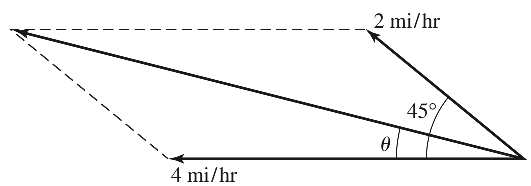
11.1.49 Let $\mathbf{w} = \langle 0, -5\rangle$ represent the water, and let $\mathbf{v_w} = \langle 40, 0\rangle$ represent the boat relative to the water. Then relative to the shore we have $\mathbf{v_s} = \mathbf{w} + \mathbf{v_w} = \langle 40, -5\rangle$, which has magnitude $\sqrt{1625} \approx 40.3$ km/hr.

11.1.50 Let $\mathbf{p}$ represent the vector's terminal velocity vector. $\mathbf{p} = \langle 10, -40\rangle$, and $|\mathbf{p}| = \sqrt{100 + 1600} = 10\sqrt{17}$. $\theta = \tan^{-1}(10/40) \approx .245$ radians, or 14.037 degrees. Thus, the speed is $10\sqrt{17}$ meters per second, and the direction is about 14.037 degrees east of vertical.



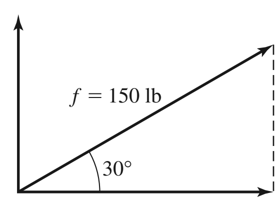
11.1.51 The plane's vector is given by $\mathbf{u} = -320\mathbf{i} + -20\sqrt{2}(\mathbf{i}+\mathbf{j}) = (-320 - 20\sqrt{2})\mathbf{i} - 20\sqrt{2}\mathbf{j}$. The magnitude of $\mathbf{u}$ is $\sqrt{(-320 - 20\sqrt{2})^2 + (-20\sqrt{2})^2} \approx 349.43$ miles per hour. $\theta = \tan^{-1}\left(\frac{20\sqrt{2}}{320+20\sqrt{2}}\right) \approx .0810$ radians, or about 4.64 degrees south of west.

11.1.52 The canoe's vector $\mathbf{u}$ is given by $\mathbf{u} = -4\mathbf{i} + (-\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}) = -(4 + \sqrt{2})\mathbf{i} + \sqrt{2}\mathbf{j}$. The magnitude of $\mathbf{u}$ is given by $\sqrt{(4 + \sqrt{2})^2 + (\sqrt{2})^2} \approx 5.5958$ miles per hour. $\theta = \tan^{-1}\left(\frac{\sqrt{2}}{(4+\sqrt{2})}\right) \approx 0.2555$ radians, or about 14.64 degrees. The canoe has speed about 5.6 miles per hour in the direction 14.64 degrees north of west.



11.1.53 Let $\mathbf{u} = \mathbf{i}$ represent the current and $\mathbf{v} = \sqrt{3}\cos(\pi/6)\mathbf{i} + \sqrt{3}\sin(\pi/6)\mathbf{j} = \frac{3}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ represent the boat relative to land. If $\mathbf{w}$ represents the wind, then $\mathbf{u} + \mathbf{w} = \mathbf{v}$, so $\mathbf{w} = \mathbf{v} - \mathbf{u} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$. Then $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$, or 60 degrees. The speed of the wind is 1 meter per second in the direction 60 degrees north of east (or 30 degrees east of north).

11.1.54 $\mathbf{F} = 150\cos(\pi/6)\mathbf{i} + 150\sin(\pi/6)\mathbf{j} = 75\sqrt{3}\mathbf{i} + 75\mathbf{j}$. The horizontal component of the force is $75\sqrt{3}$ pounds, and the vertical component is 75 pounds.



11.1.55

a. $\mathbf{F} = 40\cos(\pi/3)\mathbf{i} + 40\sin(\pi/3)\mathbf{j} = 20\mathbf{i} + 20\sqrt{3}\mathbf{j}$, so the horizontal component is 20 and the vertical is $20\sqrt{3}$.

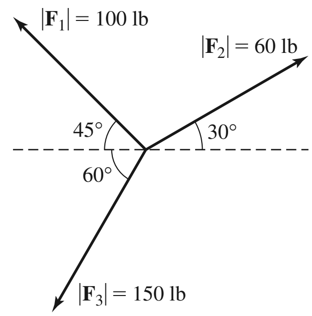
b. Yes. If it is 45 degrees, the horizontal component would be $40\cos(\pi/4) = 20\sqrt{2} > 20$.

c. No. If it is 45 degrees, the vertical component would be $40\sin(\pi/4) = 20\sqrt{2} < 20\sqrt{3}$.

11.1.56 Let $\mathbf{F_1} = 100\cos(\pi/3)\mathbf{i} + 100\sin(\pi/3)\mathbf{j} = 50\mathbf{i} + 50\sqrt{3}\mathbf{j}$, and let $\mathbf{F_2} = 60\cos(\pi/6)\mathbf{i} + 60\sin(\pi/6)\mathbf{j} = 30\sqrt{3}\mathbf{i} + 30\mathbf{j}$. Note that $\mathbf{F_2}$ has a greater horizontal component, because $30\sqrt{3} \approx 51.96 > 50$.

11.1.57 Let the magnitude of the force on the two chains be f. Let $\mathbf{F_1} = (-\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j})f$ and let $\mathbf{F_2} = (\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j})f$. Then $\mathbf{F_1} + \mathbf{F_2} - 500\mathbf{j} = 0$, and solving for f yields $f = 250\sqrt{2}$ pounds.

11.1.58 $\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} = \sqrt{2}(-50\mathbf{i} + 50\mathbf{j}) + (30\sqrt{3}\mathbf{i} + 30\mathbf{j}) + (-75\mathbf{i} - 75\sqrt{3}\mathbf{j}) = (-50\sqrt{2} + 30\sqrt{3} - 75)\mathbf{i} + (50\sqrt{2} + 30 - 75\sqrt{3})\mathbf{j}$. Thus $|\mathbf{F}|$ is about $\sqrt{(-50\sqrt{2} + 30\sqrt{3} - 75)^2 + (50\sqrt{2} + 30 - 75\sqrt{3})^2}$ which is about 98.19 pounds. $\theta = \tan^{-1}\left(\frac{50\sqrt{2}+30-75\sqrt{3}}{-50\sqrt{2}+30\sqrt{3}-75}\right) \approx 17.3$ degrees. The magnitude of the net force is about 98 pounds in the direction 17.3 degrees south of west.



11.1.59

a. True. This follows because $(\mathbf{u}+\mathbf{v})+\mathbf{w} = (\mathbf{w}+\mathbf{u})+\mathbf{v}$ (vector addition is commutative and associative.)

b. True. This is because $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

c. False. For example, if $\mathbf{u} = \langle 3,4\rangle$ and $\mathbf{v} = \langle -3,-1\rangle$, then $|\mathbf{u}+\mathbf{v}| = |\langle 0,3\rangle| = 3$, while $|\mathbf{u}| = 5$.

d. False. For example, if $\mathbf{u} = \langle 3,4\rangle$ and $\mathbf{v} = \langle -1,-4\rangle$, then $|\mathbf{u}+\mathbf{v}| = |\langle 2,0\rangle| = 2$, while $|\mathbf{u}|+|\mathbf{v}| = 5+\sqrt{17}$.

e. False. For example, if $\mathbf{u} = \langle 3,0\rangle$ and $\mathbf{v} = \langle 6,0\rangle$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel, but have different lengths.

f. False. For example, given $A(0,0)$, $B(3,4)$, $C(1,1)$ and $D(4,5)$, we have $\overrightarrow{AB} = \langle 3,4\rangle$ and $\overrightarrow{CD} = \langle 3,4\rangle$, but $A \neq C$ and $B \neq D$.

g. False. For example, $\mathbf{u} = \langle 0,1\rangle$ and $\mathbf{v} = \langle -1,0\rangle$ are perpendicular, but $|\mathbf{u}+\mathbf{v}| = \sqrt{2}$, while $|\mathbf{u}|+|\mathbf{v}| = 2$.

h. True. Suppose $\mathbf{v} = k\mathbf{u}$ with $k > 0$. Then

$$|\mathbf{u}+\mathbf{v}| = |\mathbf{u}+k\mathbf{u}| = |(1+k)\mathbf{u}| = (1+k)\,|\mathbf{u}| = |\mathbf{u}| + k\,|\mathbf{u}| = |\mathbf{u}| + |k\mathbf{u}| = |\mathbf{u}| + |\mathbf{v}|.$$

11.1.60

a. $\overrightarrow{AB} = \langle 6,16\rangle - \langle -2,0\rangle = \langle 8,16\rangle$.

b. $\overrightarrow{AC} = \langle 1,4\rangle - \langle -2,0\rangle = \langle 3,4\rangle$.

c. $\overrightarrow{EF} = \langle 3\sqrt{2}, -4\sqrt{2}\rangle - \langle \sqrt{2}, \sqrt{2}\rangle = \langle 2\sqrt{2}, -5\sqrt{2}\rangle$.

d. $\overrightarrow{CD} = \langle 5,4\rangle - \langle 1,4\rangle = \langle 4,0\rangle$.

11.1.61

a. Because the magnitude of $\mathbf{v}$ is $\sqrt{36+64} = 10$, the two desired vectors are $\langle 6/10, -8/10\rangle = \langle 3/5, -4/5\rangle$ and $\langle -3/5, 4/5\rangle$.

b. If the magnitude of $\mathbf{v}$ is 1, then $\sqrt{\frac{1}{9} + b^2} = 1$, so $b^2 = \frac{8}{9}$, so $b = \pm\frac{2\sqrt{2}}{3}$.

c. If the magnitude of $\mathbf{w}$ is 1, then $\sqrt{a^2 + \frac{a^2}{9}} = 1$, so $\frac{10a^2}{9} = 1$, so $a = \pm\frac{3}{\sqrt{10}}$.

11.1.62 $\overrightarrow{AB} = \langle 3, 6\rangle$ and $\overrightarrow{CD} = \langle b - a + 2, a - b - 7\rangle$. The system of linear equations $-a + b + 2 = 3$, $a - b - 7 = 6$ has no solution, so there are no values for a and b which will make these vectors equal.

11.1.63 $10\langle a, b\rangle = \langle 2, -3\rangle$, so $10a = 2$, and $a = 1/5$. Also, $10b = -3$, so $b = -3/10$. Thus $\mathbf{x} = \langle 1/5, -3/10\rangle$.

11.1.64 $2\langle a, b\rangle + \langle 2, -3\rangle = \langle -4, 1\rangle$, so $\langle a, b\rangle = \langle -3, 2\rangle = \mathbf{x}$.

11.1.65 $3\langle a, b\rangle - 4\langle 2, -3\rangle = \langle -4, 1\rangle$, so $\langle a, b\rangle = \frac{1}{3}\langle 4, -11\rangle = \mathbf{x}$.

11.1.66 $-4\langle a, b\rangle = \langle 2, -3\rangle - 8\langle -4, 1\rangle = \langle 34, -11\rangle$, so $\langle a, b\rangle = \frac{1}{4}\langle -34, 11\rangle = \mathbf{x}$.

11.1.67 $\langle 4, -8\rangle = 4\mathbf{i} + -8\mathbf{j}$.

11.1.68 Suppose $\langle 4, -8\rangle = c_1\langle 1, 1\rangle + c_2\langle -1, 1\rangle$. Then $c_1 - c_2 = 4$ and $c_1 + c_2 = -8$. Adding these two equations to each other yields $2c_1 = -4$, so $c_1 = -2$. And thus $c_2 = -6$. We have $\langle 4, -8\rangle = -2\mathbf{u} - 6\mathbf{v}$.

11.1.69 Let $\langle a, b\rangle = c_1\mathbf{u} + c_2\mathbf{v}$. Then $c_1 - c_2 = a$ and $c_1 + c_2 = b$. Adding these two equations to each other yields $2c_1 = a + b$, so $c_1 = \frac{a+b}{2}$. And thus $c_2 = \frac{b-a}{2}$. We have $\langle a, b\rangle = \frac{a+b}{2}\mathbf{u} + \frac{b-a}{2}\mathbf{v}$.

11.1.70 Because $2\mathbf{u} = \mathbf{i}$ and $2(\mathbf{u} - 4\mathbf{v}) = 2\mathbf{j}$, we can conclude that $8\mathbf{v} = \mathbf{i} - 2\mathbf{j}$ (by substracting.) Thus $\mathbf{u} = \frac{1}{2}\mathbf{i}$ and $\mathbf{v} = \frac{1}{8}\mathbf{i} - \frac{1}{4}\mathbf{j}$.

11.1.71 Because $2\mathbf{u} + 3\mathbf{v} = \mathbf{i}$ and $-2(\mathbf{u} - \mathbf{v}) = -2\mathbf{j}$, we can conclude that $3\mathbf{v} + 2\mathbf{v} = \mathbf{i} - 2\mathbf{j}$ (by adding), so $\mathbf{v} = \frac{1}{5}\mathbf{i} - \frac{2}{5}\mathbf{j}$. It then follows that $\mathbf{u} = \mathbf{v} + \mathbf{j} = \frac{1}{5}\mathbf{i} - \frac{2}{5}\mathbf{j} + \mathbf{j} = \frac{1}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$.

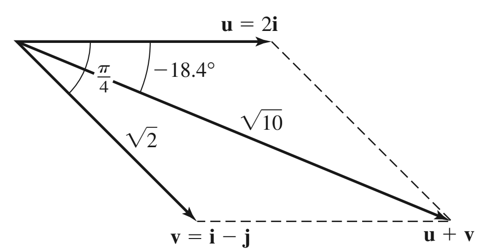
11.1.72 $\mathbf{u} = 3\langle 3, -5\rangle - 9\langle 6, 0\rangle = \langle -45, -15\rangle$.

11.1.73 $\mathbf{u} = 3\frac{\langle 5, -12\rangle}{\sqrt{25+144}} = \frac{3}{13}\langle 5, -12\rangle$.

11.1.74 $\mathbf{u} = -\frac{\langle 6, -8\rangle}{10} \cdot 10 = \langle -6, 8\rangle$.

11.1.75 $\mathbf{u} = \mathbf{u_1} + \mathbf{u_2} = \langle 4, -6\rangle + \langle 5, 9\rangle = \langle 9, 3\rangle$.

11.1.76 Let $\mathbf{u}$ represent the motion of the ant on the paper, and let $\mathbf{v}$ represent the motion of the paper. $\mathbf{u} + \mathbf{v} = 2\mathbf{i} + (\mathbf{i} - \mathbf{j}) = 3\mathbf{i} - \mathbf{j}$. $|\mathbf{u} + \mathbf{v}| = \sqrt{9 + 1} = \sqrt{10}$. $\theta = \tan^{-1}(-1/3) = -18.4$ degrees. The ant moves in the direction 18.4 degrees south of east with speed $\sqrt{10}$ miles per hour.



11.1.77

a. The sum is **0** because each vector has exactly one additive inverse in the set among the 12 vectors.

b. The 6:00 vector, because the others cancel in pairs, but this vector remains.

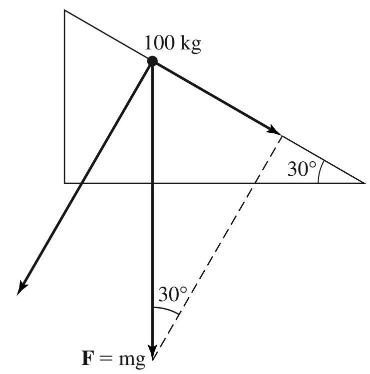
c. If we remove the 1:00 through 6:00 vectors, the sum is as large as possible, because all the vectors are pointing toward the left side of the clock. Removing any 6 consecutive vectors gives a sum whose magnitude is as large as possible.

d. Let $\mathbf{w}$ be the vector that points from 12:00 toward 6:00 but which has length r equal to the radius of the clock. The sum we are seeking is $12\mathbf{w}$. The sum of the vectors pointing to 1:00 and 11:00 add up to $(2 - \sqrt{3})\mathbf{w}$, the sum of the vectors pointing to 2:00 and 10:00 is $\mathbf{w}$, the vectors pointing to 3:00 and 9:00 add up to $2\mathbf{w}$, the vectors pointing to 4:00 and 8:00 add up to $3\mathbf{w}$, and the vectors pointing to 5:00 and 7:00 add up to $(\sqrt{3} + 2)\mathbf{w}$. Finally, the single vector pointing to 6:00 is $2\mathbf{w}$. The sum of all of these is $12\mathbf{w}$.

11.1.78 Because the ring doesn't move, $\mathbf{F}_3$ is the opposite of $\mathbf{F_1}+\mathbf{F}_2$, so $\mathbf{F_3} = \langle 50\sqrt{2}-30\sqrt{3}, -50\sqrt{2}-30\rangle$. Thus $|\mathbf{F}_3| \approx 102.44$ pounds, and the direction is given by $\alpha = \tan^{-1}\left(\frac{50\sqrt{2}+30}{50\sqrt{2}-30\sqrt{3}}\right) \approx 79.45$ degrees south of east.

11.1.79 The magnitude of the net force is $|\mathbf{F}| = \sqrt{40^2+30^2} = 50$ pounds. $\alpha = \tan^{-1}(\frac{3}{4}) \approx .6435$ radians or 36.87 degrees. The net force has magnitude 50 pounds in the direction 36.87 degrees north of east.

11.1.80 The component parallel to the plane is $mg\sin(30°) = 490$ kg $\cdot$ m/s^2. The component perpendicular to the plane is $mg\cos(30°) = 848.7$ kg $\cdot$ m/s^2.



11.1.81 $\mathbf{u}+\mathbf{v} = \langle u_1,u_2\rangle + \langle v_1,v_2\rangle = \langle u_1+v_1, u_2+v_2\rangle = \langle v_1+u_1, v_2+u_2\rangle = \langle v_1,v_2\rangle + \langle u_1,u_2\rangle = \mathbf{v}+\mathbf{u}$.

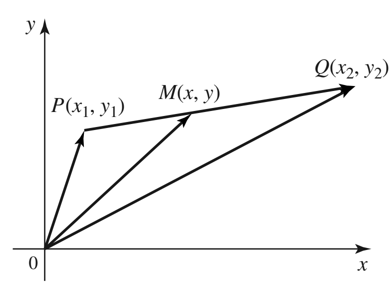
11.1.82 $(\mathbf{u}+\mathbf{v})+\mathbf{w} = (\langle u_1,u_2\rangle + \langle v_1,v_2\rangle) + \langle w_1,w_2\rangle = \langle u_1+v_1, u_2+v_2\rangle + \langle w_1,w_2\rangle = \langle (u_1+v_1)+w_1, (u_2+v_2)+w_2\rangle = \langle u_1+(v_1+w_1), u_2+(v_2+w_2)\rangle = \langle u_1,u_2\rangle + \langle v_1+w_1, v_2+w_2\rangle = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.

11.1.83 $a(c\mathbf{v}) = a(c\langle v_1,v_2\rangle) = a\langle cv_1, cv_2\rangle = \langle a(cv_1), a(cv_2)\rangle = \langle (ac)v_1, (ac)v_2\rangle = (ac)\langle v_1,v_2\rangle = (ac)\mathbf{v}$.

11.1.84 $a(\mathbf{u}+\mathbf{v}) = a(\langle u_1,u_2\rangle + \langle v_1,v_2\rangle) = a(\langle u_1+v_1, u_2+v_2\rangle) = \langle a(u_1+v_1), a(u_2+v_2)\rangle = \langle au_1+av_1, au_2+av_2\rangle = \langle au_1,au_2\rangle + \langle av_1,av_2\rangle = a\mathbf{u}+a\mathbf{v}$.

11.1.85 $(a+c)\mathbf{v} = (a+c)\langle v_1,v_2\rangle = \langle (a+c)v_1, (a+c)v_2\rangle = \langle av_1+cv_1, av_2+cv_2\rangle = \langle av_1,av_2\rangle + \langle cv_1,cv_2\rangle = a\mathbf{v}+c\mathbf{v}$.

11.1.86 Let $M(x,y)$ be the midpoint. Because $\overrightarrow{OM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ}$, we have $\langle x,y\rangle = \langle x_1,y_1\rangle + \frac{1}{2}(\langle x_2,y_2\rangle - \langle x_1,y_1\rangle) = \langle x_1,y_1\rangle + \langle \frac{1}{2}x_2, \frac{1}{2}y_2\rangle + \langle -\frac{1}{2}x_1, -\frac{1}{2}y_1\rangle = \langle \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\rangle$.



11.1.87 $|c\mathbf{v}| = |c\langle v_1,v_2\rangle| = |\langle cv_1,cv_2\rangle| = \sqrt{(cv_1)^2+(cv_2)^2} = \sqrt{c^2}\sqrt{v_1^2+v_2^2} = |c|\,|\mathbf{v}|$.

11.1.88 Yes. Because $\overrightarrow{PQ} = \overrightarrow{RS}$, we have that these two vectors are parallel and have the same magnitude. Thus the quadrilateral $RSQP$ is a parallelogram. Hence, $\overrightarrow{PR}$ is parallel to $\overrightarrow{QS}$ and they have the same magnitude, and are thus equal.

11.1.89

a. Note that $-6\mathbf{u} = \mathbf{v}$, so $\{\mathbf{u},\mathbf{v}\}$ is linearly dependent. But there is no scalar c so that $c\mathbf{u} = \mathbf{w}$, nor any scalar d so that $d\mathbf{v} = \mathbf{w}$ so $\{\mathbf{u},\mathbf{w}\}$ is linearly independent and $\{\mathbf{v},\mathbf{w}\}$ is linearly independent.

b. Two nonzero vectors are linearly independent when they are not parallel, and are linearly dependent when they are parallel.

c. Suppose $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Consider the equation $c_1\mathbf{u}+c_2\mathbf{v}=\mathbf{w}$ for a given vector $\mathbf{w}$. We are seeking a solution for the system of linear equatons $c_1u_1+c_2v_1=w_1$ and $c_1u_2+c_2v_2=w_2$. The solution for this system is given by $c_1=\frac{1}{u_1v_2-u_2v_1}(v_2w_1-v_1w_2)$ and $c_2=\frac{1}{u_1v_2-u_2v_1}(-u_2w_1+u_1w_2)$, provided $u_1v_2-u_2v_1\neq 0$. This condition is equivalent to saying that $\mathbf{v}$ is not a multiple of $\mathbf{u}$. Thus a solution to the system of linear equations exists exactly when the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

11.1.90 Suppose that $u_1v_1+u_2v_2=0$. Then $\frac{u_2}{u_1}\cdot\frac{v_2}{v_1}=-1$. Let $m_1=\frac{u_2}{u_1}$, and note that this is the slope of the line containing the vector $\mathbf{u}$. Likewise let $m_2=\frac{v_2}{v_1}$, and note that this is the slope of the line containing the vector $\mathbf{v}$. Because $m_1\cdot m_2=-1$, the vectors are perpendicular.

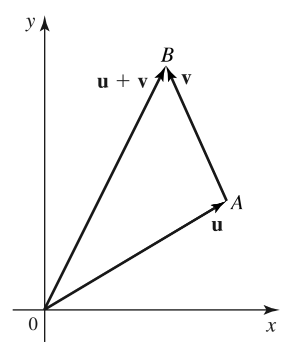
11.1.91

a. If $\mathbf{u}$ and $\mathbf{v}$ are parallel, we must have $\frac{a}{2}=\frac{5}{6}$, so $a=\frac{5}{3}$.

b. If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, we must have $2a+30=0$, so $a=-15$.

11.1.92

a. The triangle rule states that in any triangle, the length of one side is less than or equal to the sum of the lengths of the other two sides. Suppose that $\mathbf{u}=\overrightarrow{OA}$ and $\mathbf{v}=\overrightarrow{AB}$. Then $\mathbf{u}+\mathbf{v}=\overrightarrow{OB}$. The triangle rule applied to triangle OAB assures us that $\left|\overrightarrow{OB}\right|\le\left|\overrightarrow{OA}\right|+\left|\overrightarrow{AB}\right|$, so $|\mathbf{u}+\mathbf{v}|\le|\mathbf{u}|+|\mathbf{v}|$.



b. Equality occurs when $\mathbf{u}$ and $\mathbf{v}$ are parallel and in the same direction, so that $\mathbf{v}=k\mathbf{u}$ where $k>0$. Then $|\mathbf{u}+\mathbf{v}|=|(1+k)\mathbf{u}|=(1+k)\,|\mathbf{u}|=|\mathbf{u}|+|k\mathbf{u}|=|\mathbf{u}|+|\mathbf{v}|$.

11.2 Vectors in Three Dimensions

11.2.1 Starting at the origin $(0,0,0)$, move 3 units in the positive x-direction, 2 units in the negative y-direction, and 1 unit in the positive z-direction, to arrive at the point $(3,-2,1)$.

11.2.2 Every point in the xz-plane has a y-coordinate of 0.

11.2.3 The plane $x=4$ is parallel to the yz-plane, but contains all of the points with x-coordinate 4. It is perpendicular to the x-axis.

11.2.4 $\mathbf{u}=\langle 0-3,-6-5,3-(-2)\rangle=\langle -3,-11,5\rangle$.

11.2.5 $\mathbf{u}+\mathbf{v}=\langle 3+6,5+(-5),-7+1\rangle=\langle 9,0,-6\rangle$. $3\mathbf{u}-\mathbf{v}=\langle 9,15,-21\rangle-\langle 6,-5,1\rangle=\langle 3,20,-22\rangle$.

11.2.6 $\left|\overrightarrow{PQ}\right|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

11.2.7 Because $\sqrt{3^2+(-1)^2+2^2}=\sqrt{14}<\sqrt{0^2+0^2+(-4)^2}=4$, the point $(0,0,-4)$ is further from the origin.

11.2.8 $\overrightarrow{PQ} = \langle 1-(-1), 3-(-4), -6-(6)\rangle = \langle 2, 7, -12\rangle = 2\mathbf{i} + 7\mathbf{j} - 12\mathbf{k}$.

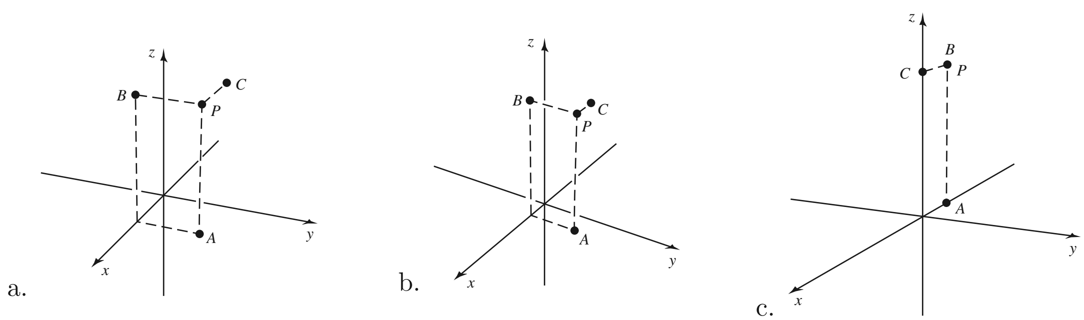
11.2.9 $A(3,0,5)$, $B(3,4,0)$, $C(0,4,5)$.

11.2.10 $A(5,0,10)$, $B(0,8,0)$, $C(5,8,10)$.

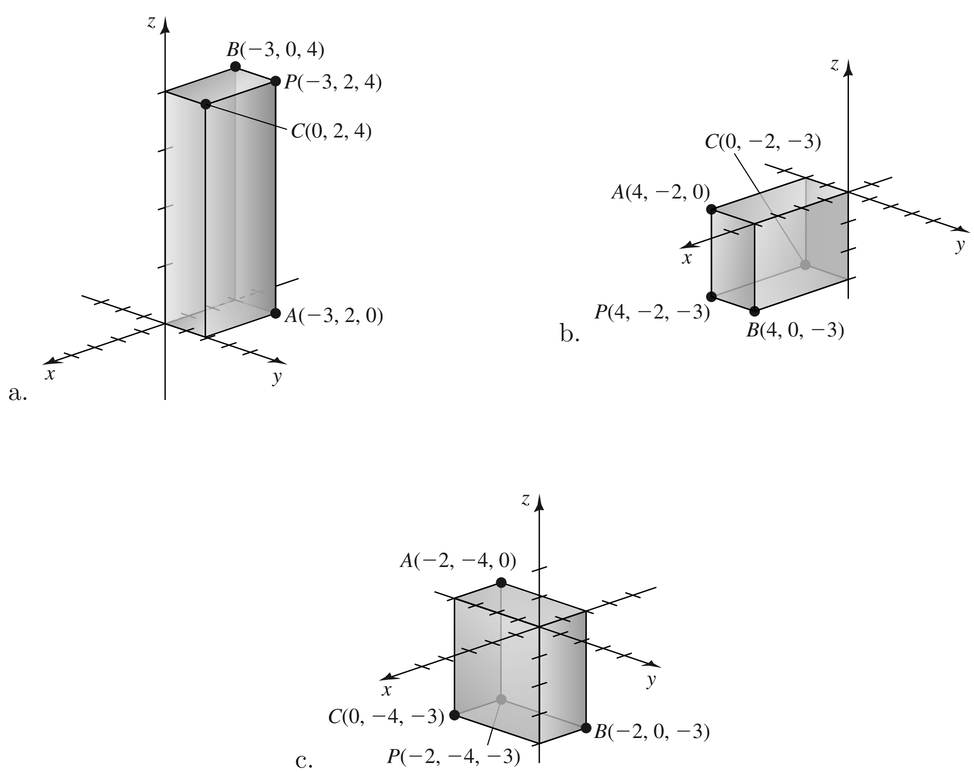
11.2.11 $A(3,-4,5)$, $B(0,-4,0)$, $C(0,-4,5)$.

11.2.12 $A(-3,-3,0)$, $B(0,0,3)$, $C(-3,-3,3)$.

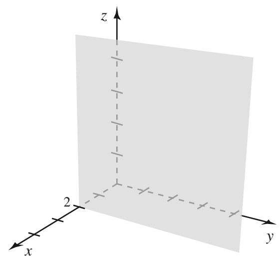
11.2.13



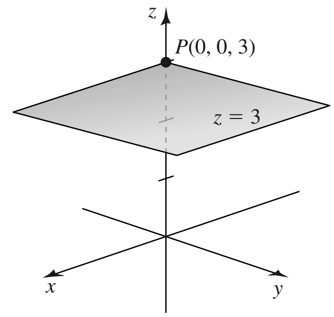
11.2.14



11.2.15



11.2.16



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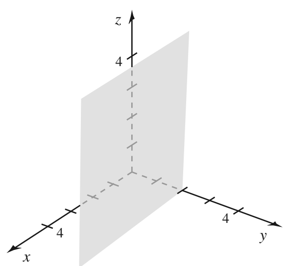
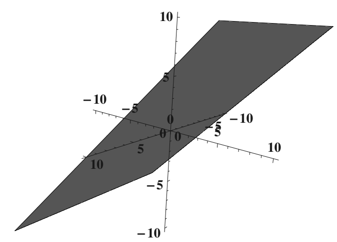


11.2.18



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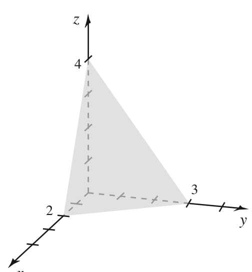
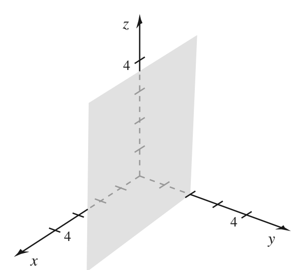
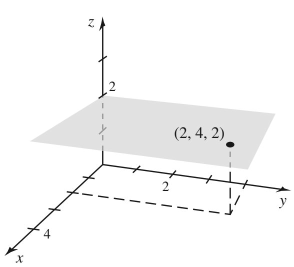


11.2.20

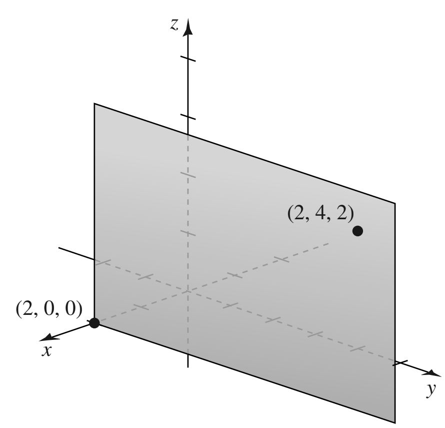


11.2.21



The plane $z = 2$.

11.2.22



The plane $x = 2$.

11.2.23 $(x-1)^2+(y-2)^2+(z-3)^2=16$.

11.2.24 Note that the radius r of this sphere would be $r=\sqrt{(3-1)^2+(4-2)^2+5^2}=\sqrt{33}$. The equation of the sphere is thus $(x-1)^2+(y-2)^2+z^2=33$.

11.2.25 $(x+2)^2+y^2+(z-4)^2\le 1$.

11.2.26 Note that the radius of the ball is given by $r=\sqrt{1^2+(4-(-2))^2+(8-6)^2}=\sqrt{41}$. The ball is thus described by the inequality $x^2+(y+2)^2+(z-6)^2\le 41$.

11.2.27 The midpoint of the line segment $\overline{PQ}$ is $(\frac{1+2}{2},\frac{0+3}{2},\frac{5+9}{2})=(3/2,3/2,7)$. The radius of the sphere is $r=\frac{1}{2}\sqrt{(2-1)^2+(3-0)^2+(9-5)^2}=\frac{\sqrt{26}}{2}$. The equation of the sphere is therefore $(x-3/2)^2+(y-3/2)^2+(z-7)^2=\frac{13}{2}$.

11.2.28 The midpoint of the line segment $\overline{PQ}$ is $(\frac{-4+0}{2},\frac{2+2}{2},\frac{3+7}{2})=(-2,2,5)$. The radius of the sphere is $r=\frac{1}{2}\sqrt{(0-(-4))^2+(2-2)^2+(7-3)^2}=\sqrt{8}$. The equation of the sphere is therefore $(x+2)^2+(y-2)^2+(z-5)^2=8$.

11.2.29 This is a sphere centered at $(1,0,0)$ of radius 3.

11.2.30 Completing the square, we have $(x+1)^2+(y^2-2y+1)+z^2=25$, which can be written as $(x+1)^2+(y-1)^2+z^2=5^2$. This is a sphere centered at $(-1,1,0)$ with radius 5.

11.2.31 Completing the squares, we have $x^2+(y^2-2y+1)+(z^2-4z+4)=4+5$, so we have $x^2+(y-1)^2+(z-2)^2=3^2$, which describes a sphere of radius 3 centered at $(0,1,2)$.

11.2.32 Completing the squares, we have $(x^2-6x+9)+(y^2+6y+9)+(z^2-8z+16)=2+9+9+16=36$, so we have $(x-3)^2+(y+3)^2+(z-4)^2=6^2$, which describes a sphere of radius 6 centered at $(3,-3,4)$.

11.2.33 Completing the square, we have $x^2+(y^2-14y+49)+z^2\ge -13+49=36$, which can be written as $x^2+(y-7)^2+z^2\ge 6^2$. This is the outside of a ball centered at $(0,7,0)$ with radius 6. (Including the sphere itself.)

11.2.34 Completing the square, we have $x^2+(y^2-14y+49)+z^2\le -13+49=36$, which can be written as $x^2+(y-7)^2+z^2\le 6^2$. This is a ball centered at $(0,7,0)$ with radius 6.

11.2.35 Completing the squares, we have $(x^2-8x+16)+(y^2-14y+49)+(z^2-18z+81)\le 79+16+49+81=225$, which can be written as $(x-4)^2+(y-7)^2+(z-9)^2\le 15^2$. This is a ball centered at $(4,7,9)$ with radius 15.

11.2.36 Completing the squares, we have $(x^2-8x+16)+(y^2+14y+49)+(z^2-18z+81)\ge 65+16+49+81=211$, which can be written as $(x-4)^2+(y+7)^2+(z-9)^2\ge 211$. This is the outside of a ball centered at $(4,-7,9)$ with radius $\sqrt{211}$. (Including the sphere itself.)

11.2.37 Completing the squares, we have $(x^2-2x+1)+(y^2+6y+9)+z^2=-10+1+9=0$, or $(x-1)^2+(y+3)^2+z^2=0$. This is the single point $(1,-3,0)$.

11.2.38 Completing the squares, we have $(x^2-4x+4)+(y^2+6y+9)+z^2=-14+4+9=-1$. This can be written as $(x-2)^2+(y+3)^2+z^2=-1$, and no real numbers satisfy this equations.

11.2.39

a. $3\langle 4,-3,0\rangle+2\langle 0,1,1\rangle=\langle 12,-7,2\rangle$.

b. $4\langle 4,-3,0\rangle-\langle 0,1,1\rangle=\langle 16,-13,-1\rangle$.

c. $|\langle 4,-3,0\rangle+3\langle 0,1,1\rangle|=|\langle 4,0,3\rangle|=\sqrt{16+0+9}=5$.

11.2.40

a. $3\langle -2,-3,0\rangle + 2\langle 1,2,1\rangle = \langle -4,-5,2\rangle$.

b. $4\langle -2,-3,0\rangle - \langle 1,2,1\rangle = \langle -9,-14,-1\rangle$.

c. $|\langle -2,-3,0\rangle + 3\langle 1,2,1\rangle| = |\langle 1,3,3\rangle| = \sqrt{1+9+9} = \sqrt{19}$.

11.2.41

a. $3\langle -2,1,-2\rangle + 2\langle 1,1,1\rangle = \langle -4,5,-4\rangle$.

b. $4\langle -2,1,-2\rangle - \langle 1,1,1\rangle = \langle -9,3,-9\rangle$.

c. $|\langle -2,1,-2\rangle + 3\langle 1,1,1\rangle| = |\langle 1,4,1\rangle| = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$.

11.2.42

a. $3\langle -5,0,2\rangle + 2\langle 3,1,1\rangle = \langle -9,2,8\rangle$.

b. $4\langle -5,0,2\rangle - \langle 3,1,1\rangle = \langle -23,-1,7\rangle$.

c. $|\langle -5,0,2\rangle + 3\langle 3,1,1\rangle| = |\langle 4,3,5\rangle| = \sqrt{16+9+25} = \sqrt{50} = 5\sqrt{2}$.

11.2.43

a. $3\langle -7,11,8\rangle + 2\langle 3,-5,-1\rangle = \langle -15,23,22\rangle$.

b. $4\langle -7,11,8\rangle - \langle 3,-5,-1\rangle = \langle -31,49,33\rangle$.

c. $|\langle -7,11,8\rangle + 3\langle 3,-5,-1\rangle| = |\langle 2,-4,5\rangle| = \sqrt{4+16+25} = \sqrt{45} = 3\sqrt{5}$.

11.2.44

a. $3\langle -4,-8\sqrt{3},2\sqrt{2}\rangle + 2\langle 2,3\sqrt{3},-\sqrt{2}\rangle = \langle -8,-18\sqrt{3},4\sqrt{2}\rangle$.

b. $4\langle -4,-8\sqrt{3},2\sqrt{2}\rangle - \langle 2,3\sqrt{3},-\sqrt{2}\rangle = \langle -18,-35\sqrt{3},9\sqrt{2}\rangle$.

c. $\left|\langle -4,-8\sqrt{3},2\sqrt{2}\rangle + 3\langle 2,3\sqrt{3},-\sqrt{2}\rangle\right| = \left|\langle 2,\sqrt{3},-\sqrt{2}\rangle\right| = \sqrt{4+3+2} = \sqrt{9} = 3$.

11.2.45

a. $\overrightarrow{PQ} = \langle 3-1,11-5,2-0\rangle = \langle 2,6,2\rangle = 2\mathbf{i}+6\mathbf{j}+2\mathbf{k}$.

b. $|\langle 2,6,2\rangle| = \sqrt{4+36+4} = \sqrt{44} = 2\sqrt{11}$.

c. $\langle 1/\sqrt{11},3/\sqrt{11},1/\sqrt{11}\rangle$ and $\langle -1/\sqrt{11},-3/\sqrt{11},-1/\sqrt{11}\rangle$.

11.2.46

a. $\overrightarrow{PQ} = \langle 1-5,14-11,13-12\rangle = \langle -4,3,1\rangle = -4\mathbf{i}+3\mathbf{j}+1\mathbf{k}$.

b. $|\langle -4,3,1\rangle| = \sqrt{16+9+1} = \sqrt{26}$.

c. $\langle -4/\sqrt{26},3/\sqrt{26},1/\sqrt{26}\rangle$ and $\langle 4/\sqrt{26},-3/\sqrt{26},-1/\sqrt{26}\rangle$.

11.2.47

a. $\overrightarrow{PQ} = \langle -3+3,-4-1,1-0\rangle = \langle 0,-5,1\rangle = -5\mathbf{j}+1\mathbf{k}$.

b. $|\langle 0,-5,1\rangle| = \sqrt{25+1} = \sqrt{26}$.

c. $\langle 0,-5/\sqrt{26},1/\sqrt{26}\rangle$ and $\langle 0,5/\sqrt{26},-1/\sqrt{26}\rangle$.

11.2.48

a. $\overrightarrow{PQ} = \langle 3-3, 9-8, 11-12\rangle = \langle 0,1,-1\rangle = \mathbf{j}-\mathbf{k}$.

b. $|\langle 0,1,-1\rangle| = \sqrt{1+1} = \sqrt{2}$.

c. $\langle 0, 1/\sqrt{2}, -1/\sqrt{2}\rangle$ and $\langle 0, -1/\sqrt{2}, 1/\sqrt{2}\rangle$.

11.2.49

a. $\overrightarrow{PQ} = \langle -2-0, 4-0, 0-2\rangle = \langle -2,4,-2\rangle = -2\mathbf{i}+4\mathbf{j}-2\mathbf{k}$.

b. $|\langle -2,4,-2\rangle| = \sqrt{4+16+4} = 2\sqrt{6}$.

c. $\langle -1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}\rangle$ and $\langle 1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6}\rangle$.

11.2.50

a. $\overrightarrow{PQ} = \langle 1-a, 1-b, -1-c\rangle = (1-a)\mathbf{i}+(1-b)\mathbf{j}+(-1-c)\mathbf{k}$.

b. $|\langle (1-a),(1-b),(-1-c)\rangle| = \sqrt{(1-a)^2+(1-b)^2+(-1-c)^2}$.

c. $\frac{\pm 1}{\sqrt{(1-a)^2+(1-b)^2+(-1-c)^2}}\langle 1-a, 1-b, -1-c\rangle$.

11.2.51

a. The airplane's velocity vector (without wind) is given by $\langle 0,20,0\rangle$, the wind's is given by $\langle 20,0,0\rangle$ and the downdraft's is $\langle 0,0,-10\rangle$. The sum of these is $\langle 20,20,-10\rangle$.

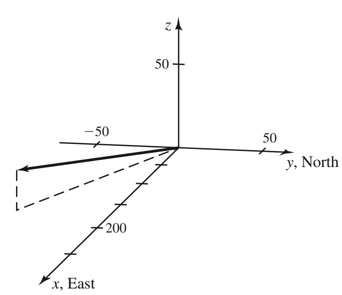
b. The speed is $|\langle 20,20,-10\rangle| = \sqrt{400+400+100} = 30$ mi/hr

11.2.52

a. The airplane's velocity vector (without wind) is given by $\langle 10,0,0\rangle$, the wind's is given by $\langle 0,-5,0\rangle$ and the updraft's is $\langle 0,0,5\rangle$. The sum of these is $\langle 10,-5,5\rangle$.

b. The speed is $|\langle 10,-5,5\rangle| = \sqrt{100+25+25} = 5\sqrt{6} \approx 12.2$ mi/hr

11.2.53 The airplane's velocity is $\mathbf{v_1} = 250\mathbf{i}$. The crosswind is blowing $\mathbf{v_2} = -25\sqrt{2}\mathbf{i} - 25\sqrt{2}\mathbf{j}$. The updraft is $\mathbf{v_3} = 30\mathbf{k}$. We have $|\mathbf{v_1}+\mathbf{v_2}+\mathbf{v_3}| = \left|\langle 250-25\sqrt{2}, -25\sqrt{2}, 30\rangle\right| \approx 219.596$ miles per hour. The direction is sketched in the diagram—it is slightly south of east and upward.



11.2.54 $\mathbf{F} = \mathbf{F_1}+\mathbf{F_2}+\mathbf{F_3} = \langle 60,20,30\rangle$. $|\mathbf{F}| = 10\sqrt{36+4+9} = 70$. The direction can be described using the angles α, β, and γ defined by $\alpha = \cos^{-1}(6/7) \approx 31$ degrees, $\beta = \cos^{-1}(20/70) \approx 73.4$ degrees, and $\gamma = \cos^{-1}(3/7) \approx 64.62$ degrees.

11.2.55 The component in the east direction is $(20\cos 30°)(\cos 45°) = 5\sqrt{6}$ knots. In the north direction, it is $(20\cos 30°)(\sin 45°) = 5\sqrt{6}$ knots. In the vertical direction, it is $20\sin 30° = 10$ knots.

11.2.56 $\mathbf{F_1}+\mathbf{F_2} = \langle 10,6,3\rangle + \langle 0,4,9\rangle = \langle 10,10,12\rangle$, so we need $\mathbf{F_3} = \langle -10,-10,-12\rangle$.

11.2.57

a. False. For example, let $\mathbf{u} = \langle 1, 0, 0 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$ and $\mathbf{w} = \langle 1, 1, 0 \rangle$. Then both $\mathbf{u}$ and $\mathbf{v}$ make a 45 degree angle with $\mathbf{w}$, but $\mathbf{u} + \mathbf{v} = \mathbf{w}$ makes a zero degree angle with $\mathbf{w}$.

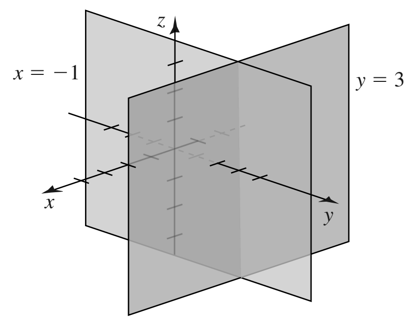
b. False. For example, $\mathbf{i}$ and $\mathbf{j}$ form a 90 degree angle with $\mathbf{k}$, as does $\mathbf{i} + \mathbf{j}$.

c. False. $\mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 1, 1, 1 \rangle \neq \langle 0, 0, 0 \rangle$.

d. True. They intersect at the point $(1, 1, 1)$.

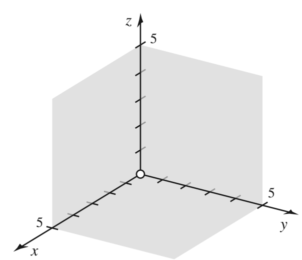
11.2.58

This is the set of all points which lie either on the plane $x = -1$ or the plane $y = 3$ or both.



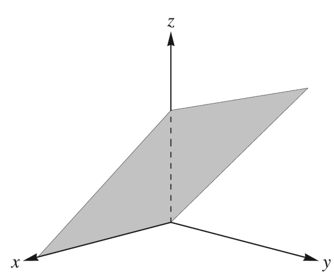
11.2.59

This represents all the points in 3-space, excluding the three axes.



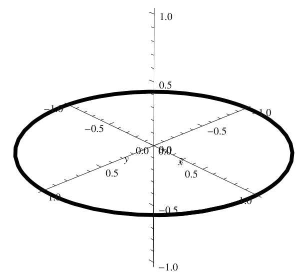
11.2.60

This represents the plane which is perpendicular to the yz-plane, and which intersects the yz-plane in the line $y = z$.



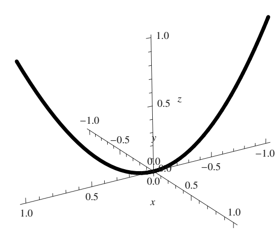
11.2.61

This represents a circle of radius 1 centered at $(0, 0, 0)$ in the xy-plane.



11.2.62

This represents a parabola in the xz-plane.



11.2.63 If $z = 1$, then $x^2 + y^2 + 1 = 5$, so $x^2 + y^2 = 4$. We have a circle of radius 2 centered at $(0, 0, 1)$ in the plane $z = 1$.

11.2.64 If $z = 6$ and $x^2 + y^2 + z^2 = 36$, then $x^2 + y^2 = 0$, so $x = 0$ and $y = 0$. This consists of the single point $(0, 0, 6)$.

11.2.65 Planes parallel to the xz-plane have the form $y = c$ for a constant c, so we must have $y = 4$. Thus, we must have $(x - 2)^2 + (z - 1)^2 = 9$ and $y = 4$.

11.2.66 The intersection of the planes $y = -5$ and $z = 1$ is a line parallel to the x-axis that contains the given point.

11.2.67 Because the magnitude of $\mathbf{v}$ is $\sqrt{36 + 64 + 0} = 10$, the desired vectors are $\pm 20\langle .6, -.8, 0\rangle = \pm\langle 12, -16, 0\rangle$.

11.2.68 Because the magnitude of $\mathbf{v}$ is $\sqrt{9 + 4 + 36} = 7$, the desired vectors are $\pm 10\langle 3/7, -2/7, 6/7\rangle = \pm\langle 30/7, -20/7, 60/7\rangle$.

11.2.69 Because the magnitude of $\mathbf{v}$ is $\sqrt{1 + 1 + 1} = \sqrt{3}$, the desired vectors are $\pm 3\langle -1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}\rangle = \pm\langle -\sqrt{3}, -\sqrt{3}, \sqrt{3}\rangle$.

11.2.70 Because the magnitude of $\mathbf{v}$ is $\sqrt{1 + 1 + 0} = \sqrt{2}$, the desired vectors are $\pm 3\langle 1/\sqrt{2}, -1/\sqrt{2}, 0\rangle$.

11.2.71

a. Because $\overrightarrow{PQ} = \langle 1, -1, 2\rangle$ and $\overrightarrow{PR} = \langle 3, -3, 6\rangle = 3\langle 1, -1, 2\rangle$ they are collinear. Q is between P and R.

b. Because $\overrightarrow{PQ} = \langle 4, 8, -8\rangle$ and $\overrightarrow{PR} = \langle -1, -2, 2\rangle = -\frac{1}{4}\langle 4, 8, -8\rangle$ they are collinear. P is between Q and R.

c. Because $\overrightarrow{PQ} = \langle 1, -5, 3\rangle$ and $\overrightarrow{PR} = \langle 2, -3, 6\rangle$ are not parallel, the given points are not collinear.

d. Because $\overrightarrow{PQ} = \langle 2, 13, 3\rangle$ and $\overrightarrow{PR} = \langle -3, -2, -1\rangle$ are not parallel, the given points are not collinear.

11.2.72 In order for the points to be collinear, we would require $\langle 4-1, 7-2, 1-3\rangle = k\langle x-1, y-2, 2-3\rangle$. Thus we seek a solution to the system of equations $3 = k(x-1)$, $5 = k(y-2)$, $-2 = -k$. Thus $k = 2$, $x = \frac{5}{2}$ and $y = \frac{9}{2}$.

11.2.73 The diagonal of the box has magnitude $\sqrt{2^2+3^2+4^2} = \sqrt{29}$, so the longest rod that will fit in the box has length $\sqrt{29}$ feet.

11.2.74

a. $|\mathbf{W}_{\text{par}}| = 100\sin\theta = 100\sin(20^\circ) \approx 34.2$. $|\mathbf{W}_{\text{perp}}| = 100\cos(20^\circ) \approx 93.97$.

b. If $\mu = .65$, we have $\mu \cdot |\mathbf{W}_{\text{perp}}| \approx .65 * 93.97 \approx 61.08 > |\mathbf{W}_{\text{par}}| \approx 34.2$, so the block does not slide.

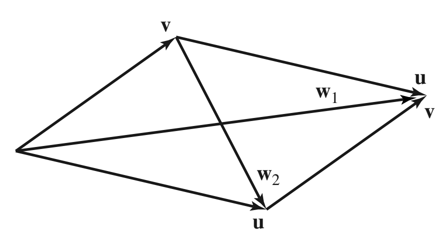
c. We seek θ so that $\mu\cos\theta = \sin\theta$, so $\tan\theta = \mu$, so $\theta = \tan^{-1}(\mu) \approx 33$ degrees.

11.2.75 Let $P(1, -\sqrt{3}, 0)$, $Q(1, \sqrt{3}, 0)$, $R(-2, 0, 0)$, and $S(0, 0, -2\sqrt{3})$ be the given points. Note that $\overrightarrow{PS} = \langle -1, \sqrt{3}, -2\sqrt{3}\rangle$, $\overrightarrow{QS} = \langle -1, -\sqrt{3}, -2\sqrt{3}\rangle$, $\overrightarrow{RS} = \langle 2, 0, -2\sqrt{3}\rangle$. Let $x(\overrightarrow{PS} + \overrightarrow{QS} + \overrightarrow{RS}) = -500\mathbf{k}$, then $-6\sqrt{3}x = -500$, so $x = \frac{250}{3\sqrt{3}}$. Then $x\overrightarrow{PS} = \frac{250}{3\sqrt{3}}\langle -1, \sqrt{3}, -2\sqrt{3}\rangle = \frac{250}{3}\langle -1/\sqrt{3}, 1, -2\rangle$. $x\overrightarrow{QS} = \frac{250}{3\sqrt{3}}\langle -1, -\sqrt{3}, -2\sqrt{3}\rangle = \frac{250}{3}\langle -1/\sqrt{3}, -1, -2\rangle$. $x\overrightarrow{RS} = \frac{250}{3\sqrt{3}}\langle 2, 0, -2\sqrt{3}\rangle = \frac{250}{3}\langle 2/\sqrt{3}, 0, -2\rangle$.

11.2.76 Let $A(2, 0, 0)$, $B(0, 2, 0)$, $C(-2, 0, 0)$, $D(0, -2, 0)$, and $E(0, 0, -4)$ be the given points. Note that $\overrightarrow{AE} = \langle -2, 0, -4\rangle$, $\overrightarrow{BE} = \langle 0, -2, -4\rangle$, $\overrightarrow{CE} = \langle 2, 0, -4\rangle$, $\overrightarrow{DE} = \langle 0, 2, -4\rangle$. We are seeking x so that $x(\overrightarrow{AE} + \overrightarrow{BE} + \overrightarrow{CE} + \overrightarrow{DE}) = -500\mathbf{k}$. Thus we require $-16x = -500$, so $x = \frac{125}{4}$. Then $x\overrightarrow{AE} = \frac{125}{4}\langle -2, 0, -4\rangle$. $x\overrightarrow{BE} = \frac{125}{4}\langle 0, -2, -4\rangle$. $x\overrightarrow{CE} = \frac{125}{4}\langle 2, 0, -4\rangle$, and $x\overrightarrow{DE} = \frac{125}{4}\langle 0, 2, -4\rangle$.

11.2.77 Let $R(x, y, z)$ be the fourth vertex. Then perhaps $\overrightarrow{OQ} = \overrightarrow{RP}$, so $\langle 2, 4, 3\rangle = \langle 1-x, 4-y, 6-z\rangle$, so $x = -1$, $y = 0$, and $z = 3$, so $R(-1, 0, 3)$ is one possible desired vertex. We could also have $\overrightarrow{RP} = -\overrightarrow{OQ}$, in which case $\langle -2, -4, -3\rangle = \langle 1-x, 4-y, 6-z\rangle$, so $R(3, 8, 9)$ is the other vertex. We could also have $\overrightarrow{OP} = \overrightarrow{RQ}$, so $\langle 1, 4, 6\rangle = \langle 2-x, 4-y, 3-z\rangle$ and $R(1, 0, -3)$ is the desired point.

11.2.78 Suppose that the parallelogram has vertices $RTDS$, and $\mathbf{u} = \overrightarrow{RS}$ and $\mathbf{v} = \overrightarrow{RT}$ are two sides. Then we also have $\mathbf{u} = \overrightarrow{TD}$, and $\mathbf{v} = \overrightarrow{SD}$. But then $\mathbf{u} + \mathbf{v} = \overrightarrow{RS} + \overrightarrow{SD} = \overrightarrow{RD}$ is one diagonal and $\mathbf{u} - \mathbf{v} = \overrightarrow{TD} + \overrightarrow{DS} = \overrightarrow{TS}$ is the other diagonal.



11.2.79 Let $M(x, y, z)$ be the midpoint. Because $\overrightarrow{OM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ}$, we have $\langle x, y, z\rangle = \langle x_1, y_1, z_1\rangle + \frac{1}{2}(\langle x_2, y_2, z_2\rangle - \langle x_1, y_1, z_1\rangle) = \langle x_1, y_1, z_1\rangle + \langle \frac{1}{2}x_2, \frac{1}{2}y_2, \frac{1}{2}z_2\rangle + \langle -\frac{1}{2}x_1, -\frac{1}{2}y_1, -\frac{1}{2}z_1\rangle = \langle \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\rangle$.

11.2.80 We complete the squares by adding $a^2 + b^2 + c^2$ to both sides of the given equation, giving

$$(x^2 - 2ax + a^2) + (y^2 - 2by + b^2) + (z^2 - 2cz + c^2) = d + a^2 + b^2 + c^2,$$

which can be written $(x-a)^2 + (y-b)^2 + (z-c)^2 = d + a^2 + b^2 + c^2$. This represents the set of points whose distance from (a, b, c) is $\sqrt{d + a^2 + b^2 + c^2}$, so it represents a sphere as long as $d + a^2 + b^2 + c^2 > 0$. The center is (a, b, c) and the radius is $\sqrt{d + a^2 + b^2 + c^2}$.

11.2.81

a. $\mathbf{u}+\mathbf{v}=-\mathbf{w}$ (by the geometric definition of vector addition), so $\mathbf{u}+\mathbf{v}+\mathbf{w}=\mathbf{0}$.

b. Let $\mathbf{M_1}=\overrightarrow{EB}$, $\mathbf{M_2}=\overrightarrow{FO}$, and $\mathbf{M_3}=\overrightarrow{GA}$. Consider triangle EAB. We have $\overrightarrow{EA}+\overrightarrow{AB}+\overrightarrow{BE}=\mathbf{0}$, so $\frac{1}{2}\mathbf{u}+\mathbf{v}=-\overrightarrow{BE}=\overrightarrow{EB}=\mathbf{M_1}$. Using similar arguments, we have $\mathbf{M_2}=\frac{1}{2}\mathbf{v}+\mathbf{w}$ and $\mathbf{M_3}=\frac{1}{2}\mathbf{w}+\mathbf{u}$.

c. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be the vectors from O to the points 1/3 of the way along $\mathbf{M_1}$, $\mathbf{M_2}$ and $\mathbf{M_3}$ respectively. Because $-\mathbf{w}=\mathbf{u}+\mathbf{v}$, we have $\frac{\mathbf{u}-\mathbf{w}}{3}=\frac{\mathbf{u}}{3}+\frac{\mathbf{u}+\mathbf{v}}{3}=\frac{2}{3}\mathbf{u}+\frac{1}{3}\mathbf{v}$. Also, $\mathbf{a}=\frac{1}{2}\mathbf{u}+\frac{1}{3}\mathbf{M_1}=\frac{1}{2}\mathbf{u}+\frac{1}{3}\left(\frac{1}{2}\mathbf{u}+\mathbf{v}\right)=\frac{1}{2}\mathbf{u}+\frac{1}{6}\mathbf{u}+\frac{1}{3}\mathbf{v}=\frac{2}{3}\mathbf{u}+\frac{1}{3}\mathbf{v}$. Thus $\mathbf{a}=\frac{\mathbf{u}-\mathbf{w}}{3}$. Also, $\mathbf{b}=-\frac{2}{3}\mathbf{M_2}=-\frac{2}{3}(\frac{1}{2}\mathbf{v}+(-\mathbf{u}-\mathbf{v}))=\frac{2}{3}\mathbf{u}+\frac{1}{3}\mathbf{v}$. We also have $\mathbf{c}=-\frac{1}{2}\mathbf{w}+\frac{1}{3}\mathbf{M_3}=-\frac{1}{2}\mathbf{w}+\frac{1}{3}(\frac{1}{2}\mathbf{w}+\mathbf{u})=\frac{1}{3}\mathbf{u}+-\frac{1}{3}(-\mathbf{u}-\mathbf{v})=\frac{2}{3}\mathbf{u}+\frac{1}{3}\mathbf{v}$. Thus $\mathbf{a}=\mathbf{b}=\mathbf{c}$.

d. Because $\mathbf{a}=\mathbf{b}=\mathbf{c}$, the medians all meet at a point that divides each median in a 2:1 ratio.

11.2.82

a. The coordinates of M_1 are $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},0\right)$, and thus $\overrightarrow{RM}_1=\langle\frac{x_1+x_2}{2}-x_3,\frac{y_1+y_2}{2}-y_3,-z_3\rangle$.

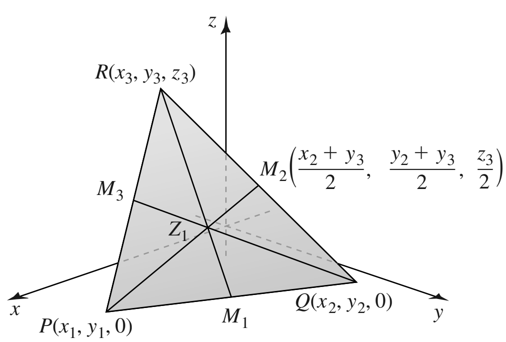
b. $\overrightarrow{OZ}_1=\overrightarrow{OR}+\frac{2}{3}\overrightarrow{RM}_1=\langle x_3,y_3,z_3\rangle+\langle\frac{x_1+x_2-2x_3}{3},\frac{y_1+y_2-2y_3}{3},-\frac{2z_3}{3}\rangle=\langle\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_3}{3}\rangle$.

c. The coordinates of M_2 are $\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2},\frac{z_3}{2}\right)$. We have $\overrightarrow{PM}_2=\langle\frac{x_2+x_3-2x_1}{2},\frac{y_2+y_3-2y_1}{2},\frac{z_3}{2}\rangle$, and $\overrightarrow{OZ}_2=\overrightarrow{OP}+\frac{2}{3}\overrightarrow{PM}_2=\langle x_1,y_1,0\rangle+\langle\frac{x_2+x_3-2x_1}{3},\frac{y_2+y_3-2y_1}{3},\frac{z_3}{3}\rangle=\langle\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_3}{3}\rangle$.

d. The coordinates of M_3 are $\left(\frac{x_1+x_3}{2},\frac{y_1+y_3}{2},\frac{z_3}{2}\right)$. We have $\overrightarrow{QM}_3=\langle\frac{x_1+x_3-2x_2}{2},\frac{y_1+y_3-2y_2}{2},\frac{z_3}{2}\rangle$, and $\overrightarrow{OZ}_3=\overrightarrow{OQ}+\frac{2}{3}\overrightarrow{QM}_3=\langle x_2,y_2,0\rangle+\langle\frac{x_1+x_3-2x_2}{3},\frac{y_1+y_3-2y_2}{3},\frac{z_3}{3}\rangle=\langle\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_3}{3}\rangle$.

e. The medians all intersect at the point $Z_1=Z_2=Z_3=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_3}{3}\right)$.

f. The intersection point is $\left(\frac{2+4+6}{3},\frac{4+1+3}{3},\frac{4}{3}\right)=(4,8/3,4/3)$.



11.2.83

a. $\mathbf{u}+\mathbf{v}=\overrightarrow{PR}$ and $\mathbf{w}+\mathbf{x}=\mathbf{x}+\mathbf{w}=\overrightarrow{PR}$, so $\mathbf{u}+\mathbf{v}=\mathbf{w}+\mathbf{x}$.

b. $\frac{1}{2}\mathbf{u}+\frac{1}{2}\mathbf{v}=\mathbf{m}=\frac{1}{2}(\mathbf{u}+\mathbf{v})$.

c. $\frac{1}{2}\mathbf{x}+\frac{1}{2}\mathbf{w}=\mathbf{n}=\frac{1}{2}(\mathbf{x}+\mathbf{w})$.

d. We have $\mathbf{n}=\frac{1}{2}(\mathbf{x}+\mathbf{w})=\frac{1}{2}(\mathbf{u}+\mathbf{v})=\mathbf{m}$.

e. Because $\mathbf{m}$ and $\mathbf{n}$ are equal, they are parallel. A similar argument will show that the other two sides are parallel as well.

11.2.84

The midpoints are $A(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},0)$, $B(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2},0)$, $C(\frac{x_3+x_4}{2},\frac{y_3+y_4}{2},\frac{z_4}{2})$, and $D(\frac{x_4+x_1}{2},\frac{y_4+y_1}{2},\frac{z_4}{2})$. So $\overrightarrow{AB}=\langle\frac{x_3-x_1}{2},\frac{y_3-y_1}{2},0\rangle=\overrightarrow{DC}$, and $\overrightarrow{BC}=\langle\frac{x_4-x_2}{2},\frac{y_4-y_2}{2},\frac{z_4}{2}\rangle=\overrightarrow{AD}$, so quadrilateral $ABCD$ is a parallelogram.

11.3 Dot Products

11.3.1 $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$, where θ is the angle between the two vectors.

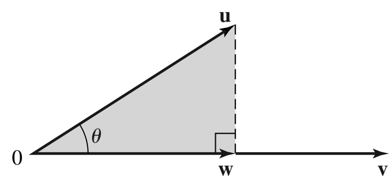
11.3.2 If $\mathbf{u} = \langle a, b, c \rangle$ and $\mathbf{v} = \langle r, s, t \rangle$, then $\mathbf{u} \cdot \mathbf{v} = ar + bs + ct$.

11.3.3 $\langle 2, 3, -6 \rangle \cdot \langle 1, -8, 3 \rangle = 2 \cdot 1 + 3 \cdot (-8) + (-6) \cdot 3 = -40$.

11.3.4 The dot product of two orthogonal vectors is 0.

11.3.5 Given non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, the angle between them is $\cos^{-1}(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|})$.

11.3.6 The projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector in the direction of $\mathbf{v}$ whose length is $|\mathbf{u}| \cos\theta$ where θ is the angle between the vectors. This length represents the length of the "shadow" that $\mathbf{u}$ casts on $\mathbf{v}$.

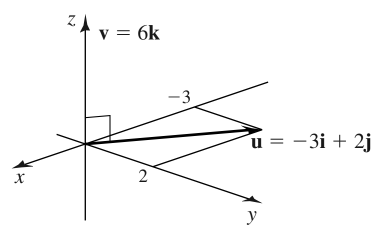


11.3.7 The scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is the number $|\mathbf{u}| \cos\theta$ where θ is the angle between the vectors. This number represents the signed length of the "shadow" that $\mathbf{u}$ casts on $\mathbf{v}$. Thus, referring to the diagram in the previous problem, the scalar projection is the length of the base of the shaded triangle.

11.3.8 The work done by a force $\mathbf{F}$ in moving an object along a displacement vector $\mathbf{d}$ is $w = \mathbf{F} \cdot \mathbf{d}$.

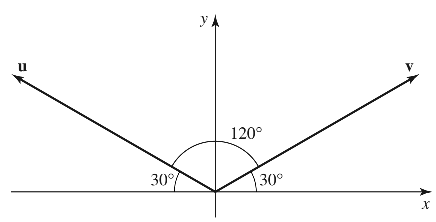
11.3.9 $\mathbf{u} \cdot \mathbf{v} = 4 \cdot 6 \cdot \cos(\pi/2) = 0$.

11.3.10 Because these vectors are perpendicular, we have $\cos\theta = 0$ where θ is the angle between them, so their dot product is zero.



11.3.11 The angle between these vectors is $\pi/4$. Thus, their dot product is $10 \cdot 10\sqrt{2} \cdot \frac{\sqrt{2}}{2} = 100$.

11.3.12 The angle between these vectors is 120°, so their dot product is $|\mathbf{u}|\,|\mathbf{v}| \cos(120°) = 2 \cdot 2 \cdot -\frac{1}{2} = -2$.



11.3.13 $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta = 1 \cdot 1 \cdot \cos(\pi/3) = 1/2$.

11.3.14 $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta = 1 \cdot 2 \cdot \cos(3\pi/4) = 2 \cdot (-\sqrt{2}/2) = -\sqrt{2}$.

11.3.15 $\mathbf{u} \cdot \mathbf{v} = 1 - 1 = 0$, so $\theta = \cos^{-1}(0) = \pi/2$.

11.3.16 $\mathbf{u}\cdot\mathbf{v} = -50+0 = -50$, so $\theta = \cos^{-1}\left(\frac{-50}{10\sqrt{50}}\right) = \cos^{-1}(-\sqrt{2}/2) = 3\pi/4$.

11.3.17 $\mathbf{u}\cdot\mathbf{v} = 1+0 = 1$, so $\theta = \cos^{-1}\left(\frac{1}{2}\right) = \pi/3$.

11.3.18 $\mathbf{u}\cdot\mathbf{v} = -2-2 = -4$, so $\theta = \cos^{-1}\left(-\frac{4}{4}\right) = \pi$.

11.3.19 $\mathbf{u}\cdot\mathbf{v} = 4\cdot 4 + 3\cdot(-6) = -2$. The angle between the vectors is thus

$$\cos^{-1}\left(-\frac{2}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(-\frac{2}{5\cdot 2\sqrt{13}}\right) \approx 1.627\,\text{radians}.$$

11.3.20 $\mathbf{u}\cdot\mathbf{v} = 3\cdot 0 + 4\cdot 4 + 0\cdot 5 = 16$. The angle between the vectors is thus

$$\cos^{-1}\left(\frac{16}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{16}{5\cdot\sqrt{41}}\right) \approx 1.047\,\text{radians}.$$

11.3.21 $\mathbf{u}\cdot\mathbf{v} = -10+0+12 = 2$. The angle between the vectors is thus

$$\cos^{-1}\left(\frac{2}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{116}\cdot\sqrt{14}}\right) \approx 1.521\,\text{radians}.$$

11.3.22 $\mathbf{u}\cdot\mathbf{v} = -27-25+2 = -50$. The angle between the vectors is thus

$$\cos^{-1}\left(-\frac{50}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(-\frac{50}{\sqrt{38}\cdot\sqrt{107}}\right) \approx 2.472\,\text{radians}.$$

11.3.23 $\mathbf{u}\cdot\mathbf{v} = 2+0-6 = -4$. The angle between the vectors is thus

$$\cos^{-1}\left(-\frac{4}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(-\frac{4}{\sqrt{13}\cdot\sqrt{21}}\right) \approx 1.815\,\text{radians}.$$

11.3.24 $\mathbf{u}\cdot\mathbf{v} = 2+16-12 = 6$. The angle between the vectors is thus

$$\cos^{-1}\left(\frac{6}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{6}{\sqrt{53}\cdot\sqrt{24}}\right) \approx 1.402\,\text{radians}.$$

11.3.25 $\text{proj}_{\mathbf{v}}\mathbf{u} = 3\mathbf{i}$. $\text{scal}_{\mathbf{v}}\mathbf{u} = 3$.

11.3.26 $\text{proj}_{\mathbf{v}}\mathbf{u} = -3\mathbf{i}$. $\text{scal}_{\mathbf{v}}\mathbf{u} = -3$.

11.3.27 $\text{proj}_{\mathbf{v}}\mathbf{u} = 3\mathbf{j}$. $\text{scal}_{\mathbf{v}}\mathbf{u} = 3$.

11.3.28 $\text{proj}_{\mathbf{v}}\mathbf{u} = -\mathbf{v} = \langle -2,-2\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = -2\sqrt{2}$.

11.3.29 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{12}{20}\langle -4,2\rangle = \langle -12/5, 6/5\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{12}{\sqrt{20}} = \frac{6}{\sqrt{5}}$.

11.3.30 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{50}{40}\langle 2,6\rangle = \langle 5/2, 15/2\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{50}{\sqrt{40}} = \frac{25}{\sqrt{10}}$.

11.3.31 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = -\frac{6}{6}\langle 1,-1,2\rangle = \langle -1,1,2\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = -\frac{6}{\sqrt{6}} = -\sqrt{6}$.

11.3.32 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = -\frac{26}{26}\langle 4,-1,-3\rangle = \langle -4,1,3\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = -\frac{26}{\sqrt{26}} = -\sqrt{26}$.

11.3.33 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = -\frac{14}{19}\langle 1,3,-3\rangle = \langle -14/19, -42/19, 42/19\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = -\frac{14}{\sqrt{19}}$.

11.3.34 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = -\frac{30}{20}\langle 0,4,-2\rangle = \langle 0,-6,3\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = -\frac{30}{\sqrt{20}} = -3\sqrt{5}$.

11.3.35 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{6}{6}\langle -1,1,-2\rangle = \langle -1,1,-2\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{6}{\sqrt{6}} = \sqrt{6}$.

11.3.36 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{0}{24}\langle 2,-4,2\rangle = \langle 0,0,0\rangle$. $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{0}{\sqrt{24}} = 0$.

11.3.37 $w = 30\cdot 50\cos\pi/6 = 750\sqrt{3}$ foot-pounds.

11.3.38 $w = 10\cdot 20\cdot\cos 15^\circ \approx 193.185$ J.

11.3.39 $w = 10\cdot 5\cdot\cos 45^\circ = 25\sqrt{2}$ J.

11.3.40 $w = \langle 4,3,2\rangle\cdot\langle 8,6,0\rangle = 50$ J.

11.3.41 $w = (40\mathbf{i}+30\mathbf{j})\cdot 10\mathbf{i} = 400$ J.

11.3.42 $w = \langle 2,4,1\rangle\cdot\langle 2,4,6\rangle = 4+16+6 = 26$.

11.3.43 Parallel to: use $\mathbf{v} = \langle\sqrt{2}/2,-\sqrt{2}/2\rangle$. $\text{proj}_{\mathbf{v}}\mathbf{F} = \frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = 5\sqrt{2}\langle\sqrt{2}/2,-\sqrt{2}/2\rangle = \langle 5,-5\rangle$.
Normal to: $\mathbf{N} = \langle 0,-10\rangle - \langle 5,-5\rangle = \langle -5,-5\rangle$.

11.3.44 Parallel to: use $\mathbf{v} = \langle\sqrt{3}/2,-1/2\rangle$. $\text{proj}_{\mathbf{v}}\mathbf{F} = \frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = 5\langle\sqrt{3}/2,-1/2\rangle = \langle 5\sqrt{3}/2,-5/2\rangle$.
Normal to: $\mathbf{N} = \langle 0,-10\rangle - \langle 5\sqrt{3}/2,-5/2\rangle = \langle -5\sqrt{3}/2,-15/2\rangle$.

11.3.45 Parallel to: use $\mathbf{v} = \langle 1/2,-\sqrt{3}/2\rangle$. $\text{proj}_{\mathbf{v}}\mathbf{F} = \frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = 5\sqrt{3}\langle 1/2,-\sqrt{3}/2\rangle = \langle 5\sqrt{3}/2,-15/2\rangle$.
Normal to: $\mathbf{N} = \langle 0,-10\rangle - \langle 5\sqrt{3}/2,-15/2\rangle = \langle -5\sqrt{3}/2,-5/2\rangle$.

11.3.46 Parallel to: use $\mathbf{v} = \langle 5/\sqrt{41},-4/\sqrt{41}\rangle$.

$$\text{proj}_{\mathbf{v}}\mathbf{F} = \frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{40}{\sqrt{41}}\langle 5/\sqrt{41},4/\sqrt{41}\rangle = \langle 200/41,160/41\rangle.$$

Normal to: $\mathbf{N} = \langle 0,-10\rangle - \langle 200/41,160/41\rangle = \langle -200/41,-570/41\rangle$.

11.3.47

a. False. One is a vector in the same direction as $\mathbf{u}$ and the other is a vector in the direction of $\mathbf{v}$, so if these vectors aren't in the same direction, they can't be equal.

b. True. This follows because $\mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) = |\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{v}$ and $\mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{v}\cdot\mathbf{u} + |\mathbf{v}|^2$, and these are equal if $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude.

c. True. Let $\mathbf{u} = \langle a,b,c\rangle$. Then $(\mathbf{u}\cdot\mathbf{i})^2 + (\mathbf{u}\cdot\mathbf{j})^2 + (\mathbf{u}\cdot\mathbf{k})^2 = a^2+b^2+c^2 = |\mathbf{u}|^2$.

d. False. For example, consider $\mathbf{u} = \langle 1,0\rangle$, $\mathbf{v} = \langle 0,1\rangle$, and $\mathbf{w} = \langle 2,0\rangle$.

e. False. Consider $\langle 1,-1,0\rangle$, $\langle 2,-1,-1\rangle$ and $\langle 3,-2,-1\rangle$. These are all orthogonal to $\langle 1,1,1\rangle$, but don't all lie in the same line.

f. True. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$, and this can't be zero unless $\mathbf{u}\cdot\mathbf{v} = 0$.

11.3.48 Let $\mathbf{u} = \langle a,b,c\rangle$ be orthogonal to $\mathbf{v}$. Then $3a+4b = 0$, so $b = -\frac{3a}{4}$. So any unit vector of the form $\langle a,-3a/4,c\rangle$ has the desired property. These have the form $\frac{1}{\sqrt{a^2+\frac{9}{16}a^2+c^2}}\langle a,-3a/4,c\rangle$ for all numbers a and c where we don't have both $a = 0$ and $c = 0$.

11.3.49 We must have $4-8a+2b = 0$, so $b = 4a-2$. These vectors have the form $\langle 1,a,4a-2\rangle$ where a can be any real number.

11.3.50 Such a vector $\langle a,b,c\rangle$ must satisfy $a+b+c = 0$, so $c = -a-b$. Thus, it must have the form $\frac{1}{\sqrt{a^2+b^2+(a+b)^2}}\langle a,b,-a-b\rangle$ for real numbers a and b, where a and b aren't both zero.

11.3.51 Let $\mathbf{u} = \pm\langle\sqrt{2}/2,\sqrt{2}/2,0\rangle$, $\mathbf{v} = \pm\langle -\sqrt{2}/2,\sqrt{2}/2,0\rangle$ and $\mathbf{w} = \pm\langle 0,0,1\rangle$.

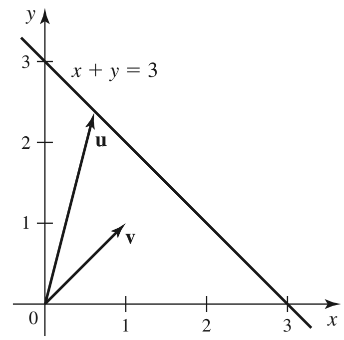
11.3.52 The two other vectors could be $\langle 0,1,-1\rangle$ and $\langle 1,0,0\rangle$.

11.3.53

a. $\text{proj}_{\mathbf{k}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{k}}{\mathbf{k}\cdot\mathbf{k}}\mathbf{k} = \frac{|\mathbf{u}||\mathbf{k}|\cos\theta}{\mathbf{k}\cdot\mathbf{k}}\mathbf{k} = \frac{1}{2}\mathbf{k}$, which is independent of $\mathbf{u}$.

b. Yes, because the scalar projection is the length of the vector projection. In fact, using the above result, it is equal to $1/2$.

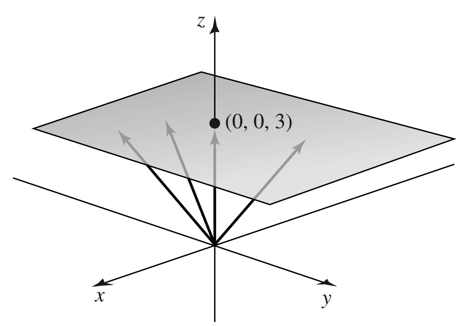
11.3.54 $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{3}{2}\langle 1,1\rangle$. Let $\mathbf{z} = \langle x,y\rangle$. Then $\text{proj}_{\mathbf{v}}\mathbf{z} = \frac{x+y}{2}\langle 1,1\rangle$, so we need $x+y=3$. All vectors from the origin whose terminal point is on the line $x+y=3$ will have the same projection onto $\mathbf{v}$. For example, $\langle 3,0\rangle$ will work.



11.3.55 Using the idea from the last problem, any vector of the form $\langle x,y\rangle$ with $x+y=3$ will work, so any vector of the form $\langle x,3-x\rangle$.

11.3.56 Note that $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle 1,2,3\rangle\cdot\langle 1,1,1\rangle}{3}\langle 1,1,1\rangle = \langle 2,2,2\rangle$. We are seeking $\langle x,y,z\rangle$ so that $\frac{x+y+z}{3}\langle 1,1,1\rangle = \langle 2,2,2\rangle$, so we require $x+y+z=6$. For example, the vector $\langle 3,2,1\rangle$ will work.

11.3.57 Note that $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle 1,2,3\rangle\cdot\langle 0,0,1\rangle}{1}\langle 0,0,1\rangle = \langle 0,0,3\rangle$. We are seeking $\langle x,y,z\rangle$ so that $\frac{z}{1}\langle 0,0,1\rangle = \langle 0,0,3\rangle$, so we require $z=3$. Any vector of the form $\langle x,y,3\rangle$ will suffice.



11.3.58 Let $\mathbf{p} = \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{7}{2}\langle 1,1\rangle$. Then let $\mathbf{n} = \mathbf{u}-\mathbf{p} = \langle 4,3\rangle - \frac{7}{2}\langle 1,1\rangle = \langle 1/2,-1/2\rangle$.

11.3.59 Let $\mathbf{p} = \text{proj}_{\mathbf{v}}\mathbf{u} = -\frac{2}{5}\langle 2,1\rangle$. Then let $\mathbf{n} = \mathbf{u}-\mathbf{p} = \langle -2,2\rangle - -\frac{2}{5}\langle 2,1\rangle = \langle -6/5,12/5\rangle$.

11.3.60 Let $\mathbf{p} = \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{7}{3}\langle 1,1,1\rangle$. Then let $\mathbf{n} = \mathbf{u}-\mathbf{p} = \langle 4,3,0\rangle - \frac{7}{3}\langle 1,1,1\rangle = \langle 5/3,2/3,-7/3\rangle$.

11.3.61 Let $\mathbf{p} = \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{1}{2}\langle 2,1,1\rangle$. Then let $\mathbf{n} = \mathbf{u}-\mathbf{p} = \langle -1,2,3\rangle - \frac{1}{2}\langle 2,1,1\rangle = \langle -2,3/2,5/2\rangle$.

11.3.62

a. $\mathbf{v} = \langle 1,3\rangle$.

b. $\mathbf{u} = \langle 2,-5\rangle$.

c. $\text{proj}_{\mathbf{v}}\mathbf{u} = -\frac{13}{10}\langle 1,3\rangle$.

d. $\mathbf{w} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 2, -5\rangle - -\frac{13}{10}\langle 1, 3\rangle = \langle 33/10, -11/10\rangle$. Note that $\mathbf{w}\cdot\mathbf{v} = 0$, and has length equal to the distance between P and l.

e. $|\mathbf{w}| = \frac{1}{10}\sqrt{(33)^2 + (-11)^2} = \frac{11}{10}\sqrt{10}$. $|\mathbf{w}|$ is the component of $\mathbf{u}$ orthogonal to $\mathbf{v}$, so it is the distance from P to l.

11.3.63

a. $\mathbf{v} = \langle 1, 2\rangle$.

b. $\mathbf{u} = \langle -12, 4\rangle$.

c. $\text{proj}_{\mathbf{v}}\mathbf{u} = -\frac{4}{5}\langle 1, 2\rangle$.

d. $\mathbf{w} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \langle -12, 4\rangle - \langle -4/5, -8/5\rangle = \langle -56/5, 28/5\rangle$. Note that $\mathbf{w}\cdot\mathbf{v} = 0$, and has length equal to the distance between P and l.

e. $|\mathbf{w}| = \frac{1}{5}\sqrt{(56)^2 + (28)^2} = \frac{28\sqrt{5}}{5}$. $|\mathbf{w}|$ is the component of $\mathbf{u}$ orthogonal to $\mathbf{v}$, so it is the distance from P to l.

11.3.64

a. $\mathbf{v} = \langle 3, 0, -4\rangle$.

b. $\mathbf{u} = \langle 0, 2, 6\rangle$.

c. $\text{proj}_{\mathbf{v}}\mathbf{u} = -\frac{24}{25}\langle 3, 0, -4\rangle$.

d. $\mathbf{w} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 0, 2, 6\rangle - -\frac{24}{25}\langle 3, 0, -4\rangle = \langle 72/25, 2, 54/25\rangle$. Note that $\mathbf{w}\cdot\mathbf{v} = 0$, and has length equal to the distance between P and l.

e. $|\mathbf{w}| = \frac{1}{25}\sqrt{(72)^2 + (50)^2 + (54)^2} = \frac{10\sqrt{106}}{25} = \frac{2\sqrt{106}}{5}$. $|\mathbf{w}|$ is the component of $\mathbf{u}$ orthogonal to $\mathbf{v}$, so it is the distance from P to l.

11.3.65

a. $\mathbf{v} = \langle -6, 8, 3\rangle$.

b. $\mathbf{u} = \langle 1, 1, -1\rangle$.

c. $\text{proj}_{\mathbf{v}}\mathbf{u} = -\frac{1}{109}\langle -6, 8, 3\rangle$.

d. $\mathbf{w} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 1, 1, -1\rangle - -\frac{1}{109}\langle -6, 8, 3\rangle = \langle 103/109, 117/109, -106/109\rangle$. Note that $\mathbf{w}\cdot\mathbf{v} = 0$, and has length equal to the distance between P and l.

e. $|\mathbf{w}| = \frac{1}{109}\sqrt{(103)^2 + (117)^2 + (-106)^2} = \sqrt{\frac{326}{109}}$. $|\mathbf{w}|$ is the component of $\mathbf{u}$ orthogonal to $\mathbf{v}$, so it is the distance from P to l.

11.3.66 $\mathbf{I}\cdot\mathbf{J} = -\frac{1}{2} + \frac{1}{2} = 0$.
Also, $|\mathbf{I}| = \sqrt{1/2 + 1/2} = 1 = |\mathbf{J}|$.

11.3.67 $\mathbf{I} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. $\mathbf{J} = \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. $\mathbf{i} = \langle 1, 0\rangle = \frac{\sqrt{2}}{2}(\mathbf{I} - \mathbf{J})$. $\mathbf{j} = \frac{\sqrt{2}}{2}(\mathbf{I} + \mathbf{J})$.

11.3.68 $\langle 2, -6\rangle = 2\mathbf{i} - 6\mathbf{j} = 2\cdot\frac{\sqrt{2}}{2}(\mathbf{I} - \mathbf{J}) - 6\cdot\frac{\sqrt{2}}{2}(\mathbf{I} + \mathbf{J}) = -2\sqrt{2}\mathbf{I} - 4\sqrt{2}\mathbf{J}$.

11.3.69

a. $|\mathbf{I}| = \sqrt{1/4 + 1/4 + 1/2} = 1$. $|\mathbf{J}| = \sqrt{1/2 + 1/2 + 0} = 1$. $|\mathbf{K}| = \sqrt{1/4 + 1/4 + 1/2} = 1$.

b. $\mathbf{I}\cdot\mathbf{J} = -1/2\sqrt{2} + 1/2\sqrt{2} = 0$. $\mathbf{I}\cdot\mathbf{K} = 1/4 + 1/4 - 1/2 = 0$, and $\mathbf{J}\cdot\mathbf{K} = -1/2\sqrt{2} + 1/2\sqrt{2} = 0$.

c. Let $\langle 1,0,0\rangle = a\mathbf{I}+b\mathbf{J}+c\mathbf{K}$. Then $\frac{1}{2}a-\frac{1}{\sqrt{2}}b+\frac{1}{2}c=1$, $\frac{1}{2}a+\frac{1}{\sqrt{2}}b+\frac{1}{2}c=0$, and $\frac{1}{\sqrt{2}}a-\frac{1}{\sqrt{2}}c=0$. Solving this system of linear equations yields $a=\frac{1}{2}$, $b=-\frac{1}{\sqrt{2}}$, and $c=\frac{1}{2}$. Thus, $\langle 1,0,0\rangle = \frac{1}{2}\mathbf{I}+-\frac{1}{\sqrt{2}}\mathbf{J}+\frac{1}{2}\mathbf{K}$.

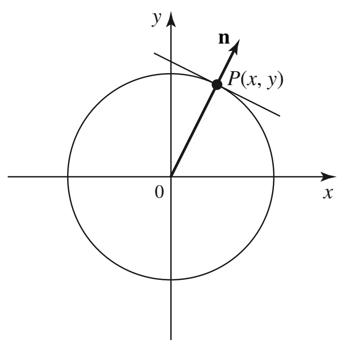
11.3.70 Note that $\overrightarrow{PQ}=\langle 1,11\rangle$, $\overrightarrow{QR}=\langle -4,-5\rangle$, and $\overrightarrow{RP}=\langle 3,-6\rangle$, and these vectors have lengths $\sqrt{122}$, $\sqrt{41}$ and $\sqrt{45}$, respectively.

The angle at P measures $\cos^{-1}(\frac{63}{\sqrt{122}\sqrt{45}})\approx 31.76$ degrees. The angle at Q measures $\cos^{-1}(\frac{59}{\sqrt{122}\sqrt{41}})\approx 33.47$ degrees, and the angle at R measures $\cos^{-1}(-\frac{18}{\sqrt{45}\sqrt{41}})\approx 114.78$ degrees.

11.3.71 Note that $\overrightarrow{PQ}=\langle 2,3,-2\rangle$, $\overrightarrow{QR}=\langle -4,0,3\rangle$, and $\overrightarrow{RP}=\langle 2,-3,-1\rangle$, and these have lengths $\sqrt{17}$, 5 and $\sqrt{14}$ respectively.

The angle at P measures $\cos^{-1}(\frac{3}{\sqrt{17}\sqrt{14}})\approx 78.8$ degrees. The angle at Q measures $\cos^{-1}(\frac{14}{5\sqrt{17}})\approx 47.2$ degrees, and the angle at R measures $\cos^{-1}(\frac{11}{5\sqrt{14}})\approx 54$ degrees.

11.3.72



a. The equation of the circle is $x^2+y^2=1$, so $2x+2y\frac{dy}{dx}=0$, and we have $\frac{dy}{dx}=-\frac{x}{y}$. The slope of the line through the origin containing the vector $\langle x,y\rangle$ is $\frac{y}{x}$, and $\frac{y}{x}\cdot-\frac{x}{y}=-1$. Thus, the vector $\langle x,y\rangle$ is normal to the tangent vector.

b. Let $x=\cos\theta$ and $y=\sin\theta$. Then the previous part shows that $\mathbf{n}=\langle x,y\rangle=\langle\cos\theta,\sin\theta\rangle$.

c. The velocity is normal where $\langle x,y\rangle = k\langle 1,2\rangle$. Then $k^2+4k^2=1$, so $k=\frac{\pm1}{\sqrt{5}}$, and the points are $(\pm1/\sqrt{5},\pm2/\sqrt{5})$.

d. We need $\langle 1,2\rangle\cdot\langle x,y\rangle=0$, so $x+2y=0$. Because we also have $x^2+y^2=1$, we see that $4y^2+y^2=1$, so $y=\frac{\pm1}{\sqrt{5}}$, and $x=\frac{\mp2}{\sqrt{5}}$. The two points are $(2/\sqrt{5},-1/\sqrt{5})$ and $(-2/\sqrt{5},1/\sqrt{5})$.

e. The component of $\mathbf{v}=\langle 1,2\rangle$ normal to C is $\text{proj}_{\mathbf{n}}\mathbf{v}=\frac{x+2y}{x^2+y^2}\langle x,y\rangle=\langle x^2+2xy,xy+2y^2\rangle$. In terms of θ, this is $\langle\cos^2\theta+2\cos\theta\sin\theta,\sin\theta\cos\theta+2\sin^2\theta\rangle$.

f. The net flow is 0, so there is no accumulation inside the circle.

11.3.73

a. The faces on $y=0$ and $z=0$.

b. The faces on $y=1$ and $z=1$.

c. The faces on $x=0$ and $x=1$.

d. Because $\mathbf{Q}$ is tangential on this face, the scalar component of $\mathbf{Q}$ normal to the face is 0.

e. The scalar component of $\mathbf{Q}$ normal to $z=1$ is 1. Note that a vector normal to $z=1$ is $\langle 0,0,1\rangle$.

f. The scalar component of $\mathbf{Q}$ normal to $y=0$ is 2. Note that a vector normal to $y=0$ is $\langle 0,1,0\rangle$.

11.3.74

a. $\mathbf{r}_{0j} = \langle 2\cos(\frac{(j-1)\pi}{3}), 2\sin(\frac{(j-1)\pi}{3})\rangle$ for $j = 1, 2, \ldots, 6$.

b. $\mathbf{r}_{12} = \langle 2\cos(2\pi/3), 2\sin(2\pi/3)\rangle = \langle -1, \sqrt{3}\rangle$.
$\mathbf{r}_{34} = \langle 2\cos(4\pi/3), 2\sin(4\pi/3)\rangle = \langle -1, -\sqrt{3}\rangle$.
$\mathbf{r}_{61} = \langle 2\cos(\pi/3), 2\sin(\pi/3)\rangle = \langle 1, \sqrt{3}\rangle$.

c. $\mathbf{r}_{07} = \mathbf{r}_{01} + \mathbf{r}_{02} = \langle 3, \sqrt{3}\rangle$.
$\mathbf{r}_{17} = \mathbf{r}_{02} = \langle 2\cos(\pi/3), 2\sin(\pi/3)\rangle = \langle 1, \sqrt{3}\rangle$.
$\mathbf{r}_{47} = \langle 4, 0\rangle + \mathbf{r}_{17} = \langle 5, \sqrt{3}\rangle$.
$\mathbf{r}_{75} = \mathbf{r}_{70} + \mathbf{r}_{05} = -2\langle 2, \sqrt{3}\rangle$.

11.3.75

a. Let the coordinates of R be (x, y, z). By symmetry, we have $y = 0$. We must have $x^2 + y^2 + z^2 = (x - \sqrt{3})^2 + (y+1)^2 + z^2 = (x-\sqrt{3})^2 + (y-1)^2 + z^2 = 4$. The first equality gives $x^2 + z^2 = x^2 - 2\sqrt{3}x + 3 + 1 + z^2$, so $2\sqrt{3}x = 4$, and $x = \frac{2}{\sqrt{3}}$. It then follows that $z = \frac{2\sqrt{2}}{\sqrt{3}}$.

b. We have $\mathbf{r}_{OP} = \langle \sqrt{3}, -1, 0\rangle$, $\mathbf{r}_{OQ} = \langle \sqrt{3}, 1, 0\rangle$, $\mathbf{r}_{PQ} = \langle 0, 2, 0\rangle$, $\mathbf{r}_{OR} = \langle 2/\sqrt{3}, 0, 2\sqrt{2}/\sqrt{3}\rangle$, and $\mathbf{r}_{PR} = \langle -\sqrt{3}/3, 1, 2\sqrt{2}/\sqrt{3}\rangle$.

11.3.76 $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$, so $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,|\cos\theta| \le |\mathbf{u}|\,|\mathbf{v}|$, because $|\cos\theta| \le 1$.

11.3.77 $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \mathbf{v} \cdot \mathbf{u}$.

11.3.78 $c(\mathbf{u} \cdot \mathbf{v}) = c(u_1v_1 + u_2v_2 + u_3v_3) = cu_1v_1 + cu_2v_2 + cu_3v_3 = (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 = (c\mathbf{u}) \cdot \mathbf{v}$. Using this result and the previous, we have $c(\mathbf{u} \cdot \mathbf{v}) = c(\mathbf{v} \cdot \mathbf{u}) = (c\mathbf{v}) \cdot \mathbf{u} = \mathbf{u} \cdot (c\mathbf{v})$.

11.3.79 $\mathbf{u}(\mathbf{v} + \mathbf{w}) = \langle u_1, u_2, u_3\rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3\rangle = u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3) = u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + u_3v_3 + u_3w_3 = (u_1v_1 + u_2v_2 + u_3v_3) + (u_1w_1 + u_2w_2 + u_3w_3) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$.

11.3.80

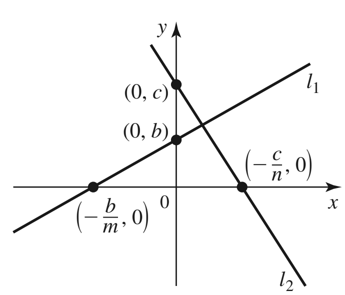
a. Using the previous results, we have $(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) = (\mathbf{u}+\mathbf{v}) \cdot \mathbf{u} + (\mathbf{u}+\mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{u}+\mathbf{v}) + \mathbf{v} \cdot (\mathbf{u}+\mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = |\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2$.

b. If the two vectors are perpendicular, then $\mathbf{u} \cdot \mathbf{v} = 0$, so the above reduces to $(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) = |\mathbf{u}|^2 + |\mathbf{v}|^2$.

c. Using the previous results, we have $(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) = (\mathbf{u}+\mathbf{v}) \cdot \mathbf{u} - (\mathbf{u}+\mathbf{v}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{u}+\mathbf{v}) - \mathbf{v} \cdot (\mathbf{u}+\mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} = |\mathbf{u}|^2 - |\mathbf{v}|^2$.

11.3.81 The statement is true. We have $\text{proj}_{\langle ka, kb\rangle}\langle c, d\rangle = \frac{\langle c,d\rangle \cdot \langle ka,kb\rangle}{(ka)^2+(kb)^2}\langle ka, kb\rangle = \frac{k(\langle a,b\rangle \cdot \langle c,d\rangle)}{k^2(a^2+b^2)} \cdot k\langle a, b\rangle = \frac{\langle a,b\rangle \cdot \langle c,d\rangle}{(a^2+b^2)}\langle a, b\rangle = \text{proj}_{\langle a,b\rangle}\langle c, d\rangle$.

11.3.82 Let $\mathbf{u}$ be a directional vector for the line $y = mx+b$, so $\mathbf{u} = k_1\langle b/m, b\rangle$ where $k_1 \ne 0$. Let $\mathbf{v}$ be a directional vector for the line $y = nx+c$, so $\mathbf{v} = k_2\langle c/n, c\rangle$ where $k_2 \ne 0$. Then $\mathbf{u} \cdot \mathbf{v} = k_1k_2\left(\frac{b}{m} \cdot \frac{c}{n} + b \cdot c\right)$ is zero exactly when $\frac{bc}{mn} + bc = 0$, which occurs for $\frac{1}{mn} + 1 = 0$, or $mn = -1$.



11.3.83

a. $\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}$, $\cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}$, and $\cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$. Thus, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \frac{a^2}{a^2+b^2+c^2}+\frac{b^2}{a^2+b^2+c^2}+\frac{c^2}{a^2+b^2+c^2} = 1$.

b. We require $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \frac{1}{2}+\frac{1}{2}+\cos^2\gamma = 1$, so $\gamma = 90°$. The vector could be $\langle 1,1,0\rangle$; it makes a 90 degree angle with $\mathbf{k}$.

c. We require $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \frac{1}{4}+\frac{1}{4}+\cos^2\gamma = 1$, so $\gamma = 45°$. The vector could be $\langle 1,1,\sqrt{2}\rangle$; it makes a 45 degree angle with $\mathbf{k}$.

d. No. If so, we would have $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \frac{3}{4}+\frac{3}{4}+\cos^2\gamma = 1$, which would imply that $\cos^2\gamma = -\frac{1}{2}$, which can't occur.

e. If $\alpha=\beta=\gamma$, then $3\cos^2\alpha = 1$, and $\alpha = \cos^{-1}(\sqrt{3}/3) \approx 54.7356$ degrees. The vector could be $\langle 1,1,1\rangle$.

11.3.84 $|\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,|\cos\theta| = |\mathbf{u}|\,|\mathbf{v}|$ if and only if $\cos\theta = \pm 1$, which occurs only when $\mathbf{u}$ and $\mathbf{v}$ are parallel, or if either $\mathbf{u}=\mathbf{0}$ or $\mathbf{v}=\mathbf{0}$.

11.3.85 $\mathbf{u}\cdot\mathbf{v} = -24-15+6 = -33$. $|\mathbf{u}| = \sqrt{3^2+(-5)^2+6^2} = \sqrt{70}$. $|\mathbf{v}| = \sqrt{(-8)^2+3^2+1^2} = \sqrt{74}$. Note that

$$33 < \sqrt{70}\sqrt{74},$$

so $|\mathbf{u}\cdot\mathbf{v}| < |\mathbf{u}|\,|\mathbf{v}|$.

11.3.86 We have $|\mathbf{u}\cdot\mathbf{v}| = \sqrt{ab}+\sqrt{ab} = 2\sqrt{ab} \le |\mathbf{u}|\,|\mathbf{v}| = \sqrt{a+b}\sqrt{b+a} = a+b$. Thus,

$$\sqrt{ab} \le \frac{a+b}{2}.$$

11.3.87

a. We have $|\mathbf{u}+\mathbf{v}|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = (\mathbf{u}+\mathbf{v})\cdot\mathbf{u}+(\mathbf{u}+\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{u}+\mathbf{v})+\mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} = |\mathbf{u}|^2+2(\mathbf{u}\cdot\mathbf{v})+|\mathbf{v}|^2$

b. Note that $2(\mathbf{u}\cdot\mathbf{v}) \le 2\,|\mathbf{u}\cdot\mathbf{v}| \le 2\,|\mathbf{u}|\,|\mathbf{v}|$. Thus (using the previous part) we have,

$$|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2+2(\mathbf{u}\cdot\mathbf{v})+|\mathbf{v}|^2 \le |\mathbf{u}|^2+2\,|\mathbf{u}|\,|\mathbf{v}|+|\mathbf{v}|^2 \le (|\mathbf{u}|+|\mathbf{v}|)^2.$$

c. Taking square roots of the previous result, and using the fact that the square root function is strictly increasing, we have $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}|+|\mathbf{v}|$.

d. Because the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ form a triangle, we can interpret this as meaning that the sum of the lengths of any two sides of a triangle is greater than or equal to the length of the other side.

11.3.88 Let $\mathbf{v} = \langle u_2,u_3,u_1\rangle$. Then $\mathbf{u}\cdot\mathbf{v} = u_1u_2+u_2u_3+u_1u_3$ and $|\mathbf{u}|\,|\mathbf{v}| = u_1^2+u_2^2+u_3^2$. So the Cauchy-Schwarz inequality gives $u_1u_2+u_2u_3+u_3u_1 \le |\mathbf{u}|^2$. Thus,

$$(u_1+u_2+u_3)^2 = u_1^2+u_2^2+u_3^2+2(u_1u_2+u_2u_3+u_3u_1) \le u_1^2+u_2^2+u_3^2+2\,|\mathbf{u}|^2 = 3\,|\mathbf{u}|^2.$$

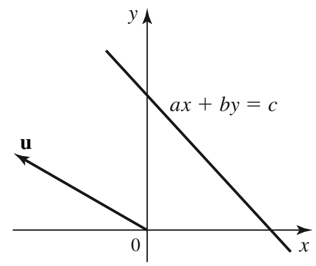
11.3.89

a. One diagonal consists of the sum of one side ($\mathbf{u}$) and the side opposite the side adjacent to $\mathbf{u}$, but because it is a parallelogram, the side opposite $\mathbf{v}$ is also $\mathbf{v}$. So the diagonal is $\mathbf{u}+\mathbf{v}$. The other diagonal is the difference of two adjacent sides, so it is $\mathbf{u}-\mathbf{v}$.

b. The two diagonals are equal when $|\mathbf{u}+\mathbf{v}| = |\mathbf{u}-\mathbf{v}|$. Squaring both sides, we see that this is equivalent to requiring $|\mathbf{u}|^2+2(\mathbf{u}\cdot\mathbf{v})+|\mathbf{v}|^2 = |\mathbf{u}|^2-2(\mathbf{u}\cdot\mathbf{v})+|\mathbf{v}|^2$, which would imply that $2(\mathbf{u}\cdot\mathbf{v}) = -2(\mathbf{u}\cdot\mathbf{v})$, or $4(\mathbf{u}\cdot\mathbf{v}) = 0$. So if the diagonals are equal, the vectors are orthogonal. These steps are reversible, so the converse is also true.

c. $|\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 + |\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 = 2(|\mathbf{u}|^2 + |\mathbf{v}|^2)$.

11.3.90 Note that $\mathbf{v} = \langle -b, a\rangle$ is a vector in the direction of the given line. A position vector $\mathbf{u}$ corresponding to the point P is $\mathbf{u} = \langle x_0, y_0\rangle$. Note that $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{-bx_0+ay_0}{a^2+b^2}\langle -b, a\rangle$. Let $\mathbf{w} = \mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \langle x_0, y_0\rangle - \frac{-bx_0+ay_0}{a^2+b^2}\langle -b, a\rangle = \frac{ax_0+by_0}{a^2+b^2}\langle a, b\rangle$. Note that the distance between the point and the line is $|\mathbf{w}| = \frac{|ax_0+by_0|}{\sqrt{a^2+b^2}}$.



11.4 Cross Products

11.4.1 $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

11.4.2 $\mathbf{u}\times\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, and points in the direction dictated by the right-hand rule.

11.4.3 Two parallel vectors have $\sin\theta = 0$ where θ is the angle between them. Thus, $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = 0$.

11.4.4 Two perpendicular vectors have $\sin\theta = 1$ where θ is the angle between them. Thus, $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = |\mathbf{u}|\,|\mathbf{v}|$.

11.4.5 If $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, then $\mathbf{u}\times\mathbf{v}$ can be thought of as the determinant of the matrix

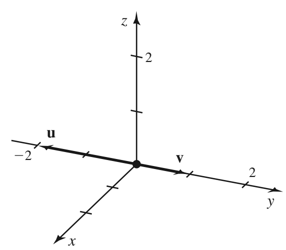
$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

11.4.6 The torque produced by the force $\mathbf{F}$ about the head of vector $\mathbf{r}$ is $\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$.

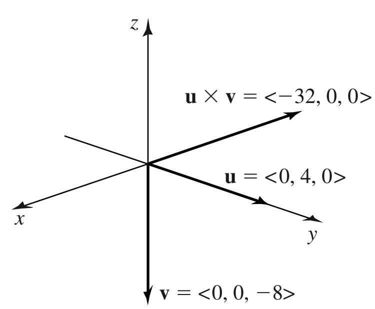
11.4.7 $\mathbf{u}\times\mathbf{v} = \langle 3,0,0\rangle \times \langle 0,5,0\rangle = \langle 0,0,15\rangle$.

11.4.8 $\mathbf{u}\times\mathbf{v} = \langle -4,0,0\rangle \times \langle 0,0,2\rangle = \langle 0,8,0\rangle$.

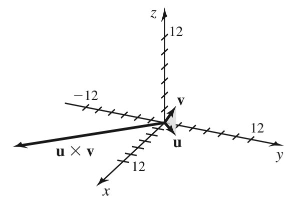
11.4.9 $\mathbf{u}\times\mathbf{v} = \langle 0,0,0\rangle$, so $|\mathbf{u}\times\mathbf{v}| = 0$.



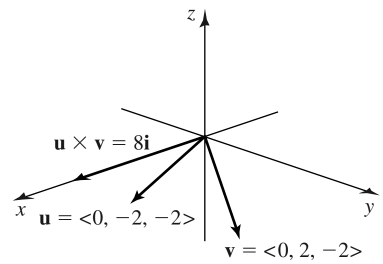
11.4.10 $\mathbf{u}\times\mathbf{v} = \langle -32,0,0\rangle$, so $|\mathbf{u}\times\mathbf{v}| = 32$.



11.4.11 $\mathbf{u} \times \mathbf{v} = \langle 9\sqrt{2}, -9\sqrt{2}, 0\rangle$, so $|\mathbf{u} \times \mathbf{v}| = 18$.



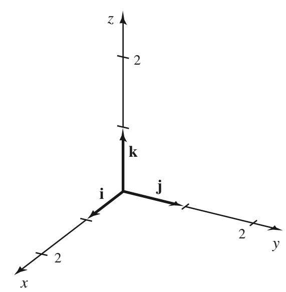
11.4.12 $\mathbf{u} \times \mathbf{v} = \langle 8, 0, 0\rangle$, so $|\mathbf{u} \times \mathbf{v}| = 8$.



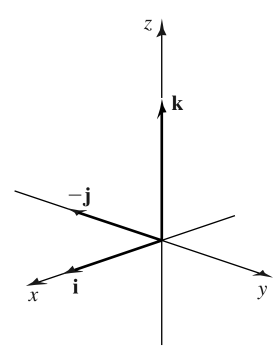
11.4.13 $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin(\pi/4) = \sqrt{2}/2$.

11.4.14 $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin(2\pi/3) = 12 \cdot (\sqrt{3}/2) = 6\sqrt{3}$.

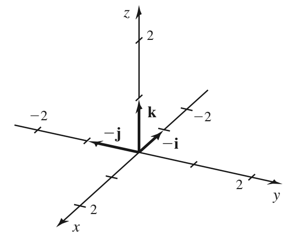
11.4.15 $\mathbf{j} \times \mathbf{k} = \mathbf{i}$.



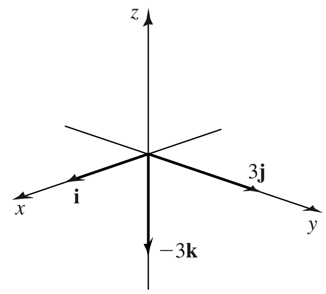
11.4.16 $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.



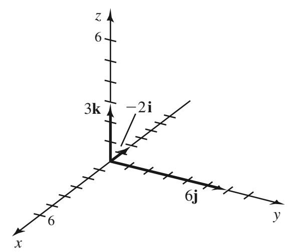
11.4.17 $-\mathbf{j} \times \mathbf{k} = -\mathbf{i}$.



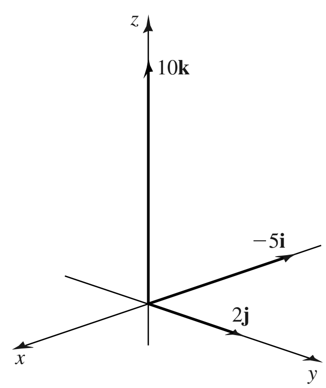
11.4.18 $3\mathbf{j} \times \mathbf{i} = -3\mathbf{k}$.



11.4.19 $-2\mathbf{i} \times 3\mathbf{k} = 6\mathbf{j}$.



11.4.20 $2\mathbf{j} \times -5\mathbf{i} = 10\mathbf{k}$.



11.4.21 $|\mathbf{u} \times \mathbf{v}| = |\langle -2, -6, 9\rangle| = \sqrt{4+36+81} = 11$.

11.4.22 $|\mathbf{u} \times \mathbf{v}| = |\langle -2, 5, -3\rangle| = \sqrt{4+9+25} = \sqrt{38}$.

11.4.23 $|\mathbf{u} \times \mathbf{v}| = |\langle 5, -4, 7\rangle| = \sqrt{25+16+49} = 3\sqrt{10}$.

11.4.24 $|\mathbf{u} \times \mathbf{v}| = |\langle 4, 26, 28\rangle| = \sqrt{16+676+784} = \sqrt{1476} = 6\sqrt{41}$.

11.4.25 $\frac{1}{2} \cdot \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2} \cdot |\langle 3,0,1\rangle \times \langle 1,1,0\rangle| = \frac{1}{2} \cdot |\langle -1,1,3\rangle| = \frac{\sqrt{11}}{2}$.

11.4.26 $\frac{1}{2} \cdot \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2} \cdot |\langle 4,-1,2\rangle \times \langle 1,1,0\rangle| = \frac{1}{2} \cdot |\langle -2,2,5\rangle| = \frac{\sqrt{33}}{2}$.

11.4.27 $\frac{1}{2} \cdot \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2} \cdot |\langle 2,10,2\rangle \times \langle 1,1,1\rangle| = \frac{1}{2} \cdot |\langle 8,0,-8\rangle| = 4\sqrt{2}$.

11.4.28 $\frac{1}{2} \cdot \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2} \cdot |\langle -2,3,2\rangle \times \langle 1,0,2\rangle| = \frac{1}{2} \cdot |\langle 6,6,-3\rangle| = \frac{9}{2}$.

11.4.29 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & 0 \\ 0 & 3 & -6 \end{vmatrix} = \langle -30,18,9\rangle$. $\mathbf{v} \times \mathbf{u} = \langle 30,-18,-9\rangle$.

11.4.30 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \langle -2,-4,-4\rangle$. $\mathbf{v} \times \mathbf{u} = \langle 2,4,4\rangle$.

11.4.31 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -9 \\ -1 & 1 & -1 \end{vmatrix} = \langle 6,11,5\rangle$. $\mathbf{v} \times \mathbf{u} = \langle -6,-11,-5\rangle$.

11.4.32 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 6 \\ 1 & 2 & -1 \end{vmatrix} = \langle -8,9,10\rangle$. $\mathbf{v} \times \mathbf{u} = \langle 8,-9,-10\rangle$.

11.4.33 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & -2 \\ 1 & 3 & -2 \end{vmatrix} = \langle 8,4,10\rangle$. $\mathbf{v} \times \mathbf{u} = \langle -8,-4,-10\rangle$.

11.4.34 $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -10 & 15 \\ .5 & 1 & -.6 \end{vmatrix} = \langle -9,8.7,7\rangle$. $\mathbf{v} \times \mathbf{u} = \langle 9,-8.7,-7\rangle$.

11.4.35 Let $\mathbf{u} = \langle 0,1,2\rangle$ and $\mathbf{v} = \langle -2,0,3\rangle$. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ -2 & 0 & 3 \end{vmatrix} = \langle 3,-4,2\rangle$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

11.4.36 Let $\mathbf{u} = \langle 1,2,3\rangle$ and $\mathbf{v} = \langle -2,4,-1\rangle$. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 4 & -1 \end{vmatrix} = \langle -14,-5,8\rangle$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

11.4.37 Let $\mathbf{u} = \langle 8,0,4\rangle$ and $\mathbf{v} = \langle -8,2,1\rangle$. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 0 & 4 \\ -8 & 2 & 1 \end{vmatrix} = \langle -8,-40,16\rangle$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

11.4.38 Let $\mathbf{u} = \langle 6, -2, 4\rangle$ and $\mathbf{v} = \langle 1, 2, 3\rangle$. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -2 & 4 \\ 1 & 2 & 3 \end{vmatrix} = \langle -14, -14, 14\rangle$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

11.4.39 $|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = \frac{1}{4} \cdot 20 \cdot \frac{\sqrt{2}}{2} = \frac{5\sqrt{2}}{2}\,\mathrm{N} \cdot \mathrm{m}$.

11.4.40 $|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = \frac{5}{6} \cdot \frac{3}{2} \cdot \sin(\pi/2) = 1.25\,\mathrm{ft} \cdot \mathrm{lb}$.

11.4.41 $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 20 & 0 & 0 \end{vmatrix} = \langle 0, 20, -20\rangle$.

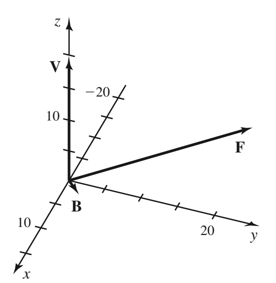
11.4.42 $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 10 & 10 & 0 \end{vmatrix} = \langle -20, 20, 20\rangle$.

11.4.43 $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 0 & 0 \\ 5 & 0 & -5 \end{vmatrix} = \langle 0, 50, 0\rangle$ has magntitude 50, while $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 0 & 0 \\ 4 & -3 & 0 \end{vmatrix} = \langle 0, 0, -30\rangle$ has magnitude 30, so the first force has greater magnitude.

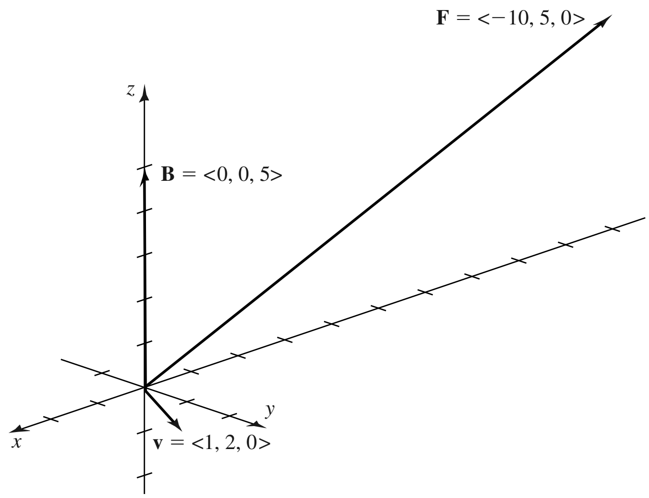
11.4.44 $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & -5 \\ 1 & 0 & -10 \end{vmatrix} = \langle 0, 45, 0\rangle$. The magnitude is 45 and the direction is in the positive y direction.

11.4.45 $\mathbf{F} = 1 \cdot (\mathbf{v} \times \mathbf{B}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 20 \\ 1 & 1 & 0 \end{vmatrix} = \langle -20, 20, 0\rangle$.

The magnitude of $\mathbf{F}$ is $20\sqrt{2}$ and the angle of the force is 135 degrees with the positive x axis in the xy-plane.



11.4.46 $\mathbf{F} = -1 \cdot (\mathbf{v} \times \mathbf{B}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 0 & 0 & 5 \end{vmatrix} = \langle -10, 5, 0\rangle$. The magnitude of $\mathbf{F}$ is $5\sqrt{5}$ and the angle of the force is about 153.435 degrees with the positive x axis in the xy-plane.



11.4.47 $|\mathbf{F}| = |q(\mathbf{v} \times \mathbf{B})| = \left|-1.6 \cdot 10^{-19}\right| \text{ C } \cdot 2 \cdot 10^5 \cdot 2 \cdot \sin 45^\circ = 4.53 \cdot 10^{-14}\,\text{kg} \cdot \text{m/s}^2$.

11.4.48 $\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) = 5 \cdot 10^{-12}\mathbf{k}$, $\mathbf{v} = 2 \cdot 10^6\mathbf{j}$. We must have $\mathbf{B} = b\mathbf{i}$ for some scalar b. We have

$$1.6 \cdot 10^{-19}(2 \cdot 10^6\mathbf{j} \times b\mathbf{i}) = 5 \cdot 10^{-12}\mathbf{k},$$

so $b = -\frac{5 \cdot 10^{-12}}{1.6 \cdot 10^{-19} \cdot 2 \cdot 10^6} = -\frac{125}{8}$. So $\mathbf{B} = -15.625\mathbf{i}$.

11.4.49

a. False. For example $\mathbf{i} \times \mathbf{i} = \langle 0, 0, 0 \rangle$, even though $\mathbf{i} \neq \langle 0, 0, 0 \rangle$.

b. False. For example, $2\mathbf{i} \times 4\mathbf{j} = 8\mathbf{k}$ has magnitude 8, while $2\mathbf{i}$ has magnitude 2 and $4\mathbf{j}$ has magnitude 4.

c. False. If the compass directions are thought to lie in a plane, $\mathbf{u} \times \mathbf{v}$ doesn't lie in that plane, so it can't be a compass direction.

d. True. If both were nonzero, the first statement implies that the vectors are parallel, and the second that they are perpendicular, which can't both occur. So at least one of the vectors must be the zero vector.

e. False. $\mathbf{i} \times 2\mathbf{i} = \langle 0, 0, 0 \rangle = \mathbf{i} \times 3\mathbf{i}$, but $2\mathbf{i} \neq 3\mathbf{i}$.

11.4.50 $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 2, 2, 6 \rangle \times \langle 6, 6, 18 \rangle = \langle 0, 0, 0 \rangle$, so the points are collinear.

11.4.51 $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 4, 6, 6 \rangle \times \langle 7, 12, 13 \rangle \neq \langle 0, 0, 0 \rangle$, so the points are not collinear.

11.4.52 Note that $\langle a, a, 2 \rangle \times \langle 1, a, 3 \rangle = \langle a, 2 - 3a, -a + a^2 \rangle$. So $a = 2$.

11.4.53 Note that $\langle a, b, a \rangle \times \langle b, a, b \rangle = \langle -a^2 + b^2, 0, a^2 - b^2 \rangle$. This is the zero vector when $a = \pm b$, so the vectors are parallel when $a = \pm b$, $a, b \neq 0$.

11.4.54 The area is $\frac{1}{2}|\mathbf{u} \times \mathbf{v}| = \frac{1}{2}\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 0 \\ 4 & 4 & 4 \end{vmatrix} \right\| = \frac{1}{2}|\langle 24, 0, -24 \rangle| = 12\sqrt{2}$.

11.4.55 The area is $\frac{1}{2}|\mathbf{u} \times \mathbf{v}| = \frac{1}{2}\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 3 \\ 6 & 0 & 6 \end{vmatrix} \right\| = \frac{1}{2}|\langle 18, 0, -18 \rangle| = 9\sqrt{2}$.

11.4.56 Two of the sides are $\mathbf{u} = \langle 2, 4, 6\rangle$ and $\mathbf{v} = \langle 3, 5, 7\rangle$.

The area is $\frac{1}{2}|\mathbf{u}\times\mathbf{v}| = \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{vmatrix}\right\| = \frac{1}{2}|\langle -2, 4, -2\rangle| = \sqrt{6}$.

11.4.57 Two of the sides are $\mathbf{u} = \langle 1, 2, 3\rangle$ and $\mathbf{v} = \langle 6, 5, 4\rangle$.

The area is $\frac{1}{2}|\mathbf{u}\times\mathbf{v}| = \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 6 & 5 & 4 \end{vmatrix}\right\| = \frac{1}{2}|\langle -7, 14, -7\rangle| = \frac{7\sqrt{6}}{2}$.

11.4.58 If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular unit vectors, then $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\pi/2 = 1$. Or more generally, if $\sin\theta = \frac{1}{|\mathbf{u}||\mathbf{v}|}$.

11.4.59 Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$. Then we have

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ u_1 & u_2 & u_3 \end{vmatrix} = \langle -1, -1, 2\rangle,$$

so $u_3 - u_2 = -1$, $u_1 - u_3 = -1$, and $u_2 - u_1 = 2$. The solutions to this system of linear equation can be characterized by letting u_1 be arbitrary, and by letting $u_2 = u_1 + 2$ and $u_3 = u_1 + 1$. Thus, $\mathbf{u} = \langle u_1, u_1 + 2, u_1 + 1\rangle$ for any real number u_1.

11.4.60 Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$. Then we have

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ u_1 & u_2 & u_3 \end{vmatrix} = \langle 0, 0, 1\rangle,$$

so $u_3 - u_2 = 0$, $u_1 - u_3 = 0$, and $u_2 - u_1 = 1$. This system of linear equations has no solutions, so there are no vectors $\mathbf{u}$ which satisfy the given equation.

11.4.61 Two of the sides of the triangle are $\mathbf{u} = \langle -a, b, 0\rangle$ and $\mathbf{v} = \langle -a, 0, c\rangle$.

The area is $\frac{1}{2}|\mathbf{u}\times\mathbf{v}| = \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}\right\| = \frac{1}{2}|\langle bc, ac, ab\rangle| = \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}$.

11.4.62 $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \langle u_1, u_2, u_3\rangle \cdot \langle v_2w_3 - v_3w_2, w_1v_3 - v_1w_3, v_1w_2 - w_1v_2\rangle = u_1(v_2w_3 - v_3w_2) + u_2(w_1v_3 - v_1w_3) + u_3(v_1w_2 - w_1v_2)$. Note that this is exactly the expression for the determinant of the given matrix, as can be seen by expanding by cofactors across the top row.

11.4.63 $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |\mathbf{u}|\,|\mathbf{v}\times\mathbf{w}|\,|\cos\theta|$. Because $|\mathbf{v}\times\mathbf{w}|$ represents the area of the base, we just need to see that the height of the parallelepiped is $|\mathbf{u}|\,|\cos\theta|$. Note that the height is given by the scalar projection of $\mathbf{u}$ on $\mathbf{v}\times\mathbf{w}$, which has value $|\cos\theta|\,|\mathbf{u}|$. Thus the given expression represents the volume of the parallelepiped.

11.4.64 Using one of the results above, we see that $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ and similarly we have $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$. These two determinants are equal, as can be seen by fact that the underlying matrices differ by an even number of row transpositions.

11.4.65 Note that $\mathbf{r} = .66\mathbf{k}$, and $\mathbf{F} = 40\mathbf{j}$. $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.66 \\ 0 & 40 & 0 \end{vmatrix} = \langle -26.4, 0, 0 \rangle$. The magnitude of the torque is 26.4 Newton-meters and the direction is on the negative x axis.

11.4.66

a. The torque of the shoulder has magnitude $2 \cdot 20 = 40$ ft lbs and the direction is perpendicular to $\mathbf{r}$ and $\mathbf{F}$, into the page.

b. The torque of the elbow has magnitude $1 \cdot 20 = 20$ foot pounds, and the direction is the same as the torque of the shoulder, into the page.

11.4.67 Because $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$, we have $|\mathbf{F}| = |q|\,|\mathbf{v}|\,|\mathbf{B}| \sin\theta$. Thus, $\frac{m|\mathbf{v}|^2}{R} = |q|\,|\mathbf{v}|\,|\mathbf{B}| \sin \pi/2$. Therefore,

$$|\mathbf{v}| = \frac{R\,|q|\,|\mathbf{B}|}{m} = \frac{0.002 \cdot 1.6 \cdot 10^{-19} \cdot .05}{9 \cdot 10^{-31}} \approx 1.758 \cdot 10^7 \text{ m/s}.$$

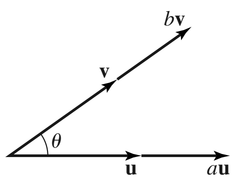
11.4.68

a. $|\mathbf{u} \times \mathbf{u}| = |\mathbf{u}|^2 \sin 0 = 0$.

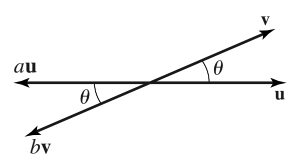
b. $\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = \langle 0, 0, 0 \rangle$.

c. Because $\mathbf{u} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{u})$, we must have $2(\mathbf{u} \times \mathbf{u}) = 0$, so $\mathbf{u} \times \mathbf{u} = 0$.

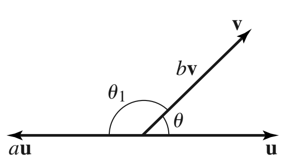
11.4.69 The result is trivial if either $a = 0$ or $b = 0$, so assume $ab \neq 0$. Note that the sine of the angle between $a\mathbf{u}$ and $b\mathbf{v}$ is the same as the sine of the angle between $\mathbf{u}$ and $\mathbf{v}$, as is demonstrated in the following diagrams.



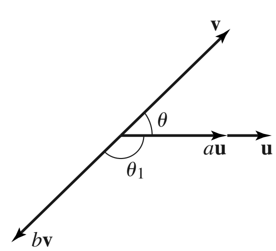
$a > 0,\ b > 0$



$a < 0,\ b < 0$



$a < 0,\ b > 0$



$a > 0,\ b < 0$

By the definition: $|(a\mathbf{u}) \times (b\mathbf{v})| = |a\mathbf{u}|\,|b\mathbf{v}| \sin\theta$, where θ is the angle between $a\mathbf{u}$ and $b\mathbf{v}$. But this is equal to $|a|\,|\mathbf{u}|\,|b|\,|\mathbf{v}| \sin\theta = |ab|\,(|\mathbf{u}|\,|\mathbf{v}| \sin\theta) = |ab|\,(|\mathbf{u} \times \mathbf{v}|)$. When a and b have the same sign, the directions are also the same, because they are determined by the right-hand rule (see diagrams above.) When a and b have opposite signs, the directions are opposite, but then $ab < 0$.

Using the determinant formula:

$$(a\mathbf{u}) \times (b\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ au_1 & au_2 & au_3 \\ bv_1 & bv_2 & bv_3 \end{vmatrix} = ab \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = ab(\mathbf{u} \times \mathbf{v}).$$

11.4.70 False. For example, $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} \neq \mathbf{0}$.

11.4.71 True. $(\mathbf{u} - \mathbf{v}) \times (\mathbf{u} + \mathbf{v}) = \mathbf{u} \times \mathbf{u} + \mathbf{u} \times \mathbf{v} - (\mathbf{v} \times \mathbf{u}) - (\mathbf{v} \times \mathbf{v}) = 2(\mathbf{u} \times \mathbf{v}) = (2\mathbf{u} \times \mathbf{v})$.

11.4.72 This follows from number 64 above, together with the commutativity of the dot product.

11.4.73

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_2w_3 - v_3w_2 & v_3w_1 - v_1w_3 & v_1w_2 - v_2w_1 \end{vmatrix} \\ &= \langle u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3), u_3(v_2w_3 - v_3w_2) \\ &\quad - u_1(v_1w_2 - v_2w_1), u_1(v_3w_1 - v_1w_3) - u_2(v_2w_3 - v_3w_2)\rangle \\ &= \langle v_1(u_2w_2 + u_3w_3) - w_1(u_2v_2 + u_3v_3), v_2(u_1w_1 + u_3w_3) \\ &\quad - w_2(u_1v_1 + u_3v_3), v_3(u_1w_1 + u_2w_2) - w_3(u_1v_1 + u_2v_2)\rangle \\ &= \langle v_1(\mathbf{u} \cdot \mathbf{w}) - w_1(\mathbf{u} \cdot \mathbf{v}), v_2(\mathbf{u} \cdot \mathbf{w}) - w_2(\mathbf{u} \cdot \mathbf{v}), v_3(\mathbf{u} \cdot \mathbf{w}) - w_3(\mathbf{u} \cdot \mathbf{w})\rangle \\ &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{w})\mathbf{w}. \end{aligned}$$

11.4.74 Consider the quantity $(\mathbf{w} \times \mathbf{x})$ as a single vector. Using the result of exercise 56, we have $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{x}) = \mathbf{u} \cdot (\mathbf{v} \times (\mathbf{w} \times \mathbf{x}))$. Now applying the result of exercise 65, this is equal to $\mathbf{u} \cdot ((\mathbf{v} \cdot \mathbf{x})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{x}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{x}) - (\mathbf{u} \cdot \mathbf{x})(\mathbf{v} \cdot \mathbf{w})$ as desired.

11.4.75

a. Suppose $\mathbf{u} \times \mathbf{z} = \mathbf{v}$. Then $\mathbf{v} \times (\mathbf{u} \times \mathbf{z}) = \mathbf{v} \times \mathbf{v} = \langle 0, 0, 0\rangle$. Now $\mathbf{v} \times (\mathbf{u} \times \mathbf{z}) = \mathbf{u}(\mathbf{v} \cdot \mathbf{z}) - \mathbf{z}(\mathbf{v} \cdot \mathbf{u})$ by exercise 65. If $\mathbf{u} \cdot \mathbf{v} = 0$, then we have $\mathbf{u}(\mathbf{v} \cdot \mathbf{z}) = \langle 0, 0, 0\rangle$. Any vector $\mathbf{z}$ which is perpendicular to $\mathbf{v}$ is a solution to this equation.

   Now suppose that the equation $\mathbf{u} \times \mathbf{z} = \mathbf{v}$ has a nonzero solution. Because the cross product of any two vectors is perpendicular to both of the vectors, we must have that $\mathbf{u} \times \mathbf{z} \cdot \mathbf{u} = 0$. But this means that $\mathbf{v} \cdot \mathbf{u} = 0$, as desired.

b. If there exists a vector $\mathbf{z}$ so that $\mathbf{u} \times \mathbf{z} = \mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ must be perpendicular. If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular nonzero vectors, then there must be a plane which contains $\mathbf{u}$ and a nonzero vector $\mathbf{z}$ so that $\mathbf{u} \times \mathbf{z} = \mathbf{v}$.

11.5 Lines and Curves in Space

11.5.1 It has one, namely t.

11.5.2 It has three, namely $x = f(t)$, $y = g(t)$, and $z = h(t)$.

11.5.3 For every real number t that is put into the function, the output is a vector $\mathbf{r}(t)$.

11.5.4 Subtract componentwise to obtain the vector $\mathbf{d} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$.

11.5.5 Let $\mathbf{d}$ be the direction vector as in the previous problem. Then $\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + t\mathbf{d}$.

11.5.6 It lies in the xz-plane.

11.5.7 Compute $\lim_{t\to a} f(t) = L_1$, $\lim_{t\to a} g(t) = L_2$, and $\lim_{t\to a} h(t) = L_3$. Then $\lim_{t\to a} \mathbf{r}(t) = \langle L_1, L_2, L_3\rangle$.

11.5.8 It is continuous at a exactly when the three component functions $x = f(t)$, $y = g(t)$ and $z = h(t)$ are continuous at a.

11.5.9 The line is $\mathbf{r}(t) = \langle 0, 0, 1\rangle + t\langle 4, 7, 0\rangle$.

11.5.10 The line is $\mathbf{r}(t) = \langle -3, 2, -1\rangle + t\langle 1, -2, 0\rangle$.

11.5.11 The direction is $\langle 0, 1, 0\rangle$, so the line l_1 is $\mathbf{r}(t) = \langle 0, 0, 1\rangle + t\langle 0, 1, 0\rangle$.

11.5.12 The direction is $\langle 1, 0, 0\rangle$ so the line l_2 is $\langle 0, 0, 1\rangle + t\langle 1, 0, 0\rangle$.

11.5.13 The direction is $\langle 1, 2, 3\rangle$, so the line is $\mathbf{r}(t) = t\langle 1, 2, 3\rangle$.

11.5.14 The direction is $\langle 2, -3, 2\rangle$, so the line is $\mathbf{r}(t) = \langle 1, 0, 1\rangle + t\langle 2, -3, 2\rangle$.

11.5.15 The direction is $\langle 8, -5, -6\rangle$, so the line is $\mathbf{r}(t) = \langle -3, 4, 6\rangle + t\langle 8, -5, -6\rangle$.

11.5.16 The direction is $\langle 10, -9, -12\rangle$, so the line is $\mathbf{r}(t) = \langle 0, 4, 8\rangle + t\langle 10, -9, -12\rangle$.

11.5.17 The direction is $\langle -2, 8, -4\rangle$, so the line is $\mathbf{r}(t) = t\langle -2, 8, -4\rangle$.

11.5.18 The direction is $\langle 4, -1, 0\rangle$, so the line is $\mathbf{r}(t) = \langle 1, -3, 4\rangle + t\langle 4, -1, 0\rangle$.

11.5.19 The direction is $\langle 1, 0, 2\rangle \times \langle 0, 1, 1\rangle = \langle -2, -1, 1\rangle$, so the line is $\mathbf{r}(t) = t\langle -2, -1, 1\rangle$.

11.5.20 The direction is $\langle 1, 1, -5\rangle \times \langle 0, 4, 0\rangle = \langle 20, 0, 4\rangle$, so the line is $\mathbf{r}(t) = \langle -3, 4, 2\rangle + t\langle 20, 0, 4\rangle$.

11.5.21 The direction is $\langle 1, 1, 2\rangle \times \langle 1, 0, 0\rangle = \langle 0, 2, -1\rangle$, so the line is $\mathbf{r}(t) = \langle -2, 5, 3\rangle + t\langle 0, 2, -1\rangle$.

11.5.22 The direction is $\langle 4, 3, -5\rangle \times \langle 0, 0, 1\rangle = \langle 3, -4, 0\rangle$, so the line is $\mathbf{r}(t) = \langle 0, 2, 1\rangle + t\langle 3, -4, 0\rangle$.

11.5.23 The direction is $\langle -2, 8, -4\rangle \times \langle -2, 1, -1\rangle = \langle -4, 6, 14\rangle$, so the line is $\mathbf{r}(t) = \langle 1, 2, 3\rangle + t\langle -4, 6, 14\rangle$.

11.5.24 The direction is $\langle 2, 3, -4\rangle \times \langle 1, 1, -1\rangle = \langle 1, -2, -1\rangle$, so the line is $\langle 1, 0, -1\rangle + t\langle 1, -2, -1\rangle$.

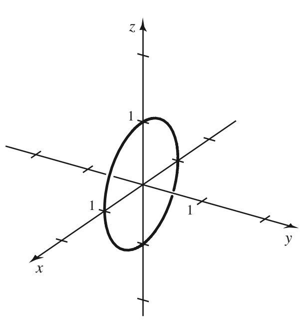
11.5.25 The line segment is $\mathbf{r}(t) = t\langle 1, 2, 3\rangle$, where $0 \le t \le 1$.

11.5.26 The line segment is $\mathbf{r}(t) = \langle 1, 0, 1\rangle + t\langle -1, -2, 0\rangle$, where $0 \le t \le 1$.

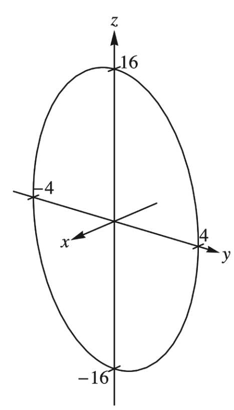
11.5.27 The line segment is $\mathbf{r}(t) = \langle 2, 4, 8\rangle + t\langle 5, 1, -5\rangle$, where $0 \le t \le 1$.

11.5.28 The line segment is $\mathbf{r}(t) = \langle -1, -8, 4\rangle + t\langle -8, 13, -7\rangle$, where $0 \le t \le 1$.

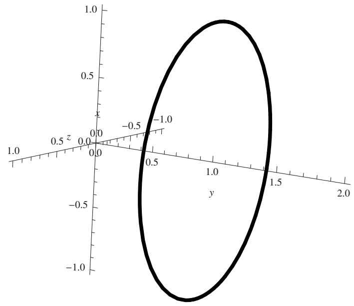
11.5.29



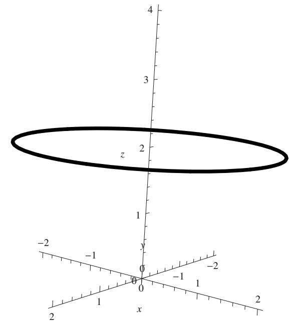
11.5.30



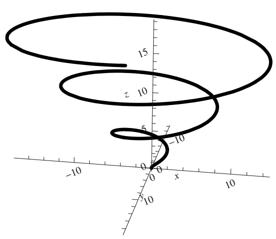
11.5.31



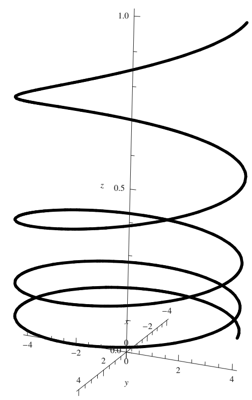
11.5.32



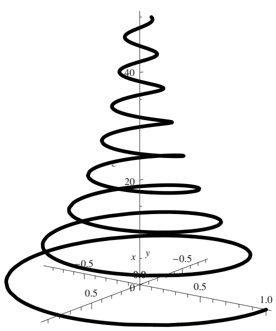
11.5.33



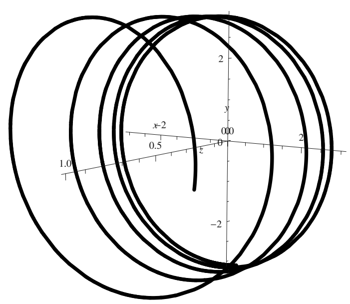
11.5.34



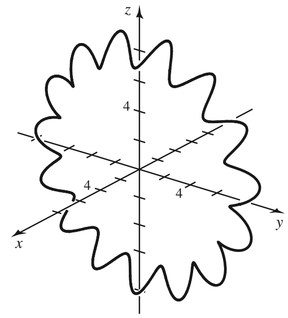
11.5.35



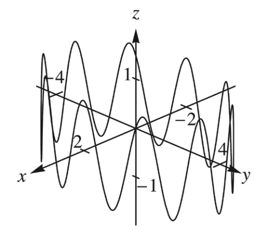
11.5.36



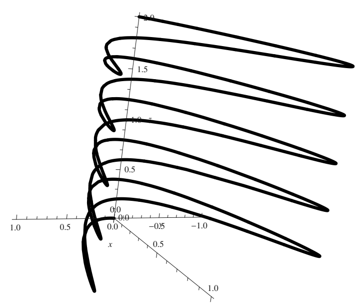
11.5.37 Note that the curve is closed (the initial point and the terminal point coincide), and is very "wavy."



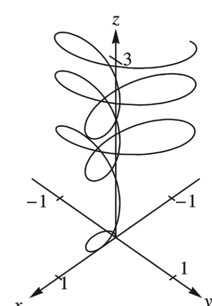
11.5.38 The projection onto the xy-plane is an ellipse elongate in the y direction. The curve oscillates in a sinusoidal wave in the z direction.



11.5.39 When viewed from the top, the curve looks parabolic.



11.5.40 When viewed from the top, the curve appears as a 3-petaled rose.



11.5.41 $\lim_{t\to\pi/2}\langle \cos 2t, -4\sin t, \frac{2t}{\pi}\rangle = \langle \cos\pi, -4\sin\pi/2, \frac{2\cdot\pi/2}{\pi}\rangle = \langle -1, -4, 1\rangle$.

11.5.42 $\lim_{t\to\ln 2}\langle 2e^t, 6e^{-t}, -4e^{-2t}\rangle = \langle 2e^{\ln 2}, 6e^{-\ln 2}, -4e^{-2\ln 2}\rangle = \langle 4, 3, -1\rangle$.

11.5.43 $\lim_{t\to\infty}\langle e^{-t}, -\frac{2t}{t+1}, \tan^{-1} t\rangle = \langle 0, -2, \pi/2\rangle$.

11.5.44 $\lim_{t\to 2}\langle \frac{t}{t^2+1}, -4e^{-t}\sin \pi t, \frac{1}{\sqrt{4t+1}}\rangle = \langle \frac{2}{5}, 0, \frac{1}{3}\rangle$.

11.5.45 Using l'Hôpital's rule (once in the first two components, twice in the third component):
$\lim_{t\to 0}\langle \frac{\sin t}{t}, -\frac{e^t - t - 1}{t}, \frac{\cos t + t^2/2 - 1}{t^2}\rangle = \lim_{t\to 0}\langle \cos t, 1 - e^t, \frac{-\cos t + 1}{2}\rangle = \langle 1, 0, 0\rangle = \mathbf{i}$.

11.5.46 Using l'Hôpital's rule (once in the first two components, but not in the 3rd component):
$\lim_{t\to 0}\langle \frac{\tan t}{t}, \frac{-3t}{\sin t}, \sqrt{t+1}\rangle = \lim_{t\to 0}\langle \sec^2 t, \frac{-3}{\cos t}, \sqrt{t+1}\rangle = \langle 1, -3, 1\rangle = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

11.5.47

a. True. This curve passes through the origin at $t = -1/2$.

b. False. For example, the x axis is not parallel to the line $\langle 0,0,1\rangle + t\langle 0,1,0\rangle$, but neither do they intersect.

c. True. The first component function approaches 0 as $t \to \infty$, while the others are periodic. The parametric equations $y = \sin t$ and $z = -\cos t$ form a circle in the yz-plane.

d. True. Both have limit $\langle 0,0,0\rangle$.

11.5.48 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and solving the resulting system of linear equations gives $t = 0$ and $s = 4$, so the point of intersection of the lines occurs when for $\mathbf{r}(0) = \mathbf{R}(4) = (-2,0,0)$. The direction of the line perpendicular to both of these is $\langle 3,2,3\rangle \times \langle 1,2,3\rangle = \langle 0,-6,4\rangle$. The line we are seeking is therefore $\langle -2,0,0\rangle + t\langle 0,-6,4\rangle$.

11.5.49 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and solving the resulting system of linear equations gives $t = 1$ and $s = 5$, so the point of intersection of the lines occurs when for $\mathbf{r}(1) = \mathbf{R}(5) = (4,3,3)$. The direction of the line perpendicular to both of these is $\langle 4,2,3\rangle \times \langle 1,2,3\rangle = \langle 0,-9,6\rangle$. The line we are seeking is therefore $\langle 4,3,3\rangle + t\langle 0,-9,6\rangle$.

11.5.50 Because the direction vectors are multiples of each other ($\langle 4,-6,4\rangle = -2\langle -2, 3-2\rangle$), the lines are parallel. (Note also that the lines don't coincide.)

11.5.51 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and solving the resulting system of linear equations gives $t = 0$ and $s = -3$, so the point of intersection of the lines occurs when for $\mathbf{r}(0) = \mathbf{R}(-3) = (1,3,2)$

11.5.52 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and attempting to solve the corresponding system of linear equations yields no solution. The lines aren't parallel since $\langle 5,-2,3\rangle$ is not a multiple of $\langle 10,4,6\rangle$. Therefore the lines are skew.

11.5.53 Setting $\mathbf{r}(t) = \mathbf{R}(s)$ and attempting to solve the corresponding system of linear equations yields no solution. The lines aren't parallel since $\langle 0,-1,1\rangle$ is not a multiple of $\langle -7,4,-1\rangle$. Therefore the lines are skew.

11.5.54 Because the direction vectors are multiples of each other ($\langle 1,-2,3\rangle = \frac{-1}{7}\langle -7,14,-21\rangle$), the lines are parallel. (Note also that the lines don't coincide.)

11.5.55 These equations represent the same line. (So they are parallel and intersecting.) Note that $\mathbf{r}(3t-5) = \langle 1+2(3t-5), 7-3(3t-5), 6+(3t-5)\rangle = \langle -9+6t, 22-9t, 1+3t\rangle = \mathbf{R}(t)$.

11.5.56 The first component function has domain $(-\infty,1)\cup(1,\infty)$, and the second has domain $(-\infty,-2)\cup(-2,\infty)$, so the domain of $\mathbf{r}(t)$ is $(-\infty,-2)\cup(-2,1)\cup(1,\infty)$.

11.5.57 The first component function has domain $[-2,\infty)$ and the second has domain $(-\infty,2]$, so the domain of $\mathbf{r}(t)$ is $[-2,2]$.

11.5.58 The first component function is defined everywhere, the second has domain $[0,\infty)$, and the third has domain $(-\infty,0)\cup(0,\infty)$, so the domain of $\mathbf{r}(t)$ is $(0,\infty)$.

11.5.59 The first component function has domain $[-2, 2]$, the second has domain $[0, \infty)$, and the third has domain $(-1, \infty)$, so the domain of $\mathbf{r}(t)$ is $[0, 2]$.

11.5.60 The line and plane intersect for $x = t = 3$, so the point of intersection is $(3, 3, 3)$.

11.5.61 The intersection occurs for $z = 4 = t - 6$, so for $t = 10$. The point of intesection is $(21, -6, 4)$.

11.5.62 The intersection occurs for $y = -2 = -t + 4$, so for $t = 6$. The point of intersection is $(13, -2, 0)$.

11.5.63 The intersection occurs for $z = -8 = -2t + 4$, so for $t = 6$. The point of intersection is $(16, 0, -8)$.

11.5.64 The intersection occurs for $y = 1 = 2 \sin t$, so for $t = \pi/6$ and $t = 5\pi/6$. The points of intersection are $(5\sqrt{3}, 1, 1)$ and $(-5\sqrt{3}, 1, 1)$.

11.5.65 The intersection occurs for $z = 16 = 4 + 3t$, so for $t = 4$. The point of intersection is $(4, 8, 16)$.

11.5.66 The intersection occurs for $x + y = \cos t + \sin t = 0$, so for $t = 3\pi/4$, $t = 7\pi/4$, $t = 11\pi/4$, and $t = 15\pi/4$. The points of intersection are $(-\sqrt{2}/2, \sqrt{2}/2, 3\pi/4)$, $(\sqrt{2}/2, -\sqrt{2}/2, 7\pi/4)$, $(-\sqrt{2}/2, \sqrt{2}/2, 11\pi/4)$, and $(\sqrt{2}/2, -\sqrt{2}/2, 15\pi/4)$.

11.5.67

a. This matches graph E. (It is a straight line.)

b. This matches graph D. (It is parabolic-like.)

c. This matches graph F. (It is a circle.)

d. This matches graph C. (It is a circular helix, elongated along the x-axis.)

e. This matches graph A. (It is a closed curve which isn't a circle.)

f. This matches graph B. (It is a circular helix, elongated along the y-axis.)

11.5.68

a. If they are to intersect, then we must have $2 + 2t = 6 + s$, $8 + t = 10 - 2s$, and $10 + 3t = 16 - s$. This system of linear equations has the solution $t = 2$ and $s = 0$. So the lines intersect at $(6, 10, 16)$.

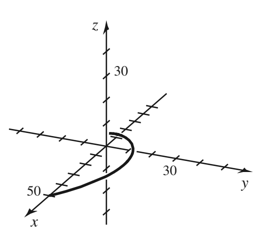
b. They do not collide, because they arrive at the intersection point at different times.

11.5.69

a. $\mathbf{r}(0) = \langle 50, 0, 0 \rangle$.

b. $\displaystyle\lim_{t\to\infty} \frac{50 \cos t}{e^t} = 0$ by the squeeze theorem, and likewise $\displaystyle\lim_{t\to\infty} \frac{50 \sin t}{e^t} = 0$. Also, $\displaystyle\lim_{t\to\infty} \left(5 - \frac{5}{e^t}\right) = 5$. Thus we have $\displaystyle\lim_{t\to\infty} \mathbf{r}(t) = \langle 0, 0, 5 \rangle$.

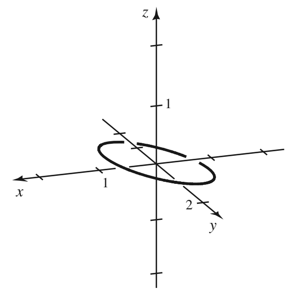
c.



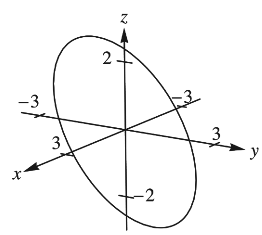
d. Let $x = 50e^{-t} \cos t$ and $y = 50e^{-t} \sin t$ and $z = 5 - 5e^{-t}$. Note that $x^2 + y^2 = 2500e^{-2t}$, so $r = 50e^{-t}$. We have $z = 5 - 5e^{-t} = 5 - \frac{r}{10}$.

11.5.70 If $x = a\cos t + b\sin t$ and $y = c\cos t + d\sin t$ and $z = e\cos t + f\sin t$ then $x^2 + y^2 + z^2 = (a\cos t + b\sin t)^2 + (c\cos t + d\sin t)^2 + (e\cos t + f\sin t)^2 = (a^2+c^2+e^2)\cos^2 t + (b^2+d^2+f^2)\sin^2 t + 2(ab+cd+ef)\sin t\cos t$. If $ab + cd + ef = 0$ and if $a^2 + c^2 + e^2 = R^2 = b^2 + d^2 + f^2$, then we have $x^2 + y^2 + z^2 = R^2$, so all the points on the curve lie at a distance of R from the origin, so (because the curve lies in a plane) it is a circle of radius R centered at the origin.

11.5.71 This has the form mentioned in exercise 52, with $a = 1/\sqrt{2}$, $b = 1/\sqrt{3}$, $c = -1/\sqrt{2}$, $d = 1/\sqrt{3}$, $e = 0$, and $f = 1/\sqrt{3}$. Note that $a^2 + c^2 + e^2 = \frac{1}{2} + \frac{1}{2} + 0 = 1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = b^2 + d^2 + f^2$, and $ab + cd + ef = 0$. So this is a circle of radius 1 centered at the origin.



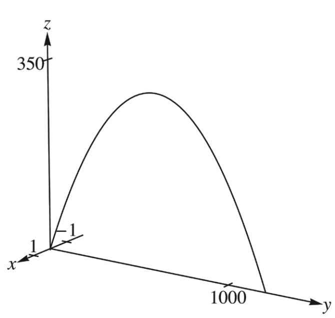
11.5.72 Using the notation of exercise 52, we have $ab + cd + ef = 0$. However, we have $a^2 + c^2 + e^2 = 6$ and $b^2 + d^2 + e^2 = 12$. Thus $x^2 + y^2 + z^2 = 6\cos^2 t + 12\sin^2 t = 6 + 6\sin^2 t$, The curve is an oval-shaped closed curve.



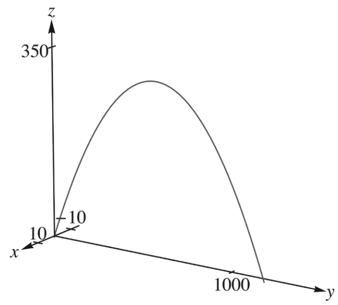
11.5.73 Note that $\mathbf{r}(0) = \langle a, c, e\rangle$ and $\mathbf{r}(\pi/2) = \langle b, d, f\rangle$, and $\mathbf{r}(\pi) = \langle -a, -c, -e\rangle$ have their terminal points on the curve. So $\langle a, c, e\rangle - \langle -a, -c, -e\rangle = \langle 2a, 2c, 2e\rangle = 2\mathbf{r}(0)$ lies in the plane containing the curve, which implies that $\mathbf{r}(0)$ lies in the plane containing the curve, and that implies that the point $(0, 0, 0)$ is in the plane containing the curve. So a normal to the curve is $\mathbf{r}(0)\times\mathbf{r}(\pi/2) = \langle a, c, e\rangle \times \langle b, d, f\rangle = \langle cf - de, be - af, ad - bc\rangle$.

11.5.74

a.



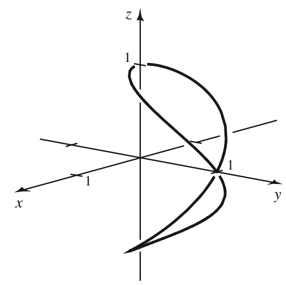
With $a = 0$, we have $\mathbf{r}(t) = \langle 0, 75t, -5t^2 + 80t\rangle$. Note that $z = 0$ when $t = 0$ and when $-5t + 80 = 0$, or $t = 16$. At this time, $y = 75 \cdot 16 = 1200$ feet.

b. 

With $a = .2$, we have $\mathbf{r}(t) = \langle .2t, (75-.02)t, -5t^2+80t\rangle$. Note that $z = 0$ when $t = 0$ and when $-5t + 80 = 0$, or $t = 16$. At this time, $y = (75 - .02) \cdot 16 = 1199.68$ feet.

c. with $a = 2.5$, the ball travels $(75 - .25) * 16 = 1196$ feet.

11.5.75 First note that $x = \frac{1}{2}\sin 2t = \sin t \cos t$ and $y = \frac{1}{2}(1 - \cos 2t) = \sin^2 t$. Then $x^2 + y^2 + z^2 = \sin^2 t \cos^2 t + \sin^4 t + \cos^2 t = \sin^2 t(\cos^2 t + \sin^2 t) + \cos^2 t = \sin^2 t \cdot 1 + \cos^2 t = 1$. So all points on the curve are equidistant from the origin, so they lie on the sphere of radius one centered at the origin.



11.5.76 Suppose $a^2 = b^2$ and $c^2 = a^2 + b^2$.

We have $x^2+y^2+z^2 = a^2\sin^2 mt \cdot \cos^2 nt + b^2 \sin^2 mt \cdot \sin^2 nt + c^2\cos^2 mt = a^2(\cos^2 nt + \sin^2 nt)\sin^2 mt + c^2\cos^2 mt = a^2\sin^2 mt + c^2 \cos^2 mt = a^2 \sin^2 mt + a^2\cos^2 mt + a^2 \cos^2 mt = a^2 + \frac{z^2}{2}$. Thus we have $x^2+y^2+z^2 = a^2 + \frac{z^2}{2}$, or $x^2 + y^2 + \frac{1}{2}z^2 = a^2$. So the curve lies on an ellipsoid.

11.5.77 In order for $\sin(mt+mT)\cos(nt+nT) = \sin mt \cos nt$ and $\sin(mt+mT)\sin(nt+nT) = \sin mt \sin nt$ and $\cos(mt + mT) = \cos mt$ we would need $T = \frac{2\pi}{m}$ or a multiple of it, and then it would be necessary for $\sin(nt + nT) = \sin nt$, which would require $T = \frac{2\pi}{n}$, or a multiple of it. Thus, the smallest such T would be $\frac{2\pi}{(m,n)}$, where (m, n) represents the greatest common factor of m and n.

11.5.78

a. Assume that $\lim_{t\to a} \mathbf{r}(t) = \mathbf{L} = \langle L_1, L_2, L_3\rangle$. Then for any $\epsilon > 0$, there exists $\delta > 0$ so that $|\mathbf{r}(t) - \mathbf{L}| = \sqrt{(f(t) - L_1)^2 + (g(t) - L_2)^2 + (h(t) - L_3)^2} < \epsilon$ for $|t - a| < \delta$.

   Note that $|f(t) - L_1| \le \sqrt{(f(t) - L_1)^2 + (g(t) - L_2)^2 + (h(t) - L_3)^2}$, and likewise for $|g(t) - L_2|$ and $|h(t) - L_3|$. So given $\epsilon > 0$, choosing the δ mentioned in the paragraph above guarantees that the absolute values of the differences of the corresponding coordinate functions and the L_i are less than epsilon whenever $|t - a| < \delta$, as desired.

b. Assume $\lim_{t\to a} f(t) = L_1$, $\lim_{t\to a} g(t) = L_2$, and $\lim_{t\to a} h(t) = L_3$. Then for any $\epsilon > 0$, there exists $\delta_1 > 0$ so that $|f(t) - L_1| < \epsilon/\sqrt{3}$ whenever $|t - a| < \delta_1$, and there exists δ_2 so that $|g(t) - L_2| < \epsilon/\sqrt{3}$ whenever $|t - a| < \delta_2$, and there exists δ_3 so that $|h(t) - L_3| < \epsilon/\sqrt{3}$ whenever $|t - a| < \delta_3$.

   Now let $\epsilon > 0$ be given, and let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then

$$|\mathbf{r}(t) - \langle L_1, L_2, L_3\rangle| = \sqrt{(f(t) - L_1)^2 + (g(t) - L_2)^2 + (h(t) - L_3)^2} \le \sqrt{3 \cdot \epsilon^2/3} = \sqrt{\epsilon^2} = \epsilon,$$

   as desired.

11.5.79 Consider the vector $\mathbf{v}$ placed geometrically so that its tail is at point P, and let the head of $\mathbf{v}$ be R so that the triangle PRQ has one side as $\mathbf{u} = \overrightarrow{PQ}$ and one side as $\mathbf{v}$. By Theorem 11.3, $\dfrac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{v}|} = |\mathbf{u}| \sin\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Now Let R' on $\mathbf{r}$ be the foot of the perpendicular dropped from Q as in ordinary geometry, so that the length of $\overline{R'Q}$ is the distance from Q to the line. By the trigonometry of the right triangle $PR'Q$, we have that the length of $\overline{R'Q}$ is $|\mathbf{u}| \sin\theta$ where θ is as before. Thus $R = R'$ and the distance between Q and $\mathbf{r}$ is $|\mathbf{u}| \sin\theta = \dfrac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{v}|}$.

11.5.80 Let the point on the line be $P(1, 3, 1)$, so that $\mathbf{u} = \overrightarrow{PQ} = \langle 4, 3, 0\rangle$. Then

$$d = \frac{|\langle 4, 3, 0\rangle \times \langle 3, -4, 1\rangle|}{|\langle 3, -4, 1\rangle|} = \frac{|\langle 3, -4, 25\rangle|}{|\langle 3, -4, 1\rangle|} = \frac{5\sqrt{26}}{\sqrt{26}} = 5.$$

11.5.81 Let the point on the line be $P(7, 2, 4)$, so that $\mathbf{u} = \overrightarrow{PQ} = \langle -12, 0, 5\rangle$. Then

$$d = \frac{|\langle -12, 0, 5\rangle \times \langle 5, -1, 12\rangle|}{|\langle 5, -1, 12\rangle|} = \frac{|\langle 5, 169, 12\rangle|}{|\langle 5, -1, 12\rangle|} = \frac{13\sqrt{170}}{\sqrt{170}} = 13.$$

11.5.82 Let the point on the line be $P(0, 0, 4)$, so that $\mathbf{u} = \overrightarrow{PQ} = \langle 6, 6, 3\rangle$. Then

$$d = \frac{|\langle 6, 6, 3\rangle \times \langle 3, -3, 0\rangle|}{|\langle 3, -3, 0\rangle|} = \frac{|\langle 9, 9, -36\rangle|}{|\langle 3, -3, 0\rangle|} = \frac{27\sqrt{2}}{3\sqrt{2}} = 9.$$

11.6 Calculus of Vector-Valued Functions

11.6.1 It is $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$.

11.6.2 $\mathbf{r}'(t)$ is a vector tangent to the curve $\mathbf{r}(t)$.

11.6.3 Divide the vector by its length, so if the vector is $\mathbf{r}'(t)$, form $\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

11.6.4 $\mathbf{r}'(t) = \langle 10t^9, 8, -\sin t\rangle$, so $\mathbf{r}''(t) = \langle 90t^8, 0, -\cos t\rangle$.

11.6.5 Compute the indefinite integral of each of the component functions, and then

$$\int \mathbf{r}(t)\,dt = \langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt\rangle.$$

11.6.6 $\int_a^b \mathbf{r}(t)\,dt = \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt\rangle$.

11.6.7 $\mathbf{r}'(t) = \langle -\sin t, 2t, \cos t\rangle$.

11.6.8 $\mathbf{r}'(t) = \langle 4e^t, 0, 1/t\rangle$.

11.6.9 $\mathbf{r}'(t) = \langle 6t, 3/\sqrt{t}, -3/t^2\rangle$.

11.6.10 $\mathbf{r}'(t) = \langle 0, -6\sin 2t, 6\cos 3t\rangle$.

11.6.11 $\mathbf{r}'(t) = \langle e^t, -2e^{-t}, -8e^{2t}\rangle$.

11.6.12 $\mathbf{r}'(t) = \langle \sec^2 t, \sec t \tan t, -\sin 2t\rangle$.

11.6.13 $\mathbf{r}'(t) = \langle e^{-t}(1-t), 1 + \ln t, \cos t - t\sin t\rangle$.

11.6.14 $\mathbf{r}'(t) = \langle -(t+1)^{-2}, (t^2+1)^{-1}, (t+1)^{-1}\rangle$.

11.6.15 $\mathbf{r}'(t) = \langle 1, 6t, 3t^2\rangle$, so $\mathbf{r}'(1) = \langle 1, 6, 3\rangle$.

11.6.16 $\mathbf{r}'(t) = \langle e^t, 3e^{3t}, 5e^{5t}\rangle$, so $\mathbf{r}'(0) = \langle 1, 3, 5\rangle$.

11.6.17 $\mathbf{r}'(t) = \langle 1, -2\sin 2t, 2\cos t\rangle$, so $\mathbf{r}'(\pi/2) = \langle 1, 0, 0\rangle$.

11.6.18 $\mathbf{r}'(t) = \langle 2\cos t, -3\sin t, \frac{1}{2}\cos(t/2)\rangle$, so $\mathbf{r}'(\pi) = \langle -2, 0, 0\rangle$.

11.6.19 $\mathbf{r}'(t) = \langle 8t^3, 9\sqrt{t}, -10/t^2\rangle$, so $\mathbf{r}'(1) = \langle 8, 9, -10\rangle$.

11.6.20 $\mathbf{r}'(t) = \langle 2e^t, -2e^{-2t}, 8e^{2t}\rangle$, so $\mathbf{r}'(\ln 3) = \langle 6, -2/9, 72\rangle$.

11.6.21 $\mathbf{r}'(t) = \langle 2, 2, 1\rangle$, so
$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{3}\langle 2, 2, 1\rangle = \langle 2/3, 2/3, 1/3\rangle.$$

11.6.22 $\mathbf{r}'(t) = \langle -\sin t, \cos t, 0\rangle$, so
$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{\sin^2 t + \cos^2 t + 0}}\langle -\sin t, \cos t, 0\rangle = \langle -\sin t, \cos t, 0\rangle.$$

11.6.23 $\mathbf{r}'(t) = \langle 0, -2\sin 2t, 4\cos 2t\rangle$, so
$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{4\sin^2 2t + 16\cos^2 2t}}\langle 0, -2\sin 2t, 4\cos 2t\rangle = \frac{1}{\sqrt{1 + 3\cos^2 2t}}\langle 0, -\sin 2t, 2\cos 2t\rangle.$$

11.6.24 $\mathbf{r}'(t) = \langle \cos t, -\sin t, -\sin t\rangle$, so
$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{\cos^2 t + \sin^2 t + \sin^2 t}}\langle \cos t, -\sin t, -\sin t\rangle = \frac{1}{\sqrt{1 + \sin^2 t}}\langle \cos t, -\sin t, -\sin t\rangle.$$

11.6.25 $\mathbf{r}'(t) = \langle 1, 0, -2/t^2\rangle$, so
$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1 + (4/t^4)}}\langle 1, 0, -2/t^2\rangle = \frac{1}{\sqrt{t^4 + 4}}\langle t^2, 0, -2\rangle.$$

11.6.26 $\mathbf{r}'(t) = \langle 2e^{2t}, 4e^{2t}, -6e^{-3t}\rangle$, so
$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{20e^{4t} + 36e^{-6t}}}\langle 2e^{2t}, 4e^{2t}, -6e^{-3t}\rangle = \frac{1}{\sqrt{5e^{4t} + 9e^{-6t}}}\langle e^{2t}, 2e^{2t}, -3e^{-3t}\rangle.$$

11.6.27 $\mathbf{r}'(t) = \langle -2\sin 2t, 0, 6\cos 2t\rangle$, so at $t = \pi/2$, we have $\mathbf{r}'(\pi/2) = \langle 0, 0, -6\rangle$. Thus, the unit tangent at $\pi/2$ is $\langle 0, 0, -1\rangle$.

11.6.28 $\mathbf{r}'(t) = \langle \cos t, -\sin t, -e^{-t}\rangle$, so at $t = 0$, we have $\mathbf{r}'(0) = \langle 1, 0, -1\rangle$. Thus, the unit tangent at 0 is $\langle 1/\sqrt{2}, 0, -1/\sqrt{2}\rangle$.

11.6.29 $\mathbf{r}'(t) = \langle 6, 0, -3/t^2\rangle$, so at $t = 1$, we have $\mathbf{r}'(1) = \langle 6, 0, -3\rangle$. Thus, the unit normal at 1 is $\langle 2/\sqrt{5}, 0, -1/\sqrt{5}\rangle$.

11.6.30 $\mathbf{r}'(t) = \langle \sqrt{7}e^t, 3e^t, 3e^t\rangle$, so at $t = \ln 2$, we have $\mathbf{r}'(\ln 2) = \langle 2\sqrt{7}, 6, 6\rangle$. Thus, the unit normal at $\ln 2$ is $\langle \sqrt{7}/5, 3/5, 3/5\rangle$.

11.6.31 $(t^{12} + 3t)\mathbf{u}'(t) + \mathbf{u}(t)(12t^{11} + 3) = (t^{12} + 3t)\langle 6t^2, 2t, 0\rangle + \langle 2t^3, (t^2 - 1), -8\rangle(12t^{11} + 3) = \langle 30t^{14} + 24t^3, 14t^{13} - 12t^{11} + 9t^2 - 3, -96t^{11} - 24\rangle$.

11.6.32 $(4t^8 - 6t^3)\mathbf{v}'(t) + \mathbf{v}(t)(32t^7 - 18t^2) = (4t^8 - 6t^3)\langle e^t, -2e^{-t}, -2e^{2t}\rangle + \langle e^t, 2e^{-t}, -e^{2t}\rangle(32t^7 - 18t^2) = \langle (4t^8 + 32t^7 - 6t^3 - 18t^2)e^t, (64t^7 - 36t^2 - 8t^8 + 12t^3)e^{-t}, (-8t^8 - 32t^7 + 12t^3 + 18t^2)e^{2t}\rangle$.

11.6.33 $\mathbf{u}'(t^4-2t)\cdot(4t^3-2) = \langle 6(t^4-2t)^2, 2(t^4-2t), 0\rangle(4t^3-2) = \langle 6(t^4-2t)^2(4t^3-2), 2(t^4-2t)(4t^3-2), 0\rangle = 4t(2t^3-1)(t^3-2)\langle 3t(t^3-2), 1, 0\rangle$.

11.6.34 $\mathbf{v}'(\sqrt{t})\cdot\frac{1}{2\sqrt{t}} = \langle e^{\sqrt{t}}, -2e^{-\sqrt{t}}, -2e^{2\sqrt{t}}\rangle\cdot\frac{1}{2\sqrt{t}} = \langle \frac{e^{\sqrt{t}}}{2\sqrt{t}}, -\frac{1}{\sqrt{t}e^{\sqrt{t}}}, -\frac{e^{2\sqrt{t}}}{\sqrt{t}}\rangle$.

11.6.35 $\mathbf{u}(t)\cdot\mathbf{v}'(t)+\mathbf{v}(t)\cdot\mathbf{u}'(t) = \langle 2t^3, (t^2-1), -8\rangle\cdot\langle e^t, -2e^{-t}, -2e^{2t}\rangle + \langle e^t, 2e^{-t}, -e^{2t}\rangle\cdot\langle 6t^2, 2t, 0\rangle = 2t^3e^t - 2(t^2-1)e^{-t} + 16e^{2t} + 6t^2e^t + 4te^{-t} + 0 = e^t(2t^3+6t^2) - 2e^{-t}(t^2-2t-1) + 16e^{2t}$.

11.6.36 $\mathbf{u}(t)\times\mathbf{v}'(t)+\mathbf{u}'(t)\times\mathbf{v}(t) = \langle 2t^3, (t^2-1), -8\rangle\times\langle e^t, -2e^{-t}, -2e^{2t}\rangle + \langle 6t^2, 2t, 0\rangle\times\langle e^t, 2e^{-t}, -e^{2t}\rangle =$
$\langle -2e^{2t}t^2 - 16e^{-t} + 2e^{2t}, 4e^{2t}t^3 - 8e^t, -4e^{-t}t^3 - e^tt^2 + e^t\rangle + \langle -2e^{2t}t, 6e^{2t}t^2, 12e^{-t}t^2 - 2e^tt\rangle =$
$\langle -2e^{2t}t^2 - 2e^{2t}t - 16e^{-t} + 2e^{2t}, 4e^{2t}t^3 + 6e^{2t}t^2 - 8e^t, -4e^{-t}t^3 + 12e^{-t}t^2 - e^tt^2 - 2e^tt + e^t\rangle$.

11.6.37 $\langle t^2, 2t^2, -2t^3\rangle\cdot\langle e^t, 2e^t, 3e^{-t}\rangle + \langle 2t, 4t, -6t^2\rangle\cdot\langle e^t, 2e^t, -3e^{-t}\rangle = t^2e^t + 4t^2e^t - 6t^3e^{-t} + 2te^t + 8te^t + 18t^2e^{-t} = 5t^2e^t + 10te^t - 6t^3e^{-t} + 18t^2e^{-t}$.

11.6.38 $\langle t^3, -2t, -2\rangle\times\langle 1, -2t, -3t^2\rangle + \langle 3t^2, -2, 0\rangle\times\langle t, -t^2, -t^3\rangle = \langle 6t^3-4t, 3t^5-2, 2t-2t^4\rangle + \langle 2t^3, 3t^5, 2t-3t^4\rangle = \langle 8t^3-4t, 6t^5-2, 4t-5t^4\rangle$.

11.6.39 $\langle 3t^2, \sqrt{t}, -2/t\rangle\cdot\langle -\sin t, 2\cos 2t, -3\rangle + \langle 6t, 1/(2\sqrt{t}), 2/t^2\rangle\cdot\langle \cos t, \sin 2t, -3t\rangle = -3t^2\sin t + 2\sqrt{t}\cos 2t + 6t\cos t + \sin(2t)/(2\sqrt{t})$.

11.6.40 $\langle t^3, 6, -2\sqrt{t}\rangle\times\langle 3, -24t, 12t^{-3}\rangle + \langle 3t^2, 0, -1/\sqrt{t}\rangle\times\langle 3t, -12t^2, -6t^{-2}\rangle =$
$\langle \frac{72}{t^3} - 48t^{3/2}, -6\sqrt{t} - 12, -24t^4 - 18\rangle + \langle -12t^{3/2}, 18 - 3\sqrt{t}, -36t^4\rangle = \langle \frac{72}{t^3} - 60t^{3/2}, 6 - 9\sqrt{t}, -60t^4 - 18\rangle$.

11.6.41 $\mathbf{r}'(t) = \langle 2t, 1, 0\rangle$. $\mathbf{r}''(t) = \langle 2, 0, 0\rangle$. $\mathbf{r}'''(t) = \langle 0, 0, 0\rangle$.

11.6.42 $\mathbf{r}'(t) = \langle 36t^{11} - 2t, 8t^7 + 3t^2, -4t^{-5}\rangle$. $\mathbf{r}''(t) = \langle 396t^{10} - 2, 56t^6 + 6t, 20t^{-6}\rangle$. $\mathbf{r}'''(t) = \langle 3960t^9, 336t^5 + 6, -120t^{-7}\rangle$.

11.6.43 $\mathbf{r}'(t) = \langle -3\sin 3t, 4\cos 4t, -6\sin 6t\rangle$. $\mathbf{r}''(t) = \langle -9\cos 3t, -16\sin 4t, -36\cos 6t\rangle$.
$\mathbf{r}'''(t) = \langle 27\sin 3t, -64\cos 4t, 216\sin 6t\rangle$.

11.6.44 $\mathbf{r}'(t) = \langle 4e^{4t}, -8e^{-4t}, -2e^{-t}\rangle$. $\mathbf{r}''(t) = \langle 16e^{4t}, 32e^{-4t}, 2e^{-t}\rangle$. $\mathbf{r}'''(t) = \langle 64e^{4t}, -128e^{-4t}, -2e^{-t}\rangle$.

11.6.45 $\mathbf{r}'(t) = \langle \frac{1}{2\sqrt{t+4}}, \frac{1}{(t+1)^2}, 2e^{-t^2}t\rangle$. $\mathbf{r}''(t) = \langle -\frac{1}{4(t+4)^{3/2}}, -\frac{2}{(t+1)^3}, e^{-t^2}\left(2-4t^2\right)\rangle$.
$\mathbf{r}'''(t) = \langle \frac{3}{8(t+4)^{5/2}}, \frac{6}{(t+1)^4}, 4e^{-t^2}t\left(2t^2-3\right)\rangle$.

11.6.46 $\mathbf{r}'(t) = \langle \sec^2(t), 1 - \frac{1}{t^2}, -\frac{1}{t+1}\rangle$. $\mathbf{r}''(t) = \langle 2\tan(t)\sec^2(t), \frac{2}{t^3}, \frac{1}{(t+1)^2}\rangle$.
$\mathbf{r}'''(t) = \langle 2\sec^4(t) + 4\tan^2(t)\sec^2(t), -\frac{6}{t^4}, -\frac{2}{(t+1)^3}\rangle$.

11.6.47 $\displaystyle\int \langle t^4 - 3t, 2t - 1, 10\rangle\,dt = \langle t^5/5 - 3t^2/2, t^2 - t, 10t\rangle + \mathbf{C}$.

11.6.48 $\displaystyle\int \langle 5t^{-4} - t^2, t^6 - 4t^3, 2/t\rangle\,dt = \langle (-5/3)t^{-3} - t^3/3, t^7/7 - t^4, 2\ln|t|\rangle + \mathbf{C}$.

11.6.49 $\displaystyle\int \langle 2\cos t, 2\sin 3t, 4\cos 8t\rangle\,dt = \langle 2\sin t, (-2/3)\cos 3t, (1/2)\sin 8t\rangle + \mathbf{C}$.

11.6.50 $\displaystyle\int \langle te^t, t\sin t^2, -\frac{2t}{\sqrt{t^2+4}}\rangle\,dt = \langle (t-1)e^t, -\frac{1}{2}\cos t^2, -2\sqrt{t^2+4}\rangle + \mathbf{C}$.

11.6.51 $\displaystyle\int \langle e^{3t}, \frac{1}{1+t^2}, \frac{-1}{\sqrt{2t}}\rangle\,dt = \langle e^{3t}/3, \tan^{-1} t, -\sqrt{2t}\rangle + \mathbf{C}$.

11.6.52 $\displaystyle\int \langle 2^t, \frac{1}{1+2t}, \ln t\rangle\,dt = \langle 2^t/(\ln 2), \frac{1}{2}\ln|1+2t|, t\ln t - t\rangle + \mathbf{C}$.

11.6.53 $\int \langle e^t, \sin t, \sec^2 t\rangle\, dt = \langle e^t, -\cos t, \tan t\rangle + \mathbf{C}$. Because $\mathbf{r}(0) = \langle 2,2,2\rangle = \langle 1,-1,0\rangle + \mathbf{C}$, we have $\mathbf{C} = \langle 1,3,2\rangle$. Thus, $\mathbf{r}(t) = \langle e^t, -\cos t, \tan t\rangle + \langle 1,3,2\rangle = \langle 1+e^t, 3-\cos t, 2+\tan t\rangle$.

11.6.54 $\int \langle 0,2,2t\rangle\, dt = \langle 0,2t,t^2\rangle + \mathbf{C}$. Because $\mathbf{r}(1) = \langle 4,3,-5\rangle$, we have $\mathbf{r}(1) = \langle 0,2,1\rangle + C = \langle 4,3,-5\rangle$, so $\mathbf{C} = \langle 4,1,-6\rangle$. Thus, $\mathbf{r}(t) = \langle 4, 2t+1, t^2-6\rangle$.

11.6.55 $\int \langle 1,2t,3t^2\rangle\, dt = \langle t,t^2,t^3\rangle + \mathbf{C}$. Because $\mathbf{r}(1) = \langle 4,3,-5\rangle$, we have $\langle 1,1,1\rangle + \mathbf{C} = \langle 4,3,-5\rangle$, so $\mathbf{C} = \langle 3,2,-6\rangle$, and $\mathbf{r}(t) = \langle t+3, t^2+2, t^3-6\rangle$.

11.6.56 $\int \langle \sqrt{t}, \cos \pi t, 4/t\rangle\, dt = \langle (2/3)t^{3/2}, (\sin \pi t)/\pi, 4\ln|t|\rangle + \mathbf{C}$. Because $\mathbf{r}(1) = \langle 2,3,4\rangle$, we have $\langle 2/3,0,0\rangle + \mathbf{C} = \langle 2,3,4\rangle$, so $\mathbf{C} = \langle 4/3,3,4\rangle$, and $\mathbf{r}(t) = \langle (2/3)t^{4/2} + 1/3, (\sin \pi t)/\pi + 3, 4\ln|t| + 4\rangle$.

11.6.57 $\int \langle e^{2t}, 1-2e^{-t}, 1-2e^t\rangle\, dt = \langle e^{2t}/2, t+2e^{-t}, t-2e^t\rangle + \mathbf{C}$. Because $\mathbf{r}(0) = \langle 1,1,1\rangle$, we have $\langle 1/2,2,-2\rangle + \mathbf{C} = \langle 1,1,1\rangle$, so $\mathbf{C} = \langle 1/2,-1,3\rangle$, and $\mathbf{r}(t) = \langle e^{2t}/2 + 1/2, t+2e^{-t}-1, t-2e^t+3\rangle$.

11.6.58 $\int \langle t/(t^2+1), te^{-t^2}, -\frac{2t}{\sqrt{t^2+4}}\rangle\, dt = \langle \frac{1}{2}\ln(t^2+1), -e^{-t^2}/2, -2\sqrt{t^2+4}\rangle + \mathbf{C}$. Because $\mathbf{r}(0) = \langle 1,3/2,-3\rangle$, we have $\langle 0,-1/2,-4\rangle + \mathbf{C} = \langle 1,3/2,-3\rangle$, so $\mathbf{C} = \langle 1,2,1\rangle$, and $\mathbf{r}(t) = \langle \frac{1}{2}\ln(t^2+1) + 1, -e^{-t^2}/2 + 2, -2\sqrt{t^2+4} + 1\rangle$.

11.6.59 $\int_{-1}^{1} \langle 1,t,3t^2\rangle\, dt = \langle t, t^2/2, t^3\rangle \Big|_{-1}^{1} = \langle 2,0,2\rangle$.

11.6.60 $\int_1^4 \langle 6t^2, 8t^3, 9t^2\rangle\, dt = \langle 2t^3, 2t^4, 3t^3\rangle\Big|_1^4 = \langle 128,512,192\rangle - \langle 2,2,3\rangle = \langle 126,510,189\rangle$.

11.6.61 $\int_0^{\ln 2} \langle e^t, e^t\cos \pi e^t\rangle\, dt = \langle e^t, \frac{\sin \pi e^t}{\pi}\rangle\Big|_0^{\ln 2} = \langle 2,0\rangle - \langle 1,0\rangle = \langle 1,0\rangle = \mathbf{i}$.

11.6.62

$$\int_{1/2}^{1} \langle \frac{3}{1+2t}, 0, -\pi\csc^2(\pi t/2)\rangle\, dt = \langle \frac{3}{2}\ln(1+2t), 0, 2\cot(\pi t/2)\rangle\Big|_{1/2}^{1}$$
$$= \langle (3/2)\ln 3, 0, 0\rangle - \langle (3/2)\ln 2, 0, 2\rangle = \langle (3/2)\ln(3/2), 0, -2\rangle.$$

11.6.63 $\int_{-\pi}^{\pi} \langle \sin t, \cos t, 2t\rangle\, dt = \langle -\cos t, \sin t, t^2\rangle \Big|_{-\pi}^{\pi} = \langle 0,0,0\rangle$.

11.6.64 $\int_0^{\ln 2} \langle e^{-t}, 2e^{2t}, -4e^t\rangle\, dt = \langle -e^{-t}, e^{2t}, -4e^t\rangle \Big|_0^{\ln 2} = \langle 1/2, 3, -4\rangle$.

11.6.65 $\int_0^2 \langle te^t, 2te^t, -te^t\rangle\, dt = \langle (t-1)e^t, 2(t-1)e^t, -(t-1)e^t\rangle \Big|_0^2 = \langle e^2+1, 2e^2+2, -e^2-1\rangle = (e^2+1)\langle 1,2,-1\rangle$.

11.6.66 $\int_0^{\pi/4} \langle \sec^2 t, -2\cos t, -1\rangle\, dt = \langle \tan t, -2\sin t, -t\rangle \Big|_0^{\pi/4} = \langle 1, -\sqrt{2}, -\pi/4\rangle$.

11.6.67

a. False. For example, if $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, then $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$ is not parallel to $\mathbf{r}(t)$, and is in fact perpendicular to it.

b. True. $\mathbf{r}'(t) = \langle 1, 2t-2, -\pi \sin \pi t \rangle \neq \langle 0,0,0 \rangle$. Each component function is differentiable, and the derivative is never $\langle 0,0,0 \rangle$, so the function is smooth by definition.

c. True. This follows because $\int_{-a}^{a} o(x)\,dx = 0$ for any odd function $o(x)$.

11.6.68 $\mathbf{r}'(t) = \langle e^t, 2e^{2t}, 3e^{3t} \rangle$, so $\mathbf{r}'(0) = \langle 1,2,3 \rangle$. We have $\mathbf{r}(0) = \langle 1,1,1 \rangle$, so the tangent line is given by $\langle 1+t, 1+2t, 1+3t \rangle$.

11.6.69 $\mathbf{r}'(t) = \langle -\sin t, 2\cos 2t, 1 \rangle$, so $\mathbf{r}'(\pi/2) = \langle -1, -2, 1 \rangle$. We have $\mathbf{r}(\pi/2) = \langle 2, 3, \pi/2 \rangle$, so the tangent line is given by $\langle 2-t, 3-2t, \pi/2+t \rangle$.

11.6.70 $\mathbf{r}'(t) = \langle \frac{1}{\sqrt{2t+1}}, \pi \cos \pi t, 0 \rangle$, so $\mathbf{r}'(4) = \langle 1/3, \pi, 0 \rangle$. We have $\mathbf{r}(4) = \langle 3,0,4 \rangle$, so the tangent line is given by $\langle 3+(1/3)t, \pi t, 4 \rangle$.

11.6.71 $\mathbf{r}'(t) = \langle 3, 7, 2t \rangle$, so $\mathbf{r}'(1) = \langle 3,7,2 \rangle$. We have $\mathbf{r}(1) = \langle 2,9,1 \rangle$, so the tangent line is given by $\langle 2+3t, 9+7t, 1+2t \rangle$.

11.6.72 $\mathbf{u}'(t^3) \cdot 3t^2 = 3t^2 \langle 0, 1, 2t^3 \rangle = \langle 0, 3t^2, 6t^5 \rangle$.

11.6.73 $\mathbf{v}'(e^t) \cdot e^t = e^t \langle 2e^t, -2, 0 \rangle = \langle 2e^{2t}, -2e^t, 0 \rangle$.

11.6.74 $g(t)\mathbf{v}'(t) + \mathbf{v}(t)g'(t) = (2\sqrt{t})\langle 2t, -2, 0 \rangle + \langle t^2, -2t, 1 \rangle \left(\frac{1}{\sqrt{t}} \right) = \langle 5t^{3/2}, -6\sqrt{t}, t^{-1/2} \rangle$.

11.6.75 $\mathbf{v}'(g(t))g'(t) = \langle 4\sqrt{t}, -2, 0 \rangle \left(\frac{1}{\sqrt{t}} \right) = \langle 4, -2/\sqrt{t}, 0 \rangle$.

11.6.76 $\mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t) = \langle 1, t, t^2 \rangle \cdot \langle 2t, -2, 0 \rangle + \langle 0, 1, 2t \rangle \cdot \langle t^2, -2t, 1 \rangle = (2t - 2t + 0) + 0 - 2t + 2t = 0$.

11.6.77 $\mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t) = \langle 1, t, t^2 \rangle \times \langle 2t, -2, 0 \rangle + \langle 0, 1, 2t \rangle \times \langle t^2, -2t, 1 \rangle = \langle 2t^2, 2t^3, -2t^2 - 2 \rangle + \langle 4t^2 + 1, 2t^3, -t^2 \rangle = \langle 6t^2 + 1, 4t^3, -3t^2 - 2 \rangle$

11.6.78 $\mathbf{r}'(t) = \langle -a \sin t, a \cos t \rangle$, and $\mathbf{r}(t) \cdot \mathbf{r}'(t) = \langle a \cos t, a \sin t \rangle \cdot \langle -a \sin t, a \cos t \rangle = -a^2 \cos t \sin t + a^2 \sin t \cos t = 0$. So $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal for all t.

11.6.79 $\mathbf{r}'(t) = \langle 2at, 1 \rangle$. We have $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ when $(2at)(at^2+1) + t(1) = 0$, which occurs only for $t = 0$ because $2a^2t^2 + 2a + 1 > 0$ for all t. The corresponding point on the parabola is $(1, 0)$.

11.6.80 $\mathbf{r}'(t) = \langle 1/(2\sqrt{t}), 0, 1 \rangle$. We have $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ when $1/2 + 0 + t = 0$, so for $t = -1/2$, which isn't in the domain. So the vectors $\mathbf{r}$ and $\mathbf{r}'$ are never orthogonal.

11.6.81 $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$. We have $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ when $-\sin t \cos t + \sin t \cos t + t = 0$, which occurs only for $t = 0$. So the only point on the helix where these vectors are orthogonal is at $t = 0$. This corresponds to the point $(1, 0, 0)$.

11.6.82 $\mathbf{r}'(t) = \langle -2 \sin t, 8 \cos t, 0 \rangle$. We have $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ when $-4 \sin t \cos t + 64 \sin t \cos t + 0 = 0$, which occurs when $60 \sin t \cos t = 0$, or $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$, and $t = 2\pi$.

The corresponding points are $(2, 0, 0)$, $(0, 8, 0)$, $(-2, 0, 0)$, and $(0, -8, 0)$.

11.6.83 Note that $\mathbf{r}(t) = \langle a_1 t, a_2 t, a_3 t \rangle = t \langle a_1, a_2, a_3 \rangle$ where the a_i's are real numbers has this property because $\mathbf{r}'(t) = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{r}(t)$ is a multiple of $\mathbf{r}'(t)$.

Also, $\mathbf{r}(t) = \langle a_1 e^{kt}, a_2 e^{kt}, a_3 e^{kt} \rangle$ where k is a real number has this property, as its derivative is k times itself.

11.6.84

a. This is equal to $\mathbf{u}(0)\cdot\mathbf{v}'(0)+\mathbf{u}'(0)\cdot\mathbf{v}(0)=\langle 0,1,1\rangle\cdot\langle 1,1,2\rangle+\langle 0,7,1\rangle\cdot\langle 0,1,1\rangle=1+2+7+1=11$.

b. This is equal to $\mathbf{u}(0)\times\mathbf{v}'(0)+\mathbf{u}'(0)\times\mathbf{v}(0)=\langle 0,1,1\rangle\times\langle 1,1,2\rangle+\langle 0,7,1\rangle\times\langle 0,1,1\rangle=\langle 1,1,-1\rangle+\langle 6,0,0\rangle=\langle 7,1,-1\rangle$.

c. This is equal to $-\sin(0)\cdot\mathbf{u}(0)+\cos(0)\cdot\mathbf{u}'(0)=\langle 0,7,1\rangle$.

11.6.85

a. $\mathbf{r}(t)\cdot\mathbf{r}'(t)=(a^2+b^2+c^2)t=|\mathbf{r}(t)|\,|\mathbf{r}'(t)|\cos\theta$, so $\cos\theta=\frac{(a^2+b^2+c^2)t}{\sqrt{a^2+b^2+c^2}t\cdot\sqrt{a^2+b^2+c^2}}=1$, so $\theta=0$.

b. $\mathbf{r}(t)\cdot\mathbf{r}'(t)=ax_0+by_0+cz_0+(a^2+b^2+c^2)t=|\mathbf{r}(t)|\,|\mathbf{r}'(t)|\cos\theta$, so

$$\cos\theta=\frac{ax_0+by_0+cz_0+(a^2+b^2+c^2)t}{\sqrt{(x_0+at)^2+(y_0+bt)^2+(z_0+ct)^2}\cdot\sqrt{a^2+b^2+c^2}}.$$

Because x_0, y_0, and z_0 are not all 0, $\cos\theta$ depends on t.

c. In part a, the curve is a straight line through the origin, so the position vector and the tangent vector are parallel for all t. In part b, the line is not through the origin, so the tangent vector (which is the direction vector for the line) is not parallel to the position vector.

11.6.86 $\frac{d}{dt}(\mathbf{u}(t)+\mathbf{v}(t))=\frac{d}{dt}\langle u_1(t)+v_1(t),u_2(t)+v_2(t),u_3(t)+v_3(t)\rangle=\langle u_1'(t)+v_1'(t),u_2'(t)+v_2'(t),u_3'(t)+v_3'(t)\rangle=\mathbf{u}'(t)+\mathbf{v}'(t)$.

11.6.87

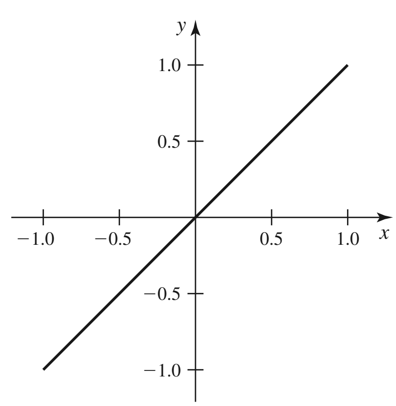
$$\begin{aligned}\frac{d}{dt}(f(t)\mathbf{u}(t))&=\frac{d}{dt}\langle f(t)u_1(t),f(t)u_2(t),f(t)u_3(t)\rangle\\&=\langle f'(t)u_1(t)+f(t)u_1'(t)+,f'(t)u_2(t)+f(t)u_2'(t),f'(t)u_3(t)+f(t)u_3'(t)\rangle\\&=f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t).\end{aligned}$$

11.6.88

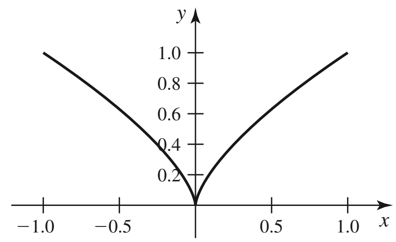
$$\begin{aligned}\frac{d}{dt}(\mathbf{u}\times\mathbf{v})&=\frac{d}{dt}\langle u_2v_3-v_2u_3,u_3v_1-v_3u_1,u_1v_2-u_2v_1\rangle\\&=\langle u_2'v_3+u_2v_3'-(v_2'u_3+v_2u_3'),u_3'v_1+u_3v_1'-(u_1'v_3+u_1v_3'),u_1'v_2+u_1v_2'-(v_1'u_2+v_1u_2')\rangle\\&=\langle u_2'v_3-v_2u_3'+u_2v_3'-v_2'u_3,u_3v_1'-u_1v_3'+u_3'v_1-u_1'v_3,u_1'v_2-v_1u_2'+u_1v_2'-v_1'u_2\rangle\\&=\langle(u_2'v_3-v_2u_3'),(u_3v_1'-u_1v_3'),(u_1'v_2-v_1u_2')\rangle+\langle(u_2v_3'-v_2'u_3),(u_3'v_1-u_1'v_3),(u_1v_2'-v_1'u_2\rangle\\&=(\mathbf{u}'(t)\times\mathbf{v}(t))+(\mathbf{u}(t)\times\mathbf{v}'(t))\end{aligned}$$

11.6.89

a. $\mathbf{r}'(t)=\langle 3t^2,3t^2\rangle$, so $\mathbf{r}'(0)=\langle 0,0\rangle$. There is no cusp because $\lim_{t\to 0}\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3t^2}{3t^2}=1$ exists.



b. $\mathbf{r}'(t) = \langle 3t^2, 2t\rangle$, so $\mathbf{r}'(0) = \langle 0, 0\rangle$. There is a cusp because $\lim_{t\to 0} \frac{dy}{dx} = \lim_{t\to 0} \frac{dy/dt}{dx/dt} = \lim_{t\to 0} \frac{2t}{3t^2} = \lim_{t\to 0} \frac{2}{3t}$ does not exist.



c. The curve $\mathbf{r}(t)$ for $-\infty < t < \infty$ traces out the whole curve $y = x^2$, while the curve $\mathbf{p}(t)$ only traces out the part in the first quadrant, because $x = t^2 > 0$ for all t.

d. $\mathbf{r}'(t) = \langle mt^{m-1}, nt^{n-1}\rangle$, so $\mathbf{r}'(0) = \langle 0, 0\rangle$.

Assume $m > n$. There is a cusp because

$$\lim_{t\to 0} \frac{dy}{dx} = \lim_{t\to 0} \frac{dy/dt}{dx/dt} = \lim_{t\to 0} \frac{nt^{n-1}}{mt^{m-1}} = \lim_{t\to 0} (n/m)\frac{1}{t^{m-n}},$$

which does not exist.

Now assume $m < n$. There is a cusp because

$$\lim_{t\to 0} \frac{dx}{dy} = \lim_{t\to 0} \frac{dx/dt}{dy/dt} = \lim_{t\to 0} \frac{mt^{m-1}}{nt^{n-1}} = \lim_{t\to 0} (m/n)\frac{1}{t^{n-m}},$$

which does not exist.

11.6.90 If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ lies on the sphere $x^2 + y^2 + z^2 = a^2$, then (differentiating with respect to t) we have $2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0$, so $\langle x, y, z\rangle \cdot \langle x', y', z'\rangle = 0$, so $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$.

If $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$, then $x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0$, so $2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0$, so by integrating both sides with respect to t, we have $x^2 + y^2 + z^2 = K$ for some constant $K > 0$, so the curve lies on a sphere.

11.7 Motion in Space

11.7.1 The velocity is the derivative of position, the speed is the magnitude of velocity, and the acceleration is the derivative of velocity.

11.7.2 For the circle $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$ for $a > 0$, the two vectors are orthogonal, with the velocity vector tangent to the circle.

11.7.3 $m\mathbf{a}(t) = \mathbf{F}(t)$.

11.7.4 $m\mathbf{a}(t) = \langle mx''(t), my''(t), mz''(t)\rangle = \mathbf{F}(t) = \langle 0, 0, -mg\rangle$.

11.7.5 Integrate the acceleration to find an expression for the velocity plus a constant, and then use the initial velocity condition to find the constant.

11.7.6 Integrate the velocity to find an expression for the position plus a constant, and then use the initial position condition to find the constant.

11.7.7

a. $\mathbf{v}(t) = \langle 6t, 8t\rangle$, so the speed is $\sqrt{36t^2 + 64t^2} = \sqrt{100t^2} = 10t$.

b. $\mathbf{a}(t) = \langle 6, 8\rangle$.

11.7.8

a. $\mathbf{v}(t) = \langle 5t, 12t\rangle$, so the speed is $\sqrt{25t^2 + 144t^2} = \sqrt{169t^2} = 13t$.

b. $\mathbf{a}(t) = \langle 5, 12\rangle$.

11.7.9

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2, -4\rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{20} = 2\sqrt{5}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 0\rangle$.

11.7.10

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -2t, 6t^2\rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{4t^2 + 36t^4} = 2\,|t|\,\sqrt{1 + 9t^2}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -2, 12t\rangle$.

11.7.11

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 8\cos t, -8\sin t\rangle$, so the speed is $|\mathbf{r}'(t)| = 8$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -8\sin t, -8\cos t\rangle$.

11.7.12

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\sin t, 4\cos t\rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{9\sin^2 t + 16\cos^2 t} = \sqrt{9 + 7\cos^2 t}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -3\cos t, -4\sin t\rangle$.

11.7.13

a. $\mathbf{v}(t) = \langle 2t, 2t, t\rangle$, so the speed is $\sqrt{4t^2 + 4t^2 + t^2} = 3t$.

b. $\mathbf{a}(t) = \langle 2, 2, 1\rangle$.

11.7.14

a. $\mathbf{v}(t) = \langle 4e^{2t}, 2e^{2t}, 4e^{2t}\rangle$, so the speed is $\sqrt{16e^{4t} + 4e^{4t} + 16e^{4t}} = 6e^{2t}$.

b. $\mathbf{a}(t) = \langle 8e^{2t}, 4e^{2t}, 8e^{2t}\rangle$.

11.7.15

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, -4, 6\rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{1 + 16 + 36} = \sqrt{53}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 0, 0\rangle$.

11.7.16

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 3\cos t, -5\sin t, 4\cos t\rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{25\cos^2 t + 25\sin^2 t} = 5$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -3\sin t, -5\cos t, -4\sin t\rangle$.

11.7.17

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 0, 2t, -e^{-t}\rangle$, so the speed is $|\mathbf{r}'(t)| = \sqrt{4t^2 + e^{-2t}}$.

b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 2, e^{-t}\rangle$.

11.7.18

a. $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -26\sin 2t, 24\cos 2t, 10\cos 2t\rangle$, so the speed is

$$|\mathbf{r}'(t)| = \sqrt{676\sin^2 2t + 576\cos^2 2t + 100\cos^2 2t} = 26.$$

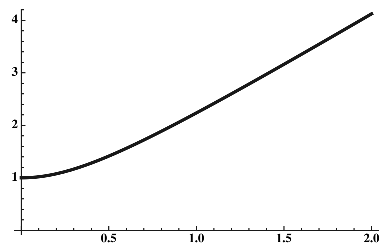
b. $\mathbf{a}(t) = \mathbf{r}''(t) = \langle -52\cos 2t, -48\sin 2t, -20\sin 2t\rangle$.

11.7.19

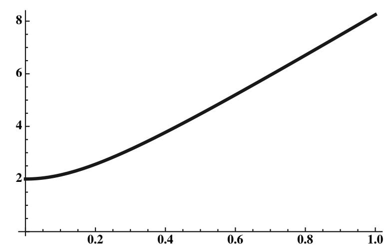
a. The interval must be shrunk by a factor of 2, so $[c, d] = [0, 1]$.

b. $\mathbf{r}'(t) = \langle 1, 2t\rangle$, and $\mathbf{R}'(t) = \langle 2, 8t\rangle$.

c. $|\mathbf{r}'(t)| = \sqrt{1 + 4t^2}$ and $|\mathbf{R}'(t)| = 2\sqrt{1 + 16t^2}$.



The speed of $\mathbf{r}(t)$.



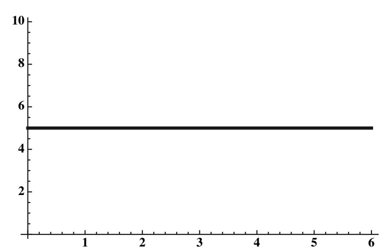
The speed of $\mathbf{R}(t)$.

11.7.20

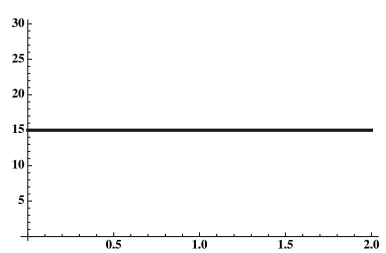
a. The interval must be shrunk by a factor of 3, so $[c, d] = [0, 2]$.

b. $\mathbf{r}'(t) = \langle 3, 4\rangle$, and $\mathbf{R}'(t) = \langle 9, 12\rangle$.

c. $|\mathbf{r}'(t)| = \sqrt{9 + 16} = 5$ and $|\mathbf{R}'(t)| = \sqrt{81 + 144} = 15$.



The speed of $\mathbf{r}(t)$.



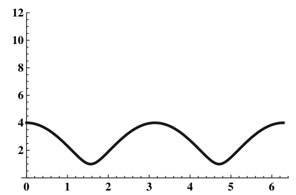
The speed of $\mathbf{R}(t)$.

11.7.21

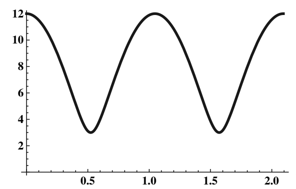
a. The interval must be shrunk by a factor of 1/3, so $[c, d] = [0, 2\pi/3]$.

b. $\mathbf{r}'(t) = \langle -\sin t, 4\cos t\rangle$, and $\mathbf{R}'(t) = \langle -3\sin 3t, 12\cos 3t\rangle$.

c. $|\mathbf{r}'(t)| = \sqrt{\sin^2 t + 16\cos^2 t}$ and $|\mathbf{R}'(t)| = 3\sqrt{\sin^2 3t + 16\cos^2 3t}$.



The speed of $\mathbf{r}(t)$.



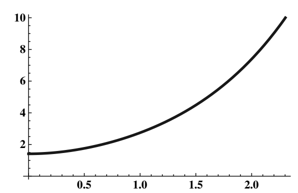
The speed of $\mathbf{R}(t)$.

11.7.22

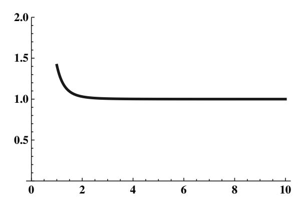
a. Because $e^0 = 1$ and $e^{\ln 10} = 10$, we have $[c, d] = [1, 10]$.

b. $\mathbf{r}'(t) = \langle -e^{-t}, e^{-t}\rangle$, and $\mathbf{R}'(t) = \langle -1, 1/t^2\rangle$.

c. $|\mathbf{r}'(t)| = \sqrt{e^{2t} + e^{-2t}} = \sqrt{2\cosh(2t)}$ and $|\mathbf{R}'(t)| = \sqrt{1 + 1/t^4}$.



The speed of $\mathbf{r}(t)$.



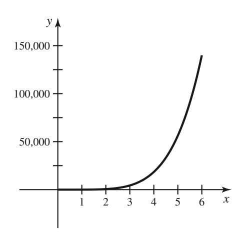
The speed of $\mathbf{R}(t)$.

11.7.23

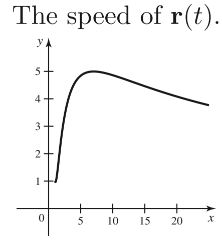
a. Because $e^{0^2} = 1$ and $e^{6^2} = e^{36}$, we have $[c, d] = [1, e^{36}]$.

b. $\mathbf{r}'(t) = \langle 2t, -8t^3, 18t^5\rangle$, and $\mathbf{R}'(t) = \langle 1/t, (-4\ln t)/t, (9\ln^2 t)/t\rangle$.

c. $|\mathbf{r}'(t)| = 2t\sqrt{1 + 16t^4 + 81t^8}$ and $|\mathbf{R}'(t)| = \frac{1}{t}\sqrt{1 + 16\ln^2 t + 81\ln^4 t}$.



The speed of $\mathbf{r}(t)$.



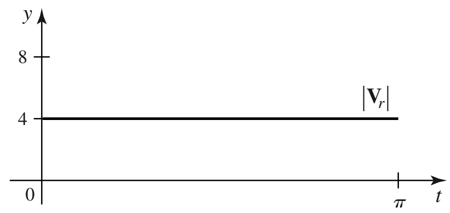
The speed of $\mathbf{R}(t)$.

11.7.24

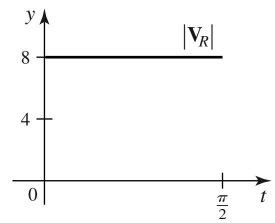
a. The interval must be shrunk by a factor of 1/2 so $[c, d] = [0, \pi/2]$.

b. $\mathbf{r}'(t) = \langle -4\sin 2t, 2\sqrt{2}\cos 2t, 2\sqrt{2}\cos 2t\rangle$, $\mathbf{R}'(t) = \langle -8\sin 4t, 4\sqrt{2}\cos 4t, 4\sqrt{2}\cos 4t\rangle$.

c. $|\mathbf{r}'(t)| = \sqrt{16\sin^2 2t + 8\cos^2 2t + 8\cos^2 2t} = \sqrt{16} = 4$, while we have $|\mathbf{R}'(t)| = \sqrt{64\sin^2 4t + 32\cos^2 4t + 32\cos^2 4t} = 8$.



The speed of $\mathbf{r}(t)$.



The speed of $\mathbf{R}(t)$.

11.7.25 Note that $x^2 + y^2 = 64$, so the trajectory lies on a circle centered at the origin of radius 8. $\mathbf{r}(t)\cdot\mathbf{r}'(t) = \langle 8\cos 2t, 8\sin 2t\rangle \cdot \langle -16\sin 2t, 16\cos 2t\rangle = -128\sin 2t\cos 2t + 128\sin 2t\cos 2t = 0$.

11.7.26 Note that $x^2 + y^2 = 16\sin^2 t + 4\cos^2 t = 4 + 12\sin^2 t$ which is not a constant, so the trajectory does not lie on a circle centered at the origin.

11.7.27 Note that $x^2 + y^2 = (\sin t + \sqrt{3}\cos t)^2 + (\sqrt{3}\sin t - \cos t)^2 = (\sin^2 t + 2\sin t\sqrt{3}\cos t + 3\cos^2 t) + (3\sin^2 t - 2\cos t\sqrt{3}\sin t + \cos^2 t) = 4$, so the trajectory lies on a circle centered at the origin of radius 2. $\mathbf{r}(t)\cdot\mathbf{r}'(t) = \langle \sin t + \sqrt{3}\cos t, \sqrt{3}\sin t - \cos t\rangle \cdot \langle \cos t - \sqrt{3}\sin t, \sqrt{3}\cos t + \sin t\rangle = (\sin t\cos t - \sqrt{3}\sin^2 t + \sqrt{3}\cos^2 t - 3\sin t\cos t) + (3\sin t\cos t + \sqrt{3}\sin^2 t - \sqrt{3}\cos^2 t - \sin t\cos t) = 0$.

11.7.28 $x^2 + y^2 + z^2 = 9\sin^2 t + 25\cos^2 t + 16\sin^2 t = 25$, so the trajectory lies on a sphere centered at the origin of radius 5. $\mathbf{r}(t)\cdot\mathbf{r}'(t) = \langle 3\sin t, 5\cos t, 4\sin t\rangle \cdot \langle 3\cos t, -5\sin t, 4\cos t\rangle = 9\sin t\cos t - 25\sin t\cos t + 16\sin t\cos t = 0$.

11.7.29 $x^2 + y^2 + z^2 = \sin^2 t + \cos^2 t + \cos^2 t = 1 + \cos^2 t$, which is not a constant, so the trajectory does not lie on a sphere centered at the origin.

11.7.30 $x^2+y^2+z^2 = (\sqrt{3}\cos t+\sqrt{2}\sin t)^2+(-\sqrt{3}\cos t+\sqrt{2}\sin t)^2+(\sqrt{2}\sin t)^2 = (3\cos^2 t+2\sqrt{6}\sin t\cos t+2\sin^2 t) + (3\cos^2 t - 2\sqrt{6}\sin t\cos t + 2\sin^2 t) + 2\sin^2 t = 6$, so the trajectory lies on a sphere centered at the origin of radius $\sqrt{6}$. $\mathbf{r}(t)\cdot\mathbf{r}'(t) = \langle\sqrt{3}\cos t + \sqrt{2}\sin t, -\sqrt{3}\cos t + \sqrt{2}\sin t, \sqrt{2}\sin t\rangle \cdot \langle -\sqrt{3}\sin t + \sqrt{2}\cos t, \sqrt{3}\sin t + \sqrt{2}\cos t, \sqrt{2}\cos t\rangle = (-3\sin t\cos t + \sqrt{6}\cos^2 t - \sqrt{6}\sin^2 t + 2\sin t\cos t) + (-3\sin t\cos t - \sqrt{6}\cos^2 t + \sqrt{6}\sin^2 t + 2\sin t\cos t) + (2\sin t\cos t) = 0$.

11.7.31 $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0, 1\rangle\,dt = \langle 0, t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 2, 3\rangle$, we have $\mathbf{C} = \langle 2, 3\rangle$. Thus, $\mathbf{v}(t) = \langle 2, t+3\rangle$.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 2, t+3\rangle\,dt = \langle 2t, t^2/2 + 3t\rangle + \mathbf{D}$. Because $\mathbf{r}(0) = \langle 0, 0\rangle$, we have $\mathbf{D} = \langle 0, 0\rangle$. Therefore, $\mathbf{r}(t) = \langle 2t, t^2/2 + 3t\rangle$.

11.7.32 $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 1, 2\rangle\,dt = \langle t, 2t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 1, 1\rangle$, we have $\mathbf{C} = \langle 1, 1\rangle$. Thus, $\mathbf{v}(t) = \langle t+1, 2t+1\rangle$.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle t+1, 2t+1\rangle\,dt = \langle t^2/2 + t, t^2 + t\rangle + \mathbf{D}$. Because $\mathbf{r}(0) = \langle 2, 3\rangle$, we have $\mathbf{D} = \langle 2, 3\rangle$. Therefore, $\mathbf{r}(t) = \langle t^2/2 + t + 2, t^2 + t + 3\rangle$.

11.7.33 $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0, 10\rangle\,dt = \langle 0, 10t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 0, 5\rangle$, we have $\mathbf{v}(t) = \langle 0, 10t + 5\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 0, 10t + 5\rangle\,dt = \langle 0, 5t^2 + 5t\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 1, -1\rangle$, we have $\mathbf{r}(t) = \langle 1, 5t^2 + 5t - 1\rangle$.

11.7.34 $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 1, t\rangle\,dt = \langle t, t^2/2\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 2, -1\rangle$, we have $\mathbf{v}(t) = \langle t+2, t^2/2-1\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle t + 2, t^2/2 - 1\rangle\,dt = \langle t^2/2 + 2t, t^3/6 - t\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 8\rangle$, we have $\mathbf{r}(t) = \langle t^2/2 + 2t, t^3/6 - t + 8\rangle$.

11.7.35 $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle \cos t, 2\sin t\rangle\,dt = \langle \sin t, -2\cos t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 0, 1\rangle$, we have $\mathbf{v}(t) = \langle \sin t, 3 - 2\cos t\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle \sin t, 3 - 2\cos t\rangle\,dt = \langle -\cos t, 3t - 2\sin t\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 1, 0\rangle$, we have $\mathbf{r}(t) = \langle 2 - \cos t, 3t - 2\sin t\rangle$.

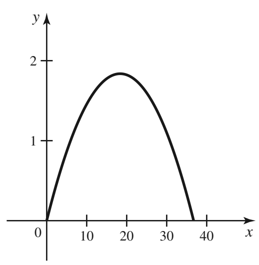
11.7.36 $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle e^{-t}, 1\rangle\,dt = \langle -e^{-t}, t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 1, 0\rangle$, we have $\mathbf{v}(t) = \langle 2 - e^{-t}, t\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 2 - e^{-t}, t\rangle\,dt = \langle 2t + e^{-t}, t^2/2\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0\rangle$, we have $\mathbf{r}(t) = \langle 2t + e^{-t} - 1, t^2/2\rangle$.

11.7.37

a. $\mathbf{v}(t) = \int \langle 0, -9.8\rangle\,dt = \langle 0, -9.8t\rangle + \mathbf{C}$, and because $\mathbf{v}(0) = \langle 30, 6\rangle$, we have $\mathbf{v}(t) = \langle 30, 6 - 9.8t\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 30, 6 - 9.8t\rangle\,dt = \langle 30t, 6t - 4.9t^2\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0\rangle$, we have $\mathbf{r}(t) = \langle 30t, 6t - 4.9t^2\rangle$.

b. 

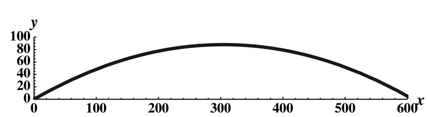
c. The ball hits the ground when $6t - 4.9t^2 = 0$, which occurs for $t = 6/4.9 \approx 1.22$ seconds. The range of the ball is approximately $30 \cdot 1.22 \approx 36.7$ meters.

d. The maximum height occurs at time $T \approx 1.22/2 = .61$ seconds, and is $6T - 4.9T^2 \approx 1.84$ meters.

11.7.38

a. $\mathbf{v}(t) = \int \langle 0, -32\rangle\,dt = \langle 0, -32t\rangle + \mathbf{C}$, and because $\mathbf{v}(0) = 150\langle \sqrt{3}/2, 1/2\rangle = \langle 75\sqrt{3}, 75\rangle$, we have $\mathbf{v}(t) = \langle 75\sqrt{3}, -32t + 75\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 75\sqrt{3}, -32t + 75\rangle\,dt = \langle 75\sqrt{3}t, -16t^2 + 75t\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0\rangle$, we have $\mathbf{r}(t) = \langle 75\sqrt{3}t, -16t^2 + 75t\rangle$.

b. 

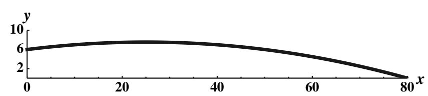
c. The ball hits the ground when $-16t^2 + 75t = 0$, which occurs for $t = 75/16 = 4.6875$ seconds. The range of the ball is approximately $75\sqrt{3}(4.6875) \approx 608.9$ feet.

d. The maximum height occurs at time $T \approx 4.6875/2 \approx 2.344$, and is $-16(2.344)^2 + 75(2.344) \approx 87.89$ feet.

11.7.39

a. $\mathbf{v}(t) = \int \langle 0, -32 \rangle \, dt = \langle 0, -32t \rangle + \mathbf{C}$, and because $\mathbf{v}(0) = \langle 80, 10 \rangle$, we have $\mathbf{v}(t) = \langle 80, 10 - 32t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) \, dt = \int \langle 80, 10 - 32t \rangle \, dt = \langle 80t, 10t - 16t^2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 6 \rangle$, we have $\mathbf{r}(t) = \langle 80t, 6 + 10t - 16t^2 \rangle$.

b. 

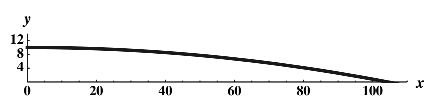
c. The ball hits the ground when $-16t^2 + 10t + 6 = -2(t-1)(8t+3) = 0$, which occurs for $t = 1$ second. The range of the ball is $80 \cdot 1 = 80$ feet.

d. The maximum height occurs at time $T \approx 10/32 \approx .3125$, and is $-16(.3125)^2 + 10(.3125) + 6 \approx 7.56$ feet.

11.7.40

a. $\mathbf{v}(t) = \int \langle 0, -32 \rangle \, dt = \langle 0, -32t \rangle + \mathbf{C}$, and because $\mathbf{v}(0) = 132\langle 1, 0 \rangle$, we have $\mathbf{v}(t) = \langle 132, -32t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) \, dt = \int \langle 132, -32t \rangle \, dt = \langle 132t, -16t^2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 10 \rangle$, we have $\mathbf{r}(t) = \langle 132t, -16t^2 + 10 \rangle$.

b. 

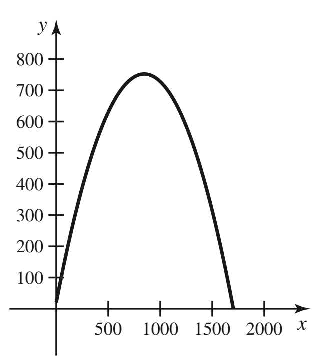
c. The ball hits the ground when $-16t^2 + 10 = 0$, which occurs for $t = \sqrt{10/16} \approx .79$ second. The range of the ball is approximately $132(.79) \approx 104.36$ feet.

d. The maximum height occurs at time $T = 0$, and is 10 feet.

11.7.41

a. $\mathbf{v}(t) = \int \langle 0, -32 \rangle \, dt = \langle 0, -32t \rangle + \mathbf{C}$, and because $\mathbf{v}(0) = 250\langle 1/2, \sqrt{3}/2 \rangle = \langle 125, 125\sqrt{3} \rangle$, we have $\mathbf{v}(t) = \langle 125, 125\sqrt{3} - 32t \rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t) \, dt = \int \langle 125, 125\sqrt{3} - 32t \rangle \, dt = \langle 125t, 125\sqrt{3}t - 16t^2 \rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 20 \rangle$, we have $\mathbf{r}(t) = \langle 125t, 20 + 125\sqrt{3}t - 16t^2 \rangle$.



b.

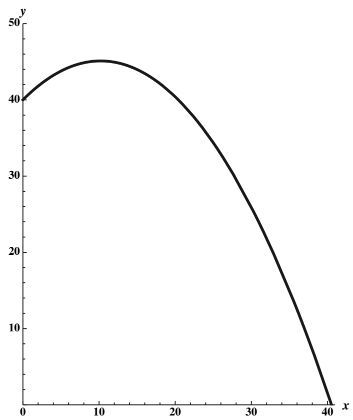
c. The ball hits the ground when $20 + 125\sqrt{3}t - 16t^2 = 0$, which occurs for $t \approx 13.62$ seconds. The range of the ball is approximately $125 \cdot 13.62 \approx 1702$ feet.

d. The maximum height occurs when $125\sqrt{3} - 32t = 0$, which is when $t \approx 6.77$ and it is about $20 + 125\sqrt{3}(6.77) - 16(6.77)^2 \approx 752.4$ feet.

11.7.42

a. $\mathbf{v}(t) = \int \langle 0, -9.8\rangle\, dt = \langle 0, -9.8t\rangle + \mathbf{C}$, and because $\mathbf{v}(0) = 10\sqrt{2}\langle\sqrt{2}/2, \sqrt{2}/2\rangle = \langle 10, 10\rangle$, we have $\mathbf{v}(t) = \langle 10, 10 - 9.8t\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int \langle 10, 10 - 9.8t\rangle\, dt = \langle 10t, 10t - 4.9t^2\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 40\rangle$, we have $\mathbf{r}(t) = \langle 10t, 40 + 10t - 4.9t^2\rangle$.

b. 

c. The ball hits the ground when $40+10t-4.9t^2 = 0$, which occurs when $t \approx 4.06$ seconds. The range of the ball is approximately $10 \cdot 4.06 = 40.6$ meters.

d. The maximum height occurs at the time when $10-9.8t = 0$, which is when $t = 10/9.8 \approx 1.02$. The height at this time is about 45.1 meters.

11.7.43 $\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int \langle 0, 0, 10\rangle\, dt = \langle 0, 0, 10t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 1, 5, 0\rangle$, we have $\mathbf{v}(t) = \langle 1, 5, 10t\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int \langle 1, 5, 10t\rangle\, dt = \langle t, 5t, 5t^2\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 5, 0\rangle$, we have $\mathbf{r}(t) = \langle t, 5t + 5, 5t^2\rangle$.

11.7.44 $\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int \langle 1, t, 4t\rangle\, dt = \langle t, t^2/2, 2t^2\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 20, 0, 0\rangle$, we have $\mathbf{v}(t) = \langle t + 20, t^2/2, 2t^2\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int \langle t+20, t^2/2, 2t^2\rangle\, dt = \langle t^2/2+20t, t^3/6, 2t^3/3\rangle+\mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, we have $\mathbf{r}(t) = \langle t^2/2 + 20t, t^3/6, 2t^3/3\rangle$.

11.7.45 $\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int \langle \sin t, \cos t, 1\rangle\, dt = \langle -\cos t, \sin t, t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 0, 2, 0\rangle$, we have $\mathbf{v}(t) = \langle 1 - \cos t, \sin t + 2, t\rangle$.

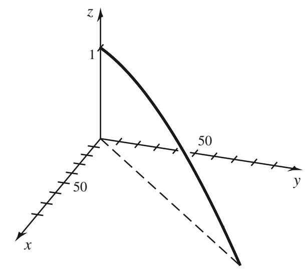
Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int \langle 1 - \cos t, \sin t + 2, t\rangle\, dt = \langle t - \sin t, -\cos t + 2t, t^2/2\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, we have $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t + 2t, t^2/2\rangle$.

11.7.46 $\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int \langle t, e^{-t}, 1\rangle\, dt = \langle t^2/2, -e^{-t}, t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 0, 0, 1\rangle$, we have $\mathbf{v}(t) = \langle t^2/2, 1 - e^{-t}, t + 1\rangle$.

Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int \langle t^2/2, 1-e^{-t}, t+1\rangle\, dt = \langle t^3/6, t+e^{-t}, t^2/2+t\rangle+\mathbf{D}$, and because $\mathbf{r}(0) = \langle 4, 0, 0\rangle$, we have $\mathbf{r}(t) = \langle t^3/6 + 4, t + e^{-t} - 1, t^2/2 + t\rangle$.

11.7.47

a. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0, 0, -9.8\rangle\,dt = \langle 0, 0, -9.8t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 200, 200, 0\rangle$, we have $\mathbf{v}(t) = \langle 200, 200, -9.8t\rangle$.
Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 200, 200, -9.8t\rangle\,dt = \langle 200t, 200t, -4.9t^2\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 1\rangle$, we have $\mathbf{r}(t) = \langle 200t, 200t, -4.9t^2 + 1\rangle$.

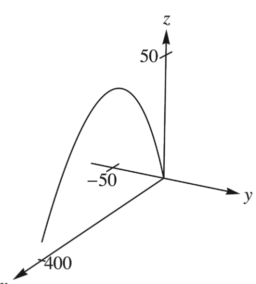
b. 

c. The bullet hits the ground when $-4.9t^2 + 1 = 0$, which occurs for $t \approx .452$ seconds. At this time, the bullet is approximately at the point $(200 \cdot 0.452, 200 \cdot 0.452, 0) \approx (90.35, 90.35, 0)$. So its range is approximately $\sqrt{90.35^2 + 90.35^2} \approx 127.8$ meters.

d. The maximum height of the bullet is its initial height of 1 meter.

11.7.48

a. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0, -.8, -9.8\rangle\,dt = \langle 0, -.8t, -9.8t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 50, 0, 30\rangle$, we have $\mathbf{v}(t) = \langle 50, -.8t, -9.8t + 30\rangle$.
Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 50, -.8t, -9.8t + 30\rangle\,dt = \langle 50t, -.4t^2, -4.9t^2 + 30t\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, we have $\mathbf{r}(t) = \langle 50t, -.4t^2, -4.9t^2 + 30t\rangle$.

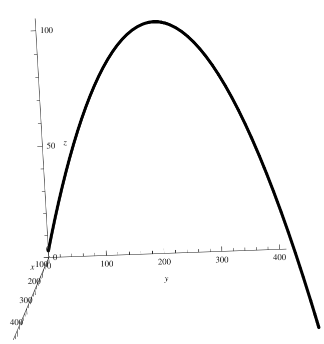
b. 

c. The ball hits the ground when $-4.9t^2 + 30t = 0$, which occurs for $t \approx 30/4.9 \approx 6.12$ seconds. At this time, the ball is at the point $((50)(6.12), -.4(6.12)^2, 0) \approx (306, -14.98, 0)$. So its range is approximately $\sqrt{306^2 + 14.98^2} \approx 306.4$ meters.

d. The maximum height of the ball occurs when $-9.8t + 30 = 0$, or when $t = 30/9.8 \approx 3.06$ seconds. At this time the ball's height is about $30(3.06) - 4.9(3.06)^2 \approx 45.92$ meters.

11.7.49

a. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 10, 0, -32\rangle\,dt = \langle 10t, 0, -32t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 60, 80, 80\rangle$, we have $\mathbf{v}(t) = \langle 10t + 60, 80, -32t + 80\rangle$.
Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 10t + 60, 80, -32t + 80\rangle\,dt = \langle 5t^2 + 60t, 80t, -16t^2 + 80t\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 3\rangle$, we have $\mathbf{r}(t) = \langle 5t^2 + 60t, 80t, -16t^2 + 80t + 3\rangle$.

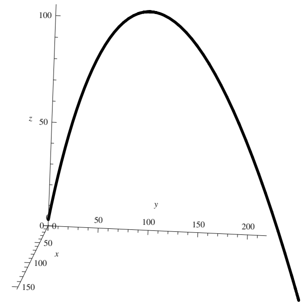
b. 

c. The ball hits the ground when $-16t^2+80t+3=0$, which occurs for $t \approx 5.04$ seconds. At this time, the ball is at the point $(5(5.04)^2+60(5.04), 80(5.04), 0) \approx (429.4, 403.2, 0)$. So its range is approximately $\sqrt{429.4^2+403.2^2} \approx 589$ feet.

d. The maximum height of the ball occurs when $-32t+80=0$, or when $t = 80/32 = 2.5$ seconds. At this time the ball's height is about $-16(2.5)^2+80(2.5)+3 = 103$ feet.

11.7.50

a. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0, 5, -32\rangle\,dt = \langle 0, 5t, -32t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 30, 30, 80\rangle$, we have $\mathbf{v}(t) = \langle 30, 5t+30, -32t+80\rangle$. Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 30, 5t+30, -32t+80\rangle\,dt = \langle 30t, 5t^2/2+30t, -16t^2+80t\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 3\rangle$, we have $\mathbf{r}(t) = \langle 30t, 5t^2/2+30t, -16t^2+80t+3\rangle$.

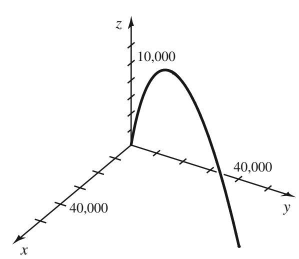
b. 

c. The ball hits the ground when $-16t^2+80t+3=0$, which occurs for $t \approx 5.04$ seconds. At this time, the ball is at the point $(30(5.04), 5(5.04)^2/2+30(5.04), 0) \approx (151.2, 214.7, 0)$. So its range is approximately $\sqrt{151.2^2+214.7^2} \approx 263$ feet.

d. The maximum height of the ball occurs when $-32t+80=0$, or when $t = 80/32 = 2.5$ seconds. At this time the ball's height is about $-16(2.5)^2+80(2.5)+3 = 103$ feet.

11.7.51

a. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 0, 2.5, -9.8\rangle\,dt = \langle 0, 2.5t, -9.8t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 300, 400, 500\rangle$, we have $\mathbf{v}(t) = \langle 300, 2.5t+400, 500-9.8t\rangle$. Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 300, 2.5t+400, 500-9.8t\rangle\,dt = \langle 300t, 1.25t^2+400t, 500t-4.9t^2\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0, 0, 10\rangle$, we have $\mathbf{r}(t) = \langle 300t, 1.25t^2+400t, 10+500t-4.9t^2\rangle$.

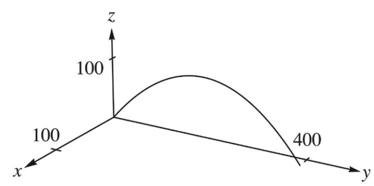
b. 

c. The rocket hits the ground when $-4.9t^2+500t+10=0$, which occurs for $t \approx 102.1$ seconds. At this time, the rocket is at the point $(30630, 53870.5, 0)$. So its range is approximately $\sqrt{30630^2+53870.5^2} \approx 61969.6$ meters.

d. The maximum height of the rocket occurs when $-9.8t+500=0$, or when $t = 500/9.8 \approx 51.02$ seconds. At this time the rocket's height is about $10+500(51.02)-4.9(51.02)^2 \approx 12,765$ meters.

11.7.52

a. $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \langle 1.2, 0, -32\rangle\,dt = \langle 1.2t, 0, -32t\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 0, 80, 80\rangle$, we have $\mathbf{v}(t) = \langle 1.2t, 80, 80-32t\rangle$. Also, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle 1.2t, 80, 80 - 32t\rangle\,dt = \langle .6t^2, 80t, 80t - 16t^2\rangle + \mathbf{D}$, and because $\mathbf{r}(0) = \langle 0,0,0\rangle$, we have $\mathbf{r}(t) = \langle .6t^2, 80t, 80t - 16t^2\rangle$.

b. 

c. The ball hits the ground when $-16t^2 + 80t = 0$, which occurs for $t = 80/16 = 5$ seconds. At this time, the ball is at the point $(15, 400, 0)$. So its range is approximately $\sqrt{15^2 + 400^2} \approx 400.28$ feet.

d. The maximum height of the ball occurs when $-32t + 80 = 0$, or when $t = 80/32 = 2.5$ seconds. At this time the ball's height is about $-16(2.5)^2 + 80 \cdot 2.5 = 100$ feet.

11.7.53

a. False. For example, if $\mathbf{v}(t) = \langle \cos t, \sin t\rangle$, then its speed is constantly 1 even though its components aren't constant.

b. True. They both generate $\{(x, y) \,|\, x^2 + y^2 = 1\}$.

c. False. For example, $\langle t, t, t\rangle$ has variable magnitude but constant direction.

d. True. If $\mathbf{a}(t) = \langle 0,0,0\rangle$, then $\mathbf{v}(t) = \langle 0,0,0\rangle + \mathbf{C}$ for a constant vector $\mathbf{C}$.

e. False. Recall that for two-dimensional motion the range is given by $\frac{|\mathbf{v}_0|^2 \sin 2\alpha}{g}$, so doubling the speed should quadruple the range.

f. True. The time of flight is given by $T = \frac{2|\mathbf{v}_0|\sin\alpha}{g}$, so doubling the speed doubles the time of flight.

g. True. For example, if $\mathbf{v}(t) = \langle e^t, e^t, e^t\rangle$, then $\mathbf{a}(t) = \langle e^t, e^t, e^t\rangle$ as well.

11.7.54 The time of flight is $T = \frac{2|\mathbf{v}_0|\sin\alpha}{g} = \frac{2\cdot 20}{32} = 1.25$ seconds.
The range of the flight is $\frac{|\mathbf{v}_0|^2 2\sin\alpha\cos\alpha}{g} = \frac{400}{32} = 12.5$ feet.
The maximum height is given by $\frac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \frac{400}{64} = 6.25$ feet.

11.7.55 Note that $\langle u_0, v_0\rangle = 75\langle\sqrt{3}, 1\rangle$
The time of flight is $T = \frac{2|\mathbf{v}_0|\sin\alpha}{g} = \frac{2\cdot 75}{9.8} \approx 15.3$ seconds.
The range of the flight is $\frac{|\mathbf{v}_0|^2 2\sin\alpha\cos\alpha}{g} = \frac{11250\sqrt{3}}{9.8} \approx 1988.3$ meters.
The maximum height is given by $\frac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \frac{75^2}{19.6} \approx 287$ meters.

11.7.56 The time of flight is $T = \frac{2|\mathbf{v}_0|\sin\alpha}{g} = \frac{2\cdot 80}{9.8} \approx 16.33$ seconds.
The range of the flight is $\frac{|\mathbf{v}_0|^2 2\sin\alpha\cos\alpha}{g} = \frac{6400}{9.8} \approx 653$ meters.
The maximum height is given by $\frac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \frac{80^2}{19.6} \approx 326.5$ meters.

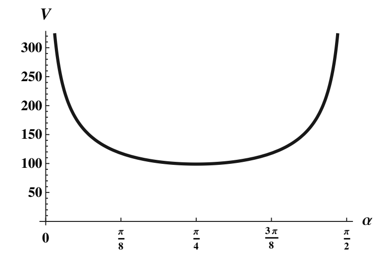
11.7.57 Note that $\langle u_0, v_0\rangle = 200\langle 1, \sqrt{3}\rangle$ The time of flight is $T = \frac{2|\mathbf{v}_0|\sin\alpha}{g} = \frac{2\cdot 200\sqrt{3}}{32} = 21.65$ seconds.
The range of the flight is $\frac{|\mathbf{v}_0|^2 2\sin\alpha\cos\alpha}{g} = \frac{80{,}000\sqrt{3}}{32} = 4330.13$ feet.
The maximum height is given by $\frac{(|\mathbf{v}_0|\sin\alpha)^2}{2g} = \frac{(200\sqrt{3})^2}{64} = 1875$ feet.

11.7.58 Each of these values on the moon would be 6 times the corresponding value on the earth, because the factor of g in the denominator would result in an extra factor of 6 if g were replaced by $g/6$.

11.7.59 We desire $\frac{|\mathbf{v}_0|^2 2\sin\alpha\cos\alpha}{g} = 300$ meters, so we require $\sin 2\alpha = 300 \cdot \frac{9.8}{60^2} \approx .81666$. So $2\alpha = \sin^{-1}(.81666)$, and $\alpha \approx 27.4$ degrees or $\alpha \approx 62.62$ degrees.

11.7.60

a. Let V stand for the initial speed. The range is given by $\frac{V^2 \sin 2\alpha}{g}$, so we require $\frac{9800}{\sin 2\alpha} = V^2$, so $V = \sqrt{9800 \csc 2\alpha}$.



b. The speed is minimized when $\frac{dV}{d\alpha} = \frac{-9800 \csc 2\alpha \cot 2\alpha}{\sqrt{9800 \csc 2\alpha}} = 0$, which occurs when $\cos 2\alpha = 0$, or $\alpha = \pi/4$. At this value of α, the value of V is about 99 meters per second.

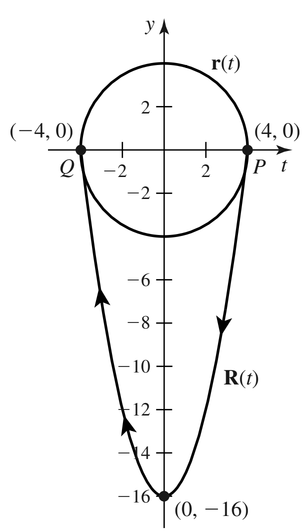
c. The flight time is given by $T = \frac{2|V|\sin\alpha}{g}$, so if $V = \sqrt{9800 \csc 2\alpha}$, this would be $T = \frac{2\sqrt{9800 \csc 2\alpha}\sin\alpha}{9.8} = c\sqrt{\tan\alpha}$. for a positive constant c. Thus T is an increasing function on $(0, \pi/2)$, so smaller angles give a shorter flight time, but no minimum exists on $(0, \pi/2)$.

11.7.61

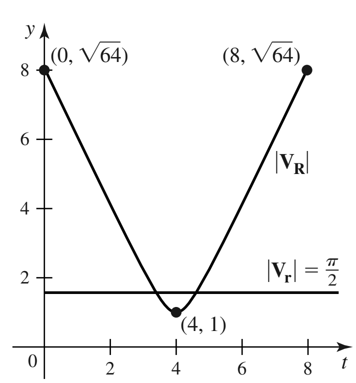
a. If $t_1 > t_0$ are two values of t, we have $\mathbf{r}(t_1) - \mathbf{r}(t_0) = (f(t_1) - f(t_0))\langle a, b, c\rangle$, which is always a vector in the same direction, regardless of the values of t_1 and t_0.

b. $\mathbf{r}'(t) = f'(t)\langle a, b, c\rangle$ is a multiple of $\langle a, b, c\rangle$, so the tangent vector is always a multiple of the vector $\langle a, b, c\rangle$, so the motion of the object doesn't vary in direction, although it might vary in speed.

11.7.62



a.



b.

c. Both travelers arrive at $t = 8$.

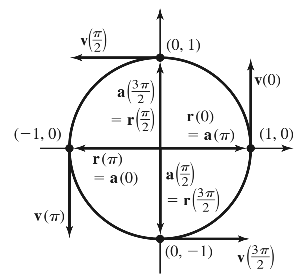
11.7.63

a. The object traverses the circle once over the interval $[0, 2\pi/\omega]$.

b. The velocity is $\mathbf{v}(t) = \langle -A\omega\sin(\omega t), A\omega\cos(\omega t)\rangle$. The velocity is not constant in direction, but it is constant in speed, because the speed is $|A\omega|$.

c. The acceleration is $\mathbf{a}(t) = -A\omega^2\langle\cos\omega t, \sin\omega t\rangle$.

d. The position and velocity are orthogonal. The position and acceleration are in opposite directions.



11.7.64

a. Note that $\mathbf{r}(t) = \langle(-7/5)t + 1, (6/5)t + 2, (6/5)t + 4\rangle$ has $\mathbf{r}(0) = \langle 1, 2, 4\rangle$ and $\mathbf{r}(5) = \langle -6, 8, 10\rangle$.

b. Consider $\mathbf{r}(t) = e^t\langle -\frac{7}{11}, \frac{6}{11}, \frac{6}{11}\rangle + \langle\frac{13}{6}, 1, 3\rangle$ for $\ln(11/6) \le t \le \ln(77/6)$. Note that

$$\mathbf{r}(\ln(11/6)) = \frac{11}{6}\langle -7/11, 6/11, 6/11\rangle + \langle 13/6, 1, 3\rangle = \langle -7/6, 1, 1\rangle + \langle 13/6, 1, 3\rangle = \langle 1, 2, 4\rangle$$

and

$$\mathbf{r}(\ln(77/6)) = \frac{77}{6}\langle -7/11, 6/11, 6/11\rangle + \langle 13/6, 1, 3\rangle = \langle -49/6, 7, 7\rangle + \langle 13/6, 1, 3\rangle = \langle -6, 8, 10\rangle.$$

Also $|\mathbf{r}'(t)| = |\langle e^t(-7/11), e^t(6/11), e^t(6/11)\rangle| = e^t\sqrt{\frac{49+36+36}{11^2}} = e^t$. The fact that $\mathbf{r}(t)$ traverses a straight line follows from exercise 49.

11.7.65

a. Consider $\mathbf{r}(t) = \langle 5\sin(\pi t/6), 5\cos(\pi t/6)\rangle$ for $0 \le t \le 12$. Note that the speed is the constant $5\pi/6$, and that $\mathbf{r}(0) = (0, 5) = \mathbf{r}(12)$.

b. Consider $\mathbf{r}(t) = \langle 5\sin((1-e^{-t})/5), 5\cos((1-e^{-t})/5)\rangle$ for $-\ln(10\pi+1) \le t \le 0$. Note that $\mathbf{r}(-\ln(10\pi + 1)) = (0, 5) = \mathbf{r}(0)$. Also note that the speed is $|\mathbf{r}'(t)| = |e^{-t}\langle\cos((1-e^{-t})/5), -\sin((1-e^{-t})/5)\rangle| = e^{-t}$.

11.7.66

a. Let $\mathbf{r}(t) = \langle\cos 6t, \sin 6t, 8t\rangle$. Then $\mathbf{r}$ moves along the circular helix, but the speed is

$$|\langle -6\sin 6t, 6\cos 6t, 8\rangle| = \sqrt{36\sin^2 6t + 36\cos^2 6t + 64} = 10.$$

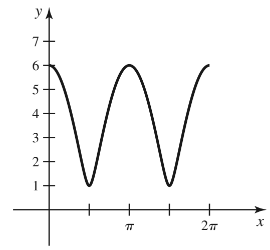
b. Let $\mathbf{r}(t) = \langle\cos(t^2/(2\sqrt{2})), \sin(t^2/(2\sqrt{2})), t^2/(2\sqrt{2})\rangle$. Then $\mathbf{r}$ moves along the circular helix, but the speed is

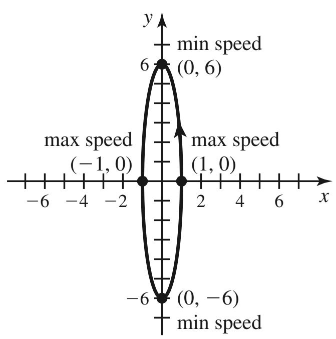
$$\left|\langle -\frac{t}{\sqrt{2}}\sin(t^2/(2\sqrt{2})), \frac{t}{\sqrt{2}}\cos(t^2/(2\sqrt{2})), t/\sqrt{2}\rangle\right| = \sqrt{\frac{t^2}{2}\sin^2(t^2/(2\sqrt{2})) + \frac{t^2}{2}\cos^2(t^2/(2\sqrt{2})) + \frac{t^2}{2}} = t$$

for $t \ge 0$.

11.7.67

a. The velocity is $\mathbf{v}(t) = \langle -a\sin t, b\cos t\rangle$ and the speed is $\sqrt{a^2\sin^2 t + b^2\cos^2 t}$.

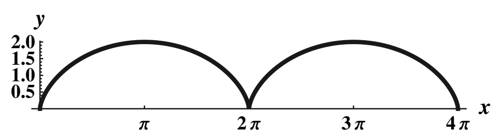
b. 

c. 

Yes, as the diagram indicates.

d. Assume $a > b > 0$. Then the maximum speed occurs at $\pi/2$ and is equal to a, while the minimum speed occurs at π and is equal to b. So the ratio is $\frac{a}{b}$. In the case $b > a > 0$, the ratio is $\frac{b}{a}$.

11.7.68

a. 

b. The velocity is $\langle 1 - \cos t, \sin t\rangle$ and the speed is $\sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2 - 2\cos t}$. The speed is maximal when $t = \pi, 3\pi$ and minimal when $t = 0,\ 2\pi,\ 4\pi$.

c. The acceleration is $\langle \sin t, \cos t\rangle$ and $|\mathbf{a}(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$.

d. At $t = 2\pi$, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1-\cos t}$ doesn't exist, and the limit as $t \to 2\pi^+ > 0$ while the limit as $t \to 2\pi^- < 0$.

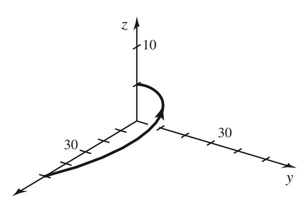
11.7.69

a. The initial point is $\mathbf{r}(0) = \langle 50, 0, 0\rangle$, and the "terminal" point is $\langle 0, 0, 5\rangle$ because $\lim_{t\to\infty} e^{-t} = 0$, while $\sin t$ and $\cos t$ are bounded between -1 and 1 as $t \to \infty$.

b. The speed is given by

$$5e^{-t}\left|\langle -10(\cos t + \sin t), 10(\cos t - \sin t), 1\rangle\right|,$$

which can be written as $5e^{-t}\sqrt{201}$.

c. 

11.7.70 Let the angle be α. Then $\mathbf{v}(t) = \langle 150\cos\alpha, 150\sin\alpha - 32t\rangle$, and $\mathbf{r}(t) = \langle 150t\cos\alpha, 150t\sin\alpha - 16t^2\rangle$. Because we require the ball to land in the hole, we need the point $(390, 40)$ to be on the curve. So $150t\cos\alpha =$

390 and $150t\sin\alpha - 16t^2 = 40$. Thus $t = \frac{390}{150\cos\alpha}$, and therefore $150 \cdot \frac{390\cdot\sin\alpha}{150\cos\alpha} - 16\left(\frac{390}{150\cos\alpha}\right)^2 = 40$. This can be written $390\tan\alpha - \frac{2704}{25}\sec^2\alpha - 40 = 0$, or

$$390\tan\alpha - \frac{2704}{25}(1+\tan^2\alpha) - 40 = -\frac{2704}{25}\tan^2\alpha + 390\tan\alpha - \frac{3704}{25} = 0.$$

By the quadratic formula, we have

$$\tan\alpha = \frac{1}{208}\left(375 - \sqrt{81361}\right) \text{ and } \tan\alpha = \frac{1}{208}\left(375 + \sqrt{81361}\right).$$

Applying the inverse tangent and then writing the answer in degrees, we obtain $\alpha = 72.51$ degrees and $\alpha = 23.34$ degrees

11.7.71 Let the angle be α. Then $\mathbf{v}(t) = \langle 120\cos\alpha, 120\sin\alpha - 32t\rangle$, and $\mathbf{r}(t) = \langle 120t\cos\alpha, 120t\sin\alpha - 16t^2\rangle$. Because we require the ball to land in the hole, we need the point $(420, -50)$ to be on the curve. So $120t\cos\alpha = 420$ and $120t\sin\alpha - 16t^2 = -50$. Thus $t = \frac{420}{120\cos\alpha}$, and therefore $120 \cdot \frac{420\cdot\sin\alpha}{120\cos\alpha} - 16\left(\frac{420}{120\cos\alpha}\right)^2 = -50$. This can be written $420\tan\alpha - 196\sec^2\alpha + 50 = -196\tan^2\alpha + 420\tan\alpha - 146 = 0$. By the quadratic formula, we have

$$\tan\alpha = \frac{1}{14}\left(15 - \sqrt{79}\right) \text{ and } \tan\alpha = \frac{1}{14}\left(15 + \sqrt{79}\right).$$

Applying the inverse tangent and then writing the answer in degrees, we obtain $\alpha = 59.63$ degrees and $\alpha = 23.58$ degrees.

11.7.72 Let s be the initial speed of the ball. Note that $\mathbf{v}(t) = \langle s\sqrt{2}/2, s\sqrt{2}/2 - 32t\rangle$ and $\mathbf{r}(t) = \langle st\sqrt{2}/2, st\sqrt{2}/2 - 16t^2\rangle$. Because we want the second coordinate to be 40 when the first coordinate is 390, we have $st\sqrt{2}/2 = 390$ and $st\sqrt{2}/2 - 16t^2 = 40$. Solving the first equation for t yields $t = \frac{780}{\sqrt{2}s}$. Putting this value into the second equation yields $s \cdot \frac{780}{\sqrt{2}s} \cdot \frac{\sqrt{2}}{2} - 16\left(\frac{780}{\sqrt{2}s}\right)^2 = 40$. Solving this last equation for s yields $s = \frac{\sqrt{8}\cdot 780}{\sqrt{350}} \approx 117.9$.

11.7.73 Let s be the initial speed of the ball. Note that $\mathbf{v}(t) = \langle s\sqrt{3}/2, s/2 - 32t\rangle$ and $\mathbf{r}(t) = \langle st\sqrt{3}/2, st/2 - 16t^2\rangle$. Because we want the second coordinate to be -50 when the first coordinate is 420, we have $st\sqrt{3}/2 = 420$ and $st/2 - 16t^2 = -50$. Solving the first equation for t yields $t = \frac{840}{\sqrt{3}s}$. Putting this value into the second equation yields $s \cdot \frac{840}{\sqrt{3}s} \cdot \frac{1}{2} - 16\left(\frac{840}{\sqrt{3}s}\right)^2 = -50$. Solving this last equation for s yields $s \approx 113.4$.

11.7.74

a. $\mathbf{v}(t) = \langle 40, -9.8t\rangle$ and $\mathbf{r}(t) = \langle 40t, 8 - 4.9t^2\rangle$. Let $x = 40t$ and $y = 8 - 4.9t^2$. Then the equation of trajectory is $y = 8 - 4.9\left(\frac{x}{40}\right)^2$. The equation of the outrun surface is $y = -\frac{1}{\sqrt{3}}x$. These curves intersect when $8 - 4.9\left(\frac{x}{40}\right)^2 = -\frac{1}{\sqrt{3}}x$, which when solved for x yields 201.487, and then $y = -116.329$. The distance from the origin to the landing points is $\sqrt{201.487^2 + (-116.329)^2} \approx 232.657$ meters.

b. In this scenario, $\mathbf{v}(t) = \langle 40 - .15t, -9.8t\rangle$ and $\mathbf{r}(t) = \langle 40t - .075t^2, 8 - 4.9t^2\rangle$. Let $x = 40t - .075t^2$ and $y = 8 - 4.9t^2$. Then $\frac{8-y}{4.9} = t^2$, so $x = 40\sqrt{\frac{8-y}{4.9}} - .075(\frac{8-y}{4.9})$. Because we also have $x = -\sqrt{3}y$, we are looking for the solution to the equation $-\sqrt{3}y = 40\sqrt{\frac{8-y}{4.9}} - .075(\frac{8-y}{4.9})$. This results in $y = -114.29$ and so $x = 197.95$. Then the length of the jump is $\sqrt{(-114.29)^2 + 197.95^2} \approx 228.575$ meters.

c. $\mathbf{v} = \langle 40\cos\theta, 40\sin\theta - 9.8t\rangle$, and $\mathbf{r}(t) = \langle 40t\cos\theta, 8 + 40t\sin\theta - 4.9t^2\rangle$. Let $x = 40t\cos\theta$ and $y = 8 + 40t\sin\theta - 4.9t^2$. Then the equation of trajectory is $y = 8 + x\tan\theta - 4.9\left(\frac{x}{40\cos\theta}\right)^2$. The equation of the outrun surface is $y = -\frac{1}{\sqrt{3}}x$. These intersect when $8 + x\tan\theta - 4.9\left(\frac{x}{40\cos\theta}\right)^2 = -\frac{1}{\sqrt{3}}x$. Solving for x in terms of θ (using a computer algebra system) gives

$$x = \frac{0.5\left(1.04785\sqrt{\tan^2\theta + 1.05164\tan\theta + 0.392835} + \tan\theta + 0.57735\right)}{0.0030625\tan^2\theta + 0.0030625}.$$

This is maximized for $\theta \approx 29.41$ degrees.

11.7.75

a. $\mathbf{v} = \langle 130, 0, -3 - 32t\rangle$ and $\mathbf{r}(t) = \langle 130t, 0, 6 - 3t - 16t^2\rangle$. When $x = 60$, $t = 6/13$, so $z = 6 - 3(6/13) - 16(6/13)^2 \approx 1.207$ feet. The flight lasts $t = 6/13$ seconds.

b. Suppose that the initial velocity is $\langle 130, 0, b\rangle$, so that $\mathbf{v}(t) = \langle 130t, 0, 6 + bt - 16t^2\rangle$. So $z = 3 = 6 + b(6/13) - 16(6/13)^2$, which when solved for b gives $b = .8846$.

c. In this scenario, we have $\mathbf{v}(t) = \langle 130, 8t, -3 - 32t\rangle$ and $\mathbf{r}(t)\langle 130t, 4t^2, 6 - 3t - 16t^2\rangle$. As before, $x = 60$ when $t = 6/13$, and at that time $y = 4(6/13)^2 \approx .8521$ feet.

d. It moves more in the second half, because of the factor of t^2. This makes life more difficult for the batter!

e. In this case $\mathbf{v}(t) = \langle 130, ct, -3 - 32t\rangle$ and $\mathbf{r}(t) = \langle 130t, -3 + ct^2/2, 6 - 3t - 16t^2\rangle$. Again, we have $t = 6/13$, and so we require $-3 + c\left(\frac{6}{13}\right)^2 \cdot \frac{1}{2} = 0$, so $c \approx 28.17$.

11.7.76

a. Let $|\mathbf{v}_0| = v_0$. We have $\mathbf{v}(t) = \langle v_0 \cos\alpha, v_0 \sin\alpha - gt\rangle$ and $\mathbf{r}(t) = \langle v_0 t\cos\alpha, v_0 t \sin\alpha - (1/2)gt^2\rangle$. Let the point where the object strikes the ground be $(a, -a\tan\theta)$. If T is the time of the flight, we have $a = v_0 T\cos\alpha$ and $-a\tan\theta = v_0 T \sin\alpha - \frac{1}{2}gT^2$. Eliminating a from these two equations gives $T = \frac{2v_0}{g}(\cos\alpha\tan\theta + \sin\alpha)$. Eliminating T gives $a = \frac{2v_0^2\cos\alpha}{g}(\cos\alpha\tan\theta + \sin\alpha)$. The maximum height occurs when $v_0\sin\alpha - gt = 0$, or $t = \frac{v_0\sin\alpha}{g}$. The value of the maximum height is $y = \frac{v_0^2\sin^2\alpha}{g} - \frac{g}{2}\frac{v_0^2\sin^2\alpha}{g^2} = \frac{v_0^2\sin^2\alpha}{2g}$.

b. Again we have $\mathbf{v}(t) = \langle v_0\cos\alpha, v_0\sin\alpha - gt\rangle$ and $\mathbf{r}(t) = \langle v_0 t\cos\alpha, v_0 t\sin\alpha - (1/2)gt^2\rangle$. Let the point where the object strikes the ground be $(a, a\tan\theta)$. If T is the time of the flight, we have $a = v_0 T\cos\alpha$ and $a\tan\theta = v_0 T\sin\alpha - \frac{1}{2}gT^2$. Eliminating a from these two equations gives $T = \frac{2v_0}{g}(-\cos\alpha\tan\theta + \sin\alpha)$. Eliminating T gives $a = \frac{2v_0^2\cos\alpha}{g}(-\cos\alpha\tan\theta + \sin\alpha)$. The maximum height occurs when $v_0\sin\alpha - gt = 0$, or $t = \frac{v_0\sin\alpha}{g}$. The value of the maximum height is $y = \frac{v_0^2\sin^2\alpha}{g} - \frac{g}{2}\frac{v_0^2\sin^2\alpha}{g^2} = \frac{v_0^2\sin^2\alpha}{2g}$.

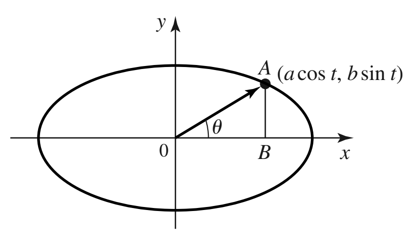
11.7.77 We have $\mathbf{v}(t) = \langle v_0\cos\alpha, v_0\sin\alpha - gt\rangle$, and $\mathbf{r}(t) = \langle v_0 t\cos\alpha, y_0 + v_0 t\sin\alpha - \frac{1}{2}gt^2\rangle$. Suppose that object hits the ground at $(a, 0)$. Then $a = v_0 T\cos\alpha$ and $0 = y_0 + v_0 T\sin\alpha - \frac{1}{2}gT^2$ where T is the time of the flight. So by the quadratic formula, $T = \frac{v_0\sin\alpha + \sqrt{v_0^2\sin^2\alpha + 2gy_0}}{g}$. Thus $a = v_0 T\cos\alpha = v_0(\cos\alpha)\left(\frac{v_0\sin\alpha + \sqrt{v_0^2\sin^2\alpha + 2gy_0}}{g}\right)$ is the range. Because the maximum height when $y_0 = 0$ is $\frac{v_0^2\sin^2\alpha}{2g}$, the maximum height in this scenario is $y_0 + \frac{v_0^2\sin^2\alpha}{2g}$.

11.7.78 We have $x = u_0 t + x_0$, and $y = -\frac{gt^2}{2} + v_0 t + y_0$. Eliminating t gives $y = -\frac{g}{2}((x - x_0)/u_0)^2 + v_0((x - x_0)/u_0) + y_0$, which is a segment of a parabola. If $y(T) = 0$, then we have $-\frac{g}{2}T^2 + v_0 T + y_0 = 0$, so $T^2 - \frac{2v_0}{g}T - \frac{2y_0}{g} = 0$, so by the quadratic formula we have $T = \frac{v_0}{g} + \sqrt{\left(\frac{v_0}{g}\right)^2 + \frac{2y_0}{g}} = \frac{v_0 + \sqrt{v_0^2 + 2gy_0}}{g}$.

11.7.79 Note that $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, and $z = cy$, so this curve is the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $z = cy$, which results in an ellipse in that plane.

11.7.80

a. In right triangle ABO, we have $\tan\theta = \frac{AB}{OB} = \frac{b\sin t}{a\cos t} = \frac{b}{a}\tan t$.



b. $\theta = \tan^{-1}((b/a)\tan t)$, so $\theta'(t) = \frac{1}{1+((b/a)\tan t)^2} \cdot \frac{b}{a}\sec^2 t = \frac{ab}{a^2\cos^2 t + b^2\sin^2 t}$.

c. $\frac{dA}{dt} = \frac{dA}{d\theta} \cdot \frac{d\theta}{dt} = \frac{1}{2}\left|\mathbf{r}(\theta(t))\right|^2 \cdot \frac{ab}{a^2\cos^2 t + b^2\sin^2 t} = \frac{1}{2}ab$.

d. Because $\frac{dA}{dt}$ is a constant, the object sweeps out equal areas in equal times as it moves about the ellipse.

11.7.81 $\mathbf{r}(t)$ can be written $\langle R\cos\phi\cos t, R\sin\phi\cos t, R\sin t\rangle$ where R is the radius of the sphere and ϕ is a real constant. Note that

$$\mathbf{v}(t) = \langle -R\cos\phi\sin t, -R\sin\phi\sin t, R\cos t\rangle$$

and $\mathbf{a}(t) = \langle -R\cos\phi\cos t, -R\sin\phi\cos t, -R\sin t\rangle$, and that $\mathbf{r}(t)\cdot\mathbf{a}(t) = -R^2\cos^2\phi\cos^2 t - R^2\sin^2\phi\cos^2 t - R^2\sin^2 t = -R^2(\cos^2\phi + \sin^2\phi)\cos^2 t - R^2\sin^2 t = -R^2(\cos^2 t + \sin^2 t) = -R^2 = -\left|\mathbf{v}(t)\right|^2$.

11.7.82

a. $\left|\mathbf{r}(t)\right|^2 = (a\cos t + b\sin t)^2 + (c\cos t + d\sin t)^2 = (a^2+c^2)\cos^2 t + (2ab+2cd)\sin t\cos t + (b^2+d^2)\sin^2 t$. In order for the path to be a circle, it would be sufficient that $a^2 + c^2 = b^2 + d^2$ and that $ab + cd = 0$.

b. In order for the path to be an ellipse, it would be sufficient that $ab + cd = 0$.

11.7.83

a. Consider the vector $\mathbf{r}(0) = \langle a, c, e\rangle$. For any t so that $0 < t < 2\pi$, consider the vector $\mathbf{r}(t)$. We will show that all such vectors lie in the same plane by showing that when crossed with $\mathbf{r}(0)$, we always get a multiple of the same vector.

Computing $\mathbf{r}(t) \times \langle a, c, e\rangle$ we obtain

$$\langle de\sin(t) - cf\sin(t), af\sin(t) - be\sin(t), bc\sin(t) - ad\sin(t)\rangle = \sin t\langle de - cf, af - be, bc - ad\rangle.$$

So for any $t \in (0, 2\pi)$, we have that $\mathbf{r}(t) \times \langle a, c, e\rangle$ is a multiple of the constant vector $\langle de - cf, af - be, bc - ad\rangle$. This can only happen if all the vectors $\mathbf{r}(t)$ lie in the same plane.

b. $\left|\mathbf{r}(t)\right|^2 = (a\cos t + b\sin t)^2 + (c\cos t + d\sin t)^2 + (e\cos t + f\sin t)^2 = (a^2+c^2+e^2)\cos^2 t + (2ab + 2cd + 2ef)\sin t\cos t + (b^2+d^2+f^2)\sin^2 t$ In order for the path to be a circle, it would be sufficient that $a^2 + c^2 + e^2 = b^2 + d^2 + f^2$ and that $ab + cd + ef = 0$. In order for the path to be an ellipse, it would be sufficient that $ab + cd + ef = 0$.

11.8 Lengths of Curves

11.8.1 $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_a^b \sqrt{1+4}\,dt = \sqrt{5}(b-a)$.

11.8.2 To compute the length of the curve, compute $L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt$.

11.8.3 The arc length is $L = \int_a^b \left|\mathbf{r}'(t)\right|\,dt = \int_a^b \left|\mathbf{v}(t)\right|\,dt$.

11.8.4 It travels a distance equal to the length of the path it traces in space: $\int_a^b \left|\mathbf{r}'(t)\right|\,dt = \int_a^b \left|\mathbf{v}(t)\right|\,dt$.

11.8.5 $L = \int_0^\pi \left|\langle -20\sin(2t), 20\cos(2t)\rangle\right|\,dt = 20\pi$.

11.8.6 $L = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$.

11.8.7 If the parameter t used to describe a trajectory also measures the arc length s of the curve that is generated, then we say that the curve is parametrized by its arc length. This occurs when $\left|\mathbf{v}(t)\right| = 1$.

11.8.8 Note that $\left|\mathbf{v}(t)\right| = \sqrt{\sin^2 t + \cos^2 t} = 1$, so it is parametrized by arc length.
The arc length is $s = \int_0^t \left|\mathbf{v}(t)\right|\,dt = \int_0^t 1\,dt = t$.

11.8.9 $L = \int_0^1 \sqrt{36t^2 + 64t^2}\,dt = \int_0^1 10t\,dt = 5t^2\Big|_0^1 = 5.$

11.8.10 $L = \int_0^1 \sqrt{9+16+1}\,dt = \sqrt{26}t\,\Big|_0^1 = \sqrt{26}.$

11.8.11 $L = \int_0^\pi \sqrt{(-3\sin t)^2 + (3\cos t)^2}\,dt = \int_0^\pi 3\,dt = 3\pi.$

11.8.12 $L = \int_0^{2\pi/3} \sqrt{(-12\sin(3t))^2 + (12\cos(3t))^2}\,dt = 12(2\pi/3) = 8\pi.$

11.8.13 Note that $(\cos t + t\sin t)' = -\sin t + \sin t + t\cos t = t\cos t$, and $(\sin t - t\cos t)' = \cos t - (\cos t - t\sin t) = t\sin t$. $L = \int_0^{\pi/2} \sqrt{(t\cos t)^2 + (t\sin t)^2}\,dt = \int_0^{\pi/2} t\,dt = \frac{t^2}{2}\,\Big|_0^{\pi/2} = \frac{\pi^2}{8}.$

11.8.14 Note that $(\cos t + \sin t)' = -\sin t + \cos t$, and $(\cos t - \sin t)' = -\sin t - \cos t$.

$$L = \int_0^{2\pi} \sqrt{(-\sin t + \cos t)^2 + (-\sin t - \cos t)^2}\,dt = \int_0^{2\pi} \sqrt{2 - 2\cos t\sin t + 2\cos t\sin t}\,dt = \sqrt{2}(2\pi).$$

11.8.15 $L = \int_1^6 \sqrt{3^2 + (-4)^2 + 3^2}\,dt = \sqrt{34}(6-1) = 5\sqrt{34}.$

11.8.16 $L = \int_0^{6\pi} \sqrt{16\sin^2 t + 16\cos^2 t + 9}\,dt = 5(6\pi) = 30\pi.$

11.8.17 $L = \int_0^{4\pi} \sqrt{1 + 64\cos^2 t + 64\sin^2 t}\,dt = \sqrt{65}(4\pi).$

11.8.18 $L = \int_0^2 \sqrt{t^2 + (2t+1)}\,dt = \int_0^2 (t+1)\,dt = \left(\frac{t^2}{2} + t\right)\Big|_0^2 = 4.$

11.8.19 $L = \int_0^{\ln 2} \sqrt{4e^{4t} + 16e^{4t} + 16e^{4t}}\,dt = \int_0^{\ln 2} 6e^{2t}\,dt = 3e^{2t}\,\Big|_0^{\ln 2} = 12 - 3 = 9.$

11.8.20 $L = \int_0^4 \sqrt{4t^2 + 9t^4}\,dt = \int_0^4 t\sqrt{4 + 9t^2}\,dt$. Let $u = 4 + 9t^2$ so that $du = 18t\,dt$. Then $L = \frac{1}{18}\int_4^{148} \sqrt{u}\,du = \frac{1}{27}\,u^{3/2}\,\Big|_4^{148} = \frac{8}{27}(37\sqrt{37} - 1).$

11.8.21 $L = \int_0^{\pi/2} \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t}\,dt = \int_0^{\pi/2} 3\sin t\cos t\,dt = \frac{3}{2}\,\sin^2 t\,\Big|_0^{\pi/2} = \frac{3}{2}.$

11.8.22 $L = \int_0^{2\pi} \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t}\,dt = 5(2\pi) = 10\pi.$

11.8.23 The speed is $\sqrt{36t^4 + 9t^4 + 225t^4} = \sqrt{270}\,t^2$. The length is thus $L = \int_0^4 \sqrt{270}\,t^2\,dt = \sqrt{270}\,\frac{t^3}{3}\,\Big|_0^4 = \sqrt{30}(64 - 0) = 64\sqrt{30}.$

11.8.24 The speed is $\sqrt{100t^2\sin^2 t^2 + 100t^2\cos^2 t^2 + 24^2t^2}\,dt = \sqrt{100t^2 + 576t^2} = 26t$. The length is thus $L = \int_0^2 26t\,dt = 13t^2\Big|_0^2 = 52.$

11.8.25 The speed is $2\sqrt{(13\cos 2t)^2 + (-12\sin 2t)^2 + (-5\sin 2t)^2} = 2\cdot\sqrt{169} = 26$. The length is thus $L = \int_0^\pi 26\,dt = 26\pi.$

11.8.26 The speed is $e^t\sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2 + 1^2} = e^t\sqrt{3}$. The length is thus $L = \int_0^{\ln 2} \sqrt{3}e^t\,dt = \sqrt{3}\,e^t\,\Big|_0^{\ln 2} = \sqrt{3}(2-1) = \sqrt{3}.$

11.8.27 $L = \int_0^{2\pi} \sqrt{4\sin^2 t + 16\cos^2 t}\, dt = \int_0^{2\pi} \sqrt{4 + 12\cos^2 t}\, dt = 2\int_0^{2\pi} \sqrt{1 + 3\cos^2 t}\, dt \approx 19.38.$

11.8.28 $L = \int_0^{2\pi} \sqrt{4\sin^2 t + 16\cos^2 t + 36\sin^2 t}\, dt = \int_0^{2\pi} \sqrt{16\sin^2 t + 16\cos^2 t + 24\sin^2 t}\, dt = 2\int_0^{2\pi} \sqrt{4 + 6\sin^2 t}\, dt \approx 32.85.$

11.8.29 $L = \int_{-2}^{2} \sqrt{1 + 64t^2}\, dt \approx 32.5.$

11.8.30 $L = \int_0^{\ln 3} \sqrt{e^{2t} + 4e^{-2t} + 1}\, dt \approx 2.73.$

11.8.31 Note that the diameter of the circle is a, and that the complete circle is traversed for $0 \le t \le \pi$. $L = \int_0^{\pi} \sqrt{(a\sin\theta)^2 + (a\cos\theta)^2}\, d\theta = \int_0^{\pi} a\, d\theta = \pi a.$

11.8.32 $L = \int_0^{2\pi} \sqrt{(2 - 2\sin\theta)^2 + 4\cos^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{8 - 8\sin\theta}\, d\theta$. By symmetry, this is

$$2\sqrt{8}\int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin\theta} \cdot \frac{\sqrt{1 + \sin\theta}}{\sqrt{1 + \sin\theta}}\, d\theta = 2\sqrt{8}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{1 + \sin\theta}}\, d\theta.$$

Let $u = 1 + \sin\theta$ so that $du = \cos\theta\, d\theta$. Then we have $L = 2\sqrt{8}\int_0^2 u^{-1/2}\, du = 4\sqrt{8} \cdot \sqrt{u}\Big|_0^2 = 4\sqrt{8} \cdot \sqrt{2} = 16.$

11.8.33 $L = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta$. Let $u = \theta^2 + 4$, so that $du = 2\theta\, d\theta$. Substituting gives $\frac{1}{2}\int_4^{4\pi^2+4} \sqrt{u}\, du = \frac{1}{2} \cdot \frac{2}{3}\left(u^{3/2}\right)\Big|_4^{4\pi^2+4} = \frac{1}{3}\left(8(\pi^2 + 1)^{3/2} - 8\right) = \frac{8}{3}\left((\pi^2 + 1)^{3/2} - 1\right).$

11.8.34 $L = \int_0^{2\pi n} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta = \sqrt{2}\int_0^{2\pi n} e^{\theta}\, d\theta = \sqrt{2} \cdot e^{\theta}\Big|_0^{2\pi n} = \sqrt{2}(e^{2\pi n} - 1).$

11.8.35 Using symmetry, $L = 8\int_0^{\pi} \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2}\, d\theta = 8\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta = 8\sqrt{2}\int_0^{\pi} \sqrt{1 + \cos\theta} \cdot \frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}}\, d\theta = 8\sqrt{2}\int_0^{\pi} \frac{\sin\theta}{\sqrt{1-\cos\theta}}\, d\theta$. Let $u = 1 - \cos\theta$ so that $du = \sin\theta\, d\theta$. Then $L = 8\sqrt{2}\int_0^2 \frac{1}{\sqrt{u}}\, du = 16\sqrt{2} \cdot \sqrt{u}\Big|_0^2 = 32.$

11.8.36 $L = \int_0^6 \sqrt{(4\theta^2)^2 + 64\theta^2}\, d\theta = \int_0^6 4\theta\sqrt{\theta^2 + 4}\, d\theta$. Let $u = \theta^2 + 4$ so that $du = 2\theta\, d\theta$. Then $L = 2\int_4^{40} u^{1/2}\, du = \frac{4}{3} \cdot u^{3/2}\Big|_4^{40} = \frac{4}{3}(40^{3/2} - 8) = \frac{32}{3}(10\sqrt{10} - 1).$

11.8.37 $L = \int_0^{\ln 8} \sqrt{4e^{4\theta} + 16e^{4\theta}}\, d\theta = 2\sqrt{5}\int_0^{\ln 8} e^{2\theta}\, d\theta = \sqrt{5} \cdot e^{2\theta}\Big|_0^{\ln 8} = \sqrt{5}(64 - 1) = 63\sqrt{5}.$

11.8.38 $L = \int_0^{\pi} \sqrt{\sin^4(\theta/2) + \sin^2(\theta/2)\cos^2(\theta/2)}\, d\theta = \int_0^{\pi} \sqrt{\sin^2(\theta/2)}\, d\theta = -2\cos(\theta/2)\Big|_0^{\pi} = 2.$

11.8.39 $L = \int_0^{\pi/2} \sqrt{\sin^6(\theta/3) + \sin^4(\theta/3)\cos^2(\theta/3)}\, d\theta = \int_0^{\pi/2} \sin^2(\theta/3)\, d\theta = \frac{1}{2}\int_0^{\pi/2} 1 - \cos(2\theta/3)\, d\theta = \frac{1}{2}\left(\theta - \frac{3}{2}\sin(2\theta/3)\right)\Big|_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} - \frac{3\sqrt{3}}{4}\right) = \frac{2\pi - 3\sqrt{3}}{8}.$

11.8.40

$$L = \int_0^{\pi/2} \sqrt{\frac{2}{(1 + \cos\theta)^2} + \left(\frac{\sqrt{2}\sin\theta}{(1 + \cos\theta)^2}\right)^2}\, d\theta = \int_0^{\pi/2} \sqrt{\frac{2(1 + \cos\theta)^2 + 2\sin^2\theta}{(1 + \cos\theta)^4}}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{\frac{4 + 4\cos\theta}{(1 + \cos\theta)^4}}\, d\theta = 2\int_0^{\pi/2} (1 + \cos\theta)^{-3/2}\, d\theta.$$

Now note that $(\sqrt{1+\cos\theta})^{-3} = (\sqrt{2}\cos(\theta/2))^{-3}$, so $L = \frac{2}{2\sqrt{2}}\int_0^{\pi/2}\sec^3(\theta/2)\,d\theta = \frac{2}{\sqrt{2}}\int_0^{\pi/4}\sec^3 u\,du = \frac{1}{\sqrt{2}}\left(\sec u\tan u + \ln|\sec u + \tan u|\right)\Big|_0^{\pi/4} = \frac{\sqrt{2}}{2}\left(\sqrt{2}+\ln(\sqrt{2}+1)\right) = 1 + \frac{\sqrt{2}}{2}\ln(\sqrt{2}+1)$.

11.8.41 Note that $|\mathbf{v}| = \sqrt{0+\cos^2 t+\sin^2 t} = 1$, so it does use arc length as its parameter.

11.8.42 Note that $|\mathbf{v}| = \sqrt{1/3+1/3+1/3} = 1$, so it does use arc length as its parameter.

11.8.43 Note that $|\mathbf{v}| = \sqrt{1+4} \neq 1$, so it doesn't use arc length as a parameter.
Consider $\mathbf{r}(s) = \langle s/\sqrt{5}, 2s/\sqrt{5}\rangle$ for $0 \le s \le 3\sqrt{5}$. This has $|\mathbf{r}'(s)| = \frac{1}{\sqrt{5}}\sqrt{1+4} = 1$, so it does use arc length as its parameter.

11.8.44 Note that $|\mathbf{v}| = \sqrt{1+4+36} \neq 1$, so it doesn't use arc length as a parameter.
Consider $\mathbf{r}(s) = \langle (s/\sqrt{41})+1, (2s/\sqrt{41})-3, 6s/\sqrt{41}\rangle$ for $0 \le s \le 10\sqrt{41}$. This has $|\mathbf{r}'(s)| = \frac{1}{\sqrt{41}}\sqrt{1+4+36} = 1$, so it does use arc length as its parameter.

11.8.45 Note that $|\mathbf{v}| = \sqrt{4\sin^2 t+4\cos^2 t} \neq 1$, so it doesn't use arc length as a parameter.
Consider $\mathbf{r}(s) = \langle 2\cos(s/2), 2\sin(s/2)\rangle$ for $0 \le s \le 4\pi$. This has $|\mathbf{r}'(s)| = \sqrt{\sin^2(s/2)+\cos^2(s/2)} = 1$, so it does use arc length as its parameter.

11.8.46 Note that $|\mathbf{v}| = \sqrt{25\sin^2 t+9\sin^2 t+16\sin^2 t} = \sqrt{25} \neq 1$, so it doesn't use arc length as a parameter.
Consider $\mathbf{r}(s) = \langle 5\cos(s/5), 3\sin(s/5), 4\sin(s/5)\rangle$ for $0 \le s \le 5\pi$. This has

$$|\mathbf{r}'(s)| = \sqrt{\sin^2(s/5)+(9/25)\cos^2(s/5)+(16/25)\cos^2(s/5)} = 1,$$

so it does use arc length as its parameter.

11.8.47 Note that $|\mathbf{v}| = \sqrt{4t^2\sin^2(t^2)+4t^2\cos^2(t^2)} = 2t \neq 1$, so it doesn't use arc length as a parameter.
Consider $\mathbf{r}(s) = \langle \cos s, \sin s\rangle$ for $0 \le s \le \pi$. This has $|\mathbf{r}'(s)| = \sqrt{\sin^2 s+\cos^2 s} = 1$, so it does use arc length as its parameter.

11.8.48 Note that $|\mathbf{v}| = \sqrt{4t^2+16t^2+64t^2} = \sqrt{84}t = 2\sqrt{21}t \neq 1$, so it doesn't use arc length as a parameter.
Consider $\mathbf{r}(s) = \langle (1/\sqrt{21})s, (2/\sqrt{21})s, (4/\sqrt{21})s\rangle$ for $\sqrt{21} \le s \le 16\sqrt{21}$. This has $|\mathbf{r}'(s)| = \frac{1}{\sqrt{21}}\sqrt{1+4+16} = 1$, so it does use arc length as its parameter.

11.8.49 Note that $|\mathbf{v}| = \sqrt{e^{2t}+e^{2t}+e^{2t}} = \sqrt{3}e^t \neq 1$, so it doesn't use arc length as its parameter.
Consider $\mathbf{r}(s) = \langle \frac{s}{\sqrt{3}}+1, \frac{s}{\sqrt{3}}+1, \frac{s}{\sqrt{3}}+1\rangle$ for $s \ge 0$. This has $|\mathbf{r}'(s)| = 1$, so it does use arc length as its parameter.

11.8.50 Note that $|\mathbf{v}| = \sqrt{(1/2)\sin^2 t+(1/2)\sin^2/2+\cos^2 t} = 1$, so it does use arc length as its parameter.

11.8.51

a. True. $L = \int_a^b S\,dt = S(b-a)$.

b. True. Both have length $L = \int_a^b \sqrt{f'(t)^2+g'(t)^2}\,dt = \int_a^b \sqrt{g'(t)^2+f'(t)^2}\,dt$.

c. True. Both have length $L = \int_a^b \sqrt{f'(t)^2+g'(t)^2}\,dt = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{(f'(u^2)(2u))^2+(g'(u^2)(2u))^2}\,du$. The equality can be seen via the substitution $u^2 = t$.

d. False. It is not the case that $|\mathbf{r}'(t)| = 1$ for all t, because $|\mathbf{r}'(t)| = \sqrt{1+40t^2}$.

11.8.52

a. $x = x_0 + t(x_1 - x_0)$, $y = y_0 + t(y_1 - y_0)$, $z = z_0 + t(z_1 - z_0)$, where $0 \le t \le 1$.

b. $L = \int_0^1 \sqrt{x'^2 + y'^2 + z'^2}\, dt = \int_0^1 \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}\, dt$. This is equal to $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$.

c. The distance formula also gives the length of this line segment to be

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

11.8.53

a. Let $x = a\cos t$, $y = b\sin t$ and $z = c\sin t$. Then $x^2 + y^2 + z^2 = a^2\cos^2 t + b^2\sin^2 t + c^2\sin^2 t = a^2\cos^2 t + (b^2 + c^2)\sin^2 t = a^2$, assuming $a^2 = b^2 + c^2$. So the curve lies on a sphere, but also note that $cy = bz$, so the curve also lies in the plane $cy - bz = 0$. So the curve is a circle centered at the origin.

b. The circle has arc length $L = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t + c^2\cos^2 t}\, dt = \sqrt{a^2}\int_0^{2\pi}\sqrt{\sin^2 t + \cos^2 t}\, dt = 2a\pi$.

c. As in exercise 69a from section 11.7, the curve describes a circle when $a^2 + c^2 + e^2 = b^2 + d^2 + f^2 = R^2$ and $ab + cd + ef = 0$. Note that $|\mathbf{r}(t)|^2 = (a\cos t + b\sin t)^2 + (c\cos t + d\sin t)^2 + (e\cos t + f\sin t)^2 = (a^2 + c^2 + e^2)\cos^2 t + (2ab + 2cd + 2ef)\sin t\cos t + (b^2 + d^2 + f^2)\sin^2 t$, so if the conditions are met, then the curve describes a circle of radius R and $L = \int_0^{2\pi}\sqrt{R^2}\, dt = 2\pi R$.

11.8.54 Assume $m \ne 0$. $\mathbf{r}'(t) = \langle mt^{m-1}, mt^{m-1}, (3m/2)t^{(3m/2-1)}\rangle$, so $|\mathbf{r}'(t)| = \frac{3|m|t^{m-1}}{2}\sqrt{\frac{4}{9} + \frac{4}{9} + t^m}$. So

$$L = \frac{3\,|m|}{2}\int_a^b t^{m-1}\sqrt{(8/9) + t^m}\, dt.$$

Let $u = \frac{8}{9} + t^m$, so that $\pm du = |m|\, t^{m-1}\, dt$. Then if $m > 0$ we have $L = \frac{3}{2}\int_{(8/9)+a^m}^{(8/9)+b^m}\sqrt{u}\, du = (8/9 + b^m)^{3/2} - (8/9 + a^m)^{3/2}$, and if $m < 0$, we have $L = \frac{3}{2}\int_{(8/9)+b^m}^{(8/9)+a^m}\sqrt{u}\, du = (8/9 + a^m)^{3/2} - (8/9 + b^m)^{3/2}$. Note that in the case $m = 0$, the curve is the constant $\mathbf{r}(t) = \langle 1, 1, 1\rangle$, so $L = 0$.

11.8.55

a. $\mathbf{r}'(t) = h'(t)\langle A, B\rangle$, so $|\mathbf{r}'(t)| = |h'(t)|\sqrt{A^2 + B^2}$. Thus, $L = \sqrt{A^2 + B^2}\int_a^b |h'(t)|\, dt$

b. $L = \sqrt{2^2 + 5^2}\int_0^4 3t^2\, dt = \sqrt{29}\, t^3\Big|_0^4 = 64\sqrt{29}$.

c. $L = \sqrt{4^2 + 10^2}\int_1^8 \left|-1/t^2\right|\, dt = \sqrt{116}\,\frac{1}{t}\Big|_8^1 = \frac{7\sqrt{29}}{4}$.

11.8.56

a. $L = \int_0^{\sqrt{8}}\sqrt{16\theta^2 + 16}\, d\theta = 4\int_0^{\sqrt{8}}\sqrt{\theta^2 + 1}\, d\theta \approx 20.5$.

b. $L(\theta) = 4\int_0^{\theta}\sqrt{\theta^2 + 1}\, d\theta = 2(\theta\sqrt{1 + \theta^2} + \ln(\theta + \sqrt{\theta^2 + 1}))$.

c. By the Fundamental Theorem of Calculus, $L'(\theta) = 4\sqrt{\theta^2 + 1} > 0$. $L''(\theta) = \frac{4\theta}{\sqrt{1+\theta^2}} \ge 0$ if $\theta \ge 0$. The arc length is increasing at an increasing rate.

11.8.57 $r'(\theta) = -ae^{-a\theta}$. Thus,

$$\begin{aligned} L = \int_0^\infty \sqrt{e^{-2a\theta} + a^2 e^{-2a\theta}}\, d\theta &= \sqrt{1+a^2}\int_0^\infty e^{-a\theta}\, d\theta \\ &= \lim_{b\to\infty} \sqrt{1+a^2}\int_0^b e^{-a\theta}\, d\theta = \frac{\sqrt{1+a^2}}{-a}\lim_{b\to\infty} e^{-a\theta}\Big|_0^b \\ &= \frac{\sqrt{1+a^2}}{-a}(0-1) = \frac{\sqrt{1+a^2}}{a}. \end{aligned}$$

11.8.58 $\int_0^\pi \sqrt{4\cos^2(3\theta) + 36\sin^2(3\theta)}\, d\theta = \int_0^\pi 2\sqrt{1+8\sin^2\theta}\, d\theta \approx 13.36.$

11.8.59 Using symmetry, we are seeking 4 times the curve traversed for $0 \le \theta \le \pi/4$. Thus, $L = 4\int_0^{\pi/4} \sqrt{6\sin 2\theta + 6\cot 2\theta \cos 2\theta}\, d\theta = 12.85.$

11.8.60 $L = \int_0^{2\pi} \sqrt{(2-4\sin\theta)^2 + 16\cos^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{20 - 16\sin\theta}\, d\theta = 26.73.$

11.8.61 $L = \int_0^{2\pi} \sqrt{(4-2\cos\theta)^2 + 4\sin^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{20 - 16\cos\theta}\, d\theta = 26.73.$

11.8.62 $x'(t) = a(1-\cos t)$ and $y'(t) = a\sin t$. So $\sqrt{x'(t)^2 + y'(t)^2} = a\sqrt{2-2\cos t} = 2a\,|\sin(t/2)|$. Thus, $L = \int_0^{2\pi} 2a\sin(t/2)\, dt = 2a\,(-2\cos(t/2))\Big|_0^{2\pi} = 8a.$

11.8.63

a. $y = -4.9t^2 + 25t$ is 0 for $t > 0$ when $-4.9t + 25 = 0$, or $t = 25/4.9 \approx 5.102$ seconds.

b. $L \approx \int_0^{5.102} \sqrt{400 + (25-9.8t)^2}\, dt.$

c. Let $u = -9.8t + 25$ so that $du = -9.8dt$. Then $L \approx \frac{1}{-9.8}\int_{25}^{-24.9996} \sqrt{400+u^2}\, du \approx 124.43$ meters.

d. $x = u_0(5.102) = 20(5.102) = 102.04$ meters.

11.8.64

a. Because $x^2 + y^2 = 1$, the path is the unit circle. The particle is at $(0,1)$ at $x = 0$, and returns there at $t = \sqrt{2\pi}$.

b. The length of the path is $2\pi = \int_0^{\sqrt{2\pi}} \sqrt{(2t\cos t^2)^2 + (2t\sin t^2)^2}\, dt = \int_0^{\sqrt{2\pi}} 2t\, dt.$

c. This particle traces out the same circle as that one, but it does it faster, completing the circle in $\sqrt{2\pi}$ time units rather than 2π time units. Note that the speed of this particle is $2t$ rather than the constant 1 which is the speed of the other particle.

d. This also traces out the same circle, over the time interval $[0, \sqrt[n]{2\pi}]$.

e. It is also 2π because the arc is the same circle.

f. The second runner would win the race. The runners occupy the same position at $t = 1$ (namely the point $(\sin(1), \cos(1))$), so that is when one passes the other.

11.8.65 Recall that a curve is parametrized by arc length exactly when $|\mathbf{v}| = 1$. We have $\mathbf{v} = \langle a, b, c\rangle$, so $|\mathbf{v}| = \sqrt{a^2+b^2+c^2}$, which is equal to one if and only if $a^2 + b^2 + c^2 = 1$.

11.8.66 Suppose $a^2 = b^2 + c^2$. Recall that a curve is parametrized by arc length exactly when $|\mathbf{v}| = 1$. We have $\mathbf{v} = \langle -a\sin t, b\cos t, c\cos t\rangle$, so

$$|\mathbf{v}| = \sqrt{a^2\cos^2 t + b^2\sin^2 t + c^2\sin^2 t} = \sqrt{a^2\cos^2 t + a^2\sin^2 t} = \sqrt{a^2},$$

which is equal to one if and only if $a^2 = b^2 + c^2 = 1$.

11.8.67 $\int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{(cf'(t))^2 + (cg'(t))^2}\, dt = |c| \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt = |c|\, L.$

11.8.68 If $r = f(\theta)$, then parametrically we have $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$. Note that $\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$ and $\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$. Thus

$$\sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2},$$

which can be written

$$\sqrt{(f'(\theta)\cos\theta)^2 + (f'(\theta)\sin\theta)^2 + (f(\theta)\sin\theta)^2 + (f(\theta)\cos\theta)^2} = \sqrt{(f'(\theta))^2 + (f(\theta))^2}.$$

Thus $L = \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta$ as desired.

11.8.69 The curve can be parametrized by $x = t$ and $y = f(t)$. Then $\mathbf{r}'(t) = \langle 1, f'(t)\rangle$, so

$$|\mathbf{v}(t)| = \sqrt{1 + f'(t)^2}.$$

Then

$$L = \int_a^b \sqrt{1 + f'(t)^2}\, dt = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

11.8.70

a. Let t_0 be a number on $[a, b]$, so that $\mathbf{r}(t_0) = \langle f(t_0), g(t_0), h(t_0)\rangle$ is a point on $\mathbf{r}$. Then $u^{-1}(t_0) = u_0$ is a number between $u^{-1}(a)$ and $u^{-1}(b)$. Note that $\mathbf{R}(u_0) = \langle f(u(u^{-1}(t_0))), g(u(u^{-1}(t_0))), h(u(u^{-1}(t_0)))\rangle$ $= \langle f(t_0), g(t_0), h(t_0)\rangle = \mathbf{r}(t_0)$. So for every point on $\mathbf{r}$ there is a corresponding point on $\mathbf{R}$.

   Now suppose that z_0 is a number between $u^{-1}(a)$ and $u^{-1}(b)$, so that $\mathbf{R}(z_0) = \langle f(u(z_0)), g(u(z_0)), h(u(z_0))\rangle$ is a point on $\mathbf{R}$. Then $u(z_0) = s_0$ is a number between a and b, and $\mathbf{r}(s_0) = \langle f(s_0), g(s_0), h(s_0)\rangle = \langle f(u(z_0)), g(u(z_0)), h(u(z_0))\rangle = \mathbf{R}(z_0)$. So for every point on $\mathbf{R}$ there is a corresponding point on $\mathbf{r}$.

   Thus the two curves represent the same sets of points.

b. Assume $u'(t) > 0$. The case $u'(t) < 0$ is similar. The length of $\mathbf{R}$ is

$$L = \int_{u^{-1}(a)}^{u^{-1}(b)} \sqrt{(f'(u)u'(t))^2 + (g'(u)u'(t))^2 + (h'(u)u'(t))^2}\, dt.$$

   Let $s = u(t)$ so that $ds = u'(t)\, dt$. Then we have $L = \int_a^b \sqrt{f'(s)^2 + g'(s)^2 + h'(s)^2}\, ds$, which is the length of $\mathbf{r}$.

11.9 Curvature and Normal Vectors

11.9.1 A straight line has zero curvature.

11.9.2 The curvature is a measure of the magnitude of the rate of change of the unit tangent vector with respect to arc length. It is a scalar function.

11.9.3 $\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$ or $\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3}$.

11.9.4 The principal unit normal vector of a curve is a vector function whose value for any point on the curve is the vector perpendicular to the tangent to the curve, having unit length, and pointing to the inside of the curve.

11.9.5 $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$.

11.9.6 $\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where $a_N = \kappa\,|\mathbf{v}|^2 = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|}$ and $a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|}$.

11.9.7 $\mathbf{B}$ is a length one vector mutually perpendicular to $\mathbf{T}$ and $\mathbf{N}$. The three vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ form a right-handed coordinate system.

11.9.8 $\mathbf{B} = \mathbf{T}\times\mathbf{N}$.

11.9.9 The torsion is the rate at which the curve moves out of the osculating lane.

11.9.10 $\tau = \frac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|r}$.

11.9.11 $\mathbf{r}'(t) = \langle 2,4,6\rangle$, so $\mathbf{T} = \frac{1}{2\sqrt{14}}\langle 2,4,6\rangle = \frac{1}{\sqrt{14}}\langle 1,2,3\rangle$.
So $\frac{d\mathbf{T}}{dt} = \langle 0,0,0\rangle$, and $\kappa = 0$.

11.9.12 $\mathbf{r}'(t) = \langle -2\sin t, -2\cos t\rangle$, so $\mathbf{T}(t) = \langle -\sin t, -\cos t\rangle$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{8}\left|\langle -2\cos t, 2\sin t, 0\rangle \times \langle -2\sin t, -2\cos t, 0\rangle\right| = \frac{1}{2}.$$

11.9.13 $\mathbf{r}'(t) = \langle 2, 4\cos t, -4\sin t\rangle$, so $\mathbf{T}(t) = \frac{1}{\sqrt{5}}\langle 1, 2\cos t, -2\sin t\rangle$. So

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{2\sqrt{5}}\left|\frac{1}{\sqrt{5}}\langle 0, -2\sin t, -2\cos t\rangle\right| = \frac{1}{5}.$$

11.9.14 $\mathbf{r}'(t) = \langle -2t\sin t^2, 2t\cos t^2\rangle$, so $\mathbf{T}(t) = \langle -\sin t^2, \cos t^2\rangle$. So

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{2t}\left|\langle -2t\cos(t^2), -2t\sin(t^2)\rangle\right| = 1.$$

11.9.15 $\mathbf{r}'(t) = \langle\sqrt{3}\cos t, \cos t, -2\sin t\rangle$, so $|\mathbf{v}(t)| = \sqrt{3\cos^2 t + \cos^2 t + 4\sin^2 t} = 2$.
Thus, $\mathbf{T}(t) = \frac{1}{2}\langle\sqrt{3}\cos t, \cos t, -2\sin t\rangle$. So

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{2}\left|\frac{1}{2}\langle -\sqrt{3}\sin t, -\sin t, -2\cos t\rangle\right| = \frac{1}{2}\cdot\frac{1}{2}\cdot 2 = \frac{1}{2}.$$

11.9.16 $\mathbf{r}'(t) = \langle 1, -\tan t\rangle$, so $|\mathbf{v}(t)| = \sqrt{1+\tan^2 t} = \sec t$. So $\mathbf{T}(t) = \langle\cos t, -\sin t\rangle$.

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \cos t\,|\langle -\sin t, -\cos t\rangle| = \cos t.$$

11.9.17 $\mathbf{r}'(t) = \langle 1, 4t\rangle$, so $|\mathbf{v}(t)| = \sqrt{16t^2+1}$. Thus, $\mathbf{T}(t) = \frac{1}{\sqrt{16t^2+1}}\langle 1, 4t\rangle$.

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{(16t^2+1)^{3/2}}\left|\langle 0,4,0\rangle\times\langle 1,4t,0\rangle\right| = \frac{1}{(16t^2+1)^{3/2}}\left|\langle 0,0,-4\rangle\right| = \frac{4}{(16t^2+1)^{3/2}}.$$

11.9.18 $\mathbf{r}'(t) = \langle -3\cos^2 t\sin t, 3\sin^2 t\cos t\rangle$, so $|\mathbf{v}(t)| = 3\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t} = 3\sqrt{\cos^2 t\sin^2 t} = 3\,|\cos t\sin t|$. Thus $\mathbf{T}(t) = \langle -\cos t, \sin t\rangle$.

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{3\,|\sin t\cos t|}\,|\langle\sin t, \cos t\rangle| = \frac{1}{3}\,|\sec t\csc t|$$

11.9.19 $\mathbf{r}'(t) = \langle \cos(\pi t^2/2), \sin(\pi t^2/2)\rangle$, so $|\mathbf{v}(t)| = 1$. So $\mathbf{T}(t) = \mathbf{v}(t) = \langle \cos(\pi t^2/2), \sin(\pi t^2/2)\rangle$.

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = 1 \cdot \left|\langle -\pi t \sin(\pi t/^2 2), \pi t \cos(\pi t^2/2)\rangle\right| = \pi t.$$

11.9.20 $\mathbf{r}'(t) = \langle \cos(t^2), \sin(t^2)\rangle$, so $|\mathbf{v}(t)| = 1$. So $\mathbf{T}(t) = \mathbf{v}(t) = \langle \cos(t^2), \sin(t^2)\rangle$.

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = 1 \cdot \left|\langle -2t \sin(t^2), 2t \cos(t^2)\rangle\right| = 2t.$$

11.9.21 $\mathbf{r}'(t) = \langle 3\sin t, 3\cos t\rangle$ and $\mathbf{r}''(t) = \langle 3\cos t, -3\sin t\rangle$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{27}\left|\langle 3\cos t, -3\sin t, 0\rangle \times \langle 3\sin t, 3\cos t, 0\rangle\right| = \frac{1}{27}\left|\langle 0,0,9\rangle\right| = \frac{1}{3}.$$

11.9.22 $\mathbf{r}'(t) = \langle 4, 3\cos t, -3\sin t\rangle$ and $\mathbf{r}''(t) = \langle 0, -3\sin t, -3\cos t\rangle$.

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{125}\left|\langle 0, -3\sin t, -3\cos t\rangle \times \langle 4, 3\cos t, -3\sin t\rangle\right| = \frac{1}{125}\left|\langle -9, 12\cos t, 12\sin t\rangle\right| = \frac{15}{125} = \frac{3}{25}.$$

11.9.23 $\mathbf{r}'(t) = \langle 2t, 1\rangle$ and $\mathbf{r}''(t) = \langle 2, 0\rangle$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{(4t^2+1)^{3/2}}\left|\langle 2,0,0\rangle \times \langle 2t,1,0\rangle\right| = \frac{1}{(4t^2+1)^{3/2}}\left|\langle 0,0,2\rangle\right| = \frac{2}{(4t^2+1)^{3/2}}.$$

11.9.24 $\mathbf{r}'(t) = \langle \sqrt{3}\cos t, \cos t, -2\sin t\rangle$ and $\mathbf{r}''(t) = \langle -\sqrt{3}\sin t, -\sin t, -2\cos t\rangle$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{8}\left|\langle -\sqrt{3}\sin t, -\sin t, -2\cos t\rangle \times \langle \sqrt{3}\cos t, \cos t, -2\sin t\rangle\right| = \frac{1}{8}\left|\langle 2, -2\sqrt{3}, 0\rangle\right| = \frac{4}{8} = \frac{1}{2}.$$

11.9.25 $\mathbf{r}'(t) = \langle -4\sin t, \cos t, -2\sin t\rangle$ and $\mathbf{r}''(t) = \langle -4\cos t, -\sin t, -2\cos t\rangle$.

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{|\langle 2,0,-4\rangle|}{\left(\sqrt{20\sin^2(t)+\cos^2(t)}\right)^3} = \frac{2\sqrt{5}}{\left(20\sin^2(t)+\cos^2(t)\right)^{3/2}}.$$

11.9.26 $\mathbf{r}'(t) = \langle e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t\rangle$ and $\mathbf{r}''(t) = \langle -2e^t\sin t, 2e^t\cos t, e^t\rangle$. $|\mathbf{v}(t)| = \sqrt{3}e^t$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{1}{3\sqrt{3}e^{3t}}\left|\langle -2e^t\sin t, 2e^t\cos t, e^t\rangle \times \langle e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t\rangle\right|$$

$$= \frac{1}{3\sqrt{3}e^{3t}}\left|e^{2t}\langle \cos t - \sin t, \cos t + \sin t, -2\rangle\right| = \frac{\sqrt{6}e^{2t}}{3\sqrt{3}e^{3t}} = \frac{\sqrt{2}}{3e^t}.$$

11.9.27 $\mathbf{r}'(t) = \langle 2\cos t, -2\sin t\rangle$, so $\mathbf{T} = \langle \cos t, -\sin t\rangle$ and $|\mathbf{T}| = 1$. $\mathbf{N}(t) = \langle -\sin t, -\cos t\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = -\sin t\cos t + \sin t \cos t = 0$.

11.9.28 $\mathbf{r}'(t) = \langle 4\cos t, 4\sin t, 10\rangle$, and $|\mathbf{r}'(t)| = 2\sqrt{29}$, so $\mathbf{T} = \frac{1}{\sqrt{29}}\langle 2\cos t, -2\sin t, 5\rangle$ and $|\mathbf{T}| = 1$. $\mathbf{N}(t) = \langle -\sin t, -\cos t, 0\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

11.9.29 $\mathbf{r}'(t) = \langle t, -3, 0\rangle$, and $|\mathbf{r}'(t)| = \sqrt{t^2+9}$, so $\mathbf{T} = \frac{1}{\sqrt{t^2+9}}\langle t, -3, 0\rangle$ and $|\mathbf{T}| = 1$. We have $\mathbf{T}'(t) = \langle \frac{9}{(\sqrt{t^2+9})^3}, \frac{3t}{(\sqrt{t^2+9})^3}, 0\rangle$, so $\mathbf{N}(t) = \frac{1}{\sqrt{t^2+9}}\langle 3, t, 0\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

11.9.30 $\mathbf{r}'(t) = \langle t, t^2\rangle$, and $|\mathbf{r}'(t)| = |t|\sqrt{t^2+1}$, so $\mathbf{T} = \frac{1}{\sqrt{t^2+1}}\langle 1, t\rangle$ and $|\mathbf{T}| = 1$.
$\mathbf{T}'(t) = \langle -\frac{t}{(\sqrt{t^2+1})^3}, \frac{1}{(\sqrt{t^2+1})^3}\rangle$, so $\mathbf{N}(t) = \frac{1}{\sqrt{t^2+1}}\langle -t, 1\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

11.9.31 $\mathbf{r}'(t) = \langle -2t\sin t^2, 2t\cos t^2\rangle$, and $|\mathbf{r}'(t)| = 2t$, so $\mathbf{T} = \langle -\sin t^2, \cos t^2\rangle$ and $|\mathbf{T}| = 1$. $\mathbf{T}'(t) = \langle -2t\cos t^2, -2t\sin t^2\rangle$, so $\mathbf{N}(t) = \langle -\cos t^2, -\sin t^2\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

11.9.32 $\mathbf{r}'(t) = \langle -3\cos^2 t\sin t, 3\sin^2 t\cos t\rangle$, and $|\mathbf{r}'(t)| = 3|\sin t\cos t|$, so $\mathbf{T} = \langle -\cos t, \sin t\rangle$ and $|\mathbf{T}| = 1$. $\mathbf{T}'(t) = \langle \sin t, \cos t\rangle$, so $\mathbf{N}(t) = \langle \sin t, \cos t\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

11.9.33 $\mathbf{r}'(t) = \langle 2t, 1\rangle$, and $|\mathbf{r}'(t)| = \sqrt{4t^2+1}$, so $\mathbf{T} = \frac{1}{\sqrt{4t^2+1}}\langle 2t, 1\rangle$ and $|\mathbf{T}| = 1$.
$\mathbf{T}'(t) = \langle \frac{2}{(\sqrt{4t^2+1})^3}, -\frac{4t}{(\sqrt{4t^2+1})^3}\rangle$, so $\mathbf{N}(t) = \frac{1}{\sqrt{4t^2+1}}\langle 1, -2t\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

11.9.34 $\mathbf{r}'(t) = \langle 1, -\tan t\rangle$, and $|\mathbf{r}'(t)| = \sec t$, so $\mathbf{T} = \langle \cos t, -\sin t\rangle$ and $|\mathbf{T}| = 1$. $\mathbf{T}'(t) = \langle -\sin t, -\cos t\rangle$, so $\mathbf{N}(t) = \langle -\sin t, -\cos t\rangle$. Note that $|\mathbf{N}| = 1$ and $\mathbf{T}\cdot\mathbf{N} = 0$.

11.9.35 $\mathbf{r}'(t) = \langle 1, 4, -6\rangle$ and $\mathbf{r}''(t) = \langle 0, 0, 0\rangle$. So $\kappa = 0$ and thus $a_N = 0$. Also, $a_T = 0$. We have $\mathbf{a} = 0\cdot\mathbf{T} + 0\cdot\mathbf{N} = \langle 0, 0, 0\rangle$.

11.9.36 $\mathbf{r}'(t) = \langle -10\sin t, -10\cos t\rangle$ and $\mathbf{r}''(t) = \langle -10\cos t, 10\sin t\rangle$. Note that $\mathbf{a}\cdot\mathbf{v} = 0$, so $a_T = 0$. $\mathbf{a}\times\mathbf{v} = \langle 0, 0, 100\rangle$, so $a_N = \frac{100}{10} = 10$. We have $\mathbf{a} = \langle -10\cos t, 10\sin t\rangle = 0\cdot\mathbf{T} + 10\mathbf{N}$.

11.9.37 In problem 26 above, we computed $|\mathbf{v}(t)| = \sqrt{3}e^t$ and $\kappa = \frac{\sqrt{2}}{3e^t}$. Also, $\mathbf{v}(t)\cdot\mathbf{a}(t) = 3e^{2t}$. Thus $a_T = \frac{3e^{2t}}{\sqrt{3}e^t} = \sqrt{3}e^t$ and $a_N = \frac{\sqrt{2}}{3e^t}\cdot 3e^{2t} = \sqrt{2}e^t$.

11.9.38 $\mathbf{r}'(t) = \langle 1, 2t\rangle$ and $\mathbf{r}''(t) = \langle 0, 2\rangle$. Note that $\mathbf{a}\cdot\mathbf{v} = 4t$, so $a_T = \frac{4t}{\sqrt{4t^2+1}}$. $\mathbf{a}\times\mathbf{v} = \langle 0, 0, -2\rangle$, so $a_N = \frac{2}{\sqrt{4t^2+1}}$. We have $\mathbf{a} = \langle 0, 2\rangle = \frac{4t}{\sqrt{4t^2+1}}\cdot\mathbf{T} + \frac{2}{\sqrt{4t^2+1}}\cdot\mathbf{N}$.

11.9.39 $\mathbf{r}'(t) = \langle 3t^2, 2t\rangle$ and $\mathbf{r}''(t) = \langle 6t, 2\rangle$. Note that $\mathbf{a}\cdot\mathbf{v} = 18t^3 + 4t$, so $a_T = \frac{18t^3+4t}{\sqrt{9t^4+4t^2}} = \frac{18t^2+4}{\sqrt{9t^2+4}}$. $\mathbf{a}\times\mathbf{v} = \langle 0, 0, 6t^2\rangle$, so $a_N = \frac{6t^2}{t\sqrt{9t^2+4}} = \frac{6t}{\sqrt{9t^2+4}}$. We have $\mathbf{a} = \langle 6t, 2\rangle = \frac{18t^2+4}{\sqrt{9t^2+4}}\cdot\mathbf{T} + \frac{6t}{\sqrt{9t^2+4}}\cdot\mathbf{N}$.

11.9.40 $\mathbf{r}'(t) = \langle -20\sin t, 20\cos t, 30\rangle$ and $\mathbf{r}''(t) = \langle -20\cos t, -20\sin t, 0\rangle$. Note that $\mathbf{a}\cdot\mathbf{v} = 0$,so $a_T = 0$. $\mathbf{a}\times\mathbf{v} = \langle -600\sin t, 600\cos t, -400\rangle$, so $a_N = \frac{200\sqrt{13}}{10\sqrt{13}} = 20$. We have $\mathbf{a} = \langle -20\cos t, -20\sin t, 0\rangle = 0\cdot\mathbf{T} + 20\mathbf{N}$.

11.9.41 $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \langle \cos t, -\sin t, 0\rangle\times\langle -\sin t, -\cos t, 0\rangle = \langle 0, 0, -1\rangle$. Because $\mathbf{B}$ is constant, $\tau = 0$.

11.9.42 $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \langle \frac{2\cos(t)}{\sqrt{29}}, \frac{-2\sin(t)}{\sqrt{29}}, \frac{5}{\sqrt{29}}\rangle\times\langle -\sin t, -\cos t, 0\rangle = \frac{1}{\sqrt{29}}\langle 5\cos t, -5\sin t, -2\rangle$. Also, $\tau = \frac{-d\mathbf{B}/dt}{ds/dt}\cdot\mathbf{N} = \langle \frac{5\sin(t)}{58}, \frac{5\cos(t)}{58}, 0\rangle\cdot\langle -\sin t, -\cos t, 0\rangle = -\frac{5}{58}$.

11.9.43 $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \langle \frac{t}{\sqrt{t^2+9}}, -\frac{3}{\sqrt{t^2+9}}, 0\rangle\times\langle \frac{3}{\sqrt{t^2+9}}, \frac{t}{\sqrt{t^2+9}}, 0\rangle = \langle 0, 0, 1\rangle$. Because $\mathbf{B}$ is constant, $\tau = 0$.

11.9.44 $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \langle \frac{1}{\sqrt{t^2+1}}, \frac{t}{\sqrt{t^2+1}}, 0\rangle\times\langle -\frac{t}{\sqrt{t^2+1}}, \frac{1}{\sqrt{t^2+1}}, 0\rangle = \langle 0, 0, 1\rangle$. Because $\mathbf{B}$ is constant, $\tau = 0$.

11.9.45 $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, -1\rangle$, so $\mathbf{T} = \langle (-2/\sqrt{5})\sin t, (2/\sqrt{5})\cos t, -1/\sqrt{5}\rangle$.

$\mathbf{r}''(t) = \langle -2\cos t, -2\sin t, 0\rangle$, so $\mathbf{N} = \langle -\cos t, -\sin t, 0\rangle$. Thus, $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \frac{1}{\sqrt{5}}\langle -\sin t, \cos t, 2\rangle$. Also, $\tau = \frac{-d\mathbf{B}/dt}{ds/dt}\cdot\mathbf{N} = -\frac{1}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}}(\langle -\cos t, -\sin t, 0\rangle\cdot\langle -\cos t, -\sin t, 0\rangle = -\frac{1}{5}$.

11.9.46 $\mathbf{r}'(t) = \langle 1, \sinh t, -\cosh t\rangle$, so $\mathbf{r}''(t) = \langle 0, \cosh t, -\sinh t\rangle$. $\mathbf{r}'(t)\times\mathbf{r}''(t) = \langle 1, \sinh t, \cosh t\rangle$. Note that $|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{1+\sinh^2 t+\cosh^2 t} = \sqrt{2}\cosh t$. Therefore, $\mathbf{B} = \frac{1}{\sqrt{2}}\langle \operatorname{sech} t, \tanh t, 1\rangle$.

Now $\mathbf{r}'''(t) = \langle 0, \sinh t, -\cosh t\rangle$ so $(\mathbf{r}'(t)\times\mathbf{r}''(t))\cdot\mathbf{r}'''(t) = \langle 1, \sinh t, \cosh t\rangle\cdot\langle 0, \sinh t, -\cosh t\rangle = -1$. Thus $\tau = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2} = -\frac{1}{2\cosh^2 t} = -\frac{1}{2}\operatorname{sech}^2 t$.

11.9.47 $\mathbf{r}'(t) = \langle 12, -5\sin t, 5\cos t\rangle$, so $\mathbf{r}''(t) = \langle 0, -5\cos t, -5\sin t\rangle$. Thus, $\mathbf{T} = \langle 12/13, (-5/13)\sin t, (5/13)\cos t\rangle$ and $\mathbf{N} = \langle 0, -\cos t, -\sin t\rangle$.
So $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \langle (5/13), (12/13)\sin t, (-12/13)\cos t\rangle$.

Also, $\tau = -\frac{d\mathbf{B}/dt}{ds/dt}\cdot\mathbf{N} = -\frac{1}{13}\cdot\frac{1}{13}(\langle 0, 12\cos t, 12\sin t\rangle\cdot\langle 0, -\cos t, -\sin t\rangle = -\frac{1}{169}\cdot(-12) = \frac{12}{169}$.

11.9.48 $\mathbf{r}'(t) = \langle t\sin t, t\cos t, 1\rangle$, and $\mathbf{r}''(t) = \langle \sin t + t\cos t, \cos t - t\sin t, 0\rangle$. Then $\mathbf{r}'(t)\times\mathbf{r}''(t) = \langle t\sin t - \cos t, \sin t + t\cos t, -t^2\rangle$. We then have $\mathbf{B} = \frac{1}{\sqrt{1+t^2+t^4}}\langle t\sin t - \cos t, \sin t + t\cos t, -t^2\rangle$.

Note that $\mathbf{r}'''(t) = \langle 2\cos t - t\sin t, -2\sin t - t\cos t, 0\rangle$. Then we have $\tau = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2} = -\frac{(2+t^2)}{1+t^2+t^4}$.

11.9.49

a. False. For example, consider $\mathbf{r}(t) = \langle \cos t, \sin t, 1\rangle$. Then $\mathbf{T} = \langle -\sin t, \cos t, 0\rangle$ and also $\mathbf{N} = \langle -\cos t, -\sin t, 0\rangle$. Note that $\mathbf{T}$ and $\mathbf{N}$ lie in the xy-plane, but $\mathbf{r}$ doesn't.

b. False. $\mathbf{T}$ does depend on the orientation, but $\mathbf{N}$ doesn't. Reversing the orientation changes $\mathbf{T}$ to $-\mathbf{T}$, but leaves $\mathbf{N}$ alone.

c. False. Note that $|\mathbf{T}|$ is independent of orientation, so $\left|\frac{d\mathbf{T}}{ds}\right|$ is too.

d. True. As we have already seen for the circle, $\mathbf{v}\cdot\mathbf{a} = 0$. Thus $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} = 0\cdot\mathbf{T} + \kappa\,|\mathbf{v}|^2\,\mathbf{N} = \frac{1}{R}\mathbf{N}$.

e. False. For example, if the car's motion is given by $\mathbf{r}(t) = \langle 60\cos t, 60\sin t\rangle$, then the speed is a constant 60, but $\mathbf{a} = \langle -60\cos t, -60\sin t\rangle \neq \langle 0, 0\rangle$.

f. False. If it lies in the xy-plane, it will have zero torsion.

g. False. If it lies in the xy-plane, it might have very large curvature but zero torsion.

11.9.50 Let $\mathbf{r}(t) = \langle t, f(t), 0\rangle$. Then $\mathbf{r}'(t) = \langle 1, f'(t), 0\rangle$ and $\mathbf{r}''(t) = \langle 0, f''(t), 0\rangle$. Note that $|\mathbf{v}(t)| = \sqrt{1 + f'(t)^2}$, and $\mathbf{a}\times\mathbf{v} = \langle 0, 0, -f''(t)\rangle$, so $\kappa = \frac{|f''(t)|}{(1+f'(t)^2)^{3/2}}$.

11.9.51 $f'(x) = 2x$ and $f''(x) = 2$, so

$$\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$$

11.9.52 $f'(x) = -\frac{x}{\sqrt{a^2-x^2}}$ and $f''(x) = -\frac{a^2}{(a^2-x^2)^{3/2}}$, so

$$\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{\left|-\frac{a^2}{(a^2-x^2)^{3/2}}\right|}{\left(1+\left(-\frac{x}{\sqrt{a^2-x^2}}\right)^2\right)^{3/2}} = \frac{a^2}{a^3} = \frac{1}{a}.$$

11.9.53 $f'(x) = 1/x$ and $f''(x) = -1/x^2$, so

$$\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{1/x^2}{(1+(1/x)^2)^{3/2}} = \frac{x}{(x^2+1)^{3/2}}.$$

11.9.54 $f'(x) = -\tan x$ and $f''(x) = -\sec^2 x$, so

$$\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{\sec^2 x}{(1+\tan^2 x)^{3/2}} = \frac{\sec^2 x}{\sec^3 x} = \cos x.$$

11.9.55 $\mathbf{r}'(t) = \langle f'(t), g'(t)\rangle$ and $\mathbf{r}''(t) = \langle f''(t), g''(t)\rangle$. So $\mathbf{a}\times\mathbf{v} = \langle f''(t), g''(t), 0\rangle \times \langle f'(t), g'(t), 0\rangle = \langle 0, 0, g'(t)f''(t) - f'(t)g''(t)\rangle$. Because $|\mathbf{v}| = \sqrt{(f'(t))^2 + (g'(t))^2}$, we have

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{|f'g'' - g'f''|}{((f')^2+(g')^2)^{3/2}}.$$

11.9.56 $f'(t) = a\cos t$ and $f''(t) = -a\sin t$, while $g'(t) = -a\sin t$ and $g''(t) = -a\cos t$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{|f'g'' - g'f''|}{((f')^2+(g')^2)^{3/2}} = \frac{\left|-a^2\cos^2 t - a^2\sin^2 t\right|}{(a^2\cos^2 t + a^2\sin^2 t)^{3/2}} = \left|\frac{a^2}{a^3}\right| = \frac{1}{|a|}.$$

11.9.57 $f'(t) = a\cos t$ and $f''(t) = -a\sin t$, while $g'(t) = -b\sin t$ and $g''(t) = -b\cos t$. So

$$\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{|f'g'' - g'f''|}{((f')^2+(g')^2)^{3/2}} = \frac{\left|-ab\cos^2 t - ab\sin^2 t\right|}{(a^2\cos^2 t + b^2\sin^2 t)^{3/2}} = \frac{|ab|}{(a^2\cos^2 t + b^2\sin^2 t)^{3/2}}.$$

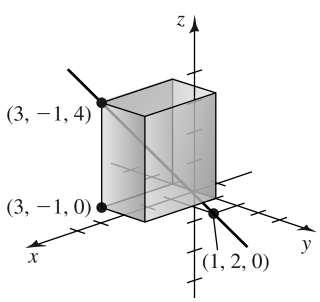
11.9.58 $f'(t) = -3a\cos^2 t \sin t$ and $f''(t) = 3a\cos t(3\sin^2 t - 1)$, while $g'(t) = 3a\sin^2 t\cos t$ and $g''(t) = 3a\sin t(3\cos^2 t - 1)$. So

$$\kappa = \frac{|f'g'' - g'f''|}{((f')^2 + (g')^2)^{3/2}} = \frac{\left|9a^2\cos^2 t\sin^2 t(3\cos^2 t - 1 + 3\sin^2 t - 1)\right|}{27a^3\left|\cos^3 t\sin^3 t\right|} = \frac{1}{3a\left|\cos t\sin t\right|}.$$

11.9.59 $f'(t) = 1$ and $f''(t) = 0$, while $g'(t) = 2at$ and $g''(t) = 2a$. So

$$\kappa = \frac{|f'g'' - g'f''|}{((f')^2 + (g')^2)^{3/2}} = \frac{|2a - 0|}{(1 + 4a^2t^2)^{3/2}} = \frac{2\,|a|}{(1 + 4a^2t^2)^{3/2}}.$$

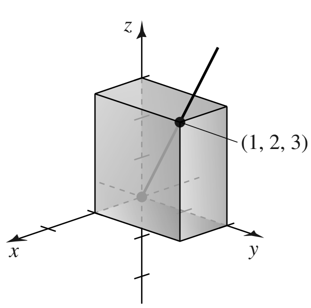
11.9.60

a. 

b. For line A: $\mathbf{v}(t) = \langle 2, -3, 4\rangle$ and $\mathbf{a}(t) = \langle 0, 0, 0\rangle$.
For line B: $\mathbf{v}(t) = \langle 6, -9, 12\rangle$ and $\mathbf{a}(t) = \langle 0, 0, 0\rangle$.
B has 3 times the velocity of A, but the same acceleration.

c. For both lines, we have $a_N = 0$ and $a_T = 0$. $\mathbf{a} = \langle 0, 0, 0\rangle = 0 \cdot \mathbf{T} = 0 \cdot \mathbf{N}$.

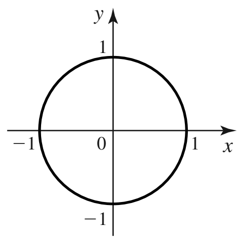
11.9.61

a. 

b. For line A: $\mathbf{v}(t) = \langle 1, 2, 3\rangle$ and $\mathbf{a}(t) = \langle 0, 0, 0\rangle$.
For line B: $\mathbf{v}(t) = \langle 2t, 4t, 6t\rangle$ and $\mathbf{a}(t) = \langle 2, 4, 6\rangle$.
A has constant velocity and zero acceleration, while B has linearly increasing velocity and constant acceleration.

c. For A, we have $a_T = a_N = 0$. For B, we have $a_T = \frac{56t}{\sqrt{56t}} = 2\sqrt{14}$ and $a_N = 0$ (because $\mathbf{v}$ and $\mathbf{a}$ are in the same direction).

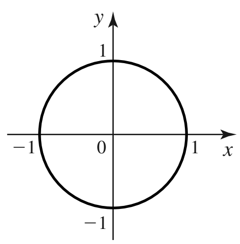
11.9.62

a. 

b. For curve A: $\mathbf{v}(t) = \langle -\sin t, \cos t\rangle$ and $\mathbf{a}(t) = \langle -\cos t, -\sin t\rangle$.
For curve B: $\mathbf{v}(t) = \langle -3\sin 3t, 3\cos 3t\rangle$ and $\mathbf{a}(t) = \langle -9\cos 3t, -9\sin 3t\rangle$.
A has constant velocity 1 while B has constant velocity 3. The magnitude of the acceleration on B is 9 times that on A.

c. For A, we have $\mathbf{a} = \mathbf{N}$, so $a_T = 0$ and $a_N = 1$. For B, we have $\mathbf{a} = 9\mathbf{N}$, so $a_T = 0$ and $a_N = 9$.

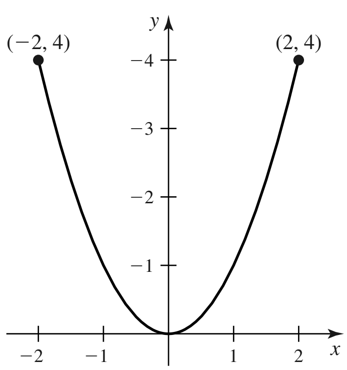
11.9.63

a. 

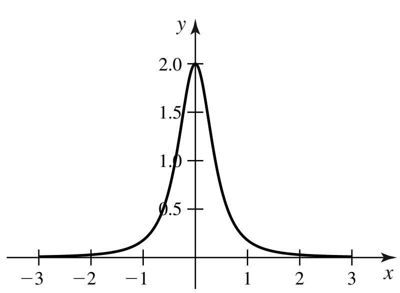
b. For curve A: $\mathbf{v}(t) = \langle -\sin t, \cos t\rangle$ and $\mathbf{a}(t) = \langle -\cos t, -\sin t\rangle$.
For curve B: $\mathbf{v}(t) = 2t\langle -\sin t^2, \cos t^2\rangle$ and $\mathbf{a}(t) = \langle -4t^2\cos t^2 - 2\sin t^2, -4t^2\sin t^2 + 2\cos t^2\rangle$.
A has constant velocity 1 while B does not have constant velocity.

c. For A, we have $\mathbf{a} = \mathbf{N}$, so $a_T = 0$ and $a_N = 1$. For B, we have $a_T = 2$ and $a_N = 4t^2$.

11.9.64

a. 

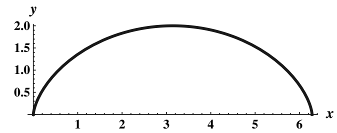
b. $\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$.

c. 

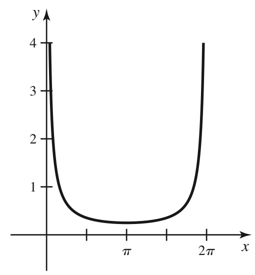
d. $\kappa'(x) = -\frac{24x}{(1+4x^2)^{5/2}}$. κ has a maximum at 0. $\kappa''(x) = \frac{24(16x^2-1)}{(1+4x^2)^{7/2}}$. κ has inflection points at $x = \pm 1/4$.

e. Symmetry of $y = x^2$ implies symmetry in κ, which does occur. The parabola appears to have greater curvature near 0 and less near the endpoints, which is consistent with the graph of κ.

11.9.65

a. 

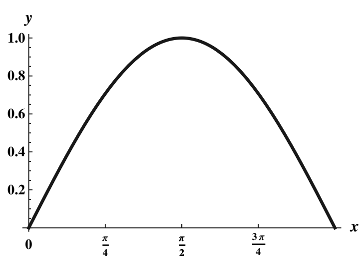
b. $\kappa = \frac{|f'g''-f''g'|}{(f'(x)^2+g'(x)^2)^{3/2}} = \frac{|(1-\cos t)\cos t - \sin t \sin t|}{((1-\cos t)^2+\sin^2 t)^{3/2}} = \frac{1-\cos t}{2\sqrt{2}(1-\cos t)^{3/2}} = \frac{1}{2\sqrt{2}\sqrt{1-\cos t}}$.

c. 

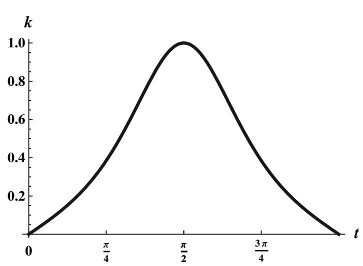
d. $\kappa'(t) = -\frac{\sin t}{4\sqrt{2}(1-\cos t)^{3/2}}$. κ has a minimum at π. $\kappa''(t) = \frac{\sin^2\left(\frac{t}{2}\right)(\cos t+3)}{4\sqrt{2}(1-\cos(t))^{5/2}}$. κ has no inflection points on the given interval.

e. Symmetry of the given curve about π (on the interval $(0, 2\pi)$ implies symmetry in κ, which does occur. The curve is flatter near π and more curved near 0 and 2π.

11.9.66

a. 

b. $\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{\sin x}{(1+\cos^2 x)^{3/2}}$.

c. 

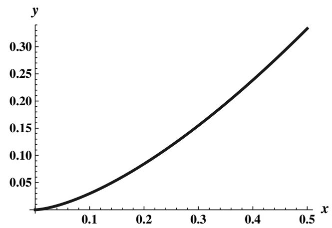
d. $\kappa'(x) = -\frac{\cos(x)(\cos(2x)-3)}{(\cos^2(x)+1)^{5/2}}$. κ has a maximum at $\pi/2$.

$$\kappa''(x) = -\frac{2\sqrt{2}(-4\sin(x) - 19\sin(3x) + \sin(5x))}{(\cos(2x)+3)^{7/2}}.$$

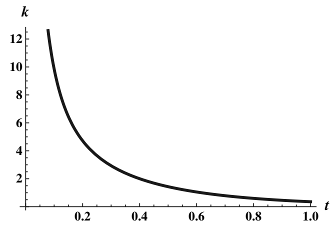
κ has inflection points at approximately 1.122 and $\pi - 1.122$.

e. Symmetry of the given curve about $\pi/2$ (on the interval $(0, \pi)$ implies symmetry in κ, which does occur. The curve is flatter near 0 and π and more curved near $\pi/2$.

11.9.67

a. 

b. $\kappa = \frac{|f'g''-f''g'|}{(f'(x)^2+g'(x)^2)^{3/2}} = \frac{|t\cdot 2t-1\cdot t^2|}{(t^2+t^4)^{3/2}} = \frac{1}{t(1+t^2)^{3/2}}$.

c. 

d. $\kappa'(t) = \frac{-4t^2-1}{t^2(t^2+1)^{5/2}}$. κ has no extrema. $\kappa''(t) = \frac{20t^4+7t^2+2}{t^3(t^2+1)^{7/2}}$. κ has no inflection points.

e. The curve gets flatter as $t \to \infty$.

11.9.68 $y'(x) = 1/x$ and $y''(x) = -1/x^2$. So $\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{1/x^2}{(1+(1/x^2))^{3/2}} = \frac{x}{(x^2+1)^{3/2}}$.

$\kappa'(x) = \frac{1-2x^2}{(x^2+1)^{5/2}}$, which is 0 for $x > 0$ only for $x = \sqrt{2}/2$. The first derivative test shows that this is where the maximum curvature exists, and the value of the maximum curvature is $\kappa(\sqrt{2}/2) = 2\sqrt{3}/9$.

11.9.69 $y'(x) = e^x$ and $y''(x) = e^x$. So $\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \frac{e^x}{(1+(e^{2x}))^{3/2}}$.

$\kappa'(x) = \frac{e^x-2e^{3x}}{(e^{2x}+1)^{5/2}}$, which is 0 for $x = -\frac{\ln 2}{2}$. The first derivative test shows that this is where the maximum curvature exists, and the value of the maximum curvature is $\kappa((-\ln 2)/2) = 2\sqrt{3}/9$.

11.9.70 By example 3, the curvature of a circle centered at the origin is the reciprocal of the radius of the circle. Because curvature is independent of the coordinate system, the curvature of the circle of curvature is the reciprocal of its radius, but by definition, its curvature is κ, so its radius is $\frac{1}{\kappa}$.

11.9.71 $\mathbf{r}'(t) = \langle 1, 2t\rangle$ and $\mathbf{r}''(t) = \langle 0, 2\rangle$. Thus $\kappa = \frac{2}{(1+4t^2)^{3/2}}$. At $t = 0$, we have $\kappa = 2$. So we are seeking a circle of radius $1/2$. The center of the osculating circle is the point along the normal line to the curve at $(0,0)$ which is $1/2$ unit from $(0,0)$ and is on the inside of the curve, so it is $(0, 1/2)$. The equation is $x^2 + (y-(1/2))^2 = \frac{1}{4}$.

11.9.72 $y' = 1/x$ and $y'' = -1/x^2$, so $\kappa(x) = \frac{|1/x^2|}{(1+(1/x)^2)^{3/2}} = \frac{x}{(x^2+1)^{3/2}}$. At $x = 1$ we have $\kappa = \frac{1}{2\sqrt{2}}$, so the radius of the osculating circle is $2\sqrt{2}$. The center of the osculating circle is the point along the the normal line to the curve at $(1,0)$ which is $2\sqrt{2}$ units from $(1,0)$ and is on the inside of the curve, so it is $(3,-2)$. The equation of the osculating circle is $(x-3)^2 + (y+2)^2 = 8$.

11.9.73 $\mathbf{r}'(t) = \langle 1-\cos t, \sin t\rangle$ and $\mathbf{r}''(t) = \langle \sin t, -\cos t\rangle$. Thus $\kappa(\pi) = \frac{(2-0)}{(2^2+0^2)^{3/2}} = \frac{2}{8} = \frac{1}{4}$, so the radius of the osculating circle is 4. The center of the osculating circle is the point along the normal line to the curve at $(\pi, 2)$ which is 4 units from $(\pi, 2)$ and is on the inside of the curve, so it is $(\pi, -2)$. The equation of the osculating circle is $(x-\pi)^2 + (y+2)^2 = 16$.

11.9.74 $y' = \cos x$ and $y'' = -\sin x$, so $\kappa(\pi/2) = \frac{|-\sin(\pi/2)|}{(1+(\cos(\pi/2))^2)^{3/2}} = 1$. Thus the radius of the osculating circle is 1. The center of the osculating circle is the point along the the normal line to the curve at $(\pi/2, 1)$ which is 1 unit from $(\pi/2, 1)$ and is on the inside of the curve, so it is $(\pi/2, 0)$. The equation of the osculating circle is $(x-\pi/2)^2 + y^2 = 1$.

11.9.75 $y' = n\cos nx$ and $y'' = -n^2 \sin nx$, so $\kappa(\pi/2n) = \frac{\left|-n^2\sin(\pi/2)\right|}{(1+(n^2\cos^2(\pi/2))^{3/2}} = \frac{n^2}{(1+0)^{3/2}} = n^2$. This increases as n increases.

11.9.76 Note that $\mathbf{r}'(t) = \langle 1, 2t\rangle$, so $\mathbf{T} = \frac{1}{\sqrt{1+4t^2}}\langle 1, 2t\rangle$. So $\mathbf{T}'(t) = \langle -\frac{4t}{(4t^2+1)^{3/2}}, \frac{2}{(4t^2+1)^{3/2}}\rangle$, so $\mathbf{N} = \frac{1}{\sqrt{4t^2+1}}\langle -2t, 1\rangle$. Then

$$\mathbf{a} = \frac{2}{\sqrt{1+4t^2}}\left(\langle -\frac{2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}}\rangle + 2t\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}\rangle\right) = \langle 0, 2\rangle.$$

11.9.77

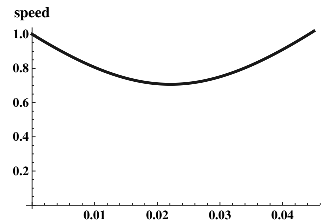
a. $\mathbf{r}'(t) = \langle V_0\cos\alpha, V_0\sin\alpha - gt\rangle$, so the speed is

$$\sqrt{V_0^2\cos^2\alpha + V_0^2\sin^2\alpha - 2gtV_0\sin\alpha + g^2t^2},$$

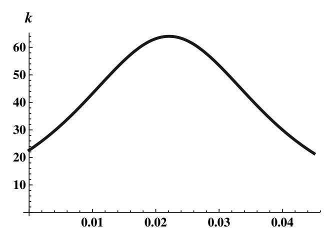
which can be written

$$\sqrt{V_0^2 - 2gtV_0\sin\alpha + g^2t^2}.$$

The graph shown is for $V_0 = 1$, $g = 32$, and $\alpha = 45$ degrees.



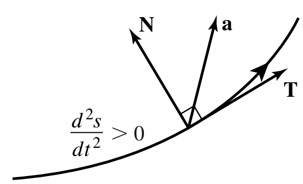
b. $\mathbf{a} = \langle 0, -g\rangle$. We have $\mathbf{a}\times\mathbf{v} = \langle 0, 0, gV_0\cos\alpha\rangle$. So $\kappa = \frac{|gV_0\cos\alpha|}{(V_0^2 - 2gtV_0\sin\alpha + g^2t^2)^{3/2}}$. The graph shown is for $V_0 = 1$, $g = 32$, and $\alpha = 45$ degrees.



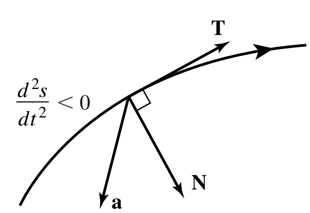
c. The speed has a minimum and the curvature has a maximum at the halfway time of the flight, namely at $\frac{V_0\sin\alpha}{g}$.

11.9.78

Suppose $\frac{d^2s}{dt^2} > 0$. Recall that $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, and that $a_T = \frac{d^2s}{dt^2} > 0$. Also $a_N = \kappa|\mathbf{v}|^2 > 0$. So $\mathbf{a}$ is a linear combination of $\vec{T}$ and $\mathbf{N}$ with positive coefficients, so it lies between the two of them.



Suppose $\frac{d^2s}{dt^2} < 0$. Recall that $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, and that $a_T = \frac{d^2s}{dt^2} < 0$. Also $a_N = \kappa|\mathbf{v}|^2 > 0$. So $\mathbf{a}$ is a linear combination of $\mathbf{T}$ and $\mathbf{N}$ with a negative coefficient on $\mathbf{T}$ and a positive coefficient on $\mathbf{N}$, so it does not lie between the two of them.



11.9.79 $\mathbf{r}'(t) = \langle pbt^{p-1}, pdt^{p-1}, pft^{p-1}\rangle = pt^{p-1}\langle b, d, f\rangle$. $\mathbf{r}''(t) = p(p-1)t^{p-2}\langle b, d, f\rangle$. Because for any t, $\mathbf{r}''(t)$ and $\mathbf{r}'(t)$ are multiples of $\langle b, d, f\rangle$, their cross product is $\langle 0, 0, 0\rangle$. Thus $\kappa = \frac{0}{|\mathbf{v}|^3} = 0$. The given curve represents a straight line, which has zero curvature.

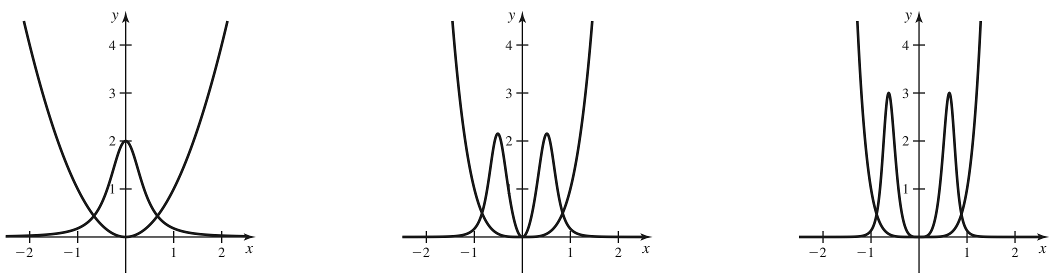
11.9.80 $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \frac{d\mathbf{T}}{ds}\left|\mathbf{r}'(t)\right|$. So

$$\frac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{\frac{d\mathbf{T}}{ds}\left|\mathbf{r}'(t)\right|}{\left|\frac{d\mathbf{T}}{ds}\left|\mathbf{r}'(t)\right|\right|} = \mathbf{N}.$$

11.9.81

a. $f_n'(x) = 2nx^{2n-1}$ and $f_n''(x) = (2n)(2n-1)x^{2n-2}$. $\kappa = \frac{\left|2n(2n-1)x^{2n-2}\right|}{(1+4n^2x^{4n-2})^{3/2}}$. So $\kappa_1(x) = \frac{2}{(1+4x^2)^{3/2}}$, $\kappa_2(x) = \frac{12x^2}{(1+16x^6)^{3/2}}$, and $\kappa_3(x) = \frac{30x^4}{(1+36x^{10})^{3/2}}$.

b.



$\kappa_1(x)$ $\kappa_2(x)$ $\kappa_3(x)$

Note that the curves are symmetric about the y-axis.

c. $\kappa'(x) = -\frac{4n(2n-1)x^{2n-1}\left(2n^2(4n-1)x^{4n}-(n-1)x^2\right)}{(4n^2x^{4n}+x^2)^2\sqrt{4n^2x^{4n-2}+1}}$. By symmetry, we can concentrate on the critical points for $x > 0$. We have $\kappa'(x) = 0$ for $x > 0$ and $n > 1$, when $x = 2^{\frac{1}{2-4n}}\left(\frac{\sqrt{n^2(4n-1)}}{\sqrt{n-1}}\right)^{\frac{1}{1-2n}}$. For $n = 1$, the maximum occurs at 0. For $n = 2$, it occurs at $\frac{1}{\sqrt{2}\cdot 7^{1/6}}$. For $n = 3$, it occurs at $\frac{1}{3^{1/5}11^{1/10}}$.

d. If the maximum curvature for f_n occurs at $\pm z_n$, then $\lim_{n\to\infty} z_n = 1$.

11.9.82

a. Write $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$. Then $\mathbf{T}\times\mathbf{a} = \mathbf{T}\times(a_T\mathbf{T}+a_N\mathbf{N}) = a_t(\mathbf{T}\times\mathbf{T}) + a_N(\mathbf{T}\times\mathbf{N}) = 0 + a_N\mathbf{B}$. Then we have $\frac{\mathbf{v}}{|\mathbf{v}|}\times\mathbf{a} = \kappa\left|\mathbf{v}\right|^2\mathbf{B}$, so $\mathbf{v}\times\mathbf{a} = \kappa\left|\mathbf{v}\right|^3\mathbf{B}$.

b. Taking norms of the last equation in part a, we have $|\mathbf{v}\times\mathbf{a}| = \kappa\left|\mathbf{v}\right|^3\cdot 1$, so $\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$.

11.9.83 We have that $\mathbf{B} = \frac{\mathbf{v}\times\mathbf{a}}{\kappa|\mathbf{v}|^3}$ and $\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$, so $\mathbf{B} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$.

11.9.84 $\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = -\frac{d\mathbf{B}/dt}{ds/dt}\cdot\mathbf{N} = -\frac{d\mathbf{B}/dt}{|\mathbf{v}|}\cdot\mathbf{N}$.

Chapter Eleven Review

1

a. True. Addition of vectors is commutative.

b. False. For example, the vector in the direction of $\mathbf{i}$ with the length of $\mathbf{j}$ is $\mathbf{i}$, but the vector in the direction of $\mathbf{j}$ with the length of $\mathbf{i}$ is $\mathbf{j}$, and $\mathbf{i} \neq \mathbf{j}$.

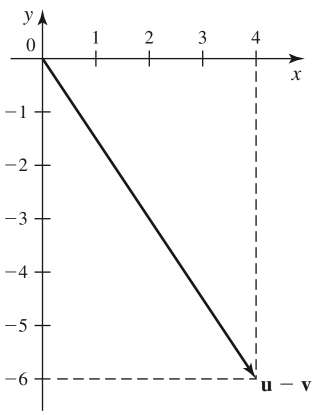
c. True, because it then follows that $\mathbf{u} = -\mathbf{v}$, so the two are parallel.

d. True. This follows because $\int \mathbf{r}'(t)\,dt = \int \langle 0,0,0\rangle\,dt = \langle 0,0,0\rangle + \mathbf{C} = \langle a,b,c\rangle$ for some real numbers a, b, and c.

e. False. Its length is $\sqrt{169\sin^2 t + 169\cos^2 t} = 13 \neq 1$.

f. False. For example, for the curve $\mathbf{r}(t) = \langle t^2, t\rangle$, we have $\mathbf{N} = \frac{1}{\sqrt{1+4t^2}}\langle 1, -2t\rangle$. So if, for example, $t = 2$, we have $\mathbf{r}(2) = \langle 4, 2\rangle$ and $\mathbf{N} = \frac{1}{\sqrt{17}}\langle 1, -4\rangle$, which aren't parallel.

2



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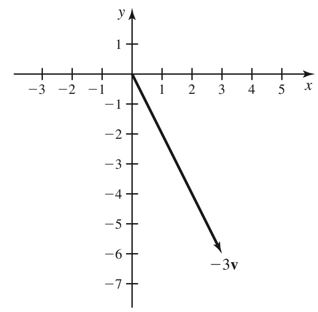

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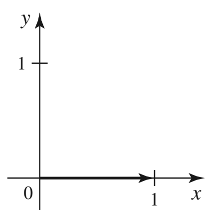

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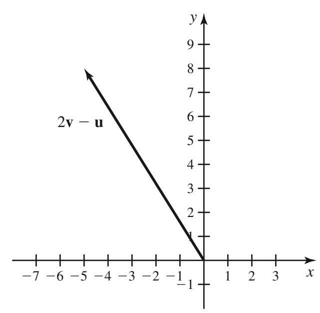


6 $\mathbf{u} - 3\mathbf{v} = \langle 2, 4, -5\rangle - \langle -18, 30, 6\rangle = \langle 20, -26, -11\rangle$.

7 $|\mathbf{u} + \mathbf{v}| = |\langle -4, 14, -3\rangle| = \sqrt{16 + 196 + 9} = \sqrt{221}$.

8 $\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{4+16+25}}\langle 2, 4, -5\rangle = \frac{1}{\sqrt{45}}\langle 2, 4, -5\rangle$.

9 $|\mathbf{v}| = \sqrt{36 + 100 + 4} = \sqrt{140}$. So the desired vector is $\frac{20}{\sqrt{140}}\langle -6, 10, 2\rangle = \frac{20}{\sqrt{35}}\langle -3, 5, 1\rangle$. The vector $-\frac{20}{\sqrt{35}}\langle -3, 5, 1\rangle$ also has the desired property.

10 $\mathbf{u} \cdot \mathbf{v} = -12 + 40 - 10 = 18$. $\cos\theta = \frac{18}{\sqrt{45}\sqrt{140}}$, so $\theta = \cos^{-1}\frac{18}{\sqrt{6300}} = 76.9$ degrees.

11 $\mathbf{u} \times \mathbf{v} = \langle 2, 4, -5\rangle \times \langle -6, 10, 2\rangle = 2\langle 29, 13, 22\rangle$. Thus, $\mathbf{v} \times \mathbf{u} = -2\langle 29, 13, 22\rangle$. The area of the indicated triangle is $\frac{1}{2} \cdot 2\sqrt{29^2 + 13^2 + 22^2} \approx 38.65$.

12 We must have $a + c = 2$, $a + b = 2$, and $b + c = 2$. Thus $a = b = c = 1$.

13

a. $\mathbf{v} = 550\langle -\sqrt{2}/2, \sqrt{2}/2\rangle = \langle -275\sqrt{2}, 275\sqrt{2}\rangle$.

b. $\mathbf{v} = 550\langle -\sqrt{2}/2, \sqrt{2}/2\rangle + \langle 0, 40\rangle = \langle -275\sqrt{2}, 275\sqrt{2} + 40\rangle$.

14

a. This is equal to $\langle 2,-8,5\rangle - \langle 2,0,6\rangle = \langle 0,-8,-1\rangle$.

b. The midpoint is $\left(\frac{2+2}{2}, \frac{-8+0}{2}, \frac{5+6}{2}\right) = (2,-4,5.5)$. The magnitude of $\overrightarrow{PM}$ is $\frac{1}{2}\,|\langle 0,-8,-1\rangle| = \frac{1}{2}\sqrt{64+1} = \frac{1}{2}\sqrt{65}$.

c. $-\frac{8}{\sqrt{65}}\langle 0,-8,-1\rangle = \frac{8}{\sqrt{65}}\langle 0,8,1\rangle$.

15 $\{(x,y,z)\,|\,(x-1)^2+y^2+(z+1)^2=16\}$.

16 $\{(x,y,z)\,|\,(x-2)^2+(y-4)^2+(z+3)^2<100\}$.

17 $\{(x,y,z)\,|\,x^2+(y-1)^2+z^2>4\}$.

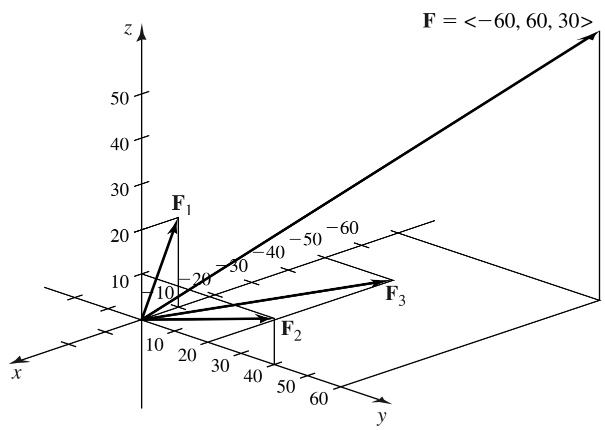
18 Completing the square gives $(x^2-6x+9)+(y^2+8y+16)+(z^2-2z+1)=23+9+16+1=49$, or $(x-3)^2+(y+4)^2+(z-1)^2=7^2$. This is a sphere of radius 7 centered at $(3,-4,1)$.

19 Completing the square gives $(x^2-x+1/4)+(y^2+4y+4)+(z^2-6z+9)<-11+1/4+4+9=\frac{9}{4}$, or $(x-1/2)^2+(y+2)^2+(z-3)^2<(3/2)^2$. This is a ball centered at $(1/2,-2,3)$ of radius $3/2$.

20 Completing the square gives $x^2+(y^2-10y+25)+(z^2-6z+9)=-34+25+9$, or $x^2+(y-5)^2+(z-3)^2=0$. This consists of the single point $(0,5,3)$.

21 Completing the square gives $(x^2-6x+9)+y^2+(z^2-20z+100)>-9+9+100$, or $(x-3)^2+y^2+(z-10)^2>10^2$. These are the points outside of a sphere of radius 10 centered at $(3,0,10)$.

22 $\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3=\langle -60,60,30\rangle$.
$|\mathbf{F}|=30\sqrt{4+4+1}=90$.



23 The magnitude of $\langle 0,4,-50\rangle$ is $\sqrt{2516}\approx 50.16$ meters per second. The direction is $\cos^{-1}(4/\sqrt{2516})\approx 85.4$ degrees below the horizontal in the northerly horizontal direction.

24 The plane's original velocity vector is $250\langle 1,0,0\rangle$, the crosswind's is $40\langle\sqrt{2}/2,\sqrt{2}/2,0\rangle$ and the downdraft's is $25\langle 0,0,-1\rangle$. The resulting velocity vector is $\langle 250+20\sqrt{2},20\sqrt{2},-25\rangle$. The speed is therefore $\sqrt{(250+20\sqrt{2})^2+(20\sqrt{2})^2+(-25)^2}\approx 280.8$ mph.

25 This is a circle of radius one centered at $(0,2,0)$ and sitting in the plane $y=2$.

26

a. $\mathbf{u}\cdot\mathbf{v}=0-3+20=17$. $|\mathbf{u}|=5$ and $|\mathbf{v}|=\sqrt{42}$. Thus the angle between the vectors is $\cos^{-1}(\frac{17}{5\sqrt{42}})\approx 1.02$ radians.

b. $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{17}{42}\langle -4, 1, 5\rangle$, and $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{17}{\sqrt{42}}$.

c. $\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{17}{25}\langle 0, -3, 4\rangle$, and $\text{scal}_{\mathbf{u}}\mathbf{v} = \frac{17}{5}$.

27

a. $\mathbf{u}\cdot\mathbf{v} = -3 + 12 + 12 = 21$, $|\mathbf{u}| = 3$ and $|\mathbf{v}| = 9$, so $\theta = \cos^{-1}\left(\frac{21}{27}\right) \approx .68$ radian.

b. $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{21}{81}\langle 3, 6, 6\rangle = \langle 7/9, 14/9, 14/9\rangle$, and $\text{scal}_{\mathbf{v}}\mathbf{u} = \frac{21}{9} = \frac{7}{3}$.

c. $\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{21}{9}\langle -1, 2, 2\rangle = \langle -7/3, 14/3, 14/3\rangle$, and $\text{scal}_{\mathbf{u}}\mathbf{v} = \frac{21}{3} = 7$.

28

a. $|\mathbf{F}_{\text{par}}| = 180\sin 30^\circ = 90$. $|\mathbf{F}_{\text{perp}}| = 180\cos 30^\circ = 90\sqrt{3}$.

b. Work $= 90\cdot 10 = 90$ foot-lbs.

29 $\langle 2, -6, 9\rangle \times \langle -1, 0, 6\rangle = \langle -36, -21, -6\rangle$. The length of this vector is $3\sqrt{12^2 + 7^2 + 2^2} = 3\sqrt{197}$. Thus the unit normals are $\frac{\pm 1}{\sqrt{197}}\langle 12, 7, 2\rangle$.

30

a. The angle is given by $\cos^{-1}(\frac{\langle 2,0,-2\rangle\cdot\langle 2,2,0\rangle}{2\sqrt{2}\cdot 2\sqrt{2}}) = \cos^{-1}(1/2) = \pi/3$.

b. The angle is also given by $\sin^{-1}(\frac{|\langle 2,0,-2\rangle\times\langle 2,2,0\rangle|}{2\sqrt{2}\cdot 2\sqrt{2}}) = \sin^{-1}(\frac{|\langle 4,-4,4\rangle|}{8}) = \sin^{-1}(4\sqrt{3}/8) = \pi/3$.

31 $T(\theta) = (.4)\cdot 98\cdot\sin\theta \approx 39.2\sin\theta$ Newton-meters. This has a maximum of 39.2 when $\sin\theta = 1$ (at $\theta = \pi/2$) and a minimum of 0 at $\theta = 0$. The direction of the torque does not change as the knee is lifted.

32 The direction is $\langle -6-2, 4-6, 0-(-1)\rangle = \langle -8, -2, 1\rangle$. So the line is described by $\langle x, y, z\rangle = \langle 2, 6, -1\rangle + t\langle 8, 2, -1\rangle$, $\infty < t < \infty$.

33 The direction of the line segment is $\langle 2-0, -8-(-3), 1-9\rangle = \langle 2, -5, -8\rangle$. The line segment is described by $\langle x, y, z\rangle = \langle 0, -3, 9\rangle + t\langle 2, -5, -8\rangle$, $0 \le t \le 1$.

34 The direction is $\langle 2, -5, 6\rangle$, so the line is given by $\langle 0, 1, 1\rangle + t\langle 2, -5, 6\rangle = \langle 2t, 1-5t, 1+6t\rangle$.

35 The direction is given by $\langle 0, -1, 3\rangle \times \langle 2, -1, 2\rangle = \langle 1, 6, 2\rangle$. Thus the line is given by $\langle 0, 1, 1\rangle + t\langle 1, 6, 2\rangle = \langle t, 1+6t, 1+2t\rangle$.

36 The direction is given by $\langle -2, 1, 7\rangle \times \langle 0, 1, 0\rangle = \langle -7, 0, -2\rangle$. Thus the line is given by $\langle 0, 1, 4\rangle + t\langle -7, 0, -2\rangle = \langle -7t, 1, 4-2t\rangle$.

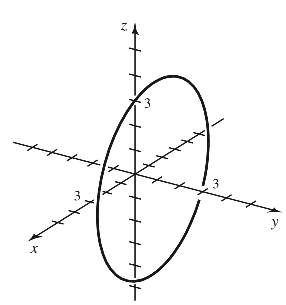
37 Two adjacent sides of the parallelogram are given by $\langle 0, -2, 3\rangle$ and $\langle 3, 0, 1\rangle$, so the area is

$$|\langle 0, -2, 3\rangle \times \langle 3, 0, 1\rangle| = |\langle -2, 9, 6\rangle| = 11.$$

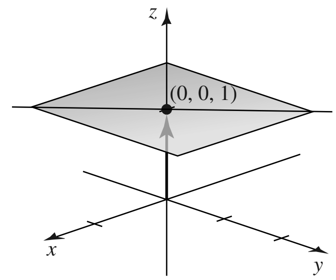
38 Two adjacent sides of the triangle are given by $\langle 4, 0, -4\rangle$ and $\langle -1, 2, -5\rangle$, so the area is

$$\frac{1}{2}\,|\langle 4, 0, -4\rangle \times \langle -1, 2, -5\rangle| = \frac{1}{2}\,|\langle 8, 24, 8\rangle| = 4\sqrt{11}.$$

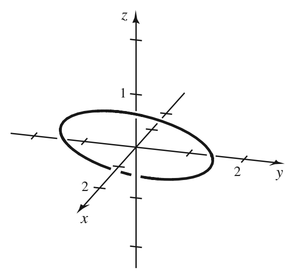
39 The curve is a circle of radius 4 with center $(0, 1, 0)$ sitting in the plane $y = 1$. It is the intersection of the plane $y = 1$ and the cylinder $x^2 + z^2 = 16$.



40 Note that if $\langle x, y, z\rangle = \langle e^t, 2e^t, 1\rangle$, then $y = 2x$ and $z = 1$. So this is a line in the plane $z = 1$.



41 Note that $x^2 + y^2 + z^2 = 2$, so this curve lies on a sphere of radius 2. Also, every point satisfies $z = x$, so it is a circle centered at the origin of radius 2, sitting in the plane $z = x$.



42

a. $\lim_{t\to 0} \mathbf{r}(t) = \langle 1, -3\rangle$ and $\lim_{t\to\infty} \mathbf{r}(t)$ does not exist.

b. $\mathbf{r}'(t) = \langle 1, 2t\rangle$ so $\mathbf{r}'(0) = \langle 1, 0\rangle$.

c. $\mathbf{r}''(t) = \langle 0, 2\rangle$.

d. $\int \mathbf{r}(t)\, dt = \langle t^2/2 + t, t^3 - 3t\rangle + \langle C_1, C_2\rangle$.

43

a. $\lim_{t\to 0} \mathbf{r}(t) = \langle 1, 0\rangle$ and $\lim_{t\to\infty} \mathbf{r}(t) = \langle 0, 1\rangle$.

b. $\mathbf{r}'(t) = \langle \frac{-2}{(2t+1)^2}, \frac{1}{(1+t)^2}\rangle$ so $\mathbf{r}'(0) = \langle -2, 1\rangle$.

c. $\mathbf{r}''(t) = \langle \frac{8}{(2t+1)^3}, \frac{-2}{(1+t)^3}\rangle$.

d. $\int \mathbf{r}(t)\, dt = \langle \frac{1}{2}\ln|2t+1|, t - \ln|t+1|\rangle + \langle C_1, C_2\rangle$.

44

a. $\lim_{t\to 0} \mathbf{r}(t) = \langle 1, 0, 0\rangle$ and $\lim_{t\to\infty} \mathbf{r}(t) = \langle 0, 0, \frac{\pi}{2}\rangle$.

b. $\mathbf{r}'(t) = \langle -2e^{-2t}, e^{-t} - te^{-t}, \frac{1}{1+t^2}\rangle$ so $\mathbf{r}'(0) = \langle -2, 1, 1\rangle$.

c. $\mathbf{r}''(t) = \langle 4e^{-2t}, te^{-t} - 2e^{-t}, \frac{2t}{(1+t^2)^2}\rangle$.

d. $\int \mathbf{r}(t)\, dt = \langle -\frac{1}{2}e^{-2t}, (1+t)e^{-t}, t\tan^{-1}(t) - \frac{1}{2}\ln\left(t^2+1\right)\rangle + \langle C_1, C_2\rangle$.

45

a. $\lim_{t\to 0} \mathbf{r}(t) = \langle 0, 3, 0\rangle$ and $\lim_{t\to\infty} \mathbf{r}(t)$ does not exist.

b. $\mathbf{r}'(t) = \langle 2\cos 2t, -12\sin 4t, 1\rangle$ so $\mathbf{r}'(0) = \langle 2, 0, 1\rangle$.

c. $\mathbf{r}''(t) = \langle -4\sin 2t, -48\cos 4t, 0\rangle$.

d. $\int \mathbf{r}(t)\,dt = \langle -\frac{1}{2}cos2t, \frac{3}{4}\sin 4t, \frac{1}{2}t^2\rangle + \langle C_1, C_2\rangle$.

46 $\mathbf{r}'(t) = \langle 0, 8\cos t, -\sin t\rangle$. The vectors $\mathbf{r}'(t)$ and $\mathbf{r}(t)$ are orthogonal when $\mathbf{r}\cdot\mathbf{r}' = 0 + 64\sin t\cos t - \sin t\cos t = 63\sin t\cos t$ is zero. This occurs when either $\sin t = 0$ or $\cos t = 0$, so at $t = 0$, $t = \pi/2$, $t = 3\pi/2$ and $t = 2\pi$, which correspond to the points $(1, 0, 1)$, $(1, 8, 0)$, $(1, 0, -1)$ and $(1, -8, 0)$.

47

a. The trajectory is given by $\mathbf{r}(t) = \langle 50t, 50t - 16t^2\rangle$. The projectile is at $y = 30$ when $-16t^2 + 50t - 30 = 0$, which occurs at $t = \frac{1}{16}\left(25 \pm \sqrt{145}\right) \approx .81$ and 2.32. At these times, $x = 50t \approx 40.5$ and 116. The first time represents when the projectile has not yet reached the cliff, while the second time represents when the projectile lands on the cliff, so the coordinates of the landing spot are approximately $(116, 30)$.

b. The maximum height occurs where $y' = 0$, which occurs for $50 - 32t = 0$, or $t = 25/16$. The maximum height is $50 \cdot \frac{25}{16} - 16\left(\frac{25}{16}\right)^2 = \frac{625}{16} \approx 39.06$ feet.

c. As mentioned above, the flight ends at $t \approx 2.32$ seconds.

d. The length of the trajectory is $\int_0^{2.32} \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{2.32} \sqrt{2500 + (50 - 32t)^2}\,dt$.

e. $L \approx 129$ feet.

f. Suppose the launch angle is α. Then $\mathbf{r}(t) = \langle 50\sqrt{2}t\cos\alpha, 50\sqrt{2}t\sin\alpha - 16t^2\rangle$. We want $y \geq 30$ when $x = 50$. We know that $x = 50$ when $t = \frac{\sec\alpha}{\sqrt{2}}$. At this time, we have $y = 50\tan\alpha - 8\sec^2\alpha$. This expression is greater than or equal to 30 for approximately $41.5 \leq \alpha \leq 79.4$.

48 The initial velocity of the ball is given by $\langle s\sqrt{3}/2, s/2\rangle$ where s is the initial speed of the ball. We have $\mathbf{r}(t) = \langle (s\sqrt{3}/2)t, -16t^2 + (s/2)t + 2\rangle$. We know that $(s\sqrt{3}/2)t = 10$ when $-16t^2 + (s/2)t + 2 = 0$. Solving the first equation for t gives $t = 20/(s\sqrt{3})$. Putting this into the second equation gives $-16(20/(s\sqrt{3}))^2 + (s/2)(20/(s\sqrt{3})) = -2$. Solving for s gives $s \approx 16.6$ feet per second.

49 The initial velocity of the ball is given by $s\langle\sqrt{2}/2, \sqrt{2}/2\rangle$ where s is the initial speed.

We have $\mathbf{r}(t) = \langle (s\sqrt{2}/2)t, -16t^2 + (s\sqrt{2}/2)t + 6\rangle$. We know that $(s\sqrt{2}/2)t = 15$ when $-16t^2 + (s\sqrt{2}/2)t + 6 = 10$. Solving the first equation for t gives $t = 30/(s\sqrt{2})$. Putting this into the second equation gives $-16(30/(s\sqrt{2}))^2 + (s\sqrt{2}/2)(30/(s\sqrt{2})) = 4$. Solving for s gives $s \approx 25.6$ feet per second.

50 $L = \int_0^2 \sqrt{81t^7 + 9t^4}\,dt = \int_0^2 3t^2\sqrt{9t^3 + 1}\,dt$. Let $u = 9t^3 + 1$ so that $du = 27t^2\,dt$. We have $L = \frac{1}{9}\int_1^{73} u^{1/2}\,du = \frac{2}{27}\left(u^{3/2}\right)\Big|_1^{73} = \frac{2}{27}(73\sqrt{73} - 1)$.

51 $L = \int_1^3 \sqrt{4t^2 + 8t + 4}\,dt = 2\int_1^3 \sqrt{(t+1)^2}\,dt = 2\int_1^3 (t+1)\,dt = 2\left(t^2/2 + t\right)\Big|_1^3 = 2(9/2 + 3 - (1/2 + 1)) = 9 + 6 - 1 - 2 = 12$.

52 $L = \int_0^{\pi/4} \sqrt{1 + \tan^2 t + \sec^2 t}\,dt = \int_0^{\pi/4} \sqrt{2}\sec t\,dt = \sqrt{2}\left(\ln|\sec t + \tan t|\right)\Big|_0^{\pi/4} = \sqrt{2}(\ln(\sqrt{2} + 1) - \ln(1 + 0)) = \sqrt{2}\ln(\sqrt{2} + 1)$.

53

a. $\mathbf{v}(t) = \int \langle 0, \sqrt{2}, 2t\rangle\,dt = \langle 0, \sqrt{2}t, t^2\rangle + \mathbf{C}$. Because $\mathbf{v}(0) = \langle 1, 0, 0\rangle$, we have $\mathbf{C} = \langle 1, 0, 0\rangle$, so $\mathbf{v}(t) = \langle 1, \sqrt{2}t, t^2\rangle$.

b. $L = \int_0^3 \sqrt{1 + 2t^2 + t^4}\,dt = \int_0^3 (t^2 + 1)\,dt = \left(t^3/3 + t\right)\Big|_0^3 = 9 + 3 = 12$.

54 $L = \int_0^{2\pi} \sqrt{(3 + 2\cos\theta)^2 + (-2\sin\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{13 + 12\cos\theta}\,d\theta \approx 21.01$.

55 $L = \int_0^{2\pi} \sqrt{(3-6\cos\theta)^2 + (6\sin\theta)^2}\,d\theta = \int_0^{2\pi} 3\sqrt{5-4\cos(\theta)}\,d\theta \approx 40.09.$

56 $|\mathbf{r}'(t)| = |\langle 4,-3\rangle| = 5$. Let $s = \int_1^t 5\,du = 5t-5$. Solving for t gives $t = \frac{s+5}{5}$. Thus, $\mathbf{r}(s) = \langle 1+4(s+5)/5, -3(s+5)/5\rangle = \langle 5+\frac{4s}{5}, -3-\frac{3s}{5}\rangle$ for $s \ge 0$.

57 $|\mathbf{r}'(t)| = \left|\langle 2t, 2\sqrt{2}t^{1/2}, 2\rangle\right| = \sqrt{4t^2+8t+4} = 2\sqrt{(t+1)^2} = 2(t+1)$. Let $s = \int_0^t 2(u+1)\,du = \left(u^2+2u\right)\Big|_0^t = t^2+2t$. Then $s+1=(t+1)^2$, so $t = \sqrt{s+1}-1$. Thus, $\mathbf{r}(s) = \langle(\sqrt{s+1}-1)^2, \frac{4\sqrt{2}}{3}(\sqrt{s+1}-1)^{3/2}, 2(\sqrt{s+1}-1)\rangle$ for $s \ge 0$.

58

a. $\mathbf{r}'(t) = \langle -3\sin t, 4\cos t\rangle$, so $\mathbf{T} = \frac{1}{\sqrt{9+7\cos^2 t}}\langle -3\sin t, 4\cos t\rangle$. Because $\mathbf{N}$ is a unit vector perpendicular to $\mathbf{T}$ pointing toward the inside of the ellipse, it is $\mathbf{N} = \frac{1}{\sqrt{9+7\cos^2 t}}\langle -4\cos t, -3\sin t\rangle$. All of these are valid for $0 \le t \le 2\pi$.

b. $|\mathbf{r}'(t)| = \sqrt{9+7\cos^2 t}$. The derivative of this is $-\frac{7\cos t\sin t}{\sqrt{9+7\cos^2 t}}$, which is 0 at $t=0$, $t=\pi/2$, $t=\pi$, and $t=3\pi/2$. The speed is maximal at $t=0$ and $t=\pi$ and minimal at $t=\pi/2$ and $t=3\pi/2$.

c. $\kappa = \frac{|\mathbf{r}''(t)\times\mathbf{r}'(t)|}{(9+7\cos^2 t)^{3/2}} = \frac{|\langle 0,0,-12\rangle|}{(9+7\cos^2 t)^{3/2}} = \frac{12}{(9+7\cos^2 t)^{3/2}}$. The curvature is maximal at $t=\pi/2$ and $t=3\pi/2$ where the denominator is minimized, and is minimal at $t=0$ and $t=\pi$ where the denominator is maximized. Note that the velocity is maximized where the curvature is minimal, and vice versa.

d. In order for $\mathbf{r}$ and $\mathbf{N}$ to be parallel , we require $3\cos t = m\cdot -\frac{4\cos t}{\sqrt{9+7\cos^2 t}}$ and $4\sin t = m\cdot -\frac{3\sin t}{\sqrt{9+7\cos^2 t}}$ for some constant m. This only occurs when either $\sin t = 0$ or $\cos t = 0$. (If $\sin t = 0$, then $m=-3$ and if $\cos t = 0$ then $m=-4$.) So the corresponding points on $\mathbf{r}$ are $(3,0)$, $(0,4)$, $(-3,0)$, and $(0,-4)$.

59

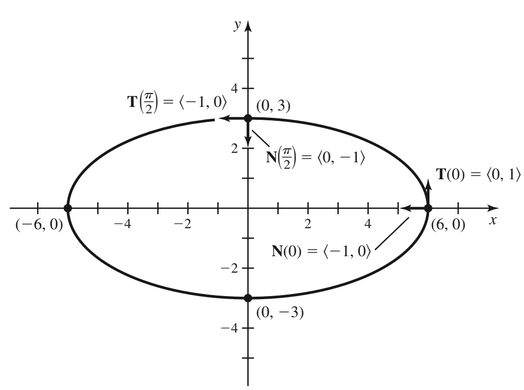
a. $\mathbf{r}'(t) = \langle -6\sin t, 3\cos t\rangle$, so $\mathbf{T} = \frac{1}{\sqrt{1+3\sin^2 t}}\langle -2\sin t, \cos t\rangle$.

b. $\kappa = \frac{|\mathbf{r}''(t)\times\mathbf{r}'(t)|}{(3\sqrt{1+3\sin^2 t})^3} = \frac{|\langle 0,0,-18\rangle|}{(3\sqrt{1+3\sin^2 t})^3} = \frac{2}{3(\sqrt{1+3\sin^2 t})^3}$.

c. Note that $\frac{1}{\sqrt{1+3\sin^2 t}}\langle -\cos t, -2\sin t\rangle$ has length one, and is perpendicular to $\mathbf{T}$ (see part [d]), and points to the inside of the curve, so it is $\mathbf{N}$.

d. $\left|\frac{1}{\sqrt{1+3\sin^2 t}}\langle -\cos t, -2\sin t\rangle\right| = \frac{\sqrt{1+3\sin^2 t}}{\sqrt{1+3\sin^2 t}} = 1$ and $\mathbf{T}\cdot\mathbf{N} = \frac{1}{\sqrt{1+3\sin^2 t}}(2\sin t\cos t - 2\sin t\cos t) = 0$.

e.



60

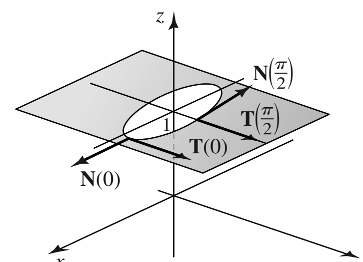
a. $\mathbf{r}'(t) = \langle -\sin t, 2\cos t, 0\rangle$, so $\mathbf{T} = \frac{1}{\sqrt{1+3\cos^2 t}}\langle -\sin t, 2\cos t, 0\rangle$.

b. $\kappa = \frac{|\mathbf{r}''(t)\times\mathbf{r}'(t)|}{(\sqrt{1+3\cos^2 t})^3} = \frac{|\langle 0,0,-2\rangle|}{(\sqrt{1+3\cos^2 t})^3} = \frac{2}{(\sqrt{1+3\cos^2 t})^3}$.

c. Note that $\frac{1}{\sqrt{1+3\cos^2 t}}\langle -2\cos t, -\sin t, 0\rangle$ has length one, and is perpendicular to $\mathbf{T}$ (see part [d]), and points to the inside of the curve, so it is $\mathbf{N}$.

d. $\left|\frac{1}{\sqrt{1+3\cos^2 t}}\langle -2\cos t, -\sin t\rangle\right| = \frac{\sqrt{1+3\cos^2 t}}{\sqrt{1+3\cos^2 t}} = 1$ and $\mathbf{T}\cdot\mathbf{N} = \frac{1}{\sqrt{1+3\cos^2 t}}(2\sin t\cos t - 2\sin t\cos t) = 0$.

e.



61

a. $\mathbf{r}'(t) = \langle -\sin t, -2\sin t, \sqrt{5}\cos t\rangle$, so

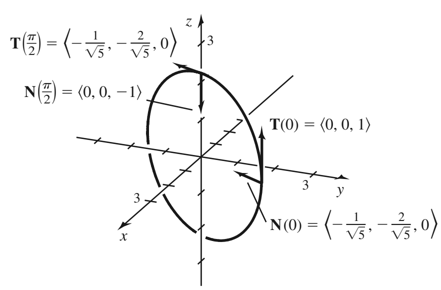
$$\mathbf{T} = \frac{1}{\sqrt{5}}\langle -\sin t, -2\sin t, \sqrt{5}\cos t\rangle.$$

b. $\kappa = \frac{|\mathbf{r}''(t)\times\mathbf{r}'(t)|}{(\sqrt{5})^3} = \frac{|\langle -2\sqrt{5},\sqrt{5},0\rangle|}{(\sqrt{5})^3} = \frac{1}{\sqrt{5}}$.

c. $\frac{dT}{dt} = \frac{1}{\sqrt{5}}\langle -\cos t, -2\cos t, -\sqrt{5}\sin t\rangle = \mathbf{N}$.

d. $\left|\frac{1}{\sqrt{5}}\langle -\cos t, -2\cos t, -\sqrt{5}\sin t\rangle\right| = \frac{\sqrt{5}}{\sqrt{5}} = 1$ and $\mathbf{T}\cdot\mathbf{N} = \frac{1}{\sqrt{5}}(\sin t\cos t + 4\sin t\cos t - 5\sin t\cos t) = 0$.

e.



62

a. $\mathbf{r}'(t) = \langle 1, -2\sin t, 2\cos t\rangle$, so

$$\mathbf{T} = \frac{1}{\sqrt{5}}\langle 1, -2\sin t, 2\cos t\rangle.$$

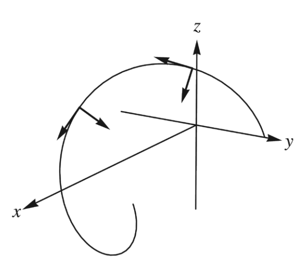
b. $\kappa = \frac{|\mathbf{r}''(t)\times\mathbf{r}'(t)|}{(\sqrt{5})^3} = \frac{|\langle -4, -2\sin(t), 2\cos(t)\rangle|}{(\sqrt{5})^3} = \frac{2}{5}$.

c. $\frac{dT}{dt} = \frac{1}{\sqrt{5}}\langle 0, -2\cos t, -2\sin t\rangle$, so

$$\mathbf{N} = \langle 0, -\cos t, -\sin t\rangle.$$

d. $|\mathbf{N}| = \sqrt{\cos^2 t + \sin^2 t} = 1$ and $\mathbf{T}\cdot\mathbf{N} = \frac{1}{\sqrt{5}}(0 + 2\sin t\cos t - 2\sin t\cos t) = 0$.

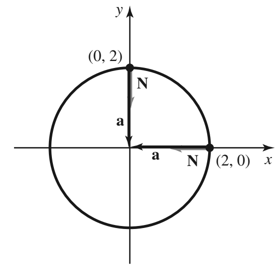
e.



63

a. $\mathbf{v}(t) = \langle -2\sin t, 2\cos t\rangle$, and $\mathbf{a}(t) = \langle -2\cos t, -2\sin t\rangle$. Because $\mathbf{v}\cdot\mathbf{a} = 0$, we have $a_T = 0$. Note that $\mathbf{a}\times\mathbf{v} = \langle -2\cos t, -2\sin t, 0\rangle \times \langle -2\sin t, 2\cos t, 0\rangle = \langle 0,0,-4\rangle$, so $a_N = \frac{4}{2} = 2$. We have $\mathbf{a} = \langle -2\cos t, -2\sin t\rangle = 2\mathbf{N} + 0\cdot\mathbf{T}$.

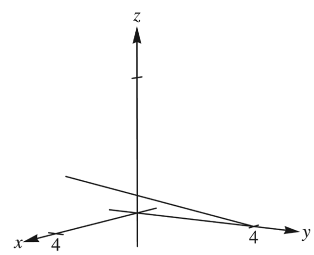
b. Note that at $t = 0$ we have $\mathbf{a} = \langle -2, 0\rangle = 2\langle -1, 0\rangle = 2\mathbf{N}$, and at $t = \pi/2$ we have $\mathbf{a} = \langle 0, -2\rangle = 2\langle 0, -1\rangle = 2\mathbf{N}$.



64

a. $\mathbf{v}(t) = \langle 3, -1, 1\rangle$, and $\mathbf{a}(t) = \langle 0, 0, 0\rangle$. Because $\mathbf{v} \cdot \mathbf{a} = 0$, we have $a_T = 0$. Note that $\mathbf{a} \times \mathbf{v} = \langle 0, 0, 0\rangle$, so $a_N = 0$. We have $\mathbf{a} = \langle 0, 0, 0\rangle = 0 \cdot \mathbf{N} + 0 \cdot \mathbf{T}$.

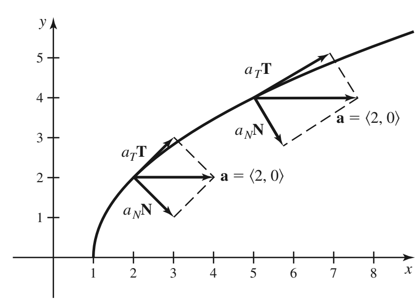
b. For all t, we have $\mathbf{a} = \langle 0, 0, 0\rangle = 0 \cdot \mathbf{N} + 0 \cdot \mathbf{T}$.



65

a. $\mathbf{v}(t) = \langle 2t, 2\rangle$, and $\mathbf{a}(t) = \langle 2, 0\rangle$. Because $\mathbf{v} \cdot \mathbf{a} = 4t$, we have $a_T = \frac{4t}{2\sqrt{t^2+1}} = \frac{2t}{\sqrt{t^2+1}}$. Note that $\mathbf{a} \times \mathbf{v} = \langle 0, 0, 4\rangle$, so $a_N = \frac{4}{2\sqrt{t^2+1}} = \frac{2}{\sqrt{t^2+1}}$. We have that $\mathbf{a} = \langle 2, 0\rangle = \frac{2}{\sqrt{t^2+1}}\mathbf{N} + \frac{2t}{\sqrt{t^2+1}}\mathbf{T}$.

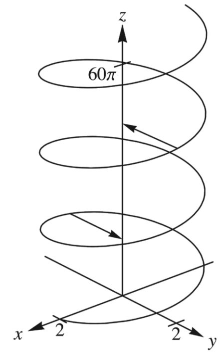
b. At $t = 1$, we have $\mathbf{a} = \langle 2, 0\rangle = \frac{2}{\sqrt{2}}\mathbf{T} + \frac{2}{\sqrt{2}}\mathbf{N} = \frac{2}{\sqrt{2}}\langle \sqrt{2}/2, \sqrt{2}/2\rangle + \frac{2}{\sqrt{2}}\langle \sqrt{2}/2, -\sqrt{2}/2\rangle$.
At $t = 2$, we have $\mathbf{a} = \langle 2, 0\rangle = \frac{4}{\sqrt{5}}\langle 2/\sqrt{5}, 1/\sqrt{5}\rangle + \frac{2}{\sqrt{5}}\langle 1/\sqrt{5}, -2/\sqrt{5}\rangle$.



66

a. $\mathbf{v}(t) = \langle -2\sin t, 2\cos t, 10\rangle$, and $\mathbf{a}(t) = \langle -2\cos t, -2\sin t, 0\rangle$. Because $\mathbf{v} \cdot \mathbf{a} = 0$, we have $a_T = 0$. Note that $\mathbf{a} \times \mathbf{v} = \langle -20\sin(t), 20\cos(t), -4\rangle$, so $a_N = \frac{4\sqrt{26}}{2\sqrt{26}} = 2$.

b. At $t = \pi/2$, we have $\mathbf{a} = \langle 0, -2, 0\rangle = 0 \cdot \mathbf{T} + 2\mathbf{N} = 2\langle 0, -1, 0\rangle$.
At $t = 3\pi/2$, we have $\mathbf{a} = \langle 0, 2, 0\rangle = 0 \cdot \mathbf{T} + 2\mathbf{N} = 2\langle 0, 1, 0\rangle$.



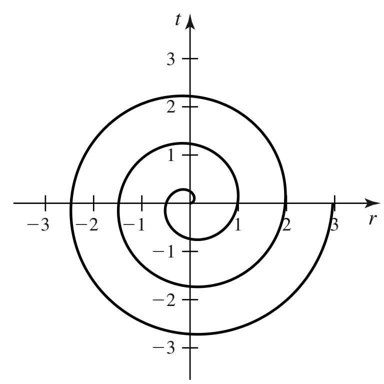
67

a. We are looking for points (x, y) so that $\langle x - x_0, y - y_0\rangle \cdot \langle a, b\rangle = 0$, so $a(x - x_0) + b(y - y_0) = 0$, or $ax + by = ax_0 + by_0$.

b. Note that $\langle a, b, 0\rangle \times \langle x - x_0, y - y_0, 0\rangle = \langle 0, 0, a(y - y_0) - b(x - x_0)\rangle$. This is equal to the zero vector when $ay - ay_0 = bx - bx_0$, or $ay - bx = ay_0 - bx_0$. So the equation of a line passing through (x_0, y_0) and parallel to $\langle a, b\rangle$ is given by $ay - bx = ay_0 - bx_0$.

68

a. The curve makes "one loop" for every 2π radians, so for $0 \le \theta \le 2\pi N$, it makes N loops. When $\theta = 2\pi N$, we have that the radius of the whole spiral is $R = tN$.



b. $L = \int_0^{2\pi N} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{2\pi N} \sqrt{\left(\frac{t\theta}{2\pi}\right)^2 + \left(\frac{t}{2\pi}\right)^2}\, d\theta = \frac{t}{2\pi}\int_0^{2\pi N} \sqrt{\theta^2 + 1}\, d\theta$.

c. Note that $\lim_{\theta\to\infty} \frac{\theta^2+1}{\theta^2} = 1$, so when θ is large $\theta^2 \approx \theta^2 + 1$. So $L \approx \frac{t}{2\pi}\int_0^{2\pi N} \theta\, d\theta = \frac{t}{2\pi} \cdot (2\pi N)^2 \cdot \frac{1}{2} = t\pi N^2$. Because $N = \frac{R}{t}$, we have $L \approx \frac{\pi R^2}{t}$.

d. Let $t\theta_1/(2\pi) = r$ and $t\theta_2/(2\pi) = R$. We have $\theta_1 = \frac{2\pi r}{t} = \frac{2\pi \cdot .025}{1.5\cdot 10^{-6}} = \frac{\pi}{3} \cdot 10^5$, and $\theta_2 = \frac{2\pi R}{t} = \frac{2\pi\cdot 0.059}{1.5\cdot 10^{-6}} \approx 2.36 \cdot \frac{\pi}{3} \cdot 10^5$.

e. $L \approx \frac{1.5\cdot 10^{-6}}{2\pi}\int_{(\pi/3)\cdot 10^5}^{(2.36\pi/3)\cdot 10^5} \theta\, d\theta \approx 5981.6$ meters. This is about 598,160 centimeters, and about 3.7 miles.

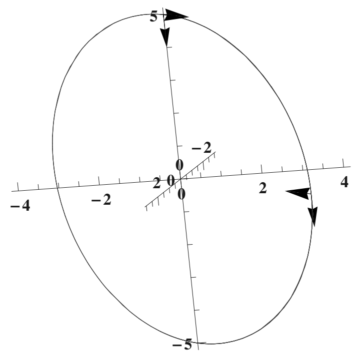
69 We have $\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$, so $\mathbf{r}'(1) = \langle 1, 2, 3\rangle$. Thus, $\mathbf{T}(1) = \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle$.
Then $\mathbf{N}(1) = \langle -\frac{11}{\sqrt{266}}, -4\sqrt{\frac{2}{133}}, \frac{9}{\sqrt{266}}\rangle$, so $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \langle \frac{3}{\sqrt{19}}, -\frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}}\rangle$. Also, $\tau = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2} = \frac{3}{19}$.

70

a. $\mathbf{r}'(t) = \langle 3\cos t, 4\cos t, -5\sin t\rangle$, and $|\mathbf{r}'(t)| = 5$, so $\mathbf{T}(t) = \frac{1}{5}\langle 3\cos t, 4\cos t, -5\sin t\rangle$.

b. $\mathbf{T}'(t) = \frac{1}{5}\langle -3\sin t, -4\sin t, -5\cos t\rangle$, which has length one, so $\mathbf{N}(t) = \frac{1}{5}\langle -3\sin t, -4\sin t, -5\cos t\rangle$.

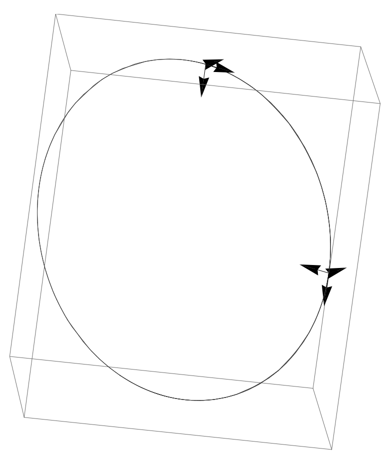
c. At $t = 0$ we have $\mathbf{T} = \langle 3/5, 4/5, 0\rangle$ and $\mathbf{N} = \langle 0, 0, -1/5\rangle$. At $t = \pi/2$ we have $\mathbf{T} = \langle 0, 0, -1\rangle$ and $\mathbf{N} = \langle -3/5, -4/5, 0\rangle$.



d. Yes.

e. $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \langle -\frac{4}{5}, \frac{3}{5}, 0\rangle$.

f.



g. One should check that $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are all of unit length and are mutually orthogonal.

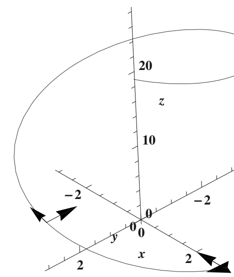
h. Because $\mathbf{B}$ is constant, $\tau = 0$.

71

a. $\mathbf{r}'(t) = \langle 3\cos t, -3\sin t, 4\rangle$, and $|\mathbf{r}'(t)| = 5$, so $\mathbf{T}(t) = \frac{1}{5}\langle 3\cos t, -3\sin t, 4\rangle$.

b. $\mathbf{T}'(t) = \frac{1}{5}\langle -3\sin t, -3\cos t, 0\rangle$, so $\mathbf{N}(t) = \langle -\sin t, -\cos t, 0\rangle$.

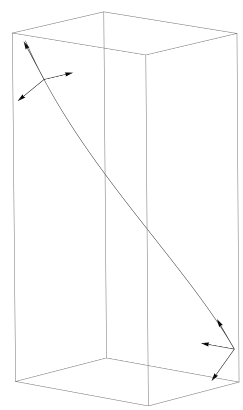
c. At $t = 0$ we have $\mathbf{T} = \langle 3/5, 0, 4/5\rangle$ and $\mathbf{N} = \langle 0, -1, 0\rangle$. At $t = \pi/2$ we have $\mathbf{T} = \langle 0, -3/5, 4/5\rangle$ and $\mathbf{N} = \langle -1, 0, 0\rangle$.



d. Yes.

e. $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{5}\langle 4\cos t, -4\sin t, -3\rangle$.

f.



g. One should check that $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are all of unit length and are mutually orthogonal.

h. $\tau = -\frac{d\mathbf{B}/dt}{ds/dt} \cdot \mathbf{N} = \frac{1}{25}\langle -4\sin t, -4\cos t, 0\rangle \cdot \langle -\sin t, -\cos t, 0\rangle = -\frac{4}{25}$.

72

a. $\mathbf{v}(t) = \langle 2a_1t + b_1, 2a_2t + b_2, 2a_3t + b_3\rangle$ and $\mathbf{a}(t) = \langle 2a_1, 2a_2, 2a_3\rangle$. Thus $\mathbf{v} \times \mathbf{a} = 2\langle a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2\rangle$, which is constant. Thus $\mathbf{B} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|}$ is a constant, so $\tau = 0$.

b. $\mathbf{a}'(t) = \langle 0, 0, 0\rangle$, so $\tau = \frac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|^2} = 0$.

73

a. First consider the case where $a_3 = b_3 = c_3 = 0$. Let $t \neq s$ be in the interval I, and consider $\mathbf{r}(t) \times \mathbf{r}(s)$. We will show that this vector is always a multiple of the same constant vector. We have

$$\begin{aligned}
&\mathbf{r}(t) \times \mathbf{r}(s)\\
&= \langle a_1f(t) + a_2g(t), b_1f(t) + b_2g(t), c_1f(t) + c_2g(t)\rangle \times \langle a_1f(s) + a_2g(s), b_1f(s) + b_2g(s), c_1f(s) + c_2g(s)\rangle\\
&= (f(t)g(s) - f(s)g(t))\langle c_2b_1 - b_2c_1, a_2c_1 - a_1c_2, a_1b_2 - a_2b_1\rangle,
\end{aligned}$$

where this last computation admittedly requires patience. Because $\mathbf{r}(t) \times \mathbf{r}(s)$ is always orthogonal to the same vector, the vectors $\mathbf{r}(t)$ must all lie in the same plane.

Now consider the case where a_3, b_3, and c_3 are not necessarily 0, and consider $\mathbf{p}(t) = \mathbf{r}(t) - \langle a_3, b_3, c_3 \rangle$. Note that $\mathbf{p}(t)$ has the form required in the argument in the previous paragraph. Using the result above, the curve $\mathbf{p}(t)$ lies in a plane, which implies that $\mathbf{r}(t) = \mathbf{p}(t) + \langle a_3, b_3, c_3 \rangle$ lies in a plane as well, because we are just translating all the vectors $\mathbf{p}(t)$ by the same constant vector.

b. If the curve lies in a plane, the $\mathbf{B}$ is always normal to the plane with length 1. Hence $\mathbf{B}$ is constant, so $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = 0$.

Chapter 12

Functions of Several Variables

12.1 Planes and Surfaces

12.1.1 One point and a normal vector determine a plane

12.1.2 The vector $n = \langle -2, -3, 4\rangle$ is normal to this plane.

12.1.3 The point (x, y, z) where this plane intersects the x-axis has $y = z = 0$; substituting in the equation of the plane gives $x = -6$. Similarly, we see that the plane meets the y-axis at $y = -4$ and the z-axis at $z = 3$.

12.1.4 Substituting in the general equation for a plane gives $(x-1)+(y-0)+(z-0) = 0$, which simplifies to $x + y + z = 1$.

12.1.5 Since z is absent from the equation $x^2 + 2y^2 = 8$, this cylinder is parallel to the z-axis. Similarly, $z^2 + 2y^2 = 8$ is parallel to the x-axis and $x^2 + 2z^2 = 8$ is parallel to the y-axis.

12.1.6 This is a cylinder consisting of all lines parallel to the y-axis that pass through the parabola $x = z^2$ in the xz-plane.

12.1.7 The traces of a surface are the sets of points at which the surface intersects a plane that is parallel to one of the coordinate planes.

12.1.8 This is an elliptic paraboloid.

12.1.9 This is an ellipsoid.

12.1.10 This is a hyperboloid of two sheets.

12.1.11 Substituting in the general equation for a plane gives $(x - 0) + (y - 2) - (z - (-2)) = 0$, which simplifies to $x + y - z = 4$.

12.1.12 Substituting in the general equation for a plane gives $(x - 1) - (y - 0) + 2\,(z - (-3)) = 0$, which simplifies to $x - y + 2z = -5$.

12.1.13 Substituting in the general equation for a plane gives $-1\,(x - 2) + 2\,(y - 3) - 3\,(z - 0) = 0$, which simplifies to $-x + 2y - 3z = 4$.

12.1.14 Substituting in the general equation for a plane gives $-1\,(x - 1) + 4\,(y - 2) - 3\,(z - (-3)) = 0$, which simplifies to $-x + 4y - 3z = 16$.

12.1.15 A vector normal to the plane is given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \langle -2, -1, 2 \rangle$. Substituting in the general equation for a plane gives $-2(x-1) - 1(y-2) + 2(z-3) = 0$, which simplifies to $2x + y - 2z = -2$.

12.1.16 A vector normal to the plane is given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 1 \\ 4 & 2 & 0 \end{vmatrix} = \langle -2, 4, 14 \rangle$. Substituting in the general equation for a plane gives $-2(x-3) + 4(y-0) + 2(z+2) = 0$, which simplifies to $x - 2y - 7z = 17$.

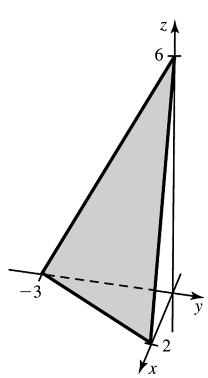
12.1.17 Let $P = (1, 0, 3)$, $Q = (0, 4, 2)$ and $R = (1, 1, 1)$. Then the vectors $\overrightarrow{PQ} = \langle -1, 4, -1 \rangle$ and $\overrightarrow{PR} = \langle 0, 1, -2 \rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 4 & -1 \\ 0 & 1 & -2 \end{vmatrix} = -7\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ is normal to the plane. The plane has equation $7(x-1) + 2(y-0) + 1(z-3) = 0$, which simplifies to $7x + 2y + z = 10$.

12.1.18 Let $P = (-1, 1, 1)$, $Q = (0, 0, 2)$ and $R = (3, -1, -2)$. Then the vectors $\overrightarrow{PQ} = \langle 1, -1, 1 \rangle$ and $\overrightarrow{PR} = \langle 4, -2, -3 \rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 4 & -2 & -3 \end{vmatrix} = 5\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$ is normal to the plane. The plane has equation $5(x - (-1)) + 7(y-1) + 2(z-1) = 0$, which simplifies to $5x + 7y + 2z = 4$.

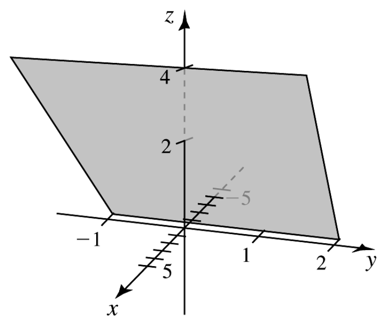
12.1.19 Let $P = (2, -1, 4)$, $Q = (1, 1, -1)$ and $R = (-4, 1, 1)$. Then the vectors $\overrightarrow{PQ} = \langle -1, 2, -5 \rangle$ and $\overrightarrow{PR} = \langle -6, 2, -3 \rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -5 \\ -6 & 2 & -3 \end{vmatrix} = 4\mathbf{i} + 27\mathbf{j} + 10\mathbf{k}$ is normal to the plane. The plane has equation $4(x-2) + 27(y - (-1)) + 10(z-4) = 0$, which simplifies to $4x + 27y + 10z = 21$.

12.1.20 Let $P = (5, 3, 1)$, $Q = (1, 3, -5)$ and $R = (-1, 3, 1)$. Then the vectors $\overrightarrow{PQ} = \langle -4, 0, -6 \rangle$ and $\overrightarrow{PR} = \langle -6, 0, 0 \rangle$ lie in the plane, s0 $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 0 & -6 \\ -6 & 0 & 0 \end{vmatrix} = 36\mathbf{j}$ is normal to the plane. The plane has equation $36(y-3) = 0$, which simplifies to $y = 3$.

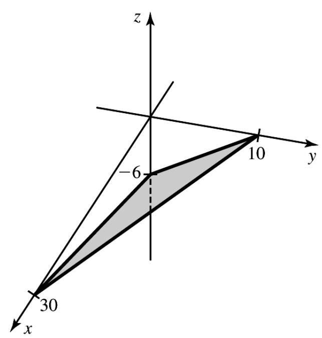
12.1.21 The x-intercept is found by setting $y = z = 0$ and solving $3x = 6$ to get $x = 2$. Similarly, we see that the y-intercept is -3 and the z-intercept is 6. The xy-trace is found by setting $z = 0$, which gives $3x - 2y = 6$. Similarly, the xz-trace is $3x + z = 6$ and the yz-trace is $-2y + z = 6$.



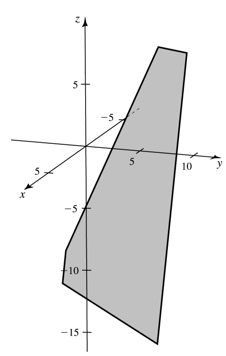
12.1.22 The x-intercept is found by setting $y = z = 0$ and solving $-4x = 16$ to get $x = -4$. Similarly, we see that the z-intercept is 2. Setting $x = z = 0$ gives $0 = 16$, so this plane does not intersect the y-axis. The xy-trace is found by setting $z = 0$, which gives $-4x = 16$ or $x = -4$. Similarly, the xz-trace is $-4x + 8z = 16$ or $x - 2z = -4$ and the yz-trace is $z = 2$.



12.1.23 The x-intercept is found by setting $y = z = 0$ which gives $x = 30$. Similarly, we see that the y-intercept is 10 and the z-intercept is -6. The xy-trace is found by setting z = 0, which gives $x + 3y = 30$. Similarly, the xz-trace is $x - 5z = 30$ and the yz-trace is $3y - 5z = 30$.



12.1.24 The x-intercept is found by setting $y = z = 0$ and solving $12x + 72 = 0$ to get $x = -6$. Similarly, we see that the y-intercept is 8 and the z-intercept is -18. The xy-trace is found by setting $z = 0$, which gives $12x - 9y = -72$ or $4x - 3y = -24$. Similarly, the xz-trace is $12x + 4z = -72$ or $3x + z = -18$ and the yz-trace is $9y - 4z = 72$.



12.1.25 The normal vectors to the planes are $\langle 1, 1, 4\rangle$ and $\langle -1, -3, 1\rangle$, and the dot product of these vectors is $-1 - 3 + 4 = 0$, so the planes are orthogonal.

12.1.26 The normal vectors to the planes are $\langle 2, 2, -3\rangle$ and $\langle -10, -10, 15\rangle$, and these are parallel because $\langle -10, -10, 15\rangle = -5\langle 2, 2, -3\rangle$. Thus, the planes are parallel.

12.1.27 The normal vectors to the planes are $\langle 3, 2, -3\rangle$ and $\langle -6, -10, 1\rangle$. These are neither parallel nor perpendicular, because one is not a multiple of the other and because their dot product is not 0. Thus, the planes are neither parallel nor perpendicular.

12.1.28 The normal vectors to the planes are $\langle 3,2,2\rangle$ and $\langle -6,-10,19\rangle$, and the dot product of these vectors is $-18-20+38=0$, so the planes are orthogonal.

12.1.29 Rewrite R and T so we have $Q: 3x-2y+z=12$, $R: 3x-2y+z=0$, $T: 3x-2y+z=12$. This shows that Q and T are identical, and Q, R and T are parallel. Note that $\langle 3,-2,1\rangle\cdot\langle -1,2,7\rangle=0$ so S is orthogonal to Q, R and T.

12.1.30 The planes Q, R, S and T have normal vectors $\langle 1,1,-1\rangle$, $\langle 0,1,1\rangle$, $\langle 1,-1,0\rangle$ and $\langle 1,1,1\rangle$ respectively. None of these vectors are scalar multiples of each other, so no two of these planes are parallel (or identical). Observe that $\langle 1,1,-1\rangle\cdot\langle 0,1,1\rangle=\langle 1,1,-1\rangle\cdot\langle 1,-1,0\rangle=0$; therefore Q is orthogonal to R and S. We also have $\langle 1,-1,0\rangle\cdot\langle 1,1,1\rangle=0$, so S and T are orthogonal. All other pairs have non-zero dot products, so no other pairs are orthogonal.

12.1.31 The plane Q has normal vector $\langle -1,2,-4\rangle$; therefore the parallel plane passing through the point $P_0\,(1,0,4)$ has equation $-1\,(x-1)+2\,(y-0)-4\,(z-4)=0$, which simplifies to $-x+2y-4z=-17$.

12.1.32 The plane Q has normal vector $\langle 2,1,-1\rangle$; therefore the parallel plane passing through the point $P_0\,(0,2,-2)$ has equation $2\,(x-0)+1\,(y-2)-1\,(z-(-2))=0$, which simplifies to $2x+y-z=4$.

12.1.33 The plane Q has normal vector $\langle 4,3,-2\rangle$; therefore the parallel plane passing through the point $P_0\,(1,-1,3)$ has equation $4\,(x-1)+3\,(y-(-1))-2\,(z-3)=0$, which simplifies to $4x+3y-2z=-5$.

12.1.34 12.1.28 The plane Q has normal vector $\langle 1,-5,-2\rangle$; therefore the parallel plane passing through the point $P_0\,(1,2,0)$ has equation $1\,(x-1)-5\,(y-2)-2\,(z-0)=0$, which simplifies to $x-5y-2z=-9$.

12.1.35 First, note that the vectors normal to the planes, $\mathbf{n}_Q=\langle -1,2,1\rangle$ and $\mathbf{n}_R=\langle 1,1,1\rangle$, are not multiples of each other; therefore these planes are not parallel and they intersect in a line ℓ. We need to find a point on ℓ and a vector in the direction of ℓ. Setting $x=0$ in the equations of the planes gives equations of the lines in which the planes intersect the yz-plane: $2y+z=1$, $y+z=0$. Solving these equations simultaneously gives $y=1$ and $z=-1$, so $(0,1,-1)$ is a point on ℓ. A vector in the direction of ℓ is $\mathbf{n}_Q\times\mathbf{n}_R=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&2&1\\1&1&1\end{vmatrix}=\mathbf{i}+2\mathbf{j}-3\mathbf{k}=\langle 1,2,-3\rangle$. Therefore ℓ has equation $\mathrm{r}\,(t)=\langle 0,1,-1\rangle+t\langle 1,2,-3\rangle=\langle t,1+2t,-1-3t\rangle$, or $x=t$, $y=1+2t$, $z=-1-3t$.

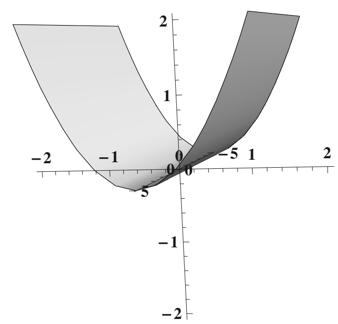
12.1.36 First, note that the vectors normal to the planes, $\mathbf{n}_Q=\langle 1,2,-1\rangle$ and $\mathbf{n}_R=\langle 1,1,1\rangle$, are not multiples of each other; therefore these planes are not parallel and they intersect in a line ℓ. We need to find a point on ℓ and a vector in the direction of ℓ. Setting $z=0$ in the equations of the planes gives equations of the lines in which the planes intersect the xy-plane: $x+2y=1$, $x+y=1$. Solving these equations simultaneously gives $x=1$ and $y=0$, so $(1,0,0)$ is a point on ℓ. A vector in the direction of ℓ is $\mathbf{n}_Q\times\mathbf{n}_R=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-1\\1&1&1\end{vmatrix}=3\mathbf{i}-2\mathbf{j}-\mathbf{k}=\langle 3,-2,-1\rangle$. Therefore ℓ has equation $\mathbf{r}\,(t)=\langle 1,0,0\rangle+t\langle 3,-2,-1\rangle=\langle 1+3t,-2t,-t\rangle$, or $x=1+3t$, $y=-2t$, $z=-t$.

12.1.37 First, note that the vectors normal to the planes, $\mathbf{n}_Q=\langle 2,-1,3\rangle$ and $\mathbf{n}_R=\langle -1,3,1\rangle$, are not multiples of each other; therefore these planes are not parallel and they intersect in a line ℓ. We need to find a point on ℓ and a vector in the direction of ℓ. Setting $z=0$ in the equations of the planes gives equations of the lines in which the planes intersect the xy-plane: $2x-y=1$, $-x+3y=4$. Solving these equations simultaneously gives $x=\frac{7}{5}$ and $y=\frac{9}{5}$, so $\left(\frac{7}{5},\frac{9}{5},0\right)$ is a point on ℓ. A vector in the direction of ℓ is $\mathbf{n}_Q\times\mathbf{n}_R=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&3\\-1&3&1\end{vmatrix}=-10\mathbf{i}-5\mathbf{j}+5\mathbf{k}=-5\langle 2,1,-1\rangle$. Therefore ℓ has equation $\mathbf{r}\,(t)=\langle\frac{7}{5},\frac{9}{5},0\rangle+t\langle 2,1,-1\rangle=\langle\frac{7}{5}+2t,\frac{9}{5}+t,-t\rangle$, or $x=\frac{7}{5}+2t$, $y=\frac{9}{5}+t$, $z=-t$.

12.1.38 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle 1, -1, -2 \rangle$ and $\mathbf{n}_R = \langle 1, 1, 1 \rangle$, are not multiples of each other; therefore these planes are not parallel and they intersect in a line ℓ. We need to find a point on ℓ and a vector in the direction of ℓ. Setting $z = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the xy-plane: $x - y = 1$, $x + y = -1$. Solving these equations simultaneously gives $x = 0$ and $y = -1$, so $(0, -1, 0)$ is a point on ℓ. A vector in the direction of ℓ is $\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k} = \langle 1, -3, 2 \rangle$. Therefore ℓ has equation $\mathbf{r}(t) = \langle 0, -1, 0 \rangle + t\langle 1, -3, 2 \rangle = \langle t, -1 - 3t, 2t \rangle$, or $x = t$, $y = -1 - 3t$, $z = 2t$.

12.1.39

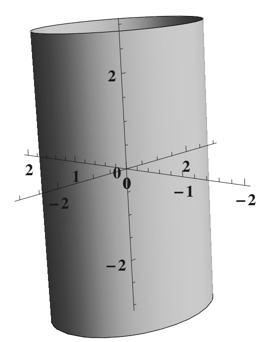
a. The cylinder is parallel to the x-axis.



b.

12.1.40

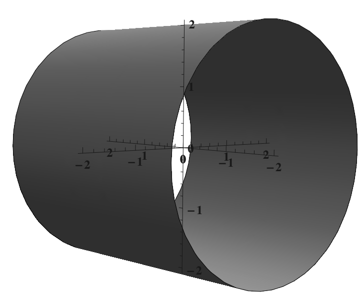
a. The cylinder is parallel to the z-axis.



b.

12.1.41

a. The cylinder is parallel to the y-axis.

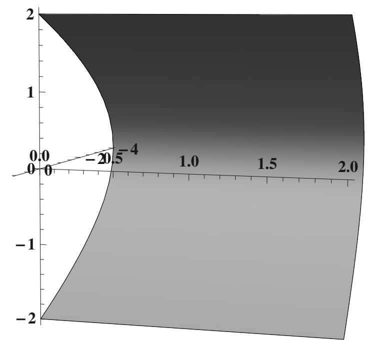


b.

12.1.42

a. The cylinder is parallel to the y-axis.

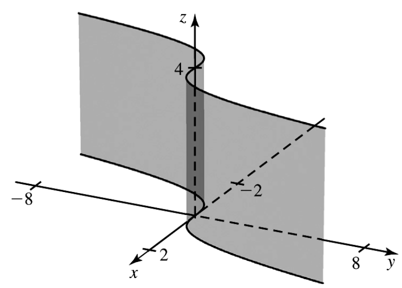
b.



12.1.43

a. The cylinder is parallel to the z-axis.

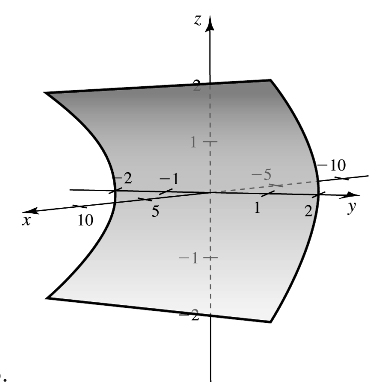
b.



12.1.44

a. The cylinder is parallel to the y-axis.

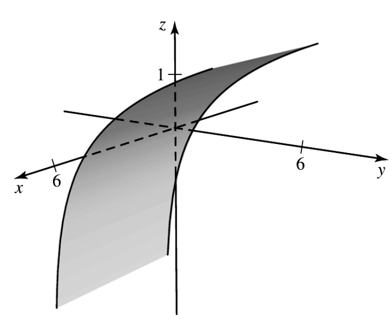
b.



12.1.45

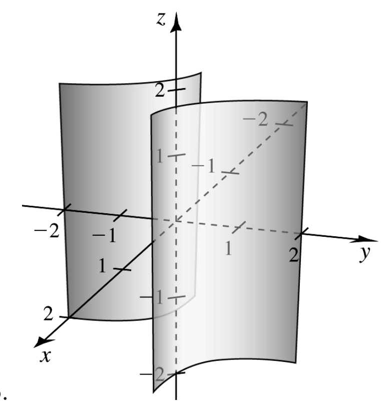
a. The cylinder is parallel to the x-axis.

b.



12.1.46

a. The cylinder is parallel to the z-axis.

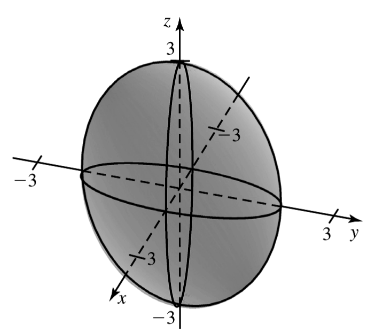
b. 

12.1.47

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 1$, so the x-intercepts are $x = \pm 1$. Similarly we see that the y-intercepts are $y = \pm 2$ and the z-intercepts are $z = \pm 3$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x^2 + \frac{y^2}{4} = 1$, $x^2 + \frac{z^2}{9} = 1$, $\frac{y^2}{4} + \frac{z^2}{9} = 1$.

c.

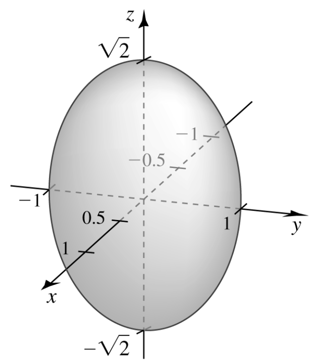


12.1.48

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $4x^2 = 1$, so the x-intercepts are $x = \pm\frac{1}{2}$. Similarly we see that the y-intercepts are $y = \pm 1$ and the z-intercepts are $z = \pm\sqrt{2}$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $4x^2 + y^2 = 1$, $4x^2 + \frac{z^2}{2} = 1$, $y^2 + \frac{z^2}{2} = 1$.

c.

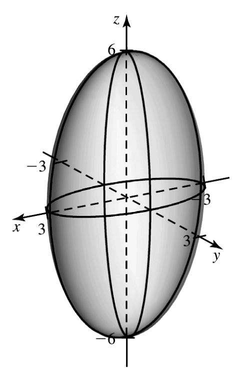


12.1.49

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 9$, so the x-intercepts are $x = \pm 3$. Similarly we see that the y-intercepts are $y = \pm 1$ and the z-intercepts are $z = \pm 6$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{x^2}{3} + 3y^2 = 3$,, $\frac{x^2}{3} + \frac{z^2}{12} = 3$, $3y^2 + \frac{z^2}{12} = 3$.

c.

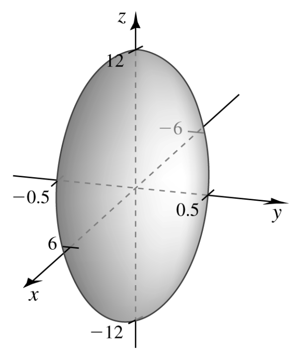


12.1.50

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 36$, so the x-intercepts are $x = \pm 6$. Similarly we see that the y-intercepts are $y = \pm \frac{1}{2}$ and the z-intercepts are $z = \pm 12$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{x^2}{6} + 24y^2 = 6$, $\frac{x^2}{6} + \frac{z^2}{24} = 6$, $24y^2 + \frac{z^2}{24} = 6$.

c.

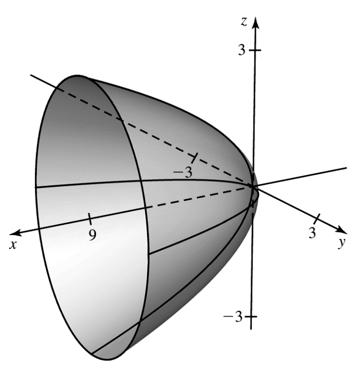


12.1.51

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x = y^2$, $x = z^2$, and $y^2 + z^2 = 0$ (which implies that $x = y = z = 0$).

c.

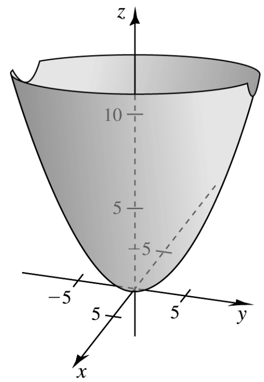


12.1.52

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $0 = \frac{x^2}{4} + \frac{y^2}{9}$ (which implies that $x = y = z = 0$), $z = \frac{x^2}{4}$, $z = \frac{y^2}{9}$.

c.

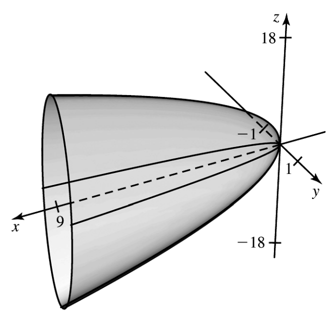


12.1.53

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x - 9y^2 = 0$, $9x - \frac{z^2}{4} = 0$, $81y^2 + \frac{z^2}{4} = 0$ (which implies that $x = y = z = 0$).

c.

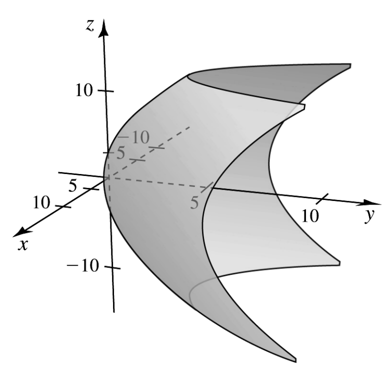


12.1.54

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $y - \frac{x^2}{16} = 0$, $\frac{x^2}{8} + \frac{z^2}{18} = 0$ (which implies that $x = y = z = 0$), $y - \frac{z^2}{36} = 0$.

c.

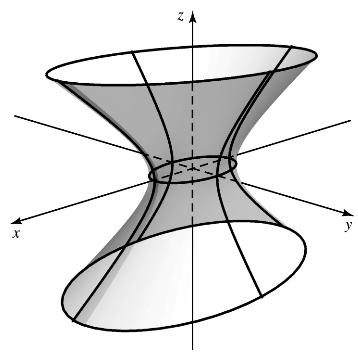


12.1.55

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 25$, so the x-intercepts are $x = \pm 5$. Similarly we see that the y-intercepts are $y = \pm 3$ and there are no z-intercepts.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{x^2}{25} + \frac{y^2}{9} = 1$, $\frac{x^2}{25} - z^2 = 1$, $\frac{y^2}{9} - z^2 = 1$.

c.

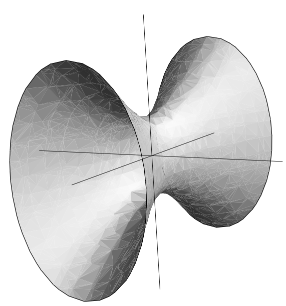


12.1.56

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which results in no intercepts. Similarly we see that the y-intercepts are $y = \pm 2$ and the z-intercepts are $z = \pm 3$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{y^2}{4} - \frac{x^2}{16} = 1$, $\frac{z^2}{9} - \frac{x^2}{16} = 1$, $\frac{y^2}{4} + \frac{z^2}{9} = 1$.

c.

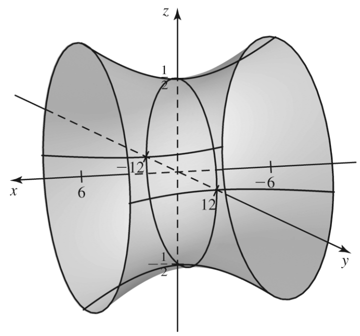


12.1.57

a. The y-intercept is found by setting $x = z = 0$ in the equation of this surface, which gives $y^2 = 144$, so the y-intercepts are $y = \pm 12$. Similarly we see that the z-intercepts are $z = \pm \frac{1}{2}$ and there are no x-intercepts.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $-\frac{x^2}{4} + \frac{y^2}{16} - 9 = 0$, $-\frac{x^2}{4} + 36z^2 - 9 = 0$, $\frac{y^2}{16} + 36z^2 - 9 = 0$.

c.

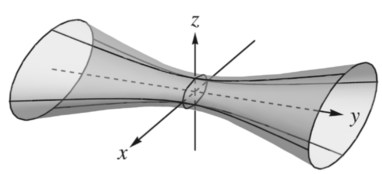


12.1.58

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x^2 = 1$, so the x-intercepts are $x = \pm 1$. Similarly we see that the z-intercepts are $z = \pm\frac{1}{3}$ and there are no y-intercepts.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x^2 - \frac{y^2}{3} - 1 = 0$, $9z^2 + x^2 - 1 = 0$, $9z^2 - \frac{y^2}{3} - 1 = 0$.

c.

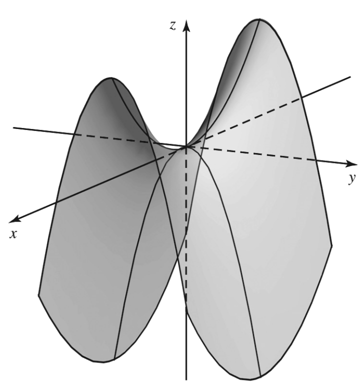


12.1.59

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{x^2}{9} - y^2 = 0$, $z = \frac{x^2}{9}$, $z = -y^2$.

c.

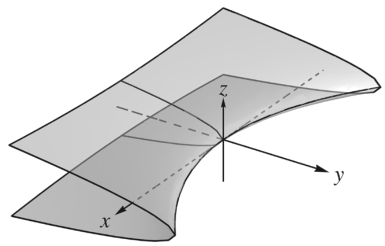


12.1.60

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $y = \frac{x^2}{16}$, $\frac{x^2}{16} - 4z^2 = 0$, $y = -4z^2$.

c.

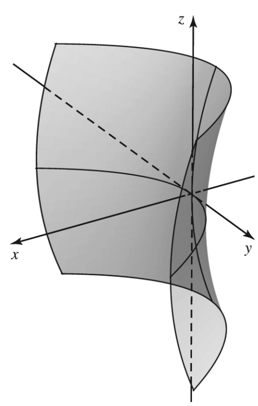


12.1.61

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $5x - \frac{y^2}{5} = 0$, $5x + \frac{z^2}{20} = 0$, $-\frac{y^2}{25} + \frac{z^2}{20} = 0$.

c.

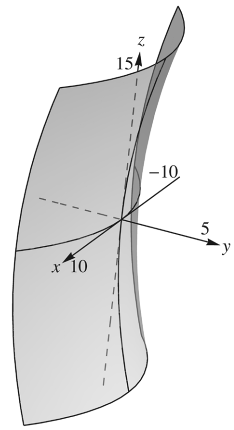


12.1.62

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $6y + \frac{x^2}{6} = 0$, $\frac{x^2}{6} - \frac{z^2}{24} = 0$, $6y - \frac{z^2}{24} = 0$.

c.

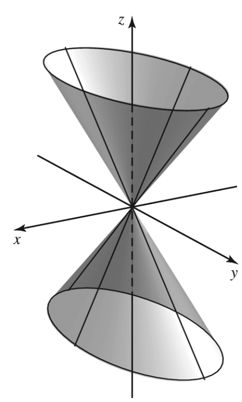


12.1.63

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $x^2 + \frac{y^2}{4} = 0$ (which implies that $x = y = z = 0$), $x^2 = z^2$, $\frac{y^2}{4} = z^2$.

c.

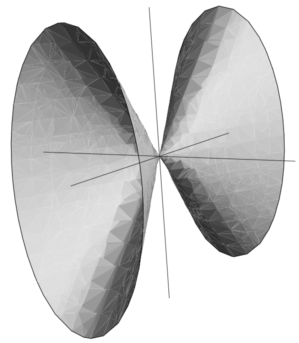


12.1.64

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $4y^2 = x^2$, $x^2 = z^2$, $4y^2 + z^2 = 0$ (which implies that $x = y = z = 0$).

c.

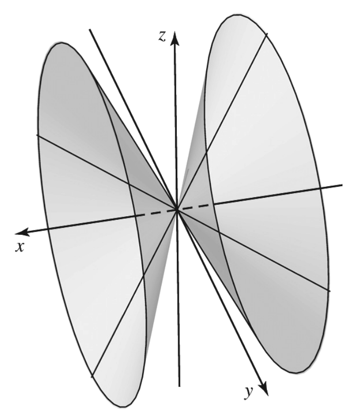


12.1.65

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{y^2}{18} = 2x^2$, $\frac{z^2}{32} = 2x^2$, $\frac{z^2}{32} + \frac{y^2}{18} = 0$ (which implies that $x = y = z = 0$).

c.

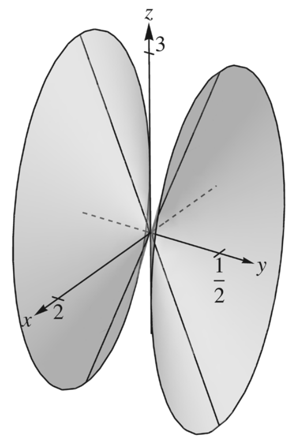


12.1.66

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $x = 0$. Similarly we see that the y-intercept is $y = 0$ and the z-intercept is $z = 0$.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $\frac{x^2}{3} = 3y^2$, $\frac{x^2}{3} + \frac{z^2}{12} = 0$, (which implies that $x = y = z = 0$), $\frac{z^2}{12} = 3y^2$.

c.

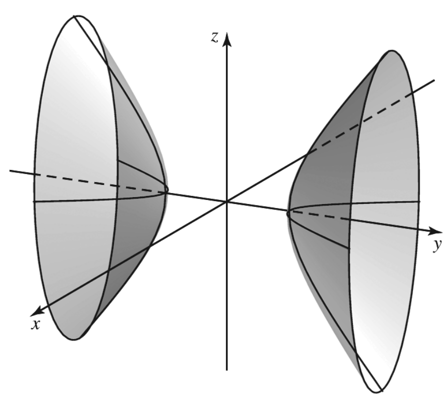


12.1.67

a. The y-intercept is found by setting $x = z = 0$ in the equation of this surface, which gives $y^2 = 4$, so the y-intercepts are $y = \pm 2$. There are no x or z-intercepts.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $-x^2 + \frac{y^2}{4} = 1$, $-x^2 - \frac{z^2}{9} = 1$ (no xz-trace), $\frac{y^2}{4} - \frac{z^2}{9} = 1$.

c.

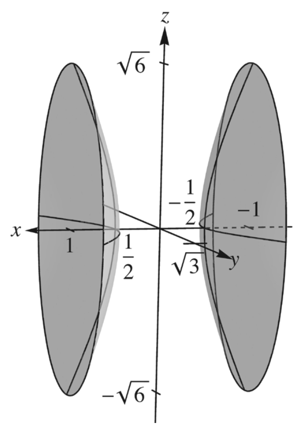


12.1.68

a. The x-intercept is found by setting $y = z = 0$ in the equation of this surface, which gives $4x^2 = 1$, so the x-intercepts are $x = \pm\frac{1}{2}$. There are no y or z-intercepts.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $1-4x^2+y^2 = 0$, $1-4x^2+\frac{z^2}{2} = 0$, $1+y^2+\frac{z^2}{2} = 0$ (no yz-trace).

c.

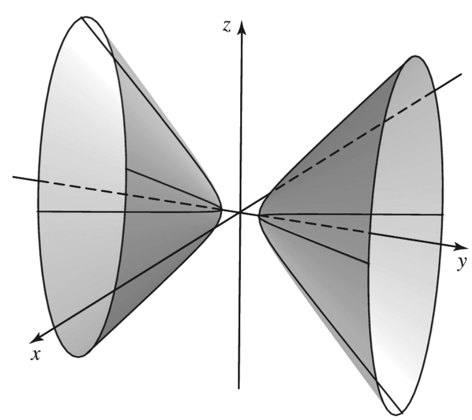


12.1.69

a. The y-intercept is found by setting $x = z = 0$ in the equation of this surface, which gives $3\ y^2 = 1$, so the y-intercepts are $y = \pm\frac{\sqrt{3}}{3}$. There are no x or z-intercepts.

b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $-\frac{x^2}{3} + 3y^2 = 1$, $-\frac{x^2}{3} - \frac{z^2}{12} = 1$ (no xz-trace), $3y^2 - \frac{z^2}{12} = 1$.

c.

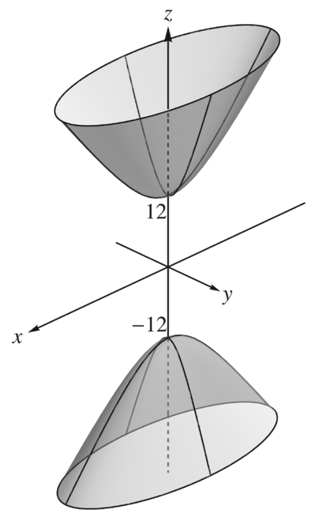


12.1.70

a. The z-intercept is found by setting $x = y = 0$ in the equation of this surface, which gives $z^2 = 24 \cdot 6$, so the z-intercepts are $x = \pm 12$. There are no x or y-intercepts.

b. b. The equations for the xy-, xz- and yz-traces are found by setting $z = 0$, $y = 0$ and $x = 0$ respectively in the equation of the surface, which gives $-\frac{x^2}{6} - 24y^2 - 6 = 0$ (no xy-trace), $-\frac{x^2}{6} + \frac{z^2}{24} - 6 = 0$, $-24y^2 + \frac{z^2}{24} - 6 = 0$.

c.



12.1.71

a. True. Observe first that these two planes are parallel since their normal vectors are parallel. The first plane has equation $1 \cdot (x-1) + 2(y-1) - 3(z-1) = 0$, which implies that $x + 2y - 3z = 0$; the point $(3, 0, 1)$ is on this plane, so the two planes are identical.

b. False. The point (1, 0, 0) is on the first plane but not the second.

c. False. There are infinite planes orthogonal to the plane Q.

d. True. Any two points on the line ℓ together with P_0 determine the same plane.

e. False. For example, the xz- and yz-coordinate planes both contain the point $(0, 0, 1)$ and are orthogonal to the xy-coordinate plane.

f. False. Two distinct lines determine a plane only if the lines are parallel or if they intersect.

g. False. Either plane S is plane P or plane S is parallel to plane P.

12.1.72

a. Observe that the points $P = (0, 0, 0)$ and $Q = (1, -1, 2)$ lie on the line ℓ. Therefore the vectors $\overrightarrow{PP_0} = \langle 1, -2, 3\rangle$ and $\overrightarrow{PQ} = \langle 1, -1, 2\rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PP_0} \times \overrightarrow{PQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 1 & -1 & 2 \end{vmatrix} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ is normal to the plane. The plane has equation $-1(x-0) + 1(y-0) + 1(z-0) = 0$, which simplifies to $x - y - z = 0$.

b. Observe that the points $P = (0, 0, 0)$ and $Q = (1, -1, -2)$ lie on the line ℓ. Therefore the vectors $\overrightarrow{PP_0} = \langle -4, 1, 2\rangle$ and $\overrightarrow{PQ} = \langle 1, -1, -2\rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PP_0} \times \overrightarrow{PQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 2 \\ 1 & -1 & -2 \end{vmatrix} = -6\mathbf{j} + 3\mathbf{k}$ is normal to the plane. The plane has equation $0(x-0) - 6(y-0) + 3(z-0) = 0$, which simplifies to $2y - z = 0$.

12.1.73 The direction of the line is $\langle 2, -4, 1\rangle$, so the line is given by $\langle 2, 1, 3\rangle + t\langle 2, -4, 1\rangle = \langle 2+2t, 1-4t, 3+t\rangle$.

12.1.74 The direction of the line is $\langle 1,0,4\rangle$, so the line is given by $\langle 0,-10,-3\rangle+t\langle 1,0,4\rangle=\langle t,-10,-3+4t\rangle$.

12.1.75 These planes have normal $\langle 2,3,0\rangle\times\langle -1,-1,2\rangle=\langle 6,-4,1\rangle$, so the planes all have an equation of the form $6x-4y+z=d$ for some real number d.

12.1.76 The normal to the plane we are seeking is $\langle 2,5,-3\rangle\times\langle -1,5,2\rangle=\langle 25,-1,15\rangle$. The equation of the plane has the form $25x-y+15z=d$ for some d. Because the point $(0,-2,4)$ is on the plane, we have $0-(-2)+60=d$, so the plane is given by $25x-y+15z=62$.

12.1.77 First we find the line of intersection of the first two planes. The direction of the line of intersection is $\langle 1,0,3\rangle\times\langle 0,1,4\rangle=\langle -3,-4,1\rangle$, and by inspection, a point on both planes is $(0,2,1)$. Thus the line of intersection is given by $\langle -3t,2-4t,1+t\rangle$. This intersects the plane $x+y+6z=9$ when $-3t+(2-4t)+6(1+t)=9$, or $8-t=9$, so $t=-1$. Thus, the intersection is the single point $(3,6,0)$.

12.1.78 First we find the line of intersection of the first two planes. The direction of the line of intersection is $\langle 1,2,2\rangle\times\langle 0,1,4\rangle=\langle 6,-4,1\rangle$, and by inspection, a point on both planes is $(3,-2,2)$. Thus the line of intersection is given by $\langle 3+6t,-2-4t,2+t\rangle$. This intersects the plane $x+2y+8z=9$ when $3+6t+2(-2-4t)+8(2+t)=9$, or $6t+15=9$, so $t=-1$. Thus, the intersection is the single point $(-3,2,1)$.

12.1.79

a. D. This surface is a cylinder parallel to the parabola $y=z^2$ in the yz-plane.

b. A. This surface is a plane.

c. E. This surface is an ellipsoid.

d. F. This surface is a hyperboloid of one sheet.

e. B. This surface is an elliptic cone.

f. C. This surface is a cylinder parallel to the graph $y=|x|$ in the xy-plane.

12.1.80 This surface is a hyperboloid of one sheet with axis the x-axis.

12.1.81 This surface is a hyperbolic paraboloid with saddle point at the origin.

12.1.82 This surface is a hyperboloid of two sheets with axis the x-axis.

12.1.83 This surface is an elliptic paraboloid with axis the y-axis.

12.1.84 Completing the square and rewriting the equation of the surface as $(x+1)^2+y^2+4z^2=1$ shows that this surface is an ellipsoid centered at the point $(-1,0,0)$.

12.1.85 Completing the square and rewriting the equation of the surface as $9x^2+(y+1)^2-4z^2=1$ shows that this surface is a hyperboloid of one sheet with axis the line ℓ: $\mathbf{r}=\langle 0,-1,t\rangle$.

12.1.86 This surface is an elliptic cylinder (the xy-trace is an ellipse).

12.1.87 This surface is a hyperbolic cylinder (the yz-trace is a hyperbola).

12.1.88 Completing the square and rewriting the equation of the surface as $-(x-3)^2-(y+4)^2+\frac{z^2}{9}=1$ shows that this surface is a hyperboloid of two sheets with axis the line ℓ: $\mathbf{r}=\langle 3,-4,t\rangle$.

12.1.89 Completing the square and rewriting the equation of the surface as $z^2=\frac{(x-4)^2}{4}+(y-5)^2+12$ shows that this surface is a hyperboloid of two sheets.

12.1.90 The point $(x,2x+1,1)$ lies on this curve $\mathbf{r}(t)$ exactly when x satisfies the equation $\left(\frac{x}{10}\right)^2+\left(\frac{2x+1}{2}\right)^2=1$, which simplifies to $101x^2+100x-75=0$. This equation has roots $x\approx 0.4988,-1.4889$; therefore the intersection points are $(0.4988,1.9976,1)$ and $(-1.4889,-1.9778,1)$.

12.1.91 The point $\left(t, t^2, 3t^2\right)$ lies on the plane $8x+y+z=60$ exactly when $8t+t^2+3t^2=60$, which can be written as $t^2+2t-15=0$, which has solutions $t=-5,3$. Therefore the intersection points are $(-5,25,75)$ and $(3,9,27)$.

12.1.92 The point $\left(1, \sqrt{t}, -t\right)$ lies on the plane $8x+15y+3z=20$ exactly when $8+15\sqrt{t}-3t=20$ which can be written as $t-5\sqrt{t}+4=0$.

Let $s=\sqrt{t}$; then s satisfies $s^2-5s+4=0$ which has roots $s=1,4$; therefore $t=1,16$ and the intersection points are $(1,1,-1)$ and $(1,4,-16)$.

12.1.93 Suppose the point (x,y,z) lies on both the curve and the plane; then $z=\frac{x}{4}$, and substituting this in the equation $2x+3y-12z=0$ gives $y=\frac{x}{3}$. We also have $x=\cos t$ and $\frac{x}{3}=4\sin t$ for some t; therefore $\left(\frac{x}{4}\right)^2+\left(\frac{x}{12}\right)^2=1$, which can be written as $10x^2=144$, which gives $x=\pm\frac{6\sqrt{10}}{5}$, so the intersection points are $\left(\frac{6\sqrt{10}}{5}, \frac{2\sqrt{10}}{5}, \frac{3\sqrt{10}}{10}\right)$ and $\left(-\frac{6\sqrt{10}}{5}, -\frac{2\sqrt{10}}{5}, -\frac{3\sqrt{10}}{10}\right)$.

12.1.94 The x-intercept is found by setting $y=z=0$ and solving for x, which gives the point $\left(\frac{d}{a},0,0\right)$, assuming $a\neq 0$. If $a=0$, then all points on the x-axis lie on this plane when $d=0$, and no points on the x-axis lie on the plane when $d\neq 0$. Similarly, the intersection of the plane with the y-axis is the point $\left(0,\frac{d}{b},0\right)$ when $b\neq 0$, the entire y-axis when $b=d=0$ and empty when $b=0$, $d\neq 0$; and the intersection of the plane with the z-axis is the point $\left(0,0,\frac{d}{c}\right)$ when $c\neq 0$, the entire z-axis when $c=d=0$ and empty when $c=0$, $d\neq 0$.

12.1.95 The angle θ between the vectors $\mathbf{n}_1=\langle 5,2,-1\rangle$ and $\mathbf{n}_2=\langle -3,1,2\rangle$ satisfies $\cos\theta=\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}=-\frac{15}{\sqrt{30}\sqrt{14}}=-\frac{\sqrt{105}}{14}$, so $\theta=\cos^{-1}\left(-\frac{\sqrt{105}}{14}\right)\approx 2.392$ rad $\approx 137°$.

12.1.96

a. The ellipsoid has equal y- and z-intercepts, so the equation is $x^2+4y^2+4z^2=1$.

b. The ellipsoid has equal x- and z-intercepts, so the equation is $x^2+4y^2+z^2=1$.

12.1.97 All of the quadric surfaces in Table 12.1 except the hyperbolic paraboloid can have circular cross-sections around a coordinate axis, and so can be generated by revolving a curve in one of the coordinate planes about a coordinate axis.

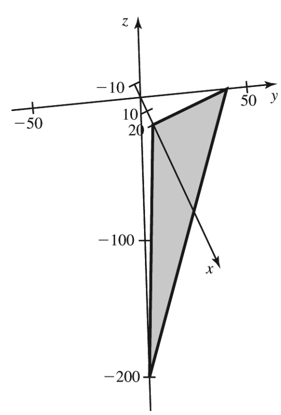
12.1.98

a. The light cone consists of all points (x,y,t) such that the distance from (x,y) to the origin is $c|t|$, where c is the speed of light; hence in these units an equation of the light cone is $x^2+y^2=t^2$.

b. The light cone consists of all points (x,y,t) such that the distance from (x,y) to the origin is $c|t|$, where c is the speed of light; hence in these units the equation of the light cone is $x^2+y^2=9\times 10^{16}t^2$.

12.1.99

a.



b. The profit is $z = 10 \cdot 20 + 5 \cdot 10 - 200 = \$50$ which is positive.

c. The profit is 0 when x and y lie on the line $2x + y = 40$.

12.1.100 The line does not meet the plane exactly when the line's direction is parallel to the plane, which is equivalent to the condition $\mathbf{v} \cdot \langle a, b, c\rangle = 0$.

12.1.101

a. Observe that any point (x, y, z) on this curve satisfies $z = cy$, so this gives the equation of the plane P.

b. Plane P has normal vector $\mathbf{n} = \langle 0, -c, 1\rangle$, so the angle θ that P makes with the xy-plane (which has normal vector $\mathbf{k}$) satisfies $\cos\theta = \frac{\mathbf{n}\cdot\mathbf{k}}{|\mathbf{n}||\mathbf{k}|} = \frac{1}{\sqrt{1+c^2}}$; hence $\theta = \tan^{-1} c$.

c. The curve can be described as the intersection of the ellipsoid given by $x^2 + \frac{y^2}{4} + \frac{z^2}{4c^2} = 1$ with the plane P, which is an ellipse in P.

12.1.102

a. Let $P\,(x, y, z)$ be the point on the plane $ax + by + cz = d$ that is closest to the origin O; then the vector $\overrightarrow{OP}$ is parallel to the normal $\mathbf{n} = \langle a, b, c\rangle$, so we can express $(x, y, z) = \lambda\,(a, b, c)$ for some scalar λ. Substituting in the equation of the plane gives $\lambda = \frac{d}{a^2+b^2+c^2} = \frac{d}{D^2}$, where $D^2 = a^2 + b^2 + c^2$, so the distance from P to the origin is $|\lambda| = \frac{|d|}{D}$.

b. Let $P\,(x, y, z)$ be the point on the plane $ax + by + cz = d$ that is closest to $P_0\,(x_0, y_0, z_0)$; then the vector $\overrightarrow{P_0P}$ is parallel to the normal $\mathbf{n} = \langle a, b, c\rangle$, so we can express $(x, y, z) = (x_0, y_0, z_0) + \lambda\,(a, b, c)$ for some scalar λ. Substituting in the equation of the plane gives $\lambda = \frac{d - ax_0 - by_0 - cz_0}{D^2}$, so the distance from P to P_0 is $|\overrightarrow{P_0P}| = |\lambda| D = \frac{|ax_0+by_0+cz_0-d|}{D}$.

12.1.103

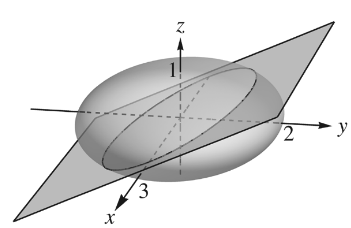
a. The length of the orthogonal projection of $\overrightarrow{PQ}$ onto the normal vector $\mathbf{n}$ is the magnitude of the scalar component of $\overrightarrow{PQ}$ in the direction of $\mathbf{n}$ which is $\frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{|\mathbf{n}|}$.

b. Let $P = (0, -1, 0)$ be a point on the plane. $\frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{|\mathbf{n}|} = \frac{|\langle 1,3,-4\rangle\cdot\langle 2,-1,3\rangle|}{\sqrt{4+1+9}} = \frac{|2-3-12|}{\sqrt{14}} = \frac{13}{\sqrt{14}}$.

12.1.104

a. Yes. The ellipsoid E is centered at the origin which is on the plane P, so the intersection of E and P is an ellipse in the plane P.

b. In this case one of the axes of symmetry for the ellipse C is the x-axis.



c. Because $z = Ax + By$, any point (x, y, z) on C satisfies $\frac{x^2}{9} + \frac{y^2}{4} + (Ax + By)^2 = 1$, which gives the equation of the projection of C on the xy-plane.

d. The equation $\frac{x^2}{9} + \frac{y^2}{4} + \left(\frac{x}{6} + \frac{y}{2}\right)^2 = 1$, can be transformed as follows: express $\frac{x^2}{9} + \frac{y^2}{4} = \left(\frac{x}{3} + \frac{y}{2}\right)^2 - \frac{xy}{3}$ and $\left(\frac{x}{6} + \frac{y}{2}\right)^2 = \left(\frac{x}{6} - \frac{y}{2}\right)^2 + \frac{xy}{3}$. Then the equation becomes $\left(\frac{x}{3} + \frac{y}{2}\right)^2 + \left(\frac{x}{6} - \frac{y}{2}\right)^2 = 1$. We can parameterize this curve by setting $\frac{x}{3} + \frac{y}{2} = \cos t$ and $\frac{x}{6} - \frac{y}{2} = \sin t$, which gives $x = 2\cos t + 2\sin t$, $y = \frac{2}{3}\cos t - \frac{4}{3}\sin t$ and $z = \frac{x}{6} + \frac{y}{2} = \frac{2}{3}\cos t - \frac{1}{3}\sin t$.

12.2 Graphs and Level Curves

12.2.1 The independent variables are x and y and the dependent variable is z.

12.2.2 The domain of f is $\mathbb{R}^2$.

12.2.3 The domain of g is $\{(x, y) : x \neq 0 \text{ or } y \neq 0\}$.

12.2.4 The domain of h is $\{(x, y) : x - y \geq 0\}$.

12.2.5 We need three dimensions to plot points $(x, y, f(x, y))$.

12.2.6 Sketch the curves $f(x, y) = z_0$ in $\mathbb{R}^2$ for several values of z_0.

12.2.7 The level curves $x^2 + y^2 = z_0$ are circles centered at $(0, 0)$ in $\mathbb{R}^2$.

12.2.8 We need three dimensions to graph the level surfaces $f(x, y, z) = w_0$.

12.2.9 The function f has 6 independent variables, so $n = 6$.

12.2.10 We can sketch level surfaces in $\mathbb{R}^3$, or use colors to code thevalues of the function at points in $\mathbb{R}^3$.

12.2.11 The domain of f is $\mathbb{R}^2$.

12.2.12 The domain of f is $\mathbb{R}^2$

12.2.13 The domain of f is $\{(x, y) : x^2 + y^2 \leq 25\}$, which is the set of all points on or within the circle of radius 5 centered at the origin.

12.2.14 The domain of f is $\{(x, y) : x^2 + y^2 > 25\}$, which is the set of all points outside the circle of radius 5 centered at the origin.

12.2.15 The domain of f is $\{(x, y) : y \neq 0\}$.

12.2.16 The domain of f is $\{(x, y) : x \neq \pm y\}$.

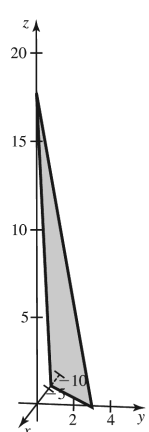
12.2.17 The domain of g is $\{(x, y) : y < x^2\}$.

12.2.18 The domain of f is $\{(x, y) : -1 \leq y - x^2 \leq 1\}$; which is the set of points lying between or on the parabolas $y = x^2 - 1$ and $y = x^2 + 1$.

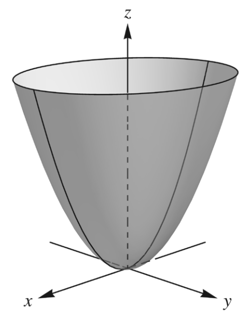
12.2.19 The domain of g is $\{(x, y) : x\,y \geq 0, (x, y) \neq (0, 0)\}$.

12.2.20 The domain of h is $\{(x, y) : x - 2y + 4 \geq 0\}$.

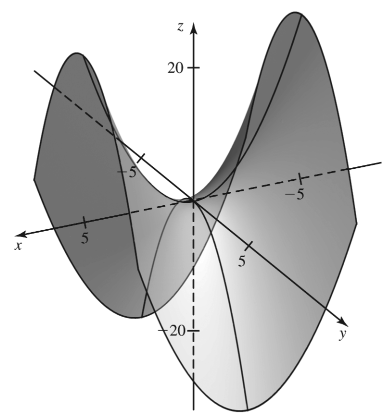
12.2.21 This surface is a plane; the function's domain is $\mathbb{R}^2$ and its range is $\mathbb{R}$.



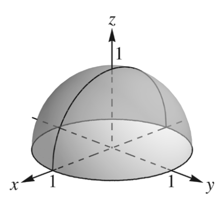
12.2.22 This surface is an elliptic paraboloid; the function's domain is $\mathbb{R}^2$ and its range is the interval $[0, \infty)$.



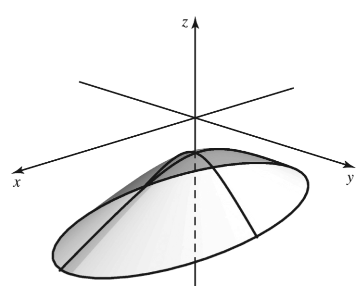
12.2.23 This surface is a hyperbolic paraboloid; the function's domain is $\mathbb{R}^2$ and its range is $\mathbb{R}$.



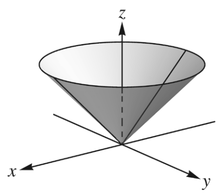
12.2.24 This surface is a hemisphere; the function's domain is $\{(x, y) : x^2 + y^2 \leq 1\}$ and its range is the interval $[0, 1]$.



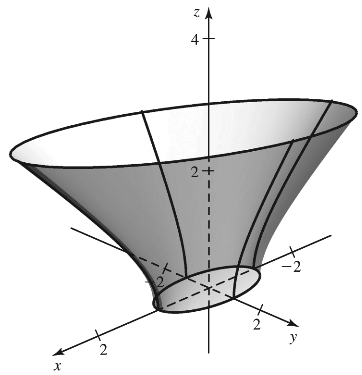
12.2.25 This surface is the lower part of a hyperboloid of two sheets; the function's domain is $\mathbb{R}^2$ and its range is the interval $(-\infty, -1]$.



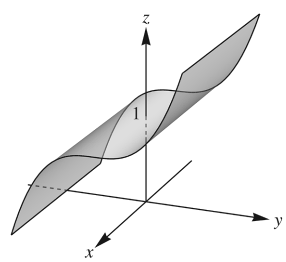
12.2.26 This surface is the upper part of a circular cone; the function's domain is $\mathbb{R}^2$ and its range is the interval $[0, \infty)$.



12.2.27 This surface is the upper part of a hyperboloid of one sheet; the function's domain is $\{(x, y) : x^2 + y^2 \geq 1\}$ and its range is the interval $[0, \infty)$.



12.2.28 This surface is the cylinder parallel to the x-axis through the curve $z = y^3 + 1$ in the yz-plane ; the function's domain is $\mathbb{R}^2$ and its range is $\mathbb{R}$.



12.2.29

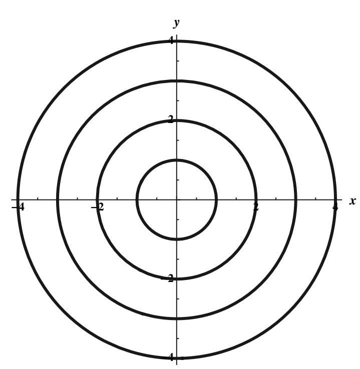
a. A. Notice that the range of the function in (A) is $[-1, 1]$.

b. D. Notice that the function in (D) becomes large and negative for(x, y) near $(0, 0)$.

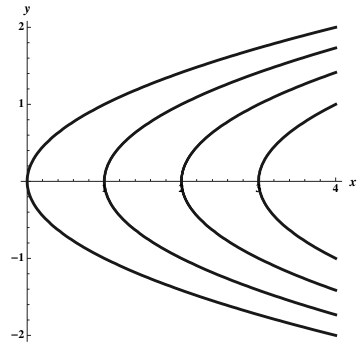
c. B. Notice that the function in (B) becomes large as you get close to $y = x$.

d. C. Notice that the function in (C) is everywhere positive.

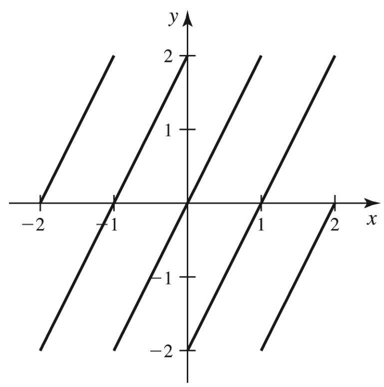
12.2.30



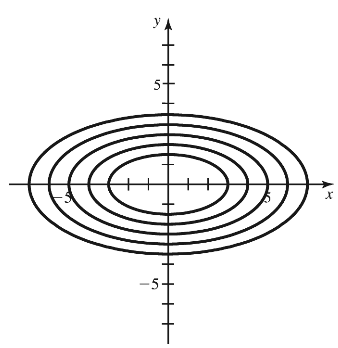
12.2.31



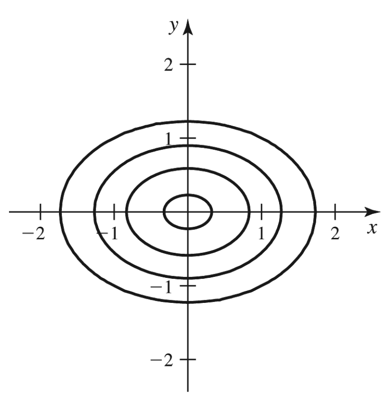
12.2.32



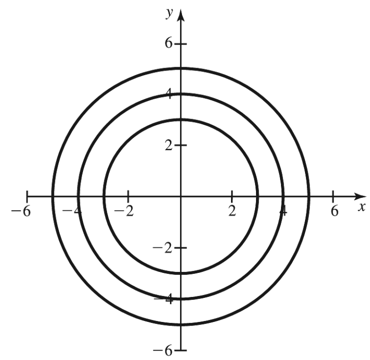
12.2.33



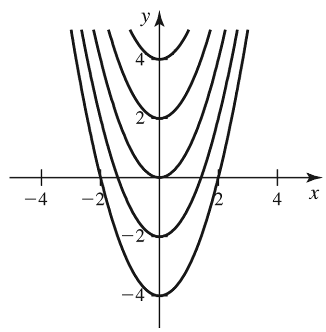
12.2.34



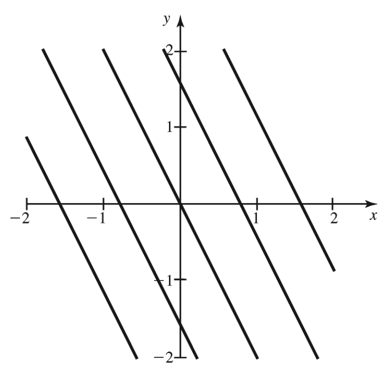
12.2.35



12.2.36



12.2.37



12.2.38

a. B. Notice that the level curves for the function consist of lines parallel to the x-axis.

b. E. Notice that the level curves for the function are hyperbolas.

c. C. Notice that the level curves are oval and are elongated along the y-axis; therefore the level sets match (C).

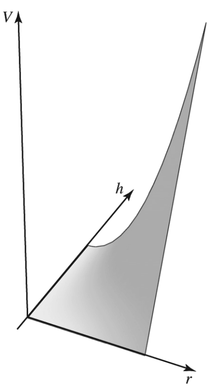
d. D. Notice that the level curves for the function are circles.

e. A. Notice that the level curves are oval and are elongated along the x-axis ; therefore the level sets match (A).

f. F. Notice that the level curves for the function are ellipses.

12.2.39

a.



b. The domain is $D = \{(r, h) : r > 0, h > 0\}$.

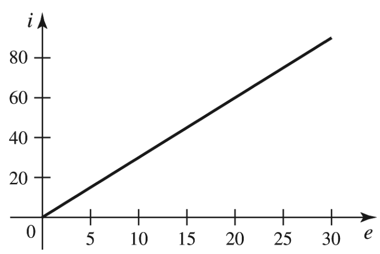
c. We have $\pi r^2 h = 300$, so $h = \frac{300}{\pi r^2}$.

12.2.40

a. His ERA was $A(24, 224) = \frac{9 \cdot 24}{224} = 0.9643$.

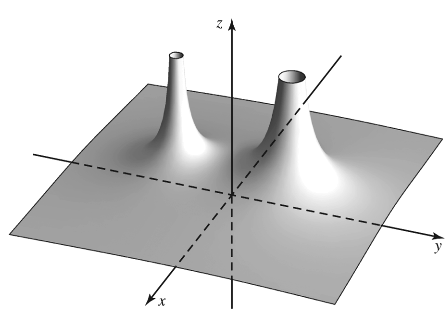
b. His ERA is $A\left(4, \frac{1}{3}\right) = \frac{9 \cdot 4}{1/3} = 108$.

c. The relationship is $e = \frac{i}{3}$, so a pitcher with an ERA of 3 gives up one run every three innings.



12.2.41

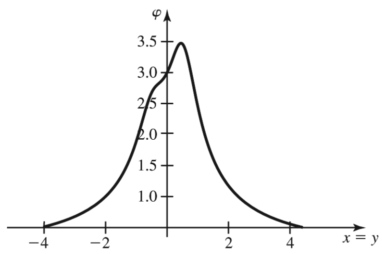
a.



b. The potential function is defined for all (x, y) in $\mathbb{R}^2$ except $(0, 1)$ and $(0, -1)$.

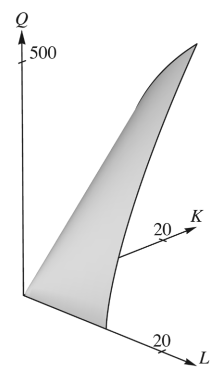
c. We have $\phi(2, 3) \approx 0.93 > \phi(3, 2) \approx 0.87$.

d.



12.2.42

a.

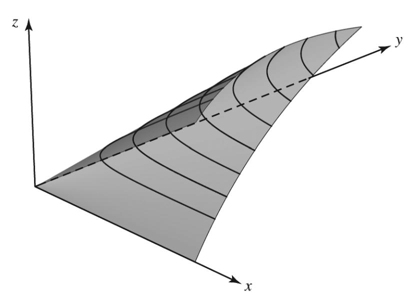


b. We have $Q = cL^a K^b = 40 \cdot (10)^{1/3} K^{2/3}$.

c. We have $Q = cL^a K^b = 40 \cdot (15)^{2/3} L^{1/3}$.

12.2.43

a.

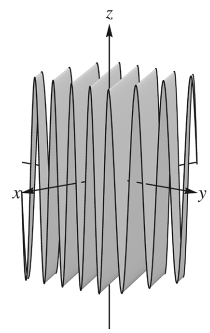


b. The maximum resistance is $R(10, 10) = 5$ ohms.

c. This means $R(x, y) = R(y, x)$.

12.2.44

a.



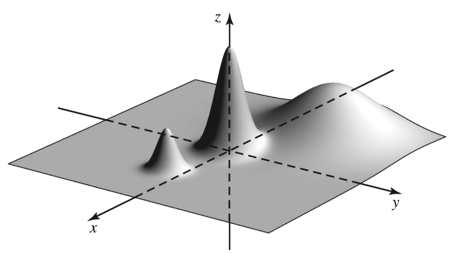
b. The domain of this function is $\mathbb{R}^2$.

c. The maximum and minimum water heights are ± 10.

d. The maximum and minimum heights occur along the lines $2x - 3y = \frac{\pi}{2} + k\pi$, where k is any integer. A vector orthogonal to these lines is $\mathrm{v} = \langle 2, -3 \rangle$.

12.2.45

a.



b. The peaks occur near the points (0, 0), (−5, 3) and (4, −1).

c. We have $f(0, 0) \approx 10.17$, $f(-5, 3) \approx 5.00$, $f(4, -1) \approx 4.00$.

12.2.46 The domain of f is $\mathbb{R}^3$.

12.2.47 The domain of g is $\{(x, y, z) : x \neq z\}$, which is all points in $\mathbb{R}^3$ not on the plane given by $x = z$.

12.2.48 The domain of p is $\{(x, y, z) : x^2 + y^2 + z^2 \geq 9\}$, which is all points in $\mathbb{R}^3$ on or outside the sphere of radius 3 centered at the origin.

12.2.49 The domain of f is $\{(x, y, z) : y \geq z\}$, which is all points in $\mathbb{R}^3$ on or below the plane given by $z = y$.

12.2.50 The domain of Q is $\mathbb{R}^3$.

12.2.51 The domain of F is $\{(x, y, z) : x^2 \leq y\}$, which is all points on the side of the vertical cylinder $y = x^2$ that contains the positive y-axis.

12.2.52 The domain of f is $\{(w, x, y, z) : w^2 + x^2 + y^2 + z^2 \leq 1\}$, which is all points in R^4 on or inside the hypersphere of radius 1 centered at the origin.

12.2.53

a. False. This function has domain $\mathbb{R}^2$.

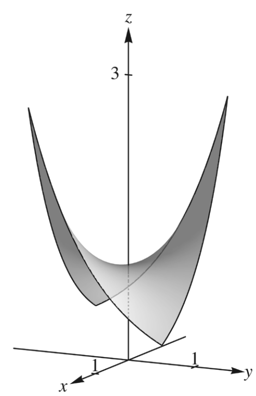
b. False. The domain of a function of 4 variables is a region in $\mathbb{R}^4$.

c. True. The level curves for the function defined by $z = 2x - 3y$ are lines of the form $2x - 3y = c$ for any constant c.

12.2.54

a. The domain is $\mathbb{R}^2$ and the range is the interval $(0, \infty)$.

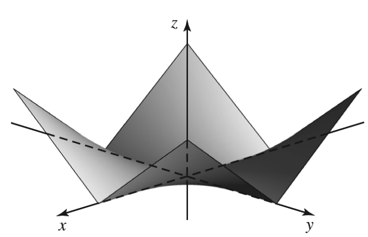
b.



12.2.55

a. The domain is $\mathbb{R}^2$ and the range is the interval $[0, \infty)$.

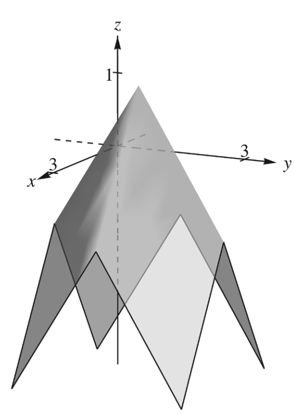
b.



12.2.56

a. The domain is $\mathbb{R}^2$ and the range is $\mathbb{R}$.

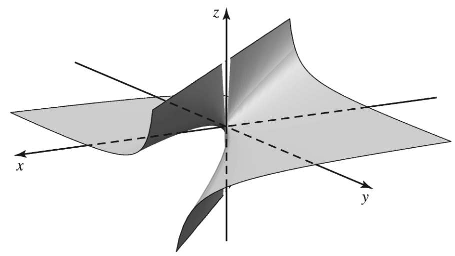
b.



12.2.57

a. The domain is $\{(x, y) : x \neq y\}$ and the range is $\mathbb{R}$.

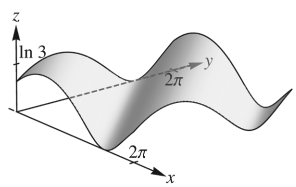
b.



12.2.58

a. The domain is $\mathbb{R}^2$ and the range is the interval $[0, \ln 3]$.

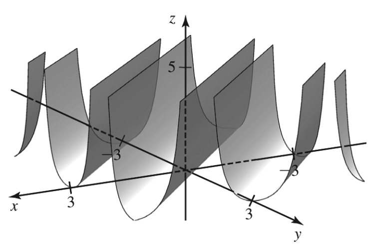
b.



12.2.59

a. The domain is $\{(x, y) : y \neq x + \frac{\pi}{2} + n\pi$ for any integer $n\}$ and the range is the interval $[0, \infty)$.

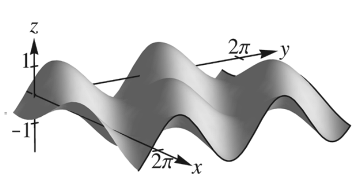
b.



12.2.60

a. The domain is $\mathbb{R}^2$ and the range is the interval $[-1, 1]$.

b.



12.2.61 This function has a peak at the origin.

12.2.62 This function has a peak at the point $\left(\frac{1}{2}, -1\right)$.

12.2.63 This function has a depression at the point (1, 0).

12.2.64 This function has a depression at the point (1, 1).

12.2.65 The level curves are the lines given by $ax + by = d - cz_0$, where z_0 is a constant; these lines all have slope $-\frac{a}{b}$ (in the case $b = 0$ the lines are all vertical).

12.2.66 If $\frac{1}{x^2+y^2+z^2} = K$, then $x^2 + y^2 + z^2 = \frac{1}{K} = C$ for a constant C. Thus, the level surfaces are spheres centered at the origin.

12.2.67 If $x^2+y^2-z = C$, then $z = x^2+y^2-C$, so the level surfaces are paraboloids with vertices $(0, 0, -C)$.

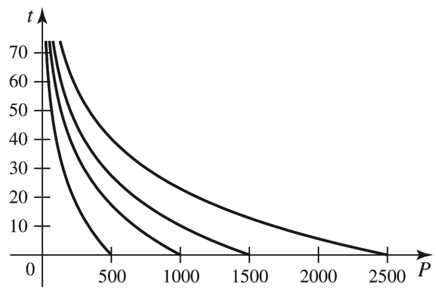
12.2.68 If $x^2 - y^2 - z = C$, then $z = x^2 - y^2 - C$, so the level surfaces are hyperbolic paraboloids with a saddle point at $(0, 0, -C)$.

12.2.69 If $\sqrt{x^2 + 2z^2} = K$, then $x^2 + 2z^2 = C$ where $C = K^2$, so the level surfaces are elliptic cylinders parallel to the y-axis.

12.2.70

a. The set of points (P, t) with $B = 2000$ is given by $Pe^{0.04t} = 2000$; solving for P gives $P = 2000e^{-0.04t}$.

b. The level curves are given by $P = Be^{0.04t}$ with $B = 500, 1000, 1500$ and 2000.

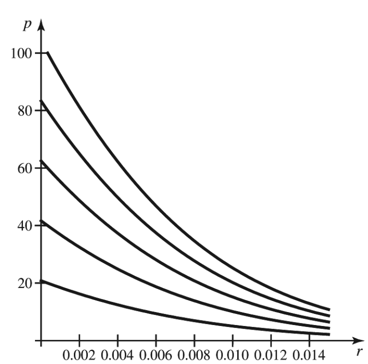


c. As t increases along a level curve, P decreases and vice versa.

12.2.71

a. Solving for P in the equation $B(P, r, t) = 20,000$ with $t = 20$ years gives $P = \frac{20,000r}{(1+r)^{240}-1}$.

b. The level curves are given by $P = \frac{Br}{(1+r)^{240}-1}$,with $B = 5000, 10,000, 15,000$ and $25,000$.

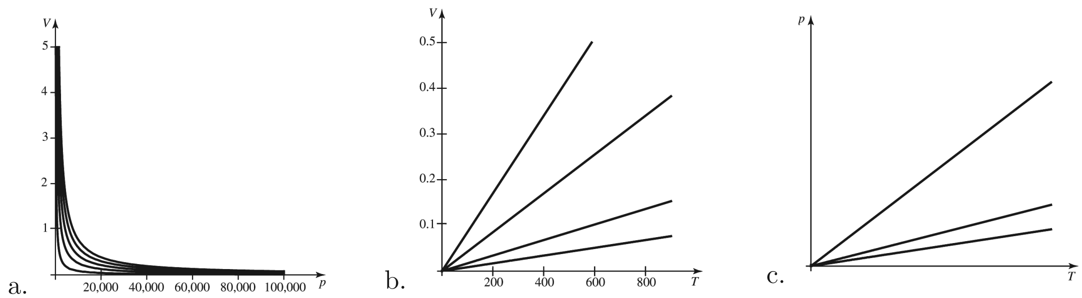


12.2.72

a. His quarterback rating is $R = \frac{50+20\cdot 67.21+80\cdot 7.07-100\cdot 1.45+100\cdot 7.78}{24} = 108.033$.

b. If c, t and y are fixed then R is a decreasing linear function of t. This makes sense since a quarterback's rating should decrease if his interception percentage increases.

12.2.73



12.2.74 Factor the equation $x^2 - (y+z)\,x + yz = (x-y)\,(x-z)$; hence the domain of g is $\{(x,\,y,\,z) : x \neq y \text{ and } x \neq z\}$. The domain consists of all points not on the planes given by $x = y$ and $x = z$.

12.2.75 The domain of f is $\{(x,\,y) : x - 1 \leq y \leq x + 1\}$. This is the region between the two parallel lines given by $y = x - 1$ and $y = x + 1$.

12.2.76 The domain of f is $\{(x,\,y,\,z) : z > x^2 + y^2 - 2x - 3\}$, which is equivalent to $\{(x,\,y,\,z) : z > (x-1)^2 + y^2 - 4\}$. This region consists of all points inside a circular paraboloid with vertex at $(1,\,0,\,-4)$.

12.2.77 Factor the equation $z^2 - xz + yz - xy = (z-x)\,(z+y)$; hence the domain of h is $\{(x,\,y,\,z) : (z-x)\,(z+y) \geq 0\}$, which is equivalent to $D = \{(x,\,y,\,z) : (x \leq z \text{ and } y \geq -z) \text{ or } (x \geq z \text{ and } y \leq -z)\}$. The domain consists of all points above or below both the planes given by $z = x$ and $z = -y$ as well as the points on either one of these planes.

12.2.78

a. This "ball" is the solid octahedron with vertices $(\pm 1,\,0,\,0)$, $(0,\,\pm 1,\,0)$ and $(0,\,0,\,\pm 1)$.

b. This "ball" is the solid cube with vertices $(x,\,y,\,z)$ where $x,\,y,\,z = \pm 1$.

12.3 Limits and Continuity

12.3.1 The values of $|f\,(x,\,y) - L|$ can be made arbitrarily small if $(x,\,y)$ is sufficiently close to $(a,\,b)$.

12.3.2 If $f\,(x,\,y)$ has a different limit as$(x,\,y)$ approaches $(a,\,b)$ along two different paths, then $\lim_{(x,\,y)\to(a,\,b)} f\,(x,\,y)$ does not exist.

12.3.3 If $f\,(x,\,y)$ is a polynomial, then $\lim_{(x,\,y)\to(a,\,b)} f\,(x,\,y) = f\,(a,\,b)$; in other words, the limit can be found by plugging in $x = a$, $y = b$ in $f\,(x,\,y)$.

12.3.4 We evaluate $\lim_{(x,\,y)\to(a,\,b)} f\,(x,\,y)$ along all paths that approach $(a,\,b)$ from within the domain of f.

12.3.5 If the limits along different paths do not agree, then the limit does not exist.

12.3.6 Evaluating $\lim_{(x,\,y)\to(a,\,b)} f\,(x,\,y)$ along a finite number of paths does not establish that the limit is the same along *all* paths that approach $(a,\,b)$.

12.3.7 The function f must be defined at $(a,\,b)$, $\lim_{(x,\,y)\to(a,\,b)} f\,(x,\,y)$ must exist, and the limit must equal $f\,(a,\,b)$.

12.3.8 Yes, (0, 0) is a boundary point for R becauseevery disc centered at (0, 0) contains points in R and points not in R (namely (0, 0)). The set R is neither open nor closed because it contains some of its boundary points (namely points on the unit circle) but not all of them (namely (0, 0)).

12.3.9 A rational function is continuous at all points where its denominator is nonzero.

12.3.10 Since xy^2z^3 is a polynomial, $\lim_{(x,\,y,\,z)\to(1,\,1,\,-1)} xy^2z^3 = 1\cdot 1^2\cdot(-1)^3 = -1$.

12.3.11 $\lim_{(x,\,y)\to(2,\,9)} 101 = 101$.

12.3.12 $\lim_{(x,\,y)\to(1,\,-3)} (3x+4y-2) = 3\cdot 1 + 4\,(-3) - 2 = -11$.

12.3.13 $\lim_{(x,\,y)\to(-3,\,3)} \left(4x^2 - y^2\right) = 4\cdot(-3)^2 - (3)^2 = 27$.

12.3.14 $\lim_{(x,\,y)\to(2,\,-1)} \left(xy^8 - 3x^2y^3\right) = 2\,(-1)^8 - 3\cdot 2^2\,(-1)^3 = 14$.

12.3.15 $\lim_{(x,\,y)\to(0,\,\pi)} \frac{\cos xy+\sin xy}{2y} = \frac{\cos 0+\sin 0}{2\pi} = \frac{1}{2\pi}$.

12.3.16 $\lim_{(x,\,y)\to(e^2,\,4)} \ln\sqrt{xy} = \ln\sqrt{4e^2} = \ln 2e = 1+\ln 2$.

12.3.17 $\lim_{(x,\,y)\to(2,\,0)} \frac{x^2-3xy^2}{x+y} = \frac{2^2-3\cdot 2\cdot 0^2}{2+0} = 2$.

12.3.18 $\lim_{(u,\,v)\to(1,\,-1)} \frac{10uv-2v^2}{u^2+v^2} = \frac{10\cdot 1(-1)-2(-1)^2}{1^2+(-1)^2} = -6$.

12.3.19 $\lim_{(x,\,y)\to(6,\,2)} \frac{x^2-3xy}{x-3y} = \lim_{(x,\,y)\to(6,\,2)} \frac{x(x-3y)}{x-3y} = \lim_{(x,\,y)\to(6,\,2)} x = 6$.

12.3.20 $\lim_{(x,\,y)\to(1,\,-2)} \frac{y^2+2xy}{y+2x} = \lim_{(x,\,y)\to(1,\,-2)} \frac{y(y+2x)}{y+2x} = \lim_{(x,\,y)\to(1,\,-2)} y = -2$.

12.3.21 $\lim_{(x,y)\to(3,1)} \frac{x^2-7xy+12y^2}{x-3y} = \lim_{(x,y)\to(3,1)} \frac{(x-3y)(x-4y)}{x-3y} = \lim_{(x,y)\to(3,1)}(x-4y) = 3-4 = -1$.

12.3.22 $\lim_{(x,y)\to(-1,1)} \frac{2x^2-xy-3y^2}{x+y} = \lim_{(x,y)\to(-1,1)} \frac{(x+y)(2x-3y)}{x+y} = \lim_{(x,y)\to(-1,1)}(2x-3y) = -5$.

12.3.23 $\lim_{(x,\,y)\to(2,\,2)} \frac{y^2-4}{xy-2x} = \lim_{(x,\,y)\to(2,\,2)} \frac{(y+2)(y-2)}{x(y-2)} = \lim_{(x,\,y)\to(2,\,2)} \frac{y+2}{x} = \frac{2+2}{2} = 2$.

12.3.24 $\lim_{(x,\,y)\to(4,\,5)} \frac{\sqrt{x+y}-3}{x+y-9} = \lim_{(x,\,y)\to(4,\,5)} \frac{\left(\sqrt{x+y}-3\right)\left(\sqrt{x+y}+3\right)}{(x+y-9)\left(\sqrt{x+y}+3\right)} = \lim_{(x,\,y)\to(4,\,5)} \frac{x+y-9}{(x+y-9)\left(\sqrt{x+y}+3\right)} = \lim_{(x,\,y)\to(4,\,5)} \frac{1}{\sqrt{x+y}+3} = \frac{1}{\sqrt{4+5}+3} = \frac{1}{6}$.

12.3.25 $\lim_{(x,\,y)\to(1,\,2)} \frac{\sqrt{y}-\sqrt{x+1}}{y-x-1} = \lim_{(x,\,y)\to(1,\,2)} \frac{\left(\sqrt{y}-\sqrt{x+1}\right)\left(\sqrt{y}+\sqrt{x+1}\right)}{(y-x-1)\left(\sqrt{y}+\sqrt{x+1}\right)} =$ $\lim_{(x,\,y)\to(1,\,2)} \frac{y-x-1}{(y-x-1)\left(\sqrt{y}+\sqrt{x+1}\right)} = \lim_{(x,\,y)\to(1,\,2)} \frac{1}{\sqrt{y}+\sqrt{x+1}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$.

12.3.26 $\lim_{(u,\,v)\to(8,\,8)} \frac{u^{1/3}-v^{1/3}}{u^{2/3}-v^{2/3}} = \lim_{(u,\,v)\to(8,\,8)} \frac{u^{1/3}-v^{1/3}}{\left(u^{1/3}-v^{1/3}\right)\left(u^{1/3}+v^{1/3}\right)} = \lim_{(u,\,v)\to(8,\,8)} \frac{1}{u^{1/3}+v^{1/3}} = \frac{1}{2+2} = \frac{1}{4}$.

12.3.27 Observe that along the line $y=0$, $\lim_{(x,\,y)\to(0,\,0)} \frac{x+2y}{x-2y} = \lim_{x\to 0} \frac{x}{x} = 1$, whereas along the line $x=0$, $\lim_{(x,\,y)\to(0,\,0)} \frac{x+2y}{x-2y} = \lim_{y\to 0} \frac{2y}{-2y} = -1$.

12.3.28 Observe that along the line $y=x$, $\lim_{(x,\,y)\to(0,\,0)} \frac{4x\,y}{3x^2+y^2} = \lim_{x\to 0} \frac{4x^2}{4x^2} = 1$, whereas along the line $y=-x$, $\lim_{(x,\,y)\to(0,\,0)} \frac{4xy}{3x^2+y^2} = \lim_{x\to 0} -\frac{4x^2}{4x^2} = -1$.

12.3.29 Observe that along the line $x=0$, $\lim_{(x,\,y)\to(0,\,0)} \frac{y^4-2x^2}{y^4+x^2} = \lim_{y\to 0} \frac{y^4}{y^4} = 1$, whereas along the line $y=0$, $\lim_{(x,\,y)\to(0,\,0)} \frac{y^4-2x^2}{y^4+x^2} = \lim_{x\to 0} -\frac{2x^2}{x^2} = -2$.

12.3.30 Observe that along the line $y = 0$, $\lim_{(x,\,y)\to(0,\,0)} \frac{x^3-y^2}{x^3+y^2} = \lim_{x\to 0} \frac{x^3}{x^3} = 1$, whereas along the line $x = 0$, $\lim_{(x,\,y)\to(0,\,0)} \frac{x^3-y^2}{x^3+y^2} = \lim_{y\to 0} -\frac{y^2}{y^2} = -1$.

12.3.31 Observe that along the line $y = x$, $\lim_{(x,\,y)\to(0,\,0)} \frac{y^3+x^3}{xy^2} = \lim_{x\to 0} \frac{2x^3}{x^3} = 2$, whereas along the line $y = -x$, $\lim_{(x,\,y)\to(0,\,0)} \frac{y^3+x^3}{xy^2} = \lim_{x\to 0} \frac{0}{-x^3} = 0$.

12.3.32 Observe that along the line $y = 0$, $\lim_{(x,\,y)\to(0,\,0)} \frac{y}{\sqrt{x^2-y^2}} = \lim_{x\to 0} \frac{0}{|x|} = 0$, whereas along the ray $x = 2y$, $y > 0$, $\lim_{(x,\,y)\to(0,\,0)} \frac{y}{\sqrt{x^2-y^2}} = \lim_{x\to 0} \frac{y}{\sqrt{3}y} = \frac{1}{\sqrt{3}}$.

12.3.33 The function f is continuous on $\mathbb{R}^2$.

12.3.34 The function f is continuous on $\mathbb{R}^2$ (the denominator of this rational function is always positive).

12.3.35 The function p is continuous at all points except the origin, where it is undefined.

12.3.36 The function S is continuous at all points except where $x^2 = y^2$, which is along the lines $x = y$ and $x = -y$.

12.3.37 The function f is continuous on $\mathbb{R}^2$ except where $x = 0$.

12.3.38 The function f is continuous on $\mathbb{R}^2$ except where $x = 0$ or $y = \pm 1$.

12.3.39 The function f is continuous on $\mathbb{R}^2$ except the origin. Note that along the line $y = x$, $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+x^2} = \frac{1}{2} \neq 0$.

12.3.40 The function f is continuous on $\mathbb{R}^2$ except the origin. Note that along the line $x = 0$, $\lim_{(x,y)\to(0,0)} \frac{y^4-2x^2}{y^4+x^2} = \lim_{(x,y)\to(0,0)} \frac{y^4}{y^4} = 1 \neq 0$.

12.3.41 The function f is continuous on $\mathbb{R}^2$.

12.3.42 The function f is continuous on $\mathbb{R}^2$.

12.3.43 The function f is continuous on $\mathbb{R}^2$.

12.3.44 The function g is continuous on its domain, which is $D = \{(x,\,y) : x > y\}$.

12.3.45 The function h is continuous on $\mathbb{R}^2$.

12.3.46 The function p is continuous on $\mathbb{R}^2$.

12.3.47 The function f is continuous on its domain, which is $D = \{(x,\,y) : (x,\,y) \neq (0,\,0)\}$.

12.3.48 The function f is continuous on its domain, which is $D = \{(x,\,y) : x^2 + y^2 \leq 4\}$.

12.3.49 The function g is continuous on $\mathbb{R}^2$.

12.3.50 The function h is continuous on its domain, which is $D = \{(x,\,y) : x \geq y\}$.

12.3.51 The function f is continuous on $\mathbb{R}^2$.

12.3.52 The function f is continuous on $\mathbb{R}^2$.

12.3.53 $\lim_{(x,\,y,\,z)\to(1,\,\ln 2,\,3)} z\, e^{xy} = 3\, e^{1\cdot \ln 2} = 6$.

12.3.54 $\lim_{(x,\,y,\,z)\to(0,\,1,\,0)} \ln(1+y)\; e^{x\,z} = (\ln 2)\; e^0 = \ln 2$.

12.3.55 $\lim_{(x,\,y,\,z)\to(1,\,1,\,1)} \frac{yz-xy-xz-x^2}{yz+xy+xz-y^2} = \frac{1-1-1-1}{1+1+1-1} = -1$.

12.3.56 $\lim_{(x,\,y,\,z)\to(1,\,1,\,1)} \frac{x-\sqrt{xz}-\sqrt{xy}+\sqrt{yz}}{x-\sqrt{xz}+\sqrt{xy}-\sqrt{yz}} = \lim_{(x,\,y,\,z)\to(1,\,1,\,1)} \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}-\sqrt{z})}{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{z})} =$
$\lim_{(x,\,y,\,z)\to(1,\,1,\,1)} \frac{(\sqrt{x}-\sqrt{y})}{(\sqrt{x}+\sqrt{y})} = \frac{1-1}{1+1} = 0.$

12.3.57 $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2+xy-xz-yz}{x-z} = \lim_{(x,y,z)\to(1,1,1)} \frac{(x-z)(x+y)}{x-z} = \lim_{(x,y,z)\to(1,1,1)}(x+y) = 2.$

12.3.58 $\lim_{(x,y,z)\to(1,-1,1)} \frac{xz+5x+yz+5y}{x+y} = \lim_{(x,y,z)\to(1,-1,1)} \frac{(x+y)(z+5)}{x+y} = \lim_{(x,y,z)\to(1,-1,1)}(z+5) = 6.$

12.3.59

a. False. The limit may be different or not exist along other paths approaching (0, 0).

b. False. We may have $f(a,\,b)$ undefined, or $f(a,\,b) \neq L$.

c. True. The limit must exist for f to be continuous at $(a,\,b)$.

d. False. For example, take $P = (0,\,0)$ and the domain of f to be $\{(x,\,y) : (x,\,y) \neq (0,\,0)\}$.

12.3.60 Observe that along the line $y = 0$, $\lim_{(x,\,y)\to(0,\,0)} \frac{y^2}{x^8+y^2} = \lim_{x\to 0} \frac{0}{x^8} = 0$, whereas along the curve $y = x^4$, $\lim_{(x,\,y)\to(0,\,0)} \frac{y^2}{x^8+y^2} = \lim_{x\to 0} \frac{x^8}{2x^8} = \frac{1}{2}$, therefore this limit does not exist.

12.3.61 $\lim_{(x,\,y)\to(0,\,1)} \frac{y\sin x}{x(y+1)} = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{y\to 1} \frac{y}{y+1}\right) = 1 \cdot \frac{1}{2} = \frac{1}{2}.$

12.3.62 $\lim_{(x,\,y)\to(1,\,1)} \frac{x^2+x\,y-2y^2}{2x^2-x\,y-y^2} = \lim_{(x,\,y)\to(1,\,1)} \frac{(x+2y)(x-y)}{(2x+y)(x-y)} = \lim_{(x,\,y)\to(1,\,1)} \frac{x+2y}{2x+y} = \frac{1+2}{2+1} = 1.$

12.3.63 $\lim_{(x,\,y)\to(1,\,0)} \frac{y\ln y}{x} = \left(\lim_{x\to 1} \frac{1}{x}\right)(\lim_{y\to 0} \ln y^y) = 1 \cdot \ln 1 = 0$

12.3.64 Observe that along the line $y = x$, $\lim_{(x,\,y)\to(0,\,0)} \frac{|\,xy\,|}{xy} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$, whereas along the line $y = -x$, $\lim_{(x,\,y)\to(0,\,0)} \frac{|\,xy\,|}{xy} = \lim_{x\to 0} \frac{x^2}{-x^2} = -1$, therefore this limit does not exist.

12.3.65 Observe that along the line $y = x$, $\lim_{(x,\,y)\to(0,\,0)} \frac{|\,x-y\,|}{|\,x+y\,|} = \lim_{x\to 0} \frac{0}{2|x|} = 0$, whereas along the line $y = 2x$, $\lim_{(x,\,y)\to(0,\,0)} \frac{|\,x-y\,|}{|\,x+y\,|} = \lim_{x\to 0} \frac{|x|}{3|x|} = \frac{1}{3}$, therefore this limit does not exist.

12.3.66 $\lim_{(u,\,v)\to(-1,\,0)} \frac{uv\,e^{-v}}{u^2+v^2} = \frac{(-1)\cdot 0\cdot e^{o}}{(-1)^2+0^2} = 0.$

12.3.67 Observe that $\lim_{(x,\,y)\to(2,\,0)} \frac{1-\cos y}{xy^2} = \left(\lim_{x\to 2} \frac{1}{x}\right)\left(\lim_{y\to 0} \frac{1-\cos y}{y^2}\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ where the y-limit is evaluated by two applications of L′Hôpital′s rule.

12.3.68 $\lim_{(x,\,y)\to(0,\,0)} \frac{x-y}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r\cos\theta - r\sin\theta}{r} = \lim_{r\to 0}(\cos\theta - \sin\theta)$, which does not exist.

12.3.69 $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r^2} = \lim_{r\to 0} r\cos^2\theta\sin\theta = 0.$

12.3.70 Note that $2+(x+y)^2+(x-y)^2 = 2+2x^2+2y^2$. Making the polar substitution yields

$$\lim_{r\to 0} \tan^{-1}\left(\frac{2+2r^2}{2e^{r^2}}\right) = \tan^{-1}(1) = \frac{\pi}{4}.$$

12.3.71 $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2+x^2y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2+r^4\cos^2\theta\sin^2\theta}{r^2} = \lim_{r\to 0}(1+r^2\cos^2\theta\sin^2\theta) = 1.$

12.3.72

a. Let $z = x+y$; then $z \to 0$ as $(x,\,y) \to (0,\,0)$ so $\lim_{(x,\,y)\to(0,\,0)} \frac{\sin(x+y)}{x+y} = \lim_{z\to 0} \frac{\sin z}{z} = 1.$

b. Let $z = x + y$, then $\frac{\sin x + \sin y}{x+y} = \frac{\sin x + \sin(z-x)}{z} = \frac{\sin x + \sin z \cos x - \cos z \sin x}{z} = \sin x\left(\frac{1-\cos z}{z}\right) + \cos x\left(\frac{\sin z}{z}\right)$. As $(x,\, y) \to (0,\, 0)$ we have $z \to 0$; $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$, so $\lim_{(x,\, y)\to(0,\, 0)} \frac{\sin x + \sin y}{x+y} = 0\cdot 0 + 1\cdot 1 = 1$.

12.3.73 Let $u = x^2 + y^1 - 1$. Then as $x^2 + y^2 \to 1$, $u \to 0$. Because $\lim_{u\to 0} \frac{\sin u}{u} = 1$, we must have $b = 1$ in order for f to be continuous everywhere.

12.3.74 Let $u = xy$. Note that as $xy \to 0$, $u \to 0$. Also, $\lim_{xy\to 0} \frac{1+2xy-\cos(xy)}{xy} = \lim_{u\to 0} \frac{1+2u-\cos u}{u} = \lim_{u\to 0} \frac{2+\sin u}{1} = 2$ by L'Hôpital's rule. Thus, we must have $a = 2$ in order for f to be continuous everywhere.

12.3.75 The limit is 0 along the lines $x = 0$ or $y = 0$. However, along the line $x = y$ we have $\lim_{(x,y)\to(0,0)} \frac{a\,x^m y^n}{b\,x^{m+n} + c\,y^{m+n}} = \lim_{x\to 0} \frac{ax^{m+n}}{bx^{n+m} + cx^{m+n}} = \frac{a}{b+c} \neq 0$ because $a \neq 0$. Therefore this limit does not exist.

12.3.76 The limit is 0 along the line $y = 0$. However, along the curve $y = x^2$ we have $\lim_{(x,y)\to(0,0)} \frac{a\,x^{2(p-n)} y^n}{b\,x^{2p} + c\,y^p} = \lim_{x\to 0} \frac{ay^p}{(b+c)y^p} = \frac{a}{b+c} \neq 0$ because $a \neq 0$. Therefore this limit does not exist.

12.3.77 Let $u = xy$; then $u \to 0$ as $(x,\, y) \to (1,\, 0)$, so $\lim_{(x,\, y)\to(1,\, 0)} \frac{\sin xy}{xy} = \lim_{u\to 0} \frac{\sin u}{u} = 1$.

12.3.78 Let $u = xy$; then $u \to 0$ as $(x,\, y) \to (4,\, 0)$, so $\lim_{(x,\, y)\to(4,\, 0)} x^2 y \ln xy = \lim_{(x,\, y)\to(4,\, 0)} x\cdot xy \ln xy = 4\lim_{u\to 0} u\ln u = 4\lim_{u\to 0} \ln u^u = 4\cdot 0 = 0$.

12.3.79 Let $u = xy$; then $u \to 0$ as $(x,\, y) \to (0,\, 2)$, so $\lim_{(x,\, y)\to(0,\, 2)} (2xy)^{xy} = \lim_{u\to 0} 2^u u^u = 1$.

12.3.80 Let $u = xy$; then $u \to 0$ as $(x,\, y) \to \left(0,\, \frac{\pi}{2}\right)$, so $\lim_{(x,\, y)\to(0,\, \pi/2)} \frac{1-\cos xy}{4x^2y^3} = \lim_{(x,\, y)\to(0,\, \pi/2)} \left(\frac{1}{4y}\cdot\frac{1-\cos xy}{x^2y^2}\right) = \frac{1}{2\pi}\lim_{u\to 0} \frac{1-\cos u}{u^2} = \frac{1}{4\pi}$. (Use L'Hôpital's rule twice to see that $\lim_{u\to 0} (1-\cos u)/u^2 = 1/2$).

12.3.81 Because $\lim_{(x,\, y)\to(0,\, 0)} e^{-1/\left(x^2+y^2\right)} = 0$, we should define $f(0,\, 0) = 0$.

12.3.82 For any $\epsilon > 0$, let $\delta = \epsilon$; then $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |\,y-b\,| < \epsilon$ because $|\,y-b\,| \leq \sqrt{(x-a)^2 + (y-b)^2}$.

12.3.83 For any $\epsilon > 0$, let $\delta = \frac{\epsilon}{2}$. Then $|\,x-a\,|,\, |\,y-b\,| \leq \sqrt{(x-a)^2 + (y-b)^2}$, so $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ which implies that $|x + y - (a+b)\,| \leq |\,x-a\,| + |\,y-b\,| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

12.3.84 Suppose $\lim_{(x,\, y)\to(a,\, b)} f(x,\, y) = L$, $\lim_{(x,\, y)\to(a,\, b)} g(x,\, y) = M$. Let $\epsilon > 0$. Then there exist $\delta_1,\, \delta_2 > 0$ such that $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_1 \implies |\,f(x,\, y) - L\,| < \frac{\epsilon}{2}$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_2 \implies |\,g(x,\, y) - M\,| < \frac{\epsilon}{2}$. Let $\delta = \min(\delta_1,\, \delta_2)$. Then $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies$ $|\,f(x,\, y) + g(x,\, y) - (L+M)\,| \leq |\,f(x,\, y) - L\,| + |\,g(x,\, y) - M\,| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

12.3.85 Observe first that this is trivial when $c = 0$, so assume $c \neq 0$ and let $\epsilon > 0$. Then there exists $\delta > 0$ such that $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |\,f(x,\, y) - L\,| < \frac{\epsilon}{|c|}$. Therefore $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |\,c\,f(x,\, y) - c\,L\,| = |c|\,|\,f(x,\, y) - L\,| < |c|\cdot\frac{\epsilon}{|c|} = \epsilon$.

12.4 Partial Derivatives

12.4.1 The slope parallel to the x-axis is $f_x(a,b)$, and the slope parallel to the y-axis is $f_y(a,b)$.

12.4.2 $f_x(x,y) = 6xy + y^3,\ f_y(x,y) = 3x^2 + 3xy^2$.

12.4.3 $f_x(x,y) = \cos(xy) + x(-\sin(xy))\,y = \cos(xy) - xy\sin(xy),\ f_y(x,y) = x(-\sin(xy))\,x = -x^2\sin(xy)$.

12.4.4 We have $f_x(x,y) = 6xy + y^3,\ f_y(x,y) = 3x^2 + 3xy^2$; therefore $f_{xx}(x,y) = 6y,\quad f_{yy}(x,y) = 6xy,\quad f_{xy}(x,y) = f_{yx}(x,y) = 6x + 3y^2$

12.4.5 Think of x and y as being fixed, and differentiate with respect to the variable z.

12.4.6 Note that $\frac{\partial V}{\partial r} = 2\pi rh > 0$, so the volume is an increasing function of the radius r if the height h is fixed.

12.4.7 $f_x = \lim_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \dfrac{5(x+h)y - 5xy}{h} = \lim_{h\to 0} \dfrac{5hy}{h} = \lim_{h\to 0} 5y = 5y.$
$f_y = \lim_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h} = \lim_{h\to 0} \dfrac{5x(y+h) - 5xy}{h} = \lim_{h\to 0} \dfrac{5xh}{h} = \lim_{h\to 0} 5x = 5x.$

12.4.8 $f_x = \lim_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \dfrac{x+h+y^2+4-(x+y^2+4)}{h} = \lim_{h\to 0} \dfrac{h}{h} = \lim_{h\to 0} 1 = 1.$
$f_y = \lim_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h} = \lim_{h\to 0} \dfrac{x+(y+h)^2+4-(x+y^2+4)}{h} = \lim_{h\to 0} \dfrac{y^2+2hy+h^2-y^2}{h} = \lim_{h\to 0} 2y + h = 2y.$

12.4.9 $f_x = \lim_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \dfrac{\frac{x+h}{y} - \frac{x}{y}}{h} = \lim_{h\to 0} \dfrac{h}{hy} = \lim_{h\to 0} \dfrac{1}{y} = \dfrac{1}{y}.$
$f_y = \lim_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h} = \lim_{h\to 0} \dfrac{\frac{x}{y+h} - \frac{x}{y}}{h} = \lim_{h\to 0} \dfrac{xy - x(y+h)}{y(y+h)h} = \lim_{h\to 0} \dfrac{-x}{y(y+h)} = -\dfrac{x}{y^2}.$

12.4.10

$$\begin{aligned} f_x &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)y} - \sqrt{xy}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{(x+h)y} - \sqrt{xy}}{h} \cdot \frac{\sqrt{(x+h)y} + \sqrt{xy}}{\sqrt{(x+h)y} + \sqrt{xy}} \\ &= \lim_{h\to 0} \frac{xy + hy - xy}{h(\sqrt{(x+h)y} + \sqrt{xy})} = \frac{y}{2\sqrt{xy}}. \end{aligned}$$

$$\begin{aligned} f_y &= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} = \lim_{h\to 0} \frac{\sqrt{x(y+h)} - \sqrt{xy}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x(y+h)} - \sqrt{xy}}{h} \cdot \frac{\sqrt{x(y+h)} + \sqrt{xy}}{\sqrt{x(y+h)} + \sqrt{xy}} \\ &= \lim_{h\to 0} \frac{xy + xh - xy}{h(\sqrt{x(y+h)} + \sqrt{xy})} = \frac{x}{2\sqrt{xy}}. \end{aligned}$$

12.4.11 $f_x(x,y) = 6x,\ f_y(x,y) = 12y^2$.

12.4.12 $f_x(x,y) = 2xy,\ f_y(x,y) = x^2$.

12.4.13 $f_x(x,y) = 6xy,\ f_y(x,y) = 3x^2$.

12.4.14 $f_x(x,y) = 12x^5 + 2y,\ f_y(x,y) = 8y^7 + 2x$.

12.4.15 $f_x(x,y)=e^y,\ f_y(x,y)=xe^y.$

12.4.16 $f_x(x,y)=(y/x)\cdot(1/y)=1/x,\ f_y(x,y)=(y/x)\cdot(-x/y^2)=-1/y.$

12.4.17 $g_x(x,y)=(-\sin(2xy))\,2y=-2y\sin(2xy),\ g_y(x,y)=(-\sin(2xy))\,2x=-2x\sin(2xy).$

12.4.18 $h_x(x,y)=\left(y^2+1\right)e^x,\ h_y(x,y)=2ye^x.$

12.4.19 $f_x(x,y)=2xye^{x^2y},\ f_y(x,y)=x^2e^{x^2y}.$

12.4.20 $f_s(s,t)=\frac{(s+t)(1)-(s-t)(1)}{(s+t)^2}=\frac{2t}{(s+t)^2}.\ f_t(s,t)=\frac{(s+t)(-1)-(s-t)(1)}{(s+t)^2}=\frac{-2s}{(s+t)^2}.$

12.4.21 $f_w(w,z)=\frac{\left(w^2+z^2\right)\cdot 1-w\cdot 2w}{(w^2+z^2)^2}=\frac{z^2-w^2}{(w^2+z^2)^2},\ f_z(w,z)=-w\left(w^2+z^2\right)^{-2}\cdot 2z=-\frac{2wz}{(w^2+z^2)^2}.$

12.4.22 $g_x(x,z)=1\cdot\ln\left(z^2+x^2\right)+x\cdot\frac{2x}{z^2+x^2}=\ln\left(z^2+x^2\right)+\frac{2x^2}{z^2+x^2},\ g_z(x,z)=x\cdot\frac{2z}{z^2+x^2}=\frac{2xz}{z^2+x^2}.$

12.4.23 $s_y(y,z)=z^2\left(\sec^2 yz\right)z=z^3\sec^2 yz,\ s_z(y,z)=2z\tan yz+z^2\left(\sec^2 yz\right)y=2z\tan yz+yz^2\sec^2 yz$

12.4.24 $F_p(p,q)=\frac{1}{2}\left(p^2+pq+q^2\right)^{-1/2}(2p+q)=\frac{2p+q}{2\sqrt{p^2+pq+q^2}},\ \ F_q(p,q)=\frac{1}{2}\left(p^2+pq+q^2\right)^{-1/2}(p+2q)=\frac{p+2q}{2\sqrt{p^2+pq+q^2}}.$

12.4.25 $G_s(s,t)=\frac{t}{2\sqrt{st}}\cdot\frac{1}{s+t}+\sqrt{st}\cdot-\frac{1}{(s+t)^2}=\frac{\sqrt{st}(s+t)-2s\sqrt{st}}{2s(s+t)^2}=\frac{\sqrt{st}(t-s)}{2s(s+t)^2},\ \ G_t(s,t)=\frac{s}{2\sqrt{st}}\cdot\frac{1}{s+t}+\sqrt{st}\cdot-\frac{1}{(s+t)^2}=\frac{\sqrt{st}(s+t)-2t\sqrt{st}}{2t(s+t)^2}=\frac{\sqrt{st}(s-t)}{2t(s+t)^2}.$

12.4.26 $h_u(u,v)=\frac{1}{2}\left(\frac{uv}{u-v}\right)^{-1/2}\left(\frac{(u-v)v-uv\cdot 1}{(u-v)^2}\right)=-\frac{1}{2}u^{-1/2}v^{3/2}(u-v)^{-3/2},$
$h_v(u,v)=\frac{1}{2}\left(\frac{uv}{u-v}\right)^{-1/2}\left(\frac{(u-v)u-uv\cdot(-1)}{(u-v)^2}\right)=\frac{1}{2}u^{3/2}v^{-1/2}(u-v)^{-3/2}$

12.4.27 $f_x(x,y)=2yx^{2y-1},\ f_y(x,y)=2x^{2y}\ln x.$

12.4.28 $f_x(x,y)=\frac{1}{2}\cdot(x^2y^3)^{-1/2}(2xy^3)=\frac{xy^3}{\sqrt{x^2y^3}},\ f_y(x,y)=\frac{1}{2}\cdot(x^2y^3)^{-1/2}(3x^2y^2)=\frac{3x^2y^2}{2\sqrt{x^2y^3}}.$

12.4.29 We have $h_x(x,y)=3x^2+y^2$, $h_y(x,y)=2xy$; therefore $h_{xx}(x,y)=6x$, $h_{yy}(x,y)=2x$, $h_{xy}(x,y)=h_{yx}(y,x)=2y$.

12.4.30 We have $f_x(x,y)=10x^4y^2+2xy$, $f_y(x,y)=4x^5y+x^2$; therefore $f_{xx}(x,y)=40x^3y^2+2y$, $f_{yy}(x,y)=4x^5$, $f_{xy}(x,y)=f_{yx}(y,x)=20x^4y+2x$.

12.4.31 We have $f_x(x,y)=2xy^3$, $f_y(x,y)=3x^2y^2$; therefore $f_{xx}(x,y)=2y^3$, $f_{yy}(x,y)=6x^2y$, $f_{xy}(x,y)=f_{yx}(x,y)=6xy^2$.

12.4.32 We have $f_x(x,y)=2(x+3y)$, $f_y(x,y)=6(x+3y)$; therefore $f_{xx}(x,y)=2$, $f_{yy}(x,y)=18$, $f_{xy}(x,y)=f_{yx}(x,y)=6$.

12.4.33 We have $f_x(x,y)=4y^3\cos 4x$, $f_y(x,y)=3y^2\sin 4x$; therefore $f_{xx}(x,y)=-16y^3\sin 4x$, $f_{yy}(x,y)=6y\sin 4x$, $f_{xy}(x,y)=f_{yx}(y,x)=12y^2\cos 4x$.

12.4.34 We have $f_x(x,y)=-y\sin xy$, $f_y(x,y)=-x\sin xy$; therefore $f_{xx}(x,y)=-y^2\cos xy$, $f_{yy}(x,y)=-x^2\cos xy$, $f_{xy}(x,y)=f_{yx}(y,x)=-\sin xy-xy\cos xy$.

12.4.35 We have $p_u(u,v)=\frac{2u}{u^2+v^2+4}$, $p_v(u,v)=\frac{2v}{u^2+v^2+4}$; therefore $p_{uu}(u,v)=\frac{\left(u^2+v^2+4\right)\cdot 2-2u\cdot 2u}{(u^2+v^2+4)^2}=\frac{-2u^2+2v^2+8}{(u^2+v^2+4)^2}$, $p_{vv}(u,v)=\frac{\left(u^2+v^2+4\right)\cdot 2-2v\cdot 2v}{(u^2+v^2+4)^2}=\frac{2u^2-2v^2+8}{(u^2+v^2+4)^2}$, $p_{uv}(u,v)=p_{vu}(u,v)=-2u\left(u^2+v^2+4\right)^{-2}\cdot 2v=-\frac{4uv}{(u^2+v^2+4)^2}$.

12.4.36 We have $Q_r(r,s) = \frac{1}{s}$, $Q_s(r,s) = -\frac{r}{s^2}$; therefore $Q_{rr}(r,s) = 0$, $Q_{ss}(r,s) = \frac{2r}{s^3}$, $Q_{rs}(r,s) = Q_{sr}(r,s) = -\frac{1}{s^2}$.

12.4.37 We have $F_r(r,\,s) = e^s$, $F_s(r,\,s) = re^s$; therefore $F_{rr}(r,\,s) = 0$, $F_{ss}(r,\,s) = re^s$, $F_{rs}(r,\,s) = F_{sr}(r,\,s) = e^s$.

12.4.38 We have $H_x(x,y) = \frac{x}{\sqrt{4+x^2+y^2}}$, $H_y(x,y) = \frac{y}{\sqrt{4+x^2+y^2}}$; therefore
$H_{xx}(x,y) = \frac{\left(\sqrt{4+x^2+y^2}\right)\cdot 1 - x\frac{x}{\sqrt{4+x^2+y^2}}}{4+x^2+y^2} = \frac{4+y^2}{(4+x^2+y^2)^{3/2}}$, $H_{yy}(x,y) = \frac{\left(\sqrt{4+x^2+y^2}\right)\cdot 1 - y\frac{y}{\sqrt{4+x^2+y^2}}}{4+x^2+y^2} = \frac{4+x^2}{(4+x^2+y^2)^{3/2}}$,
$H_{xy}(x,y) = H_{yx}(x,y) = -\frac{1}{2}x\left(4+x^2+y^2\right)^{-3/2}\cdot 2y = -\frac{xy}{(4+x^2+y^2)^{3/2}}$

12.4.39 Observe that $f_x(x,\,y) = 6x^2$, so $f_{xy}(x,\,y) = 0$; and $f_y(x,\,y) = 6y$, so $f_{yx}(x,\,y) = 0$.

12.4.40 Observe that $f_x(x,\,y) = e^y$, so $f_{xy}(x,\,y) = e^y$; and $f_y(x,\,y) = xe^y$, so $f_{yx}(x,\,y) = e^y$.

12.4.41 Observe that $f_x(x,\,y) = -y\sin xy$, so $f_{xy}(x,\,y) = -\sin xy - xy\cos xy$; and $f_y(x,\,y) = -x\sin xy$, so $f_{yx}(x,\,y) = -\sin xy - xy\cos xy$.

12.4.42 Observe that $f_x(x,\,y) = 6xy^{-1} + 2x^{-2}y^2$, so $f_{xy}(x,\,y) = -6xy^{-2} + 4x^{-2}y$; and $f_y(x,\,y) = -3x^2y^{-2} - 4x^{-1}y$, so $f_{yx}(x,\,y) = -6xy^{-2} + 4x^{-2}y$.

12.4.43 Observe that $f_x(x,\,y) = e^{x+y}$, so $f_{xy}(x,\,y) = e^{x+y}$; and $f_y(x,\,y) = e^{x+y}$, so $f_{yx}(x,\,y) = e^{x+y}$.

12.4.44 Observe that $f_x(x,\,y) = \frac{1}{2}x^{-1/2}y^{1/2}$, so $f_{xy}(x,\,y) = \frac{1}{4}x^{-1/2}y^{-1/2} = \frac{1}{\sqrt{4xy}}$; and $f_y(x,\,y) = \frac{1}{2}x^{1/2}y^{-1/2}$, so $f_{yx}(x,\,y) = \frac{1}{4}x^{-1/2}y^{-1/2} = \frac{1}{\sqrt{4xy}}$.

12.4.45 $f_x(x,\,y,\,z) = y+z$; $f_y(x,\,y,\,z) = x+z$; $f_z(x,\,y,\,z) = x+y$.

12.4.46 $g_x(x,\,y,\,z) = 4xy - 3z^4$; $f_y(x,\,y,\,z) = 2x^2 + 20yz^2$; $f_z(x,\,y,\,z) = -12xz^3 + 20y^2z$.

12.4.47 $h_x(x,\,y,\,z) = h_y(x,\,y,\,z) = h_z(x,\,y,\,z) = -\sin(x+y+z)$.

12.4.48 $Q_x(x,\,y,\,z) = yz\sec^2 xyz$; $Q_y(x,\,y,\,z) = xz\sec^2 xyz$; $Q_z(x,\,y,\,z) = xy\sec^2 xyz$.

12.4.49 $F_u(u,\,v,\,w) = \frac{1}{v+w}$; $F_v(u,\,v,\,w) = -\frac{u}{(v+w)^2}$; $F_w(u,\,v,\,w) = -\frac{u}{(v+w)^2}$.

12.4.50 $G_r(r,\,s,\,t) = \frac{s+t}{2\sqrt{rs+rt+st}}$; $G_s(r,\,s,\,t) = \frac{r+t}{2\sqrt{rs+rt+st}}$; $G_t(r,\,s,\,t) = \frac{r+s}{2\sqrt{rs+rt+st}}$.

12.4.51 $f_w(w,\,x,\,y,\,z) = 2w\,x\,y^2$; $f_x(w,\,x,\,y,\,z) = w^2y^2 + y^3z^2$; $f_y(w,\,x,\,y,\,z) = 2w^2x\,y + 3x\,y^2z^2$; $f_z(w,\,x,\,y,\,z) = 2x\,y^3z$.

12.4.52 $g_w(w,x,y,z) = g_x(w,x,y,z) = -\sin(w+x)\sin(y-z)$; $g_y(w,x,y,z) = \cos(w+x)\cos(y-z)$; $g_z(w,x,y,z) = -\cos(w+x)\cos(y-z)$.

12.4.53 $h_w(w,\,x,\,y,\,z) = \frac{z}{xy}$; $h_x(w,\,x,\,y,\,z) = -\frac{wz}{x^2y}$; $h_y(w,\,x,\,y,\,z) = -\frac{wz}{xy^2}$; $h_z(w,\,x,\,y,\,z) = \frac{w}{xy}$.

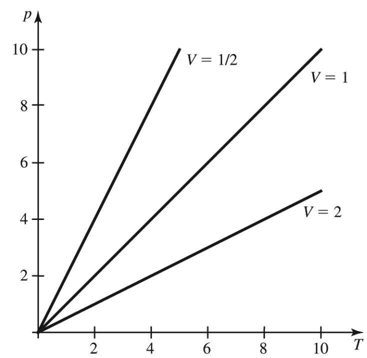
12.4.54 $F_w(w,\,x,\,y,\,z) = \sqrt{x+2y+3z}$; $F_x(w,\,x,\,y,\,z) = \frac{w}{2\sqrt{x+2y+3z}}$; $F_y(w,\,x,\,y,\,z) = \frac{w}{\sqrt{x+2y+3z}}$; $F_z(w,\,x,\,y,\,z) = \frac{3w}{2\sqrt{x+2y+3z}}$.

12.4.55

a. We have $V = \frac{kT}{P}$, so $\frac{\partial V}{\partial P} = -\frac{kT}{P^2}$. Because this partial derivative is negative, the volume decreases as the pressure increases at a fixed temperature.

b. We have $\frac{\partial V}{\partial T} = \frac{k}{P}$. Because this partial derivative is positive, the volume increases as the temperature increases at a fixed pressure.

c.



12.4.56

a. $V_x = 2xh$, $V_h = x^2$.

b. $\triangle V \approx 2xh\triangle x = 2 \cdot 0.5 \cdot 1.5 \cdot 0.01 = 0.015\,\mathrm{m}^3$.

c. $\triangle V = x^2\triangle h = (0.5)^2(-0.01) = -0.0025\,\mathrm{m}^3$ (notice that because V is a linear function of h, the linear approximation is exact).

d. Notice that for fixed height h, $\frac{\triangle V}{V} \approx \frac{2xh\triangle x}{x^2h} = 2\frac{\triangle x}{x}$. Therefore a 10% change in x will produce (approximately) a 20% change in V .

e. Notice that for fixed base x, $\frac{\triangle V}{V} = \frac{\triangle h}{h}$, so a 10% change in h will produce a 10% change in V .

12.4.57

a. Observe that as $f(x, y) = 0$ along either coordinate axis but on the line $y = x$, $f(x, y) = -\frac{x^2}{2x^2} = -\frac{1}{2}$, so $\lim_{(x,\,y)\to(0,\,0)} f(x, y)$ does not exist, and hence f is not continuous at (0, 0).

b. By Theorem 12.6, f is not differentiable at (0, 0).

c. Because f is identically 0 on the coordinate axes, $f_x(0, 0) = f_y(0, 0) = 0$.

d. We have $f_x(x, y) = -\left(\frac{(x^2+y^2)y - xy\cdot 2x}{(x^2+y^2)^2}\right) = \frac{(x^2-y^2)y}{(x^2+y^2)^2}$. Along the line $x = 2y$, $f_x(x, y) = \frac{3y^3}{25y^4} = \frac{3}{25}\cdot\frac{1}{y}$, which does not converge to 0 as $y \to 0$. Hence f_x is not continuous at (0, 0). A similar argument shows that f_y is also not continuous at (0, 0).

e. Theorem 12.5 does not apply because the partials f_x and f_y are not continuous at (0, 0), and Theorem 12.6 does not apply because f is not differentiable at (0, 0).

12.4.58

a. Observe that as $f(x, y) = 0$ along either coordinate axis but on the curve $x = y^2$, $f(x, y) = \frac{2y^4}{2y^4} = 1$, so $\lim_{(x,\,y)\to(0,\,0)} f(x, y)$ does not exist, and hence f is not continuous at (0, 0).

b. By Theorem 12.6, f is not differentiable at (0, 0).

c. Because f is identically 0 on the coordinate axes, $f_x(0, 0) = f_y(0, 0) = 0$.

d. We have $f_x(x, y) = \frac{(x^2+y^4)2y^2 - 2xy^2(2x)}{(x^2+y^4)^2} = \frac{2y^2(y^4-x^2)}{(x^2+y^4)^2}$. Along the curve $x = 2y^2$, $f_x(x, y) = -\frac{6y^6}{25y^8} = -\frac{6}{25}\cdot\frac{1}{y^2}$, which does not converge to 0 as $y \to 0$. Hence f_x is not continuous at (0, 0). We also have $f_y(x, y) = \frac{(x^2+y^4)4xy - 2xy^2(4y^3)}{(x^2+y^4)^2} = \frac{4xy(x^2-y^4)}{(x^2+y^4)^2}$. Along the curve $x = 2y^2$, $f_y(x, y) = -\frac{24y^7}{25y^8} = -\frac{24}{25}\cdot\frac{1}{y}$, which does not converge to 0 as $y \to 0$. Hence f_y is not continuous at (0, 0).

e. Theorem 12.5 does not apply because the partials f_x and f_y are not continuous at (0, 0), and Theorem 12.6 does not apply because f is not differentiable at (0, 0).

12.4.59

a. False. $\frac{\partial}{\partial x} y^{10} = 0$ because x and y are independent variables.

b. False. $\frac{\partial^2}{\partial x\, \partial y}(xy)^{1/2} = \frac{1}{2} \cdot x^{-1/2} \cdot \frac{1}{2} y^{-1/2} = \frac{1}{4\sqrt{xy}}$.

c. True. If f has continuous partial derivatives of all orders, then the order of differentiation for mixed partials can be exchanged.

12.4.60 $f_x(2,3) \approx \frac{f(2.1,3)-f(2,3)}{.1} = \frac{4.347-4.243}{.1} = 1.04$.

12.4.61 $f_y(2,3) \approx \frac{f(2,3.1)-f(2,3)}{.1} = \frac{4.384-4.243}{.1} = 1.41$.

12.4.62 $f_x(2.2, 3.4) \approx \frac{f(2.3,3.4)-f(2.2,3.4)}{.1} = \frac{5.156-5.043}{.1} = 1.13$. Answers may vary.

12.4.63 $f_y(2.4, 3.3) =\approx \frac{f(2.4,3.4)-f(2.4,3.3)}{.1} = \frac{5.267-5.112}{.1} = 1.55$. Answers may vary.

12.4.64 We have $f_x(x, y) = -\frac{y\, e^{-xy}}{1+e^{-xy}}$ and $f_y(x, y) = -\frac{x\, e^{-xy}}{1+e^{-xy}}$.

12.4.65 We have $f_x(x, y) = -\frac{2x}{1+(x^2+y^2)^2}$ and $f_y(x, y) = -\frac{2y}{1+(x^2+y^2)^2}$.

12.4.66 We have $f_x(x, y) = f_y(x, y) = 2\sin(2(x+y)) - 2\cos(x+y)\sin(x+y)$.

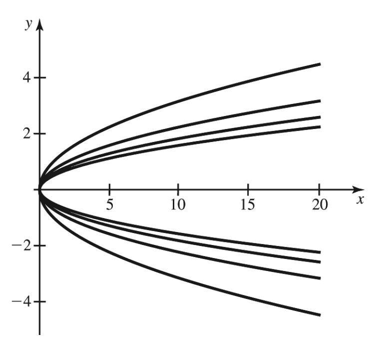
12.4.67 We have $h_x(x, y, z) = z(1+x+2y)^{z-1}$, $h_y(x, y, z) = 2z(1+x+2y)^{z-1}$, and $h_x(x, y, z) = (1+x+2y)^z \ln(1+x+2y)$.

12.4.68 Observe that for any rational function of the form $R(t) = \frac{at+b}{ct+d}$ we have $R'(t) = \frac{ad-bc}{(ct+d)^2}$; therefore $g_x(x, y, z) = \frac{4(3y-3z)-(-2y-2z)(-6)}{(3y-6x-3z)^2} = -\frac{24z}{(3y-6x-3z)^2} = -\frac{8z}{3(y-2x-z)^2}$; $g_y(x, y, z) = \frac{(-2)(-6x-3z)-(4x-2z)3}{(3y-6x-3z)^2} = \frac{12z}{(3y-6x-3z)^2} = \frac{4z}{3(y-2x-z)^2}$; $g_z(x, y, z) = \frac{(-2)(3y-6x)-(4x-2y)(-3)}{(3y-6x-3z)^2} = \frac{24x-12y}{(3y-6x-3z)^2} = \frac{8x-4y}{3(y-2x-z)^2}$.

12.4.69

a. We have $z_x = \frac{1}{y^2}$ and $z_y = -\frac{2x}{y^3}$.

b.



c. We observe that z increases at the same rate as x, which makes sense because $z_x = 1$ along this line.

d. We observe that z increases when $y < 0$, is undefined when $y = 0$ and decreases when $y > 0$, which is consistent with $z_y = -\frac{2}{y^3}$ along this line.

12.4.70

a. We have $V_h = \frac{\pi}{3}\left(3r \cdot 2h - 3h^2\right) = \pi\left(2rh - h^2\right)$ and $V_r = \pi h^2$.

b. Because $V_r = \pi h^2$, V_r is greater when $h = 0.8r$.

c. Solve $V_r = \pi h^2 = 1$ to obtain $h = \pi^{-1/2}$.

d. The maximum value of $2rh - h^2$ as a function of h occurs when $2r - 2h = 0$, which gives $h = r$.

12.4.71

a. Because $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ we have $c = \left(a^2 + b^2 - ab\right)^{1/2}$, and therefore $\frac{\partial c}{\partial a} = \frac{2a-b}{2\sqrt{a^2+b^2-ab}}$ and $\frac{\partial c}{\partial b} = \frac{2b-a}{2\sqrt{a^2+b^2-ab}}$.

b. Implicit differentiation gives $2c\frac{\partial c}{\partial a} = 2a - b$, so $\frac{\partial c}{\partial a} = \frac{2a-b}{2c}$ and $2c\frac{\partial c}{\partial b} = 2b - a$, so $\frac{\partial c}{\partial b} = \frac{2b-a}{2c}$.

c. The necessary relationship is $2a - b > 0$ or $a > \frac{b}{2}$.

12.4.72

a. We have $B_w = \frac{1}{h^2}$.

b. The partial derivative $B_w > 0$ so B is an increasing function of w.

c. We have $B_h = -\frac{2w}{h^3}$.

d. The partial derivative $B_h < 0$ so B is a decreasing function of h.

12.4.73

a. We have $\varphi_x(x, y) = -\frac{2x}{\left(x^2+(y-1)^2\right)^{3/2}} - \frac{x}{\left(x^2+(y+1)^2\right)^{3/2}}$ and $\varphi_y(x, y) = -\frac{2(y-1)}{\left(x^2+(y-1)^2\right)^{3/2}} - \frac{y+1}{\left(x^2+(y+1)^2\right)^{3/2}}$.

b. Observe that $|\varphi_x(x, y)| \le \frac{2|x|}{|x|^{3/2}} + \frac{|x|}{|x|^{3/2}} = \frac{3}{|x|^{1/2}}$ and similarly $|\varphi_y(x, y)| \le \frac{2|y-1|}{|y-1|^{3/2}} + \frac{|y+1|}{|y+1|^{3/2}} = \frac{2}{|y-1|^{1/2}} + \frac{1}{|y+1|^{1/2}}$, which both converge to 0 as $x, y \to \infty$.

c. We see that $\varphi_x(0, y) = 0$ as long as $y \neq \pm 1$. This is consistent with the observation that along horizontal lines $y = y_0$ the potential function takes its maximum at $x = 0$.

d. We see that $\varphi_y(x, 0) = \frac{1}{(x^2+1)^{3/2}}$. This implies that if we cross the x-axis at any point from below to above, the potential function is increasing.

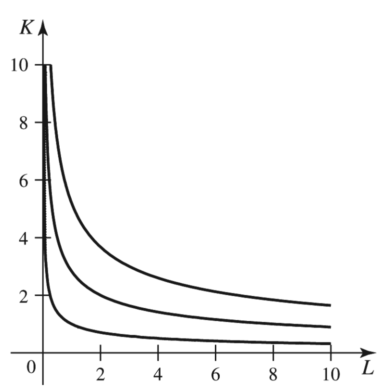
12.4.74

a. We have $Q_L = \frac{1}{3}L^{-2/3}K^{2/3}$ and $Q_K = \frac{2}{3}L^{1/3}K^{-1/3}$.

b. We have $\triangle Q \approx Q_K(10,\, 20)\,\triangle K = \frac{2}{3}(10)^{1/3}(20)^{-1/3} \cdot 0.5 \approx 0.2646$.

c. We have $\triangle Q \approx Q_L(10,\, 20)\,\triangle L = \frac{1}{3}(10)^{-2/3}(20)^{2/3} \cdot (-0.5) \approx -0.2646$.

d.



e. As we move along the vertical line $L = 2$ in the positive K-direction, Q increases, which is consistent with $Q_K > 0$.

f. As we move along the horizontal line $K = 2$ in the positive L-direction, Q increases, which is consistent with $Q_L > 0$.

12.4.75

a. Solving for R gives $R = \left(R_1^{-1} + R_2^{-1}\right)^{-1}$, so $\frac{\partial R}{\partial R_1} = -\left(R_1^{-1} + R_2^{-1}\right)^{-2}\left(-R_1^{-2}\right) = \frac{R_2^2}{(R_1+R_2)^2}$ and similarly $\frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1+R_2)^2}$.

b. We have $-R^{-2}\frac{\partial R}{\partial R_1} = -R_1^{-2} \Longrightarrow \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}$ and similarly $\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2^2}$.

c. Because $\frac{\partial R}{\partial R_1} > 0$, an increase in R_1 causes an increase in R.

d. Because $\frac{\partial R}{\partial R_2} > 0$, a decrease in R_2 causes a decrease in R.

12.4.76

a. The period T is found by solving $\frac{\pi t}{2} = 2\pi$, so $t = 4$.

b. We have $u_t = 2\sin(\pi x)\cos\left(\frac{\pi t}{2}\right) \cdot \frac{\pi}{2} = \pi \sin(\pi x)\cos\left(\frac{\pi t}{2}\right)$.

c. For fixed t, the quantity u_t is largest when $\sin(\pi x) = \pm 1$, which occurs at $x = \frac{1}{2}$ (the middle of the string).

d. For fixed x, the quantity u_t is largest when $\cos\left(\frac{\pi t}{2}\right) = \pm 1$, which occurs at $t = 2k$ for any integer k. At these times the string is in its rest position (no displacement).

e. We have $u_x = 2\pi\cos(\pi x)\cos\left(\frac{\pi t}{2}\right)$.

f. For fixed t, the slope is greatest when $\cos(\pi x) = \pm 1$, which occurs at the endpoints of the string ($x = 0$, 1).

12.4.77 Observe that $\frac{\partial^2 u}{\partial t^2} = -4c^2\cos(2(x+ct)) = c^2\frac{\partial^2 u}{\partial x^2}$.

12.4.78 Observe that $\frac{\partial^2 u}{\partial t^2} = -20c^2\cos(2(x+ct)) - 3c^2\sin(x-ct) = c^2\frac{\partial^2 u}{\partial x^2}$.

12.4.79 Observe that $\frac{\partial^2 u}{\partial t^2} = Ac^2 f''(x+ct) + Bc^2 g''(x-ct) = c^2\frac{\partial^2 u}{\partial x^2}$.

12.4.80 Observe that $u_{xx} + u_{yy} = e^{-x}\sin y + e^{-x}(-\sin y) = 0$.

12.4.81 Observe that $u_{xx} + u_{yy} = 6x - 6x = 0$.

12.4.82 Observe that $u_{xx} + u_{yy} = a^2 e^{ax}\cos ay + e^{ax}\left(-a^2\cos ay\right) = 0$

12.4.83 Observe that $u_{xx} = \frac{2(x-1)y}{\left[(x-1)^2+y^2\right]^2} - \frac{2(x+1)y}{\left[(x+1)^2+y^2\right]^2}$, and $u_{yy} = -\frac{2(x-1)y}{\left[(x-1)^2+y^2\right]^2} + \frac{2(x+1)y}{\left[(x+1)^2+y^2\right]^2}$; so $u_{xx} + u_{yy} = 0$.

12.4.84 We see that $u_t = -10e^{-t}\sin x = u_{xx}$.

12.4.85 We see that $u_t = -16e^{-4t}\cos 2x = u_{xx}$.

12.4.86 We see that $u_t = -e^{-t}(2\sin x + 3\cos x) = u_{xx}$.

12.4.87 We see that $u_t = -a^2 A e^{-a^2 t}\cos ax = u_{xx}$.

12.4.88 We have $f(0, 0) = 0$, $f_x(0, 0) = f_y(0, 0) = 1$, and $f(\triangle x, \triangle y) = 1 \cdot \triangle x + 1 \cdot \triangle y$, so we can take $\epsilon_1 = \epsilon_2 = 0$.

12.4.89 We have $f(0, 0) = 0$, $f_x(0, 0) = f_y(0, 0) = 0$, and $f(\triangle x, \triangle y) = \triangle x \cdot \triangle y$, so we can take $\epsilon_1 = \triangle y$, $\epsilon_2 = 0$ or $\epsilon_1 = 0$, $\epsilon_2 = \triangle x$.

12.4.90

a. Observe that $\lim_{(x, y)\to(0, 0)} (1 - |xy|) = 1 = f(0, 0)$, so f is continuous at $(0, 0)$.

b. Let $(a, b) = (0, 0)$; then $f(a + \triangle x, b + \triangle y) - f(a, b) = -|\triangle x||\triangle y| = \epsilon_1 \triangle x$ where $\epsilon_1 = \pm\triangle y$ (depending on the sign of x). Because $\epsilon_1 \to 0$ as $\triangle y \to 0$, we see that f is differentiable at $(0, 0)$.

c. Because f is identically equal to 1 on the coordinate axes, $f_x(0, 0) = f_y(0, 0) = 0$.

d. The partial derivative $f_x(0, y)$ does not exist for $y \neq 0$, because the function $|x|$ is not differentiable at $x = 0$. Similarly, the partial derivative $f_y(x, 0)$ does not exist for $x \neq 0$. Hence the partials f_x and f_y are not continuous at$(0, 0)$.

e. Theorem 12.5 does not apply because the partials f_x and f_y are notcontinuous at $(0, 0)$. Theorem 12.6 implies that f is continuous at $(0, 0)$, which we saw in part (a).

12.4.91

a. Observe that $\lim_{(x, y)\to(0, 0)} \sqrt{|xy|} = 0 = f(0, 0)$, so f is continuous at $(0, 0)$.

b. Let $(a, b) = (0, 0)$, and suppose that $f(a + \triangle x, b + \triangle y) - f(a, b) = \sqrt{|\triangle x \triangle y|} = \epsilon_1 \triangle x + \epsilon_2 \triangle y$. Let $\triangle x = \triangle y$; then we obtain $\sqrt{|\triangle x|^2} = |\triangle x| = (\epsilon_1 + \epsilon_2)\triangle x$ which implies that $\epsilon_1 + \epsilon_2 = \pm 1$, and so we cannot have $\epsilon_1, \epsilon_2 \to 0$, as $\triangle x, \triangle y \to 0$. Therefore f is not differentiable at $(0, 0)$.

c. Because f is identically equal to 0 on the coordinate axes, $f_x(0, 0) = f_y(0, 0) = 0$.

d. The partial derivative $f_x(0, y)$ does not exist for $y \neq 0$ because the function $\sqrt{|x|}$ is not differentiable at $x = 0$. Similarly, the partial derivative $f_y(x, 0)$ does not exist for $x \neq 0$. Hence the partials f_x and f_y are not continuous at $(0, 0)$.

e. Theorem 12.5 does not apply because the partials f_x and f_y are not continuous at $(0, 0)$, and Theorem 12.6 does not apply because f is not differentiable at $(0, 0)$.

12.4.92

a. There are 3 choices for each of the variables we differentiate with respect to, so there are 9 possible second partial derivatives: $w_{xx}, w_{yy}, w_{zz}, w_{xy}, w_{yx}, w_{xz}, w_{zx}, w_{yz}, w_{zy}$.

b. This function has continuous partial derivatives of all orders, so $w_{xy} = w_{yx}$, $w_{xz} = w_{zx}$ and $w_{yz} = w_{zy}$.

c. There are 4 choices for each of the variables we differentiate with respect to, so there are 16 possible second partial derivatives.

12.4.93

a. By the fundamental theorem of calculus, $f_x(x, y) = -\frac{\partial}{\partial x}\int_x^y h(s)\, ds = -h(x)$ and similarly $f_y(x, y) = h(y)$.

b. Let $H(s)$ be an antiderivative of $h(s)$; then $f(x, y) = H(xy) - H(1)$, so $f_x(x, y) = y\, h(xy)$, $f_y(x, y) = x\, h(xy)$.

12.4.94 Observe that $f_x(x, y) = \frac{(cx+dy)a-(ax+by)c}{(cx+dy)^2} = \frac{y(ad-bc)}{(cx+dy)^2} = 0$; similarly, $f_y(x, y) = 0$. Suppose $a \neq 0$ (other cases can be handled similarly). Then $ad = bc$ implies $d = \frac{bc}{a}$; this also shows that $c \neq 0$, otherwise both c and $d = 0$ and the function is undefined. Hence $ax + by = \frac{a}{c}\left(cx + \frac{bc}{a}y\right) = \frac{a}{c}(cx + dy)$, so $f(x, y) = \frac{a}{c}$ is a constant function which implies $f_x = f_y = 0$.

12.4.95

a. Observe that $u_x = 2x = v_y$ and $u_y = -2y = -v_x$.

b. Observe that $u_x = 3x^2 - 3y^2 = v_y$ and $u_y = -6xy = -v_x$.

c. We have $u_{xx} = v_{yx} = v_{xy} = -u_{yy}$, so $u_{xx} + u_{yy} = 0$. The proof that $v_{xx} + v_{yy} = 0$ is similar.

12.5 The Chain Rule

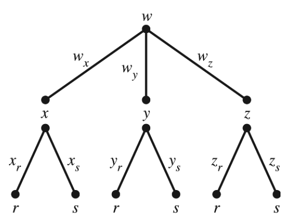
12.5.1 There is one dependent variable (z), two intermediate variables (x and y) and one independent variable (t).

12.5.2 Multiply each of the partial derivatives of z by the t-derivative of the corresponding function, and add all these expressions.

12.5.3 Multiply each of the partial derivatives of w by the t-derivative of the corresponding function, and add all these expressions.

12.5.4 Multiply each of the partial derivatives of z by the t-partial derivative of the corresponding function, and add all these expressions.

12.5.5



12.5.6 Use Theorem 12.9: $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

12.5.7 We have $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2x(2t) + 3y^2(1) = 4t^3 + 3t^2$.

12.5.8 We have $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = y^2(2t) + 2xy(1) = 2t^3 + 2t^3 = 4t^3$.

12.5.9 We have $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (\sin y)\, 2t + (x\, \cos y)\, 12t^2 = 2t\, \sin\left(4t^3\right) + 12t^4 \cos\left(4t^3\right)$.

12.5.10 We have $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left(2xy - y^3\right) 2t + \left(x^2 - 3xy^2\right)\left(-2t^{-3}\right) = 2t\left(2 - t^{-6}\right) - 2t^{-3}\left(t^4 - 3t^{-2}\right)$ $= 2t + 4t^{-5}$.

12.5.11 We have $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = (-2\sin 2x\, \sin 3y)\left(\frac{1}{2}\right) + (3\, \cos 2x\, \cos 3y)\, 4t^3 = -\sin t\, \sin 3t^4 + 12t^3 \cos t\, \cos 3t^4$.

12.5.12 We have $\frac{dz}{dt} = \frac{\partial z}{\partial r}\frac{dr}{dt} + \frac{\partial z}{\partial s}\frac{ds}{dt} = \frac{r}{\sqrt{r^2+s^2}}(-2\sin 2t) + \frac{s}{\sqrt{r^2+s^2}}(2\cos 2t) = \frac{-2\cos 2t\, \sin 2t + 2\sin 2t \cos 2t}{1} = 0$.

12.5.13 We have $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = (y\sin z)\, 2t + (x\sin z)\, 12t^2 + (xy\cos z)\cdot 1 = \left(2ty + 12t^2 x\right)\sin z + xy\cos z = 20t^4 \sin(t+1) + 4t^5 \cos(t+1)$.

12.5.14 We have $\frac{dQ}{dt} = \frac{\partial Q}{\partial x}\frac{dx}{dt} + \frac{\partial Q}{\partial y}\frac{dy}{dt} + \frac{\partial Q}{\partial z}\frac{dz}{dt} = \frac{x}{\sqrt{x^2+y^2+z^2}}\cos t + \frac{y}{\sqrt{x^2+y^2+z^2}}(-\sin t) + \frac{z}{\sqrt{x^2+y^2+z^2}}(-\sin t) = -\frac{\cos t\, \sin t}{\sqrt{1+\cos^2 t}}$.

12.5.15 We have $\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} = \frac{1}{x+y+z}\cdot 1 + \frac{1}{x+y+z}\cdot 2t + \frac{1}{x+y+z}\cdot 3t^2 = \frac{1+2t+3t^2}{t+t^2+t^3}$.

12.5.16 We have $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} = \frac{1}{y+z}\cdot 1 + \left(\frac{-x-z}{(y+z)^2}\right)\cdot 2 + \frac{y-x}{(y+z)^2}\cdot 3 = \frac{-5x+4y-z}{(y+z)^2} = \frac{-5t+8t-3t}{25t^2} = 0.$

12.5.17

a. By the chain rule, $V'(t) = 2\pi\, r(t)\, h(t)\, r'(t) + \pi[r(t)]^2 h'(t)$.

b. Substituting $r(t) = e^t$ and $h(t) = e^{-2t}$ gives $V'(t) = 2\pi\, e^t e^{-2t} e^t + \pi\, e^{2t}\left(-2e^{-2t}\right) = 0$.

c. Because $V'(t) = 0$, the volume remains constant.

12.5.18

a. By the chain rule, $V'(t) = \frac{2}{3}x(t)\; h(t)\; x'(t) + \frac{1}{3}[x(t)]^2\; h'(t)$.

b. Substituting $x(t) = \frac{t}{t+1}$ and $h(t) = \frac{1}{t+1}$ gives $V'(t) = \frac{2}{3}\frac{t}{t+1}\frac{1}{t+1}\frac{1}{(t+1)^2} + \frac{1}{3}\left(\frac{t}{t+1}\right)^2\left(-\frac{1}{(t+1)^2}\right) = \frac{1}{3}\frac{t(2-t)}{(t+1)^4}$.

c. The volume increases for $0 \le t \le 2$ and decreases for $t \ge 2$.

12.5.19 $z_s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2x\sin y + x^2\cos y\cdot 0 = 2(s-t)\sin t^2$ and $z_t = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2x\sin y)(-1) + x^2\cos y(2t) = 2(t-s)\sin t^2 + 2t(s-t)^2\cos t^2$.

12.5.20 $z_s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2\cos(2x+y)(2s) + \cos(2x+y)(2s) = 6s\cos(3s^2-t^2)$ and $z_t = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = 2\cos(2x+y)(-2t) + \cos(2x+y)(2t) = -2t\cos(2(s^2-t^2)+s^2+t^2) = -2t\cos(3s^2-t^2)$.

12.5.21 $z_s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (y-2xy)\cdot 1 + \left(x-x^2\right)\cdot 1 = s-t-2\left(s^2-t^2\right) + (s+t) - (s+t)^2 = 2s-3s^2-2st+t^2$ and $z_t = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (y-2xy)\cdot 1 + \left(x-x^2\right)\cdot(-1) = s-t-2\left(s^2-t^2\right) - (s+t) + (s+t)^2 = -s^2-2t+2st+3t^2$.

12.5.22 $z_s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \cos x\;\cos 2y - 2\sin x\;\sin 2y = \cos(s+t)\cos[2(s-t)] - 2\sin(s+t)\sin[2(s-t)]$ and $z_t = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \cos x\;\cos 2y - 2\sin x\;\sin 2y\cdot(-1) = \cos(s+t)\cos[2(s-t)] + 2\sin(s+t)\sin[2(s-t)]$

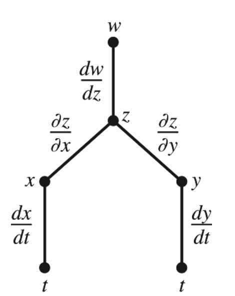
12.5.23 $z_s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = e^{x+y}\cdot t + e^{x+y}\cdot 1 = (t+1)\,e^{st+s+t}$ and $z_t = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = e^{x+y}\cdot s + e^{x+y}\cdot 1 = (s+1)\,e^{st+s+t}$.

12.5.24 $z_s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (y-2)(-\sin s) + (x+3)\cdot 0 = \sin s\,(2-\sin t)$ and $z_t = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (y-2)\cdot 0 + (x+3)\cdot\cos t = (3+\cos t)\cos t$.

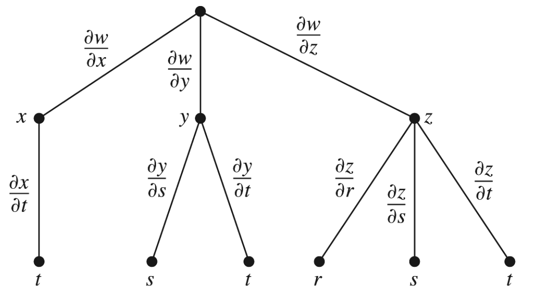
12.5.25 $w_s = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{1}{y+z}\cdot 1 + \frac{z-x}{(y+z)^2}\cdot t - \frac{x+y}{(y+z)^2}\cdot 1 = \frac{(1+t)(z-x)}{(y+z)^2} = -\frac{2t(1+t)}{(st+s-t)^2}$ and $w_t = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = \frac{1}{y+z}\cdot 1 + \frac{z-x}{(y+z)^2}\cdot s - \frac{x+y}{(y+z)^2}\cdot(-1) = \frac{(1-s)x+2y+(1+s)z}{(y+z)^2} = \frac{2s}{(st+s-t)^2}$.

12.5.26 $w_r = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = \frac{x}{\sqrt{x^2+y^2+z^2}}\cdot 0 + \frac{y}{\sqrt{x^2+y^2+z^2}}\cdot s + \frac{z}{\sqrt{x^2+y^2+z^2}}\cdot t = \frac{r\left(s^2+t^2\right)}{\sqrt{r^2s^2+r^2t^2+s^2t^2}}$. Similarly, $w_s = \frac{s\left(r^2+t^2\right)}{\sqrt{r^2s^2+r^2t^2+s^2t^2}}$ and $w_t = \frac{t\left(r^2+s^2\right)}{\sqrt{r^2s^2+r^2t^2+s^2t^2}}$.

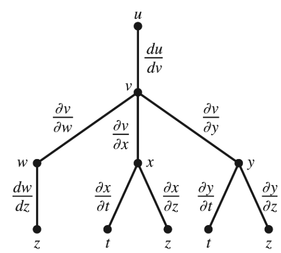
12.5.27 $\frac{dw}{dt} = \frac{dw}{dz}\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)$



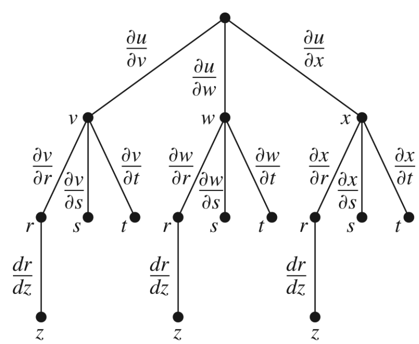
12.5.28 $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$



12.5.29 $\frac{\partial u}{\partial z} = \frac{du}{dv}\left(\frac{\partial v}{\partial w}\frac{dw}{dz} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial z}\right)$



12.5.30 $\frac{du}{dz} = \left(\frac{\partial u}{\partial v}\frac{dv}{dr} + \frac{\partial u}{\partial w}\frac{dw}{dr} + \frac{\partial u}{\partial x}\frac{dx}{dr}\right)\frac{dr}{dz}$



12.5.31 Let $F(x, y) = x^2 - 2y^2 - 1$; then by Theorem 12.9, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{-4y} = \frac{x}{2y}$.

12.5.32 Let $F(x, y) = x^3 + 3xy^2 - y^5$; then by Theorem 12.9, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2+3y^2}{6xy-5y^4} = \frac{3x^2+3y^2}{5y^4-6xy}$.

12.5.33 Let $F(x, y) = 2\sin(xy) - 1$; then by Theorem 12.9, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2y\,\cos(xy)}{2x\,\cos(xy)} = -\frac{y}{x}$.

12.5.34 Let $F(x, y) = ye^{xy} - 2$; then by Theorem 12.9, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y^2 e^{xy}}{e^{xy}+xy\,e^{xy}} = -\frac{y^2}{1+xy}$.

12.5.35 Note that we can simplify this equation to $x^2 + 2xy + y^4 = 9$, so let $F(x, y) = x^2 + 2xy + y^4 - 9$; then by Theorem 12.9, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+2y}{2x+4y^3} = -\frac{x+y}{x+2y^3}$.

12.5.36 Let $F(x,y) = y\ln\left(x^2 + y^2 + 4\right) - 3$; then by Theorem 12.9, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{-\left(\frac{2xy}{x^2+y^2+4}\right)}{\left(\ln(x^2+y^2+4)+\frac{2y^2}{x^2+y^2+4}\right)} = -\frac{2xy}{2y^2+(x^2+y^2+4)\ln(x^2+y^2+4)}$.

12.5.37 The chain rule gives $\frac{\partial s}{\partial x} = \frac{\partial s}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial s}{\partial v}\frac{\partial v}{\partial x} = \frac{u}{\sqrt{u^2+v^2}}\cdot 0 + \frac{v}{\sqrt{u^2+v^2}}\cdot(-2) = \frac{4x}{\sqrt{4(x^2+y^2)}} = \frac{2x}{\sqrt{x^2+y^2}}$ and $\frac{\partial s}{\partial y} = \frac{\partial s}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial s}{\partial v}\frac{\partial v}{\partial y} = \frac{u}{\sqrt{u^2+v^2}}\cdot 2 + \frac{v}{\sqrt{u^2+v^2}}\cdot 0 = \frac{4y}{\sqrt{4(x^2+y^2)}} = \frac{2y}{\sqrt{x^2+y^2}}$.

12.5.38 The chain rule gives $\frac{\partial s}{\partial x} = \frac{\partial s}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial s}{\partial v}\frac{\partial v}{\partial x} = \frac{u}{\sqrt{u^2+v^2}} \cdot (1-2x)(1-2y) + \frac{v}{\sqrt{u^2+v^2}} \cdot y(y-1)(-2) = \frac{x(1-x)(1-2x)(1-2y)^2-2y^2(1-2x)(y-1)^2}{\sqrt{x^2(1-x)^2(1-2y)^2+y^2(y-1)^2(1-2x)^2}}$ and
$\frac{\partial s}{\partial y} = \frac{\partial s}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial s}{\partial v}\frac{\partial v}{\partial y} = \frac{u}{\sqrt{u^2+v^2}} \cdot x(1-x)(-2) + \frac{v}{\sqrt{u^2+v^2}} \cdot (2y-1)(1-2x) = \frac{y(1-y)(1-2y)(1-2x)^2-2x^2(1-2y)(1-x)^2}{\sqrt{x^2(1-x)^2(1-2y)^2+y^2(1-y)^2(1-2x)^2}}$.

12.5.39

a. False. The correct equation is $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$.

b. False. w is a function of both s and t, so the rate of change of w with respect to t is the partial derivative $\frac{\partial w}{\partial t}$.

12.5.40

a. We have $z = \ln(te^t + e^t) = \ln e^t + \ln(t+1) = t + \ln(t+1)$, so $z'(t) = 1 + \frac{1}{t+1} = \frac{2+t}{1+t}$.

b. Using the chain rule, $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{1}{x+y}(1+t)e^t + \frac{1}{x+y}e^t = \frac{(2+t)e^t}{(1+t)e^t} = \frac{2+t}{1+t}$.

12.5.41

a. We have $z = \left(t^2+2t\right)^{-1} + \left(t^3-2\right)^{-1}$, so $z'(t) = -\frac{(2t+2)}{(t^2+2t)^2} - \frac{3t^2}{(t^3-2)^2}$.

b. Using the chain rule, $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = -\frac{(2t+2)}{x^2} - \frac{3t^2}{y^2} = -\frac{(2t+2)}{(t^2+2t)^2} - \frac{3t^2}{(t^3-2)^2}$.

12.5.42 The chain rule gives $\frac{\partial z}{\partial p} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial p} = \frac{1}{y} \cdot 1 - \frac{x}{y^2} \cdot 1 = \frac{y-x}{y^2} = -\frac{2q}{(p-q)^2}$.

12.5.43 The chain rule gives $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = yz \cdot 8t^3 + xz\left(-3t^{-2}\right) + xy\left(-12t^{-4}\right) = \frac{12}{t^4} \cdot 8t^3 + 8t\left(-\frac{3}{t^2}\right) + 6t^3\left(-\frac{12}{t^4}\right) = 0$.This can also be seen by expressing w in terms of t: $w = 2t^4 3t^{-1} 4t^{-3} = 24$, so $\frac{dw}{dt} = 0$.

12.5.44 Observe that $w = \cos(x+y) - (\cos x \cos y - \sin x \sin y) = 0$, $\frac{\partial w}{\partial x} = 0$.

12.5.45 The chain rule gives $-\frac{1}{x^2} - \frac{1}{z^2}\frac{\partial z}{\partial x} = 0$, so $\frac{\partial z}{\partial x} = -\frac{z^2}{x^2}$.

12.5.46 Observe that $z = xy - 1$, so $\frac{\partial z}{\partial x} = y$.

12.5.47

a. The chain rule gives $w'(t) = aw_x + bw_y + cw_z$.

b. Using part (a), $w'(t) = ayz + bxz + cxy = 3abct^2$.

c. Using part (a), $w'(t) = \frac{ax}{\sqrt{x^2+y^2+z^2}} + \frac{by}{\sqrt{x^2+y^2+z^2}} + \frac{cz}{\sqrt{x^2+y^2+z^2}} = \frac{ax+by+cz}{\sqrt{x^2+y^2+z^2}} = \sqrt{a^2+b^2+c^2}\,\frac{t}{|t|}$

d. Differentiate the result from part (a) one more time:
$w''(t) = a\left(aw_{xx} + bw_{xy} + cw_{xz}\right) + b\left(aw_{yx} + bw_{yy} + cw_{yz}\right) + c\left(aw_{zx} + bw_{zy} + cw_{zz}\right)$ which simplifies to $w''(t) = a^2 w_{xx} + b^2 w_{yy} + c^2 w_{zz} + 2abw_{xy} + 2acw_{xz} + 2bcw_{yz}$.

12.5.48 Differentiate $F(x, y, z(x, y)) = 0$ using the chain rule: $F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \frac{\partial z}{\partial x} = 0$, so $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$. The proof that $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$ is similar.

12.5.49 Let $F(x, y, z) = xy + xz + yz - 3$; then the result from Exercise 48 gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y+z}{x+y}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x+z}{x+y}$.

12.5.50 Let $F(x, y, z) = x^2 + 2y^2 - 3z^2 - 1$; then the result from Exercise 48 gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{-6z} = \frac{x}{3z}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{-6z} = \frac{2y}{3z}$.

12.5.51 Let $F(x,\, y,\, z) = xyz + x + y - z$; then the result from Exercise 48 gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz+1}{xy-1}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz+1}{xy-1}$.

12.5.52

a. Let $F(x,\, y,\, z) = e^{xyz} - 2$; then the result from Exercise 48 gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yze^{xyz}}{xye^{xyz}} = -\frac{z}{x}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xze^{xyz}}{xye^{xyz}} = -\frac{z}{y}$.

b. Take logarithms of both sides to obtain $xyz - \ln 2 = 0$; then the result from Exercise 48 applied to $F(x,\, y,\, z) = xyz - \ln 2$ gives $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz}{xy} = -\frac{z}{x}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz}{xy} = -\frac{z}{y}$.

c. Solve for z to obtain $z = \frac{\ln 2}{xy}$; therefore $\frac{\partial z}{\partial x} = -\frac{\ln 2}{x^2 y} = -\frac{z}{x}, \frac{\partial z}{\partial y} = -\frac{\ln 2}{xy^2} = -\frac{z}{y}$.

12.5.53

a. The chain rule gives $z'(t) = 2x\,(-\sin t) + 8y\,\cos t = 6 \sin t\, \cos t = 3 \sin 2t$.

b. Observe that for $0 \le t \le 2\pi$, $z'(t) = 3 \sin 2t > 0$ when $0 < t < \frac{\pi}{2}$or $\pi < t < \frac{3\pi}{2}$.

12.5.54

a. The chain rule gives $z'(t) = 8x\,(-\sin t) - 2y\,\cos t = -10 \sin t\, \cos t = -5 \sin 2t$.

b. Observe that for $0 \le t \le 2\pi$, $z'(t) = -5 \sin 2t > 0$ when $\frac{\pi}{2} < t < \pi$ or $\frac{3\pi}{2} < t < 2\pi$.

12.5.55

a. The chain rule gives $z'(t) = -\frac{x}{\sqrt{1-x^2-y^2}}\,(-e^{-t}) + -\frac{y}{\sqrt{1-x^2-y^2}}\,(-e^{-t}) = \frac{2e^{-2t}}{\sqrt{1-e^{-2t}}}$.

b. Observe that $z'(t) > 0$ for all t where defined, so the function $z(t)$ is increasingfor all $t \ge \frac{1}{2} \ln 2$.

12.5.56

a. The chain rule gives $z'(t) = 4x\,(-\sin t) + 2y \cos t = -4\,(1 + \cos t) \sin t + 2 \sin t \cos t = -2 \sin t\,(2 + \cos t)$.

b. Observe that $2 + \cos t > 0$ for all t, so for $0 \le t \le 2\pi$, so $z'(t) > 0$ if and only if $\sin t < 0$, which occurs when $\pi < t < 2\pi$.

12.5.57 The chain rule gives $E'(t) = m\,(uu' + vv') + mgy' = m\,(x'x'' + y'y'' + gy') =$
$m\,(u_0 \cdot 0 + y'\,(y'' + g)) = 0$. Therefore, the energy of the projectile remains constant during the motion.

12.5.58

a. The marginal utilities are $\frac{\partial U}{\partial x} = ax^{a-1}y^{1-a}$, $\frac{\partial U}{\partial y} = (1-a)\,x^a y^{-a}$.

b. Using Theorem 12.9, we find that the slope of the indifference curve $U(x,\, y) - c = 0$ is $\text{MRS} = -\frac{U_x}{U_y} = -\frac{a}{1-a}\frac{y}{x}$.

c. The result from part (b) above gives $\text{MRS} = -\frac{0.4}{0.6}\frac{12}{8} = -1$.

12.5.59

a. If r and R increase at the same rate then $R - r$ is a constant C, so $V = \frac{C^2\pi^2}{4}\,(R + r)$ is increasing.

b. Similarly, if r and R decrease at the same rate then V is decreasing.

12.5.60

a. The chain rule gives $S'(t) = \frac{1}{60}\left(\frac{w}{2\sqrt{hw}}h'(t) + \frac{h}{2\sqrt{hw}}w'(t)\right) = \frac{w\,h'(t) + h\,w'(t)}{120\sqrt{hw}}$.

b. From part (a), the condition that $S(t)$ is constant is $w\,h'(t) + h\,w'(t) = 0$.

c. If S is constant then we must have hw constant, so h and w are inversely proportional.

12.5.61

a. Consider P as a function of T and V and differentiate with respect to V : $\frac{\partial P}{\partial V}V+P\cdot 1=0$, so $\frac{\partial P}{\partial V}=-\frac{P}{V}$. Next, consider T as a function of P and V and differentiate with respect to P: $1\cdot V=k\frac{\partial T}{\partial P}$, so $\frac{\partial T}{\partial P}=\frac{V}{k}$. Lastly, consider V as a function of T and P and differentiate with respect to T: $P\frac{\partial V}{\partial T}=k$, so $\frac{\partial V}{\partial T}=\frac{k}{P}$.

b. Observe that $\frac{\partial P}{\partial V}\frac{\partial T}{\partial P}\frac{\partial V}{\partial T}=-\frac{P}{V}\frac{V}{k}\frac{k}{P}=-1$.

12.5.62

a. The chain rule gives $\frac{d\rho}{dt}=y\left(-2\sin t\right)+x\cdot 2\cos t=4\left(\cos^2 t-\sin^2 t\right)=4\cos 2t$.

b. The density as a function of t is given by $\rho\left(t\right)=4+4\cos t\sin t=4+2\sin 2t$, which attains its maximum at $t=\frac{\pi}{4},\ \frac{5\pi}{4}$. (This also follows from part (a) and the second derivative test.) The corresponding points on the plate are $\pm\left(\sqrt{2},\ \sqrt{2}\right)$.

12.5.63

a. The chain rule gives $w'\left(t\right)=\frac{yz}{z^2+1}\left(-\sin t\right)+\frac{xz}{z^2+1}\left(\cos t\right)+\frac{xy\left(1-z^2\right)}{\left(z^2+1\right)^2}\cdot 1=-\frac{(\sin t)t}{t^2+1}\sin t+\frac{(\cos t)t}{t^2+1}\cos t+\frac{(\cos t)(\sin t)\left(1-t^2\right)}{\left(t^2+1\right)^2}$.

b. The function $w\left(t\right)=\frac{t\,\cos t\,\sin t}{1+t^2}$ takes its maximum value on $[0,\infty)$ approximately at $t=0.838$, which gives the point $(0.669,\ 0.743,\ 0.838)$ on the spiral.

12.5.64

a. We have $x_r=\cos\theta$, $y_r=\sin\theta$, $x_\theta=-r\sin\theta$, $y_\theta=r\,\cos\theta$.

b. We have $r=\left(x^2+y^2\right)^{1/2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$, so $r_x=\frac{x}{\sqrt{x^2+y^2}}$, $r_y=\frac{y}{\sqrt{x^2+y^2}}$, and $\theta_x=-\frac{y}{x^2+y^2}$, $\theta_y=\frac{x}{x^2+y^2}$.

c. The chain rule gives $z_r=f_x\cos\theta+f_y\sin\theta$, $z_\theta=-rf_x\sin\theta+rf_y\cos\theta$.

d. The chain rule gives $z_x=g_r\frac{x}{\sqrt{x^2+y^2}}-g_\theta\frac{y}{x^2+y^2}$, $z_y=g_r\frac{y}{\sqrt{x^2+y^2}}+g_\theta\frac{x}{x^2+y^2}$.

e. Observe that $z_x^2+z_y^2=g_r^2\left(\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}\right)+g_\theta^2\left(\frac{x^2}{(x^2+y^2)^2}+\frac{y^2}{(x^2+y^2)^2}\right)=g_r^2+\frac{1}{r^2}g_\theta^2=z_r^2+\frac{1}{r^2}z_\theta^2$.

12.5.65

a. From problem 64 part (d) we have $z_x=\frac{x}{r}z_r-\frac{y}{r^2}z_\theta$, $z_y=\frac{y}{r}z_r+\frac{x}{r^2}z_\theta$

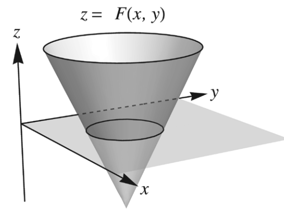
b. Differentiating the equation for z_x in part (a) with respect to x gives $z_{xx}=\frac{1}{r}z_r+x\left(-\frac{1}{r^2}\right)r_xz_r+\frac{x}{r}\left(z_r\right)_x+\frac{2y}{r^3}r_xz_\theta-\frac{y}{r^2}\left(z_\theta\right)_x=\frac{x}{r}\left(z_r\right)_x-\frac{y}{r^2}\left(z_\theta\right)_x+\left(\frac{r^2}{r^3}-\frac{x^2}{r^3}\right)z_r+\frac{2xy}{r^4}z_\theta=\frac{x}{r}\left(\frac{x}{r}z_{rr}-\frac{y}{r^2}z_{r\theta}\right)-\frac{y}{r^2}\left(\frac{x}{r}z_{\theta r}-\frac{y}{r^2}z_{\theta\theta}\right)+\frac{y^2}{r^3}z_r+\frac{2xy}{r^4}z_\theta=\frac{x^2}{r^2}z_{rr}+\frac{y^2}{r^4}z_{\theta\theta}-\frac{2xy}{r^3}z_{r\theta}+\frac{y^2}{r^3}z_r+\frac{2xy}{r^4}z_\theta$.

c. Differentiating the equation for z_y in part (a) with respect to y gives $z_{yy}=\frac{1}{r}z_r+y\left(-\frac{1}{r^2}\right)r_yz_r+\frac{y}{r}\left(z_r\right)_y-\frac{2x}{r^3}r_yz_\theta+\frac{x}{r^2}\left(z_\theta\right)_y=\frac{y}{r}\left(z_r\right)_y+\frac{x}{r^2}\left(z_\theta\right)_y+\left(\frac{r^2}{r^3}-\frac{y^2}{r^3}\right)z_r-\frac{2xy}{r^4}z_\theta=\frac{y}{r}\left(\frac{y}{r}z_{rr}+\frac{x}{r^2}z_{r\theta}\right)+\frac{x}{r^2}\left(\frac{y}{r}z_{\theta r}+\frac{x}{r^2}z_{\theta\theta}\right)+\frac{x^2}{r^3}z_r-\frac{2xy}{r^4}z_\theta=\frac{y^2}{r^2}z_{rr}+\frac{x^2}{r^4}z_{\theta\theta}+\frac{2xy}{r^3}z_{r\theta}+\frac{x^2}{r^3}z_r-\frac{2xy}{r^4}z_\theta$.

d. Adding the results from (b) and (c) gives $z_{xx}+z_{yy}=z_{rr}+\frac{1}{r}z_r+\frac{1}{r^2}z_{\theta\theta}$.

12.5.66

a. The tangent plane to the surface $z=F\left(x,\ y\right)$ has normal vector $\mathrm{n}=\left\langle F_x,\ F_y,\ -1\right\rangle$, so we see that the projection of n into the xy-plane is orthogonal to the curve $F\left(x,\ y\right)=0$.



b. The curve $F(x, y) = 0$ has a vertical tangent at a point where $F_y(x, y) = 0$.

12.5.67

a. Assuming y is fixed, the chain rule gives $F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \left(\frac{\partial z}{\partial x}\right)_y = 0$, so $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$.

b. Similarly we find that $\left(\frac{\partial y}{\partial z}\right)_x = -\frac{F_z}{F_y}$ and $\left(\frac{\partial x}{\partial y}\right)_z = -\frac{F_y}{F_x}$

c. From (a) and (b) we see that $\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial x}{\partial y}\right)_z = -1$.

d. Let $\left(\frac{\partial w}{\partial x}\right)_{y,z}$ denote the partial derivative of w with respect to x holding y and z constant, with similar notation for the other possible pairs of variables. A similar derivation as in part (a) and (b) above for $F(w, x, y, z) = 0$ shows that $\left(\frac{\partial w}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial y}\right)_{w,z}\left(\frac{\partial y}{\partial z}\right)_{w,x}\left(\frac{\partial z}{\partial w}\right)_{x,y} = \left(-\frac{F_x}{F_w}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)\left(-\frac{F_w}{F_z}\right) = 1$.

12.5.68

a. We know that $y'(x) = -\frac{f_x}{f_y}$; differentiating again with respect to x gives $y''(x) = -\frac{f_y (f_x)' - f_x (f_y)'}{f_y^2} = -\frac{f_y\left(f_{xx} + f_{xy}\, y'\right) - f_x\left(f_{yx} + f_{yy}\, y'\right)}{f_y^2} = -\frac{f_{xx} f_y^2 - 2 f_x f_y f_{xy} + f_{yy} f_x^2}{f_y^3}$.

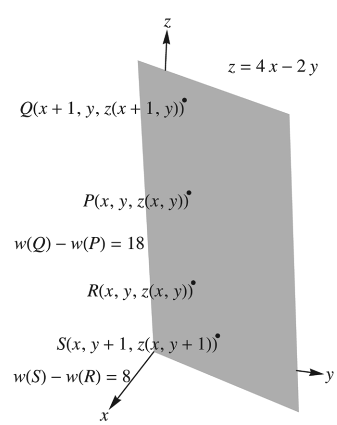
b. For $f(x, y) = xy - 1$ we have $f_x = y$, $f_y = x$, $f_{xy} = 1$, $f_{xx} = f_{yy} = 0$, so we obtain $y''(x) = --\frac{2xy}{x^3} = \frac{2}{x^3}$ because $xy = 1$. In this case we can solve to find $y(x) = \frac{1}{x}$, so $y''(x) = \frac{2}{x^3}$ is correct.

12.5.69

a. We have $\left(\frac{\partial w}{\partial x}\right)_y = f_x + f_z \frac{dz}{dx} = 2 + 4 \cdot 4 = 18$.

b. Rewrite $z = 4x - 2y$ as $y = 2x - \frac{z}{2}$; therefore $\left(\frac{\partial w}{\partial x}\right)_z = f_x + f_y \frac{dy}{dx} = 2 + 3 \cdot 2 = 8$.

c.



d. Hold x constant; then $\left(\frac{\partial w}{\partial y}\right)_x = f_y + f_z \frac{dz}{dy} = 3 + 4(-2) = -5$. Hold z constant; then $\left(\frac{\partial w}{\partial y}\right)_z = f_x \frac{dx}{dy} + f_y = 2 \cdot \frac{1}{2} + 3 = 4$. Hold x constant; then $\left(\frac{\partial w}{\partial z}\right)_x = f_y \frac{dy}{dz} + f_z = 3\left(-\frac{1}{2}\right) + 4 = \frac{5}{2}$. Hold y constant; then $\left(\frac{\partial w}{\partial z}\right)_y = f_x \frac{dx}{dz} + f_z = 2 \cdot \frac{1}{4} + 4 = \frac{9}{2}$.

12.6 Directional Derivatives and the Gradient

12.6.1 Take the dot product of the unit direction vector u and the gradient of the function.

12.6.2 The gradients $\nabla f = \langle f_x, f_y \rangle$ and $\nabla f = \langle f_x, f_y, f_z \rangle$.

12.6.3 The direction of the gradient vector is the direction in which the function is increasing the most (steepest ascent).

12.6.4 The magnitude of the gradient vector is the largest possible directional derivative of the function at a point.

12.6.5 The gradient is perpendicular to the level curves.

12.6.6 The direction must be perpendicular to the level curves and in a direction in which the function is increasing; therefore the gradients have directions $\pm\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ at the points $\pm(1,1)$ respectively.

12.6.7

a. Note that $f_x = -x$ and $f_y = -2y$. So $\nabla f(2,0) = \langle -2, 0\rangle$, $\nabla f(0,2) = \langle 0, -4\rangle$, and $\nabla f(1,1) = \langle -1, -2\rangle$.

	$(a,b) = (2,0)$	$(a,b) = (0,2)$	$(a,b) = (1,1)$
$\mathbf{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$	$-\sqrt{2}$	$-2\sqrt{2}$	$-3\sqrt{2}/2$
$\mathbf{v} = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$	$\sqrt{2}$	$-2\sqrt{2}$	$-\sqrt{2}/2$
$\mathbf{w} = \langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$	$\sqrt{2}$	$2\sqrt{2}$	$3\sqrt{2}/2$

b. The function is decreasing at $(2,0)$ in the direction of **u** and increasing at $(2,0)$ in the direction of **v** and **w**.

12.6.8

a. Note that $f_x = 4x$ and $f_y = 2y$. So $\nabla f(1,0) = \langle 4, 0\rangle$, $\nabla f(1,1) = \langle 4, 2\rangle$, and $\nabla f(1,2) = \langle 4, 4\rangle$.

	$(a,b) = (1,0)$	$(a,b) = (1,1)$	$(a,b) = (1,2)$
$\mathbf{u} = \langle 1, 0 \rangle$	4	4	4
$\mathbf{v} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$	$2\sqrt{2}$	$3\sqrt{2}$	$4\sqrt{2}$
$\mathbf{w} = \langle 0, 1 \rangle$	0	2	4

b. The function is increasing at $(1,0)$ in the direction of **u** and **v** and is constant in the direction of **w**.

12.6.9 $\nabla f(x,y) = \langle 6x, -10y\rangle$ so $\nabla f(2,-1) = \langle 12, 10\rangle$.

12.6.10 $\nabla f(x,y) = \langle 8x - 2y, -2x + 2y\rangle$, so $\nabla f(-1,-5) = \langle 2, -8\rangle$.

12.6.11 $\nabla g(x,y) = \langle 2x - 8xy - 8y^2, -4x^2 - 16xy\rangle$, so $\nabla g(-1,2) = \langle -18, 28\rangle$.

12.6.12 $\nabla p(x,y) = \langle -\frac{4x}{\sqrt{12-4x^2-y^2}}, -\frac{y}{\sqrt{12-4x^2-y^2}}\rangle$, so $\nabla p(-1,-1) = \langle \frac{4}{\sqrt{7}}, \frac{1}{\sqrt{7}}\rangle$.

12.6.13 $\nabla f(x,y) = \langle 2xye^{2xy} + e^{2xy}, 2x^2e^{2xy}\rangle$, so $\nabla f(1,0) = \langle 1, 2\rangle$.

12.6.14 $\nabla f(x,y) = \langle 3\cos(3x+2y), 2\cos(3x+2y)\rangle$, so $\nabla f(\pi, 3\pi/2) = \langle 3, 2\rangle$.

12.6.15 $\nabla F(x,y) = \langle -2xe^{-x^2-2y^2}, -4ye^{-x^2-2y^2}\rangle$, so $\nabla F(-1,2) = 2e^{-9}\langle 1, -4\rangle$.

12.6.16 $\nabla h(x,y) = \langle \frac{2x}{1+x^2+2y^2}, \frac{4y}{1+x^2+2y^2}\rangle$, so $\nabla h(2,-3) = \frac{2}{23}\langle 2, -6\rangle$.

12.6.17 $\nabla f(x,y) = \langle 2x, -2y\rangle$, so $\nabla f(-1,-3) = \langle -2, 6\rangle$. We have $D_u f(-1,-3) = \langle -2,6\rangle \cdot \langle 3/5, -4/5\rangle = -6/5 - 24/5 = -30/5 = -6$.

12.6.18 $\nabla f(x,y) = \langle 6x, 3y^2\rangle$, so $\nabla f(3,2) = \langle 18, 12\rangle$. We have $D_u f(3,2) = \langle 18,12\rangle \cdot \langle 5/13, 12/13\rangle = 90/13 + 144/13 = 234/13 = 18$.

12.6.19 $\nabla f(x,y) = \langle -6x, y^3\rangle$, so $\nabla f(2,-3) = \langle -12, -27\rangle$. We have $D_u f\,(2,-3) = \langle -12,-27\rangle \cdot \langle \frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle = \frac{27}{2} - 6\sqrt{3}$.

12.6.20 $\nabla g\,(x,y) = \langle 2\pi\cos\pi\,(2x-y)\,, -\pi\cos\pi\,(2x-y)\rangle$, so $\nabla g(-1,-1) = \langle -2\pi, \pi\rangle$. We have $D_u g\,(-1,-1) = \langle -2\pi, \pi\rangle \cdot \langle \frac{5}{13}, -\frac{12}{13}\rangle = -\frac{22}{13}\pi$.

12.6.21 $\nabla f(x,y) = \langle -\frac{x}{\sqrt{4-x^2-2y}}, -\frac{1}{\sqrt{4-x^2-2y}}\rangle$, so $\nabla f(2,-2) = \langle -1, -\frac{1}{2}\rangle$. Thus, $D_u f\,(2,-2) = \langle -1, -\frac{1}{2}\rangle \cdot \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle = -\frac{2}{\sqrt{5}}$.

12.6.22 $\nabla f(x,y) = \langle 13ye^{xy}, 13xe^{xy}\rangle$, so $\nabla f(1,0) = \langle 0, 13\rangle$. A unit vector in the direction given is $\langle 5/13, 12/13\rangle$, so $D_u f = \langle 0,13\rangle \cdot \langle 5/13, 12/13\rangle = 12$.

12.6.23 $\nabla f(x,y) = \langle 6x, 2\rangle$, so $\nabla f(1,2) = \langle 6, 2\rangle$. A unit vector in the direction given is $\langle -3/5, 4/5\rangle$, so $D_u f = \langle 6,2\rangle \cdot \langle -3/5, 4/5\rangle = -2$.

12.6.24 $\nabla h(x,y) = \langle -e^{-x-y}, -e^{-x-y}\rangle$, so $\nabla h(\ln 2, \ln 3) = -\langle \frac{1}{6}, \frac{1}{6}\rangle$ A unit vector in the direction given is $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$. We have $D_u h\,(\ln 2, \ln 3) = -\langle \frac{1}{6}, \frac{1}{6}\rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle = -\frac{2}{6\sqrt{2}} = -\frac{1}{3\sqrt{2}}$.

12.6.25 $\nabla g(x,y) = \langle \frac{2x}{4+x^2+y^2}, \frac{2y}{4+x^2+y^2}\rangle$, so $\nabla g(-1,2) = \langle -\frac{2}{9}, \frac{4}{9}\rangle$. A unit vector in the direction given is $\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$. We have $D_u g\,(-1,2) = \langle -\frac{2}{9}, \frac{4}{9}\rangle \cdot \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle = 0$.

12.6.26 $\nabla f(x,y) = \langle -\frac{y}{(x-y)^2}, \frac{x}{(x-y)^2}\rangle$, so $\nabla f(4,1) = \langle -\frac{1}{9}, \frac{4}{9}\rangle$. A unit vector in the direction given is $\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$. We have $D_u f\,(4,1) = \langle -\frac{1}{9}, \frac{4}{9}\rangle \cdot \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle = \frac{1}{\sqrt{5}}$.

12.6.27

a. At the point $(1,-2)$ the value of the gradient is $\nabla f\,(1,-2) = \langle 2x, -8y\rangle\Big|_{(1,-2)} = \langle 2, 16\rangle$. Therefore, the direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{65}}\langle 1, 8\rangle$ and the direction of steepest descent is $-\mathbf{u}$.

b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{65}}\langle -8, 1\rangle$.

12.6.28

a. At the point $(2,1)$ the value of the gradient is $\nabla f\,(2,1) = \langle 2x+4y, 4x-2y\rangle\Big|_{(2,1)} = \langle 8, 6\rangle$. Therefore, the direction of steepest ascent is $\mathbf{u} = \langle 4/5, 3/5\rangle$ and the direction of steepest descent is $-\mathbf{u}$.

b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \langle -3/5, 4/5\rangle$.

12.6.29

a. At the point $(-1,1)$ the value of the gradient is $\nabla f\,(-1,1) = \langle 4x^3 - 2xy, -x^2 + 2y\rangle\Big|_{(-1,1)} = \langle -2, 1\rangle$. Therefore, the direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{5}}\langle -2, 1\rangle$ and the direction of steepest descent is $-\mathbf{u}$.

b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle$.

12.6.30

a. At the point $(1,2)$ the value of the gradient is $\nabla P(1,2) = \left\langle \frac{x+y}{\sqrt{20+x^2+2xy-y^2}}, \frac{x-y}{\sqrt{20+x^2+2xy-y^2}} \right\rangle\Big|_{(1,2)} = \langle \frac{3}{\sqrt{21}}, -\frac{1}{\sqrt{21}} \rangle$. Therefore, the direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{10}}\langle 3,-1\rangle$ and the direction of steepest descent is $-\mathbf{u}$.

b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{10}}\langle 1,3\rangle$.

12.6.31

a. At the point $(-1,1)$ the value of the gradient is $\nabla f(-1,1) = \left\langle -xe^{-x^2/2-y^2/2}, -ye^{-x^2/2-y^2/2} \right\rangle\big|_{(-1,1)} = e^{-1}\langle 1,-1\rangle$. Therefore, the direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1,-1\rangle$ and the direction of steepest descent is $-\mathbf{u}$.

b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{2}}\langle 1,1\rangle$.

12.6.32

a. At the point $(0,\pi)$ the value of the gradient is $\nabla f(0,\pi) = \langle 4\cos(2x-3y), -6\cos(2x-3y)\rangle\big|_{(0,\pi)} = \langle -4,6\rangle$. Therefore, the direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{13}}\langle -2,3\rangle$ and the direction of steepest descent is $-\mathbf{u}$.

b. Take any vector perpendicular to $\mathbf{u}$; for example, $\mathbf{v} = \frac{1}{\sqrt{13}}\langle 3,2\rangle$.

12.6.33

a. The gradient of $f$at P is $\nabla f(3,2) = \langle -4x, -6y\rangle\Big|_{(3,2)} = \langle -12,-12\rangle$.

b. The direction of steepest ascent is$\mathbf{u} = \frac{1}{\sqrt{2}}\langle -1,-1\rangle$ which makes angle $\theta = \frac{5\pi}{4}$ with the x-axis; therefore, the angle of maximum decrease is $\theta = \frac{\pi}{4}$ and the angles of zero change are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

c. We have $g(\theta) = \langle -12,-12\rangle \cdot \langle \cos\theta, \sin\theta\rangle = -12\cos\theta - 12\sin\theta$.

d. The critical points for $g(\theta)$ satisfy $g'(\theta) = 12(\sin\theta - \cos\theta) = 0$, which gives $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$. By inspection we see that the maximum occurs at $\frac{5\pi}{4}$, and we have $g\left(\frac{5\pi}{4}\right) = 12\sqrt{2}$.

e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that $|\nabla f(3,2)| = 12\sqrt{2} = g\left(\frac{5\pi}{4}\right)$.

12.6.34

a. The gradient of f at p is $\nabla f(-3,-1) = \langle 2x, 6y\rangle\Big|_{(-3,-1)} = \langle -6,-6\rangle$.

b. The direction of steepest ascent is $\mathbf{u} = \frac{1}{\sqrt{2}}\langle -1,-1\rangle$ which makes angle $\theta = \frac{5\pi}{4}$ with the x-axis; therefore, the angle of maximum decrease is $\theta = \frac{\pi}{4}$ and the angles of zero change are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

c. We have $g(\theta) = \langle -6,-6\rangle \cdot \langle \cos\theta, \sin\theta\rangle = -6\cos\theta - 6\sin\theta$.

d. The critical points for $g(\theta)$ satisfy $g'(\theta) = 6(\sin\theta - \cos\theta) = 0$, which gives $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$. By inspection we see that the maximum occurs at $\frac{5\pi}{4}$, and we have $g\left(\frac{5\pi}{4}\right) = 6\sqrt{2}$.

e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that$|\nabla f(-3,-1)| = 6\sqrt{2} = g\left(\frac{5\pi}{4}\right)$.

12.6.35

a. The gradient of f at P is $\nabla f\left(\sqrt{3},1\right) = \left\langle \frac{x}{\sqrt{2+x^2+y^2}}, \frac{y}{\sqrt{2+x^2+y^2}} \right\rangle\Big|_{(\sqrt{3},1)} = \left\langle \frac{\sqrt{3}}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle = \frac{\sqrt{6}}{6}\left\langle \sqrt{3}, 1\right\rangle$.

b. The direction of steepest ascent is $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ which makes angle $\theta = \frac{\pi}{6}$ with the x-axis; therefore, the angle of maximum decrease is $\theta = \frac{7\pi}{6}$ and the angles of zero change are $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

c. We have $g(\theta) = \frac{\sqrt{6}}{6}\left\langle \sqrt{3}, 1 \right\rangle \cdot \langle \cos\theta, \sin\theta \rangle = \frac{\sqrt{18}}{6}\cos\theta + \frac{\sqrt{6}}{6}\sin\theta$.

d. The critical points for $g(\theta)$ satisfy $g'(\theta) = \frac{\sqrt{6}}{6}\left(-\sqrt{3}\sin\theta + \cos\theta\right) = 0$, which gives $\tan\theta = \frac{1}{\sqrt{3}}$, so $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$. By inspection we see that the maximum occurs at $\frac{\pi}{6}$, and we have $g\left(\frac{\pi}{6}\right) = \frac{\sqrt{6}}{3}$.

e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that $|\nabla f(3,1)| = \frac{\sqrt{6}}{3} = g\left(\frac{\pi}{6}\right)$.

12.6.36

a. The gradient of f at P is $\nabla f\left(-1, -\frac{1}{\sqrt{3}}\right) = \left\langle -\frac{x}{\sqrt{12-x^2-y^2}}, -\frac{y}{\sqrt{12-x^2-y^2}} \right\rangle\Big|_{\left(-1,-1/\sqrt{3}\right)} = \frac{1}{4\sqrt{2}}\left\langle \sqrt{3}, 1 \right\rangle$.

b. The direction of steepest ascent is $\mathrm{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ which makes angle $\theta = \frac{\pi}{6}$ with the x-axis; therefore, the angle of maximum decrease is $\theta = \frac{7\pi}{6}$ and the angles of zero change are $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

c. We have $g(\theta) = \frac{1}{4\sqrt{2}}\left\langle \sqrt{3}, 1 \right\rangle \cdot \langle \cos\theta, \sin\theta \rangle = \frac{1}{4\sqrt{2}}\left(\sqrt{3}\cos\theta + \sin\theta\right)$.

d. The critical points for $g(\theta)$ satisfy $g'(\theta) = \frac{1}{4\sqrt{2}}\left(-\sqrt{3}\sin\theta + \cos\theta\right) = 0$,which gives $\tan\theta = \frac{1}{\sqrt{3}}$, so $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$. By inspection we see that the maximum occurs at $\frac{\pi}{6}$, and we have $g\left(\frac{\pi}{6}\right) = \frac{\sqrt{2}}{4}$.

e. Observe that the maximum value of $g(\theta)$ occurs at theangle found in part (d), and that $\left|\nabla f\left(-1, -\frac{1}{\sqrt{3}}\right)\right| = \frac{\sqrt{2}}{4} = g\left(\frac{\pi}{6}\right)$.

12.6.37

a. The gradient of f at P is $\nabla f(-1,0) = \left\langle -2xe^{-x^2-2y^2}, -4ye^{-x^2-2y^2} \right\rangle\Big|_{(-1,0)} = \left\langle 2e^{-1}, 0 \right\rangle$.

b. The direction of steepest ascent is $\mathbf{u} = \langle 1, 0 \rangle$ which makes angle $\theta = 0$ with the x-axis; therefore the angle of maximum decrease is $\theta = \pi$ and the angles of zero change are $\pm\frac{\pi}{2}$.

c. We have $g(\theta) = \frac{2}{\mathrm{e}}\langle 1, 0 \rangle \cdot \langle \cos\theta, \sin\theta \rangle = \frac{2}{\mathrm{e}}\cos\theta$.

d. The maximum value of $g(\theta)$ occurs at $\theta = 0$, and we have $g(0) = \frac{2}{\mathrm{e}}$.

e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that $|\nabla f(-1,0)| = \frac{2}{\mathrm{e}} = g(0)$.

12.6.38

a. The gradient of f at P is $\nabla f\left(\frac{3}{4}, -\sqrt{3}\right) = \langle \frac{4x}{1+2x^2+3y^2}, \frac{6y}{1+2x^2+3y^2} \rangle\Big|_{\left(3/4,-\sqrt{3}\right)} = \frac{24}{89}\left\langle 1, -2\sqrt{3} \right\rangle$.

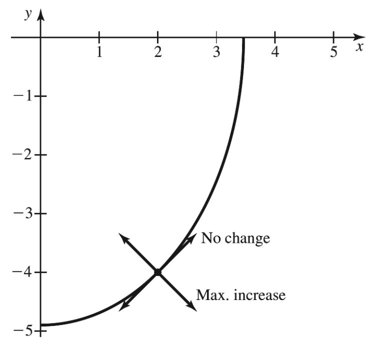
b. The direction of steepest ascent is $\mathrm{u} = \left\langle \frac{1}{\sqrt{13}}, -\frac{2\sqrt{3}}{\sqrt{13}} \right\rangle$ which makes angle $\theta = -\tan^{-1}\left(2\sqrt{3}\right)$ with the x-axis; therefore, the angle of maximum decrease is $\theta = \pi - \tan^{-1}\left(2\sqrt{3}\right)$ and the angles of zero change are $\pm\frac{\pi}{2} - \tan^{-1}\left(2\sqrt{3}\right)$.

c. We have $g(\theta) = \frac{24}{89}\left\langle 1, -2\sqrt{3} \right\rangle \cdot \langle \cos\theta, \sin\theta \rangle = \frac{24}{89}\left(\cos\theta - 2\sqrt{3}\sin\theta\right)$.

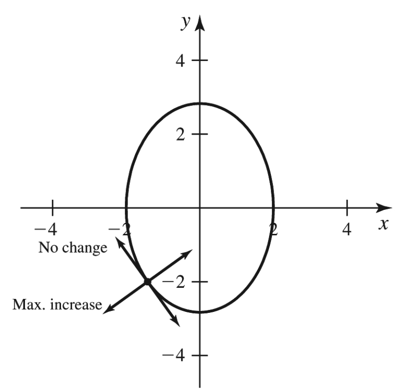
d. The critical points for $g(\theta)$ satisfy $g'(\theta) = \frac{24}{89}\left(-\sin\theta - 2\sqrt{3}\cos\theta\right) = 0$, which gives $\tan\theta = -2\sqrt{3}$, so $\theta = -\tan^{-1}\left(2\sqrt{3}\right), \pi - \tan^{-1}\left(2\sqrt{3}\right)$. By inspection we see that the maximum occurs at $-\tan^{-1}\left(2\sqrt{3}\right)$, and we have $g\left(-\tan^{-1}\left(2\sqrt{3}\right)\right) = \frac{24\sqrt{13}}{89}$.

e. Observe that the maximum value of $g(\theta)$ occurs at the angle found in part (d), and that $\left|\nabla f\left(\frac{3}{4}, -\sqrt{3}\right)\right| = \frac{24\sqrt{13}}{89} = g\left(-\tan^{-1}\left(2\sqrt{3}\right)\right)$.

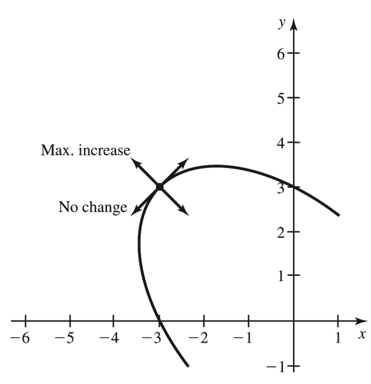
12.6.39 The gradient of f at P is $\nabla f(2,-4) = \langle 8x, 4y\rangle\Big|_{(2,-4)} = \langle 16, -16\rangle$, which gives the direction of maximum increase.



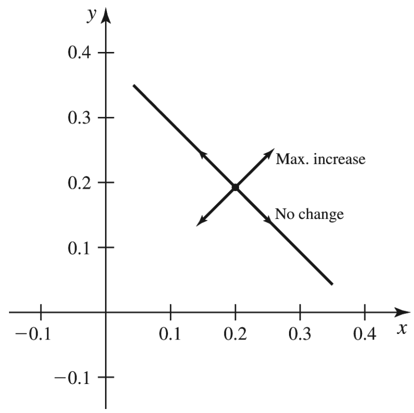
12.6.40 The gradient of f at P is $\nabla f(-1,-2) = \langle 12x, 6y\rangle\Big|_{(-1,-2)} = \langle -12, -12\rangle$, which gives the direction of maximum increase.



12.6.41 The gradient of f at P is $\nabla f(-3,3) = \langle 2x+y, x+2y\rangle\Big|_{(-3,3)} = \langle -3, 3\rangle$, which gives the direction of maximum increase.



12.6.42 The gradient of f at P is $\nabla f\left(\frac{\pi}{16}, \frac{\pi}{16}\right) = \left\langle 2\sec^2(2x+2y), 2\sec^2(2x+2y)\right\rangle\Big|_{(\pi/16,\pi/16)} = \langle 4, 4\rangle$, which gives the direction of maximum increase.



12.6.43 The slope of the level curves for $f(x,y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line has slope 0 at $(0,16)$. The gradient of f at this point is $\nabla f(0,16) = \left\langle -\frac{x}{2}, -\frac{y}{8}\right\rangle\Big|_{(0,16)} = \langle 0,-2\rangle$, which is perpendicular to the tangent line.

12.6.44 The slope of the level curves for $f(x,y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, which is undefined at $(8,0)$, so the tangent line is vertical at $(8,0)$. The gradient of f at this point is $\nabla f(8,0) = \left\langle -\frac{x}{2}, -\frac{y}{8}\right\rangle\Big|_{(8,0)} = \langle -4,0\rangle$, which is perpendicular to the tangent line.

12.6.45 The slope of the level curves for $f(x,y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, which is undefined at $(4,0)$, so the tangent line is vertical at $(4,0)$. The gradient of f at this point is $\nabla f(4,0) = \left\langle -\frac{x}{2}, -\frac{y}{8}\right\rangle\Big|_{(4,0)} = \langle -2,0\rangle$, which is perpendicular to the tangent line.

12.6.46 The slope of the level curves for $f(x,y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line has slope $-2\sqrt{3}$ at $\left(2\sqrt{3},4\right)$, and its direction is parallel to the vector $\left\langle 1,-2\sqrt{3}\right\rangle$. The gradient of f at this point is $\nabla f\left(2\sqrt{3},4\right) = \left\langle -\frac{x}{2}, -\frac{y}{8}\right\rangle\Big|_{(2\sqrt{3},4)} = \left\langle -\sqrt{3}, -\frac{1}{2}\right\rangle$, which is perpendicular to the tangent direction.

12.6.47 Let $z = f(x,y)$. Then $z^2 = 1 - \frac{x^2}{4} - \frac{y^2}{16}$, so $2zz_x = -\frac{x}{2}$, which implies that $z_x = f_x = -\frac{x}{4z}$. Also $2zz_y = -\frac{y}{8}$, which implies that $z_y = f_y = -\frac{y}{16z}$. The slope of the level curves for $f(x,y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line has slope $-\frac{2}{\sqrt{3}}$ at $\left(\frac{1}{2},\sqrt{3}\right)$, and its direction is parallel to the vector $\langle 1, -\frac{2}{\sqrt{3}}\rangle$. The gradient of f at this point is $\nabla f\left(\frac{1}{2},\sqrt{3}\right) = \left\langle -\frac{x}{4z}, -\frac{y}{16z}\right\rangle\Big|_{(1/2,\sqrt{3})} = -\frac{1}{8}\langle \frac{2}{\sqrt{3}}, 1\rangle$, which is perpendicular to the tangent direction.

12.6.48 Let $z = f(x,y)$. Then $z^2 = 1 - \frac{x^2}{4} - \frac{y^2}{16}$, so $2zz_x = -\frac{x}{2}$, which implies that $z_x = f_x = -\frac{x}{4z}$. Also $2zz_y = -\frac{y}{8}$, which implies that $z_y = f_y = -\frac{y}{16z}$. The slope of the level curves for $f(x,y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line has slope 0 at $\left(0,\sqrt{8}\right)$, and its direction is parallel to the vector $\langle 1,0\rangle$. The gradient of f at this point is $\nabla f\left(0,\sqrt{8}\right) = \left\langle -\frac{x}{4z}, -\frac{y}{16z}\right\rangle\Big|_{(0,\sqrt{8})} = \langle 0, -\frac{1}{4}\rangle$,which is perpendicular to the tangent direction.

12.6.49 Let $z = f(x,y)$. Then $z^2 = 1 - \frac{x^2}{4} - \frac{y^2}{16}$, so $2zz_x = -\frac{x}{2}$, which implies that $z_x = f_x = -\frac{x}{4z}$. Also, $2zz_y = -\frac{y}{8}$, which implies that $z_y = f_y = -\frac{y}{16z}$.The slope of the level curves for $f(x,y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line is vertical at $\left(\sqrt{2},0\right)$, and its direction is parallel to the vector $\langle 0,1\rangle$. The gradient of f at this point is $\nabla f\left(\sqrt{2},0\right) = \left\langle -\frac{x}{4z}, -\frac{y}{16z}\right\rangle\Big|_{(\sqrt{2},0)} = \langle -\frac{1}{2}, 0\rangle$, which is perpendicular to the tangent direction.

12.6.50 Let $z = f(x,y)$. Then $z^2 = 1 - \frac{x^2}{4} - \frac{y^2}{16}$, so $2zz_x = -\frac{x}{2}$, which implies that $z_x = f_x = -\frac{x}{4z}$. Also, $2zz_y = -\frac{y}{8}$, which implies that $z_y = f_y = -\frac{y}{16z}$.The slope of the level curves for $f(x,y)$ is given by $y'(x) = -\frac{f_x}{f_y} = -\frac{4x}{y}$, so the tangent line has slope -2 at $(1,2)$, and its direction is parallel to the vector $\langle 1,-2\rangle$. The gradient of f at this point is $\nabla f(1,2) = \left\langle -\frac{x}{4z}, -\frac{y}{16z}\right\rangle\Big|_{(1,2)} = -\frac{\sqrt{2}}{8}\langle 2,1\rangle$, which is perpendicular to the tangent direction.

12.6.51

a. We have $\nabla f = \langle f_x, f_y\rangle = \langle 1,0\rangle$.

b. Let $(x(t), y(t))$ be the projection into the xy-plane of the path of steepest descent starting at $(x(0), y(0)) = (4,4)$. We solve $(x'(t), y'(t)) = -\nabla f = \langle -1, 0\rangle$ which together with the initial conditions gives $x = 4-t$, $y = 4$ for $t \geq 0$.

12.6.52

a. We have $\nabla f = \langle f_x, f_y\rangle = \langle 1,1\rangle$.

b. Let $(x(t), y(t))$ be the projection into the xy-plane of the path of steepest descent starting at $(x(0), y(0)) = (2,2)$. We solve $(x'(t), y'(t)) = -\nabla f = \langle -1,-1\rangle$ which together with the initial conditions gives $x = 2-t$, $y = 2-t$ for $t \geq 0$.

12.6.53

a. We have $\nabla f = \langle f_x, f_y\rangle = \langle -2x, -4y\rangle$.

b. Let $(x(t), y(t))$ be the projection into the xy-plane of the path of steepest descent starting at $(x(0), y(0)) = \left(\frac{\pi}{2}, 1\right)$. We solve $(x'(t), y'(t)) = -\nabla f = \langle 2x, 4y\rangle$ with the initial conditions $x(0) = 1$ and $y(0) = 1$. The differential equation $\frac{dx}{dt} = -2x$ is separable: we have $\int \frac{1}{x}\,dx = \int 2\,dt$ which implies that $\ln|x| = 2t + C$. The initial condition $x(0) = 1$ gives $C = 0$ so $x(t) = e^{2t}$. Similarly, $y(t) = e^{4t}$. Thus, $y = x^2$ for $x \geq 1$.

12.6.54

a. We have $\nabla f = \langle f_x, f_y\rangle = \langle -x^{-2}, 1\rangle$.

b. Let $(x(t), y(t))$ be the projection into the xy-plane of the path of steepest descent starting at $(x(0), y(0)) = (1,2)$. We solve $(x'(t), y'(t)) = -\nabla f = \langle x^{-2}, -1\rangle$ with the initial conditions $x(0) = 1$ and $y(0) = 2$. The differential equation $\frac{dx}{dt} = \frac{1}{x^2}$ is separable: we have $\int x^2\,dx = \int dt$, which implies that $\frac{x^3}{3} = t + C$; the initlal condition gives $C = \frac{1}{3}$ so $x(t) = (3t+1)^{1/3}$ and $y(t) = 2-t$ for $t \geq 0$.

12.6.55

a. We have $\nabla f = 2x\mathbf{i} + 4y\mathbf{j} + 8z\mathbf{k}$; $\nabla f(1,0,4) = 2\mathbf{i} + 32\mathbf{k}$.

b. The vector $2\mathbf{i} + 32\mathbf{k}$ has length $2\sqrt{257}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{257}}(\mathbf{i} + 16\mathbf{k})$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(1,0,4)| = 2\sqrt{257}$.

d. The vector $\mathbf{u} = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f(1,0,4) = \langle 2,0,32 \rangle \cdot \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle = \frac{34}{\sqrt{2}} = 17\sqrt{2}$.

12.6.56

a. We have $\nabla f = -2x\mathbf{i} + 6y\mathbf{j} + z\mathbf{k}$; $\nabla f(0,2,-1) = 12\mathbf{j} - \mathbf{k}$.

b. The vector $12\mathbf{j} - \mathbf{k}$ has length $\sqrt{145}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{145}}(12\mathbf{j} - \mathbf{k})$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(0,2,-1)| = \sqrt{145}$.

d. The vector $\mathbf{u} = \langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f(0,2,-1) = \langle 0,12,-1 \rangle \cdot \langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle = \frac{13}{\sqrt{2}}$.

12.6.57

a. We have $\nabla f = 4yz\mathbf{i} + 4xz\mathbf{j} + 4xy\mathbf{k}$; $\nabla f(1,-1,-1) = 4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$.

b. The vector $4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ has length $4\sqrt{3}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} - \mathbf{k})$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(1,-1,-1)| = 4\sqrt{3}$.

d. The vector $\mathbf{u} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f(1,-1,-1) = \langle 4,-4,-4 \rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle = \frac{4}{\sqrt{3}}$.

12.6.58

a. We have $\nabla f = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$; $\nabla f(2,-2,1) = -\mathbf{i} + 3\mathbf{j}$.

b. The vector $-\mathbf{i} + 3\mathbf{j}$ has length 10, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{10}}(-\mathbf{i} + 3\mathbf{j})$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f(2,-2,1)| = \sqrt{10}$.

d. The vector $\mathbf{u} = \langle 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f(2,-2,1) = \langle -1,3,0 \rangle \cdot \langle 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle = -\frac{3}{\sqrt{2}}$.

12.6.59

a. We have $\nabla f = \cos(x+2y-z)(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$; $\nabla f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right) = \cos\left(\frac{2\pi}{3}\right)(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -\frac{1}{2}\mathbf{i} - \mathbf{j} + \frac{1}{2}\mathbf{k}$.

b. The vector $-\frac{1}{2}\mathbf{i} - \mathbf{j} + \frac{1}{2}\mathbf{k}$ has length $\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{6}}(-\mathbf{i} - 2\mathbf{j} + \mathbf{k})$.

c. The rate of change of f in the direction of maximum increase at P is $\left|\nabla f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right)\right| = \frac{\sqrt{6}}{2}$.

d. The vector $\mathbf{u} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right) = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle \cdot \langle -\frac{1}{2}, -1, \frac{1}{2} \rangle = -\frac{1}{2}$.

12.6.60

a. We have $\nabla f = e^{xyz-1}(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k})$; $\nabla f(0,1,-1) = -e^{-1}\mathbf{i}$.

b. The unit vector in the direction of $-e^{-1}\mathbf{i}$ is $\mathbf{u} = -\mathbf{i}$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f\,(0,1,-1)\,| = e^{-1}$.

d. The vector $\mathbf{u} = \langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f\,(0,1,-1) = \langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\rangle \cdot \langle -e^{-1}, 0, 0\rangle = \frac{2}{3e}$.

12.6.61

a. We have $\nabla f = \frac{2}{1+x^2+y^2+z^2}\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$; $\nabla f\,(1,1,-1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$.

b. The vector $\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$ has length $\frac{\sqrt{3}}{2}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{3}}\,(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

c. The rate of change of f in the direction of maximum increase at$P$ is $|\nabla f\,(1,1,-1)\,| = \frac{\sqrt{3}}{2}$.

d. The vector $\mathbf{u} = \langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f\,(1,1,-1) = \langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\rangle \cdot \langle \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\rangle = \frac{5}{6}$.

12.6.62

a. We have $\nabla f = \frac{1}{(y-z)^2}\,((y-z)\,\mathbf{i} + (z-x)\,\mathbf{j} + (x-y)\,\mathbf{k})$; $\nabla f\,(3,2,-1) = \frac{1}{3}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{1}{9}\mathbf{k}$.

b. The vector $\frac{1}{3}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{1}{9}\mathbf{k}$ has length $\frac{\sqrt{26}}{9}$, so the unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{26}}\,(3\mathbf{i} - 4\mathbf{j} + \mathbf{k})$.

c. The rate of change of f in the direction of maximum increase at P is $|\nabla f\,(3,2,-1)\,| = \frac{\sqrt{26}}{9}$.

d. The vector $\mathbf{u} = \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle$ is a unit vector, so the directional derivative at P in this direction is $D_u f\,(3,2,-1) = \langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle \cdot \langle \frac{1}{3}, -\frac{4}{9}, \frac{1}{9}\rangle = -\frac{7}{27}$.

12.6.63

a. False. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$.

b. False. The gradient is a vector, so it does not make sense to say that it is positive.

c. False. f is a function of three variables, so ∇f has three components.

d. True. This is because $f_x = f_y = f_z = 0$.

12.6.64

a. We can express $F\,(x,y,z) = f\,(g\,(x,y,z))$ where $g\,(x,y,z) = xyz$ and $f\,(w) = e^w$.

b. Observe that $\nabla F\,(x,y,z) = e^{xyz}\,(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) = e^{xyz}\nabla g = f'\,(g\,(x,y,z))\,\nabla g\,(x,y,z)$.

12.6.65 Observe that $\nabla f\,(x,y) = -8x\mathbf{i} - 2y\mathbf{j}$, so $\nabla f\,(1,2) = -8\mathbf{i} - 4\mathbf{j}$. A vector perpendicular to $\nabla f\,(1,2)$ is $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$, so the unit vectors perpendicular to $\nabla f\,(1,2)$ are $\mathbf{u} = \pm\frac{1}{\sqrt{5}}\,(\mathbf{i} - 2\mathbf{j})$.

12.6.66 Observe that $\nabla f\,(x,y) = 2x\mathbf{i} - 8y\mathbf{j}$, so $\nabla f\,(4,1) = 8\mathbf{i} - 8\mathbf{j}$. A vector perpendicular to $\nabla f\,(4,1)$ is $\mathbf{v} = \mathbf{i} + \mathbf{j}$, so the unit vectors perpendicular to $\nabla f\,(4,1)$ are $\mathbf{u} = \pm\frac{1}{\sqrt{2}}\,(\mathbf{i} + \mathbf{j})$.

12.6.67 Observe that $\nabla f\,(x,y) = \frac{1}{\sqrt{3+2x^2+y^2}}\,(2x\mathbf{i} + y\mathbf{j})$, so $\nabla f\,(1,-2) = \frac{2}{3}\,(\mathbf{i} - \mathbf{j})$. A vector perpendicular to $\nabla f\,(1,-2)$ is $\mathbf{v} = \mathbf{i} + \mathbf{j}$, so the unit vectors perpendicular to $\nabla f\,(1,-2)$ are $\mathbf{u} = \pm\frac{1}{\sqrt{2}}\,(\mathbf{i} + \mathbf{j})$.

12.6.68 Observe that $\nabla f\,(x,y) = -e^{1-xy}\,(y\mathbf{i} + x\mathbf{j})$, so $\nabla f\,(1,0) = -e\mathbf{j}$, so the unit vectors perpendicular to $\nabla f\,(1,0)$ are $\mathbf{u} = \pm\mathbf{i}$.

12.6.69 The function $f(x,y) = ax + by + c$ has $\nabla f\,(x,y) = a\mathbf{i} + b\mathbf{j}$, so the path in the xy-plane corresponding to the path of steepest ascent on the plane is given by $x = x_0 + at$, $y = y_0 + bt$.

12.6.70

a. The function $z = f(x, y)$ satisfies the relation $z^2 = (x-a)^2 + (y-b)^2$, which is the equation of a cone with vertex at $(a, b, 0)$.

b. We have $\nabla f(x, y) = \frac{1}{\sqrt{(x-a)^2+(y-b)^2}}((x-a)\,\mathbf{i} + (y-b)\,\mathbf{j})$, which implies that $|\nabla f(x, y)| = 1$ for all points $(x, y) \neq (a, b)$.

c. At any point $(x, y) \neq (a, b)$, the greatest rate of change of f is 1, which occurs in the direction of the vector from (a, b) to (x, y).

12.6.71

a. We have $\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle$, so $\nabla f(1, 1, 1) = \langle 2, 2, 2 \rangle$.

b. Points (x, y, z) on the plane P satisfy $\langle x-1, y-1, z-1 \rangle \cdot \langle 2, 2, 2 \rangle = 0$, so $x + y + z = 3$ is an equation for P.

12.6.72

a. We have $\nabla f(x, y, z) = \langle -yz, -xz, -xy \rangle$, so $\nabla f(2, 2, 2) = \langle -4, -4, -4 \rangle$.

b. Points (x, y, z) on the plane P satisfy $\langle x-2, y-2, z-2 \rangle \cdot \langle -4, -4, -4 \rangle = 0$, so $x + y + z = 6$ is an equation for P.

12.6.73

a. We have $\nabla f(x, y, z) = e^{x+y-z}\langle 1, 1, -1 \rangle$, so $\nabla f(1, 1, 2) = \langle 1, 1, -1 \rangle$.

b. Points (x, y, z) on the plane P satisfy $\langle x-1, y-1, z-2 \rangle \cdot \langle 1, 1, -1 \rangle = 0$, so $x + y - z = 0$ is an equation for P.

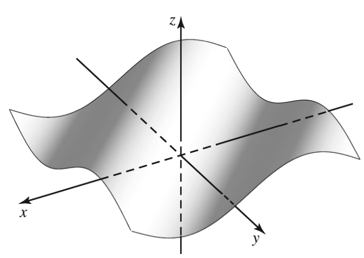
12.6.74

a. We have $\nabla f(x, y, z) = \langle y + z, x - z, x - y \rangle$, so $\nabla f(1, 1, 1) = \langle 2, 0, 0 \rangle$.

b. Points (x, y, z) on the plane P satisfy $\langle x-1, y-1, z-1 \rangle \cdot \langle 2, 0, 0 \rangle = 0$, so $x = 1$ is an equation for P.

12.6.75

a.



b. The height function has gradient $\nabla z = \cos(x-y)\langle 1, -1 \rangle$, so the directions in which the height function has zero change are $\mathbf{v} = \pm\langle 1, 1 \rangle$.

c. The direction $\mathbf{v}$ would be the opposite of the direction of ∇z, so $\mathbf{v} = \pm\langle 1, -1 \rangle$.

d. These answers are consistent with the graph in part (a).

12.6.76

a. The height function has gradient $\nabla z = \cos(ax - b)\langle a, -b\rangle$, so the directions in which the height function has zero change are $\mathbf{v} = \pm\langle b, a\rangle$.

b. The direction $\mathbf{v}$ would be the opposite of the direction of ∇z,so $\mathbf{v} = \pm\langle a, -b\rangle$.

12.6.77

a. Observe that $\varphi_x = -\frac{kQ}{r^2} r_x = -\frac{kQ}{r^2}\frac{x}{\sqrt{x^2+y^2+z^2}} = -\frac{kQx}{r^3}$; similarly $\varphi_y = -\frac{kQy}{r^3}$ and $\varphi_z = -\frac{kQz}{r^3}$. Therefore $\mathrm{E}(x, y, z) = -\nabla\varphi(x, y, z) = kQ\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle$.

b. We have $|\mathrm{E}| = \frac{kQ}{r^3}|\langle x, y, z\rangle| = \frac{kQ}{r^2}$. Therefore the magnitude of the electric field is inversely proportional to the square of the distance to the point charge.

12.6.78 Observe that $\varphi_x = \frac{GMm}{r^2} r_x = \frac{GMm}{r^2}\frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{GMmx}{r^3}$, Similarly, $\varphi_y = \frac{GMmy}{r^3}$ and $\varphi_z = \frac{GMmz}{r^3}$. Therefore $\mathrm{F}(x, y, z) = -\nabla\varphi(x, y, z) = -\frac{GMm}{r^3}\langle x, y, z\rangle$, and we have $|\mathrm{F}| = \frac{GMm}{r^3}|\langle x, y, z\rangle| = \frac{GMm}{r^2}$. Therefore the magnitude of the gravitational field is inversely proportional to the square of the distance between the two objects.

12.6.79 We have $\langle u, v\rangle = \nabla\varphi = \langle \pi\cos\pi x\sin 2\pi y, 2\pi\sin\pi x\cos 2\pi y\rangle$.

12.6.80 Observe that $\nabla f(x, y) = A\mathbf{i} + B\mathbf{j}$, which is a constant vector. Therefore the direction and magnitude of the greatest directional derivative of $f(x, y, z)$ is the same at all points (x, y).

12.6.81 We give the proofs for functions on $\mathbb{R}^2$. The proofs for functions on $\mathbb{R}^3$ are similar.

a. $\nabla(cf) = \langle (cf)_x, (cf)_y\rangle = \langle cf_x, cf_y\rangle = c\nabla f$.

b. $\nabla(f + g) = \langle (f+g)_x, (f+g)_y\rangle = \langle f_x + g_x, f_y + g_y\rangle = \nabla f + \nabla g$.

c. $\nabla(fg) = \langle (fg)_x, (fg)_y\rangle = \langle f_x g + f g_x, f_y g + f g_y\rangle = (\nabla f) g + f\nabla g$.

d. $\nabla\left(\frac{f}{g}\right) = \langle \left(\frac{f}{g}\right)_x, \left(\frac{f}{g}\right)_y\rangle = \langle \frac{gf_x - fg_x}{g^2}, \frac{gf_y - fg_y}{g^2}\rangle = \frac{g(\nabla f) - f\nabla g}{g^2}$.

e. $\nabla(f \circ g) = \langle f'(g) g_x, f'(g) g_y\rangle = f'(g)\nabla g$.

12.6.82 Using the product and chain rules from Exercise 81 gives $\nabla(xy\cos(xy)) = \cos(xy)\nabla(xy) + xy(-\sin(xy))\nabla(xy) = (\cos(xy) - xy\sin(xy))\langle y, x\rangle$.

12.6.83 Using the quotient rule from Exercise 81 gives $\nabla\left(\frac{x+y}{x^2+y^2}\right) = \frac{(x^2+y^2)\nabla(x+y) - (x+y)\nabla(x^2+y^2)}{(x^2+y^2)^2} = \frac{1}{(x^2+y^2)^2}\left((x^2+y^2)\langle 1, 1\rangle - (x+y)\langle 2x, 2y\rangle\right) = \frac{1}{(x^2+y^2)^2}\langle y^2 - x^2 - 2xy, x^2 - y^2 - 2xy\rangle$.

12.6.84 Using the chain rule from Exercise 81 gives $\nabla\left(\ln\left(1 + x^2 + y^2\right)\right) = \frac{1}{1+x^2+y^2}\nabla\left(1 + x^2 + y^2\right) = \frac{1}{1+x^2+y^2}\langle 2x, 2y\rangle$.

12.6.85 Using the chain rule from Exercise 81 gives $\nabla\left(\sqrt{25 - x^2 - y^2 - z^2}\right) = \frac{1}{2\sqrt{25-x^2-y^2-z^2}}\nabla\left(25 - x^2 - y^2 - z^2\right) = \frac{1}{2\sqrt{25-x^2-y^2-z^2}}\langle -2x, -2y, -2z\rangle = -\frac{1}{\sqrt{25-x^2-y^2-z^2}}\langle x, y, z\rangle$.

12.6.86 Using the product and chain rules from Exercise 81 gives $\nabla\left((x+y+z)e^{xyz}\right) = e^{xyz}\nabla(x+y+z) + (x+y+z)\nabla e^{xyz} = e^{xyz}\left(\langle 1, 1, 1\rangle + (x+y+z)\langle yz, xz, xy\rangle\right) = e^{xyz}\langle 1 + (x+y+z)yz, 1 + (x+y+z)xz, 1 + (x+y+z)xy\rangle$.

12.6.87 Using the quotient rule from Exercise 81 gives $\nabla\left(\frac{x+yz}{y+xz}\right) = \frac{(y+xz)\nabla(x+yz) - (x+yz)\nabla(y+xz)}{(y+xz)^2} = \frac{1}{(y+xz)^2}\left((y+xz)\langle 1, z, y\rangle - (x+yz)\langle z, 1, x\rangle\right) = \frac{1}{(y+xz)^2}\langle y\left(1 - z^2\right), x\left(z^2 - 1\right), y^2 - x^2\rangle$.

12.7 Tangent Planes and Linear Approximation

12.7.1 The gradient of F is a multiple of n.

12.7.2 Let $F(x,y,z) = z - f(x,y) = z - xy^2 - x^2y + 10$.

12.7.3 The tangent plane has equation $F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$.

12.7.4 The tangent plane has equation $z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$.

12.7.5 Multiply the change in x by f_x and the change in y by f_y and add both terms to $f(a,b)$.

12.7.6 The change in f is approximately $f_x(a,b)\triangle x + f_y(a,b)\triangle y$.

12.7.7 In terms of differentials, $dz = f_x(a,b)\,dx + f_y(a,b)\,dy$.

12.7.8 The differential for the function $w = f(x,y,z)$ at the point (a,b,c) is

$$dw = f(a,b,c)\,dx + f_y(a,b,c)\,dy + f_z(a,b,c)\,dz.$$

12.7.9 $\nabla F = \langle 2x,1,1\rangle$, so $\nabla F(1,1,1) = \langle 2,1,1\rangle$. The tangent plane at $(1,1,1)$ has equation $2(x-1)+(y-1)+(z-1)=0$, or $2x+y+z=4$. Also, $\nabla F(2,0,-1) = \langle 4,1,1\rangle$, so the equation of the tangent plane there is $4(x-2)+y+(z+1)=0$, or $4x+y+z=7$.

12.7.10 $\nabla F = \langle 2x, 3y^2, 4z^3\rangle$, so $\nabla F(1,0,1) = \langle 2,0,4\rangle$. The tangent plane at $(1,0,1)$ is given by $2(x-1)+4(z-1)=0$, or $x+2z=3$. Also, $\nabla F(-1,0,1) = \langle -2,0,4\rangle$, so the tangent plane there is given by $-2(x+1)+4(z-1)=0$, or $x-2z=-3$.

12.7.11 $\nabla F = \langle y+z, x+z, x+y\rangle$, so $\nabla F(2,2,2) = \langle 4,4,4\rangle$. The tangent plane at $(2,2,2)$ is given by $4(x-2)+4(y-2)+4(z-2)=0$, or $x+y+z=6$. At the point $(2,0,6)$ we have $\nabla F(2,0,6) = \langle 6,8,2\rangle$, so the equation of the tangent plane is $6(x-2)+8y+2(z-6)=0$, or $3x+4y+z=12$.

12.7.12 $\nabla F = \langle 2x, 2y, -2z\rangle$, so $\nabla F(3,4,6) = \langle 6,8,-10\rangle$ and the tangent plane at $(3,4,6)$ has equation $8(x-3)+8(y-4)-10(z-5)=0$, or $3x+4y-5z=0$. $\nabla F(-4,-3,5) = \langle -8,-6,-10\rangle$ so the tangent plane at $(-4,-3,5)$ has equation $-8(x+4)-6(y+3)-10(z+5)=0$ or $4x+3y+5z=0$.

12.7.13 $\nabla F = \langle y\sin z, x\sin z, yz\cos z\rangle$, so $\nabla F\left(1,2,\frac{\pi}{6}\right) = \langle 1, \frac{1}{2}, \sqrt{3}\rangle$ and the tangent plane at $\left(1,2,\frac{\pi}{6}\right)$ has equation $(x-1)+\frac{1}{2}(y-2)+\sqrt{3}\left(z-\frac{\pi}{6}\right)=0$, or $x+\frac{1}{2}y+\sqrt{3}z = 2+\frac{\sqrt{3}\pi}{6}$. Also, $\nabla F\left(-2,-1,\frac{5\pi}{6}\right) = \langle -\frac{1}{2}, -1, -\sqrt{3}\rangle$, so the tangent plane at $\left(-2,-1,\frac{5\pi}{6}\right)$ has equation $-\frac{1}{2}(x+2)-(y+1)-\sqrt{3}\left(z-\frac{5\pi}{6}\right)=0$ or $\frac{1}{2}x+y+\sqrt{3}z = \frac{5\sqrt{3}\pi}{6}-2$.

12.7.14 $\nabla F = e^{xz}\langle yz^2, z, y(1+x)z\rangle$, so $\nabla F(0,2,4) = \langle 32,4,8\rangle$ and the tangent plane at $(0,2,4)$ has equation $32x+4(y-2)+8(z-4)=0$, or $8x+y+2z=10$. Also, $\nabla F(0,-8,-1) = \langle -8,-1,8\rangle$, so the tangent plane at $(0,-8,-1)$ has equation $-8x-(y+8)+8(z+1)=0$ or $8x+y-8z=0$.

12.7.15 $\nabla F = \langle -\frac{x}{8}, -\frac{2y}{9}, 2z\rangle$, so $\nabla F\left(4,3,-\sqrt{3}\right) = \langle -\frac{1}{2}, -\frac{2}{3}, -2\sqrt{3}\rangle$ and the tangent plane at $\left(4,3,-\sqrt{3}\right)$ has equation $-\frac{1}{2}(x-4)-\frac{2}{3}(y-3)-2\sqrt{3}\left(z+\sqrt{3}\right)=0$, or $\frac{1}{2}x+\frac{2}{3}y+2\sqrt{3}z=-2$. Also, $\nabla F\left(-8,9,\sqrt{14}\right) = \left\langle 1,-2,2\sqrt{14}\right\rangle$, so the tangent plane at $\left(-8,9,\sqrt{14}\right)$ has equation $(x+8)-2(y-9)+2\sqrt{14}\left(z-\sqrt{14}\right)=0$ or $x-2y+2\sqrt{14}z=2$.

12.7.16 $\nabla F = \langle 2, 2y, -2z\rangle$, so $\nabla F(0,1,1) = \langle 2,2,-2\rangle$ and the tangent plane at $(0,1,1)$ has equation $2x+2(y-1)-2(z-1)=0$, or $x+y-z=0$. Also, $\nabla F(4,1,-3) = \langle 2,2,6\rangle$, so the tangent plane at $(4,1,-3)$ has equation $2(x-4)+2(y-1)+6(z+3)=0$ or $x+y+3z=-4$.

12.7.17 We have $f_x = -4x$, $f_y = -2y$, so the tangent plane at $(2,2,-8)$ has equation $z = f(2,2) + f_x(2,2)(x-2)+f_y(2,2)(y-2) = -8-8(x-2)-4(y-2)$, or $z=-8x-4y+16$, and the equation of the tangent plane at $(-1,-1,1)$ is $z = f(-1,-1)+f_x(-1,-1)(x+1)+f_y(-1,-1)(y+1) = 1+4(x+1)+2(y+1)$, or $z=4x+2y+7$.

12.7.18 We have $f_x = 4x$, $f_y = y$, so the tangent plane at $\left(-\frac{1}{2}, 1, 3\right)$ has equation $z = f\left(-\frac{1}{2}, 1\right) + f_x\left(-\frac{1}{2}, 1\right)\left(x + \frac{1}{2}\right) + f_y\left(-\frac{1}{2}, 1\right)(y-1) = 3 - 2\left(x + \frac{1}{2}\right) + (y-1)$, or $z = -2x + y + 1$, and the equation of the tangent plane at $(3, -2, 22)$ is $z = f(3,-2) + f_x(3,-2)(x-3) + f_y(3,-2)(y+2) = 22 + 12(x-3) - 2(y+2)$, or $z = 12x - 2y - 18$.

12.7.19 We have $f_x = ye^{xy}$, $f_y = xe^{xy}$, so the tangent plane at $(1,0,1)$ has equation $z = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y) = 1 + 0 + 1(y)$, or $z = 1 + y$. The equation of the tangent plane at $(0,1,1)$ is $z = 1 + 1(x) + 0(y-1)$, or $z = 1 + x$.

12.7.20 We have $f_x = y\cos(xy)$ and $f_y = x\cos(xy)$, so the tangent plane at $(1,0,2)$ has equation $z = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y) = 2 + 0(x-1) + 1(y)$, or $z = 2 + y$. The equation of the tangent plane at $(0,5,2)$ is $z = 2 + 5(x) + 0(y-5)$, or $z = 2 + 5x$.

12.7.21 We have $f_x = \left(2x + x^2\right)e^{x-y}$, $f_y = -x^2e^{x-y}$, so the tangent plane at $(2,2,4)$ has equation $z = f(2,2) + f_x(2,2)(x-2) + f_y(2,2)(y-2) = 4 + 8(x-2) - 4(y-2)$, or $z = 8x - 4y - 4$, and the equation of the tangent plane at $(-1,-1,1)$ is $z = f(-1,-1) + f_x(-1,-1)(x+1) + f_y(-1,-1)(y+1) = 1 - (x+1) - (y+1)$, or $z = -x - y - 1$.

12.7.22 We have $f_x = \frac{y}{1+xy}$, $f_y = \frac{x}{1+xy}$, so the tangent plane at $(1, 2, \ln 3)$ has equation $z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = \ln 3 + \frac{2}{3}(x-1) + \frac{1}{3}(y-2)$, or $z = \frac{2}{3}x + \frac{1}{3}y + \ln 3 - \frac{4}{3}$, and the equation of the tangent plane at $(-2,-1,\ln 3)$ is $z = f(-2,-1) + f_x(-2,-1)(x+2) + f_y(-2,-1)(y+1) = \ln 3 - \frac{1}{3}(x+2) - \frac{2}{3}(y+1)$, or $z = -\frac{1}{3}x - \frac{2}{3}y + \ln 3 - \frac{4}{3}$.

12.7.23 We have $f_x = \frac{y^2+2xy-x^2}{(x^2+y^2)^2}$, $f_y = \frac{y^2-2xy-x^2}{(x^2+y^2)^2}$, so the tangent plane at $\left(1, 2, -\frac{1}{5}\right)$ has equation $z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = -\frac{1}{5} + \frac{7}{25}(x-1) - \frac{1}{25}(y-2)$, or $z = \frac{7}{25}x - \frac{1}{25}y - \frac{2}{5}$, and the equation of the tangent plane at $\left(2, -1, \frac{3}{5}\right)$ is $z = f(2,-1) + f_x(2,-1)(x-2) + f_y(2,-1)(y+1) = \frac{3}{5} - \frac{7}{25}(x-2) + \frac{1}{25}(y+1)$, or $z = -\frac{7}{25}x + \frac{1}{25}y + \frac{6}{5}$.

12.7.24 We have $f_x = -2\sin(x-y)$, $f_y = 2\sin(x-y)$, so the tangent plane at $\left(\frac{\pi}{6}, -\frac{\pi}{6}, 3\right)$ has equation $z = f\left(\frac{\pi}{6}, -\frac{\pi}{6}\right) + f_x\left(\frac{\pi}{6}, -\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) + f_y\left(\frac{\pi}{6}, -\frac{\pi}{6}\right)\left(y + \frac{\pi}{6}\right) = 3 - \sqrt{3}\left(x - \frac{\pi}{6}\right) + \sqrt{3}\left(y + \frac{\pi}{6}\right)$, or $z = \sqrt{3}(y-x) + 3 + \frac{\sqrt{3}\pi}{3}$, and the equation of the tangent plane at $\left(\frac{\pi}{3}, \frac{\pi}{3}, 4\right)$ is $z = f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) + f_x\left(\frac{\pi}{3}, \frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f_y\left(\frac{\pi}{3}, \frac{\pi}{3}\right)\left(y - \frac{\pi}{3}\right) = 4$, or $z = 4$.

12.7.25

a. We have $f_x = y + 1$, $f_y = x - 1$, so the linear approximation for f at $(2,3)$ is $L(x,y) = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3) = 5 + 4(x-2) + (y-3)$,or $L(x,y) = 4x + y - 6$.

b. We have $L(2.1, 2.99) = 5.39$.

12.7.26

a. We have $f_x = -8x$, $f_y = -16y$, so the linear approximation for f at $(-1,4)$ is $L(x,y) = f(-1,4) + f_x(-1,4)(x+1) + f_y(-1,4)(y-4) = -120 + 8(x+1) - 64(y-4)$ or $L(x,y) = 8x - 64y + 144$.

b. We have $L(-1.05, 3.95) = -117.20$.

12.7.27

a. We have $f_x = -2x$, $f_y = 4y$, so the linear approximation for f at $(3,-1)$ is $L(x,y) = f(3,-1) + f_x(3,-1)(x-3) + f_y(3,-1)(y+1) = -7 - 6(x-3) - 4(y+1)$ or $L(x,y) = -6x - 4y + 7$.

b. We have $L(3.1, -1.04) = -7.44$.

12.7.28

a. We have $f_x = \frac{x}{\sqrt{x^2+y^2}}$, $f_y = \frac{y}{\sqrt{x^2+y^2}}$, so the linear approximation for f at $(3,-4)$ is $L(x,y) = f(3,-4) + f_x(3,-4)(x-3) + f_y(3,-4)(y+4) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y+4)$,or $L(x,y) = \frac{3}{5}x + \frac{4}{5}y$.

b. We have $L(3.06, -3.92) = 4.972$.

12.7.29

a. We have $f_x = \frac{1}{1+x+y}$, $f_y = \frac{1}{1+x+y}$, so the linear approximation for f at $(0,0)$ is $L(x,y) = f(0,0) + f_x(0,0)\,x + f_y(0,0)\,y = x + y$.

b. We have $L(0.1, -0.2) = -0.1$.

12.7.30

a. We have $f_x = -\frac{2y}{(x-y)^2}$, $f_y = \frac{2x}{(x-y)^2}$, so the linear approximation for f at $(3,2)$ is $L(x,y) = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2) = 5 - 4(x-3) + 6(y-2)$,or $L(x,y) = -4x + 6y + 5$.

b. We have $L(2.95, 2.05) = 5.50$.

12.7.31 We have $f_x = 2 - 2y$, $f_y = -3 - 2x$, and $dx = 0.1$, $dy = -0.1$, so $dz = f_x(1,4)\,dx + f_y(1,4)\,dy = -6dx - 5dy = -0.1$.

12.7.32 We have $f_x = -2x$, $f_y = 6y$, and $dx = -0.05$, $dy = -0.1$, so $dz = f_x(-1,2)\,dx + f_y(-1,2)\,dy = 2dx + 12dy = -1.30$.

12.7.33 We have $f_x = e^{x+y}$, $f_y = e^{x+y}$, and $dx = 0.1$, $dy = -0.05$, so $dz = f_x(0,0)\,dx + f_y(0,0)\,dy = dx + dy = 0.05$.

12.7.34 We have $f_x = \frac{1}{1+x+y}$, $f_y = \frac{1}{1+x+y}$, and $dx = -0.1$, $dy = 0.03$, so $dz = f_x(0,0)\,dx + f_y(0,0)\,dy = dx + dy = -0.07$.

12.7.35

a. If r increases and R decreases then $R^2 - r^2$ decreases, so S decreases.

b. If both R and r increase, then it is impossible to say whether $R^2 - r^2$ increases or decreases.

c. We have $dS = 8\pi^2(RdR - rdr) = 8\pi^2(5.50 \cdot 0.15 - 3 \cdot 0.05) = 5.4\pi^2 \approx 53.296$.

d. We have $dS = 8\pi^2(RdR - rdr) = 8\pi^2(7 \cdot 0.04 - 3 \cdot (-0.05)) = 3.44\pi^2 \approx 33.951$.

e. The surface area is approximately unchanged when $RdR = rdr$.

12.7.36

a. We have $dV = \frac{\pi}{3}\left(2rhdr + r^2dh\right) = \frac{\pi}{3}\left(2 \cdot 6.5 \cdot 4.2 \cdot 0.1 + 6.5^2(-0.05)\right) \approx 3.505$.

b. We have $dV = \frac{\pi}{3}\left(2rhdr + r^2dh\right) = \frac{\pi}{3}\left(2 \cdot 5.4 \cdot 12(-0.03) + 6.5^2(-0.04)\right) \approx -5.293$.

12.7.37 Observe that $dA = \pi(bda + adb)$ so $\frac{dA}{A} = \frac{da}{a} + \frac{db}{b}$, and hence the percentage increase in the area is approximately 2% + 1.5%= 3.5%.

12.7.38 Observe that $dV = (\pi/2)\left(2rhdr + r^2dh\right)$ so $\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$, and hence the percentage decrease in the area is approximately $2(-1.5\%) + 2.2\% = -0.8\%$.

12.7.39 We have $dw = \left(y^2 + 2xz\right)dx + \left(2xy + z^2\right)dy + \left(x^2 + 2yz\right)dz$.

12.7.40 We have $dw = \cos(x + y - z)(dx + dy - dz)$.

12.7.41 We have $dw = \frac{dx}{y+z} - \frac{u+x}{(y+z)^2}dy - \frac{u+x}{(y+z)^2}dz + \frac{du}{y+z}$.

12.7.42 We have $dw = \frac{q}{rs}dp + \frac{p}{rs}dq - \frac{pq}{r^2s}dr - \frac{pq}{rs^2}ds$.

12.7.43

a. Observe that $2cdc = (2a - 2b\cos\theta)\,da + (2b - 2a\cos\theta)\,db + 2ab\sin\theta\,d\theta$ so $dc = \frac{a-b\cos\theta}{c}da + \frac{b-a\cos\theta}{c}db + \frac{ab\sin\theta}{c}d\theta$. We have $a = 2$, $b = 4$, $\theta = \frac{\pi}{3}$ which gives $c = \sqrt{12}$, and $da = 0.03$, $db = -0.04$, $d\theta = \frac{\pi}{90}$; substituting in the equation above gives $dc \approx 0.035$.

b. We have $a = 2$, $b = 4$, $da = 0.03$, $db = -0.04$, $d\theta = 0$, so $dc = -\frac{0.01+0.04\cos\theta}{c}$; comparing the cases $\theta = \frac{\pi}{20}$ and $\theta = \frac{9\pi}{20}$ we see that c is smaller and $\cos\theta$ is larger in the first case; therefore the change in c is greater when $\theta = \frac{\pi}{20}$.

12.7.44

a. The cost function is obtained by multiplying the distance L by the cost per mile, which is $\frac{p}{m}$.

b. We have $C_L = \frac{p}{m}$, $C_m = -\frac{Lp}{m^2}$, $C_p = \frac{L}{m}$.This shows that C is an increasing function of L and p and a decreasing function of m, which makes sense.

c. We have $dC = \frac{p}{m}dL - \frac{Lp}{m^2}dm + \frac{L}{m}dp \approx \$10.29$.

d. Observe that $\frac{dC}{C} = \frac{dL}{L} - \frac{dm}{m} + \frac{dp}{p}$; therefore the cost is equally sensitive to a 1% change in any of the variables.

12.7.45

a. True. This is because the function $F(x,y,z) = x^2 + z^2$ has $F_y = 0$.

b. True. As $z > 0$ decreases, $\frac{1}{z}$ increases.

c. False. The gradient $\nabla F(a,b,c)$ is perpendicular to the tangent plane for the surface $F(x,y,z) = 0$ at (a,b,c).

12.7.46 Let $f(x,y) = \tan^{-1}(x+y)$; then $f_x(x,y) = f_y(x,y) = \frac{1}{1+(x+y)^2}$ and the tangent plane at $(0,0,0)$ has equation $z = f_x(0,0)\,x + f_y(0,0)\,y + f(0,0) = x + y$.

12.7.47 Let $f(x,y) = \tan^{-1}(xy)$; then $f_x(x,y) = \frac{y}{1+(xy)^2}$, $f_y(x,y) = \frac{x}{1+(xy)^2}$ and the tangent plane at $\left(1,1,\frac{\pi}{4}\right)$ has equation $z = f_x(1,1)(x-1) + f_y(1,1)(y-1) + f(1,1) = \frac{1}{2}x + \frac{1}{2}y + \frac{\pi}{4} - 1$.

12.7.48 Rewrite this equation as $x + z = 2(y - z)$ or equivalently, $x - 2y + 3z = 0$; therefore the surface is a plane and hence is identical to its tangent plane at any point.

12.7.49 The branch of the surface $\sin(xyz) = \frac{1}{2}$ at the point $\left(\pi, 1, \frac{1}{6}\right)$ can be described more simply by the equation $F(x,y,z) = xyz = \frac{\pi}{6}$. We have $F_x(x,y,z) = yz$, $F_y(x,y,z) = xz$, $Fz(x,y,z) = xy$ so the tangent plane at $\left(\pi,1,\frac{1}{6}\right)$ has equation $F_x\left(\pi,1,\frac{1}{6}\right)(x-\pi) + F_y\left(\pi,1,\frac{1}{6}\right)(y-1) + Fz\left(\pi,1,\frac{1}{6}\right)\left(z-\frac{1}{6}\right) = 0$, or $\frac{1}{6}(x-\pi) + \frac{\pi}{6}(y-1) + \pi\left(z - \frac{1}{6}\right) = 0$.

12.7.50 Let $f(x,y) = \sin(x-y)$; then $f_x(x,y) = \cos(x-y)$, $f_y(x,y) = -\cos(x-y)$ so the tangent plane at $(a,b,\sin(a-b))$ is horizontal at all points where $b - a = \frac{\pi}{2} + k\pi$ for some integer k.

12.7.51 Let $F(x,y,z) = x^2 + 2y^2 + z^2 - 2x - 2y + 3$; then $F_x(x,y,z) = 2x - 2$, $F_y(x,y,z) = 2y + 2$. The tangent plane to the surface $F(x,y,z) = 0$ at (a,b,c) is horizontal if and only if $F_x(a,b,c) = F_y(a,b,c) = 0$, which gives $a = 1$, $b = -1$. This implies $c^2 = 1$, so the points are $(1,-1,1)$ and $(1,-1,-1)$.

12.7.52 Let $F(x,y,z) = x^2 + 2y^2 + z^2 - 2x - 2z - 2$; then $F_x(x,y,z) = 2x - 2$, $F_y(x,y,z) = 4y$. The tangent plane to the surface $F(x,y,z) = 0$ at (a,b,c) is horizontal if and only if $F_x(a,b,c) = F_y(a,b,c) = 0$, which gives $a = 1$, $b = 0$. This implies $c^2 - 2c - 3 = 0$, which gives $c = 3, -1$ so the points are $(1,0,3)$ and $(1,0,-1)$.

12.7.53 Let $f(x,y) = \cos 2x \sin y$; then $f_x(x,y) = -2\sin 2x \sin y$, $f_y(x,y) = \cos 2x \cos y$; so the tangent plane at $(a,b,\cos 2a \sin b)$ is horizontal at all points where $\sin 2a = \cos b = 0$ or $\sin b = \cos 2a = 0$. In the region $-\pi \le x \le \pi$, $-\pi \le y \le \pi$ the points are $a = 0, \pm\frac{\pi}{2}, \pm\pi$ and $b = \pm\frac{\pi}{2}$, or $a = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ and $b = 0, \pm\pi$.

12.7.54

a. Apply logarithmic differentiation to $A^2 = s(s-a)(s-b)(s-c)$: $\frac{2A_a}{A} = \frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}}{s-a} + \frac{\frac{1}{2}}{s-b} + \frac{\frac{1}{2}}{s-c}$, so $A_a = \frac{A}{4}\left(\frac{1}{s} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s-a}\right)$, and similarly $A_b = \frac{A}{4}\left(\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-c} - \frac{1}{s-b}\right)$, and $A_c = \frac{A}{4}\left(\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right)$.

b. The change is approximately $dA = A_a da + A_b db + A_c dc \approx -0.5573$.

c. In this case $s = \frac{3}{2}a$, $s-a = s-b = s-c = \frac{1}{2}a$,and $A_a = A_b = A_c = \frac{2A}{3a}$ so if all sides change by the same amount da, then $\frac{dA}{A} = 3 \cdot \frac{2}{3a} da = 2\frac{da}{a}$. Therefore if all sides increase by $p\%$, the area will increase by approximately $2p\%$. (This result can also be obtained more directly using the formula $A = \frac{\sqrt{3}a^2}{4}$ for the area of an equilateral triangle with side a.)

12.7.55

a. We have $S_r = \pi\sqrt{r^2+h^2} + \frac{\pi r^2}{\sqrt{r^2+h^2}}$, $S_h = \frac{\pi r h}{\sqrt{r^2+h^2}}$; using the values $r = 2.5$, $h = 0.6$, $dr = 0.05$, $dh = -0.02$ gives $dS = S_r dr + S_h dh = \frac{\pi}{\sqrt{r^2+h^2}}\left(\left(2r^2+h^2\right)dr + rhdh\right) \approx 0.749$.

b. If $r = 100$, $h = 200$ then $dS = 40\sqrt{5}\pi\,(3dr + dh)$, so the surface area is more sensitive to small changes in r.

12.7.56

a. The vector $\mathbf{n}_1 = \langle 1, 1, -1\rangle$ is normal to this plane.

b. Rewrite the equation of the paraboloid as $F(x,y,z) = z - x^2 - 3y^2 = 0$; then $\nabla F(2,1,7) = \langle 4, 6, -1\rangle$, so $\mathbf{n}_2 = \langle 4, 6, -1\rangle$ is normal to the tangent plane.

c. The line tangent to C lies in both of these planes and hence must be orthogonal to both vectors n_1 and n_2. Therefore a direction vector for this line is given by $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \left|\mathbf{ijk}11-146-1\right| = 5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

d. The tangent line has parametric equation $\mathbf{r}(t) = \langle 2,1,7\rangle + t\langle 5,-3,2\rangle = \langle 2+5t, 1-3t, 7+2t\rangle$, or $x = 2+5t, y = 1-3t$, $z = 7+2t$.

12.7.57

a. The differential of A is given by $dA = \frac{dx}{y} - \frac{xdy}{y^2}$; substituting $x = 60$, $y = 175$, $dx = 2$ and $dy = 5$ gives $dA = \frac{2}{1225} \approx 0.00163$.

b. If the batter fails to get a hit, the average decreases by $\frac{x}{y} - \frac{x}{y+1} = \frac{x}{y(y+1)} = \frac{A}{y+1}$, whereas if the batter gets a hit, the average increases by $\frac{x+1}{y+1} - \frac{x}{y} = \frac{y-x}{y(y+1)} = \frac{1-A}{y+1}$. If $A = 0.350$ the second of these quantities is larger so the answer is no; the batting average changes more if the batter gets a hit than if he fails to get a hit.

c. The answer depends on whether A is less than or greater than 0.500.

12.7.58 The volume of the tank is $V = \frac{\pi r^2 h}{3}$, so $dV = \frac{2\pi}{3} rhdr + \frac{\pi}{3} r^2 dh$ or equivalently $\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$. If the water level drops from 2.00 to 1.95, then the radius changes proportionately so $\frac{dV}{V} = 3\frac{dh}{h} = -0.075$, which gives $dV \approx -0.039\mathrm{m}^2$.

12.7.59

a. The centerline velocity is given by $V = \frac{R^2}{L}$ so $dV = \frac{2R}{L} dR - \frac{R^2}{L^2} dL$; evaluate this with $R = 3$, $dR = 0.05$, $L = 50$, $dL = 0.5$ to obtain $dV = \frac{21}{5000} = 0.0042\mathrm{cm}^3$.

b. Rewrite the formula for dV as $\frac{dV}{V} = \frac{2dR}{R} - \frac{dL}{L}$; hence if R decreases 1% and L increases 2%, then V will decrease by approximately 4%.

c. If the radius of the cylinder increases by $p\%$, then the length of the cylinder must decrease by approximately $2p\%$ in order for the velocity to remain constant.

12.7.60

a. Let $z = xy$; then $dz = ydx + xdy$, so the absolute error is at most $(|x|+|y|)\cdot 10^{-16}$ and the percentage error at most $\frac{100\cdot(|x|+|y|)\cdot 10^{-16}}{|xy|}\% = \left(\frac{1}{|x|}+\frac{1}{|y|}\right)\cdot 10^{-14}\%$.

b. Let $z = \frac{x}{y}$; then $dz = \frac{ydx-xdy}{y^2}$, so the absolute error is at most $\left(\frac{|x|+|y|}{y^2}\right)\cdot 10^{-16}$ and the percentage error at most $100\cdot\left(\frac{|x|+|y|}{y^2}\right)\cdot\frac{|y|}{|x|}\cdot 10^{-16}\% = \left(\frac{1}{|x|}+\frac{1}{|y|}\right)\cdot 10^{-14}\%$.

c. Let $w = xyz$; then $dw = yzdx + xzdy + xydz$, so the absolute error is at most $(|xy|+|yz|+|xz|)\cdot 10^{-16}$ and the percentage error at most $\frac{100\cdot(|xy|+|yz|+|xz|)\cdot 10^{-16}}{|xyz|}\% = \left(\frac{1}{|x|}+\frac{1}{|y|}+\frac{1}{|z|}\right)\cdot 10^{-14}\%$.

d. Let $z = \frac{x/y}{z} = \frac{x}{yz}$; then $dz = \frac{yzdx-xzdy-xydz}{y^2z^2}$, so the absolute error is at most $\left(\frac{|xy|+|yz|+|xz|}{y^2z^2}\right)\cdot 10^{-16}$ and the percentage error at most $100\cdot\left(\frac{|xy|+|yz|+|xz|}{y^2z^2}\right)\cdot\frac{|yz|}{|x|}\cdot 10^{-16}\% = \left(\frac{1}{|x|}+\frac{1}{|y|}+\frac{1}{|z|}\right)\cdot 10^{-14}\%$.

12.7.61

a. We have $f_r = n\,(1-r)^{n-1}$ and $f_n = -\,(1-r)^n\ln(1-r)$.

b. We have $\triangle P \approx f\,(20, 0.1)\cdot 0.01 = 20\cdot(0.9)^{19}(0.01) \approx 0.027$.

c. We have $\triangle P \approx f\,(20, 0.9)\cdot 0.01 = 20\cdot(0.1)^{19}(0.01) \approx 2\times 10^{-20}$.

d. Small changes in the flu rate have a greater effect on the probability of catching the flu when the flu rate is small compared to when the flu rate is large.

12.7.62

a. We have $R = \frac{R_1R_2}{R_1+R_2}$ so $dR = \frac{R_2^2dR_1+R_1^2dR_2}{(R_1+R_2)^2} = R^2\left(\frac{dR_1}{R_1^2}+\frac{dR_2}{R_2^2}\right)$. Substituting $R_1 = 2$, $dR_1 = 0.05$,$R_2 = 3$, $dR_2 = -0.05$ gives $dR = 0.01$ ohms.

b. True; if $R_1 = R_2$ and $dR_1 = -dR_2$ then $dR = 0$.

c. Yes, because in this case $dR > 0$.

d. The formula in (a) shows that if $R_1 > R_2$ then R is more sensitive to changes in R_2, so if R_1 increases by the same small amount that R_2 decreases, then R will decrease.

12.7.63 From the equation $\frac{1}{R} = \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$ we obtain $-\frac{1}{R^2}\frac{\partial R}{\partial R_1} = -\frac{1}{R_1^2}$, which implies that $\frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}$, with similar formulas for the other partials. Hence $dR = R^2\left(\frac{dR_1}{R_1^2}+\frac{dR_2}{R_2^2}+\frac{dR_3}{R_3^2}\right)$. Substituting the values $R_1 = 2$, $dR_1 = 0.05$, $R_2 = 3$, $dR_2 = -0.05$, $R_3 = 1.5$, $dR_3 = 0.05$ gives $R = \frac{2}{3}$ and $dR = \frac{7}{540} \approx 0.0130$ ohms.

12.7.64 Observe that $f_x = ax^{a-1}y^b$ and $f_y = bx^ay^{b-1}$, so $\frac{dz}{z} = \frac{ax^{a-1}y^bdx+bx^ay^{b-1}dy}{x^ay^b} = a\frac{dx}{x}+b\frac{dy}{y}$.

12.7.65

a. Suppose f is a function of x and y; then $d\,(\ln f) = (\ln f)_x\,dx + (\ln f)_y\,dy = \frac{f_x}{f}dx + \frac{f_y}{f}dy = \frac{df}{f}$.

b. The absolute change in ln f is approximately $d\,(\ln f)$ and the relative change in f is approximately $\frac{df}{f}$; from part (a) these agree.

c. Observe that $df = ydx + xdy$, so $\frac{df}{f} = \frac{dx}{x}+\frac{dy}{y}$.

d. Observe that $df = \frac{ydx-xdy}{y^2}$, so $\frac{df}{f} = \frac{dx}{x}-\frac{dy}{y}$.

e. If $f = x_1 x_2 \cdots x_n$ then $\ln f = \ln x_1 + \ln x_2 + \cdots + \ln x_n$ and therefore $\frac{df}{f} = \frac{dx_1}{x_1} + \frac{dx_2}{x_2} + \cdots + \frac{dx_n}{x_n}$.

12.7.66

a. Let $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$; then $\nabla F = \langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \rangle$ so the tangent plane at (p, q, r) has equation $\frac{p}{a^2}(x - p) + \frac{q}{b^2}(y - q) + \frac{r}{c^2}(z - r) = 0$, which simplifies to $\frac{px}{a^2} + \frac{qy}{b^2} + \frac{rz}{c^2} = 1$.

b. The vectors $\langle A, B, C \rangle$ and $\langle \frac{p}{a^2}, \frac{q}{b^2}, \frac{r}{c^2} \rangle$ must be proportional; therefore we can express $\langle p, q, r \rangle = \lambda \langle Aa^2, Bb^2, Cc^2 \rangle$ for some scalar λ. Substituting in the equation of the ellipsoid gives $\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} = \lambda^2 \left(A^2a^2 + B^2b^2 + C^2c^2 \right) = 1$, so $\lambda = \pm\frac{1}{m}$ and $(p, q, r) = \pm\left(Aa^2, Bb^2, Cc^2 \right)$ where $m = \sqrt{A^2a^2 + B^2b^2 + C^2c^2}$.

c. The distance of the plane $Ax + By + Cz = 1$ to the origin is $d = \frac{1}{\sqrt{A^2+B^2+C^2}} = h$ (this formula is derived in 12.1 Exercise 102).

d. The tangent plane at $(p, q, r) = \pm\left(Aa^2, Bb^2, Cc^2 \right)$ has equation $Ax + By + Cz = \pm m$, which has distance to the origin $d = \frac{m}{\sqrt{A^2+B^2+C^2}} = mh$ using the formula in 12.1 Exercise 102.

e. The plane P does not intersect the ellipsoid if and only if the two tangent planes parallel to P are closer to the origin than P; this is equivalent to the condition $m < 1$.

12.8 Maximum/Minimum Problems

12.8.1 It is locally the highest point on the surface; you cannot get toa higher point in any direction.

12.8.2 The surface looks similar to the hyperbolic paraboloid $z = x^2 - y^2$ at the origin.

12.8.3 The partial derivatives are both zero or one or both do not exist.

12.8.4 No; for example f may have a saddle point at (a, b).

12.8.5 The discriminant of f at (a, b) is the determinant given by $D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - f_{xy}(a, b)^2$.

12.8.6 The second derivative test may be used to determine whether a critical point is a local maximum, minimum or saddle point (see Theorem 12.14).

12.8.7 The function f has an absolute minimum value at $(a, b) \in R$ if $f(x, y) \geq f(a, b)$ for all $(x, y) \in R$.

12.8.8 Determine the values of f at all critical points in the closed bounded domain R, and find the maximum/minimum values of f on the boundary of R; the greatest/smallest of these values is the absolute maximum/minimum of f on R.

12.8.9 We have $f_x = 2x$, $f_y = 2y$ so $(0, 0)$ is the only critical point of f.

12.8.10 We have $f_x = 2x - 6$, $f_y = 2y + 8$ so $(3, -4)$ is the only critical point of f.

12.8.11 We have $f_x = 6(3x - 2)$, $f_y = 2(y - 4)$ so $\left(\frac{2}{3}, 4\right)$ is the only critical point of f.

12.8.12 We have $f_x = 6x$, $f_y = -8y$ so $(0, 0)$ is the only critical point of f.

12.8.13 We have $f_x = 4x^3 - 16y$, $f_y = 4y^3 - 16x$; solving $f_x = f_y = 0$ gives $y = \frac{x^3}{4}$ and $x = \frac{y^3}{4}$; therefore $x = \frac{x^9}{4^4}$ which gives $x = 0, \pm 2$, and the critical points are $(0, 0)$, $(2, 2)$ and $(-2, -2)$.

12.8.14 We have $f_x = x^2 + 3y$, $f_y = -y^2 + 3x$; solving $f_x = f_y = 0$ gives $y = -\frac{x^2}{3}$ and $x = \frac{y^2}{3}$; therefore $x = \frac{x^4}{3^3}$ which gives $x = 0, 3$, and the critical points are $(0, 0)$ and$(3, -3)$.

12.8.15 We have $f_x = 4x^3 - 4x$ and $f_y = 2y - 4$; solving $f_x = f_y = 0$ gives $x = 0, \pm 1$ and $y = 2$; and the critical points of are $(0, 2)$ and $(\pm 1, 2)$.

12.8.16 We have $f_x = 2x + y - 2$ and $f_y = x - 1$; solving $f_x = f_y = 0$ gives $x = 1$, and therefore $y = 0$ so $(1, 0)$ is the only critical point of f.

12.8.17 We have $f_x = 2x + 6$ and $f_y = 2y$; solving $f_x = f_y = 0$ gives $x = -3$ and $y = 0$ so $(-3, 0)$ is the only critical point of f.

12.8.18 We have $f_x = (2xy^2 - 2y^2)e^{x^2y^2 - 2xy^2 + y^2}$ and $f_y = (2x^2y - 4xy + 2y)e^{x^2y^2 - 2xy^2 + y^2}$. Solving $f_x = f_y = 0$ gives that either $x = 1$ and y is arbitrary, or $y = 0$ and x is arbitrary. Thus the points $(1, y)$ and $(x, 0)$ for any x and any y are critical points.

12.8.19 We have $f_x = 4x$, $f_y = 6y$; therefore $(0, 0)$ is the only critical point. We also have $f_{xx} = 4$, $f_{yy} = 6$ and $f_{xy} = 0$; hence $D(0, 0) = 4 \cdot 6 - 0^2 = 24 > 0$ and $f_{xx}(0, 0) > 0$, which by the Second Derivative Test implies that f has a local minimum at $(0, 0)$.

12.8.20 We have $f_x = 8(4x - 1)$, $f_y = 4(2y + 4)$; therefore, $\left(\frac{1}{4}, -2\right)$ is the only critical point. We also have $f_{xx} = 32$, $f_{yy} = 8$ and $f_{xy} = 0$; hence $D\left(\frac{1}{4}, -2\right) = 32 \cdot 8 > 0$ and $f_{xx}\left(\frac{1}{4}, -2\right) > 0$, which by the Second Derivative Test implies that f has a local minimum at $\left(\frac{1}{4}, -2\right)$.

12.8.21 We have $f_x = -8x$, $f_y = 16y$; therefore, $(0, 0)$ is the only critical point. We also have $f_{xx} = -8$, $f_{yy} = 16$ and $f_{xy} = 0$; hence $D(0, 0) = -8 \cdot 16 < 0$, which by the Second Derivative Test implies that f has a saddle point at $(0, 0)$.

12.8.22 We have $f_x = 4x^3 - 4$, $f_y = 4y^3 - 32$; therefore, $(1, 2)$ is the only critical point. We also have $f_{xx} = 12x^2$, $f_{yy} = 12y^2$ and $f_{xy} = 0$; hence $D(1, 2) = 12 \cdot 48 > 0$ and $f_{xx}(1, 2) > 0$, which by the Second Derivative Test implies that f has a local minimum at $(1, 2)$.

12.8.23 We have $f_x = 4x^3 - 4y$, $f_y = 4y - 4x$; therefore, the critical points satisfy $y = x$ and $y = x^3$, which gives $x^3 = x$ and therefore $x = 0, \pm 1$, so the critical points are $(0, 0)$, $(1, 1)$ and $(-1, -1)$. We also have $f_{xx} = 12x^2$, $f_{yy} = 4$ and $f_{xy} = -4$; hence $D(x, y) = 48x^2 - 16$. Thus $D(0, 0) = -16 < 0$, so f has a saddle point at $(0, 0)$; and $D(1, 1) = D(-1, -1) = 32 > 0$, $f_{xx}(1, 1) = f_{xx}(-1, -1) = 12 > 0$ so f has a local minimum at $(1, 1)$ and $(-1, -1)$.

12.8.24 We have $f_x = (1 - x)\, ye^{-x-y}$, $f_y = (1 - y)\, xe^{-x-y}$; therefore, the critical points are $(0, 0)$ and$(1, 1)$. We also have $f_{xx} = (x - 2)\, ye^{-x-y}$, $f_{yy} = (y - 2)\, xe^{-x-y}$ and $f_{xy} = (1 - x)(1 - y)\, e^{-x-y}$; hence $D(0, 0) = -1 < 0$ so f has a saddle point at $(0, 0)$; and $D(1, 1) = e^{-2} > 0$, $f_{xx}(1, 1) = -e^{-2} < 0$ so f has a local maximum at $(1, 1)$.

12.8.25 Note that $f(x, y)$ has the same critical points as the simpler function

$$g(x, y) = x^2 + y^2 - 4x + 5 = (x - 2)^2 + y^2 + 1.$$

We have $g_x = 2(x - 2)$, $g_y = 2y$; therefore, $(2, 0)$ is the only critical point. We also have $g_{xx} = 2$, $g_{yy} = 2$ and $g_{xy} = 0$; hence $D(2, 0) = 4 > 0$ and $g_{xx}(2, 0) > 0$, which by the Second Derivative Test implies that g (and hence f) has a local minimum at $(2, 0)$.

12.8.26 We have $f_x = \frac{y}{1+x^2y^2}$, $f_y = \frac{x}{1+x^2y^2}$; therefore, $(0, 0)$ is the only critical point. We also have $f_{xx} = -\frac{2xy^3}{(1+x^2y^2)^2}$, $f_{yy} = -\frac{2x^3y}{(1+x^2y^2)^2}$ and $f_{xy} = \frac{1-x^2y^2}{(1+x^2y^2)^2}$;hence $D(0, 0) = -1 < 0$, which by the Second Derivative Test implies that f has a saddle point at $(0, 0)$.

12.8.27 We have $f_x = 2\left(1 - 2x^2\right) ye^{-x^2-y^2}$, $f_y = 2\left(1 - 2y^2\right) xe^{-x^2-y^2}$. Therefore, the critical points are $(0, 0)$, $\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Also, $f_{xx} = 4\left(2x^2 - 3\right) xye^{-x^2-y^2}$, $f_{yy} = 4\left(2y^2 - 3\right) xye^{-x^2-y^2}$ and $f_{xy} = 2\left(1 - 2x^2\right)\left(1 - 2y^2\right) e^{-x^2-y^2}$. Hence $D(0, 0) = -4 < 0$, so f has a saddle point at $(0, 0)$ by the Second Derivative Test. We also see that $D(x, y) > 0$ at the four other critical points; also $f_{xx}\left(\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) < 0$ so f has a local maximum at $\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and $f_{xx}\left(\pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) > 0$ so f has a local minimum at $\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

12.8.28 We have $f_x = 2x + y^2 - 2$, $f_y = 2xy$. Solving $f_x = f_y = 0$ gives critical points at $(1, 0)$ and $(0, \pm\sqrt{2})$. We have $f_{xx} = 2$, $f_{yy} = 2x$ and $f_{xy} = 2y$. Hence, $D(1, 0) = 4$, so f has a local minimum at $(1, 0)$. Also, $D(0, \pm\sqrt{2}) = 0 - (2\sqrt{2}) < 0$, so there are saddle points at $(0, \pm\sqrt{2})$.

12.8.29 We have $f_x = \frac{1+y^2-x^2}{(x^2+y^2+1)^2}$ and $f_y = \frac{-2xy}{(x^2+y^2+1)^2}$, so the critical points are $(\pm 1, 0)$. We have $f_{xx} = \frac{2x\left(x^2-3\left(y^2+1\right)\right)}{(x^2+y^2+1)^3}$, so $f_{xx}(\pm 1, 0) = \mp\frac{1}{2}$. We have $f_{yy} = -\frac{2x\left(x^2-3y^2+1\right)}{(x^2+y^2+1)^3}$, so $f_{yy}(\pm 1, 0) = \mp\frac{1}{2}$. Also, $f_{xy} = -\frac{2y\left(-3x^2+y^2+1\right)}{(x^2+y^2+1)^3}$, so $f_{xy}(\pm 1, 0) = 0$. Thus, $D(\pm 1, 0) = \frac{1}{4}$, so there is a local maximum at $(1, 0)$ and a local minimum at $(-1, 0)$.

12.8.30 We have $f_x = \frac{-x^2+2x+y^2}{(x^2+y^2)^2}$ and $f_y = -\frac{2(x-1)y}{(x^2+y^2)^2}$. The only critical point in the domain of f is $(2, 0)$. Note that $f_{xx} = \frac{2\left(x^3-3x^2-3xy^2+y^2\right)}{(x^2+y^2)^3}$, so $f_{xx}(2, 0) = -1/8$. Also, $f_{yy} = -\frac{2(x-1)\left(x^2-3y^2\right)}{(x^2+y^2)^3}$ and $f_{yy}(2, 0) = -1/8$. Also, $f_{xy} = -\frac{2y\left(-3x^2+4x+y^2\right)}{(x^2+y^2)^3}$ and $f_{xy}(2, 0) = 0$. Thus $D(2, 0) > 0$, so there is a local maximum at $(2, 0)$.

12.8.31 We have $f_x = 4x^3 + 8x(y - 2)$ and $f_y = 4x^2 + 16(y - 1)$. Solving $f_x = f_y = 0$ yields critical points $(0, 1)$ and $(\pm 2, 0)$. We have $f_{xx} = 12x^2 + 8(y - 2)$, $f_{yy} = 16$, and $f_{xy} = 8x$. Thus $D(0, 1) < 0$, $D(\pm 2, 0) > 0$. There is a saddle point at $(0, 1)$ and local minimums at $(\pm 2, 0)$.

12.8.32 We have $f_x = e^{(-x-y)}(\sin y) - xe^{(-x-y)}(\sin y) = e^{(-x-y)}(\sin y)(1 - x)$ and $f_y = xe^{(-x-y)}\cos y - xe^{(-x-y)}(\sin y) = xe^{(-x-y)}(\cos y - \sin y)$. We have critical points at $(1, \pi/4)$ and $(0, 0)$. $f_{xx} = (x - 2)e^{(-x-y)}\sin y$, $f_{yy} = -2xe^{(-x-y)}\cos y$ and $f_{xy} = (x - 1)\left(-e^{(-x-y)}\right)(\cos y - \sin y)$. We have $D(0, 0) = -1 < 0$ so there is a saddle point at $(0, 0)$. $D(1, \pi/4) > 0$ and $f_{xx}(1, \pi/4) < 0$, so there is a local maximum at $(1, \pi/4)$.

12.8.33 We have $f_x = ye^x$, $f_y = e^x - e^y$; therefore, the critical points must satisfy $y = 0$ and $y = x$, so $(0, 0)$ is the only critical point. We also have $f_{xx} = ye^x$, $f_{yy} = -e^y$ and $f_{xy} = e^x$; hence $D(0, 0) = -1 < 0$ so f has a saddle point at $(0, 0)$.

12.8.34 We have $f_x = 2\pi\cos(2\pi x)\ \cos(\pi y)$, $f_y = -\pi\sin(2\pi x)\ \sin(\pi y)$, so the critical points must satisfy $\cos(2\pi x) = \sin(\pi y) = 0$, or $\sin(2\pi x) = \cos(\pi y) = 0$; hence, the only critical points in the interior of the domain of f are $\pm\left(\frac{1}{4}, 0\right)$. We also have $f_{xx} = -4\pi^2\sin(2\pi x)\ \cos(\pi y)$, $f_{yy} = -\pi^2\sin(2\pi x)\ \cos(\pi y)$, $f_{xy} = -2\pi^2\cos(2\pi x)\ \sin(\pi y)$; hence $D\left(\pm\left(\frac{1}{4}, 0\right)\right) > 0$. We also have $f_{xx}\left(\frac{1}{4}, 0\right) < 0$ and $f_{xx}\left(-\frac{1}{4}, 0\right) > 0$, so f has a local maximum at $\left(\frac{1}{4}, 0\right)$ and a local minimum at $\left(-\frac{1}{4}, 0\right)$.

12.8.35 Let x, $y \geq 0$ be the dimensions of the base of the box; then for any given x, y the maximum allowable height is given by $h = 96 - 2x - 2y$, and we must have $h \geq 0$ which implies $x + y \leq 48$. The volume of the box is given by $V = 2xy(48 - x - y)$, which we must maximize over the domain R given by x, $y \geq 0$ and $x + y \leq 48$. The critical points of V satisfy $V_x = 2(48 - 2x - y)y = 0$, $V_y = 2(48 - x - 2y)x = 0$; hence the critical points in the interior of R satisfy $2x + y = 2y + x = 48$, which gives $x = y = 16$. Furthermore $V(x, y) = 0$ on the boundary of R, so the maximum volume must occur at the point $(16, 16)$. Therefore the box with largest volume has height 32 in and base 16 in×16 in with a volume of 8192 in^3.

12.8.36 Let x, $y \geq 0$ be the dimensions of the base and $h \geq 0$ be the height of the box; then the four sides of the box plus the base have total area $2xh + 2yh + xy = 2$, so $h = \frac{2-xy}{2x+2y}$. The volume of the box is given by $V = xyh = \frac{1}{2}\frac{(2-xy)xy}{x+y}$, which we must maximize over the domain R given by x, $y \geq 0$ and $xy \leq 2$. The critical points of V satisfy $V_x = \frac{1}{2}\frac{\left(2-x^2-2xy\right)y^2}{(x+y)^2} = 0$, $V_y = \frac{1}{2}\frac{\left(2-y^2-2xy\right)x^2}{(x+y)^2} = 0$; hence the critical points in R satisfy $x = y$ and $3x^2 = 2$, which gives $x = y = \frac{\sqrt{6}}{3}$, and the corresponding height is $\frac{\sqrt{6}}{6}$. Furthermore $V(x, y) \to 0$ as (x, y) approaches the boundary of R, so the maximum volume must occur at the point$\left(\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3}\right)$. Therefore, the box with largest volume has height $\frac{\sqrt{6}}{6}$ m and base $\frac{\sqrt{6}}{3} \times \frac{\sqrt{6}}{3}$ m.

12.8.37 Let x, $y \geq 0$ be the dimensions of the base and $h \geq 0$ be the height of the box; then the volume of the box is $xyh = 4$, so $h = \frac{4}{xy}$. The four sides of the box plus the base have total area $A = 2xh + 2yh + xy = \frac{8}{x} + \frac{8}{y} + xy$, which we must maximize over the domain R given by x, $y \geq 0$. The critical points of A satisfy $A_x = y - \frac{8}{x^2} = 0, A_y = x - \frac{8}{y^2} = 0$; hence the critical points in R satisfy $x^2y = xy^2 = 8$, which gives $x = y$ and hence $x, y = 2$; the corresponding height is 1. Furthermore $V(x, y) \to \infty$ as (x, y) approaches the boundary of R or as either $x, y \to \infty$, so the minimum area must occur at the point $(2, 2)$. Therefore the box with smallest area has dimensions 2m × 2m × 1m.

12.8.38 Let x, $y \geq 0$ be the dimensions of the base and $z \geq 0$ be the height of the box; then $z = 2 - \frac{x}{3} - \frac{2y}{3}$. The box has volume $V = xyz = xy\left(2 - \frac{x}{3} - \frac{2y}{3}\right)$, which we must maximize over the domain R given by x, $y \geq 0$ and $x + 2y \leq 6$. The critical points of V satisfy $V_x = 2y - \frac{2xy}{3} - \frac{2y^2}{3} = 0$, $V_y = 2x - \frac{x^2}{3} - \frac{4xy}{3} = 0$; hence the critical points in the interior of R satisfy $x + y = 3$ and $x + 4y = 6$, which gives $x = 2$ and $y = 1$; the corresponding height is $z = \frac{2}{3}$. Furthermore $V(x, y) = 0$ on the boundary of R, so the maximum area must occur at the point $(2, 1)$. Therefore the box with largest volume has dimensions $x = 2$, $y = 1$, $z = \frac{2}{3}$.

12.8.39 Observe that $f_x = 4x^3$, $f_y = 12y^3$ and $f_{xx} = 12x^2$, $f_{yy} = 36y^2$ and $f_{xy} = 0$; therefore $D(0, 0) = 0$ and the Second Derivative Test is inconclusive. Observe that both x^4 and $3y^4$ have absolute minima at 0; therefore, f has an absolute minimum at $(0, 0)$.

12.8.40 Observe that $f_x = 2xy$, $f_y = x^2$ and $f_{xx} = 2y$, $f_{yy} = 0$ and $f_{xy} = 2x$; therefore $D(0, 0) = 0$ and the Second Derivative Test is inconclusive. Observe that x^2y takes on both negative and positive values in any disc centered at $(0, 0)$; therefore, f has a saddle at $(0, 0)$.

12.8.41 Observe that $f_x = 4x^3y^2$, $f_y = 2x^4y$ and $f_{xx} = 12x^2y^2$, $f_{yy} = 2x^4$ and $f_{xy} = 8x^3y$. Note that every point on the x-axis and on the y-axis is a critical point, and that $D = 24x^6y^2 - 64x^6y^2 = -40x^6y^2$, which has value 0 at every point on the axes, so the Second Derivative Test is inconclusive. Note that the function is always nonnegative, so the value of 0 along the coordinate axes represents the minimum value.

12.8.42 Observe that $f_x = 2xy^2\cos\left(x^2y^2\right)$, $f_y = 2x^2y\cos\left(x^2y^2\right)$, $f_{xx} = 2y^2\cos\left(x^2y^2\right) - 4x^2y^4\sin\left(x^2y^2\right)$, $f_{yy} = 2x^2\cos\left(x^2y^2\right) - 4x^4y^2\sin\left(x^2y^2\right)$ and $f_{xy} = 4xy\cos\left(x^2y^2\right) - 4x^3y^3\sin\left(x^2y^2\right)$. Therefore, $D(0, 0) = 0$ and the Second Derivative Test is inconclusive. Observe that both x^2,$y^2 \geq 0$ so $\sin\left(x^2y^2\right) \geq 0$ in sufficiently small discs centered at $(0, 0)$; therefore, f has an absolute minimum at $(0, 0)$ (and in fact along both coordinate axes).

12.8.43 First find the values of f at all critical points in the interior of $R = \{x^2 + y^2 \leq 4\}$; we have $f_x = 2x$, $f_y = 2y - 2$, so $(0, 1)$ is the only critical point in R, and $f(0, 1) = 0$. Next, find the minimum and maximum values of f on the boundary of R, which we can parameterize by $x = 2\cos\theta$, $y = 2\sin\theta$ for $0 \leq \theta \leq 2\pi$. Then $f(2\cos\theta, 2\sin\theta) = 5 - 4\sin\theta$, which has maximum value 9 at $\theta = \frac{3\pi}{2}$ and minimum value 1 at $\theta = \frac{\pi}{2}$. Therefore, the maximum value of f on R is $f(0, -2) = 9$ and the minimum value is $f(0, 1) = 0$.

12.8.44 First, find the values of f at all critical points in the interior of R. We have $f_x = 4x$ and $f_y = 2y$, so the only critical point in the interior of R is $(0, 0)$ and the value of f there is 0 (which is clearly the absolute minimum for f). We parameterize the boundary by letting $x = 4\cos t$ and $y = 4\sin t$ for $0 \leq t \leq 2\pi$. On the boundary we have $f(x, y) = f(t) = 32\cos^2 t + 16\sin^2 t = 16\cos^2 t + 16$. Note that $f(t)$ has a maximum value of 32 at $t = 0$ and $t = \pi$, which corresponds to the points $(\pm 4, 0)$ in R. Thus, the absolute minimum of f is 0 at $(0, 0)$ and the absolute maximum is 32 at $(\pm 4, 0)$.

12.8.45 First find the values of f at all critical points in the interior of R; we have $f_x = 4x$, $f_y = 2y$, so $(0, 0)$ is the only critical point in R, and $f(0, 0) = 4$. Next, find the minimum and maximum values of f on the boundary of R, which is a square. On the sides $y = \pm 1$, $-1 \leq x \leq 1$ we have $f(x, \pm 1) = 4 + 2x^2 + 1 = 2x^2 + 5$, which has extreme values 5 and 7 on $[-1, 1]$. On the sides $x = \pm 1$, $-1 \leq y \leq 1$ we have$f(\pm 1, y) = 4 + 2 + y^2 = y^2 + 6$,which has extreme values 6 and 7 on $[-1, 1]$. Therefore, the maximum value of f on R is 7 and the minimum value is 4.

12.8.46 First find the values of f at all critical points in the interior of R; we have $f_x = -2x$, $f_y = -8y$, so $(0,0)$ is the only critical point in R, and $f(0,0) = 6$. Next, find the minimum and maximum values of f on the boundary of R, which is a rectangle. On the sides $y = \pm 1$, $-2 \le x \le 2$ we have $f(x, \pm 1) = 6 - x^2 - 4 = 2 - x^2$, which has extreme values ± 2 on $[-2, 2]$. On the sides $x = \pm 2$, $-1 \le y \le 1$ we have $f(\pm 2, y) = 6 - 4 - 4y^2 = 2 - 4y^2$, which has extreme values ± 2 on $[-1, 1]$. Therefore, the maximum value of f on R is 6 and the minimum value is -2.

12.8.47 First find the values of f at all critical points in the interior of R; we have $f_x = 4x - 4$ and $f_y = 6y$, so the only critical point is $(1, 0)$. Note that $f(1, 0) = 0$. Parameterize the boundary of R by letting $x = \cos t + 1$ and $y = \sin t$ for $0 \le t \le 2\pi$. Note that $f(x, y) = (2x^2 - 4x + 2) + 3y^2 = 2(x-1)^2 + 3y^2$, so $f(t) = 2\cos^2 t + 3\sin^2 t = 2 + \sin^2 t$, and f has a maximum of 3 on the boundary of R at $(1, \pm 1)$ and an absolute minimum of 2 on the boundary of R. Thus, the absolute maximum of f on all of R is 3 and the absolute minimum is 0.

12.8.48 First find the values of f at all critical points in the interior of R; we have $f_x = 2x - 2$, $f_y = 2y - 2$, so $(1,1)$ is the only critical point for f, and hence there are no critical points in the interior of R (the point $(1,1)$ is on the boundary of R). Next, find the minimum and maximum values of f on the boundary of R, which is a triangle. On the side $y = 0$, $0 \le x \le 2$ we have $f(x, 0) = x^2 - 2x = (x-1)^2 - 1$ which has extreme values -1, 0 on $[0, 2]$. Similarly on the side $x = 0$, $0 \le y \le 2$ we have $f(0, y) = y^2 - 2y = (y-1)^2 - 1$, which is the same function as above. The third side is given by $y = 2 - x$ where $0 \le x \le 2$; we have $f(x, 2-x) = x^2 + (2-x)^2 - 2x - 2(2-x) = 2(x-1)^2 - 2$, which has extreme values -2, 0 on $[0, 2]$. Therefore, the maximum value of f on R is 0 and the minimum value is -2.

12.8.49 First find the values of f at all critical points in the interior of R; we have $f_x = -4x + 4$ and $f_y = -6y - 6$, so the only critical point in the interior of R is $(1, -1)$. Note that $f(1, -1) = 4$. We parameterize the boundary of R by letting $x = 1 + \cos t$ and $y = -1 + \sin t$ for $0 \le t \le 2\pi$. Note that $f(x, y) = -2(x^2 - 2x + 1) + -3(y^2 + 2y + 1) - 1 + 2 + 3 = -2(x-1)^2 + -3(y+1)^2 + 4$. Thus $f(t) = -2\cos^2 t - 3\sin^2 t + 3\cos^2 t + 3\sin^2 t + 1 = \cos^2 t + 1$. This has a maximum of 2 at $t = 0$ and $t = \pi$ and a minimum of 1 at $t = \pi/2$ and $t = 3\pi/2$. The original function therefore has an absolute maximum of 4 at $(1, -1)$ and an absolute minimum of 1 at $(1, 0)$ and at $(1, -2)$.

12.8.50 Observe that it is easier to find the extreme values of the function $g(x, y) = x^2 + y^2 - 2x + 2$; then the extreme values of f on R will be the square roots of the extreme values of g on R. We first find the values of g at all critical points in the interior of R; we have $g_x = 2x - 2$, $g_y = 2y$, so $(1, 0)$ is the only critical point for g, and hence there are no critical points in the interior of R (the point $(1, 0)$ is on the boundary of R). Next, find the minimum and maximum values of g on the boundary of R. On the side $y = 0$, $-2 \le x \le 2$ we have $g(x, 0) = x^2 - 2x + 2 = (x-1)^2 + 1$, which has extreme values 1, 10 on $[-2, 2]$. We can parameterize the semicircular part of the boundary by $x = 2\cos\theta$, $y = 2\sin\theta$ for $0 \le \theta \le \pi$. Then $g(2\cos\theta, 2\sin\theta) = 4 - 4\cos\theta + 2 = 6 - 4\cos\theta$, which has extreme values 2, 10 on $[0, \pi]$. Therefore, the maximum value of g on R is 10 and the minimum value is 1 hence, the maximum and minimum values of f are $\sqrt{10}$ and 1.

12.8.51 $f_x = -\frac{x(2y^4+1)}{(x^2y^2+1)^2}$, which is 0 for $x = 0$, and $f_y = \frac{(x^4+2)y}{(x^2y^2+1)^2}$, which is 0 for $y = 0$. On the boundary $y = 2$ we have $f(x, y) = \frac{8-x^2}{2+8x^2}$ which is maximized on $[1, 2]$ at $x = 1$ (with value $f(1, 2) = \frac{7}{9}$) and minimized at $x = 2$ (with value $f(2, 2) = \frac{1}{3}$). On the boundary $y = x$ we have $f(x, y) = \frac{x^2}{2+2x^2}$ which has minimum value 0 at $(0, 0)$ and maximum value $\frac{1}{4}$ at $(1, 1)$. On the boundary $y = 2x$ we have $f(x, y) = \frac{7x^2}{2+8x^4}$ which has derivative $\frac{7x-28x^5}{(4x^4+1)^2}$. This is zero for $x = 0$ and for $x = \frac{1}{\sqrt{2}}$. These points lead to a minimum of 0 at $(0, 0)$ and a maximum of $\frac{7}{8}$ at $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$. Therefore the global minimum of f on the given set is $0 = f(0, 0)$ and the global maximum is $\frac{7}{8} = f\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$.

12.8.52 $f_x = \frac{x}{\sqrt{x^2+y^2}}$ and $f_y = \frac{y}{\sqrt{x^2+y^2}}$, so the critical point in the interior of the ellipse is $(0, 0)$ which yields the value $f(0, 0) = 0$ which is clearly the minimum for the function. We can parameterize the boundary with $x = 2\cos\theta$, $y = \sin\theta$ for $0 \le \theta \le 2\pi$. Then $f(x,y) = \sqrt{x^2+y^2} = \sqrt{4\cos^2\theta + \sin^2\theta} = \sqrt{3\cos^2\theta+1}$. This has derivative $\frac{-3\cos\theta\sin\theta}{\sqrt{3\cos^2\theta+1}}$ which is 0 for $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$. Checking these points, the maximum for f is 2 which occurs for $\theta = 0$ and $\theta = \pi$, which correspond with the points $(-2,0)$ and $(2,0)$ on the ellipse.

12.8.53 Observe that $f(0,0) = -4$ and $f(x,y) \ge -4$ for all points $(x,y) \in R$; hence the absolute minimum value of f on R is -4. The function f on R takes on all values in the interval $[-4,0)$; therefore f has no absolute maximum on R.

12.8.54 Because $-1 < x < 1$ and $-2 < y < 2$ on R, the function $f(x,y) = x + 3y$ takes on all values between $-1 + 3(-2) = -7$ and $1 + 3 \cdot 2 = 7$; in other words, the range of f on R is the interval $(-7,7)$, and we see that f has neither an absolute minimum or maximum on R.

12.8.55 Observe that $f(0,0) = 2$ and $f(x,y) \le 2$ for all points $(x,y) \in R$; hence, the absolute maximum value of f on R is 2. The function f on R takes on all values in the interval $(0,2]$; therefore, f has no absolute minimum on R.

12.8.56 Because $-1 < x, y < 1$ on R, the function $f(x,y) = x^2 - y^2$ takes on all values between -1 and 1; in other words, the range of f on R is the interval $(-1,1)$, and we see that f has neither an absolute minimum or maximum on R.

12.8.57 The equation of the plane can be written as $z = 4 - x - y$; so it suffices to minimize the square of the distance from $(x, y, 4-x-y)$ to $(0,3,6)$, which is given by $w = x^2 + (y-3)^2 + (x+y+2)^2$. We have $w_x = 2(2x+y+2)$, $w_y = 2(x+2y-1)$, so the critical points of w satisfy $2x + y = -2$, $x + 2y = 1$ which gives $(x,y,z) = \left(-\frac{5}{3}, \frac{4}{3}, \frac{13}{3}\right)$. Because we know that there is some point on the plane closest to $(0,3,6)$, this critical point must be that point.

12.8.58 It suffices to minimize the square of the distance from points (x,y,z) on the cone to $(1,4,0)$, which is given by $w = (x-1)^2 + (y-4)^2 + z^2 = (x-1)^2 + (y-4)^2 + x^2 + y^2$. We have $w_x = 2(2x-1)$, $w_y = 2(2y-4)$, so $\left(\frac{1}{2}, 2\right)$ is the only critical point of w; the corresponding points on the cone are $\left(\frac{1}{2}, 2, \pm\frac{\sqrt{17}}{2}\right)$. Because there is some point on the cone closest to $(1,4,0)$, this critical point provides two points.

12.8.59 Consider the square of the distance between the points (x, x^2) and $(a, a-1)$ which is $f(a,x) = (x-a)^2 + (x^2-a+1)^2$. Note that $f_a = -2(x-a) - 2(x^2-a+1)$, and $f_x = 2(x-a) + 4x(x^2-a+1)$. The critical points satisfy $x - a = -(x^2-a+1)$ and $x - a = -2x(x^2-a+1)$. This has a real solution only when $x = 1/2$ and $a = 7/8$, and this gives rise to the closest point on the parabola being $(1/2, 1/4)$ and the point on the line it is closest to being $(7/8, -1/8)$.

12.8.60 Let $x, y > 0$ be the dimensions of the base and $z > 0$ be the height of the box; then $xyz = 10$ so $z = \frac{10}{xy}$. The cost to produce the box is $C = 2 \cdot 10 \cdot xy + 2 \cdot 1 \cdot xz + 2 \cdot 1 \cdot yz = 20xy + \frac{20}{x} + \frac{20}{y}$, which we must minimize over the domain R given by $x, y > 0$. The critical points of C satisfy $C_x = 20y - \frac{20}{x^2} = 0$, $C_y = 20x - \frac{20}{y^2} = 0$; hence the critical points satisfy $x^2y = xy^2 = 1$, which gives $x = y = 1$; the corresponding height is $z = 10$. Furthermore $C(x,y) \to 0$ as (x,y) approaches the boundary of R or as either x or y approach ∞, so the minimum cost must occur at the critical point. Therefore, the box with least cost has dimensions $1\text{m} \times 1\text{m} \times 10\text{m}$.

12.8.61

a. True. This is because our definition of saddle point is a critical point which is neither a local maximum or local minimum.

b. False. A necessary condition for a local maximum at (a,b) is that both $f_x = f_y = 0$ at (a,b), assuming both partials exist.

c. True. This is because f may take on its absolute maximum or minimum at a point on the boundary of its domain.

d. True. The equation of the tangent plane at a critical point (a,b) is $z = f(a,b)$.

12.8.62 This function has a local maximum at $\left(\frac{1}{2}, -1\right)$ and saddle points at $(0,0)$, $(1,0)$, $(-2,0)$ and $(1,-2)$.

12.8.63 This function has a local minimum near $(0.3, -0.3)$ and a saddle point at $(0,0)$.

12.8.64 Let (x,y,z) be the point on the ellipsoid; then the box has volume $V = xyz$. We have the relation $36x^2 + 4y^2 + 9z^2 = 36$ which implies that $z^2 = 4\left(1 - x^2 - \frac{y^2}{9}\right)$, and it suffices to maximize the square of the volume, which is given by $w = 4x^2y^2\left(1 - x^2 - \frac{y^2}{9}\right)$ over the region R given by x, $y \geq 0$ and $x^2 - \frac{y^2}{9} \leq 1$. The critical points in the interior of R satisfy $w_x = 8xy^2\left(1 - 2x^2 - \frac{y^2}{9}\right) = 0$, $w_y = 8x^2y\left(1 - x^2 - \frac{2y^2}{9}\right) = 0$, which gives $x^2 = \frac{1}{3}$, $y^2 = 3$ so $x = \frac{\sqrt{3}}{3}$, $y = \sqrt{3}$; The corresponding value of $z = \frac{2\sqrt{3}}{3}$. Notice that $w = 0$ on the boundary of the closed and bounded region R; therefore the critical point we found must give the absolute maximum value of w, so the box with maximum volume has dimensions $\frac{\sqrt{3}}{3} \times \sqrt{3} \times \frac{2\sqrt{3}}{3}$.

12.8.65 The plane has equation $z = 2 - x + y$; the distance from a point $(x, y, 2-x+y)$ on the plane to the point $(1,1,1)$ is given by $d^2 = (x-1)^2 + (y-1)^2 + (x-y-1)^2$. It suffices to minimize the function $f(x,y) = (x-1)^2 + (y-1)^2 + (x-y-1)^2$ on R^2. We have $f_x = 2(x-1+x-y-1) = 2(2x-y-2)$, $f_y = 2(y-1+y-x+1) = 2(-x+2y)$, so the critical point of f satisfies $2x - y = 2$, $x - 2y = 0$ which gives $x = \frac{4}{3}$, $y = \frac{2}{3}$. The corresponding point on the plane is $\left(\frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right)$. Because there is a point on the plane closest to the point $(1,1,1)$, this must be the point we found.

12.8.66 The level curves of a linear function $f(x,y)$ are a family of parallel lines; therefore at any interior point of a region R, one can move to a nearby level curve in such a way as to either increase or decrease the value of f. This shows that the absolute maximum or minimum of f on R cannot occur at an interior point, and hence must occur along the boundary of R.

12.8.67

a. Using the relation $z = 200 - x - y$, we see that it suffices to minimize the function $f(x,y) = x^2 + y^2 + (200-x-y)^2$ over the closed bounded region R given by $x, y \geq 0$ and $x + y \leq 200$. We have $f_x = 2(x + x + y - 200) = 2(2x + y - 200)$, $f_y = 2(y + x + y - 200) = 2(x + 2y - 200)$; therefore, the critical point of f satisfies $2x + y = 200$, $x + 2y = 200$ which gives $x = y = \frac{200}{3}$ (the corresponding value of $z = \frac{200}{3}$ as well), and we have $f\left(\frac{200}{3}, \frac{200}{3}\right) = \frac{40{,}000}{3}$. We must also find the extreme values of f on the boundary of R. Along the segment $y = 0$, $0 \leq x \leq 200$ we have $f(x,0) = x^2 + (200-x)^2$ which has range $[20{,}000, 40{,}000]$, and we get the same result along the other two segments that make up the boundary of R. Therefore, the minimum value of $x^2 + y^2 + z^2$ is given by $x = y = z = \frac{200}{3}$.

b. The function $\sqrt{x^2 + y^2 + z^2}$ takes its minimum at the same point that minimizes $x^2 + y^2 + z^2$, which we saw in part (a) is $x = y = z = \frac{200}{3}$.

c. Using the relation $z = 200 - x - y$, we see that it suffices to minimize the function $f(x,y) = xy(200-x-y) = 200xy - x^2y - xy^2$ over the closed bounded region R given by $x, y \geq 0$ and $x + y \leq 200$. We have $f_x = y(200 - 2x - y)$, $f_y = x(200 - x - 2y)$; therefore, the critical points of f in the interior of R satisfy $2x + y = 200$, $x + 2y = 200$ which gives $x = y = \frac{200}{3}$ (the corresponding value of $z = \frac{200}{3}$ as well), and we have $f\left(\frac{200}{3}, \frac{200}{3}\right) = \left(\frac{200}{3}\right)^2$. We also observe that $f(x,y) = 0$ on the boundary of R; therefore, the maximum value of xyz is given by $x = y = z = \frac{200}{3}$.

d. The function $x^2y^2z^2$ takes its maximum at the same point that maximizes xyz, which we saw in part (c) is $x = y = z = \frac{200}{3}$.

12.8.68

a. Using the relation $z = 1 - x - y$, we see that it suffices to minimize the function $f(x,y) = \left(1+x^2\right)\left(1+y^2\right)\left(1+z^2\right) = \left(1+x^2\right)\left(1+y^2\right)\left(1+(1-x-y)^2\right)$ over the closed bounded region R given by $x, y \geq 0$ and $x + y \leq 1$. Using the chain rule, we have $f_x = 2\left(1+y^2\right)\left(x\left(1+z^2\right) - z\left(1+x^2\right)\right)$, $f_y = 2\left(1+x^2\right)\left(y\left(1+z^2\right) - z\left(1+y^2\right)\right)$. The critical points of f in the interior of R satisfy $f_x = f_y = 0$;

the first equation gives $x\left(1+z^2\right)-z\left(1+x^2\right)=(x-z)(1-xz)=0$, and we observe that $xz=1$ is impossible, so we conclude that $x=z$. Similarly $y=z$ so the critical point is $\left(\frac{1}{3},\frac{1}{3}\right)$, and we have $f\left(\frac{1}{3},\frac{1}{3}\right)=\left(\frac{10}{9}\right)^3$. On the boundary segment given by $y=0$, $0\le x\le 1$ we have $f(x,0)=\left(1+x^2\right)\left(1+(1-x)^2\right)$ and this function has a maximum value $f(0)=f(1)=2$, and minimum value $f\left(\frac{1}{2}\right)=\left(\frac{5}{4}\right)^2$. The range of f on the other two boundary components is the same. Therefore the maximum value of f is 2, and the minimum is $\left(\frac{10}{9}\right)^3$.

b. Using the relation $z=1-x-y$, we see that it suffices to minimize the function $f(x,y)=(1+\sqrt{x})\left(1+\sqrt{y}\right)(1+\sqrt{z})=(1+\sqrt{x})\left(1+\sqrt{y}\right)\left(1+\sqrt{1-x-y}\right)$ over the closed bounded region R given by $x,y\ge 0$ and $x+y\le 1$. Using the chain rule, we have

$$f_x=(1+\sqrt{y})\left(\left(1+\frac{1}{2\sqrt{x}}\right)(1+\sqrt{z})-(1+\sqrt{x})\left(1+\frac{1}{2\sqrt{z}}\right)\right)$$

$$=\left(\frac{1+\sqrt{y}}{4\sqrt{x}\sqrt{z}}\right)\left(\left(2\sqrt{x}+1\right)\left(\sqrt{z}+1\right)-\left(2\sqrt{z}+1\right)\left(\sqrt{x}+1\right)\right)=\left(\frac{1+\sqrt{y}}{4\sqrt{x}\sqrt{z}}\right)\left(\sqrt{x}-\sqrt{z}\right),$$

and similarly $f_y=\left(\frac{1+\sqrt{x}}{4\sqrt{y}\sqrt{z}}\right)\left(\sqrt{y}-\sqrt{z}\right)$. The critical points of f in the interior of R satisfy $f_x=f_y=0$ which gives $x=y=\frac{1}{3}$, so the critical point is $\left(\frac{1}{3},\frac{1}{3}\right)$, and we have $f\left(\frac{1}{3},\frac{1}{3}\right)\approx 3.925$. On the boundary segment given by $y=0$, $0\le x\le 1$ we have $f(x,0)=(1+\sqrt{x})\left(1+\sqrt{1-x}\right)$, and this function has a maximum value $f(0)=f(1)=2$, and minimum value $f\left(\frac{1}{2}\right)\approx 2.914$. The range of f on the other two boundary components is the same. Therefore the minimum value of f is 2, and the maximum is 3.925.

12.8.69

a. The function to be minimized is $f(x,y)=x^2+y^2+(x-2)^2+y^2+(x-1)^2+(y-1)^2=3x^2-6x+3y^2-2y$; we have $f_x=6(x-1)$, $f_y=2(3y-1)$ so the optimal location is the unique critical point $\left(1,\frac{1}{3}\right)$.

b. The function to be minimized is now $f(x,y)=(x-x_1)^2+(y-y_1)^2+(x-x_2)^2+(y-y_2)^2+(x-x_3)^2+(y-y_3)^2=3\left(x^2-2\overline{x}x+y^2-2\overline{y}y\right)+\text{const}$; where $\overline{x}=\frac{x_1+x_2+x_3}{3}$, $\overline{y}=\frac{y_1+y_2+y_3}{3}$. we have $f_x=6(x-\overline{x})$, $f_y=6(y-\overline{y})$ so the optimal location is the unique critical point $(\overline{x},\overline{y})$.

c. The function to be minimized is now

$$f(x,y)=\sum_{i=1}^{n}(x-x_i)^2+\sum_{i=1}^{n}(y-y_i)^2=n\left(x^2-2\overline{x}x+y^2-2\overline{y}y\right)+\text{const}$$

where $\overline{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$, $\overline{y}=\frac{1}{n}\sum_{i=1}^{n}y_i$. We have $f_x=n(x-\overline{x})$, $f_y=n(y-\overline{y})$ so the optimal location is the unique critical point $(\overline{x},\overline{y})$.

d. The actual sum of the distances is given by the function $f(x,y)=\sqrt{x^2+y^2}+\sqrt{(x-2)^2+y^2}+\sqrt{(x-1)^2+(y-1)^2}$ We can minimize this function as follows. First, fix $y=y_0$ and consider the function $g(x)=f(x,y_0)$. Each of the three terms in this function has positive second derivative, and approaches ∞ as $x\to\pm\infty$, so the same is true for their sum; therefore $g(x)$ must have a unique absolute minimum. Observe in addition that g is symmetric in the line $x=1$, which implies that the absolute minimum must occur at $x=1$. So to minimize $f(x,y)$, we can set $x=1$ and reduce to minimizing the function $h(y)=f(1,y)=2\sqrt{1+y^2}+|y-1|$, which has absolute minimum at $y=\frac{1}{\sqrt{3}}$. Therefore, the optimal location is $\left(1,\frac{1}{\sqrt{3}}\right)$, which is different from the point found in part (a).

12.8.70 The sum of the squares of the vertical distances is $E(m,b)=[(m+b)-2]^2+[(3m+b)-5]^2+[(4m+b)-6]^2=26m^2+16mb+3b^2-82m-26b+65$. We have $E_m=52m+16b-82$, $E_b=16m+6b-26$, and solving the simultaneous equations $52m+16b=82$, $16m+6b=26$ gives $m=\frac{19}{14}$, $b=\frac{10}{14}$.

12.8.71 Given the n data points $(x_1, y_1), \ldots, (x_n, y_n)$, we seek to minimize the function $E(m,b) = \sum_{k=1}^{n}(mx_k + b - y_k)^2 = \left(\sum x_k^2\right)m^2 + 2\left(\sum x_k\right)mb + nb^2 - 2\left(\sum x_k y_k\right)m - 2\left(\sum y_k\right)b + \sum y_k^2$. We have $E_m = 2\left(\sum x_k^2\right)m + 2\left(\sum x_k\right)b - 2\left(\sum x_k y_k\right)$, $E_b = 2\left(\sum x_k\right)m + 2nb - 2\left(\sum y_k\right)$ and solving $E_m = E_b = 0$ gives $m = \frac{\left(\sum x_k\right)\left(\sum y_k\right) - n\sum x_k y_k}{\left(\sum x_k\right)^2 - n\sum x_k^2}$, $b = \frac{1}{n}\left(\sum y_k - m\sum x_k\right)$.

12.8.72 Using the result in Exercise 69, we obtain $m = \frac{6\cdot 8 - 3\cdot 26}{6^2 - 3\cdot 20} = \frac{5}{4}$, $b = \frac{1}{3}\left(8 - \frac{5}{4}\cdot 6\right) = \frac{1}{6}$ so the line has equation $y = \frac{5}{4}x + \frac{1}{6}$.

12.8.73 Using the result in Exercise 69, we obtain $m = \frac{2\cdot 14 - 3\cdot 24}{2^2 - 3\cdot 10} = \frac{22}{13}$, $b = \frac{1}{3}\left(14 - \frac{22}{13}\cdot 2\right) = \frac{46}{13}$ so the line has equation $y = \frac{22}{13}x + \frac{46}{13}$.

12.8.74 Observe that $D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2 < 0$ because $f_{xx}(a,b)\, f_{yy}(a,b) < 0$ in both cases. Therefore f has a saddle at (a,b).

12.8.75 Let $s = \frac{a+b+c}{2}$ be the semi-perimeter, which we assume is constant. Then we may express $c = 2s - a - b$ and therefore $A^2 = s(s-a)(s-b)(s-(2s-a-b)) = s(s-a)(s-b)(a+b-s)$, so it suffices to maximize the simpler function $f(a,b) = s(s-a)(s-b)(a+b-s)$ over the closed bounded region given by $0 \le a, b \le s$ and $a + b \ge s$. We have $f_a = s(s-b)(-(a+b-s) + s - a) = s(s-b)(2s - 2a - b)$ and similarly $f_b = s(s-a)(2s - a - 2b)$, so the critical points of f in the interior of R satisfy $2a + b = 2s$, $a + 2b = 2s$ which gives $a = b = \frac{2}{3}s$, and therefore $c = \frac{2}{3}s$ as well, so we get an equilateral triangle. We also observe that $f(a,b) = 0$ on the boundary of R, so this critical point must give the absolute maximum of f. We conclude that of all triangles with a given perimeter, the maximum area is obtained when all three sides are equal (in the special case of perimeter 9 units, each side length is 3 units).

12.8.76 Let (x_0, y_0, z_0) be a point on the ellipsoid; then the tangent plane P at this point has equation $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$, and this plane meets the coordinate axes in the points $\left(\frac{a^2}{x_0}, 0, 0\right)$, $\left(0, \frac{b^2}{y_0}, 0\right)$, $\left(0, 0, \frac{c^2}{z_0}\right)$. Therefore, the tetrahedron T has base area $A = \frac{a^2 b^2}{2x_0 y_0}$ and height $h = \frac{c^2}{z_0}$, so its volume is $V = \frac{a^2 b^2 c^2}{6x_0 y_0 z_0}$. Minimizing this function is equivalent to maximizing its reciprocal, or equivalently the quantity $Q = \frac{x_0^2}{a^2}\cdot\frac{y_0^2}{b^2}\cdot\frac{z_0^2}{c^2}$. Now if we let $u = \frac{x_0^2}{a^2}$, $v = \frac{y_0^2}{b^2}$, $w = \frac{z_0^2}{c^2}$, then an equivalent (but simpler) problem is to maximize $Q = uvw$ where u, v,$w \ge 0$ and $u + v + w = 1$. The maximum occurs when $u = v = w = \frac{1}{3}$ (see problem 65 c), so the tetrahedron of minimal volume is given by $\frac{x_0^2}{a^2} = \frac{y_0^2}{b^2} = \frac{z_0^2}{c^2} = \frac{1}{3}$ which gives $V = \frac{abc}{6}\left(\frac{a}{x_0}\right)^3 = \frac{abc\sqrt{3}}{2}$.

12.8.77

a. We have $d_1(x,y) = \sqrt{(x-x_1)^2 + (y-y_1)^2}$, so $\nabla d_1(x,y) = \frac{x-x_1}{d_1(x,y)}\mathrm{i} + \frac{y-y_1}{d_1(x,y)}\mathrm{j}$; observe that this is a unit vector in the direction of the vector joining (x_1, y_1) to (x, y).

b. Similarly, we have $\nabla d_2(x,y) = \frac{x-x_2}{d_2(x,y)}\mathrm{i} + \frac{y-y_2}{d_2(x,y)}\mathrm{j}$ and $\nabla d_3(x,y) = \frac{x-x_3}{d_3(x,y)}\mathrm{i} + \frac{y-y_3}{d_3(x,y)}\mathrm{j}$.

c. Because $\nabla f = \nabla d_1 + \nabla d_2 + \nabla d_3$, the condition $f_x = f_y = 0$ is equivalent to $\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$.

d. If three unit vectors add to 0, they must make angles of $\pm\frac{2\pi}{3}$ with each other.

e. In this case the optimal point is the vertex at the large angle.

f. Solving the equations $f_x = f_y = 0$ numerically gives $(0.255457, 0.304504)$.

12.8.78 Write the equation of the plane in the form $ax + by + cz = 1$, with $a, b, c > 0$; then a, b, c must satisfy the condition $3a + 2b + c = 1$. This plane meets the coordinate axes in the points $\left(\frac{1}{a}, 0, 0\right)$, $\left(0, \frac{1}{b}, 0\right)$, $\left(0, 0, \frac{1}{c}\right)$. The region between then plane and the coordinate planes is a tetrahedron T which has base area $A = \frac{1}{2ab}$ and height $h = \frac{1}{c}$, so its volume is $V = \frac{1}{6abc}$. Minimizing this function is equivalent to maximizing its reciprocal, or equivalently the quantity $Q = abc$. Now if we let $u = 3a$, $v = 2b$, $w = c$, then an equivalent (but simpler) problem is to maximize $Q = uvw$ where u, v,$w \ge 0$ and $u + v + w = 1$. The maximum occurs when $u = v = w = \frac{1}{3}$ (see problem 65c), so the tetrahedron of minimal volume is given by $3a = 2b = c = \frac{1}{3}$, which gives the plane $\frac{x}{9} + \frac{y}{6} + \frac{z}{3} = 1$.

12.8.79

a. We have $f(x,y) = -2x^4 + 2x^2(e^y+1) - e^{2y} - 1$; hence $f_x = -8x^3 + 4x(e^y+1)$, $f_y = 2x^2e^y - 2e^{2y}$, so the critical points must satisfy $x^2 = e^y$, and then the equation $f_x = 0$ reduces to $e^y = 1$, so the critical points are $(\pm 1, 0)$. Next we compute$f_{xx} = -24x^2 + 4(1+e^y)$, $f_{yy} = 2x^2e^y - 4e^{2y}$, $f_{xy} = 4e^y$ so $f_{xx}f_{yy} - f_{xy}^2 = (-16)(-2) - (4)^2 = 16$ at both critical points; we also have $f_{xx} < 0$ at both critical points, so the critical points both yield local maxima.

b. We have $f_x = 8xe^y - 8x^3$, $f_y = 4x^2e^y - 4e^{4y}$, so the critical points must satisfy $x^2 = e^{3y}$, and then the equation $f_x = 0$ reduces to $e^y = 1$, so the critical points are $(\pm 1, 0)$. Next we compute $f_{xx} = 8e^y - 24x^2$, $f_{yy} = 4x^2e^y - 16e^{4y}$, $f_{xy} = 8xe^y$, so $f_{xx}f_{yy} - f_{xy}^2 = (-16)(-12) - (\pm 8)^2 = 128$ at both critical points; we also have $f_{xx} < 0$ at both critical points, so the critical points both yield local maxima.

12.8.80

a. We have $f_x = 3e^y - 3x^2$, $f_y = 3xe^y - 3e^{3y}$, so the critical points must satisfy $x = e^{2y}$, and then the equation $f_x = 0$ reduces to $e^{3y} = 1$, so the only critical point is $(1,0)$. Next we compute $f_{xx} = -6x$, $f_{yy} = 3xe^y - 9e^{3y}$, $f_{xy} = 3e^y$, so $f_{xx}f_{yy} - f_{xy}^2 = (-6)(-6) - 3^2 = 27 > 0$ at the critical point $(1,0)$; we also have $f_{xx} < 0$ at $(1,0)$, so the critical point is a local maxima, and we conclude that f has a unique local maximum. However observe that if we fix $y = 0$, then $f(x,0) = -x^3 + 3x - 1$ takes on all real values, so f does not have an absolute maximum or minimum.

b. We have $f_x = \left(2y^2 - y^4\right)e^x - \left(2y^2 - y^4 - 1\right)\frac{2x}{(1+x^2)^2}$, $f_y = \left(4y - 4y^3\right)\left(e^x + \frac{1}{1+x^2}\right)$, so the critical points must satisfy $y = 0, \pm 1$ from the equation $f_y = 0$. Substituting $y = 0$ into the equation $f_x = 0$ gives $x = 0$, whereas substituting $y = \pm 1$ into $f_x = 0$ gives no solutions; therefore, $(0,0)$ is the only critical point. We compute $f_{xx}(0,0) = 2$, $f_{yy}(0,0) = 8$, $f_{xy} = 0$, so $D(0,0) > 0$, and $f_{xx}(0,0) > 0$ implies that f has a local minimum at $(0,0)$. However, observe that if we fix $y = 1$, then $f(x,1) = e^x$ takes on all positive values, and if we fix $y = 2$, then $f(x,2) = -8e^x - \frac{9}{1+x^2}$ takes on all negative values. Hence f must take on all real values on R^2, and therefore has no absolute maximum or minimum.

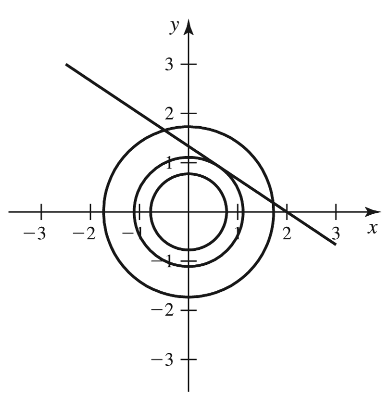
12.9 Lagrange Multipliers

12.9.1 The level curves of f must be tangential to the level curves of g at the optimal point; thus, the gradients are parallel.

12.9.2 We have $\nabla f = \langle 2x, 2y \rangle$, $\nabla g = \langle 2, 3 \rangle$ so the Lagrange multiplier conditions are $2x = 2\lambda$, $2y = 3\lambda$, $2x + 3y - 4 = 0$.

12.9.3 We have $\nabla f = \langle 2x, 2y, 2z \rangle$, $\nabla g = \langle 2, 3, -5 \rangle$ so the Lagrange multiplier conditions are $2x = 2\lambda$, $2y = 3\lambda$, $2z = -5\lambda$, $2x + 3y - 5z + 4 = 0$.

12.9.4 The function $f(x,y) = x^2 + y^2$ attains a minimum value of $\frac{16}{13}$ at the point $\left(\frac{8}{13}, \frac{12}{13}\right)$ along the constraint line $2x + 3y - 4 = 0$.



12.9.5 We have $\nabla f = \langle 1, 2\rangle$, $\nabla g = \langle 2x, 2y\rangle$ so the Lagrange multiplier conditions are $1 = 2\lambda x$, $2 = 2\lambda y$, $x^2+y^2-4=0$. Hence $\frac{1}{x} = 2\lambda = \frac{2}{y} \Longrightarrow y = 2x$; substituting this in the constraint gives $5x^2 = 4$, so $x = \pm\frac{2}{\sqrt{5}}$ and the extreme values of f on the circle $x^2+y^2=4$ must occur at the points $\pm\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$. We see that $f\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right) = 2\sqrt{5}$, $f\left(-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right) = -2\sqrt{5}$, so these are the maximum and minimum values.

12.9.6 We have $\nabla f = \langle y^2, 2xy\rangle$, $\nabla g = \langle 2x, 2y\rangle$ so the Lagrange multiplier conditions are $y^2 = 2\lambda x$, $2xy = 2\lambda y$, $x^2+y^2-1=0$. One possible solution is $y = \lambda = 0$, which gives the points $(\pm1, 0)$. All other solutions will have $x,\ y,\ \lambda \neq 0$, so we can eliminate λ and obtain $2\lambda = \frac{y^2}{x} = 2x \Longrightarrow y^2 = 2x^2$; substituting this in the constraint gives $3x^2 = 1$, so $x = \pm\frac{1}{\sqrt{3}}$, hence $y = \pm\frac{\sqrt{2}}{\sqrt{3}}$ and the extreme values of f on the circle $x^2+y^2=1$ must occur at the points $(\pm1, 0)$, $\left(\frac{1}{\sqrt{3}}, \pm\frac{\sqrt{2}}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \pm\frac{\sqrt{2}}{\sqrt{3}}\right)$. We see that $f(\pm1, 0) = 0$, $f\left(\frac{1}{\sqrt{3}}, \pm\frac{\sqrt{2}}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{9}$, $f\left(-\frac{1}{\sqrt{3}}, \pm\frac{\sqrt{2}}{\sqrt{3}}\right) = -\frac{2\sqrt{3}}{9}$, so the maximum and minimum values are $\pm\frac{2\sqrt{3}}{9}$.

12.9.7 We have $\nabla f = \langle 1, 1\rangle$, $\nabla g = \langle 2x-y, 2y-x\rangle$, so the Lagrange multiplier conditions are $2x-y = 1/\lambda$, $2y-x = 1/\lambda$, $x^2-xy+y^2 = 1$. Subtracting the first two equations gives $3x-3y=0$, so $x = y = 1/\lambda$, which yields the points $(\pm1, \pm1)$. The maximum value is 2 at $(1, 1)$ and the minimum is -2 at $(-1, -1)$.

12.9.8 We have $\nabla f = \langle 2x, 2y\rangle$, $\nabla g = \langle 4x+3y, 4y+3x\rangle$, so the Lagrange multiplier conditions are $2x = \lambda(4x+3y)$, $2y = \lambda(4y+3x)$, $2x^2+3xy+2y^2 = 7$. Multiplying the first equation by y and the second by x and subtracting gives $0 = \lambda(3y^2-3x^2)$. Note that if $\lambda = 0$, then $x = 0$ and $y = 0$ which does not satisfy the constraint. If $\lambda \neq 0$, we have $x = \pm y$. If $x = y$ we obtain the points $x = y = \pm1$, if $x = -y$, we obtain the point $(\pm\sqrt{7}, \mp\sqrt{7})$. The minimum value of f is 2 at $(\pm1, \pm1)$ and the maximum is 14 at $(\pm\sqrt{7}, \mp\sqrt{7})$.

12.9.9 We have $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 2x-y, 2y-x\rangle$, so the Lagrange multiplier conditions are $y = \lambda(2x-y)$, $x = \lambda(2y-x)$, $x^2+y^2-xy = 9$. Multiplying the first equation by x and the second by y and subtracting leads to $\lambda(2x^2-2y^2) = 0$, so either $x = \pm y$ or $\lambda = 0$. If $\lambda = 0$, we have $x = y = 0$ which doesn't meet the constraint. If $x = y$, we have $x = \pm3 = y$, if $x = -y$, we have $x = \pm\sqrt{3}$, $y = \mp\sqrt{3}$. There is a minimum value of -3 at $(\pm\sqrt{3}, \mp\sqrt{3})$ and a maximum value of 9 at $(\pm3, \pm3)$.

12.9.10 We have $\nabla f = \langle 1, -1\rangle$ and $\nabla g = \langle 2x-3y, 2y-3x\rangle$, so the Lagrange multiplier conditions are $2x-3y = 1/\lambda$, $2y-3x = -1/\lambda$, $x^2+y^2-3xy = 20$. Adding the first two equations gives $-y-x = 0$, so $x = -y$ and we have the points $(\pm2, \mp2)$. There is a maximum value of 4 at $(2, -2)$ and a minimum value of -4 at $(-2, 2)$.

12.9.11 We have $\nabla f = \langle 2ye^{2xy}, 2xe^{2xy}\rangle$ and $\nabla g = \langle 2x, 2y\rangle$, so the Lagrange multiplier conditions are $2ye^{2xy} = 2\lambda x$, $2xe^{2xy} = 2\lambda y$, $x^2+y^2 = 16$. If we multiply the first equation by y and the second by x and subtract, we have $(2y^2-2x^2)e^{2xy} = 0$, so $y = \pm x$. If $y = x$, we obtain the points $(\pm2\sqrt{2}, \pm2\sqrt{2})$, and if $y = -x$, we obtain $(\pm2\sqrt{2}, \mp2\sqrt{2})$. There is a maximum of e^{16} at $(\pm2\sqrt{2}, \pm2\sqrt{2})$ and a minimum of e^{-16} at $(\pm2\sqrt{2}, \mp2\sqrt{2})$.

12.9.12 We have $\nabla f = \langle 2x, 2y\rangle$, $\nabla g = \langle 6x^5, 6y^5\rangle$ so the Lagrange multiplier conditions are $2x = 6\lambda x^5$, $2y = 6\lambda y^5$, $x^6+y^6-1 = 0$. If $x,\ y,\ \lambda \neq 0$ we can eliminate λ and conclude that $x^4 = y^4$ or $y = \pm x$; the constraint gives $2x^6 = 1$, so $x = \pm2^{-1/6}$ and we get the four points $\pm\left(2^{-1/6}, \pm2^{-1/6}\right)$; the value of f at all of these is the same: $f\left(2^{-1/6}, 2^{-1/6}\right) = 2\cdot2^{-1/3} = 2^{2/3} \approx 1.587$. We also have solutions with x or $y = 0$, which give the points $(\pm1, 0)$ and $(0, \pm1)$; the value of f at all of these is the same: $f(1, 0) = 1$. Hence the minimum and maximum values of f on the closed bounded set given by $x^6+y^6 = 1$ are 1 and $2^{2/3}$.

12.9.13 We have $\nabla f = \langle -8x, 2y\rangle$, $\nabla g = \langle 2x, 4y\rangle$ so the Lagrange multiplier conditions are $-8x = 2\lambda x$, $2y = 4\lambda y$, $x^2+2y^2-4 = 0$. If $x,\ y \neq 0$ the first equation gives $\lambda = -4$, whereas the second gives $\lambda = \frac{1}{2}$ which is a contradiction; hence we must have x or $y = 0$, which gives the points $(\pm2, 0)$ and $\left(0, \pm\sqrt{2}\right)$, and $f(\pm2, 0) = -16$, $f\left(0, \pm\sqrt{2}\right) = 2$. Hence the minimum and maximum values of f on the closed bounded set given by $x^2+2y^2 = 4$ are -16 and 2.

12.9.14 We have $\nabla f = \langle y+1, x+1\rangle$, $\nabla g = \langle 2xy^2, 2yx^2\rangle$ so the Lagrange multiplier conditions are $y+1 = \lambda(2xy^2)$, $x+1 = \lambda(2yx^2)$, $x^2y^2 = 4$. If we multiply the first equation by x and the second by y and subtract, we obtain $x - y = 0$, so $x = y$. This yields the points $(\pm\sqrt{2}, \pm\sqrt{2})$, so there is a maximum value of $2+2\sqrt{2}$ at $(\sqrt{2}, \sqrt{2})$ and a minimum value of $2-2\sqrt{2}$ at $(-\sqrt{2}, -\sqrt{2})$.

12.9.15 We have $\nabla f = \langle 1, 3, -1\rangle$, $\nabla g = \langle 2x, 2y, 2z\rangle$ so the Lagrange multiplier conditions are $1 = 2\lambda x$, $3 = 2\lambda y$, $-1 = 2\lambda z$, $x^2+y^2+z^2-4 = 0$. These equations imply x, y, $z \neq 0$, so we can eliminate λ and obtain $y = 3x$, $z = -x$. Then the constraint gives $11x^2 = 4$, so $x = \pm\frac{2}{\sqrt{11}}$ and the solutions are the points $\pm\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$. We compute $f\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) = 2\sqrt{11}$, $f\left(-\frac{2}{\sqrt{11}}, -\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right) = -2\sqrt{11}$, and hence the minimum and maximum values of f on the closed bounded set given by $x^2+y^2+z^2 = 4$ are $-2\sqrt{11}$ and $2\sqrt{11}$.

12.9.16 We have $\nabla f = \langle yz, xz, xy\rangle$, $\nabla g = \langle 2x, 4y, 8z\rangle$ so the Lagrange multiplier conditions are $yz = 2\lambda x$, $xz = 4\lambda y$, $xy = 8\lambda z$, $x^2+2y^2+4z^2-9 = 0$. Assume first that x, y, $z \neq 0$, so we can eliminate λ and obtain $\frac{xyz}{2\lambda} = x^2 = 2y^2 = 4z^2$. Then using the constraint we obtain $3x^2 = 9$ so $x = \pm\sqrt{3}$, $y = \pm\frac{\sqrt{3}}{\sqrt{2}}$ and $z = \pm\frac{\sqrt{3}}{2}$. The value of xyz at any of these points is $\pm\sqrt{3}\cdot\frac{\sqrt{3}}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} = \pm\frac{3\sqrt{6}}{4}$. We also observe that if any of x, y, $z = 0$ then $xyz = 0$. Hence, the minimum and maximum values of f on the closed bounded set given by $x^2+2y^2+4z^2 = 9$ are $\pm\frac{3\sqrt{6}}{4}$.

12.9.17 We have $\nabla f = \langle 1, 0, 0\rangle$ and $\nabla g = \langle 2x, 2y, 2z-1\rangle$. The Lagrange multiplier conditions are $1 = 2x\lambda$, $0 = 2y\lambda$, $0 = (2z-1)\lambda$, and $x^2+y^2+z^2-z = 1$. Clearly $\lambda \neq 0$, so we must have $y = 0$ and $z = 1/2$. Solving the constraint for x gives $x = \pm\sqrt{5}/2$. There is a maximum of $\sqrt{5}/2$ at $(\sqrt{5}/2, 0, 1/2)$ and a minimum of $-\sqrt{5}/2$ at $(-\sqrt{5}/2, 0, 1/2)$.

12.9.18 We have $\nabla f = \langle 1, 0, -1\rangle$ and $\nabla g = \langle 2x, 2y-1, 2z\rangle$. The Lagrange multiplier conditions are $1 = 2x\lambda$, $0 = (2y-1)\lambda$, and $-1 = 2z\lambda$. Clearly $\lambda \neq 0$, so we must have $y = 1/2$. Also, $\frac{1}{x} = \frac{-1}{z} = 2\lambda$, so $x = -z$. The constraint equation then gives $x = \pm 3/(2\sqrt{2})$ and $z = \mp 3/(2\sqrt{2})$. There is a minimum of $-3/\sqrt{2}$ at $(-3/(2\sqrt{2}), 1/2, 3/(2\sqrt{2}))$ and a maximum of $3/\sqrt{2}$ at $(3/(2\sqrt{2}), 1/2, -3/(2\sqrt{2}))$.

12.9.19 We have $\nabla f = \langle 2x, 2y, 2z\rangle$ and $\nabla g = \langle 2x-4y, 2y-4x, 2z\rangle$. The Lagrange multiplier conditions are $2x = \lambda(2x-4y)$, $2y = \lambda(2y-4x)$, $2z = \lambda(2z)$, and $x^2+y^2+z^2-4xy = 1$. Suppose $z \neq 0$. Then $\lambda = 1$ and $x = y = 0$, so $z = \pm 1$. If $z = 0$, then multiplying the first equation by y and the second by x and subtract to obtain $0 = \lambda(4x^2-4y^2)$, and since we can't have $\lambda = 0$ (because then $x = y = z = 0$), we must have $x = \pm y$. When $x = y$, the constraint becomes $2x^2 - 4x^2 = 1$ which can't occur. So we must have $x = -y$, which yields $x = \pm 1/\sqrt{6}$ and $y = \mp 1/\sqrt{6}$. There is a maximum of 1 at $(0, 0, \pm 1)$ and a minimum of 1/3 at $(\pm 1/\sqrt{6}, \mp 1/\sqrt{6}, 0)$.

12.9.20 We have $\nabla f = \langle 1, 1, 1\rangle$ and $\nabla g = \langle 2x-2, 2y-2, 2z\rangle$. The Lagrange multiplier conditions are $1 = \lambda(2x-2)$, $1 = \lambda(2y-2)$, $1 = 2\lambda z$, and $x^2+y^2+z^2-2x-2y = 1$. The first two equations imply that $x = y$ and the third together with the first two implies that $x = y = z+1$. The constraint then implies that $2x^2 + (x-1)^2 - 4x = 1$, or $3x^2 - 6x = 0$. So $x = 0$ or $x = 2$. There is a maximum of 5 at $(2, 2, 1)$ and a minimum of -1 at $(0, 0, -1)$.

12.9.21 $\nabla f = \langle 2, 0, 2z\rangle$ and $\nabla g = \langle 2x, 2y, 4z\rangle$. The Lagrange multiplier conditions are $2 = 2\lambda x$, $0 = 2\lambda y$, $2z = 4\lambda z$, and $x^2+y^2+2z^2 = 25$. Note that $\lambda \neq 0$ so $y = 0$ and if $z \neq 0$ then $\lambda = 1/2$ so $x = 2$ and we have the point $(2, 0, \sqrt{21/2})$. If $z = 0$, then we have $x = \pm 5$. The minimum is -10 at $(-5, 0, 0)$ and the maximum is 14.5 at $(2, 0, \sqrt{21/2})$.

12.9.22 We have $\nabla f = \langle 2x, 2y, -1\rangle$ and $\nabla g = \langle 4xy^2, 4x^2y, -1\rangle$. The Lagrange multiplier conditions are $2x = \lambda(4xy^2)$, $2y = \lambda(4x^2y)$, $-1 = \lambda(-1)$, and $2x^2y^2 - z = -1$. Clearly $\lambda = 1$, and $2xy = 4xy^3 = 4x^3y$. One possible solution is $x = y = 0$ and thus $z = 1$. If $x \neq 0$ and $y \neq 0$, then $2 = 4y^2 = 4x^2$, so $(x, y) = \pm(1/\sqrt{2}, \pm(1/\sqrt{2}))$. There is a minimum of -1 at $(0, 0, 1)$, and a maximum of $-1/2$ at any of the four points $(-1/\sqrt{2}, -1/\sqrt{2}, 3/2)$, $(-1/\sqrt{2}, 1/\sqrt{2}, 3/2)$, $(1/\sqrt{2}, -1/\sqrt{2}, 3/2)$, and $(1/\sqrt{2}, 1/\sqrt{2}, 3/2)$.

12.9.23 We have $\nabla f = \langle 2x, 2y, 2z\rangle$, $\nabla g = \langle yz, xz, xy\rangle$, so the Lagrange multiplier conditions are $2x = \lambda yz$, $2y = \lambda xz$, $2z = \lambda xy$, $xyz - 4 = 0$. The first three equations give $\frac{\lambda xyz}{2} = x^2 = y^2 = z^2$, so $y = \pm x$, $z = \pm x$. Then using the constraint we obtain $x^3 = \pm 4$, so x, y, $z = \pm\sqrt[3]{4}$. The value of $x^2 + y^2 + z^2$ at any of these points is $f\left(\pm\sqrt[3]{4}, \pm\sqrt[3]{4}, \pm\sqrt[3]{4}\right) = 3 \cdot 4^{2/3} = 6\sqrt[3]{2}$. Note that $f(x, y, z)$ is the square of the distance from (x, y, z) to the origin, so this function will have an absolute minimum but no maximum on the surface given by $xyz = 4$; therefore the minimum value of f on the surface is $6\sqrt[3]{2}$.

12.9.24 Observe first that it is equivalent to find the extreme values of the simpler function $h(x, y, z) = f(x, y, z)^2 = xyz$ subject to the constraints $g(x, y, z) = x + y + z - 1 = 0$ and x, y, $z \geq 0$. We have $\nabla h = \langle yz, xz, xy\rangle$, $\nabla g = \langle 1, 1, 1\rangle$, so the Lagrange multiplier conditions are $yz = \lambda$, $xz = \lambda$, $xy = \lambda$, $x + y + z - 1 = 0$. Assume first that x, y, $z \neq 0$; then first three equations give $x = y = z$, and the constraint implies $3x = 1$, so x, y, $z = \frac{1}{3}$. The domain of f (or g) is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, which is closed and bounded. We also note that $f = h = 0$ along any of the edges of the triangle. Therefore, the maximum value of f is $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3\sqrt{3}}$ and the minimum value is 0.

12.9.25 Let x, y, $z \geq 0$ denote the lengths of the sides of the box, with z the longest side. Then the length plus girth of the box is $2x + 2y + z$ so we must maximize the volume $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = 2x + 2y + z - 108 = 0$. We have $\nabla f = \langle yz, xz, xy\rangle$, $\nabla g = \langle 2, 2, 1\rangle$ so the Lagrange multiplier conditions are $yz = 2\lambda$, $xz = 2\lambda$, $xy = \lambda$, $2x + 2y + z - 108 = 0$. Assume that x, y, $z > 0$ (otherwise the volume is 0); then the first three equations give $\frac{xyz}{2\lambda} = x = y = \frac{z}{2}$, and the constraint gives $6x = 108$, so $x = 18$ and the box has dimensions 18 in$\times$18 in$\times$36 in. The domain of f is the triangle with vertices $(54, 0, 0)$, $(0, 54, 0)$, $(0, 0, 108)$, which is closed and bounded. We also note that $f = 0$ along any of the edges of the triangle. Therefore, the maximum value of f occurs at the point we found, and the minimum value is 0.

12.9.26 Let x, y, $z > 0$ denote the lengths of the sides of the box; then its surface area is given by $f(x, y, z) = 2xy + 2yz + 2xz$, which we must maximize subject to the constraint $g(x, y, z) = xyz - 16 = 0$. We have $\nabla f = \langle 2(y + z), 2(x + z), 2(x + y)\rangle$, $\nabla g = \langle yz, xz, xy\rangle$ so the Lagrange multiplier conditions are $2(y + z) = \lambda yz$, $2(x + z) = \lambda xz$, $2(x + y) = \lambda xy$, $xyz - 16 = 0$. The first three equations give $\frac{\lambda}{2} = \frac{1}{y} + \frac{1}{z} = \frac{1}{x} + \frac{1}{z} = \frac{1}{x} + \frac{1}{y}$ which implies $x = y = z$; the constraint gives $x^3 = 16$, so $x = y = z = 2\sqrt[3]{2}$,and the box is a cube with side length $2\sqrt[3]{2}$ ft. The domain of f is the surface in the first quadrant given by $xyz = 16$, which is unbounded. We can rewrite $f(x, y, z) = 2xy + 2yz + 2xz = 2xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 32\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$ on this surface. Now if, say, $x \to \infty$ on this surface either $y \to 0$ or $z \to 0$ (or both) because of the relation $xyz = 16$; therefore, $f(x, y, z) \to \infty$. This shows that the point we found above must give the absolute minimum value of f on the surface.

12.9.27 It suffices to find the extreme values of the function $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = x^2 + xy + 2y^2 - 1 = 0$. We have $\nabla f = \langle 2x, 2y\rangle$, $\nabla g = \langle 2x + y, 4y + x\rangle$, so the Lagrange multiplier conditions are $2x = \lambda(2x + y)$, $2y = \lambda(4y + x)$, $x^2 + xy + 2y^2 - 1 = 0$. The first two equations give $2x(4y + x) = \lambda(2x + y)(4y + x) = 2y(2x + y) \implies x^2 + 4xy = y^2 + 2xy$, or $x^2 + 2xy - y^2$. This implies that both x, $y \neq 0$, for if, say, $x = 0$ then this condition gives $y = 0$ as well, which violates the constraint (same argument for $y = 0$). Let $r = \frac{y}{x}$, and rewrite this equation as $\frac{y^2}{x^2} - \frac{2y}{x} - 1 = r^2 - 2r - 1 = 0$, which we can solve to obtain $r = 1 \pm \sqrt{2}$. We now use the constraint and the relation $y = rx$ to obtain $x^2\left(1 + r + 2r^2\right) = 1$or $x^2 = \frac{1}{2r^2 + r + 1}$. Then the values of the function f are given by $f(x, y) = f(x, rx) = x^2\left(1 + r^2\right) = \frac{1 + r^2}{2r^2 + r + 1} = \frac{6 \pm 2\sqrt{2}}{7} \approx 0.4531, 1.2612$, and the corresponding minimum and maximum distances are $\sqrt{\frac{6 \pm 2\sqrt{2}}{7}} \approx 0.6731, 1.1230$.

12.9.28 Let (x, y) be the vertex of the rectangle in the first quadrant; then the area of the rectangle is $4xy$, so it suffices to maximize the function $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + 4y^2 - 4 = 0$ and x, $y \geq 0$. We have $\nabla f = \langle y, x\rangle$, $\nabla g = \langle 2x, 8y\rangle$ so the Lagrange multiplier conditions are $y = 2\lambda x$, $x = 8\lambda y$, $x^2 + 4y^2 - 4 = 0$. The first two equations give $4y^2 = 8\lambda xy = x^2$, and substituting in the constraint gives $2x^2 = 4$ so $x = \sqrt{2}$ and $y = \frac{\sqrt{2}}{2}$. The domain given by the constraint and x, $y \geq 0$ is a closed and bounded arc, and we observe that $f = 0$ at the boundary points $(2, 0)$ and $(1, 0)$. Therefore, f takes its maximum at $\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right)$, and the corresponding area of the rectangle is $A = 4 \cdot \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 4$.

12.9.29 Let (x, y) be the vertex of the rectangle in the first quadrant; then the perimeter of the rectangle is $4(x + y)$, so it suffices to maximize the function $f(x, y) = x + y$ subject to the constraint $g(x, y) = 2x^2 + 4y^2 - 3 = 0$ and $x, y \geq 0$. We have $\nabla f = \langle 1, 1\rangle$, $\nabla g = \langle 4x, 8y\rangle$, so the Lagrange multiplier conditions are $1 = 4\lambda x$, $1 = 8\lambda y$, $2x^2 + 4y^2 - 3 = 0$. The first two equations give $4x = \frac{1}{\lambda} = 8y$; so $x = 2y$. Substituting in the constraint gives $8y^2 + 4y^2 = 3$, so $y^2 = 1/4$ and $y = 1/2$ and thus $x = 1$. Note that $f(1, 1/2) = 1.5$. The domain given by the constraint and $x, y \geq 0$ is a closed and bounded arc, and we observe that at the boundary points we have $f(0, \sqrt{3}/2) = \sqrt{3}/2 \approx .87$ and $f(\sqrt{3/2}, 0) = \sqrt{3/2} \approx 1.25$. Thus, the dimensions of the rectangle of maximum perimeter is 2×1.

12.9.30 It suffices to maximize the function $f(x, y, z) = (x + 2)^2 + (y - 5)^2 + (z - 1)^2$ subject to the constraint $g(x, y, z) = 2x + 3y + 6z - 10 = 0$. We have $\nabla f = \langle 2x + 4, 2y - 10, 2z - 2\rangle$, $\nabla g = \langle 2, 3, 6\rangle$, so the Lagrange multiplier conditions are $2x + 4 = 2\lambda$, $2y - 10 = 3\lambda$, $2z - 2 = 6\lambda$, $2x + 3y + 6z - 10 = 0$. The first three equations give $x + 2 = \lambda = \frac{2y-10}{3} = \frac{z-1}{3}$, which gives $y = \frac{3}{2}x + 8$, $z = 3x + 7$. Substituting in the constraint and solving for x gives $x = -\frac{16}{7}$, so the closest point is $\left(-\frac{16}{7}, \frac{32}{7}, \frac{1}{7}\right)$; $f\left(-\frac{16}{7}, \frac{32}{7}, \frac{1}{7}\right) = 1$ so the distance is 1.

12.9.31 It suffices to minimize the function $f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z + 3)^2$ subject to the constraint $4x + y - 1 = 0$. We have $\nabla f = \langle 2(x - 1), 2(y - 2), 2(z + 3)\rangle$ and $\nabla g = \langle 4, 1, 0\rangle$. Then the Lagrange multiplier conditions are $2x - 2 = 4\lambda$, $2y - 4 = \lambda$, $2z + 6 = 0$, $4x + y - 1 = 0$. Solving this system gives $x = -3/17$, $y = 29/17$, $z = -3$, and $\lambda = -10/17$, so the closest point on the surface to the given point is $(-3/17, 29/17, -3)$.

12.9.32 It suffices to minimize the function $f(x, y, z) = (x - 1)^2 + (y - 2)^2 + z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 - z^2 = 0$. We have $\nabla f = \langle 2x - 2, 2y - 4, 2z\rangle$, $\nabla g = \langle 2x, 2y, -2z\rangle$ so the Lagrange multiplier conditions are $2x - 2 = 2\lambda x$, $2y - 4 = 2\lambda y, 2z = -2\lambda z$, $x^2 + y^2 - z^2 = 0$. Observe that $z \neq 0$; otherwise $x = y = z = 0$, which is not a solution to the Lagrange multiplier equations. Therefore, the third equation gives $\lambda = -1$, the first two equations give $x = \frac{1}{2}$, $y = 1$ and the constraint gives $z = \pm\frac{\sqrt{5}}{2}$. Therefore, the points on the cone closest to $(1, 2, 0)$ is $\left(\frac{1}{2}, 1, \pm\frac{\sqrt{5}}{2}\right)$.

12.9.33 It suffices to find the extreme values the function $f(x, y, z) = (x - 2)^2 + (y - 3)^2 + (z - 4)^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$. We have $\nabla f = \langle 2x - 4, 2y - 6, 2z - 8\rangle$, $\nabla g = \langle 2x, 2y, 2z\rangle$ so the Lagrange multiplier conditions are $2x - 4 = 2\lambda x$, $2y - 6 = 2\lambda y$, $2z - 8 = 2\lambda z$, $x^2 + y^2 + z^2 - 9 = 0$. We can write the first three equations in the form $(1 - \lambda)\langle x, y, z\rangle = \langle 2, 3, 4\rangle$ so $\langle x, y, z\rangle = c\langle 2, 3, 4\rangle$ for some scalar c; using the constraint, we find that $c = \pm\frac{3}{\sqrt{29}}$, and hence $\langle x, y, z\rangle = \pm\frac{3}{\sqrt{29}}\langle 2, 3, 4\rangle$; the corresponding values of f are $f\left(\pm\left(\frac{6}{\sqrt{29}}, \frac{9}{\sqrt{29}}, \frac{12}{\sqrt{29}}\right)\right) = 38 \mp 6\sqrt{29} = \left(\sqrt{29} \mp 3\right)^2$, so the minimum distance is $\sqrt{29} - 3$ and the maximum distance is $\sqrt{29} + 3$.

12.9.34 Assume the cylinder is the region $x^2 + y^2 \leq r^2$, $-\frac{h}{2} \leq z \leq \frac{h}{2}$; then r and h are the radius and height respectively of the cylinder, and the points on the cylinder furthest from the origin have distance $\left(r^2 + \left(\frac{h}{2}\right)^2\right)^{1/2}$; therefore, we must maximize the function $V = f(r, h) = \pi r^2 h$ subject to the constrain $g(r, h) = r^2 + \left(\frac{h}{2}\right)^2 - 16^2 = 0$. We have $\nabla f = \pi\langle 2rh, r^2\rangle$, $\nabla g = \left\langle 2r, \frac{h}{2}\right\rangle$, so the Lagrange multiplier conditions are equivalent to $2rh = 2\lambda r$, $r^2 = \lambda\frac{h}{2}$, $r^2 + \left(\frac{h}{2}\right)^2 - 16^2 = 0$. Because we are interested in maximizing the volume we may assume that $r, h > 0$; then the first equation gives $\lambda = h$ and the second equation gives $h^2 = 2r^2$; substituting in the constraint then gives dimensions $r = \frac{16\sqrt{6}}{3}$, $h = \frac{32\sqrt{3}}{3}$.

12.9.35 Notice that the constraint is equivalent to $\ell + 2g = 6$. We have $\nabla U = 5\langle \ell^{-1/2}g^{1/2}, \ell^{1/2}g^{-1/2}\rangle$ so the Lagrange multiplier conditions are equivalent to $\ell^{-1/2}g^{1/2} = \lambda$, $\ell^{1/2}g^{-1/2} = 2\lambda$, $\ell + 2g = 6$. Eliminating λ from the first two equations gives $\ell^{1/2}g^{-1/2} = 2\ell^{-1/2}g^{1/2}$, which simplifies to $g = \frac{\ell}{2}$; substituting in the constraint then gives $\ell = 3$ and $g = \frac{3}{2}$. The value of the utility function at this point is $U = 15\sqrt{2}$.

12.9.36 Notice that the constraint is equivalent to $2\ell + g = 6$. We have $\nabla U = \frac{32}{3}\langle 2\ell^{-1/3}g^{1/3}, \ell^{2/3}g^{-2/3}\rangle$, so the Lagrange multiplier conditions are equivalent to $2\ell^{-1/3}g^{1/3} = 2\lambda$, $\ell^{2/3}g^{-2/3} = \lambda$, $2\ell + g = 6$. Eliminating λ from the first two equations gives $\ell^{-1/3}g^{1/3} = \ell^{2/3}g^{-2/3}$, which simplifies to $g = \ell$; substituting in the constraint then gives $\ell = g = 2$. The value of the utility function at this point is $U = 64$.

12.9.37 Notice that the constraint is equivalent to $5\ell + 4g = 20$. We have $\nabla U = \frac{8}{5}\langle 4\ell^{-1/5}g^{1/5}, \ell^{4/5}g^{-4/5}\rangle$, so the Lagrange multiplier conditions are equivalent to $4\ell^{-1/5}g^{1/5} = 5\lambda$, $\ell^{4/5}g^{-4/5} = 4\lambda$, $5\ell + 4g = 20$. Eliminating λ from the first two equations gives $16\ell^{-1/5}g^{1/5} = 5\ell^{4/5}g^{-4/5}$, which simplifies to $g = \frac{5\ell}{16}$; substituting in the constraint then gives $\ell = \frac{16}{5}$, $g = 1$. The value of the utility function at this point is $U = 8 \cdot \left(\frac{16}{5}\right)^{4/5} \approx 20.287$.

12.9.38 We have $\nabla U = \frac{1}{6}\langle \ell^{-5/6}g^{5/6}, 5\ell^{1/6}g^{-1/6}\rangle$, so the Lagrange multiplier conditions are equivalent to $\ell^{-5/6}g^{5/6} = 4\lambda$, $5\ell^{1/6}g^{-1/6} = 5\lambda$, $4\ell + 5g = 20$. Eliminating from the first two equations gives $\ell^{-5/6}g^{5/6} = 4\ell^{1/6}g^{-1/6}$, which simplifies to $g = 4\ell$; substituting in the constraint then gives $\ell = \frac{5}{6}$, $g = \frac{10}{3}$. The value of the utility function at this point is $U = \left(\frac{5}{6}\right)^{1/6}\left(\frac{10}{3}\right)^{5/6} \approx 2.646$.

12.9.39

a. True. This is because the tangent plane to a sphere at any point has normal vector in the direction of the line joining the point to the center of the sphere.

b. False in general. In fact, the two vectors ∇f and ∇g are in the same direction, so $\nabla f \cdot \nabla g = 0$ only if one of these vectors is zero.

12.9.40 Let x, $y \geq 0$ be the dimensions of the base of the box and h be the height; then x, y, h satisfy the constraint $2x + 2y + h = 96$. The volume of the box is $V = xyh$, so the Lagrange multiplier conditions are $yh = 2\lambda$, $xh = 2\lambda$, $xy = \lambda$, $2x + 2y + h = 96$. Eliminating λ from the first three equations gives $y = x$ and $h = 2x$; substituting in the constraint then gives $x = 16$, $y = 16$, $h = 32$ in.

12.9.41 Let x, $y > 0$ be the dimensions of the base and $h > 0$ be the height of the box; then the four sides of the box plus the base have total area $2xh + 2yh + xy = 2$, which is our constraint. The volume of the box is $V = xyh$, so the Lagrange multiplier conditions are $yh = \lambda(2h + y)$, $xh = \lambda(2h + x)$, $xy = 2\lambda(x + y)$, $2xh + 2yh + xy = 2$. The first two equations give $\frac{2h+y}{yh} = \frac{2h+x}{xh} \Longrightarrow \frac{2}{y} + \frac{1}{h} = \frac{2}{x} + \frac{1}{h}$, so $y = x$. The second and third equations give $\frac{2h+x}{xh} = \frac{2x+2y}{xy} \Longrightarrow \frac{2}{x} + \frac{1}{h} = \frac{2}{x} + \frac{2}{y} = \frac{4}{x}$, so $h = \frac{x}{2}$. Then substituting in the constraint gives $3x^2 = 2$, so $x = \frac{\sqrt{6}}{3}$. Therefore, the box with largest volume has height $\frac{\sqrt{6}}{6}$ m and base $\frac{\sqrt{6}}{3} \times \frac{\sqrt{6}}{3}$ m.

12.9.42 Let x, $y > 0$ be the dimensions of the base and $h > 0$ be the height of the box; then the volume of the box is $xyh = 4$, which is our constraint. The four sides of the box plus the base have total area $A = 2xh + 2yh + xy$, so the Lagrange multiplier conditions are $2h + y = \lambda yh$, $2h + x = \lambda xh$, $2x + 2y = \lambda xy$, $xyh = 4$. The first two equations give $\frac{2h+y}{yh} = \frac{2h+x}{xh} \Longrightarrow \frac{2}{y} + \frac{1}{h} = \frac{2}{x} + \frac{1}{h}$, so $y = x$. The second and third equations give $\frac{2h+x}{xh} = \frac{2x+2y}{xy} \Longrightarrow \frac{2}{x} + \frac{1}{h} = \frac{2}{x} + \frac{2}{y} = \frac{4}{x}$, so $h = \frac{x}{2}$. Then substituting in the constraint gives $x^3 = 8$, so $x = 2$. Therefore, the box with smallest area has dimensions 2 m×2m×1m.

12.9.43 Let x, $y \geq 0$ be the dimensions of the base and $z \geq 0$ be the height of the box; then $x + 2y + 3z = 6$ is our constraint. The box has volume $V = xyz$, so the Lagrange multiplier conditions are $yz = \lambda$, $xz = 2\lambda$, $xy = 3\lambda$, $x + 2y + 3z = 6$. The first two equations give $y = \frac{x}{2}$, the first and third equations give $z = \frac{x}{3}$, and substituting in the constraint gives $x = 2$. Therefore, the box with largest volume has dimensions $x = 2$, $y = 1$, $z = \frac{2}{3}$.

12.9.44 Let (x, y, z) be the point on the ellipsoid; then x, y, $z > 0$ and the box has volume $V = xyz$; the relation $36x^2 + 4y^2 + 9z^2 = 36$ is our constraint. The Lagrange multiplier conditions are $yz = 72\lambda x$, $xz = 8\lambda y$, $xy = 18\lambda z$, $36x^2 + 4y^2 + 9z^2 = 36$. The first two equations give $\lambda = \frac{yz}{72x} = \frac{xz}{8y}$, which implies $y^2 = 9x^2$ or $y = 3x$. Similarly the first and third equations give $z = 2x$ and then the constraint gives $3x^2 = 1$, so $x = \frac{\sqrt{3}}{3}$, and the box with maximum volume has dimensions $\frac{\sqrt{3}}{3} \times \sqrt{3} \times \frac{2\sqrt{3}}{3}$.

12.9.45 The constraint is the equation of the plane $x - y + z = 2$; the distance from a point (x, y, z) to the point $(1, 1, 1)$ is given by $d^2 = (x-1)^2 + (y-1)^2 + (x-y-1)^2$, so it suffices to minimize the function $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ subject to the constraint $x - y + z = 2$. The Lagrange multiplier conditions are $2x - 2 = \lambda$, $2y - 2 = -\lambda$, $2z - 2 = \lambda$, $x - y + z = 2$. The first two equations give $2(x+y) = 4$, so $y = 2 - x$, the first and third equations give $z = x$, and then substituting in the constraint gives $x = \frac{4}{3}$, so the closest point on the plane to $(1, 1, 1)$ is $\left(\frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right)$.

12.9.46 The function $f(x, y) = x^2 + 4y^2 + 1$ has a unique critical point at $(0, 0)$, and $f(0, 0) = 1$. On the boundary $\{x^2 + 4y^2 = 1\}$ of R, $f(x, y) = 2$ at all points. Hence, the absolute minimum and maximum values of f on R are 1 and 2.

12.9.47 We have $\nabla f = \langle 2x + y, -8y + x \rangle$; solving $2x + y = 0$ and $-8y + x = 0$ simultaneously gives unique solution $(0, 0)$, and $f(0, 0) = 0$. Next we use Lagrange multipliers to find the minimum and maximum values of f on the boundary of R given by $4x^2 + 9y^2 = 36$. The Lagrange multiplier conditions are $2x + y = 8\lambda x$, $-8y + x = 18\lambda y$, $4x^2 + 9y^2 = 36$. The first two equations give $\lambda xy = \frac{2xy + y^2}{8} = \frac{x^2 - 8xy}{18} \Longrightarrow 9y^2 + 50xy - 4x^2 = 0$; therefore, $y = rx$ where r satisfies the quadratic $9r^2 + 50r - 4 = 0$, which has roots $r = \frac{-25 \pm \sqrt{661}}{9}$. Then the constraint gives $\left(4 + 9r^2\right) x^2 = 36$, so $x^2 = \frac{36}{9r^2 + 4} = \frac{18}{8 - 25r}$ (using $9r^2 = 4 - 50r$) and hence $f(x, y) = \left(1 - 4r^2 + r\right) x^2 = \frac{-7 + 209r}{9} \cdot \frac{18}{4 - 25r} = \frac{418r - 14}{4 - 25r} = \frac{-7 \pm \sqrt{661}}{2}$, which gives the absolute minimum and maximum values of f on R.

12.9.48 We have $\nabla f = \langle 4x + 2, 2y - 3 \rangle$; therefore, f has unique critical point $\left(-\frac{1}{2}, \frac{3}{2}\right)$ which is not in R and so is irrelevant. Next we use Lagrange multipliers to find the minimum and maximum values of f on the boundary of R given by $x^2 + y^2 = 1$. The Lagrange multiplier conditions are$4x + 2 = 2\lambda x$, $2y - 3 = 2\lambda y$, $x^2 + y^2 = 1$.The first two equations give$2\lambda xy = 4xy + 2y = 2xy - 3x \Longrightarrow (2x + 2)\, y = -3x$; therefore, $y = -\frac{3x}{2x + 2}$. Then the constraint gives$x^2 + \frac{9x^2}{(2x+2)^2} = 1 \Longrightarrow 4(x+1)^2 \left(x^2 - 1\right) + 9x^2 = 0$, or $4x^4 + 8x^3 + 9x^2 - 8x - 4 = 0$.Using a numerical solver, we find that this equation has roots $x \approx -0.38, 0.76$; the corresponding values of f are $\approx -2.39, 5.05$.

12.9.49 We have $\nabla f = \langle 2(x-1), 2(y+1) \rangle$; therefore f has unique critical point $(1, -1)$ which is inside R; $f(1, -1) = 0$. Next we use Lagrange multipliers to find the minimum and maximum values of $f$on the boundary of R given by $x^2 + y^2 = 4$. The Lagrange multiplier conditions are equivalent to$x - 1 = \lambda x$, $y + 1 = \lambda y$, $x^2 + y^2 = 4$. The first two equations give $\lambda xy = xy - y = xy + x \Longrightarrow y = -x$; then the constraint gives $2x^2 = 4$, so $x = \pm\sqrt{2}$ and the solutions are $\pm\left(\sqrt{2}, -\sqrt{2}\right)$. The values of f at these points are$f\left(\sqrt{2}, -\sqrt{2}\right) = 6 - 4\sqrt{2} \approx 0.343$, $f\left(-\sqrt{2}, \sqrt{2}\right) = 6 + 4\sqrt{2} \approx 11.657$; therefore, the maximum value of f on R is $6 + 4\sqrt{2}$ and the minimum value is 0.

12.9.50 The maximum and minimum values of f along the curve $g(x, y) = 0$ occur at points where the level curves of f are tangent to the curve $g(x, y) = 0$; using this, we see that the minimum and maximum values are 2 and 6.

12.9.51 The maximum and minimum values of f along the curve $g(x, y) = 0$ occur at points where the level curves of f are tangent to the curve $g(x, y) = 0$; using this, we see that the minimum and maximum values are 1 and 8.

12.9.52

a. It suffices to maximize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x^4 + y^4 + z^4 = 1$. The Lagrange multiplier conditions give $2x = 4\lambda x^3$, $2y = 4\lambda y^3$, $2z = 4\lambda z^3$, $x^4 + y^4 + z^4 = 1$. Assume first that x, y, $z \neq 0$; then we can eliminate λ and conclude that $x^2 = y^2 = z^2$; the constraint gives $3x^4 = 1$ so $x^2 = \frac{1}{\sqrt{3}}$ and $f(x, y, z) = \sqrt{3}$. There are also solutions to the Lagrange conditions with x, y or $z = 0$, but one can check that these do not give larger values of f. Therefore, the extreme points have coordinates x, y, $z = \pm 3^{-1/4}$.

b. It suffices to maximize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x^{2n} + y^{2n} + z^{2n} = 1$. The Lagrange multiplier conditions give $2x = 2\lambda n x^{2n-1}$, $2y = 2\lambda n y^{2n-1}$,

$2z = 2\lambda n z^{2n-1}$, $x^{2n} + y^{2n} + z^{2n} = 1$. As above, assume that x, y, $z \neq 0$; then we can eliminate λ and conclude that $x^{2n-2} = y^{2n-2} = z^{2n-2}$, which implies $x^{2n} = y^{2n} = z^{2n}$; the constraint gives $3x^{2n} = 1$ so $x^2 = 3^{-1/n} \Longrightarrow f(x, y, z) = 3^{\frac{n-1}{n}}$. There are also solutions to the Lagrange conditions with x, y or $z = 0$, but one can check that these do not give larger values of f. Therefore, the extreme points have coordinates x, y, $z = \pm 3^{-\frac{1}{2n}}$.

c. As $n \to \infty$, the coordinates of the extreme points x, y, $z \to \pm 1$, and the limiting distance of these points to the origin is $\sqrt{3}$.

12.9.53 Notice that the constraint is equivalent to $2K+3L = 30$. We have $\nabla f = \frac{1}{2}\langle K^{-1/2}L^{1/2}, K^{1/2}L^{-1/2}\rangle$, so the Lagrange multiplier conditions are equivalent to $K^{-1/2}L^{1/2} = 2\lambda$, $K^{1/2}L^{-1/2} = 3\lambda$, $2K + 3L = 30$. Eliminating λ from the first two equations gives $3K^{-1/2}L^{1/2} = 2K^{1/2}L^{-1/2}$, which simplifies to $3L = 2K$; substituting in the constraint then gives $K = 7.5$ and $L = 5$. The domain over which f is to be maximized is a closed line segment; K or L is 0 at the endpoints, and hence so is f. Therefore, the values of K and L found above must maximize f.

12.9.54 Notice that the constraint is equivalent to $K+2L = 12$. We have $\nabla f = \frac{10}{3}\langle K^{-2/3}L^{2/3}, 2K^{1/3}L^{-1/3}\rangle$, so the Lagrange multiplier conditions are equivalent to $K^{-2/3}L^{2/3} = \lambda$, $2K^{1/3}L^{-1/3} = 2\lambda$, $K + 2L = 12$. Eliminating λ from the first two equations gives $K^{-2/3}L^{2/3} = K^{1/3}L^{-1/3}$, which simplifies to $K = L$; substituting in the constraint then gives $K = L = 4$. The domain over which f is to be maximized is a closed line segment; K or L is 0 at the endpoints, and hence so is f. Therefore, the values of K and L found above must maximize f.

12.9.55 We have $\nabla f = \langle aK^{a-1}L^{1-a}, (1-a)K^aL^{-a}\rangle$, so the Lagrange multiplier conditions are equivalent to $aK^{a-1}L^{1-a} = \lambda p$, $(1-a)K^aL^{-a} = \lambda q$, $pK + qL = B$. Eliminating λ from the first two equations gives $aqK^{a-1}L^{1-a} = (1-a)pK^aL^{-a}$, which simplifies to $aqL = (1-a)pK$; substituting in the constraint then gives $K = \frac{aB}{p}$ and $L = \frac{(1-a)B}{q}$. The domain over which f is to be maximized is a closed line segment; K or L is 0 at the endpoints, and hence so is f. Therefore, the values of K and L found above must maximize f.

12.9.56 We have $\nabla T = \langle 50x, 50y\rangle$ and $\nabla g = \langle 2x+y, 2y+x\rangle$ where $g(x) = x^2+y^2+xy = 1$. The Lagrange multiplier conditions are $50x = \lambda(2x+y)$, $50y = 2y+x$, and $x^2+y^2+xy = 1$. Multiplying the first equation by y and the second by x and subtracting gives $0 = y^2 - x^2$, so $x = \pm y$. If $x = y$, then the constraint gives $3x^2 = 1$, so $x = \pm\frac{1}{\sqrt{3}}$. Note that $T(\pm(1/\sqrt{3}), \pm(1/\sqrt{3})) = 50/3$. If $x = -y$, the constraint gives $x = \pm 1$, and note that $T(1,-1) = T(-1,1) = 50$. The hottest temperature on the edge of the plate is 50 and the coldest is $\frac{50}{3}$.

12.9.57 The function to be maximized is $f(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + x_4$, subject to the constraint $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$. The Lagrange multiplier conditions are $1 = 2\lambda x_1$, $1 = 2\lambda x_2$, $1 = 2\lambda x_3$, $1 = 2\lambda x_4$, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 16$. The first four equations give $x_1 = x_2 = x_3 = x_4$, and then the constraint gives $4x_1^2 = 16$, so $x_1 = \pm 2$. Therefore, the maximum of f on the closed, bounded set given by $x_1^2+x_2^2+x_3^2+x_4^2 = 16$ is $f(2,2,2,2) = 8$ (and the minimum is $f(-2,-2,-2,-2) = -8$).

12.9.58 The function to be maximized is $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$, subject to the constraint $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$. The Lagrange multiplier conditions are $1 = 2\lambda x_1$, $1 = 2\lambda x_2$, $\ldots$, $1 = 2\lambda x_n$, $x_1^2+x_2^2+\cdots+x_n^2 = c^2$. The first n equations give $x_1 = x_2 = \cdots = x_n$, and then the constraint gives $nx_1^2 = c^2$, so $x_1 = \pm\frac{c}{\sqrt{n}}$. Therefore, the maximum of f on the closed, bounded set given by $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$ is $f\left(\frac{c}{\sqrt{n}}, \ldots, \frac{c}{\sqrt{n}}\right) = c\sqrt{n}$ (and the minimum is $-c\sqrt{n}$).

12.9.59 The function to be maximized is $f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$, subject to the constraint $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. The Lagrange multiplier conditions are $a_1 = 2\lambda x_1$, $a_2 = 2\lambda x_2$, $\ldots$, $a_n = 2\lambda x_n$, $x_1^2+x_2^2+\cdots+x_n^2 = 1$. The first n equations are equivalent to $c(x_1, x_2, \ldots, x_n) = (a_1, a_2, \ldots, a_n)$ for some c, and then the constraint gives $c^2 = a_1^2+a_2^2+\cdots+a_n^2$. Therefore, the maximum of f on the closed, bounded set given by $f\left(\frac{a_1}{c}, \frac{a_2}{c}, \ldots, \frac{a_n}{c}\right) = \frac{1}{c}\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$ (and the minimum is $-\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$).

12.9.60

a. The function to be maximized is $f(x,y,z) = xyz$, subject to the constraints $x+y+z=k$ and x, y, $z>0$. The Lagrange multiplier conditions are $yz=\lambda$, $xz=\lambda$, $xy=\lambda$, $x+y+z=k$; the first three equations imply $\frac{xyz}{\lambda}=x=y=z$, and the constraint gives $x=y=z=\frac{k}{3}$. The set determined by the constraints $x+y+z=k$ and x, y, $z\geq 0$ is a triangle, and $f(x,y,z)=0$ along its edges. Therefore, the maximum of f must occur at the point we found. This means that if x, y, $z>0$ and $x+y+z=k$, then $xyz\leq\left(\frac{k}{3}\right)^3$ if and only if $(xyz)^{1/3}\leq\frac{x+y+z}{3}$.

b. Generalizing the case above, the function to be maximized is $f(x_1,\ldots,x_n)=x_1\cdots x_n$, subject to the constraints $x_1+\cdots+x_n=k$ and $x_1,\ldots,x_n>0$. The Lagrange multiplier conditions are $x_2\cdots x_n=\lambda$, $x_1x_3\cdots x_n=\lambda,\ldots,$ $x_2\cdots x_{n-1}=\lambda$, $x_1+\cdots+x_n=k$; the first n equations imply $\frac{x_1\cdots x_n}{\lambda}=x_1=x_2=\cdots=x_n$, and the constraint gives $x_1=x_2=\cdots=x_n=\frac{k}{n}$. The set determined by the constraints $x_1+\cdots+x_n=k$ and $x_1,\ldots,x_n\geq 0$ is closed and bounded, and $f(x_1,\ldots,x_n)=0$ along its boundary. Therefore, the maximum of f must occur at the point we found. This means that if $x_1,\ldots,x_n>0$ and $x_1+\cdots+x_n=k$, then $x_1\cdots x_n\leq\left(\frac{k}{n}\right)^n\iff(x_1\cdots x_n)^{1/n}\leq\frac{x_1+\cdots+x_n}{k}$.

12.9.61

a. Gradients are perpendicular to level surfaces.

b. If ∇f was not in the plane spanned by ∇g and ∇h, then f could be increased or decreased by moving the point P slightly along the curve C.

c. Because ∇f is in the plane spanned by ∇g and ∇h, we can express ∇f as a linear combination of ∇g and ∇h.

d. The gradient condition from part (c), as well as the constraints, must be satisfied.

12.9.62 It suffices to minimize the function $f(x,y,z)=x^2+y^2+z^2$ subject to the constraints $g(x,y,z)=x+2z-12=0$ and $h(x,y,z)=x+y-6=0$. Using the method described in problem 61 above, we solve the equation $\nabla f=\lambda\nabla g+\mu\nabla h$, together with the constraints. This gives the conditions $2x=\lambda+\mu$, $2y=\mu$, $2z=2\lambda$, $x+2z=12$, $x+y=6$. The first three equations imply $2x=2y+z$, and the constraints give $y=6-x$, $z=6-\frac{1}{2}x$; therefore $2x=2(6-x)+6-\frac{1}{2}x$, so $x=4$, which gives $y=2$, $z=4$. There is a point on the line closest to the origin, so this point must be $(4,2,4)$.

12.9.63 We wish to find the extreme values of the function $f(x,y,z)=xyz$ subject to the constraints $g(x,y,z)=x^2+y^2-4=0$ and $h(x,y,z)=x+y+z-1=0$. Using the method described in problem 61 above, we solve the equation $\nabla f=\lambda\nabla g+\mu\nabla h$, together with the constraints. This gives the conditions $yz=2\lambda x+\mu$, $xz=2\lambda y+\mu$, $xy=\mu$, $x^2+y^2=4$, $x+y+z=1$. The first and second equations together give $z(y-x)=2\lambda(x-y)$, which implies either $x=y$ or $z=-2\lambda$. In the former case the first constraint gives $2x^2=4$, so $x=y=\pm\sqrt{2}$, solving for z gives $z=1\mp 2\sqrt{2}$ and $f(x,y,z)=2\pm 4\sqrt{2}$. In the latter case we obtain $(x+y)z=\mu=xy$ from the first and third equations, and then solving for $z=1-x-y$ gives $(x+y)(1-x-y)-xy=0\iff x+y-3xy-4=0$, using the relation $x^2+y^2=4$. Therefore, $x-4=(3x-1)\,y\,(x-4)^2=(3x-1)^2\left(4-x^2\right)x^2-8x+16=\left(9x^2-6x+1\right)\left(4-x^2\right)$ which simplifies to $9x^4-6x^3-34x^2+16x+12=0$. This equation has roots $x\approx -0.42,-1.78,1.96,0.91$ in the interval $(-2,2)$; one can check that the corresponding solutions (x,y,z) to the Lagrange conditions do not give values larger or smaller resp. than $2+4\sqrt{2}$, $2-4\sqrt{2}$, so these are in fact the maximum and minimum values of f subject to the constraints.

12.9.64 We wish to find the extreme values of the function $f(x,y,z)=x^2+y^2+z^2$ subject to the constraints $g(x,y,z)=x^2+2y^2-z+1=0$ and $h(x,y,z)=x-y+2z-4=0$. Using the method described in problem 61 above, we solve the equation $\nabla f=\lambda\nabla g+\mu\nabla h$, together with the constraints. This gives the conditions $2x=2\lambda x+\mu$, $2y=4\lambda y-\mu$, $2z=-\lambda+2\mu$, $x^2+2y^2-z=-1$, $x-y+2z=4$. We can use the second and third equations to solve for λ and μ in terms of x, y, z; then the first equation reduces to $4xy-2xz-4yz-x-y=0$. Next, we use the second constraint equation to express $z=2-\frac{x}{2}+\frac{y}{2}$, which gives $4xy-x-y=(2x+4y)z=(x+2y)(4-x+y)$, or $x^2+5xy-2y^2-5x-9y=0$. Substitute the equation $z = 2 - \frac{x}{2} + \frac{y}{2}$ in the first constraint equation to obtain $2x^2 + 4y^2 + x - y = 2$, and double the first of these two equations and add to obtain $4x^2 + 10xy - 9x - 19y = 2 \implies y = \frac{-4x^2+9x+2}{10x-19}$. We can also solve $2x^2 + 4y^2 + x - y = 2$ for y using the quadratic formula to obtain $y = \frac{1 \pm \sqrt{-32x^2-16x+33}}{8}$. Using a numerical solver, we find that the equations $\frac{-4x^2+9x+2}{10x-19} - \frac{1 \pm \sqrt{-32x^2-16x+33}}{8} = 0$ have approximate solutions $x \approx -1.1814, 0.4982$; the corresponding points on the curve are $\approx (-1.1814, 0.4613, 2.3600)$, $(0.4982, -0.3917, 1.9468)$, which have distances to the origin $\approx 2.3235, 2.0473$ respectively.

12.9.65 We wish to find the extreme values of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $g(x, y, z) = 4x^2 + 4y^2 - z^2 = 0$ and $h(x, y, z) = 2x + 4z - 5 = 0$. Using the method described in problem 61 above, we solve the equation $\nabla f = \lambda \nabla g + \mu \nabla h$, together with the constraints. This gives the conditions $2x = 8\lambda x + 2\mu$, $2y = 8\lambda y$, $2z = -2\lambda z + 4\mu$, $z^2 = 4x^2 + 4y^2$, $2x + 4z = 5$. The second equation gives $y(1 - 4\lambda) = 0$, so either $y = 0$ or $\lambda = \frac{1}{4}$. Consider first the case $y = 0$; then the first constraint equation gives $z = \pm 2x$. If $z = 2x$ then the second constraint equation gives $x = \frac{1}{2}$, $z = 1$ and we obtain the point $\left(\frac{1}{2}, 0, 1\right)$; similarly if $z = -2x$ then we obtain the point $\left(-\frac{5}{6}, 0, \frac{5}{3}\right)$. In the case $\lambda = \frac{1}{4}$ the first equation gives $\mu = 0$ and then the third equation gives $z = 0$; but then the first of the constraints implies that $x = y = z = 0$, which violates the second constraint. Hence there are no solutions to the Lagrange conditions in this case, and the minimum and maximum values of the function f along this curve are $f\left(\frac{1}{2}, 0, 1\right) = \frac{5}{4}$ and $f\left(-\frac{5}{6}, 0, \frac{5}{3}\right) = \frac{125}{36}$.

Chapter Twelve Review

1

a. False. This equation describes a plane in R^3.

b. False. If $2x^2 - 6y^2 > 0$ then $z = \sqrt{2x^2 - 6y^2}$ or $z = -\sqrt{2x^2 - 6y^2}$.

c. False. For example $f(x, y) = x^2 y$ has $f_{xxy} = 2$, $f_{xyy} = 0$.

d. False. ∇f lies in the xy-plane.

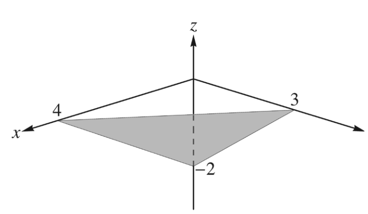
e. True. A normal vector for an orthogonal plane can be found by taking the cross product of normal vectors for the two intersecting planes.

2

a. This plane has equation $3(x - 6) + 4y - 6(z - 1) = 0$, which simplifies to $3x + 4y - 6z = 12$.

b. The x-intercept is found by setting $y = z = 0$ and solving $3x = 12$ to obtain $x = 4$. Similarly, the y and z-intercepts are $y = 3$ and $z = -2$.

c.

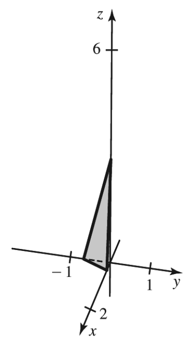


3

a. Let $P = (0, 0, 3)$, $Q = (1, 0, -6)$ and $R = (1, 2, 3)$. Then the vectors $\overrightarrow{PQ} = \langle 1, 0, -9 \rangle$ and $\overrightarrow{PR} = \langle 1, 2, 0 \rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -9 \\ 1 & 2 & 0 \end{vmatrix} = 18\mathbf{i} - 9\mathbf{j} + 2\mathbf{k}$ is normal to the plane. The plane has equation $18x - 9y + 2(z - 3) = 0$, which simplifies to $18x - 9y + 2z = 6$.

b. The x-intercept is found by setting $y = z = 0$ and solving $18x = 6$ to obtain $x = \frac{1}{3}$. Similarly, the y and z-intercepts are $y = -\frac{2}{3}$ and $z = 3$.

c.



4 First, note that the vectors normal to the planes, $\mathbf{n}_Q = \langle 2, 1, -1\rangle$and $\mathbf{n}_R = \langle -1, 1, 1\rangle$, are not multiples of each other; therefore, these planes are not parallel and they intersect in a line ℓ. We need to find a point on ℓ and a vector in the direction of ℓ. Setting $x = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the yz plane: $y - z = 0$, $y + z = 1$. Solving these equations simultaneously gives $y = z = \frac{1}{2}$, so $\left(0, \frac{1}{2}, \frac{1}{2}\right)$ is a point on ℓ. A vector in the direction of ℓ is $\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} =$ $2\mathbf{i} - \mathbf{j} + 3\mathbf{k} = \langle 2, -1, 3\rangle$. Therefore ℓ has equation $\mathbf{r}(t) = \left\langle 0, \frac{1}{2}, \frac{1}{2} \right\rangle + t\langle 2, -1, 3\rangle = \left\langle 2t, \frac{1}{2} - t, \frac{1}{2} + 3t \right\rangle$, or $x = 2t$, $y = \frac{1}{2} - t$, $z = \frac{1}{2} + 3t$.

5 First, note that the vectors normal to the planes, $\mathrm{n}_Q = \langle -3, 1, 2\rangle$ and $\mathrm{n}_R = \langle 3, 3, 4\rangle$ are not multiples of each other; therefore, these planes are not parallel and they intersect in a line ℓ. We need to find a point on ℓ and a vector in the direction of ℓ. Setting $x = 0$ in the equations of the planes gives equations of the lines in which the planes intersect the yz plane: $y + 2z = 0$, $3y + 4z = 12$. Solving these equations simultaneously gives $y = 12$, $z = -6$, so $(0, 12, -6)$ is a point on ℓ. A vector in the direction of ℓ is $\mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 2 \\ 3 & 3 & 4 \end{vmatrix} =$ $-2\mathbf{i} + 18\mathbf{j} - 12\mathbf{k} = -2\langle 1, -9, 6\rangle$. Therefore, ℓ has equation $\mathbf{r}(t) = \langle 0, 12, -6\rangle + t\langle 1, -9, 6\rangle = \langle t, 12 - 9t, -6 + 6t\rangle$, or $x = t$, $y = 12 - 9t$, $z = -6 + 6t$.

6 The line has direction $\mathbf{v} = \langle 1, 3, -3\rangle$, so the desired plane has equation $1(x - 2) + 3(y + 3) - 3(z - 1) = 0$, which simplifies to $x + 3y - 3z = -10$.

7 Let $P = (-2, 3, 1)$, $Q = (1, 1, 0)$ and $R = (-1, 0, 1)$. Then the vectors $\overrightarrow{PQ} = \langle 3, -2, -1\rangle$ and $\overrightarrow{PR} =$ $\langle 1, -3, 0\rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -1 \\ 1 & -3 & 0 \end{vmatrix} = -(3\mathbf{i} + \mathbf{j} + 7\mathbf{k})$ is normal to the plane. The plane has equation $3(x + 2) + 1(y - 3) + 7(z - 1) = 0$, which simplifies to $3x + y + 7z = 4$.

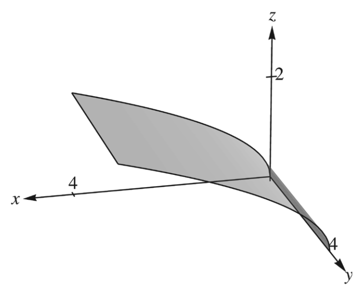
8

a. This surface is a cylinder consisting of lines parallel to the y-axis passing through the curve $z = \sqrt{x}$ in the xz-plane.

b. The xy-trace is found by setting $z = 0$ in the equation $z = \sqrt{x}$, which gives all points $(0, y, 0)$ (the y-axis). Similarly, we see that the xz-trace is the curve $z = \sqrt{x}$ and the yz-trace is the line $z = 0$.

c. The x-intercept is found by setting $y = z = 0$ in the equation $z = \sqrt{x}$, which gives the point $(0, 0, 0)$. The y-intercepts consist of all points on the y-axis, and the z-intercept is the point $(0, 0, 0)$.

d.



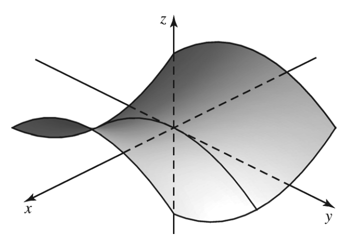
9

a. This surface is a hyperbolic paraboloid.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives $y = \pm 2x$ (two lines intersecting at the origin). Similarly, we see that the xz-trace is the parabola $z = \frac{x^2}{36}$ and the yz-trace is the parabola $z = -\frac{y^2}{144}$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y- and z-intercepts are also $(0, 0, 0)$.

d.



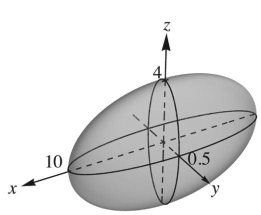
10

a. This surface is an ellipsoid.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the ellipse $\frac{x^2}{100} + 4y^2 = 1$. Similarly, we see that the xz-trace is the ellipse $\frac{x^2}{100} + \frac{z^2}{16} = 1$, and the yz-trace is the ellipse $4y^2 + \frac{z^2}{16} = 1$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the points $(\pm 10, 0, 0)$. Similarly, we see that the y-intercepts are the points $\left(0, \pm\frac{1}{2}, 0\right)$, and the z-intercepts are the points $(0, 0, \pm 4)$.

d.



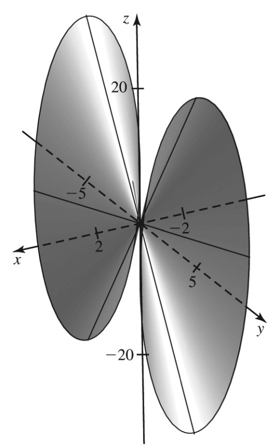
11

a. This surface is an elliptic cone.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives $y = \pm 2x$ (two lines intersecting at the origin). Similarly, we see that the xz-trace is $(0, 0, 0)$ and the yz-trace is $z = \pm 5y$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y- and z-intercepts are also $(0, 0, 0)$.

d.



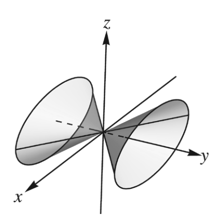
12

a. This surface is an elliptic cone.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives $y = \pm \frac{2x}{3}$ (two lines intersecting at the origin). Similarly, we see that the xz-trace is $(0, 0, 0)$, and the yz-trace is $y = \pm \frac{3z}{2}$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y- and z-intercepts are also $(0, 0, 0)$.

d.



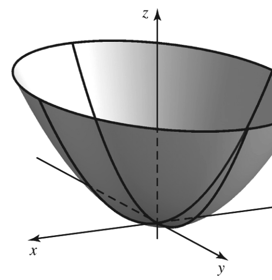
13

a. This surface is an elliptic paraboloid.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives $(0, 0, 0)$. Similarly, we see that the xz-trace is the parabola $z = \frac{x^2}{16}$, and the yz-trace is the parabola $z = \frac{y^2}{36}$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y- and z-intercepts are also $(0, 0, 0)$.

d.



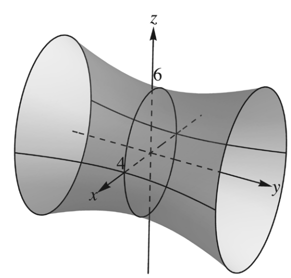
14

a. This surface is a hyperboloid of one sheet.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the hyperbola $\frac{x^2}{16} - \frac{y^2}{100} = 1$. Similarly, we see that the xz-trace is the ellipse $\frac{x^2}{16} + \frac{z^2}{36} = 1$, and the yz-trace is the hyperbola $\frac{z^2}{36} - \frac{y^2}{100} = 1$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the points $(\pm 4,\ 0,\ 0)$. Similarly, we see that there is no y-intercept, and the z-intercepts are $(0, 0, \pm 6)$.

d.



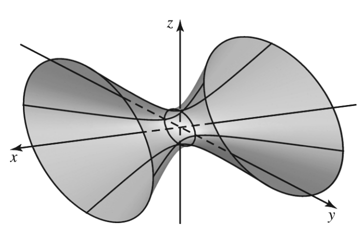
15

a. This surface is a hyperboloid of one sheet.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the hyperbola $y^2 - 2x^2 = 1$. Similarly, we see that the xz-trace is the hyperbola $4z^2 - 2x^2 = 1$, and the yz-trace is the ellipse $y^2 + 4z^2 = 1$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives no solutions. Similarly, we see that the y-intercepts are $(0, \pm 1, 0)$, and the z-intercepts are $\left(0, 0, \pm\frac{1}{2}\right)$.

d.



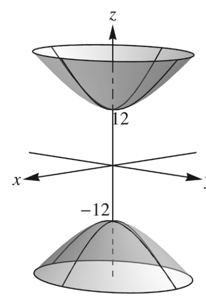
16

a. This surface is a hyperboloid of two sheets.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives no solutions. Similarly, we see that the xz-trace is the hyperbola $-\frac{x^2}{16} + \frac{z^2}{36} = 4$, and the yz-trace is the hyperbola $\frac{z^2}{36} - \frac{y^2}{25} = 4$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives no solutions. Similarly, we see that there is no y-intercept, and the z-intercepts are $(0, 0, \pm 12)$.

d.



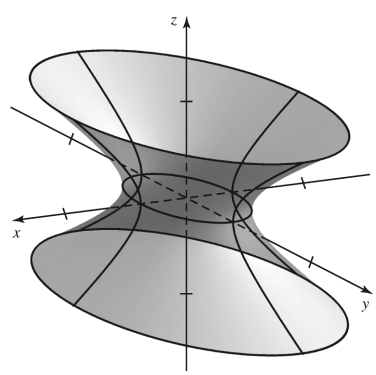
17

a. This surface is a hyperboloid of one sheet.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 4$. Similarly, we see that the xz-trace is the hyperbola $\frac{x^2}{4} - z^2 = 4$, and the yz-trace is the hyperbola $\frac{y^2}{16} - z^2 = 4$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the points $(\pm 4, 0, 0)$. Similarly, we see that the y-intercepts are $(0, \pm 8, 0)$, and there are no z-intercepts.

d.



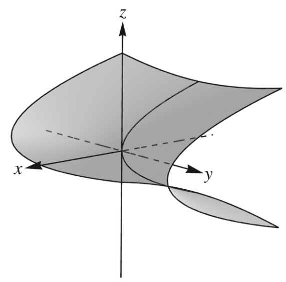
18

a. This surface is a hyperbolic paraboloid.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the parabola $x = \frac{y^2}{64}$. Similarly, we see that the xz-trace is the parabola $x = -\frac{z^2}{9}$ and the yz-trace is the $y = \pm\frac{8z}{3}$ (two lines intersecting at the origin).

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y- and z-intercepts are also $(0, 0, 0)$.

d.



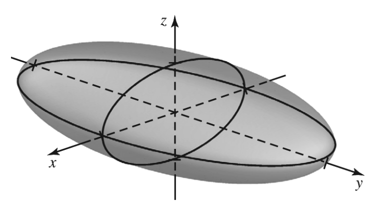
19

a. This surface is an ellipsoid.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 4$. Similarly, we see that the xz-trace is the ellipse $\frac{x^2}{4} + z^2 = 4$, and the yz-trace is the ellipse $\frac{y^2}{16} + z^2 = 4$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the points $(\pm 4, 0, 0)$. Similarly, we see that the y-intercepts are the points $(0, \pm 8, 0)$, and the z-intercepts are the points $(0, 0, \pm 2)$.

d.



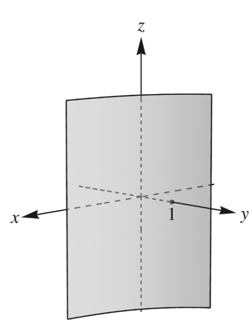
20

a. This surface is a cylinder consisting of lines parallel to the z-axis passing through the curve $y = e^{-x}$ in the xy-plane.

b. The xy-trace is found by setting $z = 0$ in the equation $y = e^{-x}$, which gives the curve $y = e^{-x}$. Similarly, we see that there is no xz-trace, and the yz-trace is the line consisting of all points $(0, 1, z)$.

c. The x-intercept is found by setting $y = z = 0$ in the equation $y = e^{-x}$, which has no solutions. Similarly, we see that the y-intercept is the point $(0, 1, 0)$, and there are no z-intercepts.

d.



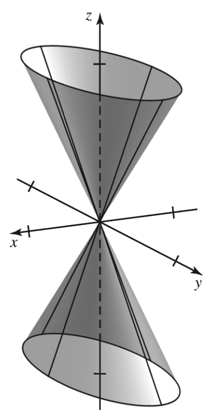
21

a. This surface is an elliptic cone.

b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the origin $(0, 0, 0)$. Similarly, we see that the xz-trace is $z = \pm\frac{8x}{3}$ (two lines intersecting at the origin), and the yz-trace is $y = \pm\frac{7z}{8}$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y and z-intercepts are also $(0, 0, 0)$.

d.



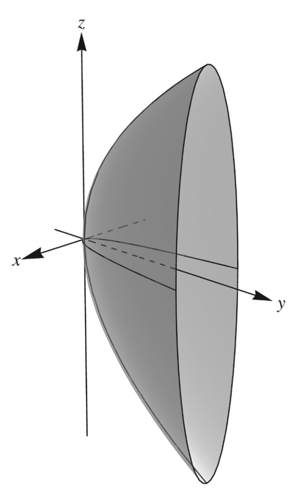
22

a. This surface is an elliptic paraboloid.

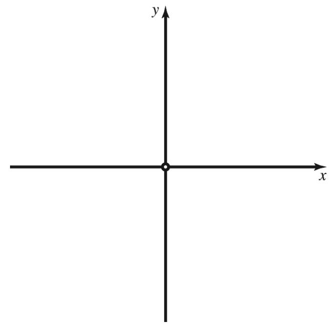
b. The xy-trace is found by setting $z = 0$ in the equation of the surface, which gives the parabola $y = 4x^2$. Similarly, we see that the xz-trace is the origin, and the yz-trace is the parabola $y = \frac{z^2}{9}$.

c. The x-intercept is found by setting $y = z = 0$ in the equation of the surface, which gives the point $(0, 0, 0)$. Similarly, we see that the y- and z-intercepts are also $(0, 0, 0)$.

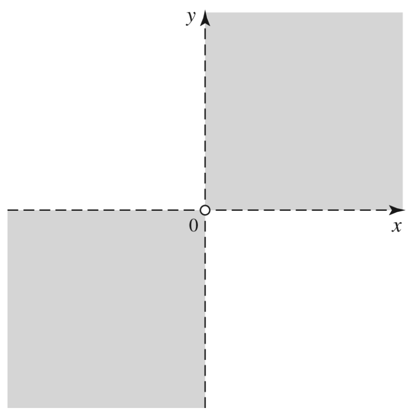
d.



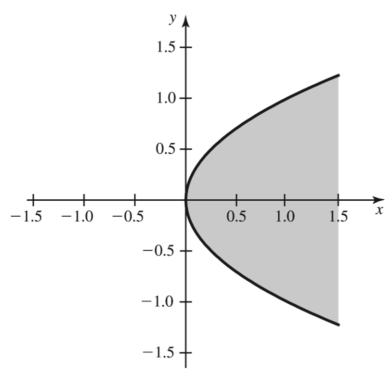
23 The domain is $D = \{(x, y) : (x, y) \neq (0, 0)\}$.



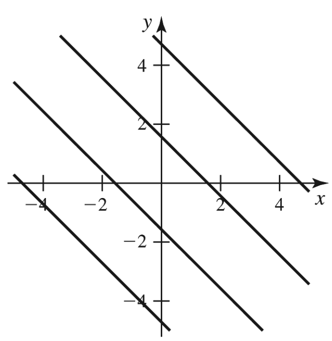
24 The domain is $D = \{(x, y) : xy > 0\}$.



25 The domain is D=$\{(x, y) : x \geq y^2\}$.



26 $D = \{(x, y) : x + y \neq \frac{\pi}{2} + k\pi\}$ where k is any integer.



27

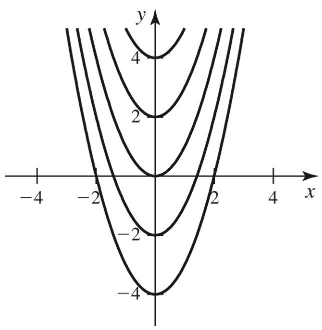
a. The graph of this function is part of a hyperboloid of two sheets and contains the origin, which matches A.

b. The graph of this function is a cylinder, which matches D.

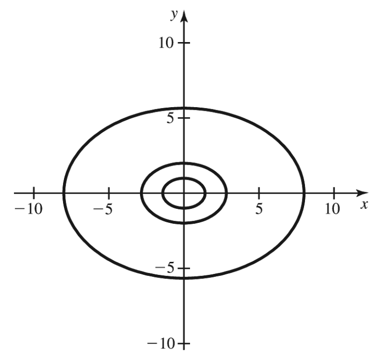
c. The graph of this function is a hyperbolic paraboloid, which matches C.

d. The graph of this function is part of a hyperboloid of one sheet, which matches B.

28



29



30

a. This matches A.

b. This matches C.

c. This matches D.

d. This matches B.

31 This limit may be evaluated directly by substitution: $\lim_{(x,y)\to(4,-2)}(10x-5y+6xy) = 10\cdot4-5\,(-2)+6\cdot4\,(-2) = 2$.

32 This limit may be evaluated directly by substitution: $\lim_{(x,y)\to(1,1)}\frac{xy}{x+y} = \frac{1\cdot1}{1+1} = \frac{1}{2}$.

33 Along the path $y=-x$ we have $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)}\frac{0}{-x^2} = 0$, but along the path $y=x$ we have $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)}\frac{2x}{x^2} = \lim_{(x,y)\to(0,0)}\frac{2}{x} \neq 0$, so the limit does not exist.

34 Along the path $y=x$ we have $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)}\frac{\sin(x^2)}{2x^2} = \frac{1}{2}$, while along the line $y=-x$ we have $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)}\frac{\sin(-x^2)}{2x^2} = \frac{-1}{2}\lim_{(x,y)\to(0,0)}\frac{\sin(x^2)}{x^2} = \frac{-1}{2}$, so the limit does not exist.

35 This limit may be evaluated by factoring the numerator and denominator and canceling the common factor $x+y$: $\lim_{(x,y)\to(-1,1)}\frac{x^2-y^2}{x^2-xy-2y^2} = \lim_{(x,y)\to(-1,1)}\frac{(x-y)(x+y)}{(x-2y)(x+y)} = \lim_{(x,y)\to(-1,1)}\frac{x-y}{x-2y} = \frac{2}{3}$.

36 This limit may be evaluated directly by substitution: $\lim_{(x,y)\to(1,2)}\frac{x^2y}{x^4+2y^2} = \frac{1\cdot2}{1+2\cdot4} = \frac{2}{9}$.

37 This limit may be evaluated directly by substitution: $\lim_{(x,y,z)\to\left(\frac{\pi}{2},0,\frac{\pi}{2}\right)} 4\cos y\sin\sqrt{xz} = 4(\cos 0)(\sin\left(\frac{\pi}{2}\right) = 4$.

38 This limit may be evaluated directly by substitution: $\lim_{(x,y,z)\to(5,2,-3)}\tan^{-1}\left(\frac{x+y^2}{z^2}\right) = \tan^{-1}1 = \frac{\pi}{4}$.

39 $f_x = 6xy^5$, $f_y = 15x^2y^4$.

40 $g_x = 4yz^2 - 3/y$, $g_y = 4xz^2 + 3x/y^2$, $g_z = 8xyz$.

41 $f_x = \frac{(x^2+y^2)(2x)-x^2(2x)}{(x^2+y^2)^2} = \frac{2xy^2}{(x^2+y^2)^2}$. $f_y = \frac{0-x^2(2y)}{(x^2+y^2)^2} = \frac{-2x^2y}{(x^2+y^2)^2}$.

42 $g_x = \frac{(x+y)yz-(xyz)}{(x+y)^2} = \frac{y^2z}{(x+y)^2}$. $g_y = \frac{(x+y)xz-(xyz)}{(x+y)^2} = \frac{x^2z}{(x+y)^2}$, $g_z = \frac{xy}{x+y}$.

43 $\frac{\partial}{\partial x}\left[xye^{xy}\right] = ye^{xy} + xy^2e^{xy} = y\left(1+xy\right)e^{xy}$, $\frac{\partial}{\partial y}\left[xye^{xy}\right] = xe^{xy} + x^2ye^{xy} = x\left(1+xy\right)e^{xy}$.

44 $\frac{\partial}{\partial u}[u\cos v - v\sin u] = \cos v - v\cos u$, $\frac{\partial}{\partial v}[u\cos v - v\sin u] = -u\sin v - \sin u$.

45 $f_x\left(x,y,z\right) = e^{x+2y+3z}$, $f_y\left(x,y,z\right) = 2e^{x+2y+3z}$, $f_z\left(x,y,z\right) = 3e^{x+2y+3z}$.

46 $H_p\left(p,q,r\right) = 2p\sqrt{q+r}$, $H_q\left(p,q,r\right) = \frac{p^2}{2\sqrt{q+r}}$, $H_r\left(p,q,r\right) = \frac{p^2}{2\sqrt{q+r}}$.

47 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y - 6y = 0$.

48 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2\left(y^2-x^2\right)}{(x^2+y^2)^2} + \frac{2\left(x^2-y^2\right)}{(x^2+y^2)^2} = 0$.

49

a. If r is held fixed and R increases then V increases, so $V_R > 0$, whereas, if R is held fixed and r increases then V decreases, so $V_r < 0$.

b. We have $V_R = 4\pi R^2 > 0$ and $V_r = -4\pi r^2 < 0$, consistent with the predictions in part (a).

c. If $R = 3$, $r = 1$ and R is increased by $\triangle R = 0.1$, then $\triangle V \approx 4\pi \cdot 3^2 \cdot 0.1 = 3.6\pi$; if r is decreased by 0.1 then $\triangle V \approx -4\pi \cdot 1^2 \cdot (-0.1) = 0.4\pi$. Therefore, the volume changes more if R is increased.

50 The chain rule gives $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = (y\sin z)\cdot 2t + (x\sin z)\cdot 12t^2 + (xy\cos z)\cdot 1 = 20t^4\sin\left(t+1\right) + 4t^5\cos\left(t+1\right)$.

51 The chain rule gives $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \frac{x}{\sqrt{x^2+y^2+z^2}}\cdot\cos t + \frac{y}{\sqrt{x^2+y^2+z^2}}\cdot(-\sin t) + \frac{z}{\sqrt{x^2+y^2+z^2}}\cdot(-\sin t) = -\frac{\cos t\sin t}{\sqrt{1+\cos^2 t}}$.

52 The chain rule gives $w_s = w_x x_s + w_y y_s + w_z z_s = yz\cdot 2t + xz\cdot t^2 + xy\cdot 2st = st^2\cdot s^2t\cdot 2t + 2st\cdot s^2t\cdot t^2 + 2st\cdot st^2\cdot 2st = 8s^3t^4$ and similarly $w_t = w_x x_t + w_y y_t + w_z z_t = yz\cdot 2s + xz\cdot 2st + xy\cdot s^2 = st^2\cdot s^2t\cdot 2s + 2st\cdot s^2t\cdot 2st + 2st\cdot st^2\cdot s^2 = 8s^4t^3$

53 The chain rule gives $w_r = w_x x_r + w_y y_r = \frac{1}{x}(st) + \frac{2}{y}(1) = \frac{1}{r} + \frac{2}{r+s} = \frac{3r+s}{r(r+s)}$. $w_s = w_x x_s + w_y y_s = \frac{1}{x}(rt) + \frac{2}{y}(1) = \frac{1}{s} + \frac{2}{r+s} = \frac{r+3s}{s(r+s)}$. $w_t = w_x x_t + w_y y_t = \frac{1}{x}(rs) + \frac{2}{y}\cdot 0 = \frac{1}{t}$.

54 Let $F\left(x,y\right) = 2x^2 + 3xy - 3y^4 - 2$; then if y is determined by $F\left(x,y\right) = 0$, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{4x+3y}{3x-12y^3} = \frac{4x+3y}{12y^3-3x}$.

55 Let $F\left(x,y\right) = y\ln\left(x^2+y^2\right) - 4$; then if y is determined by $F\left(x,y\right) = 0$, we have $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{2xy}{x^2+y^2}}{\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}} = -\frac{2xy}{2y^2+(x^2+y^2)\ln(x^2+y^2)}$.

56

a. The chain rule gives $z'(t) = 8xx'\left(t\right) + 2yy'\left(t\right) = -6\cos t\sin t = -3\sin 2t$.

b. Walking uphill corresponds to $z'(t) > 0$, which occurs when $\frac{\pi}{2} < t < \pi$ and $\frac{3\pi}{2} < t < 2\pi$.

57

a. The chain rule gives $z'(t) = 2xx'(t) - 4yy'(t) = -24\text{cos}t\text{sin}t = -12\text{sin}2t$.

b. Walking uphill corresponds to $z'(t) > 0$, which occurs when $\frac{\pi}{2} < t < \pi$ and $\frac{3\pi}{2} < t < 2\pi$.

58 Observe that $r'(t) = \frac{ar}{t}$ and $h'(t) = -\frac{bh}{t}$; hence, by the chain rule, $V'(t) = \frac{\pi}{3}\left(2rhr'(t) + r^2h'(t)\right)$ $= \frac{\pi r^2 h}{3t}(2a - b)$. Therefore, the volume of the cone remains constant if and only if $2a = b$. (This can also be seen by substituting the formulas for $r(t)$ and $h(t)$ in the formula for the volume of a cone to obtain $V(t) = \frac{\pi}{3}t^{2a-b}$.)

59

a.

	$(a,b) = (0,0)$	$(a,b) = (2,0)$	$(a,b) = (1,1)$
$\mathbf{u} = \langle\sqrt{2}/2, \sqrt{2}/2\rangle$	0	$4\sqrt{2}$	$-2\sqrt{2}$
$\mathbf{v} = \langle-\sqrt{2}/2, \sqrt{2}/2\rangle$	0	$-4\sqrt{2}$	$-6\sqrt{2}$
$\mathbf{w} = \langle-\sqrt{2}/2, -\sqrt{2}/2\rangle$	0	$-4\sqrt{2}$	$2\sqrt{2}$

b. The function is increasing at $(2,0)$ in the direction of $\mathbf{u}$ and decreasing at $(2,0)$ in the directions of $\mathbf{v}$ and $\mathbf{w}$.

60 $\nabla f = \langle 2x, 0\rangle$, so $\nabla f(1,2) = \langle 2, 0\rangle$. $D_{\mathbf{u}}f(1,2) = \langle 2,0\rangle \cdot \mathbf{u} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

61 $\nabla g = \langle 2xy^3, 3x^2y^2\rangle$, so $\nabla g(-1,1) = \langle -2, 3\rangle$. $D_u g(-1,1) = \langle -2,3\rangle \cdot \mathbf{u} = \frac{-10}{13} + \frac{36}{13} = 2$.

62 $\nabla f = \langle 1/y^2, -2x/y^3\rangle$, so $\nabla f(0,3) = \langle 1/9, 0\rangle$. $D_{\mathbf{u}}f(0,3) = \langle 0,3\rangle \cdot \mathbf{u} = \sqrt{3}/18$.

63 The gradient of h is given by $\nabla h(x,y) = h_x\mathbf{i} + h_y\mathbf{j} = \frac{1}{\sqrt{2+x^2+2y^2}}(x\mathbf{i} + 2y\mathbf{j})$; therefore, $\nabla h(2,1) = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$ and the directional derivative in the direction of $\mathbf{u}$ is given by $\nabla h(2,1) \cdot \mathbf{u} = \frac{\sqrt{2}}{10}(\mathbf{i} + \mathbf{j}) \cdot (3\mathbf{i} + 4\mathbf{j}) = \frac{7\sqrt{2}}{10} \approx 0.9899$.

64 The gradient of f is given by $\nabla f(x,y,z) = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$; therefore, $\nabla f(2,-2,1) = -\text{i}+3\text{j}$ and the directional derivative in the direction of $\mathbf{u}$ is given by $\nabla f(2,-2,1) \cdot \mathbf{u} = \frac{1}{\sqrt{2}}(-\mathbf{i} + 3\mathbf{j}) \cdot (-\mathbf{j} - \mathbf{k}) = -\frac{3\sqrt{2}}{2}$.

65 The gradient of f is given by $\nabla f(x,y,z) = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k} = \cos(x + 2y - z)(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$; therefore, $\nabla f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right) = -\frac{1}{2}(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and the directional derivative in the direction of $\mathbf{u}$ is given by $\nabla f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right) \cdot \mathbf{u} = -\frac{1}{6}(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = -\frac{1}{2}$.

66

a. The gradient of f is given by $\nabla f(x,y) = f_x\mathbf{i} + f_y\mathbf{j} = \frac{1}{1+xy}(y\mathbf{i} + x\mathbf{j})$, so $\nabla f(2,3) = \frac{1}{7}(3\mathbf{i} + 2\mathbf{j})$. The direction of steepest ascent is the unit vector in this direction, $\mathbf{u} = \frac{3}{\sqrt{13}}\text{i}+\frac{2}{\sqrt{13}}\text{j}$, and the direction of steepest descent is $-\mathbf{u}$.

b. The unit vectors that point in the direction of no change are $\text{v}= \pm\left(\frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}\right)$, because $\mathbf{u} \cdot \mathbf{v} = 0$.

67

a. The gradient of f is given by $\nabla f(x,y) = f_x\mathbf{i} + f_y\mathbf{j} = -\frac{1}{\sqrt{4-x^2-y^2}}(x\mathbf{i} + y\mathbf{j})$, so $\nabla f(-1,1) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$.

The direction of steepest ascent is the unit vector in this direction, $\mathbf{u} = \frac{\sqrt{2}}{2}\text{i}-\frac{\sqrt{2}}{2}\text{j}$, and the direction of steepest descent is $-\mathbf{u}$.

b. The unit vectors that point in the direction of no change are $\text{v}= \pm\left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}\right)$, because $\mathbf{u} \cdot \mathbf{v} = 0$.

68 If y is determined by $f(x,y)=C$ we have $\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{-4x}{-2y}=\frac{2x}{y}$. Therefore, the level curve $f(x,y)=5$ has slope $m=2$ at the point $(1,1)$, so the tangent line has direction i+2j. The gradient of f at this point is $\nabla f(1,1)=\left.(-4x\mathbf{i}-2y\mathbf{j})\right|_{(1,1)}=-4\mathbf{i}-2\mathbf{j}$, which is perpendicular to the tangent direction.

69 If x is determined by $f(x,y)=C$ we have $\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{-2y}{-4x}=\frac{y}{2x}$. Therefore, the level curve $f(x,y)=0$ has a vertical tangent at the point $(0,0)$, so the tangent line has direction j. The gradient of f at this point is $\nabla f(2,0)=\left.(-4x\mathbf{i}-2y\mathbf{j})\right|_{(2,0)}=-8\mathbf{i}$, which is perpendicular to the tangent direction.

70 The gradient of f at $(1,1)$ is given by $\nabla f(1,1)=\left.(8x\mathbf{i}-2y\mathbf{j})\right|_{(1,1)}=8\mathbf{i}-2\mathbf{j}=2(4\mathbf{i}-\mathbf{j})$, so the unit vectors in the direction of no change are $\mathbf{u}=\pm\frac{1}{\sqrt{17}}(\mathbf{i}+4\mathbf{j})$.

71 Observe that $V=-\frac{k}{2}\left(\ln\left(x^2+y^2\right)-\ln R^2\right)$; therefore $\mathbf{E}=-\nabla V=\frac{k}{2}\left(\frac{2x}{x^2+y^2}\mathbf{i}+\frac{2y}{x^2+y^2}\mathbf{j}\right)=\frac{kx}{x^2+y^2}\mathbf{i}+\frac{ky}{x^2+y^2}\mathbf{j}$.

72 Let $f(x,y,z)=2x^2+y^2-z$. $\nabla f=\langle 4x,2y,-1\rangle$ and $\nabla f(1,1,3)=\langle 4,2,-1\rangle$. The equation of the tangent plane at $(1,1,3)$ is $4(x-1)+2(y-1)-(z-3)=0$, or $4x+2y-z=3$. At the other given point we have $\nabla f(0,2,4)=\langle 0,4,-1\rangle$, so the equation of the tangent plane is $4(y-2)-(z-4)=0$, or $4y-z=4$.

73 Let $f(x,y,z)=x^2+y^2/4-z^2/9$. $\nabla f=\langle 2x,y/2,-2z/9\rangle$, so $\nabla f(0,2,0)=\langle 0,1,0\rangle$. The equation of the tangent plane at $(0,2,0)$ is given by $1(y-2)=0$, or $y=2$. At the point $(1,1,3/2)$ we have $\nabla f=\langle 2,1/2,-1/3\rangle$, so the equation of the tangent plane is $2(x-1)+(1/2)(y-1)+(-1/3)(z-3/2)=0$, or $12x+3y-2z=12$.

74 Let $F(x,y,z)=xy\sin z-1$; then $\nabla F\left(1,2,\frac{\pi}{6}\right)=\left.\langle y\sin z,\ x\sin zz,\ xy\cos z\rangle\right|_{\left(1,2,\frac{\pi}{6}\right)}=\left\langle 1,\frac{1}{2},\sqrt{3}\right\rangle$, so the tangent plane at $\left(1,2,\frac{\pi}{6}\right)$ has equation $(x-1)+\frac{1}{2}(y-2)+\sqrt{3}\left(z-\frac{\pi}{6}\right)=0$, or $6x+3y+6\sqrt{3}z-12-\pi\sqrt{3}=0$. Similarly, $\nabla F\left(-2,-1,\frac{5\pi}{6}\right)=\left.\langle y\sin z,\ x\sin z,\ xy\cos z\rangle\right|_{\left(-2,-1,\frac{5\pi}{6}\right)}=\left\langle -\frac{1}{2},-1,-\sqrt{3}\right\rangle$, so the tangent plane at $\left(-2,-1,\frac{5\pi}{6}\right)$ has equation $-\frac{1}{2}(x+2)-(y+1)-\sqrt{3}\left(z-\frac{5\pi}{6}\right)=0$, or $3x+6y+6\sqrt{3}z+12-5\pi\sqrt{3}=0$.

75 Let $F(x,y,z)=yze^{xz}-8$; then $\nabla F(0,2,4)=\left.\langle yz^2e^{xz},\ ze^{xz},\ (y+xyz)\,e^{xz}\rangle\right|_{(0,2,4)}=\langle 32,4,2\rangle$, so the tangent plane at $(0,2,4)$ has equation $32x+4(y-2)+2(z-4)=0$, or $16x+2y+z-8=0$. Similarly, $\nabla F(0,-8,-1)=\left.\langle yz^2e^{xz},\ ze^{xz},\ (y+xyz)\,e^{xz}\rangle\right|_{(0,-8,-1)}=\langle -8,-1,-8\rangle$, so the tangent plane at $(0,-8,-1)$ has equation $-8x-(y+8)-8(z+1)=0$, or $8x+y+8z+16=0$.

76 Let $f(x,y)=x^2e^{x-y}$; then $f_x(x,y)=\left(2x+x^2\right)e^{x-y}$ and $f_y(x,y)=-x^2e^{x-y}$, so the tangent plane at $(2,2,4)$ has equation $z=f(2,2)+f_x(2,2)(x-2)+f_y(2,2)(y-2)=4+8(x-2)-4(y-2)=8x-4y-4$. Similarly, the tangent plane at $(-1,-1,1)$ has equation $z=f(-1,-1)+f_x(-1,-1)(x+1)+f_y(-1,-1)(y+1)=1-(x+1)-(y+1)=-x-y-1$.

77 Let $f(x,y)=\ln(1+xy)$; then $f_x(x,y)=\frac{y}{1+xy}$ and $f_y(x,y)=\frac{x}{1+xy}$, so the tangent plane at $(1,2,\ln 3)$ has equation $z=f(1,2)+f_x(1,2)(x-1)+f_y(1,2)(y-2)=\ln 3+\frac{2}{3}(x-1)+\frac{1}{3}(y-2)=\frac{2}{3}x+\frac{1}{3}y+\ln 3-\frac{4}{3}$. Similarly, the tangent plane at $(-2,-1,\ln 3)$ has equation $z=f(-2,-1)+f_x(-2,-1)(x+2)+f_y(-2,-1)(y+1)=\ln 3-\frac{1}{3}(x+2)-\frac{2}{3}(y+1)=-\frac{1}{3}x-\frac{2}{3}y+\ln 3-\frac{4}{3}$.

78

a. The linear approximation is given by $L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)=2\sqrt{2}-8\sin\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)+4\sin\left(\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)=-4\sqrt{2}x+2\sqrt{2}y+2\sqrt{2}+\frac{\sqrt{2}\pi}{2}$.

b. This gives the estimate f(0.8, 0.8) $\approx -3.2+2\sqrt{2}+\pi\approx 2.787$ (the actual answer is 2.787 to three decimal places).

79

a. The linear approximation is given by $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = 2 + (x-2) + 5y = x + 5y$.

b. This gives the estimate $f(1.95, 0.05) \approx 2.2$ (the actual answer is 2.205 to three decimal places).

80 We have $\triangle f \approx f_x(1,-2)\triangle x + f_y(1,-2)\triangle y = 4 \cdot 0.05 + 9 \cdot 0.1 = 1.1$ (the actual change is 1.093 to three decimal places).

81 We have $dV = \pi\left(2rhdr + r^2dh\right)$, so $\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$. Therefore, if the radius decreases by 3% and the height increases by 2%, the approximate change in volume is -4%.

82 We have $dV = \pi(bcda + acdb + abdc)$, so $\frac{dV}{V} = \frac{da}{a} + \frac{db}{b} + \frac{dc}{c}$. Therefore, if a, b, c change by 2%, 1.5%, -2.5% respectively, the approximate change in the volume is 1%.

83

a. We have $dV = \pi\left(2rh - h^2\right)dh = 2\pi(-0.05) = -0.1\pi\text{m}^3$ (notice that $r = 1.50$ m is constant, so there is no contribution from the dr term in dV).

b. The surface of the water is a disc with radius $s = \sqrt{2rh - h^2}$, so the surface area is $S = \pi\left(2rh - h^2\right)$. Therefore $dS = 2\pi(r-h)dh = -0.05\pi\text{m}^2$.

84 We have $f_x = 4x^3 - 16y$, $f_y = 4y^3 - 16x$; therefore, the critical points satisfy the equations $y = \frac{x^3}{4}$ and $x = \frac{y^3}{4}$. Eliminating y gives $x^9 = 2^8x$, so $x = 0, \pm 2$, and the critical points are $(0,0)$,$\pm(2,2)$. We also have $f_{xx} = 12x^2$, $f_{yy} = 12y^2$ and $f_{xy} = -16$; hence $D(x,y) = 16\left(9x^2y^2 - 1\right)$. We see that $D(0,0) < 0$ so $(0,0)$ is a saddle. We have $D(2,2) = D(-2,-2) > 0$; $f_{xx}(2,2) = f_{xx}(-2,-2) > 0$, which by the Second Derivative Test implies that f has local minima at $\pm(2,2)$.

85 We have $f_x = x^2 + 2y$, $f_y = -y^2 + 2x$; therefore, the critical points satisfy the equations $y = -\frac{x^2}{2}$ and $x = \frac{y^2}{2}$. Eliminating y gives $x^4 = 8x$ so $x = 0, 2$, and the critical points are $(0,0)$, $(2,-2)$. We also have $f_{xx} = 2x$, $f_{yy} = -2y$ and $f_{xy} = 2$; hence $D(x,y) = -4(1 + xy)$. We see that $D(0,0) < 0$ so $(0,0)$ is a saddle. We have $D(2,-2) > 0$, $f_{xx}(2,-2) > 0$, which by the Second Derivative Test implies that f has a local minimum at $(2,-2)$.

86 We have $f_x = 2xy^2 - 6xy + 2y^2 - 6y$, $f_y = 2x^2y - 3x^2 + 4xy - 6x$; therefore, the critical points satisfy the equations $y(2xy - 6x + 2y - 6) = 0$, $x(2xy - 3x + 4y - 6) = 0$. If $y = 0$ then the second equation gives $x = 0, -2$, and if $x = 0$ then the first equation gives $y = 0, 3$. If both $x, y \neq 0$ then we can divide the equations by x, y respectively and subtract to obtain $3x + 2y = 0$, so $y = -\frac{3x}{2}$; substituting this in the first equation gives $x^2 + 3x + 2 = 0$, which has roots $x = -1, -2$. Therefore, the critical point are $(0,0)$, $(0,3)$, $(-2,0)$, $\left(-1, \frac{3}{2}\right)$ and $(-2,3)$. We also have $f_{xx} = 2y^2 - 6y$, $f_{yy} = 2x^2 + 4x$ and $f_{xy} = 4xy - 6x + 4y - 6$. We see that $D(0,0)$, $D(0,3)$, $D(-2,0)$, and $D(-2,3) < 0$ so $(0,0)$, $(0,3)$, $(-2,0)$and $(-2,3)$ are saddles. We have $D\left(-1, \frac{3}{2}\right) > 0$, $f_{xx}\left(-1, \frac{3}{2}\right) < 0$, which by the Second Derivative Test implies that f has a local maximum at $\left(-1, \frac{3}{2}\right)$.

87 We have $f_x = -3x^2 - 6x$, $f_y = -3y^2 + 6y$; therefore the critical points must have $x = 0, -2$ and $y = 0, 2$. We also have $f_{xx} = -6x - 6$, $f_{yy} = -6y + 6$ and $f_{xy} = 0$; hence $D(x,y) = 36(x+1)(y-1)$. We see that $D(0,0)$, $D(-2,2) < 0$ so $(0,0)$ and $(-2,2)$ are saddles. We have $D(0,2) > 0$, $f_{xx}(0,2) < 0$, which by the Second Derivative Test implies that f has a local maximum at $(0,2)$; $D(-2,0) > 0$, $f_{xx}(-2,0) > 0$, which by the Second Derivative Test implies that f has a local minimum at $(-2,0)$.

88 First we find the critical points of f inside the rectangle R: we have $f_x = x^2 + 2y$, $f_y = -y^2 + 2x$; the equation $f_x = 0$ gives $y = -\frac{x^2}{2}$, and then substituting this in the equation $f_y = 0$ gives $x^4 = 8x$, or $x = 0, 2$. Hence the critical points are $(0,0)$ and $(2,-2)$, neither of which is in the interior of R. Therefore, we must find the maximum and minimum values of f on the boundary of R, which consists of four segments.

On the segment $0 \le x \le 3$, $y = -1$, let $g(x) = f(x, -1) = \frac{x^3}{3} - 2x + \frac{1}{3}$: then g has a critical point at $x = \sqrt{2}$, and find that g has extreme values $\frac{1-4\sqrt{2}}{3}$ and $\frac{10}{3}$ on $[0, 3]$. Similarly, we find that on the segment $0 \le x \le 3$, $y = 1$, $g(x) = f(x, 1) = \frac{x^3}{3} + 2x - \frac{1}{3}$ has extreme values $-\frac{1}{3}$ and $\frac{44}{3}$. On the segment $-1 \le y \le 1$, $x = 0$, let $h(y) = f(0, y) = -\frac{y^3}{3}$: then h has extreme values $\pm\frac{1}{3}$ on this segment. Similarly, we find that on the segment $-1 \le y \le 1$, $x = 3$, $h(y) = f(3, y) = -\frac{y^3}{3} + 6y + 9$ has a critical point at $y = \sqrt{6}$, and h has extreme values $\frac{10}{3}$ and $4\sqrt{6} + 9$. Therefore the absolute minimum and maximum values of f on R are $f\left(\sqrt{2}, 1\right) = \frac{1-4\sqrt{2}}{3} \approx -0.886$ and $f\left(3, \sqrt{6}\right) = 4\sqrt{6} + 9 \approx 18.798$.

89 First we find the critical points of f inside the square R: we have $f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$; the equation $f_x = 0$ gives $y = x^3$, and then substituting this in the equation $f_y = 0$ gives $x^9 = x$, or $x = 0, \pm 1$. Hence, the critical points are $(0, 0)$ and $\pm(1, 1)$, which are all in the interior of R. We observe that the values of f at these points are $f(0, 0) = 1$, $f(1, 1) = f(-1, -1) = -1$. Next we must find the maximum and minimum values of f on the boundary of R, which consists of four segments. On the segment $-2 \le x \le 2$, $y = 2$, let $g(x) = f(x, 2) = x^4 - 8x + 17$: then g has a critical point at $x = \sqrt[3]{2}$, and find that g has extreme values $17 - 6\sqrt[3]{2}$ and 49 on $[0, 3]$. We also note that $f(y, x) = f(-x, -y) = f(x, y)$; therefore f takes the same values on all four segments of the square. Hence, the absolute minimum and maximum values of f on R are $f(1, 1) = f(-1, -1) = -1$ and $f(2, -2) = f(-2, 2) = 49$.

90 First we find the critical points of f inside the triangle. We have $f_x = 2xy$ and $f_y = x^2 - 3y^2$, so there are no critical points inside the triangle. Now consider the boundary triangle. On the vertical side $x = 0$, we have $f(x, y) = -y^3$ which has a maximum for $0 \le y \le 2$ of 0 and a minimum of -8. On the horizontal side $y = 0$, we have $f(x, y) = 0$, so the minimum and maximum are both 0. On the side $y = 2 - x$, we have $f(x, y) = g(x) = 2x^2 - x^3 - (2 - x)^3$. Note that $g'(x) = 4x - 3x^2 + 3(2 - x)^2 = 12 - 8x$, which is zero for $x = 3/2$. The value of g at $x = 3/2$ is $g(3/2) = 1$. Thus, the absolute maximum of f is 1 at $(3/2, 1/2)$ and the absolute minimum is -8 at $(0, 2)$.

91 First we find the critical points of f inside the semicircle. We have $f_x = y$ and $f_y = x$, so the only critical point is $(0, 0)$, where the value of f is 0. On the flat part of the semicircle where $y = 0$, the value of f is also 0. Now we parametrize the circular part of the boundary by letting $x = \cos t$, $y = \sin t$ for $0 \le t \le \pi$. Then $f(x, y) = g(t) = \cos t \sin t$, and $g'(t) = -\sin^2 t + \cos^2 t$. This is zero for $t = \pi/4$ and $t = 3\pi/4$, where the value of f is $1/2$ and $-1/2$, respectively. Thus, the absolute maximum of f is $1/2$ at $(\sqrt{2}/2, \sqrt{2}/2)$ and the absolute minimum is $-1/2$ at $(-\sqrt{2}/2, \sqrt{2}/2)$.

92 First, observe that there is a unique point on any plane that is closest to the origin. To find this point, it suffices to minimize the function $f(y, z) = x^2 + y^2 + z^2 = (8 - y - 4z)^2 + y^2 + z^2$ over all points (y, z). We have $f_y = 2(y + 4z - 8) + 2y$, $f_z = 2 \cdot 4(4z + y - 8) + 2z$ and setting $f_y = f_z = 0$ gives the equations $y + 2z = 4$, $4y + 17z = 32$; solving these simultaneously gives $y = \frac{4}{9}$, $z = \frac{16}{9}$ and hence $x = \frac{4}{9}$; therefore, the closest point is $\left(\frac{4}{9}, \frac{4}{9}, \frac{16}{9}\right)$.

93 We have $\nabla f = \langle 2, 1\rangle$ and $\nabla g = \langle 4(x - 1), 8(y - 1)\rangle$, where $g(x, y) = 2(x - 1)^2 + 4(y - 1)^2$. The Lagrange equations are thus $2 = 4\lambda(x - 1)$, $1 = 8\lambda(y - 1)$, and $2(x - 1)^2 + 4(y - 1)^2 = 1$. Solving gives $\lambda = 3/4$, $x = 5/3$, and $y = 7/6$, or $\lambda = -3/4$, $x = 1/3$, and $y = 5/6$. The maximum for f is $f(5/3, 7/6) = 29/2$ and the minimum for f is $f(1/3, 5/6) = 23/2$.

94 The Lagrange multiplier conditions are $2xy^2 = 4\lambda x$, $2x^2y = 2\lambda y$, $2x^2 + y^2 = 1$. Hence, $xy^3 = 2\lambda xy = 2x^3y \Longrightarrow xy\left(y^2 - 2x^2\right) = 0$; combining this with the constraint gives solutions $(0, \pm 1)$, $\left(\pm\frac{\sqrt{2}}{2}, 0\right)$, $\left(\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right)$. Comparing the values of f at these points, we see that the extreme values of f on the closed bounded set given by $2x^2 + y^2 = 1$ are $f(0, \pm 1) = f\left(\pm\frac{\sqrt{2}}{2}, 0\right) = 0$, $f\left(\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right) = f\left(-\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right) = \frac{1}{8}$.

95 The Lagrange multiplier conditions are $1 = 2\lambda x$, $2 = 2\lambda y$, $-1 = 2\lambda z$, $x^2 + y^2 + z^2 = 1$. Hence $2\lambda = \frac{1}{x} = \frac{2}{y} = -\frac{1}{z} \Longrightarrow y = 2x$, $z = -x$; substituting these in the constraint gives $6x^2 = 1$ so $x = \pm\frac{\sqrt{6}}{6}$ and we obtain solutions $\pm\left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}\right)$. Therefore the extreme values of f on the closed bounded set given by $x^2 + y^2 + z^2 = 1$ are $f\left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}\right) = \sqrt{6}$, $f\left(-\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right) = -\sqrt{6}$.

96 The Lagrange multiplier conditions are $2xy^2z = 4\lambda x$, $2x^2yz = 2\lambda y$, $x^2y^2 = 2\lambda z$, $2x^2 + y^2 + z^2 = 25$. If x, y or $z = 0$ then $f(x, y, z) = 0$, which is neither the maximum or minimum of f on the ellipsoid given by the constraint (f takes both positive and negative values on this set). Hence, we can assume x, y, $z \neq 0$ in the Lagrange conditions, and eliminating λ in the first three equations gives $\lambda = \frac{y^2z}{2} = x^2z = \frac{x^2y^2}{2z} \Longrightarrow y^2 = 2z^2 = 2x^2$; substituting this in the constraint gives $5x^2 = 25$, so $x = \pm\sqrt{5}$, $y = \pm\sqrt{10}$ and $z = \pm\sqrt{5}$. Therefore the maximum value of f on the closed bounded set given by $2x^2 + y^2 + z^2 = 25$ is $f\left(\pm\sqrt{5}, \pm\sqrt{10}, \sqrt{5}\right) = 50\sqrt{5}$, and the minimum value is $f\left(\pm\sqrt{5}, \pm\sqrt{10}, -\sqrt{5}\right) = -50\sqrt{5}$.

97 Let (x, y) be the corner of the rectangle in the first quadrant; then the perimeter of the rectangle is $4(x + y)$, so it suffices to find the maximum value of $x + y$ subject to the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The Lagrange multiplier conditions are $1 = \frac{2\lambda x}{a^2}$, $1 = \frac{2\lambda y}{b^2}$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Hence, $2\lambda = \frac{a^2}{x} = \frac{b^2}{y} \Longrightarrow y = \frac{b^2x}{a^2}$; substituting in the constraint gives $x^2\left(a^2 + b^2\right) = a^4$ so $x = \frac{a^2}{\sqrt{a^2+b^2}}$, $y = \frac{b^2}{\sqrt{a^2+b^2}}$, and the dimensions of the rectangle with greatest perimeter are $\frac{2a^2}{\sqrt{a^2+b^2}}$ by $\frac{2b^2}{\sqrt{a^2+b^2}}$.

98 Let r be the radius and h the height of the cylinder; then the surface area is $A = 2\pi r^2 + 2\pi rh$ and the volume is πr^2h, so it suffices to minimize $r^2 + rh$ subject to the constraint $\pi r^2h = 32$. The Lagrange multiplier conditions are $2r + h = 2\lambda rh$, $r = \lambda r^2$, $\pi r^2h = 32$. We must have r, $h > 0$, so the second equation gives $\lambda = \frac{1}{r}$, and then the first equation reduces to $2r = h$. Substituting this in the constraint gives $r^3 = \frac{16}{\pi}$, so $r = 2\sqrt[3]{\frac{2}{\pi}}$ in. and $h = 4\sqrt[3]{\frac{2}{\pi}}$ in.

99 It suffices to minimize the function $f(x, y, z) = (x-1)^2 + (y-3)^2 + (z-1)^2$ subject to the constraint $x^2 + y^2 - z^2 = 0$. The Lagrange multiplier conditions are equivalent to $x - 1 = \lambda x$, $y - 3 = \lambda y$, $z - 1 = -\lambda z$, $x^2 + y^2 - z^2 = 0$. The first two equations give $\lambda xy = (x-1)y = (y-3)x \Longrightarrow y = 3x$ and similarly, the first and third equations give $\lambda xz = (x-1)z = -x(z-1) \Longrightarrow (2x-1)z = x \Rightarrow z = \sqrt{10}x$. Substituting these equations in the constraint gives $(2x-1)^2 \cdot 10x^2 = x^2$, so either $x = 0$ (and hence $y = z = 0$ as well) or $10(2x-1)^2 = 1$, which has solutions $x = \frac{1}{2} \pm \frac{\sqrt{10}}{20}$. Therefore, there are three solutions to the Lagrange conditions: $(0, 0, 0)$, $\left(\frac{1}{2} \pm \frac{\sqrt{10}}{20}, \frac{3}{2} \pm \frac{3\sqrt{10}}{20}, \frac{1}{2} \pm \frac{\sqrt{10}}{2}\right)$. We see that $f(0, 0, 0) = 11$, $f\left(\frac{1}{2} \pm \frac{\sqrt{10}}{20}, \frac{3}{2} \pm \frac{3\sqrt{10}}{20}, \frac{1}{2} \pm \frac{\sqrt{10}}{2}\right) = \frac{11}{2} \mp \sqrt{10}$, so the closest point is $\left(\frac{1}{2} + \frac{\sqrt{10}}{20}, \frac{3}{2} + \frac{3\sqrt{10}}{20}, \frac{1}{2} + \frac{\sqrt{10}}{2}\right)$. (The function $f(x, y, z) \to \infty$ as either x, y or $z \to \infty$ on the cone; therefore, f must have minimum somewhere on the cone, which corresponds to the point we found.)

100

a. We have $d(x, y, z) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$, so $\nabla d(x, y, z) = \langle \frac{x-a}{d(x,y,z)}, \frac{y-b}{d(x,y,z)}, \frac{z-c}{d(x,y,z)} \rangle$.

b. Observe that $\nabla d(x, y, z) = \frac{1}{d(x,y,z)}\overrightarrow{PP_0}$ and $\left|\overrightarrow{PP_0}\right| = d(x, y, z)$; therefore, $\nabla d(x, y, z)$ is a unit vector.

c. The level surfaces of d are spheres centered at a, b, c, and $\nabla d(x, y, z)$ is perpendicular to these spheres (pointing outwards).

d. If $P \to P_0$ in the direction of a unit vector u, we have $\nabla d(x, y, z) = \pm$u; but because the limit must be the same in all directions, we conclude that this limit does not exist.

Chapter 13

Multiple Integration

13.1 Double Integrals over Rectangular Regions

13.1.1 $\int_0^2 \int_1^3 xy\,dy\,dx$ or $\int_1^3 \int_0^2 xy\,dx\,dy$.

13.1.2 Two solutions: $\int_0^5 \int_{-2}^4 10\ dy\ dx$ or $\int_{-2}^4 \int_0^5 10\,dx\,dy$.

13.1.3 With respect to x first: $\int_1^5 \int_{-2}^4 f\,(x,\,y)\ dx\,dy$; with respect to y first: $\int_{-2}^4 \int_1^5 f\,(x,\,y)\ dy\,dx$.

13.1.4 y is the first (inner) varaiable and has limits $-1 \le y \le 1$, x is the second (outer) variable and has limits $1 \le x \le 3$.

13.1.5 $\int_0^2 \int_0^1 (4xy)\,dx\,dy = \int_0^2 (2x^2y)\Big|_0^1 dy = \int_0^2 (2y)\,dy = (y^2)\Big|_0^2 = 4.$

13.1.6 $\int_1^2 \int_0^1 (3x^2+4y^3)\,dy\,dx = \int_1^2 (3x^2y+y^4)\Big|_0^1 dx = \int_1^2 (3x^2+1)\,dx = (x^3+x)\Big|_1^2 = 10-2 = 8.$

13.1.7 $\int_1^3 \int_0^2 x^2\,y\,dx\,dy = \int_1^3 \left(\frac{x^3y}{3}\right)\Big|_0^2 dy = \frac{8}{3}\int_1^3 y\,dy = \frac{8}{3}\left(\frac{y^2}{2}\right)\Big|_1^3 = \frac{32}{3}.$

13.1.8 $\int_0^3 \int_{-2}^1 (2x+3\,y)\ dx\,dy = \int_0^3 \left(x^2+3\,x\,y\right)\Big|_{-2}^1 dy = \int_0^3 (9y-3)\ dy = \left(\frac{9}{2}y^2-3y\right)\Big|_0^3 = \frac{63}{2}.$

13.1.9 $\int_1^3 \int_0^{\pi/2} x\ \sin y\,dy\,dx = \int_1^3 (-x\ \cos y)\Big|_0^{\pi/2} dx = \int_1^3 x\,dx = \left(\frac{x^2}{2}\right)\Big|_1^3 = 4.$

13.1.10 $\int_1^3 \int_1^2 \left(y^2+y\right)\ dx\,dy = \int_1^3 \left(x\,y^2+x\,y\right)\Big|_1^2 dy = \int_1^3 \left(y^2+y\right)\ dy = \left(\frac{y^3}{3}+\frac{y^2}{2}\right)\Big|_1^3 = \frac{38}{3}.$

13.1.11 $\int_1^4 \int_0^4 \sqrt{u\,v}\,du\,dv = \int_1^4 \left(\frac{2}{3}u^{3/2}v^{1/2}\right)\Big|_{u=0}^{u=4} dv = \int_1^4 \left(\frac{16}{3}v^{1/2}\right)\ dv = \left(\frac{32}{9}v^{3/2}\right)\Big|_{v=1}^{v=4} = \frac{224}{9}.$

13.1.12 $\int_0^{\pi/2} \int_0^1 x\ \cos x\,y\,dy\,dx = \int_0^{\pi/2} (\sin x\,y)\Big|_0^1 dx = \int_0^{\pi/2} (\sin x)\ dx = (-\cos x)\Big|_0^{\pi/2} = 1.$

13.1.13 $\int_0^{\ln 2} \int_0^1 6xe^{3y}\,dx\,dy = \int_0^{\ln 2} 3x^2e^{3y}\Big|_0^1 dy = \int_0^{\ln 2} 3e^{3y}\,dy = e^{3y}\Big|_0^{\ln 2} = 8-1 = 7.$

13.1.14 $\int_0^1 \int_0^1 \frac{y}{1+x^2}\,dx\,dy = \int_0^1 y\tan^{-1}(x)\Big|_0^1 dy = \int_0^1 \frac{\pi y}{4}\,dy = \frac{\pi y^2}{8}\Big|_0^1 = \frac{\pi}{8}.$

13.1.15 $\int_1^{\ln 5} \int_0^{\ln 3} e^{x+y}\,dx\,dy = \int_1^{\ln 5} \int_0^{\ln 3} (e^x)\ (e^y)\ dx\,dy = \int_1^{\ln 5} (e^x\cdot e^y\,)\Big|_0^{\ln 3} dy = \int_1^{\ln 5} (2e^y\,)\ dy =$ $(2e^y)\Big|_1^{\ln 5} = 10-2e.$

13.1.16 $\int_0^{\pi/4}\int_0^3(\sec\theta)r\,dr\,d\theta = \int_0^{\pi/4}\left((\sec\,\theta)\frac{r^2}{2}\right)\Big|_{r=0}^{r=3}d\theta = \int_0^{\pi/4}\left(\frac{9}{2}\sec\theta\right)\,d\theta =$ $\left(\frac{9}{2}\cdot\ln\left|\sec\,\theta+\tan\theta\right|\right)\Big|_{\theta=0}^{\theta=\pi/4} = \frac{9}{2}\left(\ln\left|\sqrt{2}+1\right|-\ln\left|1+0\right|\right) = \frac{9}{2}\ln\left(\sqrt{2}+1\right).$

13.1.17 $\iint\limits_R (x+2y)\;dA = \int_1^4\int_0^3(x+2y)\;dx\,dy = \int_1^4\left(\frac{x^2}{2}+2\,x\,y\right)\Big|_0^3 dy = \int_1^4\left(\frac{9}{2}+6y\right)\,dy =$ $\left(\frac{9}{2}y+3y^2\right)\Big|_1^4 = \frac{117}{2}.$

13.1.18 $\iint\limits_R \left(x^2+x\,y\right)\;dA = \int_{-1}^1\int_1^2\left(x^2+x\,y\right)\;dx\,dy = \int_{-1}^1\left(\frac{x^3}{3}+\frac{x^2y}{2}\right)\Big|_1^2 dy = \int_{-1}^1\left(\frac{7}{3}+\frac{3y}{2}\right)\,dy =$ $\left(\frac{7}{3}y+\frac{3}{4}y^2\right)\Big|_{-1}^1 = \frac{14}{3}.$

13.1.19 $\iint\limits_R 4x^3\cos y\,dA = \int_1^2\int_0^{\pi/2}4x^3\cos y\,dy\,dx = \int_1^2 4x^3\sin y\Big|_0^{\pi/2}dx = \int_1^2 4x^3\,dx = x^4\Big|_1^2 = 16-1 = 15.$

13.1.20 $\iint\limits_R \frac{y}{\sqrt{1-x^2}}\,dA = \int_{1/2}^{\sqrt{3}/2}\int_1^2\frac{y}{\sqrt{1-x^2}}\,dy\,dx = \int_{1/2}^{\sqrt{3}/2}\frac{y^2}{2\sqrt{1-x^2}}\Big|_1^2 = \frac{3}{2}\int_{1/2}^{\sqrt{3}/2}\frac{1}{\sqrt{1-x^2}}\,dx =$ $\frac{3}{2}\left(\sin^{-1}(x)\right)\Big|_{1/2}^{\sqrt{3}/2} = \frac{3}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{4}.$

13.1.21 $\iint\limits_R \sqrt{\frac{x}{y}}\;dA = \int_1^4\int_0^1\left(\frac{x}{y}\right)^{1/2}dx\,dy = \int_1^4\left(\frac{2}{3}\cdot\frac{x^{3/2}}{y^{1/2}}\right)\Big|_0^1 dy = \int_1^4\left(\frac{2}{3\;y^{1/2}}\right)\,dy = \left(\frac{4}{3}y^{1/2}\right)\Big|_1^4 = \frac{4}{3}.$

13.1.22 $\iint\limits_R xy\sin x^2\;dA = \int_0^{\sqrt{\pi/2}}\int_0^1 x\,y\,\sin x^2\,dy\,dx = \int_0^{\sqrt{\pi/2}}\left(\frac{x\,y^2}{2}\sin x^2\right)\Big|_0^1 dx = \frac{1}{2}\int_0^{\sqrt{\pi/2}}\left(x\,\sin x^2\right)\,dx$ $= -\frac{1}{4}\left(\cos x^2\right)\Big|_0^{\sqrt{\pi/2}} = -\frac{1}{4}\left(\cos\left(\frac{\pi}{2}\right)-1\right) = -\frac{1}{4}\left(-1\right) = \frac{1}{4}.$

13.1.23 $\iint\limits_R e^{x+2y}\;dA = \int_1^{\ln 3}\int_0^{\ln 2}\left(e^x\cdot e^{2y}\right)\,dx\,dy = \int_1^{\ln 3}\left(e^x\cdot e^{2y}\right)\Big|_0^{\ln 2}dy = \int_1^{\ln 3}e^{2y}\,dy = \left(\frac{1}{2}e^{2y}\right)\Big|_1^{\ln 3} =$ $\frac{1}{2}\left(9-e^2\right).$

13.1.24 $\iint\limits_R (x^2-y^2)^2\,dA = \int_{-1}^2\int_0^1(x^4-2x^2y^2+y^4)\,dy\,dx = \int_{-1}^2(x^4y-2x^2y^3/3+y^5/5)\Big|_0^1 dx =$ $\int_{-1}^2(x^4-2x^2/3+1/5)\,dx = (x^5/5-2x^3/9+x/5)\Big|_{-1}^2 = 32/5-16/9+2/5-(-1/5+2/9-1/5) = 26/5.$

13.1.25 $\iint\limits_R \left(x^5-y^5\right)^2\;dA = \int_{-1}^1\int_0^1\left(x^{10}-2x^5y^5+y^{10}\right)\,dx\,dy = \int_{-1}^1\left(\frac{x^{11}}{11}-\frac{1}{3}x^6y^5+x\,y^{10}\right)\Big|_0^1 dy =$ $\int_{-1}^1\left(\frac{1}{11}-\frac{1}{3}y^5+y^{10}\right)\,dy = \left(\frac{y}{11}-\frac{y^6}{18}+\frac{y^{11}}{11}\right)\Big|_{-1}^1 = \frac{4}{11}.$

13.1.26 Integrate first with respect to x. $\iint\limits_R y\cos xy\,dA = \int_0^{\pi/3}\int_0^1 y\cos xy\,dx\,dy = \int_0^{\pi/3}\sin xy\Big|_0^1 dy =$ $\int_0^{\pi/3}\sin y\,dy = -\cos y\Big|_0^{\pi/3} = -(1/2-1) = 1/2.$

13.1.27 Integrate first with respect to x. $\iint\limits_R (y+1)e^{x(y+1)}\,dA = \int_{-1}^1\int_0^1(y+1)e^{x(y+1)}\,dx\,dy =$ $\int_{-1}^1 e^{x(y+1)}\Big|_0^1 dy = \int_{-1}^1(e^{y+1}-1)\,dy = (e^{y+1}-y)\Big|_{-1}^1 = (e^2-1)-(1-(-1)) = e^2-3.$

13.1.28 Integrate first with respect to y. $\iint\limits_R x\,\sec^2(x\,y)\;dA = \int_0^{\pi/3}\int_0^1 x\,\sec^2(x\,y)\;dy\,dx =$ $\int_0^{\pi/3}\left(x\cdot\left(\frac{1}{x}\right)\tan(x\,y)\right)\Big|_0^1 dx = \int_0^{\pi/3}\tan x\,dx = (\ln|\sec\,x\,|)\Big|_0^{\pi/3} = \ln 2.$

13.1.29 Integrate first with respect to y. $\iint\limits_R 6x^5 e^{x^3 y}\, dA = \int_0^2 \int_0^2 6x^5 e^{x^3 y}\, dy\, dx = \int_0^2 6x^2 e^{x^3 y}\Big|_0^2\, dx =$ $\int_0^2 (6x^2 e^{2x^3} - 6x^2)\, dx = (e^{2x^3} - 2x^3)\Big|_0^2 = e^{16} - 16 - (1 - 0) = e^{16} - 17.$

13.1.30 Integrate first with respect to x. $\iint\limits_R y^3 \sin\left(xy^2\right)\, dA = \int_0^{\sqrt{\pi/2}} \int_0^2 y^3 \sin\left(xy^2\right)\, dx\, dy =$ $\int_0^{\sqrt{\pi/2}} \left(-\frac{y^3}{y^2}\cos\left(x\, y^2\right)\right)\Big|_0^2 dy = \int_0^{\sqrt{\pi/2}} \left(y - y\cos\left(2y^2\right)\right)\, dy = \left(\frac{y^2}{2} - \frac{1}{4}\sin\left(2y^2\right)\right)\Big|_0^{\sqrt{\pi/2}} = \frac{\pi}{4}.$

13.1.31 Integrate first with respect to y. $\iint\limits_R \frac{x}{(1+x\, y)^2}\, dA = \int_0^4 \int_1^2 \frac{x}{(1+x\, y)^2}\, dy\, dx$. Let $u = 1 + xy$, so that $du = x\, dy$. Then we have $\int_0^4 \left(-\frac{1}{1+x\, y}\right)\Big|_1^2 dx = \int_0^4 \left(\frac{1}{1+x} - \frac{1}{1+2x}\right)\, dx = \left(\ln|1+x| - \frac{1}{2}\ln|1+2x|\right)\Big|_0^4 = \ln 5 - \frac{1}{2}\ln 9 = \ln\left(\frac{5}{3}\right).$

13.1.32 $\overline{f}_{\text{ave}} = \frac{1}{\text{area of } R} \iint\limits_R f(x,\, y)\, dA = \frac{1}{4}\int_0^2 \int_0^2 (4 - x - y)\, dy\, dx = \frac{1}{4}\int_0^2 \left(4y - x\, y - \frac{y^2}{2}\right)\Big|_0^2 dx =$ $\frac{1}{4}\int_0^2 (6 - 2x)\, dx = \frac{1}{4}\left(6x - x^2\right)\Big|_0^2 = \frac{1}{4}(8) = 2.$

13.1.33 $\overline{f}_{\text{ave}} = \frac{1}{\text{area of } R} \iint\limits_R f(x,\, y)\, dA = \frac{1}{6\ln 2}\int_0^6 \int_0^{\ln 2} e^{-y}\, dy\, dx = \frac{1}{6\ln 2}\int_0^6 \left(-e^{-y}\right)\Big|_0^{\ln 2} dx =$ $\frac{1}{6\ln 2}\int_0^6 \left(\frac{1}{2}\right)\, dx = \frac{1}{6\ln 2}\left(\frac{x}{2}\right)\Big|_0^6 = \frac{1}{6\ln 2}(3) = \frac{1}{2\ln 2}.$

13.1.34 $\overline{f}_{\text{ave}} = \frac{1}{\text{area of } R} \iint\limits_R f(x,\, y)\, dA = \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \sin x \sin y\, dy\, dx = \frac{1}{\pi^2}\int_0^\pi \left(-\sin x \cos y\right)\Big|_0^\pi dx =$ $\frac{1}{\pi^2}\int_0^\pi 2\sin x\, dx = \frac{2}{\pi^2}\left(-\cos x\right)\Big|_0^\pi = \frac{2}{\pi^2}(2) = \frac{4}{\pi^2}.$

13.1.35 $\overline{f}_{\text{ave}} = \frac{1}{\text{area of } R} \iint\limits_R f(x,\, y)\, dA = \frac{1}{8}\int_{-2}^2 \int_0^2 \left(x^2 + y^2\right)\, dy\, dx = \frac{1}{8}\int_{-2}^2 \left(x^2 y + \frac{1}{3}y^3\right)\Big|_0^2 dx =$ $\frac{1}{8}\int_{-2}^2 \left(2x^2 + \frac{8}{3}\right)\, dx = \frac{1}{8}\left(\frac{2}{3}x^3 + \frac{8}{3}x\right)\Big|_{-2}^2 = \frac{8}{3}.$

13.1.36 $\overline{f}_{\text{ave}} = \frac{1}{\text{area of } R} \iint\limits_R f(x,\, y)\, dA = \frac{1}{9}\int_0^3 \int_0^3 \left((x-3)^2 + (y-3)^2\right)\, dy\, dx =$ $\frac{1}{9}\int_0^3 \left(y\,(x-3)^2 + \frac{1}{3}(y-3)^3\right)\Big|_0^3 dx = \frac{1}{9}\int_0^3 \left(3\,(x-3)^2 + 9\right)\, dx = \frac{1}{9}\left((x-3)^2 + 9x\right)\Big|_0^3 = 6.$

13.1.37

a. True. The region $R = \{(x,\, y) \mid 1 \le x \le 3,\, 4 \le y \le 6\}$ is a rectangle that has width 2 and length 2, thus is a square.

b. False. The region for $\int_4^6 \int_1^3 f(x,\, y)\, dx\, dy$ is $R = \{(x,\, y) \mid 1 \le x \le 3,\, 4 \le y \le 6\}$ is not equivalent to the region for $\int_4^6 \int_1^3 f(x,\, y)\, dy\, dx$ which is $R = \{(x,\, y) \mid 4 \le x \le 6,\, 1 \le y \le 3\}$ thus Fubini's Theorem does not apply.

c. True. The region is $R = \{(x,\, y) \mid 1 \le x \le 3,\, 4 \le y \le 6\}$.

13.1.38

a. $\iint\limits_R xy\, e^{-\left(x^2+y^2\right)}\, dA = \int_{-a}^a \int_{-b}^b xy\, e^{-\left(x^2+y^2\right)}\, dy\, dx$. Let $f(x,y) = xye^{-(x^2+y^2)}$. Note that $f(-x,y) = -f(x,y)$, so the integral over the portion of R in the first and fourth quadrants cancels the integral over the portion of R in the second and third quadrants, yielding a value of 0.

b. Let $f(x,\, y) = \frac{\sin(x-y)}{x^2+y^2+1}$. Note that $f(-x,-y) = -f(x,y)$, so the integral over the portion of R in the first quadrant cancels the integral over the portion in the third quadrant, and likewise for the second and fourth quadrants, yielding a value of 0.

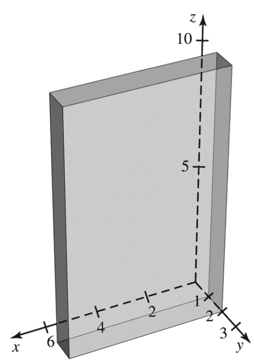
13.1.39

a. total population = (total population region 1) + (total population region 2) + . . . + (total population region 9) = (population density of region 1) × (area of region 1) + (population density of region 2) × (area of region 2) + . . . + (population density of region 9) × (area of region 9) = $(350)\left(\frac{1}{2}\right)+(300)\left(\frac{1}{4}\right)+(150)\left(\frac{3}{4}\right)+(500)\left(\frac{1}{2}\right)+(400)\left(\frac{1}{4}\right)+(250)\left(\frac{3}{4}\right)+(250)(1)+(200)\left(\frac{1}{2}\right)+(150)\left(\frac{3}{2}\right) = 1475$.

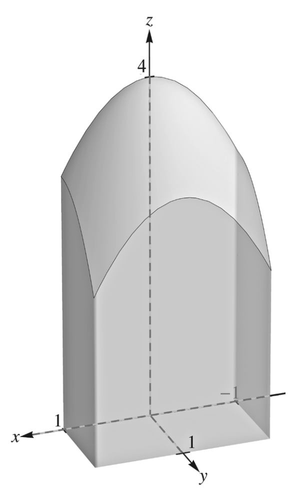
b. The calculation above could be expressed as the sum $\sum_{k=1}^{3} f(\overline{x}_k,\, \overline{y}_k)\,\Delta A_k$ where $f(\overline{x}_k,\, \overline{y}_k)$ represents the population density for each sub-region of R with area ΔA_k. This sum can be used as an approximation for a 'continuous' function $f(x,\, y)$ that represents the population density at each point in R then $\sum_{k=1}^{n} f(\overline{x}_k,\, \overline{y}_k)\,\Delta A_k \approx \iint\limits_R f(x,\, y)\; dA$.

13.1.40 V $\approx \sum_{k=1}^{n} f(\overline{x}_k,\, \overline{y}_k)\,\Delta A_k$. Also, $\Delta A_k = \Delta x_k \Delta y_k = 5 \cdot 6 = 30$. So $\sum_{k=1}^{15} f(\overline{x}_k,\, \overline{y}_k)\,\Delta A_k = 30 \cdot (f(\overline{x}_1,\, \overline{y}_1) + f(\overline{x}_2,\, \overline{y}_2) + \cdots + f(\overline{x}_{15},\, \overline{y}_{15})) =$
$30 \cdot (1 + 1.5 + 2.0 + 2.5 + 3.0 + 1 + 1.5 + 2.0 + 2.5 + 3.0 + 0.75 + 1.25 + 1.75 + 2.25 + 2.75) = 862.5\,\mathrm{m}^3$.

13.1.41 $V = \int_0^6 \int_1^2 10\,dy\,dx = \int_0^6 (10y)\Big|_1^2 dx = \int_0^6 10\,dx = (10x)\Big|_0^6 = 60$.



13.1.42 $V = \int_0^1 \int_{-1}^1 \left(4 - x^2 - y^2\right)\, dx\, dy \;\; = \int_0^1 \left(4x - \frac{x^3}{3} - x\,y^2\right)\Big|_{-1}^{1} dy =$
$\int_0^1 \left(\frac{22}{3} - 2y^2\right)\, dy = \left(\frac{22}{3}y - \frac{2}{3}y^3\right)\Big|_0^1 = \frac{20}{3}$.



13.1.43 $\int_1^2 \int_1^2 \frac{x}{x+y}\,dy\,dx = \int_1^2 \left(x\,\ln(x+y)\right)\Big|_1^2 dx = \int_1^2 (x\ln(x+2) - x\ln(x+1))\,dx = (-\frac{1}{2}x^2\ln(x+1) + \frac{1}{2}x^2\ln(x+2) + \frac{x}{2} + \frac{1}{2}\ln(x+1) - 2\ln(x+2))\Big|_1^2 = -2\ln 3 + 2\ln 4 + 1 + \frac{\ln 3}{2} - 2\ln 4 - (\frac{-\ln 2}{2} + \frac{\ln 3}{2} + \frac{1}{2} + \frac{\ln 2}{2} - 2\ln 3) = 1 - \frac{1}{2} = \frac{1}{2}$.

13.1.44 $\int_0^2 \int_0^1 x^5\,y^2 e^{x^3y^3}\,dy\,dx = \int_0^2 \left(\frac{x^5}{3x^3}\,e^{x^5y^3}\right)\Big|_0^1 dx = \frac{1}{3}\int_0^2 \left(x^2\,e^{x^3} - x^2\right)dx = \frac{1}{3}\left(\frac{1}{3}e^{x^3} - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{1}{9}\left(e^8 - 9\right)$.

13.1.45 $\int_0^1 \int_1^4 \frac{3y}{\sqrt{x+y^2}}\,dx\,dy = \int_0^1 \left(6y\sqrt{x+y^2}\right)\Big|_1^4 dy = 6\int_0^1 \left(y\sqrt{4+y^2} - y\sqrt{1+y^2}\right)dy = 6\left(\frac{1}{3}\left(4+y^2\right)^{3/2} - \frac{1}{3}\left(1+y^2\right)^{3/2}\right)\Big|_0^1 = 2\left(5\sqrt{5} - 2\sqrt{2} - 7\right)$.

13.1.46 $\int_1^4 \int_0^2 e^{y\sqrt{x}}\,dy\,dx = \int_1^4 \left(\frac{e^{y\sqrt{x}}}{\sqrt{x}}\right)\Big|_0^2 dx = \int_1^4 \left(\frac{e^{2\sqrt{x}}}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right)dx = \left(e^{2\sqrt{x}} - 2\sqrt{x}\right)\Big|_1^4 = e^4 - e^2 - 2$.

13.1.47 $\int_{-2}^2 \int_0^{\ln 4} e^{-x}\,dx\,dy = \int_{-2}^2 \left(-e^{-x}\right)\Big|_0^{\ln 4} dy = \int_{-2}^2 \frac{3}{4}\,dy = \left(\frac{3}{4}y\right)\Big|_{-2}^2 = 3$.

13.1.48 $\int_0^2 \int_0^1 (6 - x - 2y)\;dy\,dx = \int_0^2 \left(6y - x\,y - y^2\right)\Big|_0^1 dx = \int_0^2 (5 - x)\,dx = \left(5x - \frac{x^2}{2}\right)\Big|_0^2 = 8$.

13.1.49 $\int_{-1}^3 \int_0^2 (24 - 3x - 4y)\;dy\,dx = \int_{-1}^3 \left(24y - 3x\,y - 2y^2\right)\Big|_0^2 dx = \int_{-1}^3 (40 - 6x)\,dx = \left(40x - 3x^2\right)\Big|_{-1}^3 = 136$.

13.1.50 $\int_1^2 \int_0^1 \left(12 - x^2 - 2y^2\right)\,dy\,dx = \int_1^2 \left(12y - x^2y - \frac{2}{3}y^3\right)\Big|_0^1 dx = \int_1^2 \left(\frac{34}{3} - x^2\right)dx = \left(\frac{34}{3}x - \frac{x^3}{3}\right)\Big|_1^2 = 9$.

13.1.51 $\int_0^a \int_0^\pi \sin(x+y)\;dx\,dy = \int_0^a \left(-\cos(x+y)\right)\Big|_0^\pi dy = \int_0^a (\cos y - \cos(\pi + y))\;dy = (\sin y - \sin(\pi + y))\Big|_0^a = (\sin a - \sin(\pi + a)) = 1$. Solving for a yields: $(\sin a - \sin(\pi + a)) = 1$, so $2\,\sin a = 1$, so $\sin a = \frac{1}{2}$. The solutions are $a = \frac{\pi}{6}$ and $a = \frac{5\pi}{6}$.

13.1.52 $\overline{f} = \frac{1}{a^2}\int_0^a \int_0^a (x + y - 8)\;dx\,dy = \frac{1}{a^2}\int_0^a \left(\frac{x^2}{2} + x\,y - 8x\right)\Big|_0^a dy = \frac{1}{a^2}\int_0^a \left(\frac{a^2}{2} + a\,y - 8a\right)\;dy = \frac{1}{a^2}\left(\frac{a^2}{2}y + \frac{a\,y^2}{2} - 8a\,y\right)\Big|_0^a = \frac{1}{a^2}\left(\frac{a^3}{2} + \frac{a^3}{2} - 8a^2\right) = a - 8 = \overline{f}$.

Setting $\overline{f} = 0$ and solving for a yields $a - 8 = 0$, thus $a = 8$.

13.1.53 $\overline{f} = \frac{1}{a^2}\int_0^a \int_0^a \left(4 - x^2 - y^2\right)\,dy\,dx = \frac{1}{a^2}\int_0^a \left(4y - x^2y - \frac{y^3}{3}\right)\Big|_0^a dx = \frac{1}{a^2}\int_0^a \left(4a - a\,x^2 - \frac{a^3}{3}\right)\,dx = \frac{1}{a^2}\left(4a\,x - \frac{a\,x^3}{3} - \frac{a^3}{3}x\right)\Big|_0^a = \frac{1}{a^2}\left(4a^2 - \frac{2a^4}{3}\right) = 4 - \frac{2a^2}{3} = \overline{f}$. Setting $\overline{f} = 0$ and solving for a yields $4 - \frac{2a^2}{3} = 0$, so $a^2 = 6$. Because $a > 0$, we have $a = \sqrt{6}$.

13.1.54 Consider $\int_0^b \int_0^a (6 - x - 3y)\;dx\,dy = \int_0^b \left(6x - x^2/2 - 3xy\right)\Big|_0^a dy = \int_0^b \left(6a - a^2/2 - 3ay\right)\;dy = \left((6a - a^2/2)y - 3ay^2/2\right)\Big|_0^b = \left(6ab - a^2b/2 - 3ab^2/2\right)$.

Maximize $\left(6ab - a^2b/2 - 3ab^2/2\right)$ with the constraint $a = 6 - 3b$. Substituting $6 - 3b$ for a gives $9(2b - b^2)$. This is maximized for $b = 1$. Thus, the desired point is $(3, 1)$ and the value of the integral is 9.

13.1.55

a. $m = \iint\limits_R \rho(x, y)\, dA = \int_0^\pi \int_0^{\pi/2} (1 + \sin x)\, dx\, dy = \int_0^\pi (x - \cos x)\Big|_0^{\pi/2} dy =$
$\int_0^\pi \left(\frac{\pi}{2} + 1\right) dy = \left(\frac{\pi}{2} + 1\right) y\Big|_0^\pi = \frac{\pi^2}{2} + \pi.$

b. $m = \iint\limits_R \rho(x, y)\, dA = \int_0^{\pi/2} \int_0^\pi (1 + \sin y)\, dy\, dx = \int_0^{\pi/2} (y - \cos y)\Big|_0^\pi dx =$
$\int_0^\pi (\pi + 2)\, dx = (\pi + 2)\, x\Big|_0^{\pi/2} = \frac{\pi^2}{2} + \pi.$

c. $m = \iint\limits_R \rho(x, y)\, dA = \int_0^\pi \int_0^{\pi/2} (1 + \sin x \sin y)\, dx\, dy =$
$\int_0^\pi (x - \cos x \sin y)\Big|_0^{\pi/2} dy = \int_0^\pi \left(\frac{\pi}{2} + \sin y\right) dy = \left(\frac{\pi}{2} y - \cos y\right)\Big|_0^\pi = \frac{\pi^2}{2} + 2.$

13.1.56 Let the base of the shed be in the xy-plane with corners at (0, 0), (16, 0), (16, 10), and (0, 10). Let the peak of the roof be over the point (0, 0). With these assumptions the height of the roof $h(x, y)$ over any point (x, y) in the base is the lesser of these two functions $f(x, y) = 12 - \frac{x}{4}$ and $g(x, y) = 12 - \frac{2y}{5}$. Divide the base of the shed into 40 squares each of size 2 by 2 (that is, $\triangle x = \triangle y = 2$), and produce the Riemann sum for $h(x, y)$ over the base, evaluating $h(x, y)$ at the midpoint of each square. Thus $\overline{x}_1 = 1$, $\overline{x}_2 = 3$, $\overline{x}_3 = 5$, $\overline{x}_4 = 7$, $\overline{x}_5 = 9$, $\overline{x}_6 = 11$, $\overline{x}_7 = 13$, and $\overline{x}_8 = 15$. Likewise $\overline{y}_1 = 1$, $\overline{y}_2 = 3$, $\overline{y}_3 = 5$, $\overline{y}_4 = 7$, and $\overline{y}_5 = 9$. The volume will thus be approximated by the Riemann sum $\sum_{j=1}^{8} \sum_{i=1}^{5} h\left(\overline{x}_i, \overline{y}_j\right) \triangle x_i \triangle y_j = 373.7 \times 4 = 1494.8$.

13.1.57 The area of the constant cross section of S is $A = \int_a^b f(x)\, dx$ for any value of y. The volume of S can be expressed as $\int_c^d \int_a^b f(x)\, dx\, dy = \int_c^d A\, dy = A \cdot y\Big|_c^d = A(d - c)$.

13.1.58

a. $\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \int_a^b (g(x) \cdot h(y))\, dx\, dy = \int_c^d h(y) \left(\int_a^b g(x)\, dx\right) dy$, because $h(y)$ is constant relative to x. Also, $\int_a^b g(x)\, dx$ is constant relative to y, so we have $\left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right)$.

b. $\left(\int_a^b g(x)\, dx\right)^2 = \left(\int_a^b g(x)\, dx\right)\left(\int_a^b g(x)\, dx\right) = \int_a^b (g(x))^2\, dx.$

c. $\int_0^{2\pi} \int_{10}^{30} \left(\cos x \cdot e^{-4y^2}\right) dy\, dx = \left(\int_0^{2\pi} \cos x\, dx\right)\left(\int_{10}^{30} e^{-4y^2}\, dy\right) = (0)\left(\int_{10}^{30} e^{-4y^2}\, dy\right) = 0.$

13.1.59 $\iint\limits_R \frac{\partial^2 f}{\partial x\, \partial y}\, dA = \int_0^b \int_0^a \left(\frac{\partial^2 f}{\partial x\, \partial y}\right) dx\, dy = \int_0^b \left(\frac{\partial f}{\partial y}\right)\Big|_0^a dy = \int_0^b \left(\frac{\partial f}{\partial y}(a, y) - \frac{\partial f}{\partial y}(0, y)\right) dy =$
$(f(a, y) - f(0, y))\Big|_0^b = f(a, b) - f(0, b) - f(a, 0) + f(0, 0)$.

13.1.60

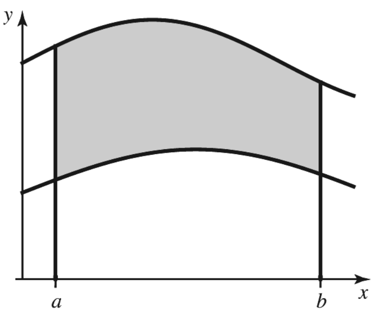
a. $\int_0^1 \int_0^1 \cos\left(x\sqrt{y}\right) dx\, dy = \int_0^1 \left(\frac{\sin\left(x\sqrt{y}\right)}{\sqrt{y}}\right)\Big|_0^1 dy = \int_0^1 \frac{\sin\left(\sqrt{y}\right)}{\sqrt{y}}\, dy = \left(-2\cos\sqrt{y}\right)\Big|_0^1 = 2(1 - \cos(1)).$

b. $\int_0^1 \int_0^1 x^3 y \cos\left(x^2 y^2\right) dy\, dx = \int_0^1 \left(\frac{x^3}{2x^2} \sin\left(x^2 y^2\right)\right)\Big|_0^1 dx = \int_0^1 \frac{1}{2} x \sin\left(x^2\right) dx = \left(-\frac{1}{4}\cos\left(x^2\right)\right)\Big|_0^1 =$
$\frac{1}{4}(1 - \cos(1)).$

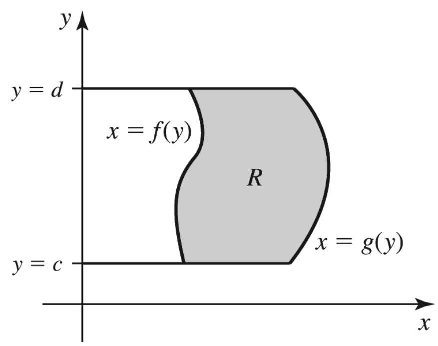
13.1.61 $\iint\limits_R x^{2r-1} y^{s-1} f(x^r \cdot y^s)\, dA = \int_0^1 \int_0^1 x^{2r-1} y^{s-1} f(x^r \cdot y^s)\, dy\, dx$. Choose $u = x^r \cdot y^s$, then $\frac{du}{s \cdot x^r} = y^{s-1}\, dy$, and $\int x^{2r-1} y^{s-1} f(x^r y^s)\, dy = \int \frac{x^{2r-1}}{s \cdot x^r} f(u)\, du = \frac{1}{s} x^{r-1} F(u) + C = \frac{1}{s} x^{r-1} F(x^r y^s) + C$. Thus, $\int_0^1 x^{2r-1} y^{s-1} f(x^r y^s)\, dy = \frac{1}{s} x^{r-1} F(x^r y^s)\Big|_0^1 = \frac{1}{s} x^{r-1} F(x^r)\, dx$, so $\int_0^1 \int_0^1 x^{2r-1} y^{s-1} f(x^r y^s)\, dy\, dx = \frac{1}{s} \int_0^1 x^{r-1} F(x^r)\, dx$. Now choose $u = x^r$. Then $\frac{du}{r} = x^{r-1}\, dx$, and $\int x^{r-1} F(x^r)\, dx = \frac{1}{r} \int F(u)\, du = \frac{1}{r} G(u) + C = \frac{1}{r} G(x^r) + C$. Thus, $\int_0^1 \int_0^1 x^{2r-1} y^{s-1} f(x^r y^s)\, dy\, dx = \frac{1}{s}\left(\frac{1}{r} G(x^r)\right)\Big|_0^1 = \frac{G(1) - G(0)}{rs}$.

13.2 Double Integrals over General Regions

13.2.1 A region is bounded below by $y = f(x)$ and above by $y = g(x)$, on the left by $x = a$ and on the right by $x = b$. $R = \{(x, y) \mid a \le x \le b,\ f(x) \le y \le g(x)\}$.



13.2.2 A region is bounded below by $y = c$ and above by $y = d$, on the left by $x = f(y)$ and on the right by $x = g(y)$. $R = \{(x, y) \mid f(y) \le x \le g(y),\ c \le y \le d$.

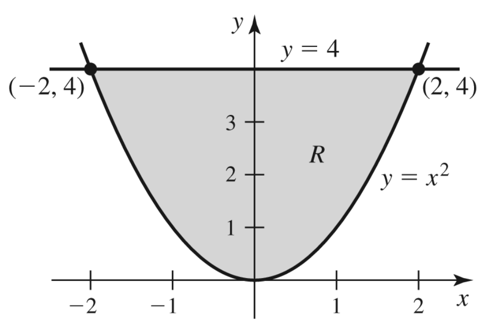


13.2.3 Integrate first with respect to x, then with respect to y. $\iint\limits_R f(x, y)\, dA = \int_0^1 \int_{y-1}^{1-y} x\, y\, dx\, dy$.

13.2.4 Integrate first with respect to x, then with respect to y. $A = \iint\limits_R (1)\, dA = \int_0^{17} \int_{\frac{y-3}{2}}^{\frac{y+4}{3}} (1)\, dx\, dy$.

13.2.5 $\int_0^1 \int_{x^2}^{\sqrt{x}} f(x, y)\, dy\, dx$.

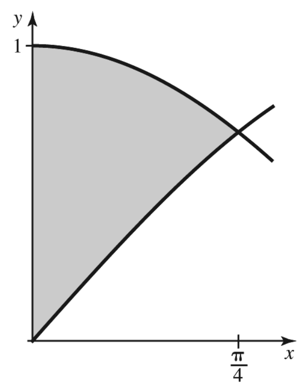
13.2.6



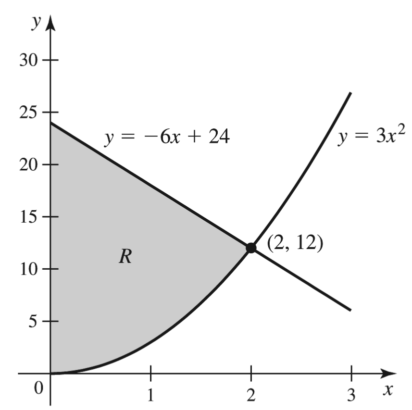
13.2.7 $\iint\limits_R f(x, y)\, dA = \int_0^2 \int_{x^2}^{4x} f(x, y)\ dy\, dx$.

13.2.8 $\iint\limits_R f(x, y)\, dA = \int_{-3}^4 \int_{2x^2}^{2x+24} f(x, y)\, dy\, dx$.

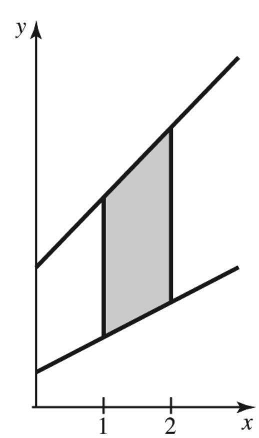
13.2.9 $\iint_R f(x, y)\, dA = \int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} f(x, y)\, dy\, dx.$



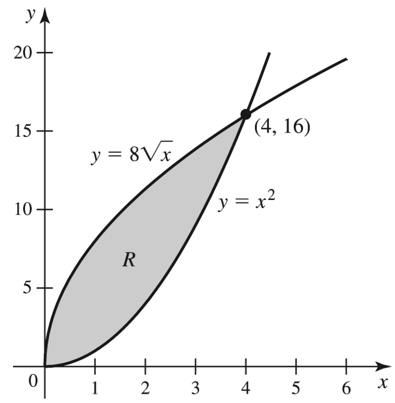
13.2.10 $\iint_R f(x, y)\, dA = \int_0^{2} \int_{3x^2}^{-6x+24} f(x, y)\, dy\, dx.$



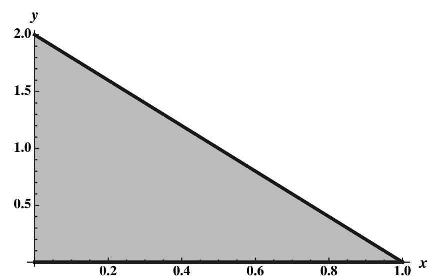
13.2.11 $\iint_R f(x, y)\, dA = \int_1^{2} \int_{x+1}^{2x+4} f(x, y)\, dy\, dx.$



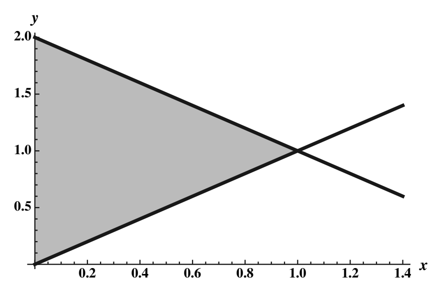
13.2.12 $\iint_R f(x, y)\, dA = \int_0^{4} \int_{x^2}^{8\sqrt{x}} f(x, y)\, dy\, dx$ (or) $= \int_0^{16} \int_{y^2/16}^{\sqrt{y}} f(x, y)\, dx\, dy.$



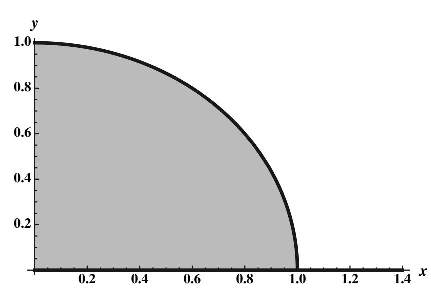
13.2.13 $\iint\limits_R f(x,y)\,dA = \int_0^1 \int_0^{-2x+2} f(x,y)\,dy\,dx.$



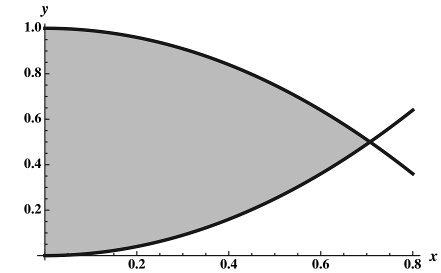
13.2.14 $\iint\limits_R f(x,y)\,dA = \int_0^1 \int_x^{-x+2} f(x,y)\,dy\,dx.$



13.2.15 $\iint\limits_R f(x,y)\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx.$



13.2.16 $\iint\limits_R f(x,y)\,dA = \int_0^{1/\sqrt{2}} \int_{x^2}^{1-x^2} f(x,y)\,dy\,dx.$



13.2.17 $\int_0^1 3y^2\Big|_x^1 dx = \int_0^1 (3-3x^2)\,dx = (3x-x^3)\Big|_0^1 = 2.$

13.2.18 $\int_0^1 5xy^3\Big|_0^{2x} dx = \int_0^1 40x^4\,dx = 8x^5\Big|_0^1 = 8.$

13.2.19 $\int_0^2 \int_{x^2}^{2x} x\,y\,dy\,dx = \int_0^2 \left(\frac{x\,y^2}{2}\right)\Big|_{x^2}^{2x} dx = \frac{1}{2}\int_0^2 \left(4x^3 - x^5\right)\,dx = \frac{1}{2}\left(x^4 - \frac{x^6}{6}\right)\Big|_0^2 = \frac{8}{3}.$

13.2.20 $\int_0^3 \int_{x^2}^{x+6} (x-1)\,dy\,dx = \int_0^3 y(x-1)\Big|_{x^2}^{x+6} dx = \int_0^3 ((x+6)(x-1) - x^2(x-1))\,dx = \int_0^3 (-x^3 + 2x^2 + 5x - 6)\,dx = (-x^4/4 + 2x^3/3 + 5x^2/2 - 6x)\Big|_0^3 = -(3^4)/4 + 2(3^3)/3 + 5(3)^2/2 - 6(3) = 9/4.$

13.2.21 $\int_{-\pi/4}^{\pi/4} \int_{\sin x}^{\cos x} dy\,dx = \int_{-\pi/4}^{\pi/4} (y)\Big|_{\sin x}^{\cos x} dx = \int_{-\pi/4}^{\pi/4} (\cos x - \sin x)\,dx = (\sin x + \cos x)\Big|_{-\pi/4}^{\pi/4} = \sqrt{2}.$

13.2.22 $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2x^2 y\,dy\,dx = \int_0^1 \left(x^2 y^2\right)\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_0^1 0\,dx = 0.$

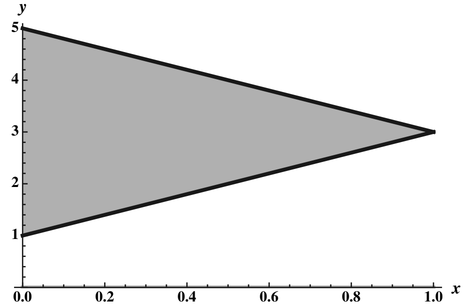
13.2.23 $\int_{-2}^2 \int_{x^2}^{8-x^2} x\,dy\,dx = \int_{-2}^2 (x\,y)\Big|_{x^2}^{8-x^2} dx = \int_{-2}^2 \left(8x - 2x^3\right)\,dx = \left(4x^2 - \frac{x^4}{2}\right)\Big|_{-2}^2 = 0.$

13.2.24 $\int_0^{\ln 2} \int_{e^x}^2 dy\,dx = \int_0^{\ln 2} (y)\Big|_{e^x}^2 dx = \int_0^{\ln 2} (2 - e^x)\,dx = (2x - e^x)\Big|_0^{\ln 2} = 2\ln 2 - 1.$

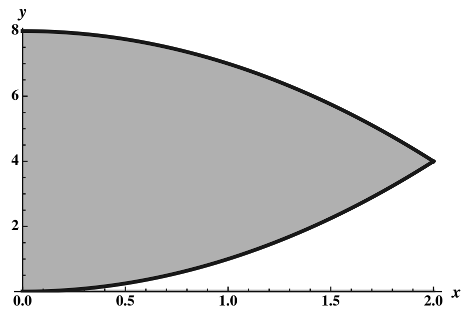
13.2.25 $\int_0^1 \int_0^x 2e^{x^2}\,dy\,dx = \int_0^1 2xe^{x^2}\,dx = e^{x^2}\Big|_0^1 = e - 1.$

13.2.26 $\int_0^{\sqrt[3]{\pi/2}} \int_0^x y\cos x^3\,dy\,dx = \int_0^{\sqrt[3]{\pi/2}} (y^2/2)\cos x^3\Big|_0^x dx = \frac{1}{2}\int_0^{\sqrt[3]{\pi/2}} x^2\cos x^3\,dx = \frac{1}{6}\left(\sin x^3\right)\Big|_0^{\sqrt[3]{\pi/2}} = \frac{1}{6}.$

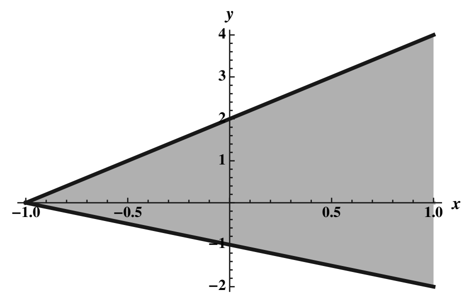
13.2.27 $\iint\limits_R xy\,dA = \int_0^1 \int_{2x+1}^{-2x+5} xy\,dy\,dx = \int_0^1 \left(\frac{xy^2}{2}\right)\Big|_{2x+1}^{-2x+5} dx = \int_0^1 \left(12x - 12x^2\right)\,dx =$
$\left(6x^2 - 4x^3\right)\Big|_0^1 = 2.$



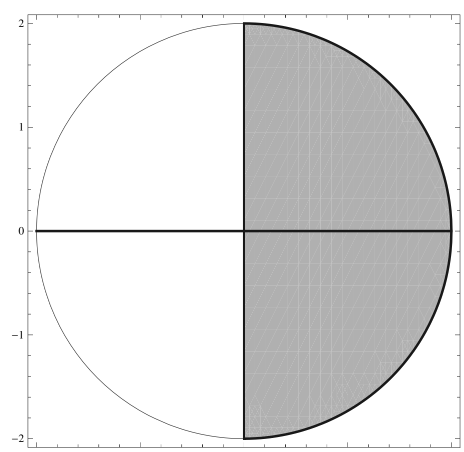
13.2.28 $\iint\limits_R (x+y)\,dA = \int_0^2 \int_{x^2}^{8-x^2} (x+y)\,dy\,dx = \int_0^2 \left(x\,y + \frac{y^2}{2}\right)\Big|_{x^2}^{8-x^2} dx =$
$\int_0^2 \left(32 + 8x - 8x^2 - 2x^3\right)\,dx = \left(32x + 4x^2 - \frac{8}{3}x^3 - \frac{x^4}{2}\right)\Big|_0^2 = \frac{152}{3}.$



13.2.29 $\iint\limits_R y^2\,dA = \int_{-1}^{1} \int_{-x-1}^{2x+2} y^2\,dy\,dx = \int_{-1}^{1} \frac{y^3}{3}\Big|_{-x-1}^{2x+2} dx = \frac{1}{3}\int_{-1}^{1} \left(8\,(x+1)^3 + (x+1)^3\right)\,dx =$
$3\int_{-1}^{1} (x+1)^3\,dx = \frac{3}{4}\,(x+1)^4\Big|_{-1}^{1} = 12.$



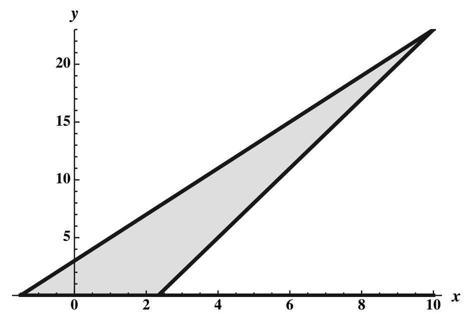
13.2.30 $\iint\limits_R x^2 y\,dA = \int_{-4}^{4} \int_0^{\sqrt{16-y^2}} x^2 y\,dx\,dy = \int_{-4}^{4} \left(\frac{x^3 y}{3}\right)\Big|_0^{\sqrt{16-y^2}} dy = \frac{1}{3}\int_{-4}^{4} y\left(16-y^2\right)^{3/2} dx$. Let $u =$ $16 - y^2$ so that $du = -2y\,dy$, Then the integral is equal to $\frac{1}{6}\int_0^0 u^{3/2}\,du = 0$.



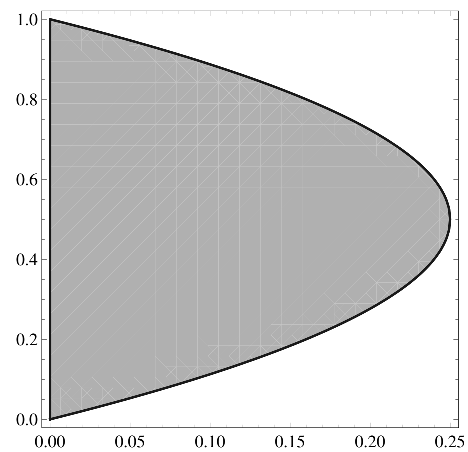
13.2.31 $\iint\limits_R f(x,\,y)\,dA = \int_0^{18} \int_{y/2}^{(y+9)/3} f(x,\,y)\,dx\,dy.$

13.2.32 $\iint\limits_R f(x,\,y)\,dA = \int_{-4}^{5} \int_{(5-y)/3}^{\sqrt{25-y^2}} f(x,\,y)\,dx\,dy.$

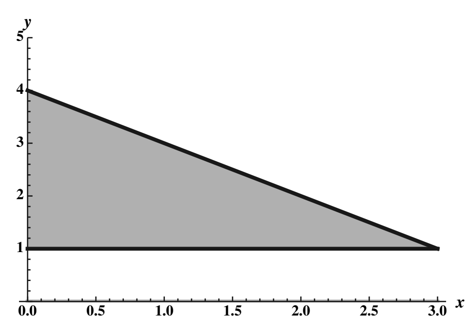
13.2.33 $\iint\limits_R f(x, y)\, dA = \int_0^{23} \int_{(y-3)/2}^{(y+7)/3} f(x, y)\, dx\, dy.$



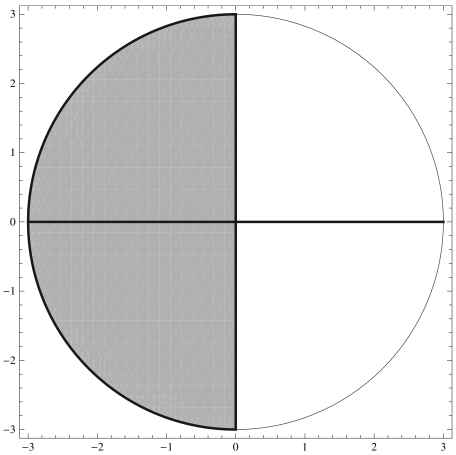
13.2.34 $\iint\limits_R f(x, y)\, dA = \int_0^1 \int_0^{y(1-y)} f(x, y)\, dx\, dy.$



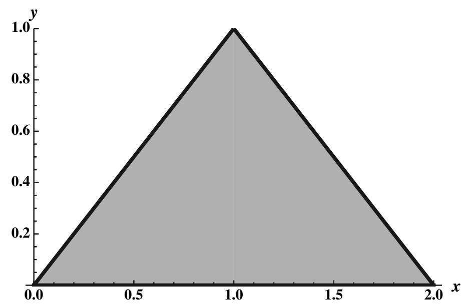
13.2.35 $\iint\limits_R f(x, y)\, dA = \int_1^4 \int_0^{4-y} f(x, y)\, dx\, dy = \int_0^3 \int_1^{4-x} f(x, y)\, dy\, dx.$



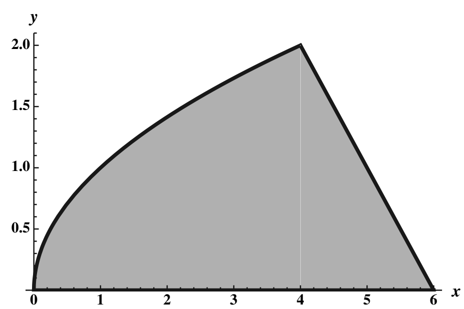
13.2.36 $\iint\limits_R f(x, y)\, dA = \int_{-3}^3 \int_{-\sqrt{9-y^2}}^0 f(x, y)\, dx\, dy.$



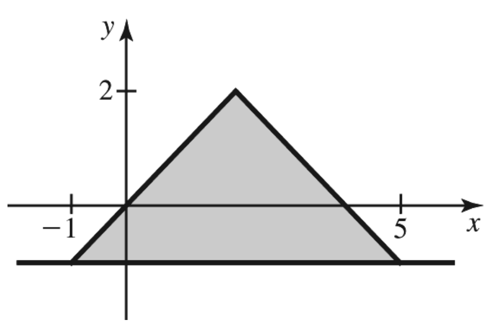
13.2.37 $\iint\limits_R f(x, y)\, dA = \int_0^1 \int_y^{2-y} f(x, y)\, dx\, dy.$



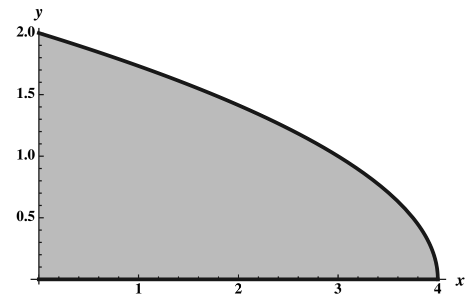
13.2.38 $\iint\limits_R f(x, y)\, dA = \int_0^2 \int_{y^2}^{6-y} f(x, y)\, dx\, dy.$



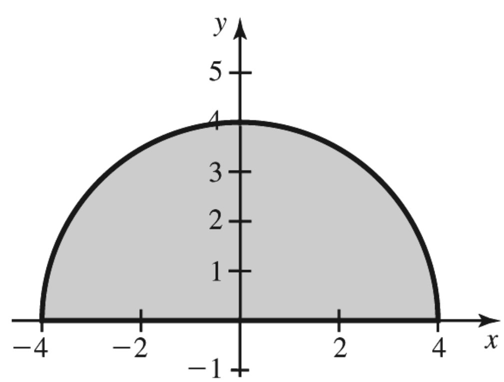
13.2.39 $R = \{(x, y) \mid y \le x \le 4 - y, -1 \le y \le 2\}$. We have $\int_{-1}^{2} \int_y^{4-y} dx\, dy = \int_{-1}^{2} (x)\Big|_y^{4-y} dy = \int_{-1}^{2} (4 - 2y)\, dy = (4 - 2y)\Big|_{-1}^{2} = 9.$



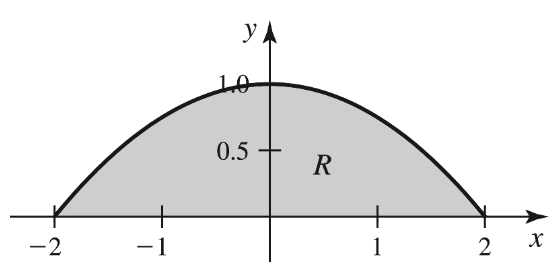
13.2.40 $R = \{(x,y)\,|\,0 \le x \le 4 - y^2,\ 0 \le y \le 2\}$. We have $\int_0^2 \int_0^{4-y^2} y\,dx\,dy = \int_0^2 (xy)\Big|_0^{4-y^2} dy = \int_0^2 (4y - y^3)\,dy = (2y^2 - y^4/4)\Big|_0^2 = 8 - 4 = 4$.



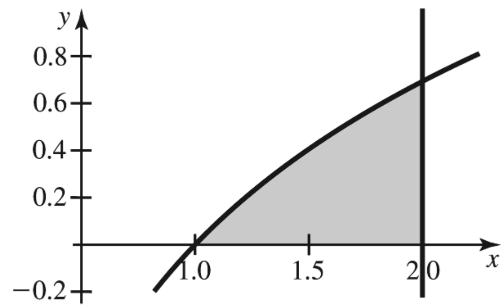
13.2.41 $R = \{(x,\,y)\ |\ -\sqrt{16-y^2} \le x \le \sqrt{16-y^2},\ 0 \le y \le 4\}$. We have $\int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 2xy\,dx\,dy = \int_0^4 \left(x^2\,y\right)\Big|_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} dy = \int_0^4 0\,dy = 0$.



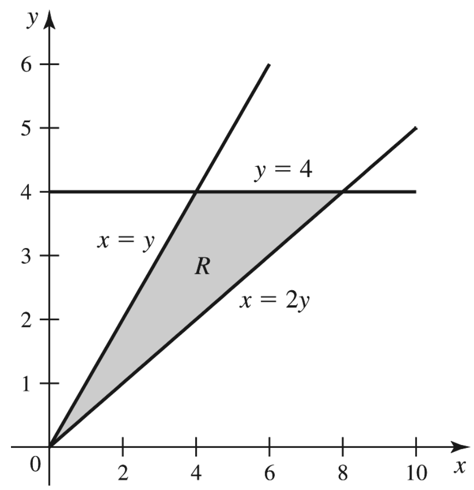
13.2.42 $R = \{(x,\,y)\ |\ -2\sqrt{1-y^2} \le x \le 2\sqrt{1-y^2},\ 0 \le y \le 1\}$. We have $\int_0^1 \int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} 2x\,dx\,dy = \int_0^1 \left(x^2\right)\Big|_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} dy = \int_0^1 0\,dy = 0$.



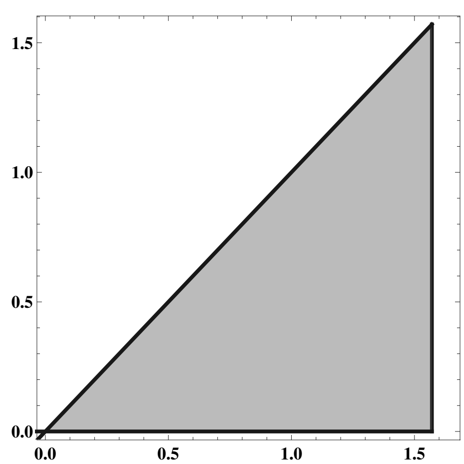
13.2.43 $R = \{(x,\,y)\ |\ e^y \le x \le 2,\ 0 \le y \le \ln 2\}$. We have $\int_0^{\ln 2} \int_{e^y}^2 \left(\frac{y}{x}\right)\,dx\,dy = \int_0^{\ln 2} \left(y \ln|\,x\,|\right)\Big|_{e^y}^2 dy = \int_0^{\ln 2} \left(y \ln 2 - y^2\right)\,dy = \left(\frac{\ln 2}{2} y^2 - \frac{y^3}{3}\right)\Big|_0^{\ln 2} = \frac{(\ln 2)^3}{6}$.



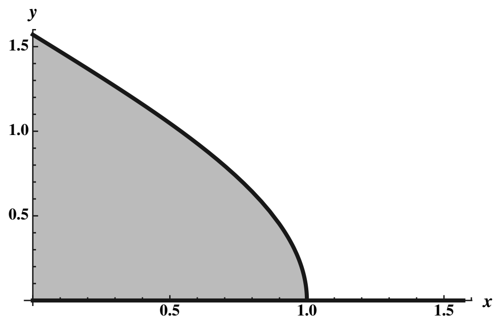
13.2.44 $R = \{(x, y) \mid y \le x \le 2y,\ 0 \le y \le 4\}$. We have $\int_0^4 \int_y^{2y} (x\,y)\,dx\,dy = \int_0^4 \left(\frac{x^2 y}{2}\right)\Big|_y^{2y} dy =$ $\int_0^4 \left(\frac{3}{2}y^3\right)\,dy = \left(\frac{3}{8}y^4\right)\Big|_0^4 = 96.$



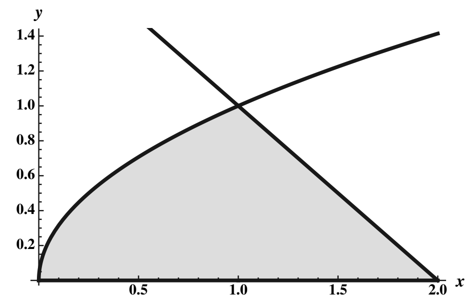
13.2.45 $R = \{(x, y) \mid y \le x \le \pi/2, 0 \le y \le \pi/2\}$. We have $\int_0^{\pi/2} \int_y^{\pi/2} 6\sin(2x-3y)\,dx\,dy = \int_0^{\pi/2}(-3\cos(2x-3y))\Big|_y^{\pi/2} dy = -3\int_0^{\pi/2}(\cos(\pi-3y) - \cos(-y))\,dy = (\sin(\pi-3y) - 3\sin(-y))\Big|_0^{\pi/2} = -1-(-3) = 2.$



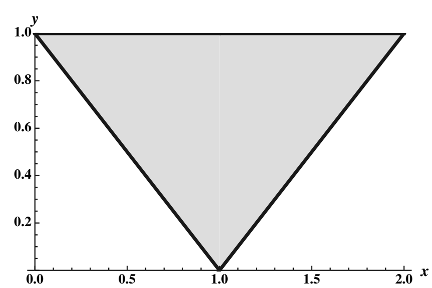
13.2.46 $R = \{(x, y) \mid 0 \le x \le \cos y, 0 \le y \le \pi/2\}$. We have
$\int_0^{\pi/2} \int_0^{\cos y} e^{\sin y}\,dx\,dy = \int_0^{\pi/2}(xe^{\sin y})\Big|_0^{\cos y} dy = \int \cos y e^{\sin y}\,dy = e^{\sin y}\Big|_0^{\pi/2} = e-1.$



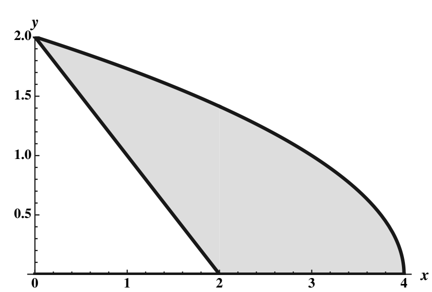
13.2.47 $\iint\limits_R 12y\,dA = \int_0^1 \int_{y^2}^{2-y} 12y\,dx\,dy = \int_0^1 (12xy)\Big|_{y^2}^{2-y}\,dy = 12\int_0^1 (2y - y^2 - y^3)\,dy = 12(y^2 - y^3/3 - y^4/4)\Big|_0^1 = 12(5/12) = 5.$



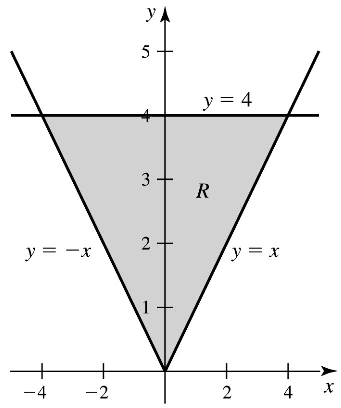
13.2.48 $\iint\limits_R y^2\,dA = \int_0^1 \int_{1-y}^{y+1} y^2\,dx\,dy = \int_0^1 (xy^2)\Big|_{1-y}^{y+1}\,dy = \int_0^1 (y^3 + y^2 - y^2 + y^3)\,dy = \int_0^1 2y^3\,dy = y^4/2\Big|_0^1 = 1/2.$



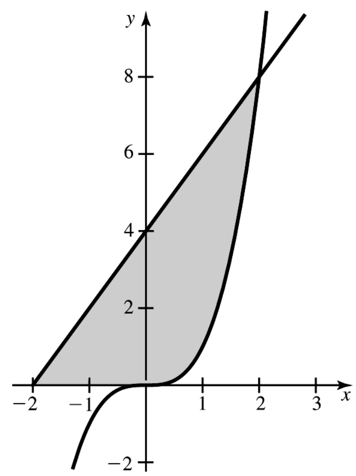
13.2.49 $\iint\limits_R 3xy\,dA = \int_0^2 \int_{2-y}^{4-y^2} 3xy\,dx\,dy = \int_0^2 (3x^2y/2)\Big|_{2-y}^{4-y^2}\,dy = \int_0^2 (3(4-y^2)^2y/2 - 3(2-y)^2y/2)\,dy = \frac{3}{2}\int_0^2 (16y - 8y^3 + y^5 - (4y - 4y^2 + y^3))\,dy = \frac{3}{2}\int_0^2 (y^5 - 9y^3 + 4y^2 + 12y)\,dy = \frac{3}{2}(y^6/6 - 9y^4/4 + 4y^3/3 + 6y^2)\Big|_0^2 = \frac{3}{2}(32/3 - 36 + 32/3 + 24) = 14.$



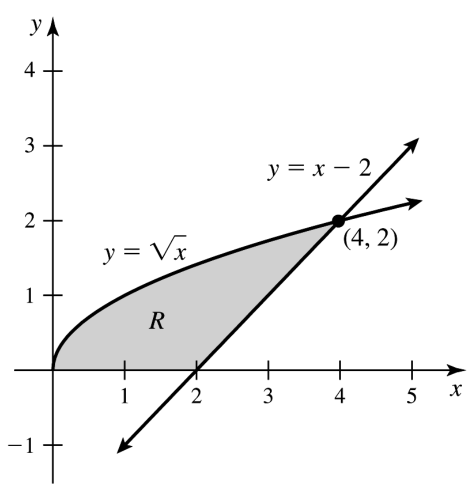
13.2.50 $\iint\limits_R (x+y)\,dA = \int_0^4 \int_{-y}^{y} (x+y)\,dx\,dy = \int_0^4 \left(\frac{x^2}{2} + xy\right)\Big|_{-y}^{y} dy = \int_0^4 \left(2y^2\right)\,dy = \left(\frac{2}{3}y^3\right)\Big|_0^4 = \frac{128}{4}.$



13.2.51 $\iint\limits_R 3x^2\,dA = \int_0^8 \int_{\frac{y-4}{2}}^{\sqrt[3]{y}} 3x^2\,dx\,dy = \int_0^8 \left(x^3\right)\Big|_{\frac{y-4}{2}}^{\sqrt[3]{y}} dy = \int_0^8 \left(y - (1/8)(y^3 - 12y^2 + 48y - 64)\right)\,dy =$ $\int_0^8 (-y^3/8 + 3y^2/2 - 5y + 8)\,dy = (-y^4/32 + y^3/2 - 5y^2/2 + 8y)\Big|_0^8 = 32.$



13.2.52 $\iint\limits_R x^2y\,dA = \int_0^2 \int_{y^2}^{y+2} x^2y\,dx\,dy = \int_0^2 \left(\frac{x^3y}{3}\right)\Big|_{y^2}^{y+2} dy = \int_0^2 \left(\frac{8}{3}y + 4y^2 + 2y^3 + \frac{y^4}{3} - \frac{y^7}{3}\right)\,dy =$ $\left(\frac{4}{3}y^2 + \frac{4}{3}y^3 + \frac{y^4}{2} + \frac{y^5}{15} - \frac{y^8}{24}\right)\Big|_0^2 = \frac{232}{15}.$



13.2.53 $V = \int_0^4 \int_0^{2-\frac{x}{2}} (8-2x-4y)\,dy\,dx = \int_0^4 \left(8y-2xy-2y^2\right)\Big|_0^{2-\frac{x}{2}} dx = \int_0^4 \left(8-4x-\frac{1}{2}x^2\right)\,dx =$ $\left(8x-2x^2-\frac{1}{6}x^3\right)\Big|_0^4 = \frac{32}{3}$, or $V = \int_0^2 \int_0^{4-2y} (8-2x-4y)\,dx\,dy = \frac{32}{3}$.

13.2.54 $\int_0^1 \int_0^{1-x^2} (1-y-x^2)\,dy\,dx = \int_0^1 (y-y^2/2-x^2y)\Big|_0^{1-x^2} dx = \int_0^1 (1-x^2-(1/2)(1-2x^2+x^4)-x^2+x^4)\,dx =$ $\frac{1}{2}\int_0^1 (x^4-2x^2+1)\,dx = \frac{1}{2}(x^5/5-2x^3/3+x)\Big|_0^1 = \frac{4}{15}$.

13.2.55 $V = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (12+x+y)\ dy\,dx = \int_{-1}^1 \left(12y+xy+\frac{1}{2}y^2\right)\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx =$ $\int_{-1}^1 \left(24\sqrt{1-x^2}+2x\sqrt{1-x^2}\right)\,dx = \left(12x\sqrt{1-x^2}+12\sin^{-1}x-\frac{2}{3}\left(1-x^2\right)^{3/2}\right)\Big|_{-1}^1 =$ $12\left(\sin^{-1}(1)-\sin^{-1}(-1)\right) = 12\pi$.

13.2.56 $V = \int_0^1 \int_0^x y^2\,dy\,dx = \int_0^1 \left(\frac{1}{3}y^3\right)\Big|_0^x dx = \frac{1}{3}\int_0^1 x^3\,dx = \frac{1}{12}\left(x^4\right)\Big|_0^1 = \frac{1}{12}$.

13.2.57 $\int_0^2 \int_{x^2}^{2x} f(x,y)\ dy\,dx = \int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\ dx\,dy$.

13.2.58 $\int_0^3 \int_0^{6-2x} f(x,y)\ dy\,dx = \int_0^6 \int_0^{\frac{6-y}{2}} f(x,y)\ dx\,dy$.

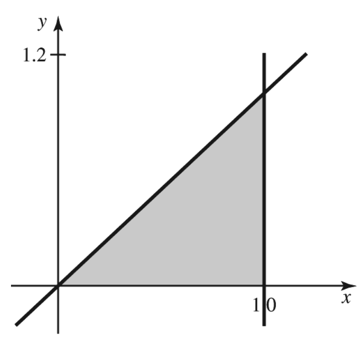
13.2.59 $\int_{1/2}^1 \int_0^{-\ln y} f(x,y)\ dx\,dy = \int_0^{\ln 2} \int_{1/2}^{e^{-x}} f(x,y)\ dy\,dx$.

13.2.60 $\int_0^1 \int_1^{e^y} f(x,y)\ dx\,dy = \int_1^e \int_{\ln x}^1 f(x,y)\ dy\,dx$.

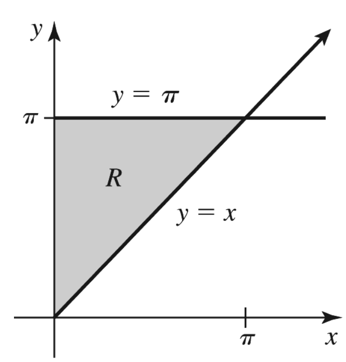
13.2.61 $\int_0^1 \int_0^{\cos^{-1} y} f(x,y)\ dx\,dy = \int_0^{\pi/2} \int_0^{\cos x} f(x,y)\ dy\,dx$.

13.2.62 $\int_1^e \int_0^{\ln x} f(x,y)\ dy\,dx = \int_0^1 \int_{e^y}^e f(x,y)\ dx\,dy$.

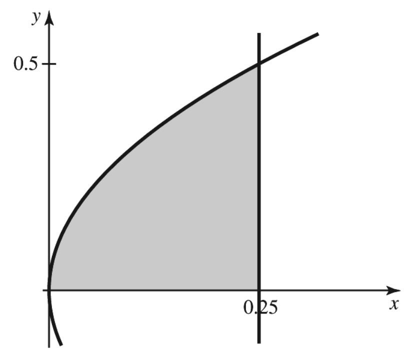
13.2.63 $\int_0^1 \int_y^1 e^{x^2}\,dx\,dy = \int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 \left(y\,e^{x^2}\right)\Big|_0^x dx = \int_0^1 \left(x\,e^{x^2}\right)\,dx = \left(\frac{1}{2}e^{x^2}\right)\Big|_0^1 = \frac{1}{2}(e-1)$.



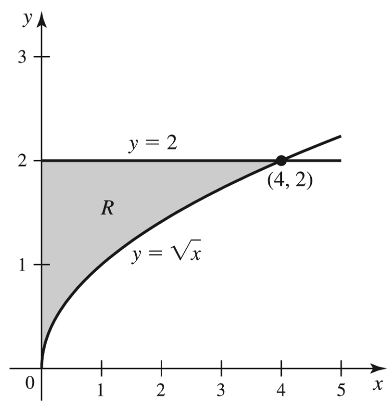
13.2.64 $\int_0^\pi \int_x^\pi \sin y^2\,dy\,dx = \int_0^\pi \int_0^y \sin y^2\,dx\,dy = \int_0^\pi \left(x\,\sin y^2\right)\Big|_0^y dy = \int_0^\pi y\,\sin y^2\,dy =$ $\left(-\frac{1}{2}\cos y^2\right)\Big|_0^\pi = \frac{1}{2}-\frac{1}{2}\cos\pi^2$.



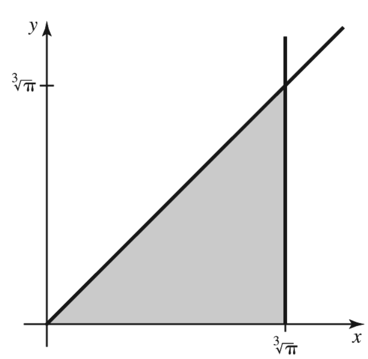
13.2.65 $\int_0^{1/2}\int_{y^2}^{1/4} y\cos\left(16\pi x^2\right)\,dx\,dy = \int_0^{1/4}\int_0^{\sqrt{x}} y\cos\left(16\pi x^2\right)\,dy\,dx = \int_0^{1/4}\left(\frac{1}{2}y^2\cos\left(16\pi x^2\right)\right)\Big|_0^{\sqrt{x}}\,dx = \int_0^{1/4}\frac{1}{2}x\cos\left(16\pi x^2\right)\,dx$. Let $u = 16\pi x^2$ so that $du = 32\pi\,dx$. We have $\int_0^{\pi}\frac{\cos u}{64\pi}\,du = \left(\frac{\sin u}{64\pi}\right)\Big|_{u=0}^{u=\pi} = 0$.



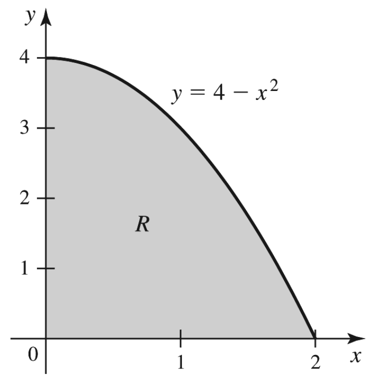
13.2.66 $\int_0^4\int_{\sqrt{x}}^2 \frac{x}{y^5+1}\,dy\,dx = \int_0^2\int_0^{y^2}\frac{x}{y^5+1}\,dx\,dy = \int_0^2\left(\frac{x^2}{2(y^5+1)}\right)\Big|_0^{y^2}\,dy = \int_0^2\frac{y^4}{2(y^5+1)}\,dy$. Let $u = y^5+1$ so that $du = 5y^4\,dy$. Substituting gives $\int_1^{33}\frac{1}{10u}\,du = \frac{1}{10}\left(\ln|u|\right)\Big|_{u=1}^{u=33} = \frac{\ln 33}{10}$.



13.2.67 $\int_0^{\sqrt[3]{\pi}}\int_y^{\sqrt[3]{\pi}} x^4\cos\left(x^2y\right)\,dx\,dy = \int_0^{\sqrt[3]{\pi}}\int_0^x x^4\cos\left(x^2y\right)\,dy\,dx = \int_0^{\sqrt[3]{\pi}}\left(x^2\sin\left(x^2y\right)\right)\Big|_0^x\,dx = \int_0^{\sqrt[3]{\pi}} x^2\sin\left(x^3\right)\,dx = -\frac{1}{3}\left(\cos x^3\right)\Big|_{u=0}^{u=\sqrt[3]{\pi}} = \frac{2}{3}$.



13.2.68 $\int_0^2\int_0^{4-x^2}\frac{x\,e^{2y}}{4-y}\,dy\,dx = \int_0^4\int_0^{\sqrt{4-y}}\frac{x\,e^{2y}}{4-y}\,dx\,dy = \int_0^4\left(\frac{x^2\,e^{2y}}{2(4-y)}\right)\Big|_0^{\sqrt{4-y}}\,dy = \int_0^4\frac{(4-y)\,e^{2y}}{2(4-y)}\,dy = \int_0^4\frac{e^{2y}}{2}\,dy = \frac{1}{4}\left(e^{2y}\right)\Big|_0^4 = \frac{1}{4}\left(e^8-1\right)$.



13.2.69 $V = \int_0^1 \int_0^{1-x} ((2 - x^2 - y^2) - (x^2 + y^2))\, dy\, dx = 2\int_0^1 \int_0^{1-x} (1 - x^2 - y^2)\, dy\, dx = 2\int_0^1 (y - x^2y - y^3/3)\Big|_0^{1-x}\, dx = \frac{2}{3}\int_0^1 ((3 - 3x) - (3x^2 - 3x^3) - (1 - 3x + 3x^2 - x^3))\, dx = \frac{2}{3}\int_0^1 (4x^3 - 6x^2 + 2)\, dx = \frac{2}{3}(x^4 - 2x^3 + 2x)\Big|_0^1 = \frac{2}{3}$.

13.2.70 $V = \int_0^1 \int_0^{1-x^2} (2 - y - 1)\, dy\, dx = \int_0^1 (y - y^2/2)\Big|_0^{1-x^2}\, dx = \frac{1}{2}\int_0^1 (2 - 2x^2 - (1 - 2x^2 + x^4))\, dx = \frac{1}{2}\int_0^1 (-x^4 + 1)\, dx = \frac{1}{2}(-x^5/5 + x)\Big|_0^1 = \frac{2}{5}$.

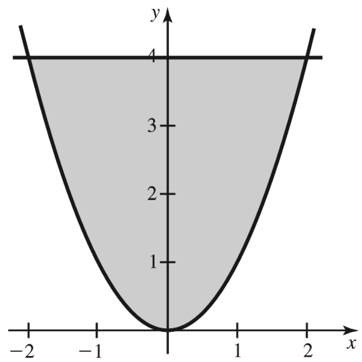
13.2.71 $V = \iint\limits_R \left(9 - x^2 - y^2\right)\, dA = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(9 - x^2 - y^2\right)\, dy\, dx =$
$\int_{-3}^{3} \left(\left(9 - x^2\right) y - \frac{1}{3}y^3\right) \Big|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\, dx = \int_{-3}^{3} \left(12\sqrt{9 - x^2} - \frac{4}{3}x^2\sqrt{9 - x^2}\right)\, dx =$
$\left(\frac{15}{2}x\sqrt{9 - x^2} - \frac{1}{3}x^2\sqrt{9 - x^2} + \frac{81}{2}\sin^{-1}\left(\frac{x}{3}\right)\right)\Big|_{-3}^{3} = \frac{81\pi}{2}$.

13.2.72 $V = \iint\limits_R \left((y + 1) - (x^2 + 1)\right)\, dA = \int_{-1}^{1} \int_{x^2}^{1} (y - x^2)\, dy\, dx = \int_{-1}^{1} \left(\frac{y^2}{2} - x^2y\right)\Big|_{x^2}^{1}\, dx =$
$\int_{-1}^{1} \left(\left(\frac{1}{2} - x^2\right) - \left(\frac{x^4}{2} - x^4\right)\right)\, dx = \int_{-1}^{1} \left(\frac{1}{2} - x^2 + \frac{x^4}{2}\right)\, dx = \left(\frac{x}{2} - \frac{x^3}{3} + \frac{x^5}{10}\right)\Big|_{-1}^{1} = \frac{8}{15}$.

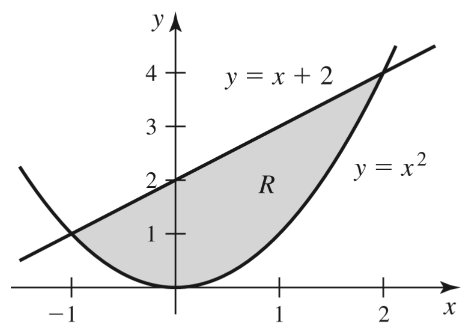
13.2.73 $V = \iint\limits_R (8 - 2x - 3y)\, dA = \int_0^1 \int_0^{2-x} (8 - 2x - 3y)\, dy\, dx = \int_0^1 \left((8 - 2x)\, y - \frac{3}{2}y^2\right)\Big|_0^{2-x}\, dx =$
$\int_0^1 \left(10 - 6x + \frac{1}{2}x^2\right)\, dx = \left(10x - 3x^2 + \frac{1}{6}x^3\right)\Big|_0^1 = \frac{43}{6}$.

13.2.74 $V = \iint\limits_R (e^{x-y} - (-e^{x-y}))\, dA = \int_0^1 \int_0^y 2e^{x-y}\, dx\, dy = \int_0^1 (2e^{x-y})\Big|_0^y\, dy = \int_0^1 2\,(1 - e^{-y})\, dy =$
$2\,(y + e^{-y})\Big|_0^1 = \frac{2}{e}$.

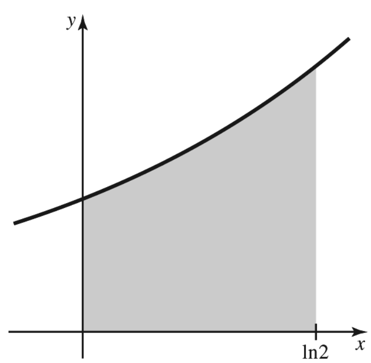
13.2.75 $A = \iint\limits_R 1\, dA = \int_{-2}^{2} \int_{x^2}^{4} 1\, dy\, dx = \int_{-2}^{2} (y)\Big|_{x^2}^{4}\, dy = \int_{-2}^{2} \left(4 - x^2\right)\, dx = \left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{2} = \frac{32}{3}$.



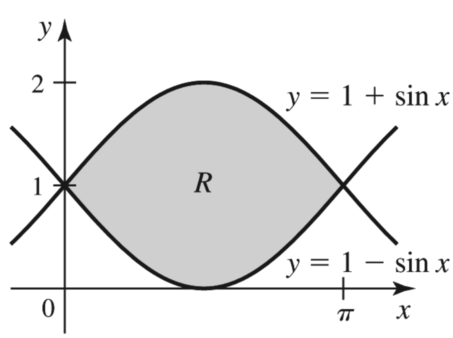
13.2.76 $A = \iint\limits_R 1\,dA = \int_{-1}^{2}\int_{x^2}^{x+2} 1\,dy\,dx = \int_{-1}^{2} (y)\Big|_{x^2}^{x+2}\,dx = \int_{-1}^{2}\left(x+2-x^2\right)\,dx =$
$\left(\frac{1}{2}x^2+2x-\frac{1}{3}x^3\right)\Big|_{-1}^{2} = \frac{9}{2}\,.$



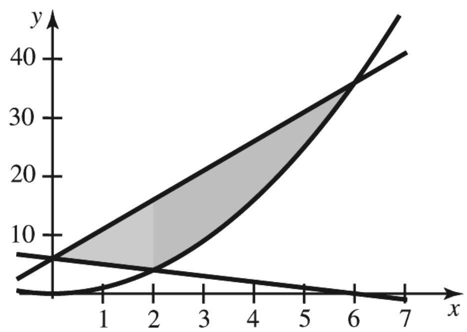
13.2.77 $A = \iint\limits_R 1\,dA = \int_0^{\ln 2}\int_0^{e^x} 1\,dy\,dx = \int_0^{\ln 2} (y)\Big|_0^{e^x}\,dx = \int_0^{\ln 2} e^x\,dx = (e^x)\Big|_0^{\ln 2} = 1.$



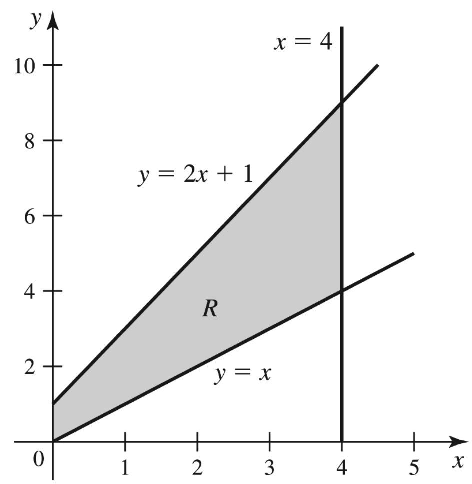
13.2.78 $A = \iint\limits_R 1\,dA = \int_0^{\pi}\int_{1-\sin x}^{1+\sin x} 1\,dy\,dx = \int_0^{\pi} (y)\Big|_{1-\sin x}^{1+\sin x}\,dx = \int_0^{\pi} 2\,\sin x\,dx = (-2\,\cos x)\Big|_0^{\pi} = 4.$



13.2.79 $A = \iint\limits_R 1\, dA. = \int_0^2 \int_{6-x}^{5x+6} 1\, dy\, dx + \int_2^6 \int_{x^2}^{5x+6} 1\, dy\, dx = \int_0^2 (y)\Big|_{6-x}^{5x+6} dx + \int_2^6 (y)\Big|_{x^2}^{5x+6} dx = \int_0^2 6x\, dx + \int_2^6 \left(5x + 6 - x^2\right) dx = \left(3x^2\right)\Big|_0^2 + \left(\frac{5}{2}x^2 + 6x - \frac{1}{3}x^3\right)\Big|_2^6 = \frac{140}{3}.$



13.2.80 $A = \iint\limits_R 1\, dA = \int_0^4 \int_x^{2x+1} 1\, dy\, dx = \int_0^4 (y)\Big|_x^{2x+1} dx = \int_0^4 (x+1)\, dx = \left(\frac{1}{2}x^2 + x\right)\Big|_0^4 = 12.$.



13.2.81

a. False. a and b must be constants or functions of y.

b. False. a and d must be constants.

c. False. Variable limits of integration are not allowed in the outermost integral.

13.2.82 $\iint\limits_R y\; dA = \int_0^{\pi/3} \int_0^{\sec x} y\, dy\, dx = \int_0^{\pi/3} \left(\frac{1}{2}y^2\right)\Big|_0^{\sec x} dx = \int_0^{\pi/3} \frac{1}{2}\sec^2 x\, dx = \left(\frac{1}{2}\tan x\right)\Big|_0^{\pi/3} = \frac{\sqrt{3}}{2}.$

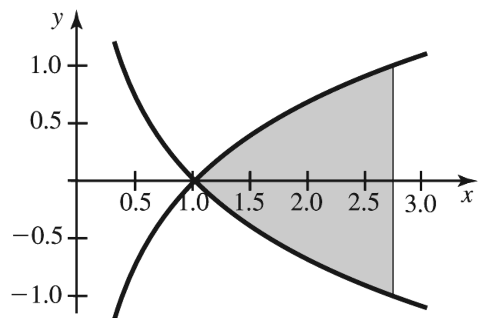
13.2.83 $\iint\limits_R (x+y)\; dA = \int_{1/2}^2 \int_{1/x}^{5/2-x} (x+y)\; dy\, dx = \int_{1/2}^2 \left(xy + \frac{1}{2}y^2\right)\Big|_{\frac{1}{x}}^{\frac{5}{2}-x} dx =$
$\int_{1/2}^2 \left(\frac{17}{8} - \frac{x^2}{2} - \frac{1}{2x^2}\right) dx = \left(\frac{17}{8}x - \frac{x^3}{6} + \frac{1}{2x}\right)\Big|_{1/2}^1 = \frac{9}{8}.$

13.2.84 $\iint\limits_R \frac{xy}{1+x^2+y^2}\; dA = \int_0^2 \int_0^x \frac{xy}{1+x^2+y^2}\, dy\, dx = \int_0^2 \frac{1}{2}x \ln\left(1 + x^2 + y^2\right)\Big|_0^x dx =$
$\frac{1}{2}\int_0^2 \left(x\ln\left(1+2x^2\right) - x\ln\left(1+x^2\right)\right) dx$. Let $u = 1 + 2x^2$ and $v = 1 + x^2$. Substituting gives $\frac{1}{8}\int_1^9 \ln u\, du - \frac{1}{4}\int_1^5 \ln v\, dv = \frac{1}{8}\left(u\ln u - u\right)\Big|_{u=1}^{u=9} - \frac{1}{4}\left(v\ln v - v\right)\Big|_{v=1}^{v=5} = \left(\frac{9}{8}\ln 9 - 1\right) - \left(\frac{5}{4}\ln 5 - 1\right) = \frac{9}{8}\ln 9 - \frac{5}{4}\ln 5.$

13.2.85 $\iint\limits_R x\sec^2 y\, dA = \int_0^{\sqrt{\pi/2}}\int_0^{x^2} x\sec^2 y\, dy\, dx = \int_0^{\sqrt{\pi/2}} x\tan y\Big|_0^{x^2}\, dx = \int_0^{\sqrt{\pi/2}} x\tan x^2\, dx =$ $\left(\frac{1}{2}\ln|\sec x^2|\right)\Big|_0^{\sqrt{\pi/2}} = \frac{1}{2}\ln\left(\sqrt{2}\right) = \frac{1}{4}\ln 2.$

13.2.86 $V = \iint\limits_R (z_{\text{top}} - z_{\text{bottom}})\; dA$ where R is enclosed by the intersection of the surfaces $z = x^2+y^2$ and $z = 1-2y$. Setting $x^2+y^2 = 1-2y$ gives the circle $x^2+(y+1)^2 = 2$.
So $V = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{-1-\sqrt{2-x^2}}^{-1+\sqrt{2-x^2}} \left((1-2y)-\left(x^2+y^2\right)\right) dy\, dx = \int_{-\sqrt{2}}^{\sqrt{2}}\left(y-y^2-x^2y-\frac{1}{3}y^3\right)\Big|_{-1-\sqrt{2-x^2}}^{-1+\sqrt{2-x^2}} dx =$
$\int_{-\sqrt{2}}^{\sqrt{2}}\left(\frac{8}{3}\sqrt{2-x^2}-\frac{4}{3}x^2\sqrt{2-x^2}\right) dx = \left(\frac{5}{3}x\sqrt{2-x^2}-\frac{1}{3}x^3\sqrt{2-x^2}+2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right)\Big|_{-\sqrt{2}}^{\sqrt{2}} = 2\pi.$

13.2.87 $\int_0^1\int_{e^y}^{e} f(x, y)\; dx\, dy + \int_{-1}^0\int_{e^{-y}}^{e} f(x, y)\; dx\, dy = \int_1^e\int_{-\ln x}^{\ln x} f(x, y)\; dy\, dx.$



13.2.88

a. $A = \int_{-1}^0\int_{-1-x}^{1+x} 1\, dy\, dx + \int_0^1\int_{x-1}^{1-x} 1\, dy\, dx = \int_{-1}^0 (2+2x)\; dy\, dx + \int_0^1 (2-2x)\; dx = \left(2x+x^2\right)\Big|_{-1}^{0} +$ $\left(2x-x^2\right)\Big|_0^1 = 1+1 = 2.$

b. $A = \int_{-1}^0\int_{-1-x}^{1+x} (12-3x-4y)\; dy\, dx + \int_0^1\int_{x-1}^{1-x} (12-3x-4y)\; dy\, dx =$
$\int_{-1}^0 \left((12-3x)\,y-2y^2\right)\Big|_{-1-x}^{1+x} dx + \int_0^1 \left((12-3x)\,y-2y^2\right)\Big|_{x-1}^{1-x} dx = \int_{-1}^0 \left(24+18x-6x^2\right) dy\, dx +$
$\int_0^1 \left(24-30x+6x^2\right)\; dx = \left(24x+9x^2-2x^3\right)\Big|_{-1}^{0} + \left(24x-15x^2+2x^3\right)\Big|_0^1 = 13+11 = 24.$

c. $V = \int_{-1}^0\int_{-1-x}^{1+x}\sqrt{1-x^2}\, dy\, dx + \int_0^1\int_{x-1}^{1-x}\sqrt{1-x^2}\, dy\, dx = 2\int_0^1\int_{x-1}^{1-x}\sqrt{1-x^2}\, dy\, dx$, where the last equality is due to symmetry. Then we have $2\int_0^1\left(y\sqrt{1-x^2}\right)\Big|_{x-1}^{1-x} dx = 2\int_0^1 (2-2x)\sqrt{1-x^2}\, dx =$
$4\int_0^1\left(\sqrt{1-x^2}-x\sqrt{1-x^2}\right)\; dx = 4\left(\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x+\frac{1}{3}\left(1-x^2\right)^{3/2}\right)\Big|_0^1 =$
$4\left(0+\frac{\pi}{4}+0-0-0-\frac{1}{3}\right) = \pi-\frac{4}{3}.$

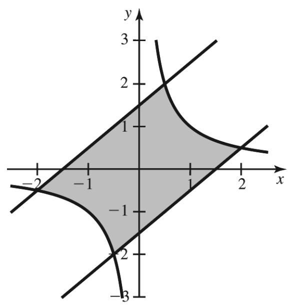
d. Use symmetry.
$V = 4\int_0^1\int_0^{1-x} 6(1-x-y)\; dy\, dx = 24\int_0^1\left((1-x)\,y-\frac{1}{2}y^2\right)\Big|_0^{1-x} dx = 24\int_0^1\frac{1}{2}(1-x)^2\; dx =$
$12\left(-\frac{1}{3}(1-x)^3\right)\Big|_0^1 = 4.$

13.2.89 $\overline{f} = \frac{1}{\text{Area of } R}\iint\limits_R f(x, y)\; dA = \frac{1}{\frac{1}{2}a^2}\int_0^a\int_0^{a-x}(a-x-y)\; dy\, dx = \frac{2}{a^2}\int_0^2\left((a-x)\,y-\frac{1}{2}y^2\right)\Big|_0^{a-x} dx$
$= \frac{2}{a^2}\int_1^9\frac{1}{2}(a-x)^2\; dx = \frac{1}{a^2}\left(-\frac{1}{3}(a-x)^3\right)\Big|_0^a = \frac{1}{a^2}\left(\frac{a^3}{3}\right) = \frac{a}{3}.$

13.2.90 $\overline{f} = \frac{1}{\text{Area of } R}\iint\limits_R f(x, y)\ dA = \frac{1}{\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(a^2 - x^2 - y^2\right)\, dy\, dx$, which by symmetry is equal to $\frac{4}{\pi a^2}\int_0^a\int_0^{\sqrt{a^2-x^2}}\left(a^2 - x^2 - y^2\right)\, dy\, dx = \frac{4}{\pi a^2}\int_0^a\left(\left(a^2 - x^2\right)y - \frac{1}{3}y^3\right)\Big|_0^{\sqrt{a^2-x^2}}\, dx =$
$\frac{4}{\pi a^2}\int_0^a \frac{2}{3}\left(a^2 - x^2\right)^{3/2}\, dx = \frac{8}{3\pi a^2}\left(-\frac{x}{8}\left(2x^2 - 5a^2\right)\sqrt{a^2 - x^2} + \frac{3a^4}{8}\sin^{-1}\left(\frac{x}{a}\right)\right)\Big|_0^a = \frac{a^2}{3}$.

13.2.91

a.

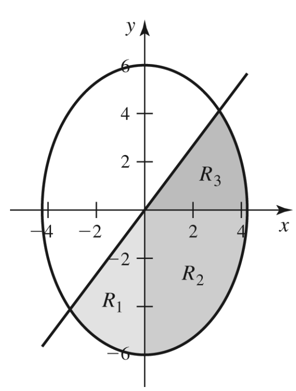


b. $A = \iint\limits_R 1\, dA = \iint\limits_{R_1} 1\, dA + \iint\limits_{R_2} 1\, dA + \iint\limits_{R_3} 1\, dA = \int_{-2}^{-1/2}\int_{1/x}^{x+3/2} 1\, dy\, dx + \int_{-1/2}^{1/2}\int_{x-3/2}^{x+3/2} 1\, dy\, dx +$
$\int_{1/2}^{2}\int_{x-3/2}^{1/x} 1\, dy\, dx = \int_{-2}^{-1/2}\left(x + \frac{3}{2} - \frac{1}{x}\right)\, dx + \int_{-1/2}^{1/2}(3)\, dx + \int_{1/2}^{2}\left(\frac{1}{x} - x + \frac{3}{2}\right)\, dx =$
$\left(\frac{1}{2}x^2 + \frac{3}{2}x - \ln|x|\right)\Big|_{-2}^{-1/2} + (3x)\Big|_{-1/2}^{1/2} + \left(\ln|x| - \frac{1}{2}x^2 + \frac{3}{2}x\right)\Big|_{1/2}^{2} = \frac{3}{8} + 2\ln 2 + 3 + \frac{3}{8} + 2\ln 2 = \frac{15}{4} + 4\ln 2.$

c. $\iint\limits_R xy\ dA = \int_{-2}^{-1/2}\int_{1/x}^{x+3/2} xy\, dy\, dx + \int_{-1/2}^{1/2}\int_{x-3/2}^{x+3/2} xy\, dy\, dx + \int_{1/2}^{2}\int_{x-3/2}^{1/x} xy\, dy\, dx =$
$\int_{-2}^{-1/2}\left(\frac{1}{2}xy^2\right)\Big|_{1/x}^{x+3/2}\, dx + \int_{-1/2}^{1/2}\left(\frac{1}{2}xy^2\right)\Big|_{x-3/2}^{x+3/2}\, dx + \int_{1/2}^{2}\left(\frac{1}{2}xy^2\right)\Big|_{x-3/2}^{1/x}\, dx =$
$\frac{1}{2}\int_{-2}^{-1/2}\left(x^3 + 3x^2 + \frac{9}{4}x - \frac{1}{x}\right)\, dx + \frac{1}{2}\int_{-1/2}^{1/2}\left(6x^2\right)\, dx + \frac{1}{2}\int_{1/2}^{2}\left(\frac{1}{x} - \frac{9}{4}x + 3x^2 - x^3\right)\, dx =$
$\frac{1}{2}\left(\frac{1}{4}x^4 + x^3 + \frac{9}{8}x^2 - \ln|x|\right)\Big|_{-2}^{-1/2} + \left(x^3\right)\Big|_{-1/2}^{1/2} + \frac{1}{2}\left(\ln|x| - \frac{9}{8}x^2 + x^3 - \frac{1}{4}x^4\right)\Big|_{1/2}^{2} =$
$\frac{1}{2}\left(-\frac{21}{64} + 2\ln 2\right) + \frac{1}{4} + \frac{1}{2}\left(-\frac{21}{64} + 2\ln 2\right) = -\frac{5}{64} + 2\ln 2.$

13.2.92

a.



b. $A = \iint\limits_R 1\,dA = \iint\limits_{R_1} 1\,dA + \iint\limits_{R_2} 1\,dA + \iint\limits_{R_3} 1\,dA$. Note: By symmmetry we see that the area of $(R_1 + R_3)$ is equal to the area of R_2. Thus $A = 2\iint\limits_{R_2} 1\,dA = 2\int_0^{3\sqrt{2}}\int_{-\sqrt{36-2x^2}}^0 1\,dy\,dx = 2\sqrt{2}\int_0^{3\sqrt{2}}\sqrt{18-x^2}\,dx =$ $2\sqrt{2}\left(\frac{1}{2}x\sqrt{18-x^2} + 9\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\right)\Big|_0^{3\sqrt{2}} = 9\sqrt{2}\pi$.

c. Again by symmetry $R_1 + R_3$ will be equivalent to R_2 but with opposite sign. $xy > 0$ in Quadrant I and III and $xy < 0$ in Quadrant II and IV. By symmetry about the origin we can deduce equivalent components of surface above the xy-plane over regions R_1 and R_3 as below the xy-plane in R_2. Thus $\iint\limits_R xy\ dA = 0$.

13.2.93 $\int_1^\infty \int_0^{e^{-x}} xy\,dy\,dx = \int_1^\infty \left(\frac{1}{2}xy^2\right)\Big|_0^{e^{-x}} dx = \int_1^\infty \frac{1}{2}x\,e^{-2x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{2}x\,e^{-2x}\,dx =$ $\lim_{b\to\infty}\frac{1}{2}\left(e^{-2x}\left(-\frac{1}{2}x - \frac{1}{4}\right)\right)\Big|_1^b = \frac{1}{4}\lim_{b\to\infty}\left(e^{-2b}\left(-b-\frac{1}{2}\right) + e^{-2}\left(\frac{3}{2}\right)\right) = \frac{1}{4}\left(0 + \frac{3}{2}e^{-2}\right) = \frac{3}{8e^2}$.

13.2.94 $\int_1^\infty \int_0^{1/x^2} \frac{2y}{x}\,dy\,dx = \int_1^\infty \left(\frac{y^2}{x}\right)\Big|_0^{1/x^2} dx = \int_1^\infty \frac{1}{x^5}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^5}\,dx =$ $\lim_{b\to\infty}\left(-\frac{1}{4x^4}\right)\Big|_1^b = \lim_{b\to\infty}\left(-\frac{1}{4b^4} + \frac{1}{4}\right) = 0 + \frac{1}{4} = \frac{1}{4}$.

13.2.95 $\int_0^\infty \int_0^\infty e^{-x-y}\,dy\,dx = \int_0^\infty \left(\lim_{b\to\infty}\int_0^b e^{-x-y}\,dy\right) dx = \int_0^\infty \left(\lim_{b\to\infty}\left(-e^{-x-y}\right)\Big|_0^b\right) dx =$ $\int_0^\infty \left(\lim_{b\to\infty}\left(e^{-x} - e^{-x-b}\right)\right) dx = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b -e^{-x}\,dx = \lim_{b\to\infty}\left(-e^{-x}\right)\Big|_0^b =$ $\lim_{b\to\infty}\left(1 - e^{-b}\right) = 1$.

13.2.96 First compute $\int_{-\infty}^\infty \frac{1}{u^2+1}\,du$. This is equal to $\int_{-\infty}^0 \frac{1}{u^2+1}\,du + \int_0^\infty \frac{1}{u^2+1}\,du = \lim_{a\to-\infty}\int_a^0 \frac{1}{u^2+1}\,du +$ $\lim_{b\to\infty}\int_0^b \frac{1}{u^2+1}\,du = \lim_{a\to-\infty}\left(\tan^{-1}u\Big|_{u=a}^{u=0}\right) + \lim_{b\to\infty}\left(\tan^{-1}u\Big|_{u=0}^{u=b}\right) =$
$\lim_{a\to-\infty} -\tan^{-1}a + \lim_{b\to\infty}\tan^{-1}b = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Now use this result to compute the inner integral: $\int_{-\infty}^\infty \frac{1}{(x^2+1)(y^2+1)}\,dy = \frac{1}{x^2+1}\int_{-\infty}^\infty \frac{1}{y^2+1}\,dy = \frac{\pi}{x^2+1}$, so the integral is $\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{(x^2+1)(y^2+1)}\,dy\,dx = \int_{-\infty}^\infty \frac{\pi}{x^2+1}\,dx = \pi^2$.

13.2.97 $V = \iint\limits_R f(x, y)\ dA$ (orient y-axis vertically) $= \iint\limits_R (y_{\text{top}} - y_{\text{bottom}})\ dzdx = \int_0^5\int_0^2\Big[(z+1) -$ $\left(\frac{-z-1}{2}\right)\Big]\ dz\,dx = \int_0^5\int_0^2\left(\frac{3}{2}z + \frac{3}{2}\right)\ dz\,dx = \int_0^5\left(\frac{3}{4}z^2 + \frac{3}{2}z\right)\Big|_0^2 dx = \int_0^5 6\,dx = (6x)\Big|_0^5 = 30$.

13.2.98 The equation of the top plane can be found by using $z - z_0 = m_x(x - x_0) + m_y(y - y_0)$, where m_x (slope in the x direction)$= -\frac{d}{a}$ and m_y (slope in the y direction)$= -\frac{d(b-a)}{ac}$. Use $(x_0,\ y_0,\ z_0) = (0,\ 0,\ d)$ to get $z = d - \frac{d}{a}x + \frac{d(b-a)}{ac}y$. Then $V = \iint\limits_R \left(d - \frac{d}{a}x + \frac{d(b-a)}{ac}y\right)\ dA = d\int_0^c\int_{cy/b}^{\frac{b-a}{c}y+a}\left(1 - \frac{x}{a} + \frac{(b-a)}{ac}y\right)dx\,dy =$ $\frac{cd}{12ab^2}\left(2a^2b^2 + 2ab^3 + 2b^4 - 2abc^2 - 4b^2c^2 + 2c^4\right)$.

13.2.99 $V = \iint\limits_R (4 - x - y)\ dA = \int_{-1}^1\int_{-1}^1 (4 - x - y)\ dy\,dx = \int_{-1}^1\left((4-x)\,y - \frac{1}{2}y^2\right)\Big|_{-1}^1 dx =$ $\int_{-1}^1 2\,(4 - x)\ dx = 2\left(4x - \frac{1}{2}x^2\right)\Big|_{-1}^1 = 16$.

13.2.100 $V = \iint\limits_R ((1-x) - (x-1))\ dA = 2\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x)\ dy\,dx =$
$\int_{-1}^1\Big[(1-x)\,y\Big]\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = 4\int_{-1}^1 (1-x)\sqrt{1-x^2}\,dx = 4\int_{-1}^1\left(\sqrt{1-x^2} - x\sqrt{1-x^2}\right)\ dx =$
$\Big[2\left(x\sqrt{1-x^2} + \sin^{-1}x\right) + \frac{4}{3}\left(1-x^2\right)^{3/2}\Big]\Big|_{-1}^1 = 2\left(\sin^{-1}(1) - \sin^{-1}(-1)\right) = 2\pi$.

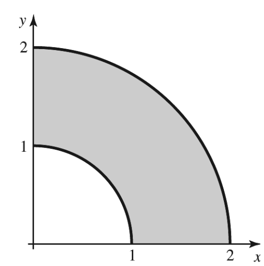
13.2.101 $V = \iint\limits_R (a(2-x) - a(x-2))\ dA = 2a\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2-x)\ dy\,dx =$
$2a\int_{-1}^{1}\left[(2-x)\,y\right]\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = 4a\int_{-1}^{1}(2-x)\sqrt{1-x^2}\,dx = 4a\int_{-1}^{1}\left(2\sqrt{1-x^2} - x\sqrt{1-x^2}\right)dx =$
$\left[4a\left(x\sqrt{1-x^2} + \sin^{-1}x\right) + \frac{4a}{3}\left(1-x^2\right)^{3/2}\right]\Big|_{-1}^{1} = 4a\left(\sin^{-1}(1) - \sin^{-1}(-1)\right) = 4a\pi.$

13.2.102 Consider the inner integral. If $m > 0$, the inner integral is $\int_0^{1/x}\frac{y^m}{x^n}\,dy = \frac{y^{m+1}}{(m+1)x^n}\Big|_0^{1/x} = \frac{1}{(m+1)x^{n+m+1}}$. If $m < 0$ and $m \neq -1$ then the inner integral is improper: $\int_0^{1/x}\frac{y^m}{x^n}\,dy = \lim_{a\to 0^+}\int_a^{1/x}\frac{y^m}{x^n}\,dy = \lim_{a\to 0^+}\left(\frac{y^{m+1}}{(m+1)x^n}\Big|_a^{1/x}\right) = \lim_{a\to 0^+}\left(\frac{1}{(m+1)x^{n+m+1}} - \frac{a^{m+1}}{(m+1)x^n}\right)$. This limit exists and equals $\frac{1}{(m+1)x^{n+m+1}}$ when $m > -1$. If $m = -1$, the inner integral is improper and diverges. Thus the inner integral $\int_0^{1/x}\frac{y^m}{x^n}\,dy = \frac{1}{(m+1)x^{n+m+1}}$ for $m > -1$. Now consider the double integral: $\int_1^{\infty}\int_0^{1/x}\frac{y^m}{x^n}\,dy\,dx = \int_1^{\infty}\frac{1}{(m+1)x^{n+m+1}}\,dx = \lim_{a\to\infty}\int_1^a\frac{x^{-n-m-1}}{m+1}\,dx = \lim_{a\to\infty}\frac{x^{-n-m}}{(m+1)(-n-m)}\Big|_1^a = \lim_{a\to\infty}\frac{a^{-n-m}-1}{(m+1)(-n-m)} = \frac{1}{(m+1)(n+m)}$ when $n+m > 0$. If $n+m < 0$, the integral diverges. If $n+m = 0$, the integral is improper and diverges. Thus $\int_1^{\infty}\int_0^{1/x}\frac{y^m}{x^n}\,dy\,dx = \frac{1}{(m+1)(n+m)}$ when $m > -1$ and $n+m > 0$.

13.2.103 $\iint\limits_{R_1} x^{-n}\ dA = \int_1^{\infty}\int_1^2 x^{-n}\,dy\,dx = \lim_{b\to\infty}\int_1^b\int_1^2 x^{-n}\,dy\,dx = \lim_{b\to\infty}\int_1^b(2-1)\,x^{-n}\,dx =$
$\lim_{b\to\infty}\int_1^b x^{-n}\,dx$ which converges if $n > 1$. $\iint\limits_{R_2} x^{-n}\ dA = \int_1^{\infty}\int_1^2 x^{-n}\,dx\,dy = \lim_{b\to\infty}\int_1^b\left(\frac{x^{-n+1}}{1-n}\right)\Big|_1^2 dy =$
$\lim_{b\to\infty}\int_1^b\left(\frac{2^{1-n}-1}{1-n}\right)dy = \lim_{b\to\infty}\left(\frac{2^{1-n}-1}{1-n}\right)y\Big|_1^b = \lim_{b\to\infty}\left(\frac{2^{1-n}-1}{1-n}\right)(b-1)$, which diverges when $n > 1$.

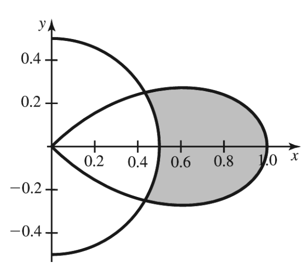
13.3 Double Integrals in Polar Coordinates

13.3.1 It is called a polar rectangle because it is analogous to a cartesian rectangle; they both have each variable constrained by constants.



13.3.2 $\iint\limits_R f(x,\,y)\ dA = \int_\alpha^\beta\int_a^b f(r\cos\theta,\, r\sin\theta)\ r\,dr\,d\theta.$

13.3.3 $R = \left\{(r,\,\theta) : \frac{1}{2} \le r \le \cos 2\theta,\ -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}\right\}$

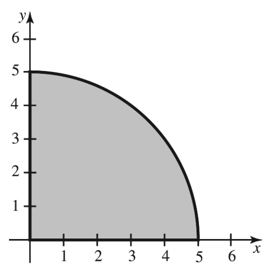


13.3.4 Geometrically $dx\,dy$ represents the length times width of a rectangular subsection of the region of integration. For equal partitions with respect to x and y each subsection has equivalent area. In polar coordinates each partition is a portion of a circular sector. For equal partitions with respect to r and θ the areas of the subsections are proportional to the distance from the origin, or r.

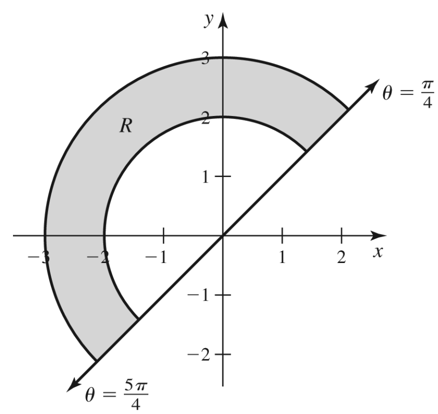
13.3.5 For $R = \left\{(r,\theta) : g(\theta) \le r \le f(\theta),\, \alpha \le \theta \le \beta\right\}$, the area of R given by $\iint\limits_R (1)\, dy\,dx$ converts to polar coordinates $\int_\alpha^\beta \int_{g(\theta)}^{f(\theta)} r\,dr\,d\theta$.

13.3.6 For $R = \left\{(r,\theta) : g(\theta) \le r \le f(\theta),\, \alpha \le \theta \le \beta\right\}$ the average value of a function $f(x,y)$ is given by $\overline{f} = \frac{1}{\text{area of } R} \iint\limits_R f(x,y)\,dy\,dx$ which converts to polar coordinates $\frac{1}{\text{area of } R} \int_\alpha^\beta \int_{g(\theta)}^{f(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$.

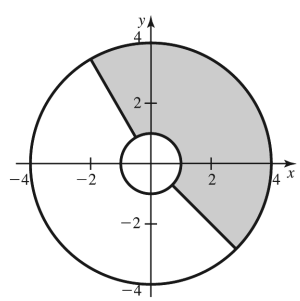
13.3.7



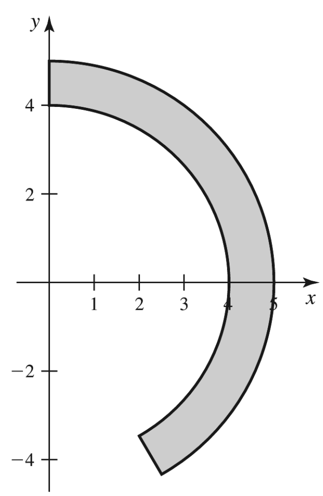
13.3.8



13.3.9



13.3.10



13.3.11 $V = \iint\limits_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta = \int_0^{2\pi} \int_0^1 \left(4 - r^2\right) r\,dr\,d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4\right)\Big|_0^1 d\theta = \int_0^{2\pi} \left(\frac{7}{4}\right) d\theta$ $= \frac{7\pi}{2}$.

13.3.12 $V = \iint\limits_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta = \int_0^{2\pi} \int_0^2 \left(4 - r^2\right) r dr\,d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4\right)\Big|_0^2 d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi$.

13.3.13 $V = \iint\limits_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta = \int_0^{2\pi} \int_1^2 \left(4 - r^2\right) r\,dr\,d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4\right)\Big|_1^2 d\theta = \int_0^{2\pi} \frac{9}{4}\,d\theta =$ $\frac{9\pi}{2}$.

13.3.14 $V = \iint\limits_R f(r\cos\theta,\, r\sin\theta)\, r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_1^2 \left(4 - r^2\right) r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \left(2r^2 - \frac{1}{4}r^4\right)\Big|_1^2\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{9}{4}\, d\theta = \frac{9\pi}{4}$.

13.3.15 $V = \iint\limits_R \left(5 - \sqrt{1+x^2+y^2}\right)\, dA = \int_0^{2\pi}\int_0^2 \left(5 - \sqrt{1+r^2}\right) r\,dr\,d\theta =$
$\int_0^{2\pi} \left(\frac{5}{2}r^2 - \frac{1}{3}\left(1+r^2\right)^{3/2}\right)\Big|_0^2 d\theta = \int_0^{2\pi}\left(10 - \frac{5\sqrt{5}-1}{3}\right) d\theta = \left(\frac{31-5\sqrt{5}}{3}\right)\cdot\theta\Big|_{\theta=0}^{\theta=2\pi} = 2\pi\left(\frac{31-5\sqrt{5}}{3}\right) = \frac{(62-10\sqrt{5})\pi}{3}$.

13.3.16 $V = \iint\limits_R \left(5 - \sqrt{1+x^2+y^2}\right)\, dA = \int_0^{\pi}\int_0^1 \left(5 - \sqrt{1+r^2}\right) r\,dr\,d\theta =$
$\int_0^{\pi} \left(\frac{5}{2}r^2 - \frac{1}{3}\left(1+r^2\right)^{3/2}\right)\Big|_0^1 d\theta = \int_0^{\pi}\left(\frac{5}{2} + \frac{1-2\sqrt{2}}{3}\right) d\theta = \frac{(17-4\sqrt{2})\pi}{6}$.

13.3.17 $V = \iint\limits_R \left(5 - \sqrt{1+x^2+y^2}\right)\, dA = \int_0^{2\pi}\int_{\sqrt{3}}^{2\sqrt{2}} \left(5 - \sqrt{1+r^2}\right) r\,dr\,d\theta =$
$\int_0^{2\pi} \left(\frac{5}{2}r^2 - \frac{1}{3}\left(1+r^2\right)^{3/2}\right)\Big|_{\sqrt{3}}^{2\sqrt{2}} d\theta = \int_0^{2\pi} (20 - 9 - (15/2 - 8/3))\, d\theta = \int_0^{2\pi}(37/6)\, d\theta = 37\pi/3$.

13.3.18 $V = \iint\limits_R \left(5 - \sqrt{1+x^2+y^2}\right)\, dA = \int_{-\pi/2}^{\pi}\int_{\sqrt{3}}^{\sqrt{15}} \left(5 - \sqrt{1+r^2}\right) r\,dr\,d\theta =$
$\int_{-\pi/2}^{\pi} \left(\frac{5}{2}r^2 - \frac{1}{3}\left(1+r^2\right)^{3/2}\right)\Big|_{\sqrt{3}}^{\sqrt{15}} d\theta = \int_{-\pi/2}^{\pi} (75/2 - 64/3 - (15/2 - 8/3))\, d\theta = \int_{-\pi/2}^{\pi}(34/3)\, d\theta =$
$\left(\frac{34}{3}\cdot\frac{3\pi}{2}\right) = 17\pi$.

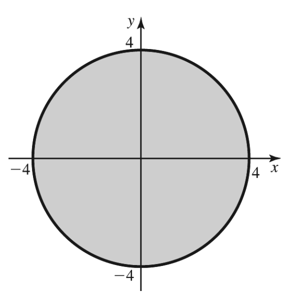
13.3.19 Let $x = r\cos\theta$ and $y = r\sin\theta$. $V = \iint\limits_R ((2-x^2-y^2)-(x^2+y^2))\, dA = \int_0^{2\pi}\int_0^1 ((2-r^2)-(r^2))r\,dr\,d\theta =$
$\int_0^{2\pi}\int_0^1 (2r - 2r^3)\, dr\,d\theta = \int_0^{2\pi}(r^2 - r^4/2)\Big|_0^1 d\theta = \frac{1}{2}\cdot 2\pi = \pi$.

13.3.20 Let $x = r\cos\theta$ and $y = r\sin\theta$. $V = \iint\limits_R ((27-x^2-2y^2)-(2x^2+y^2))\, dA = \int_0^{2\pi}\int_0^3 (27-3r^2)r\,dr\,d\theta =$
$\int_0^{2\pi}\int_0^3 (27r - 3r^3)\, dr\,d\theta = \int_0^{2\pi}(27r^2/2 - 3r^4/4)\Big|_0^3 d\theta = ((243/2) - (243/4))\cdot 2\pi = \frac{243\pi}{2}$.

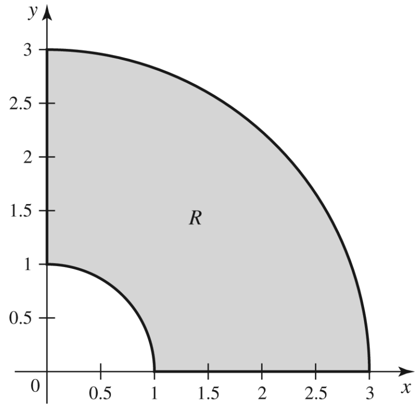
13.3.21 Let $x = r\cos\theta$ and $y = r\sin\theta$. $V = \iint\limits_R (2 - x^2 - y^2 - 1)\, dA = \int_0^{2\pi}\int_0^1 (1-r^2)r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r - r^3)\, dr\,d\theta = \int_0^{2\pi}(r^2/2 - r^4/4)\Big|_0^1 d\theta = (1/4)\cdot 2\pi = \frac{\pi}{2}$.

13.3.22 Let $x = r\cos\theta$ and $y = r\sin\theta$. $V = \iint\limits_R (8 - x^2 - 3y^2 - x^2 + y^2)\, dA = \int_0^{2\pi}\int_0^2 (8-2r^2)r\,dr\,d\theta =$
$\int_0^{2\pi}\int_0^1 (8r - 2r^3)\, dr\,d\theta = \int_0^{2\pi}(4r^2 - r^4/2)\Big|_0^2 d\theta = 8\cdot 2\pi = 16\pi$.

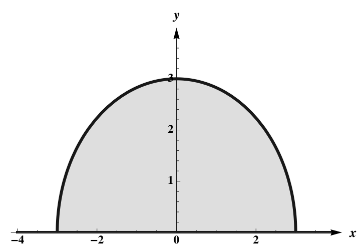
13.3.23 $\iint\limits_R \left(x^2+y^2\right)\, dA = \int_0^{2\pi}\int_0^4 \left(r^2\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^4 r^3\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{4}r^4\right)\Big|_0^4 d\theta = \int_0^{2\pi} 64\, d\theta = 128\pi$.



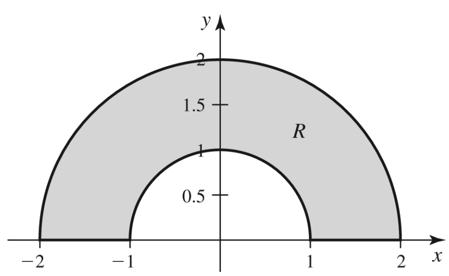
13.3.24 $\iint\limits_R (2xy)\ dA = \int_0^{\pi/2}\int_1^3 2\,(r\ \cos\theta)\,(r\ \sin\theta)\ r\,dr\,d\theta = \int_0^{\pi/2}\int_1^3 2r^3\sin\theta\ \cos\theta\,dr\,d\theta =$
$\int_0^{\pi/2}\sin\theta\ \cos\theta\ \left(\frac{1}{2}r^4\right)\Big|_1^3 d\theta = \int_0^{\pi/2} 40\ \sin\theta\ \cos\theta\,d\theta = \left(20\sin^2\ \theta\right)\Big|_{\theta=0}^{\theta=\pi/2} = 20.$



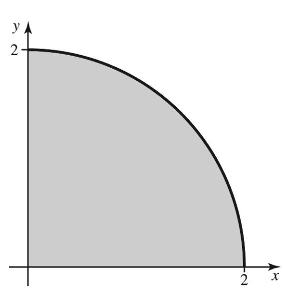
13.3.25 $\iint\limits_R (2xy)\ dA = \int_0^{\pi}\int_0^3 2\,(r\ \cos\theta)\,(r\ \sin\theta)\ r\,dr\,d\theta = \int_0^{\pi}\int_0^3 2r^3\sin\theta\ \cos\theta\,dr\,d\theta =$
$\int_0^{\pi}\cos\theta\ \sin\theta\ \left(\frac{1}{2}r^4\right)\Big|_0^3 d\theta = (81/2)\int_0^{\pi}\cos\theta\ \sin\theta\,d\theta\ = \left(\frac{81}{4}\sin^2\ \theta\right)\Big|_{\theta=0}^{\theta=\pi} = 0.$



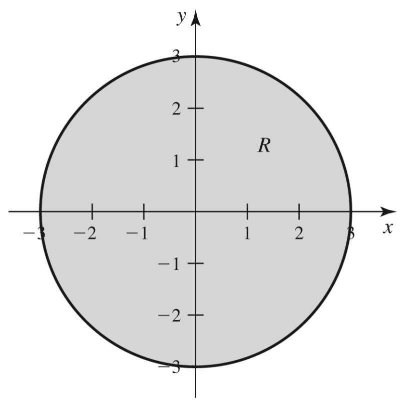
13.3.26 $\iint\limits_R \frac{1}{1+x^2+y^2}\ dA = \int_0^{\pi}\int_1^2\left(\frac{1}{1+r^2}\right)\ r\,dr\,d\theta = \int_0^{\pi}\int_1^2 \frac{r}{1+r^2}\,dr\,d\theta = \int_0^{\pi}\left(\frac{1}{2}\ln\left(1+r^2\right)\right)\Big|_1^2 d\theta =$
$\int_0^{\pi}\frac{1}{2}\,(\ln\,5 - \ln\,2)\ d\theta = \frac{\pi}{2}\ \ln\left(\frac{5}{2}\right).$



13.3.27 $\iint\limits_R \frac{1}{\sqrt{16-x^2-y^2}}\ dA\ =\ \int_0^{\pi/2}\int_0^2 \frac{1}{\sqrt{16-r^2}}\ r\,dr\,d\theta\ =\ \int_0^{\pi/2}\int_0^2 \frac{r}{\sqrt{16-r^2}}\,dr\,d\theta\ =\ \int_0^{\pi/2}\left(-\sqrt{16-r^2}\right)\Big|_0^2 d\theta\ =$
$\int_0^{\pi/2}\left(-2\sqrt{3}+4\right)\ d\theta = \pi\left(2-\sqrt{3}\right).$



13.3.28 $\iint\limits_R e^{-x^2-y^2}\ dA = \int_0^{2\pi}\int_0^3 e^{-r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^3 d\theta = \int_0^{2\pi}\left(-\frac{1}{2}e^{-9}+\frac{1}{2}\right)\,d\theta$
$= \pi\left(1-e^{-9}\right)$.



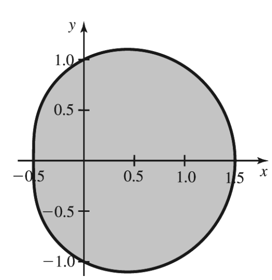
13.3.29 Setting $z \ge 0$, gives $x^2+y^2 \le 16$, and thus $0 \le r \le 4$ and $0 \le \theta \le 2\pi$. It follows that $V = \int_0^{2\pi}\int_0^4\left(e^{-r^2/8}-e^{-2}\right)r\,dr\,d\theta = \int_0^{2\pi}\int_0^4\left(r\,e^{-r^2/8}-r\,e^{-2}\right)dr\,d\theta = \int_0^{2\pi}\left(-4\,e^{-r^2/8}-\frac{1}{2}r^2\,e^{-2}\right)\Big|_0^4 d\theta = \int_0^{2\pi}\left(-12e^{-2}+4\right)\,d\theta = 8\pi\left(1-3e^{-2}\right)$.

13.3.30 Setting $z \ge 0$ gives $x^2+y^2 \le 25$, and thus $0 \le r \le 5$ and $0 \le \theta \le 2\pi$. It follows that $V = \int_0^{2\pi}\int_0^5\left(100-4r^2\right)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^5\left(100r-4r^3\right)dr\,d\theta = \int_0^{2\pi}\left(50r^2-r^4\right)\Big|_0^5 d\theta = \int_0^{2\pi} 625\,d\theta = 1250\pi$.

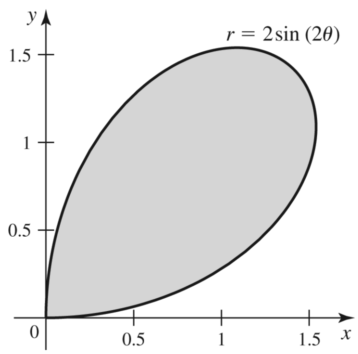
13.3.31 Setting $z \ge 0$ gives $x^2+y^2 \le 625$, and thus $0 \le r \le 25$ and $0 \le \theta \le 2\pi$. It follows that $V = \int_0^{2\pi}\int_0^{25}\left(25-\sqrt{r^2}\right)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{25}\left(25r-r^2\right)dr\,d\theta = \int_0^{2\pi}\left(\frac{25}{2}r^2-\frac{1}{3}r^3\right)\Big|_0^{25} d\theta = \int_0^{2\pi}\frac{15625}{6}\,d\theta = \frac{15625\pi}{3}$.

13.3.32 Setting $z \ge 0$ gives $x^2+y^2 \le 9$, and thus thus $0 \le r \le 3$ and $0 \le \theta \le 2\pi$. It follows that $V = \int_0^{2\pi}\int_0^3\left(\frac{20}{1+r^2}-2\right)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^3\left(\frac{20r}{1+r^2}-2r\right)dr\,d\theta = \int_0^{2\pi}\left(10\ \ln\left(1+r^2\right)-r^2\right)\Big|_0^3 d\theta =$
$\int_0^{2\pi}\left(10\ \ln\ 10-9\right)\,d\theta = 2\pi\left(10\ \ln\ 10-9\right)$.

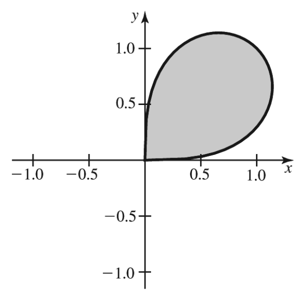
13.3.33 $\iint\limits_R f\left(r,\,\theta\right)\ dA == \int_0^{2\pi}\int_0^{1+\frac{1}{2}\cos\theta} f\left(r,\,\theta\right)\ \ r\,dr\,d\theta$.



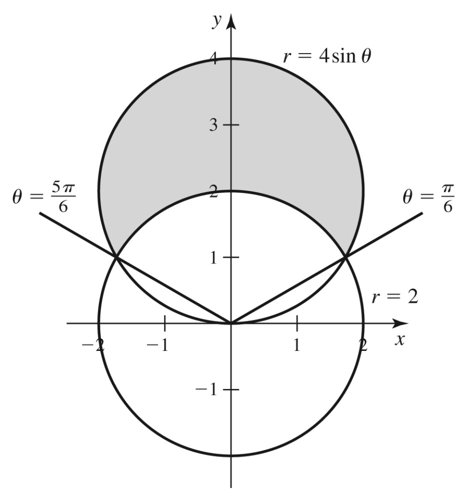
13.3.34 $\iint\limits_R f\left(r,\,\theta\right)\ dA == \int_0^{\pi/2}\int_0^{2\ \sin(2\theta)} f\left(r,\,\theta\right)\ \ r\,dr\,d\theta$.



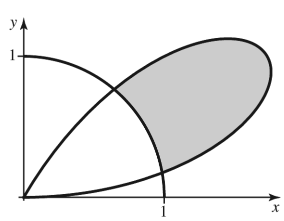
13.3.35 $\iint\limits_R f(r,\theta)\ dA == \int_0^{\pi/2}\int_0^{\sqrt{2\sin(2\theta)}} f(r,\theta)\ \ r\,dr\,d\theta.$



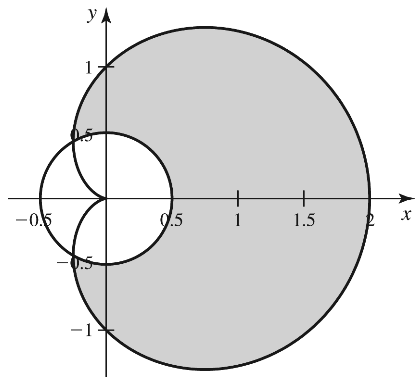
13.3.36 $\iint\limits_R f(r,\theta)\ dA == \int_{\pi/6}^{5\pi/6}\int_2^{4\sin\theta} f(r,\theta)\ \ r\,dr\,d\theta.$



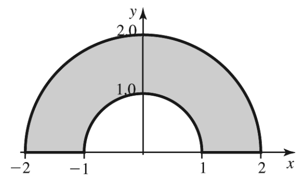
13.3.37 $\iint\limits_R f(r,\theta)\ dA == \int_{\pi/18}^{5\pi/18}\int_1^{2\sin(3\theta)} f(r,\theta)\ \ r\,dr\,d\theta.$



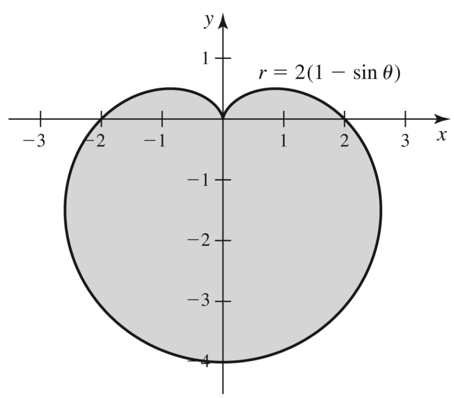
13.3.38 $\iint\limits_R f(r,\theta)\ dA == \int_{-2\pi/3}^{2\pi/3}\int_{1/2}^{1+\cos\theta} f(r,\theta)\ \ r\,dr\,d\theta.$



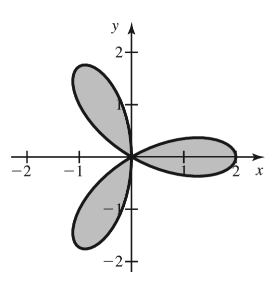
13.3.39 $A = \iint\limits_R 1\, dA = \int_0^\pi \int_1^2 r\, dr\, d\theta = \int_0^\pi \left(\frac{1}{2}r^2\right)\Big|_1^2 d\theta = \int_0^\pi \left(\frac{3}{2}\right) d\theta = \left(\frac{3}{2}\theta\right)\Big|_{\theta=0}^{\theta=\pi} = \frac{3\pi}{2}.$



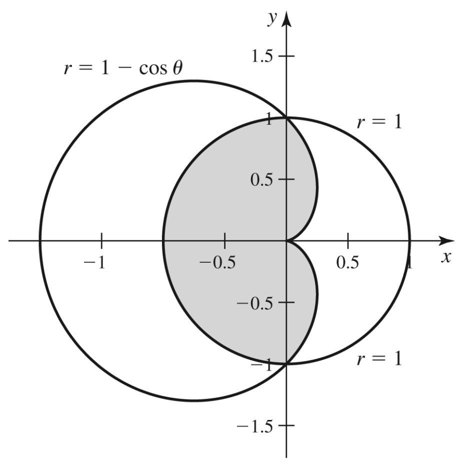
13.3.40 $A = \iint\limits_R 1\, dA = \int_0^{2\pi} \int_0^{2(1-\sin\theta)} r\, dr\, d\theta = \int_0^{2\pi} \left(\frac{1}{2}r^2\right)\Big|_0^{2(1-\sin\theta)} d\theta = \frac{1}{2}\int_0^{2\pi} 4\left(1 - 2\sin\theta + \sin^2\theta\right) d\theta$
$= 2\int_0^{2\pi} \left(\frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta\right) d\theta = 2\left(\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right)\Big|_{\theta=0}^{\theta=2\pi} = 6\pi.$



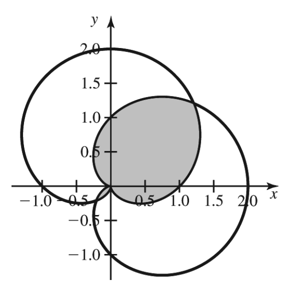
13.3.41 $A = \iint\limits_R 1\, dA = \int_0^\pi \int_0^{2\cos(3\theta)} r\, dr\, d\theta = \int_0^\pi \left(\frac{1}{2}r^2\right)\Big|_0^{2\cos(3\theta)} d\theta = \int_0^\pi \left(2\cos^2(3\theta)\right) d\theta =$
$\int_0^\pi (1 + \cos(6\theta))\, d\theta = \left(\theta + \frac{1}{6}\sin(6\theta)\right)\Big|_{\theta=0}^{\theta=\pi} = \pi.$



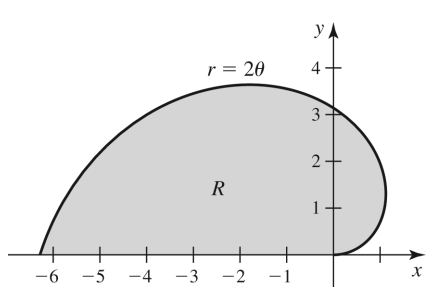
13.3.42 $A = \iint\limits_R 1\, dA = \int_{-\pi/2}^{\pi/2}\int_0^{1-\cos\theta} r\, dr\, d\theta + \int_{\pi/2}^{3\pi/2}\int_0^1 r\, dr\, d\theta$. By symmetry this is equal to $2\left(\int_0^{\pi/2}\int_0^{1-\cos\theta} r\, dr\, d\theta + \int_{\pi/2}^{\pi}\int_0^1 r\, dr\, d\theta\right) = \int_0^{\pi/2}\left(r^2\right)\Big|_0^{1-\cos\theta} d\theta + \int_{\pi/2}^{\pi}\left(r^2\right)\Big|_0^1 d\theta = \int_0^{\pi/2}(1-\cos\theta)^2\, d\theta + \int_{\pi/2}^{\pi}(1)\, d\theta = \int_0^{\pi/2}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta + \int_{\pi/2}^{\pi}(1)\, d\theta = \left(\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big|_{\theta=0}^{\theta=\pi/2} + (\theta)\Big|_{\theta=0}^{\theta=\pi/2} = \left(\frac{3\pi}{4} - 2\right) + \left(\frac{\pi}{2}\right) = \frac{5\pi}{4} - 2.$.



13.3.43 $A = \iint\limits_R 1\, dA = \int_{-3\pi/4}^{\pi/4}\int_0^{1+\sin\theta} r\, dr\, d\theta + \int_{\pi/4}^{5\pi/4}\int_0^{1+\cos\theta} r\, dr\, d\theta$. By symmetry this is equal to $2\int_{\pi/4}^{5\pi/4}\int_0^{1+\cos\theta} r\, dr\, d\theta = \int_{\pi/4}^{5\pi/4}\left(r^2\right)\Big|_0^{1+\cos\theta} d\theta = \int_{\pi/4}^{5\pi/4}\left(1 + 2\cos\theta + \cos^2\theta\right) d\theta = \int_{\pi/4}^{5\pi/4}\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta = \left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big|_{\theta=\pi/4}^{\theta=5\pi/4} = \frac{3\pi}{2} - 2\sqrt{2}.$



13.3.44 $A = \iint\limits_R 1\, dA = \int_0^{\pi}\int_0^{2\theta} r\, dr\, d\theta = \int_0^{\pi}\left(\frac{1}{2}r^2\right)\Big|_0^{2\theta} d\theta = \int_0^{\pi}\left(2\theta^2\right) d\theta = \left(\frac{2}{3}\theta^3\right)\Big|_{\theta=0}^{\theta=\pi} = \frac{2\pi^3}{3}.$



13.3.45 $\overline{f} = \frac{1}{\text{area of } R}\iint\limits_R f(r,\theta)\, dA = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a (r)\, r\, dr\, d\theta = \frac{1}{\pi a^2}\int_0^{2\pi}\left(\frac{1}{3}r^3\right)\Big|_0^a d\theta = \frac{1}{\pi a^2}\int_0^{2\pi}\left(\frac{1}{3}a^3\right) d\theta = \frac{a}{3\pi}(\theta)\Big|_{\theta=0}^{\theta=2\pi} = \frac{2a}{3}.$

13.3.46 $A = \int_0^{2\pi} \int_0^{1+\cos\theta} r\, dr\, d\theta = \int_0^{2\pi} \left(\frac{1}{2}r^2\right)\Big|_0^{1+\cos\theta} d\theta = \frac{1}{2}\int_0^{2\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta =$
$\frac{1}{2}\left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big|_{\theta=0}^{\theta=2\pi} = \frac{3\pi}{2}$.

$\overline{f} = \frac{1}{\text{area of } R} \iint\limits_R f(r, \theta)\ dA = \frac{1}{3\pi/2} \int_0^{2\pi} \int_0^{1+\cos\theta} (r)\ r\, dr\, d\theta = \frac{2}{3\pi} \int_0^{2\pi} \left(\frac{1}{3}r^3\right)\Big|_0^{1+\cos\theta} d\theta =$
$\frac{2}{9\pi} \int_0^{2\pi} \left(1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta\right) d\theta = \frac{2}{9\pi} \int_0^{2\pi} \left(\frac{5}{2} + 4\cos\theta + \frac{3}{2}\cos 2\theta - \sin^2\theta\cos\theta\right) d\theta =$
$\frac{2}{9\pi}\left(\frac{5}{2}\theta + 4\sin\theta + \frac{3}{4}\sin 2\theta - \frac{1}{3}\sin^3\theta\right)\Big|_{\theta=0}^{\theta=2\pi} = \frac{10}{9}$.

13.3.47 The square of the distance from a point to $(1, 1)$ is $(x-1)^2 + (y-1)^2 = x^2 - 2x + y^2 - 2y + 2 = r^2 - 2r\cos\theta - 2r\sin\theta + 2$.

$\overline{f} = \frac{1}{\text{area of } R} \iint\limits_R \left(x^2 - 2x + y^2 - 2y + 2\right) dA = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \left(r^2 - 2r\cos\theta - 2r\sin\theta + 2\right) r\, dr\, d\theta =$
$\frac{1}{\pi}\int_0^{2\pi} \left(\frac{1}{4}r^4 - \frac{2}{3}r^3\cos\theta - \frac{2}{3}r^3\sin\theta + r^2\right)\Big|_0^1 d\theta = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{5}{4} - \frac{2}{3}\cos\theta - \frac{2}{3}\sin\theta\right) d\theta =$
$\frac{1}{\pi}\left(\frac{5}{4}\theta - \frac{2}{3}\sin\theta + \frac{2}{3}\cos\theta\right)\Big|_{\theta=0}^{\theta=2\pi} = \frac{5}{2}$.

13.3.48 $\overline{f} = \frac{1}{\text{area of } R} \iint\limits_R f(r, \theta)\ dA = \frac{1}{12\pi}\int_0^{2\pi}\int_2^4 \frac{1}{r^2}\, r\, dr\, d\theta = \frac{1}{12\pi}\int_0^{2\pi}\int_2^4 \frac{1}{r}\, dr\, d\theta = \frac{1}{12\pi}\int_0^{2\pi} (\ln|r|)\Big|_2^4 d\theta = \frac{1}{12\pi}\int_0^{2\pi}(\ln 4 - \ln 2)\, d\theta = \frac{\ln 2}{6}$.

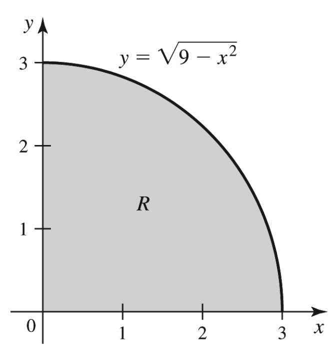
13.3.49

a. False. $\iint\limits_R \left(x^2 + y^2\right) dA = \int_0^{2\pi}\int_0^1 \left(r^2\right) r dr\, d\theta$.

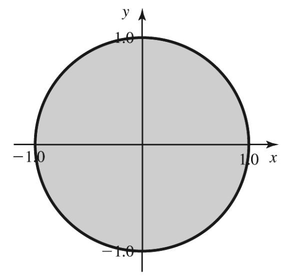
b. True. The distance from every point on a hemisphere with radius 2 to the origin is 2.

c. True. $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\, dx\, dy$ converts to $\int_0^{\pi/2}\int_0^r r\, e^{r^2}\, dr\, d\theta$.

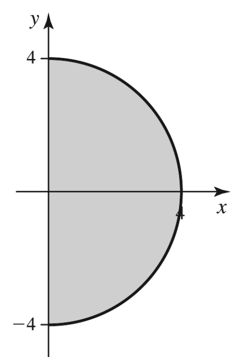
13.3.50 $\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\pi/2}\int_0^3 \sqrt{r^2}\, r\, dr\, d\theta = \int_0^{\pi/2}\left(\frac{1}{3}r^3\right)\Big|_0^3 d\theta = \int_0^{\pi/2} (9)\, d\theta = (9\theta)\Big|_{\theta=0}^{\theta=\pi/2} = \frac{9\pi}{2}$.



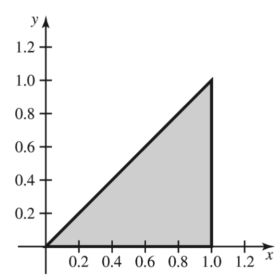
13.3.51 $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dxy = \int_0^{2\pi}\int_0^1 (r^2)^{3/2}\, r\, dr\, d\theta = \int_0^{2\pi}\left(\frac{1}{5}r^5\right)\Big|_0^1 d\theta = \int_0^{2\pi}\left(\frac{1}{5}\right) d\theta = \left(\frac{1}{5}\theta\right)\Big|_{\theta=0}^{\theta=2\pi} = \frac{2\pi}{5}$.



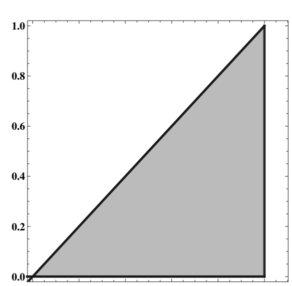
13.3.52 $\int_{-4}^{4}\int_{0}^{\sqrt{16-y^2}}(16-x^2-y^2)\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^4 (16-r^2)\ r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left(8r^2-\frac{1}{4}r^4\right)\Big|_0^4 d\theta = \int_{-\pi/2}^{\pi/2} 64\,d\theta = (64\theta)\Big|_{\theta=-\pi/2}^{\theta=\pi/2} = 64\pi.$



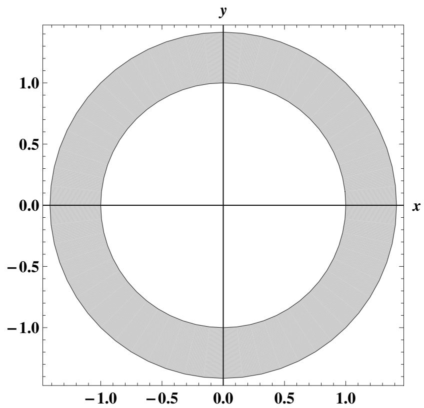
13.3.53 $\int_0^{\pi/4}\int_0^{\sec\theta}\ r^3\ dr\,d\theta = \int_0^1\int_0^x (x^2+y^2)\,dy\,dx = \int_0^1\left(x^2y+\frac{1}{3}y^3\right)\Big|_0^x dx = \int_0^1\left(\frac{4}{3}x^3\right)dx = \left(\frac{1}{3}x^4\right)\Big|_0^1 = \frac{1}{3}.$



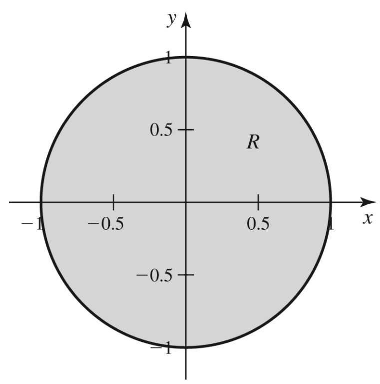
13.3.54 Let $x=\cos\theta$ and $y=\sin\theta$. We have $\int_0^{\pi/4}\int_0^{\sec\theta} r^2\,dr\,d\theta = \int_0^{\pi/4}(r^3/3)\Big|_0^{\sec\theta} d\theta = \frac{1}{3}\int_0^{\pi/4}\sec^3\theta\,d\theta = \frac{1}{6}(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4} = \frac{1}{6}(\sqrt{2}+\ln(\sqrt{2}+1)).$



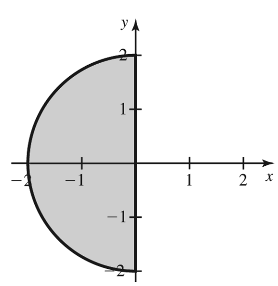
13.3.55 Let $x=\cos\theta$ and $y=\sin\theta$. The we have $\int_0^{2\pi}\int_1^2 r^2\,dr\,d\theta = \int_0^{2\pi}(r^3/3)\Big|_1^2 d\theta = \frac{7}{3}\cdot 2\pi = \frac{14\pi}{3}.$



13.3.56 $\iint\limits_R \frac{x-y}{x^2+y^2+1}\, dA = \int_0^{2\pi}\int_0^1 \frac{r\cos\theta - r\sin\theta}{r^2+1}\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \frac{r^2(\cos\theta-\sin\theta)}{r^2+1}\, dr\, d\theta =$
$\left(\int_0^{2\pi} (\cos\theta - \sin\theta)\left(\int_0^1 \left(1-\frac{1}{r^2+1}\right) dr\right) d\theta\right) = \int_0^{2\pi} (\cos\theta-\sin\theta)\left(\left(r - \tan^{-1} r\right)\Big|_0^1\right) d\theta =$
$\int_0^{2\pi} (\cos\theta-\sin\theta)\left(1-\tan^{-1}(1)\right) d\theta = \left(1-\frac{\pi}{4}\right)\int_0^{2\pi}(\cos\theta-\sin\theta)\, d\theta = \left(1-\frac{\pi}{4}\right)(\sin\theta+\cos\theta)\Big|_{\theta=0}^{2\pi} d\theta =$
$\left(1-\frac{\pi}{4}\right)\cdot 0 = 0$



13.3.57 $\iint\limits_R \frac{1}{4+\sqrt{x^2+y^2}}\, dA = \int_{\pi/2}^{3\pi/2}\int_0^2 \frac{1}{4+r}\, r\, dr\, d\theta = \int_{\pi/2}^{3\pi/2}\int_0^2 \left(1-\frac{4}{r+4}\right) dr\, d\theta =$
$\int_{\pi/2}^{3\pi/2} (r - 4\ln|r+4|)\Big|_0^2 d\theta = 2\int_{\pi/2}^{3\pi/2}\left(1-2\ln\left(\frac{3}{2}\right)\right) d\theta = 2\left(\theta\left(1-2\ln\left(\frac{3}{2}\right)\right)\right)\Big|_{\theta=\pi/2}^{\theta=3\pi/2} = 2\pi\left(1-2\ln\left(\frac{3}{2}\right)\right).$



13.3.58 For $r = 2a\cos\theta$: $A = \iint\limits_R 1\, dA = \int_0^{\pi}\int_0^{2a\cos\theta} r\, dr\, d\theta = \int_0^{\pi}\left(\frac{1}{2}r^2\right)\Big|_0^{2a\cos\theta} d\theta = \int_0^{\pi} 2a^2\cos^2\theta\, d\theta.$

$= \int_0^{\pi} a^2(1+\cos 2\theta)\, d\theta = a^2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\theta=0}^{\theta=\pi} = \pi a^2.$

For $r = 2a\sin\theta$: $A = \iint\limits_R 1\, dA = \int_0^{\pi}\int_0^{2a\sin\theta} r\, dr\, d\theta = \int_0^{\pi}\left(\frac{1}{2}r^2\right)\Big|_0^{2a\sin\theta} d\theta = \int_0^{\pi} 2a^2\sin^2\theta\, d\theta$

$= \int_0^{\pi} a^2(1-\cos 2\theta)\, d\theta = a^2\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_{\theta=0}^{\theta=\pi} = \pi a^2.$

13.3.59 Paraboloid: $V = \iint\limits_R \left(4 - (x^2+y^2)\right)\, dA = \int_0^{2\pi}\int_0^2 \left(4 - r^2\right) r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (4r - r^3)\, dr\, d\theta =$ $\int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4\right)\Big|_0^2 d\theta = \int_0^{2\pi} 4\, d\theta = 8\pi.$

Cone: $V = \iint\limits_R (4 - \sqrt{x^2+y^2})\, dA = \int_0^{2\pi}\int_0^4 (4r - r^2)\, dr\, d\theta = \int_0^{2\pi} \left(2r^2 - r^3/3\right)\Big|_0^4 d\theta = \int_0^{2\pi} \frac{32}{3}\, d\theta = \frac{64\pi}{3}.$

Hyperboloid: $V = \iint\limits_R \left(5 - \sqrt{1+x^2+y^2}\right) dA = \int_0^{2\pi}\int_0^{\sqrt{24}} \left(5 - \sqrt{1+r^2}\right) r\, dr\, d\theta =$ $\int_0^{2\pi}\int_0^{\sqrt{24}} \left(5r - r\sqrt{1+r^2}\right) dr\, d\theta = \int_0^{2\pi} \left(5r^2/2 - \frac{1}{3}\left(1+r^2\right)^{3/2}\right)\Big|_0^{\sqrt{24}} d\theta = \int_0^{2\pi} \frac{56}{3}\, d\theta = \frac{112\pi}{3}.$

The bowl that holds the most water is the hyperboloid.

13.3.60 Paraboloid: $V = \int_0^{2\pi}\int_0^{\sqrt{h}} \left(r^2\right) r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{h}} r^3\, dr\, d\theta = \int_0^{2\pi} \left(\frac{1}{4}r^4\right)\Big|_0^{\sqrt{h}} d\theta = \int_0^{2\pi} \frac{h^2}{4}\, d\theta = \frac{\pi h^2}{2}.$ Setting this equal to $\frac{176\pi}{3}$ and solving gives $h = \sqrt{\frac{352}{3}} \approx 10.83$ units.

Cone: $V = \int_0^{2\pi}\int_0^h r \cdot r\, dr\; d\theta = \int_0^{2\pi}\int_0^h r^2\, dr\; d\theta = \int_0^{2\pi} \left(\frac{1}{3}r^3\right)\Big|_0^h d\theta = \int_0^{2\pi} \frac{h^3}{3}\, d\theta = \frac{2\pi h^3}{3}.$ Setting this equal to $\frac{176\pi}{3}$ and solving gives $h = \sqrt[3]{88} \approx 4.45$ units.

13.3.61

a. $z \geq 0$ implies that $x^2 - y^2 \geq 0$, which in turn implies that $x^2 \geq y^2$. Thus $x \geq |y|$ or $x \leq -|y|$ so $R = \left\{(r,\theta)\,|-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}\right\}$.

b. $V = \iint\limits_R \left(x^2 - y^2\right)\, dA = \int_{-\pi/4}^{\pi/4}\int_0^a \left(r^2\cos^2\theta - r^2\sin^2\theta\right) r\, dr\; d\theta = \int_{-\pi/4}^{\pi/4}\int_0^a \left(\cos^2\theta - \sin^2\theta\right) r^3\, dr\, d\theta$ $= \int_{-\pi/4}^{\pi/4} \left(\cos^2\theta - \sin^2\theta\right)\left(\frac{1}{4}r^4\right)\Big|_0^a d\theta = \left(\frac{1}{4}a^4\right)\int_{-\pi/4}^{\pi/4} \cos 2\theta\, d\theta = \left(\frac{1}{4}a^4\right)\left(\frac{1}{2}\sin 2\theta\right)\Big|_{\theta=-\pi/4}^{\theta=\pi/4} = \frac{a^4}{4}.$

13.3.62

a. $V = \iint\limits_R \sqrt{16 - x^2 - y^2}\, dA = \int_0^{\pi/4}\int_0^4 \sqrt{16 - r^2}\, r\, dr\, d\theta = \int_0^{\pi/4} \left(-\frac{1}{3}\left(16 - r^2\right)^{3/2}\right)\Big|_0^4 d\theta = \int_0^{\pi/4} \frac{64}{3}\, d\theta$ $= \frac{16\pi}{3}$. Because this slice is $\frac{1}{8}$ of the hemispherical cake, the formula for the volume of a sphere can be used to confirm that the volume of the slice is $V = \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{4}{3}\pi \cdot 4^3 = \frac{16\pi}{3}$.

b. $V = \iint\limits_R \sqrt{16 - x^2 - y^2}\, dA = \int_0^{\phi}\int_0^4 \sqrt{16 - r^2}\, r\, dr\, d\theta = \int_0^{\phi} \left(-\frac{1}{3}\left(16 - r^2\right)^{3/2}\right)\Big|_0^4 d\theta = \int_0^{\phi} \frac{64}{3}\, d\theta = \frac{64\,\phi}{3}.$ By geometry, this slice is $\frac{\phi}{2\pi}$ of $\frac{1}{2}$ of a complete sphere of radius 4, thus $V = \frac{\phi}{2\pi}\left(\frac{4}{3}\pi\, 4^3\right) = \frac{64\,\phi}{3}$.

13.3.63 $\int_0^{\pi/2}\int_1^{\infty} \frac{\cos\theta}{r^3}\, r\, dr d\theta = \lim_{b\to\infty}\int_0^{\pi/2}\int_1^b \frac{\cos\theta}{r^2} dr\, d\theta = \lim_{b\to\infty}\int_0^{\pi/2} \left(-\frac{\cos\theta}{r}\right)\Big|_1^b d\theta =$ $\lim_{b\to\infty}\int_0^{\pi/2} \cos\theta\left(1 - \frac{1}{b}\right) d\theta = \lim_{b\to\infty}\left(\sin\theta\left(1 - \frac{1}{b}\right)\right)\Big|_{\theta=0}^{\theta=\pi/2} = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1.$

13.3.64 $\iint\limits_R \frac{dA}{(x^2+y^2)^{5/2}} = \int_0^{2\pi}\int_1^{\infty} \frac{1}{(r^2)^{5/2}}\, r\, dr\, d\theta = \lim_{b\to\infty}\int_0^{2\pi}\int_1^b \frac{1}{r^4} dr\, d\theta = \lim_{b\to\infty}\int_0^{2\pi} \left(-\frac{1}{3r^3}\right)\Big|_1^b d\theta =$ $\lim_{b\to\infty}\int_0^{2\pi} \frac{1}{3}\left(1 - \frac{1}{b^3}\right) d\theta = \lim_{b\to\infty}\left(\frac{1}{3}\,\theta\left(1 - \frac{1}{b^3}\right)\right)\Big|_{\theta=0}^{\theta=2\pi} = \lim_{b\to\infty} \frac{2\pi}{3}\left(1 - \frac{1}{b^3}\right) = \frac{2\pi}{3}.$

13.3.65 $\iint\limits_R e^{-x^2-y^2}\, dA = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta = \int_0^{\pi/2} \left(\lim_{b\to\infty}\int_0^b e^{-r^2}\, r\, dr\right) d\theta =$ $\int_0^{\pi/2} \left(\lim_{b\to\infty}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^b\right) d\theta = \int_0^{\pi/2} \left(\lim_{b\to\infty} \frac{1}{2}\left(1 - e^{-b^2}\right)\right) d\theta = \int_0^{\pi/2} \left(\frac{1}{2}\right) d\theta = \frac{\pi}{4}.$

13.3.66 $\iint\limits_R \frac{1}{(1+x^2+y^2)^2} dA = \int_0^{\pi/2}\int_0^{\infty} \frac{1}{(1+r^2)^2}\, r\, dr\, d\theta = \int_0^{\pi/2} \left(\lim_{b\to\infty}\int_0^b \frac{r}{(1+r^2)^2}\, dr\right) d\theta =$ $\int_0^{\pi/2} \left(\lim_{b\to\infty}\left(-\frac{1}{2(1+r^2)}\right)\Big|_0^b\right) d\theta = \int_0^{\pi/2} \left(\lim_{b\to\infty} \frac{1}{2}\left(1 - \frac{1}{1+b^2}\right)\right) d\theta = \int_0^{\pi/2} (\tfrac{1}{2})\, d\theta = \frac{\pi}{4}.$

13.3.67

a. $A = \iint_R 1\, dA = \int_0^{2\pi} \int_0^{2+\cos\theta} r\, dr\, d\theta = \int_0^{2\pi} \left(\frac{1}{2}r^2\right)\Big|_0^{2+\cos\theta} d\theta = \frac{1}{2}\int_0^{2\pi} (2+\cos\theta)^2\, d\theta =$
$\frac{1}{2}\int_0^{2\pi} \left(4 + 4\cos\theta + \cos^2\theta\right) d\theta = \frac{1}{2}\left(4\theta + 4\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_{\theta=0}^{\theta=2\pi} = \frac{9\pi}{2}.$

b. When $r = 0$, we have $\cos\theta = -\frac{1}{2}$. The outer loop is sketched for $-\frac{2\pi}{3} \le \theta \le \frac{2\pi}{3}$ and the inner loop is sketched for $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$.

Thus,

$$\begin{aligned}
A = \iint_R 1\, dA &= \int_{-2\pi/3}^{2\pi/3} \int_0^{1+2\cos\theta} r\, dr\, d\theta - \int_{2\pi/3}^{4\pi/3} \int_0^{1+2\cos\theta} r\, dr\, d\theta \\
&= \frac{1}{2}\int_{-2\pi/3}^{2\pi/3} (1+2\cos\theta)^2\, d\theta - \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1+2\cos\theta)^2\, d\theta \\
&= \frac{1}{2}\int_{-2\pi/3}^{2\pi/3} \left(1 + 4\cos\theta + 4\cos^2\theta\right) d\theta - \frac{1}{2}\int_{2\pi/3}^{4\pi/3} \left(1 + 4\cos\theta + 4\cos^2\theta\right) d\theta \\
&= \frac{1}{2}\int_{-2\pi/3}^{2\pi/3} (3 + 4\cos\theta - 2\cos 2\theta)\, d\theta - \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (3 + 4\cos\theta - 2\cos 2\theta)\, d\theta \\
&= \frac{1}{2}(3\theta + 4\sin\theta + \sin 2\theta)\Big|_{-2\pi/3}^{2\pi/3} - \frac{1}{2}(3\theta + 4\sin\theta + \sin 2\theta)\Big|_{2\pi/3}^{4\pi/3} \\
&= \frac{1}{2}\left(2\pi + \frac{3\sqrt{3}}{2}\right) - \frac{1}{2}\left(-2\pi - \frac{3\sqrt{3}}{2}\right) - \left(\frac{1}{2}\left(4\pi - \frac{3\sqrt{3}}{2}\right) - \frac{1}{2}\left(2\pi + \frac{3\sqrt{3}}{2}\right)\right) \\
&= \pi + 3\sqrt{3}
\end{aligned}$$

c. $A = \iint_R 1\, dA = \int_{2\pi/3}^{4\pi/3} \int_0^{1+2\cos\theta} r\, dr\, d\theta = \int_{2\pi/3}^{4\pi/3} \left(\frac{1}{2}r^2\right)\Big|_0^{1+2\cos\theta} d\theta = \frac{1}{2}\int_{2\pi/3}^{4\pi/3} \left(1 + 4\cos\theta + 4\cos^2\theta\right) d\theta$
$= \frac{1}{2}(3\theta + 4\sin\theta + \sin 2\theta)\Big|_{\theta=2\pi/3}^{\theta=4\pi/3} = \frac{1}{2}\left(2\pi - 4\sqrt{3} + \sqrt{3}\right) = \pi - \frac{3\sqrt{3}}{2}.$

13.3.68 The mass $m = \iint_R \rho(r, \theta)\, dA$ where $\rho(r, \theta)$ estimates the density at each point on the plate. Approximate $m = \iint_R \rho(r, \theta)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(r_i, \theta_j) \cdot r_i \triangle r \triangle\theta$. Let $\triangle r = 1$, $\triangle\theta = \frac{\pi}{4}$, then sum the upper right (clockwise outermost) corner for each subsection of region. We have $(2.0 + 2.1 + 2.2 + 2.3) \cdot (1) \cdot (1) \cdot \left(\frac{\pi}{4}\right) + (2.5 + 2.7 + 2.9 + 3.1) \cdot (2) \cdot (1) \cdot \left(\frac{\pi}{4}\right)(3.2 + 3.4 + 3.5 + 3.6) \cdot (2) \cdot (1) \cdot \left(\frac{\pi}{4}\right) \approx 56.6$ grams.

13.3.69 $m = \iint_R \rho(r, \theta)\, dA = \int_0^{\pi} \int_1^4 (4 + r\sin\theta)\, r\, dr\, d\theta = \int_0^{\pi} \left(2r^2 + \frac{1}{3}r^3\sin\theta\right)\Big|_1^4 d\theta =$
$\int_0^{\pi} (30 + 21\sin\theta)\, d\theta = (30\theta - 21\cos\theta)\Big|_{\theta=0}^{\theta=\pi} = 30\pi + 42$

13.3.70 $R = \{(r, \theta)\,|0 \le r \le g(\theta), \alpha \le \theta \le \beta\}$, $A = \iint_R 1\, dA = \int_\alpha^\beta \int_0^{g(\theta)} r\, dr\, d\theta = \int_\alpha^\beta \left(\frac{1}{2}r^2\right)\Big|_0^{g(\theta)} d\theta$. Thus $A = \frac{1}{2}\int_\alpha^\beta (g(\theta))^2\, d\theta$.

13.3.71

a. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy = \int_0^{2\pi} \int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta =$
$\int_0^{2\pi} \left(\lim_{b\to\infty} \int_0^b e^{-r^2}\, r\, dr\right) d\theta = \int_0^{2\pi} \left(\lim_{b\to\infty} \left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^b\right) d\theta = \int_0^{2\pi} \left(\lim_{b\to\infty} \frac{1}{2}\left(1 - e^{-b^2}\right)\right) d\theta$
$= \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi.$

Thus $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \pi \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(e^{-x^2}\cdot e^{-y^2}\right)dx\,dy = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\cdot\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$. Thus $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. Note that the possibility $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = -\sqrt{\pi}$ is rejected because the integrand is everywhere positive and the integral of a positive function is positive.

b. i. $\int_0^{\infty} e^{-x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

ii. Let $u = -x^2$. Then $-\frac{du}{2} = x\,dx$, and the integral becomes $\frac{1}{2}\int_{-\infty}^{0} e^u du = \left(\frac{1}{2}\right)\lim_{b\to-\infty}\int_b^0 e^u du = \left(\frac{1}{2}\right)\lim_{b\to-\infty}\left(e^u\right)\Big|_{u=b}^{u=0} = \left(\frac{1}{2}\right)\lim_{b\to-\infty}\left(1-e^b\right) = \frac{1}{2}(1-0) = \frac{1}{2}$.

iii. $\int_0^{\infty} x^2 e^{-x^2}\,dx = \lim_{b\to\infty}\int_0^b x^2 e^{-x^2}\,dx$ By parts: let $u = x$, $dv = xe^{-x^2}$, then $du = dx$, $v = -\frac{1}{2}e^{-x^2}$. We have $\lim_{b\to\infty}\left(-\frac{1}{2}x\,e^{-x^2}\Big|_{u=0}^{b} + \frac{1}{2}\int_0^b e^{-x^2}\,dx\right) = 0 + \frac{1}{2}\cdot\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}$ from (i).

13.3.72

a. $\iint_R \frac{k}{(x^2+y^2)^p}\,dA = \int_0^{2\pi}\int_1^{\infty}\frac{k}{(r^2)^p}\,r\,dr\,d\theta = \int_0^{2\pi}\left(\lim_{b\to\infty}\int_1^b \frac{k\,r}{r^{2p}}\,dr\right)d\theta = \int_0^{2\pi}\left(\lim_{b\to\infty}\int_1^b \frac{k}{r^{2p-1}}\,dr\right)d\theta$ which converges if $2p-1>1$ or $p>1$.

b. $\iint_R \frac{k}{(x^2+y^2)^p}\,dA = \int_0^{2\pi}\int_0^{1}\frac{k}{(r^2)^p}\,r\,dr\,d\theta = \int_0^{2\pi}\left(\lim_{b\to 0}\int_b^1 \frac{k\,r}{r^{2p}}\,dr\right)d\theta = \int_0^{2\pi}\left(\lim_{b\to 0}\int_b^1 \frac{k}{r^{2p-1}}\,dr\right)d\theta$ which converges if $2p-1\le 1$ or $p<1$.

13.3.73

a. $\int_0^1\int_0^1 \frac{1}{(1+x^2+y^2)^2}\,dy\,dx = \int_0^{\pi/4}\int_0^{\sec\theta}\frac{1}{(1+r^2)^2}\,r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\csc\theta}\frac{1}{(1+r^2)^2}\,r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/4}\frac{\sec^2\theta}{1+\sec^2\theta}d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2}\frac{\csc^2\theta}{1+\csc^2\theta}\,d\theta = \frac{1}{2}\int_0^{\pi/4}\frac{\sec^2\theta}{2+\tan^2\theta}\,d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2}\frac{\csc^2\theta}{2+\cot^2\theta}d\theta = \frac{1}{2}\int_0^1\frac{du}{2+u^2} + \frac{1}{2}\int_0^1\frac{du}{2+u^2} = \int_0^1\frac{du}{2+u^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)\Big|_{u=0}^{u=1} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

b. $\int_0^1\int_0^a \frac{1}{(1+x^2+y^2)^2}\,dy\,dx = \int_0^{\tan^{-1}(a)}\int_0^{\sec\theta}\frac{1}{(1+r^2)^2}\,r\,dr\,d\theta + \int_{\tan^{-1}(a)}^{\pi/2}\int_0^{a\csc\theta}\frac{1}{(1+r^2)^2}\,r\,dr\,d\theta = \frac{1}{2}\int_0^{\tan^{-1}(a)}\frac{\sec^2\theta}{1+\sec^2\theta}d\theta + \frac{1}{2}\int_{\tan^{-1}(a)}^{\pi/2}\frac{a^2\csc^2\theta}{1+a^2\csc^2\theta}d\theta = \frac{1}{2}\int_0^{\tan^{-1}(a)}\frac{\sec^2\theta}{2+\tan^2\theta}d\theta + \frac{1}{2}\int_{\tan^{-1}(a)}^{\pi/2}\frac{\csc^2\theta}{\left(\frac{1+a^2}{a^2}\right)+\cot^2\theta}d\theta = \frac{1}{2}\int_0^1\frac{du}{2+u^2} + \frac{1}{2}\int_0^{1/a}\frac{du}{\left(\frac{1+a^2}{a^2}\right)+u^2} = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)\Big|_{u=0}^{u=1} + \frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{a-u}{\sqrt{1+a^2}}\right)\Big|_{u=0}^{u=1/a} = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{a}{\sqrt{2}}\right) + \frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{1}{\sqrt{1+a^2}}\right)$.

c. $\lim_{a\to\infty}\int_0^1\int_0^a \frac{1}{(1+x^2+y^2)^2}\,dy\,dx = \lim_{a\to\infty}\left[\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{a}{\sqrt{2}}\right) + \frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{1}{\sqrt{1+a^2}}\right)\right] = \frac{1}{2\sqrt{2}}\left(\frac{\pi}{2}\right) + \frac{1}{2}(1)(0) = \frac{\pi\sqrt{2}}{8}$.

13.3.74

a. $A = \iint_R 1\,dA = \int_0^{2\pi}\int_0^{\frac{a(1-e^2)}{1+e\cos\theta}} r\,dr\,d\theta$.

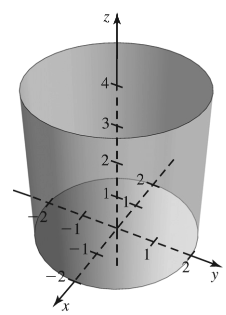
b. $A = \int_0^{2\pi}\int_0^{\frac{a(1-e^2)}{1+e\cos\theta}} r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{2}r^2\right)\Big|_0^{\frac{a(1-e^2)}{1+e\cos\theta}}d\theta = \frac{1}{2}\int_0^{2\pi}\frac{a^2(1-e^2)^2}{(1+e\cos\theta)^2}d\theta = a^2\left(1-e^2\right)^2\int_0^{\pi}\frac{1}{(1+e\cos\theta)^2}d\theta$.

If $t = \tan\left(\frac{\theta}{2}\right)$ then $\cos\theta = \frac{1-t^2}{1+t^2}$ and $d\theta = \frac{2}{1+t^2}dt$ for $-\pi<\theta<\pi$. Applying this substitution we obtain $A = 2\left(a^2\left(1-e^2\right)^2\right)\int_0^{\infty}\frac{1+t^2}{(1+e+t^2-e\,t^2)^2}dt = 2\left(a^2\left(1-e^2\right)^2\right)\lim_{b\to\infty}\int_0^b\frac{1+t^2}{(1+e+t^2-e\,t^2)^2}dt$. Apply partial fractions to separate the rational expression:

$A = 2a^2\left(1-e^2\right)^2 \lim_{b\to\infty}\int_0^b \left[\frac{\frac{2e}{e-1}}{(1+e+t^2-et^2)^2} - \frac{\frac{1}{e-1}}{1+e+t^2-e\,t^2}\right] dt = \frac{4a^2e\left(1-e^2\right)^2}{e-1}\lim_{b\to\infty}\int_0^b \frac{1}{(1+e+t^2-e\,t^2)^2}dt$ $-\frac{2a^2\left(1-e^2\right)^2}{e-1}\lim_{b\to\infty}\int_0^b \frac{1}{1+e+t^2-e\,t^2}dt$. To evaluate these integrals, use a trigonometric substitution with $t=\sqrt{\frac{1+e}{1-e}}\tan\theta$. The details are omitted. The first integral is $\lim_{b\to\infty}\int_0^b \frac{1}{(1+e+t^2-e\,t^2)^2}dt = \lim_{b\to\infty}\left[\frac{t}{2(1+e)(1+e+t^2-e\,t^2)} + \frac{1}{2(1+e)\sqrt{1-e^2}}\tan^{-1}\left(t\sqrt{\frac{1-e}{1+e}}\right)\Big|_{t=0}^{t=b}\right] =$ $\lim_{b\to\infty}\left[\frac{t}{2(1+e)(1+e+b^2-e\,b^2)} + \frac{1}{2(1+e)\sqrt{1-e^2}}\tan^{-1}\left(b\sqrt{\frac{1-e}{1+e}}\right)\right] = \frac{\pi}{4(1+e)\sqrt{1-e^2}}$. The second integral is $\lim_{b\to\infty}\int_0^b \frac{1}{1+e+t^2-e\,t^2}dt = \lim_{b\to\infty}\left[\frac{1}{\sqrt{1-e^2}}\tan^{-1}\left(t\sqrt{\frac{1-e}{1+e}}\right)\Big|_{t=0}^{t=b}\right] = \lim_{b\to\infty}\left[\frac{1}{\sqrt{1-e^2}}\tan^{-1}\left(b\sqrt{\frac{1-e}{1+e}}\right)\right] =$ $\frac{\pi}{2\sqrt{1-e^2}}$. Thus, $A = \frac{4a^2e\left(1-e^2\right)^2}{e-1}\lim_{b\to\infty}\int_0^b \frac{1}{(1+e+t^2-e\,t^2)^2}dt - \frac{2a^2\left(1-e^2\right)^2}{e-1}\lim_{b\to\infty}\int_0^b \frac{1}{1+e+t^2-e\,t^2}dt =$ $\frac{4a^2e\left(1-e^2\right)^2}{e-1}\cdot\frac{\pi}{4(1+e)\sqrt{1-e^2}} - \frac{2a^2\left(1-e^2\right)^2}{e-1}\cdot\frac{\pi}{2\sqrt{1-e^2}} = -\frac{\pi a^2e\left(1-e^2\right)^2}{(1-e^2)^{3/2}} + \frac{\pi a^2\left(1-e^2\right)^2}{(1-e)(1-e^2)^{1/2}} =$ $\frac{-\pi a^2e\left(1-e^2\right)^2+\pi a^2\left(1-e^2\right)^2(1+e)}{(1-e^2)^{3/2}} = \frac{\pi a^2\left(1-e^2\right)^2(-e+1+e)}{(1-e^2)^{3/2}} = \pi a^2\left(1-e^2\right)^{1/2} = \pi a\left(a^2\left(1-e^2\right)\right)^{1/2} = \pi ab.$

13.4 Triple Integrals

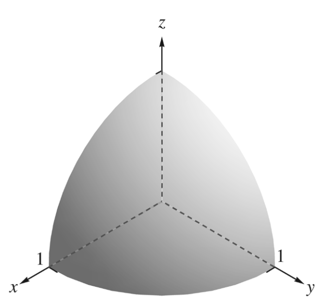
13.4.1



13.4.2 $\iiint\limits_D f(x,\,y,\,z)\;dV = \int_0^4\int_0^6\int_0^3 f(x,\,y,\,z)\;dx\,dy\,dz.$

13.4.3 $\iiint\limits_D f(x,\,y,\,z)\;dV = \int_{-9}^{9}\int_{-\sqrt{81-x^2}}^{\sqrt{81-x^2}}\int_{-\sqrt{81-x^2-y^2}}^{\sqrt{81-x^2-y^2}} f(x,\,y,\,z)\;dz\,dy\,dx.$

13.4.4



13.4.5 $\int_0^1\int_0^{\sqrt{1-z^2}}\int_0^{\sqrt{1-x^2-z^2}} f(x,\,y,\,z)\;dy\,dx\,dz.$

13.4.6 $\overline{f} = \frac{1}{\text{volume of } D}\iiint\limits_D f(x,\,y,\,z)\;dV = \frac{1}{\text{volume of } D}\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} xyz\,dz\,dy\,dx.$

13.4.7 $\int_{-2}^{2}\int_3^6\int_0^2 \;dx\,dy\,dz = \int_{-2}^{2}\int_3^6 (x)\Big|_0^2 dy\,dz = \int_{-2}^{2}\int_3^6 2\,dy\,dz = \int_{-2}^{2}(2y)\Big|_3^6\,dz = \int_{-2}^{2} 6\,dz =$ $(6z)\Big|_{-2}^{2} = 24.$

13.4.8 $\int_{-1}^{1}\int_{-1}^{2}\int_{0}^{1} 6xyz\,dy\,dx\,dz = \int_{-1}^{1}\int_{-1}^{2}\left(3xy^2z\right)\Big|_0^1 dx\,dz = \int_{-1}^{1}\int_{-1}^{2} 3xz\,dx\,dz = \int_{-1}^{1}\left(\frac{3}{2}x^2z\right)\Big|_{-1}^{2} dz =$ $\int_{-1}^{1}\frac{9}{2}z\,dz = \left(\frac{9}{4}z^2\right)\Big|_{-1}^{1} = 0.$

13.4.9 $\int_{-2}^{2}\int_{1}^{2}\int_{1}^{e}\frac{xy^2}{z}dz\,dx\,dy = \int_{-2}^{2}\int_{1}^{2} xy^2\left(xy^2\ \ln|z|\Big|_1^e\right) dx\,dy = \int_{-2}^{2}\int_{1}^{2} xy^2\,dx\,dy = \int_{-2}^{2}\left(\frac{1}{2}x^2y^2\right)\Big|_1^2 dy$ $= \int_{-2}^{2}\frac{3}{2}y^2\,dy = \left(\frac{1}{2}y^3\right)\Big|_{-2}^{2} = 8.$

13.4.10 $\int_0^{\ln 4}\int_0^{\ln 3}\int_0^{\ln 2} e^{-x+y+z}\,dx\,dy\,dz = \int_0^{\ln 4}\int_0^{\ln 3} -e^{-x+y+z}\Big|_0^{\ln 2} dy\,dz =$ $\int_0^{\ln 4}\int_0^{\ln 3}((-1/2)(e^{y+z}) + e^{y+z})\,dy\,dz = \frac{1}{2}\int_0^{\ln 4}\int_0^{\ln 3} e^{y+z}\,dy\,dz = \frac{1}{2}\int_0^{\ln 4} e^{y+z}\Big|_0^{\ln 3} dz = \frac{1}{2}\int_0^{\ln 4}(3e^z - e^z)\,dz =$ $\int_0^{\ln 4} e^z\,dz = 4 - 1 = 3.$

13.4.11 $\int_0^{\pi/2}\int_0^1\int_0^{\pi/2}\sin\pi x\cdot\cos y\cdot\sin 2z\,dy\,dx\,dz = \int_0^{\pi/2}\int_0^1\sin\pi x\ \sin 2z\left((\sin y)\Big|_0^{\pi/2}\right) dx\,dz =$ $\int_0^{\pi/2}\int_0^1\sin\pi x\ \sin 2z\ (1-0)\ dx\,dz = \int_0^{\pi/2}\int_0^1\sin\pi x\ \sin 2z\,dx\,dz = \int_0^{\pi/2}\sin 2z\left(\left(-\frac{1}{\pi}\cos\pi x\right)\Big|_0^1\right) dz =$ $\int_0^{\pi/2}\sin 2z\ \left(\frac{1}{\pi}+\frac{1}{\pi}\right)\ dz = \frac{2}{\pi}\int_0^{\pi/2}\sin 2z\,dz = \frac{2}{\pi}\left(-\frac{1}{2}\cos 2z\right)\Big|_0^{\pi/2} = \frac{2}{\pi}\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{2}{\pi}.$

13.4.12 $\int_0^2\int_1^2\int_0^1 yz\,e^x\,dx\,dz\,dy = \int_0^2\int_1^2 yz\left((e^x)\Big|_0^1\right) dz\,dy = \int_0^2\int_1^2 yz\,(e-1)\ dz\,dy =$ $(e-1)\int_0^2 y\left(\frac{1}{2}z^2\right)\Big|_1^2 dy = (e-1)\int_0^2\frac{3}{2}y\,dy = \frac{3(e-1)}{2}\left(\frac{1}{2}y^2\right)\Big|_0^2 = 3\,(e-1).$

13.4.13 $\iiint_D (xy+xz+yz)\ dV = \int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3}(xy+xz+yz)\ dz\,dy\,dx =$ $\int_{-1}^{1}\int_{-2}^{2}\left(xyz+\frac{1}{2}xz^2+\frac{1}{2}yz^2\right)\Big|_{-3}^{3} dy\,dx = \int_{-1}^{1}\int_{-2}^{2}(6xy)\ dy\,dx = \int_{-1}^{1}\left(3xy^2\right)\Big|_{-2}^{2} dx = \int_{-1}^{1}(0)\ dx = 0.$

13.4.14 $\iiint_D xyze^{-x^2-y^2}\ dV = \int_0^{\sqrt{\ln 2}}\int_0^{\sqrt{\ln 4}}\int_0^1 xyze^{-x^2-y^2}\ dz\,dy\,dx = \int_0^{\sqrt{\ln 2}}\int_0^{\sqrt{\ln 4}}\frac{1}{2}xyz^2e^{-x^2-y^2}\Big|_0^1 dy\,dx$ $= \frac{1}{2}\int_0^{\sqrt{\ln 2}}\int_0^{\sqrt{\ln 4}} xye^{-x^2-y^2}\ dy\,dx = \frac{1}{2}\int_0^{\sqrt{\ln 2}}\int_0^{\sqrt{\ln 4}} xye^{-x^2}e^{-y^2}\ dy\,dx$ $= \frac{1}{2}\int_0^{\sqrt{\ln 2}} xe^{-x^2}\left(-\frac{1}{2}e^{-y^2}\right)\Big|_0^{\sqrt{\ln 4}} dx = \frac{1}{2}\int_0^{\sqrt{\ln 2}} xe^{-x^2}\ \left(\frac{1}{2}\left(1-\frac{1}{4}\right)\right) dx = \frac{3}{16}\int_0^{\sqrt{\ln 2}} xe^{-x^2}\ dx =$ $\frac{3}{16}\left(\left(-\frac{1}{2}e^{-x^2}\right)\Big|_0^{\sqrt{\ln 2}}\right) = \frac{3}{16}\left(\frac{1}{2}\left(1-\frac{1}{2}\right)\right) = \frac{3}{64}.$

13.4.15 $V = \iiint_D 1\,dV = \int_0^6\int_0^{4-2x/3}\int_0^{2-x/3-y/2} 1\,dz\,dy\,dx = \int_0^6\int_0^{4-2x/3}(z)\Big|_0^{2-x/3-y/2} dy\,dx =$ $\int_0^6\int_0^{4-2x/3}\left(2-\frac{1}{3}x-\frac{1}{2}y\right)\ dy\,dx = \int_0^6\left(2y-\frac{1}{3}xy-\frac{1}{4}y^2\right)\Big|_0^{4-2x/3} dx =$ $\int_0^6\left(\frac{1}{9}x^2-\frac{4}{3}x+4\right)\ dx = \left(\frac{1}{27}x^3-\frac{2}{3}x^2+4x\right)\Big|_0^6 = 8.$

13.4.16 $V = \iiint_D 1\,dV = \int_0^\pi\int_x^\pi\int_0^{\sin y} 1\,dz\,dy\,dx = \int_0^\pi\int_x^\pi (z)\Big|_0^{\sin y} dy\,dx = \int_0^\pi\int_x^\pi(\sin y)\ dy\,dx =$ $\int_0^\pi(-\cos y)\Big|_x^\pi dx = \int_0^\pi(1+\cos x)\ dx = (x+\sin x)\Big|_0^\pi = \pi.$

13.4.17 $V = \iiint_D 1\,dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 1\,dz\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(z)\Big|_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dy\,dx =$ $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\sqrt{8-x^2-y^2}-\sqrt{x^2+y^2}\right) dy\,dx.$ Converting to polar coordinates gives

$\int_0^{2\pi}\int_0^2\left(\sqrt{8-r^2}-\sqrt{r^2}\right) r\,dr d\theta = \int_0^{2\pi}\int_0^2\left(r\sqrt{8-r^2}-r^2\right) dr\,d\theta = \int_0^{2\pi}\left(-\frac{1}{3}\left(8-r^2\right)^{3/2}-\frac{1}{3}r^3\right)\Big|_0^2 d\theta =$ $\int_0^{2\pi}\left(\frac{-16+16\sqrt{2}}{3}\right) d\theta = \frac{32\pi}{3}\left(\sqrt{2}-1\right).$

13.4.18 $V = \iiint\limits_D 1\,dV = \int_0^{1/2}\int_0^8\int_0^{2-4x} 1\,dz\,dy\,dx = \int_0^{1/2}\int_0^8 (z)\Big|_0^{2-4x} dy\,dx = \int_0^{1/2}\int_0^8 (2-4x)\,dy\,dx =$ $\int_0^{1/2}((2-4x)\,y)\Big|_0^8 dx = \int_0^{1/2}(16-32x)\,dx = \left(16x-16x^2\right)\Big|_0^{1/2} = 4.$

13.4.19 $V = \iiint\limits_D 1\,dV = \int_{-2}^2\int_{-\sqrt{4-x^2}}^0\int_0^{-y} 1\,dz\,dy\,dx = \int_{-2}^2\int_{-\sqrt{4-x^2}}^0 (z)\Big|_0^{-y} dy\,dx =$ $\int_{-2}^2\int_{-\sqrt{4-x^2}}^0 (-y)\,dy\,dx = \int_{-2}^2\left(-\frac{1}{2}y^2\right)\Big|_{-\sqrt{4-x^2}}^0 dx = \frac{1}{2}\int_{-2}^2\left(4-x^2\right)dx = \frac{1}{2}\left(4x-\frac{1}{3}x^3\right)\Big|_{-2}^2 = \frac{16}{3}.$

13.4.20 $V = \iiint\limits_D 1\,dV = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^3\int_0^{3-y} 1\,dz\,dy\,dx = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^3 (z)\Big|_0^{3-y} dy\,dx = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^3 (3-y)\,dy\,dx =$ $\int_{-\sqrt{3}}^{\sqrt{3}}\left(3y-\frac{1}{2}y^2\right)\Big|_{x^2}^3 dx = \int_{-\sqrt{3}}^{\sqrt{3}}\left(\frac{9}{2}-3x^2+\frac{1}{2}x^4\right)dx = \left(\frac{9}{2}x-x^3+\frac{1}{10}x^5\right)\Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{24\sqrt{3}}{5}.$

13.4.21 $V = \iiint\limits_D 1\,dV = \int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} 1\,dz\,dy\,dx = \int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (z)\Big|_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} dy\,dx =$ $\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\left(\sqrt{19-x^2-y^2}-\sqrt{1+x^2+y^2}\right) dy\,dx$. Converting to polar coordinates gives $\int_0^{2\pi}\int_0^3\left(\sqrt{19-r^2}-\sqrt{1+r^2}\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^3\left(r\sqrt{19-r^2}-r\sqrt{1+r^2}\right) dr\,d\theta =$ $\int_0^{2\pi}\left(-\frac{1}{3}\right)\left(\left(19-r^2\right)^{3/2}+\left(1+r^2\right)^{3/2}\right)\Big|_0^3 d\theta = \left(-\frac{1}{3}\right)\int_0^{2\pi}\left(20\sqrt{10}-19\sqrt{19}-1\right)d\theta =$ $\frac{1+19\sqrt{19}-20\sqrt{10}}{3}\int_0^{2\pi} d\theta = \frac{2\pi}{3}\left(1+19\sqrt{19}-20\sqrt{10}\right).$

13.4.22 $V = \iiint\limits_D 1\,dV = \int_0^{\ln 2}\int_0^1\int_1^{e^y} 1\,dz\,dx\,dy = \int_0^{\ln 2}\int_0^1 (z)\Big|_1^{e^y} dx\,dy = \int_0^{\ln 2}\int_0^1 (e^y-1)\,dx\,dy =$ $\int_0^{\ln 2}(e^y-1)\,x\Big|_0^1 dy = \int_0^{\ln 2}(e^y-1)\,dy = (e^y-y)\Big|_0^{\ln 2} = 1-\ln 2.$

13.4.23 $V = \iiint\limits_D 1\,dV = \int_{-2}^2\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}}\int_{x-3}^{3-x} 1\,dz\,dy\,dx = \int_{-2}^2\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} (z)\Big|_{x-3}^{3-x} dy\,dx =$ $\int_{-2}^2\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} (6-2x)\,dy\,dx = \int_{-2}^2 (6-2x)\,y\Big|_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} dx = \int_{-2}^2\left(6\sqrt{4-x^2}-2x\sqrt{4-x^2}\right)dx =$ $\left(3x\sqrt{4-x^2}+12\sin^{-1}\left(\frac{x}{2}\right)+\frac{2}{3}\sqrt{4-x^2}\right)\Big|_{-2}^2 = 12\pi.$

13.4.24 $V = \iiint\limits_D 1\,dV = \int_0^1\int_0^{1-z}\int_{1-y-z}^{\sqrt{(1-z)^2-y^2}} 1\,dx\,dy\,dz = \int_0^1\int_0^{1-z} (x)\Big|_{1-y-z}^{\sqrt{(1-z)^2-y^2}} dy\,dz =$ $\int_0^1\int_0^{1-z}\left(\sqrt{(1-z)^2-y^2}-1+y+z\right) dy\,dz =$ $\int_0^1\left(\frac{1}{2}y\sqrt{(1-z)^2-y^2}+\frac{1}{2}(1-z)^2\sin^{-1}\left(\frac{y}{2}\right)+\frac{1}{2}y^2-(1-z)\,y\right)\Big|_0^{1-z} dz =$ $\int_0^1\left(\frac{1}{2}(1-z)^2\sin^{-1}\left(\frac{1-z}{2}\right)-\frac{1}{2}(1-z)^2\right)dz$. After a substitution, we have $\frac{1}{2}\int_0^1\left(u^2\sin^{-1}\left(\frac{u}{2}\right)-u^2\right)dz$. Integration by parts yields $\frac{1}{2}\left(\frac{1}{3}u^3\sin^{-1}\left(\frac{u}{2}\right)+\frac{1}{9}\left(8+u^2\right)\sqrt{4-u^2}-\frac{1}{3}u^3\right)\Big|_{u=0}^{u=1} = \frac{\pi}{12}-\frac{1}{6}.$

13.4.25 $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dz\,dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}} (z)\Big|_0^{\sqrt{1-x^2}} dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-x^2}\,dy\,dx =$ $\int_0^1 y\sqrt{1-x^2}\Big|_0^{\sqrt{1-x^2}} dx = \int_0^1\left(1-x^2\right)dx = \left(x-\frac{1}{3}x^3\right)\Big|_0^1 = \frac{2}{3}.$

13.4.26 $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \left(xz^2\right)\Big|_0^{\sqrt{1-x^2-y^2}} dy\,dx =$
$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x - x^3 - xy^2\right)\,dy\,dx$. Switching the order of integration yields $\int_0^1 \int_0^{\sqrt{1-y^2}} \left(x - x^3 - xy^2\right)\,dx\,dy =$
$\int_0^1 \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 - \frac{1}{2}x^2y^2\right)\Big|_0^{\sqrt{1-y^2}} dy = \int_0^1 \left(\frac{1}{4} - \frac{1}{2}y^2 + \frac{1}{4}y^4\right)\,dy = \left(\frac{1}{4}y - \frac{1}{6}y^3 + \frac{1}{20}y^5\right)\Big|_0^1 = \frac{2}{15}$.

13.4.27 $\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\left(x^2/4\right)-y^2} dz\,dx\,dy = \int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} (z)\Big|_0^{16-\left(x^2/4\right)-y^2} dx\,dy =$
$\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \left(16 - \frac{x^2}{4} - y^2\right)\,dx\,dy = \int_0^4 \left(16x - \frac{1}{12}x^3 - xy^2\right)\Big|_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} dy =$
$\int_0^4 \left(64\sqrt{16-y^2} - \frac{4}{3}\left(16-y^2\right)^{3/2} - 4y^2\sqrt{16-y^2}\right)\,dy = \Big(32y\sqrt{16-y^2} + 512\sin^{-1}\left(\frac{y}{4}\right) -$
$\frac{40}{3}\sqrt{16-y^2} + \frac{y^3}{3}\sqrt{16-y^2} - 128\sin^{-1}\left(\frac{y}{4}\right) + 8y\sqrt{16-y^2} - y^3\sqrt{16-y^2} - 128\sin^{-1}\left(\frac{y}{4}\right)\Big)\Big|_0^4 =$
$\left(\frac{80y}{3}\sqrt{16-y^2} - \frac{2}{3}y^2\sqrt{16-y^2} + 256\sin^{-1}\left(\frac{y}{4}\right)\right)\Big|_0^4 = 128\pi$.

13.4.28 $\int_1^6 \int_0^{4-2y/3} \int_0^{12-2y-3z} \frac{1}{y}\,dx\,dz\,dy = \int_1^6 \int_0^{4-2y/3} \left(\frac{x}{y}\right)\Big|_0^{12-2y-3z} dz\,dy = \int_1^6 \int_0^{4-2y/3} \frac{12-2y-3z}{y}\,dz\,dy =$
$\int_1^6 \frac{1}{y}\left((12-2y)z - \frac{3}{2}z^2\right)\Big|_0^{4-2y/3} dy = \int_1^6 \frac{1}{y}\left(24 - 8y + \frac{2}{3}y^2\right)\,dy = \int_1^6 \left(\frac{24}{y} - 8 + \frac{2}{3}y\right)\,dy =$
$\left(24\,\ln|y| - 8y + \frac{1}{3}y^2\right)\Big|_1^6 = 24\,\ln 6 + \frac{85}{3}$.

13.4.29 $\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dy\,dx\,dz = \int_0^3 \int_0^{\sqrt{9-z^2}} \sqrt{1+x^2+z^2}\,dz\,dy$ (Use polar with $x^2+z^2=r^2$) $=$
$\int_0^{\pi/2} \int_0^3 \sqrt{1+r^2}\,r\,dr\,d\theta = \int_0^{\pi/2} \left(\frac{1}{3}\left(1+r^2\right)^{3/2}\right)\Big|_0^3 d\theta = \int_0^{\pi/2} \frac{1}{3}\left(10^{3/2}-1\right)\,d\theta = \frac{\pi}{6}\left(10\sqrt{10}-1\right)$.

13.4.30 $\int_0^\pi \int_0^\pi \int_0^{\sin x} \sin y\,dz\,dx\,dy = \int_0^\pi \int_0^\pi \sin x \cdot \sin y\,dx\,dy = \int_0^\pi \left(-\cos x \cdot \sin y\right)\Big|_0^\pi dy = \int_0^\pi 2\,\sin y\,dy =$
$\left(-2\,\cos y\right)\Big|_0^\pi = 4$.

13.4.31 $\int_1^{\ln 8} \int_1^{\sqrt{z}} \int_{\ln y}^{\ln(2y)} e^{x+y^2-z}\,dx\,dy\,dz = \int_1^{\ln 8} \int_1^{\sqrt{z}} e^{x+y^2-z}\Big|_{\ln y}^{\ln(2y)} dy\,dz =$
$\int_1^{\ln 8} \int_1^{\sqrt{z}} \left(2ye^{y^2-z} - ye^{y^2-z}\right)\,dy\,dz = \int_1^{\ln 8} \int_1^{\sqrt{z}} ye^{y^2-z}\,dy\,dz = \int_1^{\ln 8} \left(\frac{1}{2}\,e^{y^2-z}\right)\Big|_1^{\sqrt{z}} dz =$
$\frac{1}{2}\int_1^{\ln 8} \left(1 - e^{1-z}\right)\,dz = \frac{1}{2}\left(z + e^{1-z}\right)\Big|_1^{\ln 8} = \frac{1}{2}\left(\ln 8 + \frac{e}{8} - (1+1)\right) = \frac{1}{2}\ln 8 + \frac{e}{16} - 1$.

13.4.32 $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-x} 4yz\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \left(2yz^2\right)\Big|_0^{2-x} dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} 2y\left(2-x\right)^2\,dy\,dx =$
$\int_0^1 y^2\left(2-x\right)^2\Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 \left(1-x^2\right)\left(2-x\right)^2\,dx = \int_0^1 \left(4 - 4x - 3x^2 + 4x^3 - x^4\right)\,dx =$
$\left(4x - 2x^2 - x^3 + x^4 - \frac{1}{5}x^5\right)\Big|_0^1 = \frac{9}{5} - 0 = \frac{9}{5}$.

13.4.33 $\int_0^2 \int_0^4 \int_{y^2}^4 \sqrt{x}\,dz\,dx\,dy = \int_0^2 \int_0^4 z\sqrt{x}\Big|_{y^2}^4 dx\,dy = \int_0^2 \int_0^4 \left(4-y^2\right)\sqrt{x}\,dx\,dy =$
$\int_0^2 \left(4-y^2\right)\left(\frac{2}{3}x^{3/2}\right)\Big|_0^4 dy = \frac{16}{3}\int_0^2 \left(4-y^2\right)\,dy = \frac{16}{3}\left(4y - \frac{1}{3}y^3\right)\Big|_0^2 = \frac{256}{9}$.

13.4.34 $\int_0^1 \int_y^{2-y} \int_0^{2-x-y} xy\,dz\,dx\,dy = \int_0^1 \int_y^{2-y} (xyz)\Big|_0^{2-x-y} dx\,dy = \int_0^1 \int_y^{2-y} \left(2xy - x^2y - xy^2\right)\,dx\,dy =$
$\int_0^1 \left(x^2y - \frac{1}{3}x^3y - \frac{1}{2}x^2y^2\right)\Big|_y^{2-y} dy = \int_0^1 \left((2-y)^2y - \frac{1}{3}(2-y)^3y - \frac{1}{2}(2-y)^2y^2 - y^3 + \frac{1}{3}y^4 + \frac{1}{2}y^4\right)\,dy =$
$\int_0^1 \left(\frac{4}{3}y - 2y^2 + \frac{2}{3}y^4\right)\,dy = \left(\frac{2}{3}y^2 - \frac{2}{3}y^3 + \frac{2}{15}y^5\right)\Big|_0^1 = \frac{2}{15}$.

13.4.35 $V = \int_0^1 \int_0^{1-z^2} \int_0^{1-z} 1\,dy\,dx\,dz = \int_0^1 \int_0^{1-z^2} y\Big|_0^{1-z} dx\,dz = \int_0^1 \int_0^{1-z^2} (1-z)\,dx\,dz = \int_0^1 (1-z)x\Big|_0^{1-z^2} = \int_0^1 (1-z)(1-z^2)\,dz = \int_0^1 (1-z-z^2+z^3)\,dz = \left(z - \frac{z^2}{2} - \frac{z^3}{3} + \frac{z^4}{4}\right)\Big|_0^1 = \frac{5}{12}$.

13.4.36 $V = \int_0^1 \int_0^2 \int_0^{e^{-z}} 1\,dy\,dx\,dz = \int_0^1 \int_0^2 y\Big|_0^{e^{-z}} dx\,dz = \int_0^1 \int_0^2 e^{-z}\,dx\,dz = \int_0^1 2e^{-z}\,dz = (-2e^{-z})\Big|_0^1 = 2 - \frac{2}{e}$.

13.4.37 $V = \int_0^2 \int_0^4 \int_z^{z+1} 1\,dy\,dz\,dx = \int_0^2 \int_0^4 y\Big|_z^{z+1} dz\,dx = \int_0^2 \int_0^4 1\,dz\,dx = \int_0^2 4\,dx = 8$.

13.4.38 $V = \int_0^1 \int_0^{2-z-z^2} \int_{z^2}^{2-x-z} 1\,dy\,dx\,dz = \int_0^1 \int_0^{2-z-z^2} y\Big|_{z^2}^{2-x-z} dx\,dz = \int_0^1 \int_0^{2-z-z^2} (2-x-z-z^2)\,dx\,dz = \int_0^1 \left(2x - \frac{x^2}{2} - zx - z^2x\right)\Big|_0^{2-z-z^2} dz = \int_0^1 \left(\frac{z^4}{2} + z^3 - \frac{3z^2}{2} - 2z + 2\right) dz = \left(\frac{z^5}{10} + \frac{z^4}{4} - \frac{z^3}{2} - z^2 + 2z\right)\Big|_0^1 = \frac{17}{20}$.

13.4.39 We can rewrite $\int_0^5 \int_{-1}^0 \int_0^{4x+4} dy\,dx\,dz$ as $\int_0^4 \int_{(y-4)/4}^0 \int_0^5 dz\,dx\,dy$. This is then evaluated as $\int_0^4 \int_{(y-4)/4}^0 5\,dx\,dy = \int_0^4 -\frac{5}{4}(y-4)\,dy = -\frac{5}{4}\left(\frac{1}{2}y^2 - 4y\right)\Big|_0^4 = 10$.

13.4.40 We can rewrite $\int_0^1 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} dz\,dy\,dx$ as $\int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\,dz\,dx$. This can then be evaluated as $\int_0^1 \int_0^2 2\sqrt{4-z^2}\,dz\,dx = \int_0^1 \left(z\sqrt{4-z^2} + 4\sin^{-1}\left(\frac{z}{2}\right)\right)\Big|_0^2 dx = \int_0^1 (2\pi)\,dx = (2\pi x)\Big|_0^1 = 2\pi$.

13.4.41 We can rewrite $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dy\,dz\,dx$ as $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\,dy\,dx$. This is then evaluated as $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx = \int_0^1 \left(\sqrt{1-x^2}\right)\left(\sqrt{1-x^2}\right) dx = \int_0^1 \left(1-x^2\right) dx = \left(x - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{2}{3}$.

13.4.42 We can rewrite $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} dy\,dz\,dx$ as $\int_0^4 \int_0^{\sqrt{16-z^2}} \int_0^{\sqrt{16-y^2-z^2}} dx\,dy\,dz$. This can then be evaluated as $\int_0^4 \int_0^{\sqrt{16-z^2}} \sqrt{16-y^2-z^2}\,dy\,dz$. Converting to polar coordinates gives $\int_0^{\pi/2} \int_0^4 \sqrt{16-r^2}\,r\,dr\,d\theta = \int_0^{\pi/2} \left(-\frac{1}{3}\left(16-r^2\right)^{3/2}\right)\Big|_0^4 d\theta = \int_0^{\pi/2} \frac{64}{3}\,d\theta = \frac{32\pi}{3}$.

13.4.43 The average value is $\frac{1}{\text{volume of } D} \iiint_D T(x, y, z)\,dV$. This can be written as
$\frac{1}{\ln 2 \cdot \ln 4 \cdot \ln 8} \int_0^{\ln 2} \int_0^{\ln 4} \int_0^{\ln 8} 128\,e^{-x-y-z}\,dz\,dy\,dx = \frac{128}{6(\ln 2)^3} \int_0^{\ln 2} \int_0^{\ln 4} \int_0^{\ln 8} e^{-x}e^{-y}e^{-z}\,dz\,dy\,dx =$
$\frac{64}{3(\ln 2)^3} \int_0^{\ln 2} \int_0^{\ln 4} e^{-x}e^{-y}\left(-e^{-z}\right)\Big|_0^{\ln 8} dy\,dx = \frac{64}{3(\ln 2)^3} \int_0^{\ln 2} \int_0^{\ln 4} e^{-x}e^{-y}\left(1 - \frac{1}{8}\right) dy\,dx =$
$\frac{56}{3(\ln 2)^3} \int_0^{\ln 2} \int_0^{\ln 4} e^{-x}e^{-y}\,dy\,dx = \frac{56}{3(\ln 2)^3} \int_0^{\ln 2} e^{-x}\left(-e^{-y}\right)\Big|_0^{\ln 4} dx = \frac{56}{3(\ln 2)^3} \int_0^{\ln 2} e^{-x}\left(1 - \frac{1}{4}\right) dx =$
$\frac{14}{(\ln 2)^3} \int_0^{\ln 2} e^{-x}\,dx = \frac{14}{(\ln 2)^3}\left(-e^{-x}\right)\Big|_0^{\ln 2} = \frac{14}{(\ln 2)^3}\left(1 - \frac{1}{2}\right) = \frac{7}{(\ln 2)^3}$.

13.4.44 The average value is given by $\frac{1}{\text{volume of } D} \iint_D \int f(x, y, z)\,dV$ which can be written
$\frac{1}{\frac{1}{2}\left(\frac{4}{3}\pi \cdot 4^3\right)} \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} 6xyz\,dz\,dy\,dx = \frac{9}{64\pi} \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(\frac{1}{2}xyz^2\right)\Big|_0^{\sqrt{16-x^2-y^2}} dy\,dx =$
$\frac{9}{128\pi} \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} xy\left(16-x^2-y^2\right) dy\,dx = \frac{9}{128\pi} \int_{-4}^4 \left(8xy^2 - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right)\Big|_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} dx =$
$\frac{9}{128\pi} \int_{-4}^4 \Big(8x\left(16-x^2\right) - \frac{x^3}{2}\left(16-x^2\right) - \frac{x}{4}\left(16-x^2\right)^2 - 8x\left(16-x^2\right) + \frac{x^3}{2}\left(16-x^2\right) +$
$\frac{x}{4}\left(16-x^2\right)^2\Big) dx = \frac{9}{128\pi} \int_{-4}^4 0\,dx = 0$.

13.4.45 The average value is given by $\frac{1}{\text{volume of } D} \iiint_D f(x, y, z)\,dV$. This can be written as
$\frac{1}{(\pi \cdot 2^2 \cdot 2)} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^2 \left(x^2+y^2+z^2\right) dz\,dy\,dx = \frac{1}{8\pi} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(\left(x^2+y^2\right)z + \frac{1}{3}z^3\right)\Big|_0^2 dy\,dx =$

$\frac{1}{8\pi}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(2\left(x^2+y^2\right)+\frac{8}{3}\right)dy\,dx = \frac{1}{8\pi}\int_{-2}^{2}\left(2x^2y+\frac{2}{3}y^3+\frac{8}{3}y\right)\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx =$
$\frac{1}{2\pi}\int_{-2}^{2}\left(x^2\sqrt{4-x^2}+\frac{1}{3}\left(4-x^2\right)^{3/2}+\frac{4}{3}\sqrt{4-x^2}\right)dx = \frac{1}{2\pi}\Big(-\frac{x}{4}\left(x^2-2\right)\sqrt{4-x^2}+2\sin^{-1}\left(\frac{x}{2}\right)+\frac{x}{12}\left(x^2-10\right)+$
$2\sin^{-1}\left(\frac{x}{2}\right)+\frac{x}{6}\sqrt{4-x^2}+\frac{8}{3}\sin^{-1}\left(\frac{x}{2}\right)\Big)\Big|_{-2}^{2} = \frac{1}{2\pi}\left(\frac{10\pi}{3}+\frac{10\pi}{3}\right) = \frac{10}{3}.$

13.4.46 The average value is given by $\frac{1}{\text{volume of } D}\iiint_D f(x,y,z)\,dV$. The volume of D is $V = \iiint_D (1)\,dV =$ $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-x^2-y^2}(1)\,dz\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(4-x^2-y^2\right)dy\,dx$. Converting to polar coordinates gives $\int_0^{2\pi}\int_0^2\left(4-r^2\right)r\,dr\,d\theta = \int_0^{2\pi}\int_0^2\left(4r-r^3\right)dr\,d\theta = \int_0^{2\pi}\left(2r^2-\frac{1}{4}r^4\right)\Big|_0^2 d\theta = \int_0^{2\pi}4\,d\theta = 8\pi$ Thus, the average value is $\frac{1}{8\pi}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-x^2-y^2}\left(x^2+y^2+z^2\right)dz\,dy\,dx$. This can be computed as
$\frac{1}{8\pi}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\left(x^2+y^2\right)z+\frac{1}{3}z^3\right)\Big|_0^{4-x^2-y^2}dy\,dx =$
$\frac{1}{8\pi}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\left(x^2+y^2\right)\left(4-x^2-y^2\right)+\frac{1}{3}\left(4-x^2-y^2\right)^3\right)dy\,dx$. Converting to polar coordinates gives $\frac{1}{8\pi}\int_0^{2\pi}\int_0^2\left(r^2\left(4-r^2\right)+\frac{1}{3}\left(4-r^2\right)^3\right)r\,dr\,d\theta = \frac{1}{8\pi}\int_0^{2\pi}\int_0^2\left(4r^3-r^5+\frac{1}{3}r\left(4-r^2\right)^3\right)dr\,d\theta =$
$\frac{1}{8\pi}\int_0^{2\pi}\left(r^4-\frac{1}{6}r^6-\frac{1}{24}\left(4-r^2\right)^4\right)\Big|_0^2 d\theta = \frac{1}{8\pi}\int_0^{2\pi}\left(16-\frac{32}{3}-0+\frac{32}{3}\right)d\theta = \frac{2}{\pi}\int_0^{2\pi}d\theta = 4.$

13.4.47 The average value is

$$\begin{aligned}\frac{1}{\text{volume of } D}\iiint_D f(x,y,z)\,dV &= \frac{1}{\frac{1}{2}\left(\frac{4}{3}\pi\cdot 4^3\right)}\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_0^{\sqrt{16-x^2-y^2}} z\,dz\,dy\,dx\\ &= \frac{3}{128\pi}\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\left(\frac{1}{2}z^2\right)\Big|_0^{\sqrt{16-x^2-y^2}}dy\,dx\\ &= \frac{3}{256\pi}\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\left(16-x^2-y^2\right)dy\,dx.\end{aligned}$$

Converting to polar coordinates gives

$$\begin{aligned}\frac{3}{256\pi}\int_0^{2\pi}\int_0^4\left(16-r^2\right)r\,dr\,d\theta &= \frac{3}{256\pi}\int_0^{2\pi}\int_0^4\left(16r-r^3\right)dr\,d\theta\\ &= \frac{3}{256\pi}\int_0^{2\pi}\left(8r^2-\frac{1}{4}r^4\right)\Big|_0^4 d\theta = \frac{3}{256\pi}\int_0^{2\pi}(128-64)\,d\theta\\ &= \frac{3}{4\pi}\int_0^{2\pi}d\theta = \frac{3}{2}.\end{aligned}$$

13.4.48 The average value is given by $\frac{1}{\text{volume of } D}\iiint_D f(x,y,z)\,dV$, which can be evaluated by
$\frac{1}{\frac{1}{3}\pi\cdot 4^2\cdot 8}\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{2\sqrt{x^2+y^2}}^{8}\left(x^2+y^2\right)dz\,dy\,dx = \frac{3}{128\pi}\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\left(x^2+y^2\right)z\Big|_{2\sqrt{x^2+y^2}}^{8}dy\,dx =$
$\frac{3}{128\pi}\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\left(8\left(x^2+y^2\right)-2\left(x^2+y^2\right)^{3/2}\right)dy\,dx$. Converting to polar coordinates gives
$\frac{3}{128\pi}\int_0^{2\pi}\int_0^4\left(8r^2-2r^3\right)r\,dr\,d\theta = \frac{3}{128\pi}\int_0^{2\pi}\int_0^4\left(8r^3-2r^4\right)dr\,d\theta = \frac{3}{128\pi}\int_0^{2\pi}\left(2r^4-\frac{2}{5}r^5\right)\Big|_0^4 d\theta =$
$\frac{3}{128\pi}\int_0^{2\pi}\left(512-\frac{2048}{5}\right)d\theta = \frac{12}{5\pi}\int_0^{2\pi}d\theta = \frac{24}{5}.$

13.4.49

a. False. Only six iterations are possible.

b. False. The outermost limits of integration must be constants.

c. False. D is the intersection of two cylinders in first octant.

13.4.50 $\int_1^4\int_z^{4z}\int_0^{\pi^2}\frac{\sin\sqrt{yz}}{x^{3/2}}\,dy\,dx\,dz = \int_0^{\pi^2}\int_1^4\int_z^{4z}\frac{\sin\sqrt{yz}}{x^{3/2}}\,dx\,dz\,dy = \int_0^{\pi^2}\int_1^4\left(-\frac{2\sin\sqrt{yz}}{\sqrt{x}}\right)\Big|_z^{4z}\,dz\,dy =$ $\int_0^{\pi^2}\int_1^4\left(-\frac{\sin\sqrt{yz}}{\sqrt{z}}+\frac{2\sin\sqrt{yz}}{\sqrt{z}}\right)dz\,dy = \int_0^{\pi^2}\int_1^4\left(\frac{\sin\sqrt{yz}}{\sqrt{z}}\right)dz\,dy.$

Let $u=\sqrt{yz}$, $du=\frac{1}{2}(yz)^{-1/2}\cdot y\,dz$. Substituting gives $\int_0^{\pi^2}\int_{\sqrt{y}}^{2\sqrt{y}}\left(\frac{2}{\sqrt{y}}\sin u\right)du\,dy =$ $\int_0^{\pi^2}\left(-\frac{2}{\sqrt{y}}\cos u\right)\Big|_{u=\sqrt{y}}^{u=2\sqrt{y}}dy = \int_0^{\pi^2}\left(-\frac{2\cos(2\sqrt{y})}{\sqrt{y}}+\frac{2\cos(\sqrt{y})}{\sqrt{y}}\right)dy$ Let $v=\sqrt{y}$ so that $2\,dv=\frac{dy}{\sqrt{y}}$. Then we have $4\int_0^{\pi}(-\cos(2v)+\cos v)\,dv = 4\left(-\frac{1}{2}\sin(2v)+\sin v\right)\Big|_{v=0}^{v=\pi}=0.$

13.4.51 $V=\int_0^1\int_z^{z+1}\int_0^1 dx\,dy\,dz=\int_0^1\int_z^{z+1}1\,dy\,dz=\int_0^1 1\,dz=1.$

13.4.52 $V=\int_0^1\int_z^2\int_0^2 dx\,dy\,dz=\int_0^1\int_z^2 2\,dy\,dz=\int_0^1(4-2z)\,dz=\left(4z-z^2\right)\Big|_0^1=3.$

13.4.53 $V=\int_0^2\int_0^{4-2y}\int_0^{(4-z)/2}1\,dx\,dz\,dy=\int_0^2\int_0^{4-2y}\left(\frac{4-z}{2}\right)dz\,dy=\int_0^2\left(2z-\frac{1}{4}z^2\right)\Big|_0^{4-2y}dy=$ $\int_0^2\left(4-y^2\right)dy=\left(4y-\frac{1}{3}y^3\right)\Big|_0^2=\frac{16}{3}.$

13.4.54 The surfaces intersect when $z_1=z_2\Longrightarrow\sin x=\sin y\Longrightarrow x=y$ or $x=\pi-y$. By symmetry, the volume equals 4 times the volume over the region, bounded by $y=x$ and $y=\pi-x$ from $x=0$ to $x=\frac{\pi}{2}$ under $z=\sin y$. $V=4\int_0^{\pi/2}\int_x^{\pi-x}\int_0^{\sin y}1\,dz\,dy\,dx = 4\int_0^{\pi/2}\int_x^{\pi-x}\sin y\,dy\,dx = 4\int_0^{\pi/2}(-\cos y)\Big|_x^{\pi-x}dx =$ $4\int_0^{\pi/2}(\cos x-\cos(\pi-x))\,dx = 4\int_0^{\pi/2}(2\cos x)\,dx = 8(\sin x)\Big|_0^{\pi/2}=8$

13.4.55 $V=\int_{-1}^0\int_{-x-1}^{x+1}\int_0^{1-x-y}1\,dz\,dy\,dx+\int_0^1\int_{x-1}^{-x+1}\int_0^{1-x-y}1\,dz\,dy\,dx=\int_{-1}^0\int_{-x-1}^{x+1}(1-x-y)\,dy\,dx+$ $\int_0^1\int_{x-1}^{-x+1}(1-x-y)\,dy\,dx=\int_{-1}^0\left((1-x)y-\frac{1}{2}y^2\right)\Big|_{-x-1}^{x+1}dx+\int_0^1\left((1-x)y-\frac{1}{2}y^2\right)\Big|_{x-1}^{-x+1}dx=$ $\int_{-1}^0\left(2-2x^2\right)dx+\int_0^1\left(2-4x+2x^2\right)dx=\left(2x-\frac{2}{3}x^3\right)\Big|_{-1}^0+\left(2x-2x^2+\frac{2}{3}x^3\right)\Big|_0^1=\left(2-\frac{2}{3}\right)+2-2+\frac{2}{3}=2.$

13.4.56

a. $V=\iiint\limits_{D_1}1\,dV=\int_0^1\int_0^z\int_0^y 1\,dx\,dy\,dz=\int_0^1\int_0^z y\,dy\,dz=\int_0^1\frac{1}{2}z^2\,dz=\left(\frac{1}{6}z^3\right)\Big|_0^1=\frac{1}{6}.$

b. Let $D_2=\{(x,y,z):0\le y\le x\le z\le 1\}$. Then $V=\iiint\limits_{D_2}1\,dV=\int_0^1\int_0^z\int_0^x 1\,dy\,dx\,dz=\frac{1}{6}$, by swapping x and y from part a. Likewise all other "cousins" have volume $\frac{1}{6}$.

c. The union includes all points with $0\le x\le 1$, $0\le y\le 1$ and $0\le z\le 1$, which is the unit cube.

13.4.57

i. $\int_0^1\int_0^2\int_0^{1-y}dz\,dx\,dy.$

ii. $\int_0^2\int_0^1\int_0^{1-z}dy\,dz\,dx.$

iii. $\int_0^1\int_0^2\int_0^{1-z}dy\,dx\,dz.$

iv. $\int_0^1\int_0^{1-z}\int_0^2 dx\,dy\,dz.$

v. $\int_0^1\int_0^{1-y}\int_0^2 dx\,dz\,dy.$

13.4.58

i. $\int_0^1 \int_x^1 \int_0^{1-y^2} dz\,dy\,dx.$

ii. $\int_0^1 \int_0^y \int_0^{1-y^2} dz\,dx\,dy.$

iii. $\int_0^1 \int_0^{1-x^2} \int_x^{\sqrt{1-z}} dy\,dz\,dx.$

iv. $\int_0^1 \int_0^{\sqrt{1-z}} \int_x^{\sqrt{1-z}} dy\,dx\,dz.$

v. $\int_0^1 \int_0^{\sqrt{1-z}} \int_0^y dx\,dy\,dz.$

vi. $\int_0^1 \int_0^{1-y^2} \int_0^y dx\,dz\,dy.$

It is a good exercise to see that each of these evaluate to be $\frac{1}{4}$.

13.4.59 Mass is given by $\iiint\limits_D \rho(x, y, z)\,dV$.

$m_1 = \int_0^4 \int_0^{4-x} \int_0^{4-x-y} (8-z)\,dz\,dy\,dx = \int_0^4 \int_0^{4-x} \left(8z - \frac{1}{2}z^2\right)\Big|_0^{4-x-y} dy\,dx =$
$\int_0^4 \int_0^{4-x} \left(24 - 4x - \frac{x^2}{2} - 4y - xy - \frac{y^2}{2}\right) dy\,dx = \int_0^4 \left(24y - 4xy - \frac{x^2}{2}y - 2y^2 - \frac{1}{2}xy^2 - \frac{y^3}{6}\right)\Big|_0^{4-x} dx =$
$\int_0^4 \left(\frac{160}{3} - 24x + 2x^2 + \frac{x^3}{6}\right) dx = \left(\frac{160x}{3} - 12x^2 + \frac{2}{3}x^3 + \frac{x^4}{24}\right)\Big|_0^4 = \frac{224}{3}.$

$m_2 = \int_0^4 \int_0^{4-x} \int_0^{4-x-y} (4+z)\,dz\,dy\,dx = \int_0^4 \int_0^{4-x} \left(4z + \frac{1}{2}z^2\right)\Big|_0^{4-x-y} dy\;\,dx =$
$\int_0^4 \int_0^{4-x} \left(24 - 8x - \frac{x^2}{2} - 8y + xy - \frac{y^2}{2}\right) dy\,dx = \int_0^4 \left(24y - 8xy - \frac{x^2y}{2} - 4y^2 + \frac{1}{2}xy^2 - \frac{y^3}{6}\right)\Big|_0^{4-x} dx =$
$\int_0^4 \left(\frac{128}{3} - 24x + 4x^2 - \frac{x^3}{6}\right) dx = \left(\frac{128x}{3} - 12x^2 + \frac{4}{3}x^3 - \frac{x^4}{24}\right)\Big|_0^4 = \frac{160}{3}.$

Solid 1 has greater mass Because the density is greater near the bottom where the tetrahedron is wider.

13.4.60 The volume of the cheese is $\int_0^4 \int_0^4 \int_0^y 1\,dz\,dy\,dx = \int_0^4 \int_0^4 y\,dy\,dx = \int_0^4 \left(\frac{1}{2}y^2\right)\Big|_0^4 dx = \int_0^4 8\,dx =$ $(8x)\Big|_0^4 = 32$ $\frac{1}{2}(32) = \int_0^4 \int_0^a \int_0^y 1\,dz\,dy\,dx = \int_0^4 \int_0^a y\,dy\,dx =$
$\int_0^4 \left(\frac{1}{2}y^2\right)\Big|_0^a dx = \int_0^4 \left(\frac{1}{2}a^2\right) dx = \left(\left(\frac{1}{2}a^2\right)x\right)\Big|_0^4 = 2a^2$. Thus, $16 = 2a^2$, so $a = 2\sqrt{2}$.

13.4.61 The equation of a cone with height h and whose base is centered at the origin with radius r in the xy-plane is $z = h - \frac{h}{r}\sqrt{x^2+y^2}$. The volume is $V = \int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_0^{h-\frac{h}{r}\sqrt{x^2+y^2}} 1\,dz\,dy\,dx =$ $\int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \left(h - \frac{h}{r}\sqrt{x^2+y^2}\right) dy\,dx$. We switch to polar coordinates, using $x^2+y^2 = a^2$ in order to avoid confusion with the constant r. Then we have $\int_0^{2\pi} \int_0^r \left(h - \frac{h}{r}a\right) a\,da\,d\theta = \int_0^{2\pi} \int_0^r \left(ha - \frac{h}{r}a^2\right) da\,d\theta =$ $\int_0^{2\pi} \left(\frac{1}{2}ha^2 - \frac{h}{3r}a^3\right)\Big|_{a=0}^{a=r} d\theta = \int_0^{2\pi} \left(\frac{1}{2}r^2h - \frac{hr^3}{3r}\right) d\theta = \int_0^{2\pi} \frac{1}{6}r^2h\,d\theta = \frac{1}{3}\pi r^2 h.$

13.4.62 The equation of the plane through $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. The volume is $\int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} 1\,dz\,dy\,dx = \int_0^a \int_0^{b(1-x/a)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\,dx =$
$c\int_0^a \left(\left(1 - \frac{x}{a}\right)y - \frac{y^2}{2b}\right)\Big|_0^{b(1-x/a)} dx = \frac{bc}{2}\int_0^a \left(1 - \frac{x}{a}\right)^2 dx = \frac{bc}{2}\left(x - \frac{x^2}{a} + \frac{x^3}{3a^2}\right)\Big|_0^a = \frac{bc}{2}\left(a - a + \frac{a}{3}\right) = \frac{abc}{6}.$

13.4.63 The equation of a sphere with radius R is $x^2 + y^2 + z^2 = R^2$. The volume is given by

$$\int_{-\sqrt{2Rh-h^2}}^{\sqrt{2Rh-h^2}} \int_{-\sqrt{2Rh-h^2-x^2}}^{\sqrt{2Rh-h^2-x^2}} \int_{R-h}^{\sqrt{R^2-x^2-y^2}} 1\,dz\,dy\,dx$$
$$= \int_{-\sqrt{2Rh-h^2}}^{\sqrt{2Rh-h^2}} \int_{-\sqrt{2Rh-h^2-x^2}}^{\sqrt{2Rh-h^2-x^2}} \left(\sqrt{R^2-x^2-y^2} - (R-h)\right) dy\,dx.$$

Converting to polar coordinates gives $\int_0^{2\pi}\int_0^{\sqrt{2Rh-h^2}}\left(\sqrt{R^2-r^2}-(R-h)\right) r\,dr\,d\theta =$
$\int_0^{2\pi}\int_0^{\sqrt{2Rh-h^2}}\left(r\sqrt{R^2-r^2}-r(R-h)\right) dr\,d\theta = \int_0^{2\pi}\left(\left(-\frac{1}{3}\left(R^2-r^2\right)^{3/2}-\frac{1}{2}r^2(R-h)\right)\right)\Big|_0^{\sqrt{2Rh-h^2}} d\theta =$
$\int_0^{2\pi}\left[-\frac{1}{3}(R-h)^3-\frac{1}{2}\left(2Rh-h^2\right)(R-h)+\frac{1}{3}R^3\right] d\theta = \int_0^{2\pi}\left(\frac{h^2}{6}(3R-h)\right) d\theta = \frac{1}{3}\pi h^2(3R-h).$

13.4.64 There are two volumes to consider: the volume V_1 of the cylinder of radius r inside the frustum, and the volume V_2 that remains when that cylinder is removed. The first volume can be computed without calculus: $V_1 = \pi r^2 h$. Note that the base of the volume V_2 is the annulus centered at the origin with inner radius r and outer radius R. Polar coordinates may be used (with a as the radius to avoid confusion with r) to show that the equation of the given frustrum is $z = \frac{h}{R-r}(R-a)$. The volume V_2 is $\int_0^{2\pi}\int_r^R\int_0^{\frac{h}{R-r}(R-a)} a\,dz\,da\,d\theta = \int_0^{2\pi}\int_r^R \frac{ha}{R-r}(R-a)\,da\,d\theta = \frac{h}{R-r}\int_0^{2\pi}\int_r^R\left(Ra-a^2\right) da\,d\theta =$
$\frac{h}{R-r}\int_0^{2\pi}\left(\frac{Ra^2}{2}-\frac{a^3}{3}\right)\Big|_{a=r}^{a=R} d\theta = \frac{h}{R-r}\int_0^{2\pi}\left(\frac{R^3}{6}-\frac{R\,r^2}{2}+\frac{r^3}{3}\right) d\theta = \frac{h}{R-r}\cdot 2\pi\cdot\left(\frac{R^3}{6}-\frac{R\,r^2}{2}+\frac{r^3}{3}\right) =$
$\frac{h}{R-r}\cdot\frac{\pi R^3-3\pi R\,r^2+2\pi r^3}{3}$. The total volume V is $V = V_1+V_2 = \pi r^2 h + \frac{h}{R-r}\cdot\frac{\pi R^3-3\pi R\,r^2+2\pi r^3}{3} = \frac{\pi h}{3}\cdot$
$\frac{3r^2(R-r)+R^3-3R\,r^2+r^3}{R-r} = \frac{\pi h}{3}\cdot\frac{R^3-r^3}{R-r} = \frac{\pi h}{3}\left(R^2+R\,r+r^2\right)$.

13.4.65 The equation of the given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$. Using symmetry, the volume V of the ellipsoid is eight times the volume of the portion of the ellipsoid in the first octant. Thus $V =$
$8\int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} 1\,dz\,dy\,dx = 8c\int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx =$
$8c\int_0^a\left(\frac{y}{2}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}+\frac{b\left(a^2-x^2\right)}{2a^2}\sin^{-1}\left(\frac{ay}{b\sqrt{a^2-x^2}}\right)\right)\Big|_0^{b\sqrt{1-x^2/a^2}} dx = 8c\int_0^a\frac{b\pi\left(a^2-x^2\right)}{4a^2}\,dx =$
$\frac{2\pi bc}{a^2}\left(a^2x-\frac{x^3}{3}\right)\Big|_0^a = \frac{2\pi bc}{a^2}\cdot\frac{2a^3}{3} = \frac{4}{3}\pi abc$

13.4.66

a. $p(0.75) = \int_0^{0.75} 0.8e^{-0.8t}\,dt = \left(-e^{-0.8t}\right)\Big|_{t=0}^{t=0.75} = 0.4512.$

b. $p(0.75) = \int_0^{0.75}\int_0^{0.75}\left(0.8e^{-0.8t}\right)\left(0.1e^{-0.1s}\right) dt\,ds = \int_0^{0.75}\left(0.1e^{-0.1s}\right)\left(-e^{-0.8t}\right)\Big|_{t=0}^{t=0.75} ds =$
$\int_0^{0.75}\left(0.1e^{-0.1s}\right)(0.4512)\,ds = (0.4512)\left(-e^{-0.1s}\right)\Big|_{s=0}^{s=0.75} = (0.4512)(0.0723) = 0.0326.$

c. $p(0.75) = \int_0^{0.75}\int_0^{0.75}\int_0^{0.75}\left(0.8e^{-0.8t}\right)\left(0.1e^{-0.1s}\right)\left(0.05e^{-0.05u}\right) dt\,ds\,du =$
$\int_0^{0.75}\int_0^{0.75}\left(0.1e^{-0.1s}\right)\left(0.05e^{-0.05u}\right)\left(-e^{-0.8t}\right)\Big|_{t=0}^{t=0.75} ds\,du =$
$(0.4512)\int_0^{0.75}\int_0^{0.75}\left(0.1e^{-0.1s}\right)\left(0.05e^{-0.05u}\right) ds\,du = (0.4512)\int_0^{0.75}\left(0.05e^{-0.05u}\right)\left(-e^{-0.1s}\right)\Big|_{s=0}^{s=0.75} du =$
$(0.4512)(0.0723)\int_0^{0.75}\left(0.05e^{-0.05u}\right) du = (0.4512)(0.0723)\left(-e^{-0.05u}\right)\Big|_{u=0}^{u=0.75} =$
$(0.4512)(0.0723)(0.0368) = 0.0012.$

13.4.67 The Hypervolume is given by $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\int_0^{1-x-y-z} 1\,dw\,dz\,dy\,dx$. This can be computed as $\int_0^1\int_0^{1-x}\int_0^{1-x-y}(1-x-y-z)\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left((1-x-y)z-\frac{1}{2}z^2\right)\Big|_0^{-x-y} dy\,dx =$
$\int_0^1\int_0^{1-x}\left((1-x-y)^2-\frac{1}{2}(1-x-y)^2\right) dy\,dx = \frac{1}{2}\int_0^1\int_0^{1-x}(1-x-y)^2\,dy\,dx =$
$\frac{1}{2}\int_0^1\left[-\frac{1}{3}(1-x-y)^3\right]\Big|_0^{1-x} dx = \frac{1}{6}\int_0^1(1-x)^3\,dx = \frac{1}{6}\left(-\frac{1}{4}(1-x)^4\right)\Big|_0^1 = \frac{1}{24}.$

13.4.68 The region of integration for the integral $\int_0^1\int_x^1\int_x^y f(x)\,f(y)\,f(z)\,dz\,dy\,dx$ is the set of points (x, y, z) such that $0 < x < z < y < 1$. This region is one sixth of the unit cube. By rearranging x, y, and z, the other five sixths of the unit cube may be generated. Thus the integral of $f(x)\,f(y)\,f(z)$ over the unit cube is the sum of the integrals over these six regions. Now notice that the integrand

$f(x)f(y)f(z)$ does not change as x, y, and z are rearranged, so the integral over each of the six regions is the same. Thus $\int_0^1\int_0^1\int_0^1 f(x)f(y)f(z)\,dz\,dy\,dx = 6\int_0^1\int_x^1\int_x^y f(x)f(y)f(z)\,dz\,dy\,dx$. The integral of $f(x)f(y)f(z)$ over the unit cube can be calculated as follows: $\int_0^1\int_0^1\int_0^1 f(x)f(y)f(z)\,dz\,dy\,dx =$ $\int_0^1\int_0^1 f(x)f(y)\left(\int_0^1 f(z)\,dz\right)dy\,dx = \left(\int_0^1 f(z)\,dz\right)\int_0^1\int_0^1 f(x)f(y)\,dy\,dx =$ $\left(\int_0^1 f(z)\,dz\right)\int_0^1 f(x)\left(\int_0^1 f(y)\,dy\right)dx = \left(\int_0^1 f(z)\,dz\right)\left(\int_0^1 f(y)\,dy\right)\left(\int_0^1 f(x)\,dx\right)$.

Because the name of the variable is immaterial, we have $\int_0^1\int_0^1\int_0^1 f(x)f(y)f(z)\,dz\,dy\,dx =$ $\left(\int_0^1 f(z)\,dz\right)\left(\int_0^1 f(y)\,dy\right)\left(\int_0^1 f(x)\,dx\right) = \left(\int_0^1 f(x)\,dx\right)^3$.

Thus, $\int_0^1\int_x^1\int_x^y f(x)f(y)f(z)\,dz\,dy\,dx = \frac{1}{6}\int_0^1\int_0^1\int_0^1 f(x)f(y)f(z)\,dz\,dy\,dx = \frac{1}{6}\left(\int_0^1 f(x)\,dx\right)^3$.

13.5 Triple Integrals in Cylindrical and Spherical Coordinates

13.5.1 r measures the distance from the point to the z axis, θ is the angle that the segment from the point to the z-axis makes with the positive xz-plane, and z is the directed distance from the point to the xy-plane.

13.5.2 The triple (ρ, φ, θ) describes a point whose distance from the origin is ρ, such that the line from the origin to the point makes an angle of φ with the z-axis, and such that the projection of this line to the xy-plane makes an angle of θ with the positive x-axis.

13.5.3 A double cone (opening both upwards and downwards), where the radius is always 4 times the distance to the xy-plane.

13.5.4 A cone opening upwards making an angle of $\frac{\pi}{4}$ radian with the z-axis.

13.5.5 It approximates the volume of the cylindrical wedge formed by the changes Δr, $\Delta\theta$, and Δz.

13.5.6 It approximates the volume formed by the changes $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$.

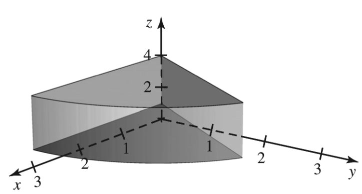
13.5.7 $\iiint\limits_D f(r, \theta, z)\,dV = \int_\alpha^\beta\int_{g(\theta)}^{h(\theta)}\int_{G(r,\theta)}^{H(r,\theta)} f(r, \theta, z)\,dz\,r\,dr\,d\theta$.

13.5.8 $\iiint\limits_D f(\rho, \varphi, \theta)\,dV = \int_\alpha^\beta\int_a^b\int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho, \varphi, \theta)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$.

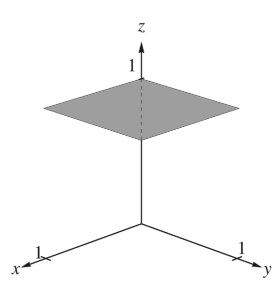
13.5.9 Cylindrical coordinates, because in cylindrical coordinates x^2+y^2 simplifies to r^2.

13.5.10 Spherical coordinates, because in spherical coordinates $x^2+y^2+z^2$ simplifies to ρ^2.

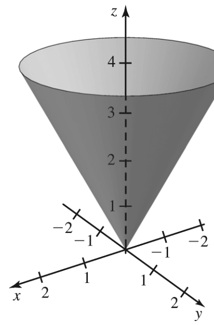
13.5.11 This is a wedge of a cylinder of radius 3 from $z=1$ to $z=4$, where the wedge angle is $\frac{\pi}{3}$.



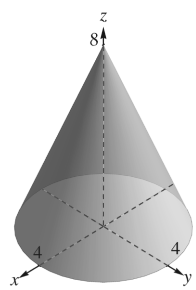
13.5.12 This is the first quadrant of the plane $z=1$ for $x, y \geq 0$.



13.5.13 This is the solid upward pointing cone given by $z = 2r$ from $z = 0$ to $z = 4$.



13.5.14 This is a solid downward-pointing cone with vertex at (0, 0, 8) between $z = 8$ and $z = 0$. Its intersection with the xy-plane is the circle $x^2 + y^2 = 16$.



13.5.15 $\int_0^{2\pi}\int_0^1\int_{-1}^1 dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\,z\Big|_{-1}^1 dr\,d\theta = \int_0^{2\pi}\int_0^1 2r\,dr\,d\theta = \int_0^{2\pi} 1\,d\theta = 2\pi$.

13.5.16 This is the integral over half of the cone whose equation is $z = 9 - 3r$, $0 \le r \le 3$. Thus $\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^{9-3\sqrt{x^2+y^2}} dz\,dx\,dy = \int_0^{\pi}\int_0^3\int_0^{9-3r} r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^3 r\,z\Big|_0^{9-3r} dr\,d\theta = \int_0^{\pi}\int_0^3 \left(9r - 3r^2\right) dr\,d\theta = \int_0^{\pi}\left(\frac{9}{2}r^2 - r^3\right)\Big|_0^3 d\theta = \int_0^{\pi}\left(\frac{81}{2} - 27\right) d\theta = \frac{27}{2}\pi$.

13.5.17 This is the integral of $\left(x^2+y^2\right)^{3/2}$ over a cylinder with radius 1 from $z = -1$ to $z = 1$. We convert to cylindrical coordinates, so the integrand becomes r^3 Because $x^2 + y^2 = r^2$. The integral is thus $\int_0^{2\pi}\int_0^1\int_{-1}^1 r^3\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2\,r^4\,dr\,d\theta = \frac{2}{5}\int_0^{2\pi} r^5\Big|_0^1 = \frac{4\pi}{5}$.

13.5.18 Converting to cylindrical coordinates, the integrand becomes $\frac{1}{1+r^2}$; the region of integration is the half-cylinder shown, so the integral is $\int_0^{\pi}\int_0^3\int_0^2 \frac{1}{1+r^2}\,dz\,r\,dr\,d\theta = \int_0^{\pi}\int_0^3 \frac{2r}{1+r^2}\,dr\,d\theta = \int_0^{\pi}\ln\left(1+r^2\right)\Big|_0^3 = \int_0^{\pi}\ln(10)\ d\theta = \pi\ \ln(10)$.

13.5.19 The region of integration is a wedge from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$ with radius 1, between $z = 0$ and $z = 4$. This can be seen by noting that the limits of integration for y range from x to the boundary of the unit circle (y-coordinate $\sqrt{1-x^2}$) and that $x = \frac{\sqrt{2}}{2}$corresponds to $\frac{\pi}{4}$. Thus the integral is $\int_{\pi/4}^{\pi/2}\int_0^1\int_0^4 e^{-r^2}\,dz\,r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\int_0^1 4r\,e^{-r^2}\,dr\,d\theta = -2\int_0^{\pi} e^{-r^2}\Big|_0^1 = -2\int_0^{\pi}\left(e^{-1}-1\right)\,d\theta = \frac{\pi}{2}\,\left(1-e^{-1}\right)$.

13.5.20 The region of integration is the volume between the cone $x^2 + y^2 = z^2$ and the plane $z = 4$. To see this, note that x and y vary over the disk $x^2 + y^2 = 16$ and that z varies from the boundary of the *cone* to 4. Converting to cylindrical coordinates, we then have $\int_0^{2\pi}\int_0^4\int_r^4 1\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^4 r\,(4-r)\,dr\,d\theta = \int_0^{2\pi}\left(2r^2 - \frac{1}{3}r^3\right)\Big|_0^4 = \int_0^{2\pi}\frac{32}{3}\,d\theta = \frac{64\pi}{3}$.

13.5.21 The region of integration is below the cone $x^2 + y^2 = z^2$ in the first octant, so the integral is $\int_0^{\pi/2}\int_0^3\int_0^r \frac{1}{r}\, dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^3 r\, dr\, d\theta = \int_0^{\pi/2} \frac{9}{2}\, d\theta = \frac{9\pi}{4}$.

13.5.22 The region of integration is a wedge from $\theta = 0$ to $\theta = \frac{\pi}{6}$ with radius 1, between $z = -1$ and $z = 1$. The integral is then $\int_0^{\pi/6}\int_0^1\int_{-1}^1 r\, dz\, r\, dr\, d\theta = \int_0^{\pi/6}\int_0^1\int_{-1}^1 r^2\, dz\, dr\, d\theta = \int_0^{\pi/6} \frac{2}{3}\, d\theta = \frac{\pi}{9}$.

13.5.23 $\int_0^{2\pi}\int_0^4\int_0^{10}\left(1+\frac{z}{2}\right) dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^4 \left(z+\frac{z^2}{4}\right)\Big|_0^{10} r\, dr\, d\theta = \int_0^{2\pi}\int_0^4 35r\, dr\, d\theta = 560\pi.$

13.5.24 $\int_0^{2\pi}\int_0^3\int_0^2 5e^{-r^2}\, dz\, r\, dr\, d\theta = -5\int_0^{2\pi}\int_0^3\left(-2re^{-r^2}\right) dr\, d\theta = 10\pi\left(1-e^{-9}\right).$

13.5.25 $\int_0^{2\pi}\int_0^6\int_0^{6-r}(7-z)\, dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^6\left(7z-\frac{1}{2}z^2\right)\Big|_0^{6-r} r\, dr\, d\theta =$
$\int_0^{2\pi}\int_0^4\left(7r(6-r)-\frac{r}{2}(6-r)^2\right) dr\, d\theta = 396\pi.$

13.5.26 $\int_0^{2\pi}\int_0^3\int_0^{9-r^2}\left(1+\frac{z}{9}\right) dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^3\left(z+\frac{z^2}{18}\right)\Big|_0^{9-r^2} r\, dr\, d\theta =$
$\int_0^{2\pi}\int_0^3\left(r\left(9-r^2\right)-\frac{r}{18}\left(9-r^2\right)^2\right) dr\, d\theta = 54\pi.$

13.5.27 The base of both surfaces in the xy-plane is the area bounded by the unit circle $r = 1$. The mass of the solid bounded by the xy-plane and $z = 4 - 4r$ is given by $\int_0^{2\pi}\int_0^1\int_0^{4-4r}(10-2z)\, dz\, r\, dr\, d\theta =$ $\int_0^{2\pi}\int_0^1\left(10z-z^2\right)\Big|_0^{4-4r} r\, dr\, d\theta = \int_0^{2\pi}\int_0^1\left(10r(4-4r)-r(4-4r)^2\right) dr\, d\theta = \frac{32\pi}{3}$. The mass of the solid bounded by the xy-plane and $z = 4 - 4r^2$ is given by $\int_0^{2\pi}\int_0^1\int_0^{4-4r^2}(10-2z)\, dz\, r\, d\, r d\theta =$
$\int_0^{2\pi}\int_0^1\left(10z-z^2\right)\Big|_0^{4-4r^2} r\, dr\, d\theta = \int_0^{2\pi}\int_0^1\left(10r\left(4-4r^2\right)-r\left(4-4r^2\right)^2\right) dr\, d\theta = \frac{44\pi}{3}$. Thus the mass of the solid bounded by the paraboloid is larger. This can also be seen by noting that for $r \le 1$, $r^2 \le r$ so that $4 - 4r^2 \ge 4 - 4r$, so that the cone is contained in the paraboloid.

13.5.28 By the argument given above, again the solid bounded by the paraboloid will have greater mass. For the solid bounded by the cone, the mass is $\int_0^{2\pi}\int_0^1\int_0^{4-4r}\left(\frac{8}{\pi}e^{-z}\right) dz\, r\, dr\, d\theta = \frac{8}{\pi}\int_0^{2\pi}\int_0^1 -r\left(e^{4r-4}-1\right) dr\, d\theta =$ $5-e^{-4}$. The mass of the solid bounded by the xy-plane and $z = 4-4r^2$ is $\int_0^{2\pi}\int_0^1\int_0^{4-4r^2}\left(\frac{8}{\pi}e^{-z}\right) dz\, r\, dr\, d\theta =$ $\frac{8}{\pi}\int_0^{2\pi}\int_0^1 -r\left(e^{4r^2-4}-1\right) dr\, d\theta = 6+2e^{-4}$.

13.5.29 The base of this solid in the xy-plane is given by $\sqrt{17} = \sqrt{1+x^2+y^2}$ so is the area bounded by the circle $x^2+y^2 = 16$ or $r = 4$. Thus $V = \int_0^{2\pi}\int_0^4\int_0^{\sqrt{17}-\sqrt{1+r^2}} 1\, dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^4\left(r\sqrt{17}-r\sqrt{1+r^2}\right) dr\, d\theta =$ $\frac{\pi\left(14\sqrt{17}+2\right)}{3}$.

13.5.30 This solid sits over the circle $r = 5$ in the xy-plane, so the volume is $\int_0^{2\pi}\int_0^5\int_{r^2}^{25} 1\, dz\, r\, dr\, d\theta =$ $\int_0^{2\pi}\int_0^5\left(25r-r^3\right) dr\, d\theta = \frac{625\pi}{2}$.

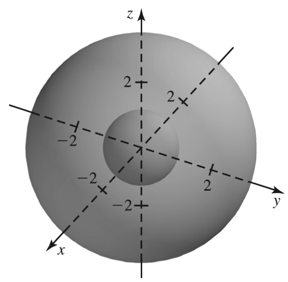
13.5.31 This solid sits over the circle $r = 5$ in the xy-plane, so the volume is $\int_0^{2\pi}\int_0^5\int_{\sqrt{4+r^2}}^{\sqrt{29}} 1\, dz\, r\, d\, r d\theta =$ $\int_0^{2\pi}\int_0^5\left(r\sqrt{29}-r\sqrt{4+r^2}\right) dr\, d\theta = \frac{\pi\left(17\sqrt{29}+16\right)}{3}$.

13.5.32 The base is swept out completely as θ ranges from 0 to π, so the volume is $\int_0^{\pi}\int_0^{2\cos\theta}\int_0^4 1\, dz\, r\, dr\, d\theta =$ $\int_0^{2\pi}\int_0^{2\cos\theta} 4r\, dr\, d\theta = 4\pi$, which is in fact the volume of a right circular cylinder of height 4 whose base is a unit circle.

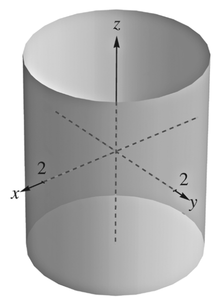
13.5.33 The first octant is determined by $0 \le \theta \le \frac{\pi}{2}$, $0 \le z$, and the condition $z = x$ in cylindrical coordinates becomes $z = r\cos\theta$, so the volume of the solid is $\int_0^{\pi/2}\int_0^1\int_0^{r\cos\theta} 1\, dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^1 r^2\cos\theta\, dr\, d\theta =$ $\frac{1}{3}$.

13.5.34 The volume is $\int_0^{2\pi}\int_1^2\int_0^{4-r(\cos\theta+\sin\theta)} 1\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_1^2 \left(4r - r^2\left(\cos\theta+\sin\theta\right)\right)\,dr\,d\theta =$ $\int_0^{2\pi}\left(2r^2 - \frac{1}{3}r^3\left(\cos\theta+\sin\theta\right)\right)\Big|_1^2 d\theta = \int_0^{2\pi}\left(6-\frac{7}{3}\left(\cos\theta+\sin\theta\right)\right)d\theta = 12\pi$

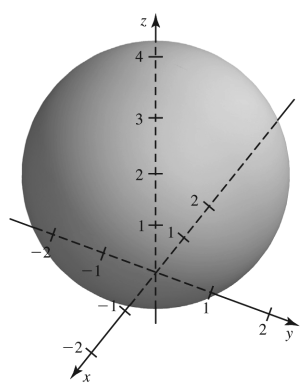
13.5.35 This is a spherical shell centered at the origin with outer radius 3 and inner radius 1.



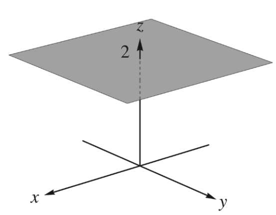
13.5.36 Because $\rho = 2\csc\varphi$, $\rho\sin\varphi = 2$. Now $x^2+y^2 = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta = \rho^2\sin^2\varphi = 4$. Thus any point on the surface satisfies $x^2+y^2=4$, so that the surface is a cylinder of radius 2 oriented along the z-axis.



13.5.37 This is a sphere of radius 2 centered at (0, 0, 2). To see this, note that $\rho = 4\cos\varphi$ implies that $\rho^2 = 4\rho\cos\varphi$. Converting to rectangular coordinates gives $x^2+y^2+z^2 = 4z$, and completing the square yields $x^2+y^2+(z-2)^2 = 4$.



13.5.38 This is the plane $z = 2$, Because $\rho = 2\sec\varphi$ implies that $z = \rho\cos\varphi = 2$.



13.5.39 $\iiint\limits_D \left(x^2+y^2+z^2\right)^{5/2} dV = \int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^5\, \rho^2 \sin\varphi\; d\varphi\, d\theta\, d\rho = \int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^7 \sin\varphi\; d\varphi\, d\theta\, d\rho =$
$\int_0^1 \int_0^{2\pi} 2\rho^7\, d\theta\, d\rho = 4\pi \int_0^1 \rho^7\, d\rho = \frac{\pi}{2}.$

13.5.40 $\iiint\limits_D e^{-\left(x^2+y^2+z^2\right)^{-3/2}} dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 e^{-\rho^3}\, \rho^2 \sin\varphi\; d\rho\, d\varphi\, d\theta = -\frac{1}{3} \int_0^{2\pi} \int_0^{\pi} e^{-\rho^3}\Big|_0^1 \sin\varphi\; d\varphi\, d\theta =$
$-\frac{\left(e^{-1}-1\right)}{3} \int_0^{2\pi} \int_0^{\pi} \sin\varphi\, d\varphi\, d\theta = -\frac{4\pi}{3}\left(e^{-1}-1\right).$

13.5.41 $\iiint\limits_D \left(x^2+y^2+z^2\right)^{-3/2} dV = \int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^{-3}\, \rho^2 \sin\varphi\; d\rho\, d\varphi\, d\theta = \int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^{-1} \sin\varphi\, d\rho\, d\varphi\, d\theta =$
$\ln 2 \int_0^{2\pi} \int_0^{\pi} \sin\varphi\, d\varphi\, d\theta = 4\pi \ln 2.$

13.5.42 $\int_0^{2\pi} \int_0^{\pi/3} \int_0^{4\sec\varphi} \rho^2 \sin\varphi\; d\rho\, d\varphi\, d\theta = \frac{64}{3} \int_0^{2\pi} \int_0^{\pi/3} \sec^3\varphi \sin\varphi\, d\varphi\, d\theta = 64\pi.$

13.5.43 $\int_0^{\pi} \int_0^{\pi/6} \int_{2\sec\varphi}^4 \rho^2 \sin\varphi\; d\rho\, d\varphi\, d\theta = \frac{1}{3} \int_0^{\pi} \int_0^{\pi/6} \left(64 - 8\sec^3\varphi\right) \sin\varphi\; d\varphi\, d\theta = \left(\frac{188}{9} - \frac{32}{3}\sqrt{3}\right)\pi.$

13.5.44 $\int_0^{2\pi} \int_0^{\pi/4} \int_1^{2\sec\varphi} \left(\rho^{-3}\right) \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \int_0^{2\pi} \int_0^{\pi/4} \sin\varphi \ln\left(2\sec\varphi\right)\; d\varphi\, d\theta =$
$2\pi\left(\ln 2\left(1 - \frac{3\sqrt{2}}{4}\right) + 1 - \frac{\sqrt{2}}{2}\right).$

13.5.45 $\int_0^{2\pi} \int_{\pi/6}^{\pi/3} \int_0^{2\csc\varphi} \rho^2 \sin\varphi\; d\rho\, d\varphi\, d\theta = \frac{8}{3} \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \csc^3\varphi \sin\varphi\; d\varphi\, d\theta = \frac{32}{9}\pi\sqrt{3}.$

13.5.46 A ball of radius a around the origin has volume given by $\int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^2 \sin\varphi\; d\rho\, d\varphi\, d\theta =$
$\frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin\varphi\; d\varphi\, d\theta = \frac{4}{3}a^3.$

13.5.47 The two spheres intersect where $2\cos\varphi = 1$, i.e. when $\varphi = \frac{\pi}{3}$. That circle is then at $z = \frac{1}{2}$ (again looking at $\cos\varphi = \frac{1}{2}$ and using the fact that the lower sphere has radius 1). The volume in question is the sum of the volumes above and below that circle of intersection; because the circle is at height $\frac{1}{2}$ and both spheres have radius 1, the volumes above and below are identical. Thus we need only compute the upper volume and double it: $2\int_0^{\pi/3} \int_{(\sec\varphi)/2}^1 \int_0^{2\pi} \rho^2 \sin\varphi\, d\theta\, d\rho\, d\varphi = \frac{4\pi}{3} \int_0^{\pi/3} \left(1 - \frac{\sec^3\varphi}{8}\right) \sin\varphi\, d\theta = \frac{5}{12}\pi.$

13.5.48 $\int_0^{2\pi} \int_0^{\pi} \int_0^{1+\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} \left(1+\cos\varphi\right)^3 \sin\varphi\, d\varphi\, d\theta = -\frac{1}{12} \int_0^{2\pi} \left(1+\cos\varphi\right)^4 \Big|_{\varphi=0}^{\varphi=\pi} d\theta$
$= \frac{4}{3} \int_0^{2\pi} d\theta = \frac{8}{3}\pi.$

13.5.49 The portion of the sphere is determined by $\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$, so the integral is
$\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{4\cos\varphi} \rho^2 \sin\varphi\; d\rho\, d\varphi\, d\theta = \frac{64}{3} \int_0^{\pi/3} \int_{\pi/4}^{\pi/2} \cos^3\varphi\, \sin\varphi d\varphi\, d\theta = \frac{8}{3}\pi.$

13.5.50 $\int_0^{2\pi} \int_{\pi/6}^{\pi/3} \int_{\csc\varphi}^{2\csc\varphi} \rho^2 \sin\varphi\; d\rho\, d\varphi\, d\theta = \frac{7}{3} \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \csc^2\ \varphi\, d\varphi\, d\theta = -\frac{7}{3} \int_0^{2\pi} \cot\varphi\Big|_{\varphi=\pi/6}^{\varphi=\pi/3} d\theta =$
$-\frac{7}{3} \int_0^{2\pi} \left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) d\theta = \frac{28}{9}\pi\sqrt{3}.$

13.5.51 The value of φ corresponding to the circle of intersection of the sphere ρ and the plane $z = 2\sqrt{3}$ is found by noting that $\cos\varphi = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$, so that $\varphi = \frac{\pi}{6}$. Similarly, the intersection with the plane $z = 2$ is where $\cos\varphi = \frac{2}{4} = \frac{1}{2}$ so that $\varphi = \frac{\pi}{3}$. We can find the volume of the required region by determining the volume of the regions above $z = 2$ and $z = 2\sqrt{3}$ and subtracting. The volume of the region above the plane $z = 2$ is $\int_0^{2\pi} \int_0^{\pi/3} \int_{2\sec\varphi}^4 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} \left(64 - 8\sec^3\varphi\right) \sin\varphi\, d\varphi\, d\theta = \frac{40}{3}\pi$, and the volume of the region above $z = 2\sqrt{3}$ is $\int_0^{2\pi} \int_0^{\pi/6} \int_{2\sqrt{3}\sec\varphi}^4 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} \left(64 - 24\sqrt{3}\sec^3\varphi\right) \sin\varphi\, d\varphi\, d\theta =$ $\frac{128}{3}\pi - 24\sqrt{3}\pi$, so that the volume of the region between the two planes is $\left(24\sqrt{3} - \frac{88}{3}\right)\pi$.

13.5.52 This cone makes an angle of $\frac{\pi}{4}$ with the z-axis. We thus want $\int_0^{2\pi} \int_0^{\pi/4} \int_{\sec\varphi}^{2\sec\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta =$ $\frac{7}{3} \int_0^{2\pi} \int_0^{\pi/3} \sec^3\varphi\, \sin\varphi\, d\varphi\, d\theta = \frac{7}{3}\pi$. A simpler method of arriving at this result is to recall that the volume of a cone is $\frac{1}{3}Ah$, where A is the area of the base and h is the height, so in this case the volume of the larger cone is $\frac{1}{3} \cdot 4\pi \cdot 2$ and of the smaller $\frac{1}{3} \cdot \pi \cdot 1$, so the difference is as above.

13.5.53

a. True. In either set of coordinates, any value of θ may be chosen.

b. True. Note that $r = z$ if and only if $\rho \sin\varphi = \rho\cos\varphi$, which happens if and only if $\varphi = \frac{\pi}{4}$.

13.5.54 $\rho^2 = \sec(2\varphi)$ implies that $1 = \rho^2\cos(2\varphi) = \rho^2\left(\cos^2\varphi - \sin^2\varphi\right) = z^2 - x^2 - y^2$ which is a hyperboloid of two sheets.

13.5.55 $\rho^2 = -\sec(2\varphi)$ implies that $-1 = \rho^2\cos(2\varphi) = \rho^2\left(\cos^2\varphi - \sin^2\varphi\right) = z^2 - x^2 - y^2$ so that $x^2 + y^2 - z^2 = 1$, which is a hyperboloid of one sheet. Because $\frac{\pi}{4} < \varphi < \frac{3\pi}{4}$, we only have the upper sheet.

13.5.56 $\int_0^{2\pi}\int_0^{\pi}\int_0^4 (1+\rho)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{3}\rho^3 + \frac{1}{4}\rho^4\right)\Big|_0^4 \sin\varphi\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{256}{3}\sin\varphi\,d\varphi\,d\theta$ $= \frac{1024}{3}\pi$.

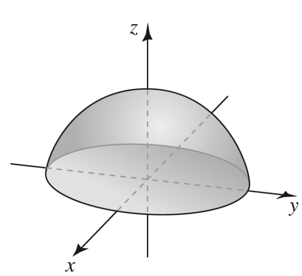
13.5.57 $\int_0^{2\pi}\int_0^{\pi}\int_0^8 2e^{-\rho^3}\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = -\frac{2}{3}\int_0^{2\pi}\int_0^{\pi}\ e^{-\rho^3}\Big|_0^8 \sin\,\varphi\,d\varphi\,d\theta =$ $-\frac{2}{3}\left(e^{-512} - 1\right)\int_0^{2\pi}\int_0^{\pi/3}\sin\varphi\,d\varphi\,d\theta = \frac{8}{3}\left(1 - e^{-512}\right)\pi$.

13.5.58 The mass is

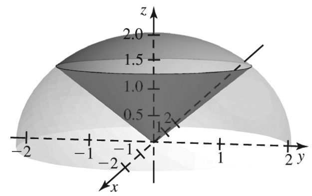
$$\begin{aligned}\int_0^{2\pi}\int_0^{4\sqrt{3}}\int_{r/\sqrt{3}}^4 (5-z)r\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^{4\sqrt{3}} (5z - z^2/2)r\Big|_{r/\sqrt{3}}^4 dr\,d\theta \\ &= \int_0^{2\pi}\int_0^{4\sqrt{3}} ((20-8) - (5r/\sqrt{3} - r^2/6))r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^{4\sqrt{3}} (12r - 5r^2/\sqrt{3} + r^3/6)\,dr\,d\theta \\ &= \int_0^{2\pi} (6r^2 - 5r^3/3\sqrt{3} + r^4/24)\Big|_0^{4\sqrt{3}} d\theta \\ &= 2\pi(288 - 320 + 96) = 128\pi\end{aligned}$$

13.5.59 Note that because of the absolute value sign, the mass is symmetric around the xy-plane, so we can compute the mass for $0 \le z \le 1$ and double it. Using cylindrical coordinates, the mass is then $2\int_0^1\int_0^{2\pi}\int_0^2 (2-z)(4-r)\,r\,dr\,d\theta\,dz = 2\int_0^1\int_0^{2\pi}\int_0^2 (2-z)\left(4r - r^2\right)\,dr\,d\theta\,dz =$ $2\int_0^1\int_0^{2\pi}(2-z)\left(2r^2 - \frac{1}{3}r^3\right)\Big|_0^2 d\theta\,dz = \frac{64\pi}{3}\int_0^1 (2-z)\,dz = 32\pi$.

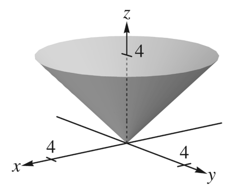
13.5.60 The cylinder $r = 1$ intersects the sphere $\rho = 5$ along the circle $x^2 + y^2 = 1$, $z = 2\sqrt{6}$. Note also that for any point (r,θ,z) on the sphere $r^2 + z^2 = x^2 + y^2 + z^2 = 25$. We have $\int_0^{2\pi}\int_1^5\int_0^{\sqrt{25-r^2}} f(r,\theta,z)\,r\,dz\,dr\,d\theta =$ $\int_0^{2\pi}\int_0^{2\sqrt{6}}\int_1^{\sqrt{25-z^2}} f(r,\,\theta,\,z)\,r\,dr\,dz\,d\theta = \int_1^5\int_0^{\sqrt{25-r^2}}\int_0^{2\pi} f(r,\,\theta,\,z)\,r\,d\theta\,dz\,dr$.



13.5.61 $\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} f(r,\theta,z)\,r\,dz\,dr\,d\theta =$ $\int_0^{2\pi}\int_0^{\sqrt{2}}\int_0^z f(r,\theta,z)\,r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_{\sqrt{2}}^2\int_0^{\sqrt{4-z^2}} f(r,\,\theta,\,z)\,r\,dr\,dz\,d\theta = \int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}}\int_0^{2\pi} f(r,\,\theta,\,z)\,r\,d\theta\,dz\,dr$.

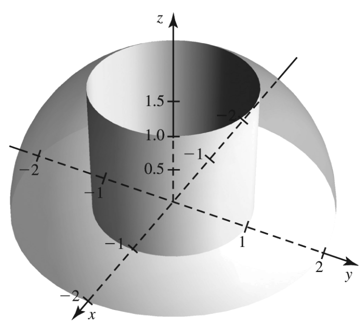


13.5.62 The region of integration is a cone with vertex at the origin making an angle of $\frac{\pi}{4}$ with the positive z-axis, from $z = 0$ to $z = 4$.



We have $\int_0^{\pi/4}\int_0^{2\pi}\int_0^{4\sec\varphi} f(\rho,\varphi,\theta)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \int_0^{\pi/4}\int_0^{4\sec\varphi}\int_0^{2\pi} f(\rho,\varphi,\theta)\,\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi$.

13.5.63 The region of integration is the solid between the upper half-sphere of radius 2 centered at the origin and a cylinder of radius 1 oriented along the z-axis.



We have $\int_{\pi/6}^{\pi/2}\int_0^{2\pi}\int_{\csc\varphi}^{2} f(\rho,\varphi,\theta)\;\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \int_{\pi/6}^{\pi/2}\int_{\csc\varphi}^{2}\int_0^{2\pi} f(\rho,\varphi,\theta)\;\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi$.

13.5.64 Using spherical coordinates, this volume is $\int_{\pi/4}^{\pi/2}\int_0^1\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{2\pi}{3}\int_{\pi/4}^{\pi/2}\sin\varphi\,d\varphi = \frac{2\pi}{3}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{3}\pi$.

13.5.65 This region is symmetric about the xy-plane, so we compute the volume of the region inside the solid cylinder for $z \ge 0$ that is below the cone $\varphi = \frac{\pi}{3}$ and double it. Use spherical coordinates. For a given value of φ, we have $0 \le r \le 2\csc\varphi$, so the volume is $2\int_{\pi/3}^{\pi/2}\int_0^{2\csc\varphi}\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{32\pi}{3}\int_{\pi/3}^{\pi/2}\csc^3\varphi\,\sin\varphi\,d\varphi = \frac{32}{9}\pi\sqrt{3}$.

13.5.66 This region is symmetric about the xy-plane, so we compute the volume of the region inside the upper half-sphere of radius 2 and below the cone $\varphi = \frac{\pi}{3}$ and double it. The volume is thus $2\int_{\pi/3}^{\pi/2}\int_0^2\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{32\pi}{3}\int_{\pi/3}^{\pi/2}\sin\varphi\,d\varphi = \frac{16\pi}{3}$.

13.5.67 Use cylindrical coordinates. Note that $x + y \ge 0$ for $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$, so this is the range of integration for θ. $\int_0^1\int_{-\pi/4}^{3\pi/4}\int_0^{r\cos\theta + r\sin\theta} r\,dz\,d\theta\,dr = \int_0^1\int_{-\pi/4}^{3\pi/4} r^2(\cos\theta + \sin\theta)\,d\theta\,dr = \int_0^1 r^2(\sin\theta - \cos\theta)\Big|_{\theta=-\pi/4}^{\theta=3\pi/4} = 2\sqrt{2}\int_0^1 r^2\,dr = \frac{2\sqrt{2}}{3}$.

13.5.68 Use cylindrical coordinates. The region is traced out for $0 \le \theta \le \pi$; however, it is symmetric around the xz-plane, so compute the volume for $0 \le \theta \le \frac{\pi}{2}$ to avoid negative values of r, and double it. $2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{4-r\cos\theta} r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{2\cos\theta} r(4 - r\cos\theta)\,dr\,d\theta =$
$2\int_0^{\pi/2}\int\left(2r^2 - \frac{1}{3}r^3\cos\theta\right)\Big|_0^{2\cos\theta} d\theta = 2\int_0^1\left(8\cos^2\theta - \frac{8}{3}\cos^4\theta\right)d\theta = 3\pi$.

13.5.69 The planes $z = x - 2$ and $z = 2 - x$ intersect when $x = 2$, which is at the boundary of the cardioid, so we can simply integrate between those two planes. Thus the volume is $\int_0^{2\pi}\int_0^{1+\cos\theta}\int_{r\cos\theta-2}^{2-r\cos\theta} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{1+\cos\theta} r\,(4-2r\cos\theta)\,d\theta\,dr = \frac{7\pi}{2}$.

13.5.70 Use cylindrical coordinates. $\int_1^2\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\,d\theta\,dz\,dr = 2\pi\int_1^2 r\sqrt{4-r^2}\,dr = 2\pi\sqrt{3}$.

13.5.71 Due to symmetry, this region is made up of eight identical pieces, one in each octant. Consider the piece in the first octant. A particle moving through this region parallel to the positive y-axis would start on the xz-plane ($y = 0$) within the unit circle $x^2 + z^2 = 1$ and would end on the cylinder that runs parallel to the x-axis ($y^2 + z^2 = 1$). The total volume is thus $V = 8\int_0^1\int_0^{\sqrt{1-z^2}}\int_0^{\sqrt{1-z^2}} 1\,dy\,dx\,dz = 8\int_0^1\int_0^{\sqrt{1-z^2}}\sqrt{1-z^2}\,dx\,dz = 8\int_0^1\left(1-z^2\right)\,dz = \frac{16}{3}$.

13.5.72 Due to symmetry, this region is made up of eight identical pieces, one in each octant. Consider the piece in the first octant. A particle moving through this region parallel to the positive y-axis would start on the xz-plane ($y = 0$) within the unit circle $x^2 + z^2 = 1$ and would either end on the cylinder that runs parallel to the x-axis ($y^2 + z^2 = 1$) or the cylinder that runs parallel to the z-axis ($x^2 + y^2 = 1$). If the particle starts at a point on the xz-plane for which $x < z$, then $\sqrt{1-z^2} < \sqrt{1-x^2}$ and the particle ends on the cylinder $y^2 + z^2 = 1$. If the particle starts at a point on the xz-plane for which $z < x$, then $\sqrt{1-x^2} < \sqrt{1-z^2}$ and the particle ends on the cylinder $x^2 + y^2 = 1$. The total volume is thus $V = 8\left(\int_0^{\sqrt{2}/2}\int_x^{\sqrt{1-x^2}}\int_0^{\sqrt{1-z^2}} 1\,dy\,dz\,dx + \int_0^{\sqrt{2}/2}\int_z^{\sqrt{1-z^2}}\int_0^{\sqrt{1-x^2}} 1\,dy\,dx\,dz\right) = 8\left(\int_0^{\sqrt{2}/2}\int_x^{\sqrt{1-x^2}}\sqrt{1-z^2}\,dz\,dx + \int_0^{\sqrt{2}/2}\int_z^{\sqrt{1-z^2}}\sqrt{1-x^2}\,dx\,dz\right) = 8\left(\left(1-\frac{\sqrt{2}}{2}\right)+\left(1-\frac{\sqrt{2}}{2}\right)\right) = 8\left(2-\sqrt{2}\right)$.

13.5.73 $\int_0^{2\pi}\int_0^2\int_0^8 r\left(1-0.05e^{-0.01r^2}\right)\,dz\,dr\,d\theta \approx 95.60362$.

13.5.74

a. $\int_1^\infty\int_0^\pi\int_0^{2\pi}\frac{2\cdot 10^{-4}}{\rho^4}\rho^2\sin\varphi\,d\theta\,d\varphi\,d\rho = 4\pi\cdot 10^{-4}\int_1^\infty\int_0^\pi\rho^{-2}\sin\varphi\,d\varphi d\rho = 8\pi\cdot 10^{-4}\int_1^\infty\rho^{-2}\,d\rho = \lim_{b\to\infty}\left(8\pi\cdot 10^{-4}\int_1^b\rho^{-2}\,d\rho\right) = 8\pi\cdot 10^{-4}\lim_{b\to\infty}\left(-\rho^{-1}\right)\Big|_1^b = 8\pi\cdot 10^{-4}\lim_{b\to\infty}\left(\frac{-1}{b}+1\right) = 8\pi\cdot 10^{-4}$.

b. $2\cdot 10^{-4}\int_0^\infty\int_0^\pi\int_0^{2\pi} e^{-0.01\rho^3}\rho^2\sin\varphi\,d\theta\,d\varphi\,d\rho = 4\pi\cdot 10^{-4}\int_0^\infty\int_0^\pi\rho^2\,e^{-0.01\rho^3}\sin\varphi\,d\varphi\,d\rho = \lim_{b\to\infty}\left(-8\pi\cdot 10^{-4}\int_0^b\rho^2\,e^{-0.01\rho^3}d\rho\right) = \lim_{b\to\infty}\left(-8\pi\cdot 10^{-4}\left(-\frac{100}{3}\,e^{-0.01\rho^3}\right)\Big|_0^b\right) = \lim_{b\to\infty}\left(\frac{8\pi}{3}\cdot 10^{-6}\left(1-e^{-0.01b^3}\right)\right) = \frac{8\pi}{3}\cdot 10^{-6}$

13.5.75

a. With $x = \cos\varphi$ we have $\sin\varphi = \sqrt{1-x^2}$ and $dx = -\sin\varphi\,d\varphi$ so that $d\varphi = -\frac{1}{\sqrt{1-x^2}}\,dx$. The limits of integration for φ become 1 to -1, so the integral then becomes

$$\begin{aligned}
F(d) &= -\frac{GMm}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\frac{(d-Rx)\sqrt{1-x^2}}{\left(R^2+d^2-2Rdx\right)^{3/2}\left(-\sqrt{1-x^2}\right)}\,dx\,d\theta \\
&= \frac{GMm}{4\pi}\int_{-1}^{1}\int_0^{2\pi}\frac{d-Rx}{\left(R^2+d^2-2Rdx\right)^{3/2}}d\theta\,dx \\
&= \frac{GMm}{2}\int_{-1}^{1}\left[\frac{d}{\left(R^2+d^2-2Rdx\right)^{3/2}}-\frac{Rx}{\left(R^2+d^2-2Rdx\right)^{3/2}}\right]dx \\
&= \frac{GMm}{2}\left(\frac{1}{R\sqrt{R^2+d^2-2Rdx}}-\frac{R^2+d^2-Rdx}{Rd^2\sqrt{R^2+d^2-2Rdx}}\right)\Big|_{-1}^{1} \\
&= \frac{GMm}{2}\frac{Rdx-R^2}{Rd^2\sqrt{R^2+d^2-2Rdx}}\Big|_{-1}^{1}.
\end{aligned}$$

Assuming $d > R$, this simplifies to $F(d) = \frac{GMm}{2}\left(\frac{Rd-R^2}{Rd^2(d-R)}+\frac{Rd+R^2}{Rd^2(d+R)}\right) = \frac{GMm}{2}\left(\frac{1}{d^2}+\frac{1}{d^2}\right) = \frac{GMm}{d^2}$.

b. Suppose $d < R$. Then we have $F(d) = \frac{GMm}{2} \frac{Rdx - R^2}{Rd^2\sqrt{R^2+d^2-2Rdx}}\Big|_{-1}^{1} = \frac{GMm}{2}\left(\frac{Rd-R^2}{Rd^2(R-d)} + \frac{Rd+R^2}{Rd^2(R+d)}\right) = \frac{GMm}{2}\left(-\frac{1}{d^2} + \frac{1}{d^2}\right) = 0$.

13.5.76 Use cylindrical coordinates with the origin in the center of one of the tank ends, the z-axis parallel to the long axis of the tank and the y-axis parallel to the surface of the water such that the positive x-axis points through the water. The lines from the origin to the point where the water meets the tank make angles of $\pm\frac{\pi}{3}$ with the x-axis Because each of these angles has cosine $\frac{1}{2}$ (the height of the water is one half the radius). Thus the volume of water (in cubic feet) is $\int_{-\pi/3}^{\pi/3}\int_{(\sec\theta)/2}^{1}\int_0^2 r\,dz\,dr\,d\theta = \int_{-\pi/3}^{\pi/3}\left(1 - \frac{1}{4}\sec^2\theta\right)d\theta = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.

This integral is perhaps just as easy to work in Cartesian coordinates: the area of the region of one end of the tank that contains water is simply $\int_{1/2}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1)\,dy\,dx = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$, which gets multiplied by 2 for the length of the tank.

13.5.77 Assume the base of the cone lies in the xy-plane and that the center of the base is at the origin, with the vertex of the cone on the positive z-axis. Then the equation of the cone in cylindrical coordinates (a, θ, z) is $z = h - \frac{h}{r}a$ where h and r are the given constants. Thus the volume is $\int_0^r\int_0^{h-\frac{h}{r}a}\int_0^{2\pi} a\,d\theta\,dz\,da = 2\pi\int_0^r a\left(h - \frac{h}{r}a\right)da = 2\pi\left(\frac{hr^2}{2} - \frac{hr^2}{3}\right) = \frac{1}{3}\pi hr^2$.

13.5.78 In spherical coordinates, $0 \le \varphi \le \cos^{-1}\left(\frac{R-h}{R}\right)$. $\int_0^{\cos^{-1}((R-h)/R)}\int_{(R-h)\sec\varphi}^{R}\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{2}{3}\int_0^{\cos^{-1}((R-h)/R)}\left(R^3 - (R-h)^3\sec^3\varphi\right)\sin\varphi\,d\varphi = \frac{1}{3}\pi h^2(3R-h)$.

13.5.79 Using similar triangles, if the frustum is extended to a complete cone, the height of the cone is $\frac{Rh}{R-r}$. Thus the equation of the cone (in cylindrical coordinates(a, θ, z) is $z = \frac{Rh}{R-r} - \frac{h}{R-r}a$ so that $a = R - z\frac{R-r}{h}$ and the volume of the frustum is $\int_0^h\int_0^{R-z\frac{R-r}{h}}\int_0^{2\pi} a\,d\theta\,da\,dz = \pi\int_0^r\left(R - z\frac{R-r}{h}\right)dz = \frac{\pi}{3}\left(R^2 + rR + r^2\right)h$.

13.5.80 The easiest way to do this problem is using Cartesian coordinates with a change of variable. The volume of the ellipsoid is 8 times the volume of the first-octant portion, so we have $8\int_0^1\int_0^{\sqrt{1-x^2/a^2}}\int_0^{\sqrt{1-x^2/a^2-y^2/b^2}} 1\,dz\,dy\,dx$. Now make a change of variables $x = au$, $y = bv$, $z = cw$ so that $dx = a\,du$, $dy = b\,dv$, $dz = c\,dw$ to get $8\int_0^1\int_0^{\sqrt{1-u^2}}\int_0^{\sqrt{1-u^2-v^2}} abc\,dw\,dv\,du =$
$8abc\int_0^1\int_0^{\sqrt{1-u^2}}\int_0^{\sqrt{1-u^2-v^2}} dw\,dv\,du$, which is just abc times the first-octant volume of a sphere of radius 1, so is equal to $\frac{4}{3}\pi abc$.

13.5.81 The two spheres are $x^2 + y^2 + z^2 = R^2$, $x^2 + y^2 + (z-r)^2 = r^2$. The equation of the second sphere simplifies to $x^2 + y^2 + z^2 - 2zr = 0$, so the two spheres meet when $R^2 - 2zr = 0$ or $z = \frac{R^2}{2r}$. This is the plane of intersection of the spheres. The volume in question now consists of two spherical caps, one on either side of this plane. The upper one is a spherical cap of the sphere of radius R with height $R - \frac{R^2}{2r} = \frac{2Rr-R^2}{2r}$, so by problem 78 has volume $\frac{\pi}{3}\left(\frac{2Rr-R^2}{2r}\right)^2\left(3R - \frac{2Rr-R^2}{2r}\right)$. The lower one is a spherical cap of the sphere of radius r with height $\frac{R^2}{2r}$, so again by problem 78 it has volume $\frac{\pi}{3}\frac{R^4}{4r^2}\left(3r - \frac{R^2}{2r}\right)$. Adding these two and simplifying gives for the volume $\frac{\pi R^3(8r-3R)}{12r}$.

13.6 Integrals for Mass Calculations

13.6.1 By definition, the system will balance when the pivot is located at the center of mass of the two people.

13.6.2 Its mass is $1\cdot 50 + 2\cdot 50 = 150\,\text{g}$. Its center of mass is given by $\frac{1}{150}\left(\int_0^{50} x\,dx + \int_{50}^{100} 2x\,dx\right) = \frac{1}{150}(1250 + 7500) = \frac{175}{3}$. So the center of mass is $\frac{175}{3}$ cm from the less dense end of the rod.

13.6.3 Integrate the density function over the region to find the mass; then integrate x times the density function and divide by the mass to get the y-coordinate and integrate y times the density function and divide by the mass to get the x-coordinate.

13.6.4 Because to compute M_x, we need to compute a weighted average of distances from the x-axis, which is y.

13.6.5 Integrate the density function over the region to find the mass. To find M_{yz}, integrate x times the density function over the region; the x-coordinate is then $\frac{M_{yz}}{M}$. Similarly for the other coordinates.

13.6.6 Because to compute M_{xz} we are finding a weighted sum of distances from the xz-plane, which is y.

13.6.7 The center of mass is $\frac{1}{13}(10\cdot 3+3(-1))=\frac{27}{13}$.

13.6.8 The center of mass is $\frac{1}{8+4+1}(8\cdot 2+4\cdot(-4)+1\cdot 0)=0$.

13.6.9 The mass is $\int_0^\pi(1+\sin x)\,dx=\pi+2$, and the center of mass is then $\frac{1}{\pi+2}\int_0^\pi x(1+\sin x)\,dx=\frac{1}{\pi+2}\left(\frac{1}{2}\pi^2+\pi\right)=\frac{\pi}{2}$

13.6.10 The mass is $\int_0^1\left(1+x^3\right)dx=\frac{5}{4}$, so the center of mass is $\frac{1}{5/4}\int_0^1 x\left(1+x^3\right)dx=\frac{4}{5}\left(\frac{1}{2}+\frac{1}{5}\right)=\frac{14}{25}$.

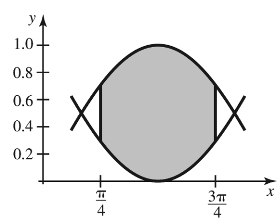
13.6.11 The mass is $\int_0^4\left(2-\frac{x^2}{16}\right)dx=8-\frac{4}{3}=\frac{20}{3}$, so the center of mass is $\frac{1}{20/3}\int_0^4 x\left(2-\frac{x^2}{16}\right)dx=\frac{3}{20}(16-4)=\frac{9}{5}$.

13.6.12 The mass is $\int_0^\pi(2+\cos x)\,dx=2\pi$, so the center of mass is $\frac{1}{2\pi}\int_0^\pi x(2+\cos x)\,dx=\frac{\pi^2-2}{2\pi}$.

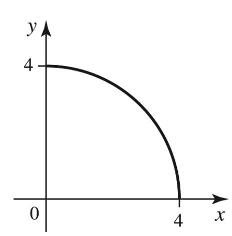
13.6.13 The mass is $1\cdot 2+\int_2^4(1+x)\,dx=10$ so that the center of mass is $\frac{1}{10}\left(\int_0^2 x\,dx+\int_2^4 x(1+x)\,dx\right)=\frac{8}{3}$.

13.6.14 The mass is $\int_0^1 x^2\,dx+\int_1^2 x(2-x)\,dx=1$, so the center of mass is $\int_0^1 x^3\,dx+\int_1^2 x^2(2-x)\,dx=\frac{7}{6}$.

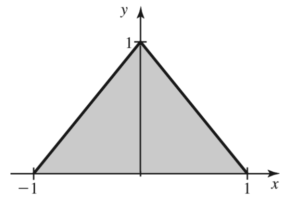
13.6.15 The region is symmetric with respect to $x=\frac{\pi}{2}$ and $y=\frac{1}{2}$, so its center of mass is at $\left(\frac{\pi}{2},\frac{1}{2}\right)$.



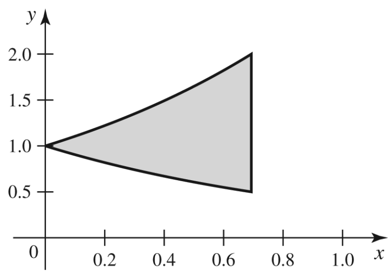
13.6.16 Assume density 1. Clearly $M_x=M_y$, so we need only compute one of them. The mass is one quarter the area of the circle, or 4π. Then using polar coordinates we have $\overline{x}=\frac{M_y}{M}=\frac{1}{4\pi}\int_0^4\int_0^{\pi/2}(r\sin\theta)\,r\,d\theta\,dr=\frac{1}{4\pi}\int_0^4 r^2\,dr=\frac{16}{3\pi}$, so that $\overline{y}=\frac{16}{3\pi}$ as well.



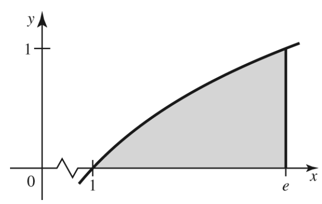
13.6.17 Assume density 1. By symmetry, $\overline{x}=0$. The mass of the region is 1, Because the two triangles together make a unit square. Thus $\overline{y}=\frac{M_x}{M}=\int_{-1}^0\int_0^{1+x}y\,dy\,dx+\int_0^1\int_0^{1-x}y\,dy\,dx=\frac{1}{3}$



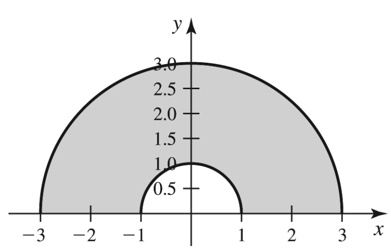
13.6.18 Assume density 1. The mass is $\int_0^{\ln 2}\int_{e^{-x}}^{e^x} 1\,dy\,dx = \int_0^{\ln 2}\left(e^x - e^{-x}\right)dx = \left(e^x + e^{-x}\right)\Big|_0^{\ln 2} = \frac{1}{2}$. Then $\overline{x} = \frac{M_y}{M} = \frac{1}{1/2}\int_0^{\ln 2}\int_{e^{-x}}^{e^x} x\,dy\,dx = 2\int_0^{\ln 2} x\left(e^x - e^{-x}\right)dx = 5\ln(2) - 3$. Also, $\overline{y} = \frac{M_x}{M} = \frac{1}{1/2}\int_0^{\ln 2}\int_{e^{-x}}^{e^x} y\,dy\,dx$ $= \int_0^{\ln 2}\left(e^{2x} - e^{-2x}\right)dx = \frac{9}{8}$, so the center of mass is $\left(5\ln(2) - 3,\ \frac{9}{8}\right)$.



13.6.19 Assume density 1. The mass is $\int_1^e\int_0^{\ln x} 1\,dy\,dx = \int_1^e \ln x\,dx = 1$. Then $\overline{x} = \frac{M_y}{M} = \frac{1}{1}\int_1^e\int_0^{\ln x} x\,dy\,dx$ $= \int_1^e x\ln x\,dx = \frac{1}{4}\left(e^2+1\right)$ and $\overline{y} = \frac{M_x}{M} = \frac{1}{1}\int_1^e\int_0^{\ln x} x\,dy\,dx = \frac{1}{2}\int_1^e(\ln x)^2\,dx = \frac{1}{2}e - 1$, so that the center of mass is $\left(\frac{1}{4}\left(e^2+1\right), \frac{1}{2}e - 1\right)$.



13.6.20 Assume density 1. The mass is $\int_0^{\pi}\int_1^3 r\,dr\,d\theta = 4\pi$. Then clearly $\overline{x} = 0$ by symmetry, and $\overline{y} = \frac{M_x}{M} =$ $\frac{1}{4\pi}\int_0^{\pi}\int_1^3 r^2\sin\theta\,dr\,d\theta = \frac{13}{6\pi}\int_0^{\pi}\sin\theta\ d\theta = \frac{13}{3\pi}$, so the center of mass is $\left(0, \frac{13}{3\pi}\right)$.



13.6.21 The mass is $\int_0^4\int_0^2\left(1+\frac{x}{2}\right)dy\,dx = 2\int_0^4\left(1+\frac{x}{2}\right)dx = 16$. Then $\overline{x} = \frac{M_y}{M} = \frac{1}{16}\int_0^4\int_0^2\left(x+\frac{x^2}{2}\right)dy\,dx$ $= \frac{1}{8}\int_0^4\left(x+\frac{x^2}{2}\right)dx = \frac{7}{3}$, and $\overline{y} = \frac{M_x}{M} = \frac{1}{16}\int_0^4\int_0^2 y\left(1+\frac{x}{2}\right)dy\,dx = \frac{1}{8}\int_0^4\left(1+\frac{x}{2}\right)dx = 1$. Thus, the center of mass is at $\left(\frac{7}{3}, 1\right)$. The density of the plate increases as you move toward the right.

13.6.22 The mass is $\int_0^5\int_0^1 2e^{-y/2}\,dx\,dy = 2\int_0^5 2e^{-y/2}\ dy = 4 - 4e^{-5/2}$. So we have $\overline{x} = \frac{M_y}{M} = \frac{1}{4-4e^{-5/2}}\int_0^5\int_0^1 2xe^{-y/2}\,dx\,dy = \frac{1}{4-4e^{-5/2}}\int_0^5 e^{-y/2}\,dy = \frac{1}{2}$. $\overline{y} = \frac{M_x}{M} = \frac{1}{4-4e^{-5/2}}\int_0^5\int_0^1 2ye^{-y/2}\,dx\,dy$ $= \frac{1}{4-4e^{-5/2}}\int_0^5 2ye^{-5y/2}\,dy = \frac{2e^{5/2}-7}{e^{5/2}-1}$, so that the center of mass is at $\left(\frac{1}{2}, \frac{2e^{5/2}-7}{e^{5/2}-1}\right)$. The density of the plate decreases as you move up the plate.

13.6.23 The mass is $\int_0^4\int_0^{4-x}(1+x+y)\,dy\,dx = \int_0^4\left(4-x+x(4-x)+\frac{(4-x)^2}{2}\right)dx = \frac{88}{3}$. By symmetry, $\overline{x}=\overline{y}$ (both the region and the density function are symmetric around $x=y$), and $\overline{x}=\frac{M_y}{M} = \frac{1}{88/3}\int_0^4\int_0^{4-x}\left(x+x^2+xy\right)dy\,dx = \frac{3}{88}\int_0^4\left(x(4-x)+x^2(4-x)+x\frac{(4-x)^2}{2}\right)dx = \frac{16}{11}$, so the center of mass is $\left(\frac{16}{11},\ \frac{16}{11}\right)$. The density of the plate increases as you move right and/or up.

13.6.24 The mass is $\int_0^\pi\int_0^2\left(r+\frac{1}{2}r^2\sin\theta\right)dr\,d\theta = \frac{6\pi+8}{3}$. The density function does not depend on x, and the region is symmetric about the y-axis, so $\overline{x}=0$. Then $\overline{y}=\frac{M_x}{M}=\frac{1}{(6\pi+8)/3}\int_0^\pi\int_0^2 r\sin\theta\left(1+\frac{1}{2}r\sin\theta\right)r\,dr\,d\theta = \frac{3\pi+16}{6\pi+8}$, so the center of mass is $\left(0,\ \frac{3\pi+16}{6\pi+8}\right)$. The density increases as you move up the plate.

13.6.25 The mass is $\int_{-3}^3\int_0^{\sqrt{9-x^2}/3}(1+y)\,dy\,dx = \int_{-3}^3\left(\frac{\sqrt{9-x^2}}{3}+\frac{9-x^2}{18}\right)dx = \frac{3}{2}\pi+2 = \frac{3\pi+4}{2}$. The density function does not depend on x, and the region is symmetric about the y-axis, so $\overline{x}=0$. Then $\overline{y}=\frac{M_x}{M} = \frac{1}{(3\pi+4)/2}\int_{-3}^3\int_0^{\sqrt{9-x^2}/3}\left(y+y^2\right)dy\,dx = \frac{3\pi+16}{12\pi+16}$, so the center of mass is $\left(0,\ \frac{3\pi+16}{12\pi+16}\right)$. The density increases as you move up the plate.

13.6.26 The mass is $\int_0^{\pi/2}\int_0^2 r\left(1+r^2\right)dr\,d\theta = 3\pi$. Because both the density function and the region are symmetric around $x=y$, $\overline{x}=\overline{y}$. We have $\overline{x}=\frac{M_y}{M}=\frac{1}{3\pi}\int_0^{\pi/2}\int_0^2 r^2\left(1+r^2\right)\cos\theta\,dr\,d\theta = \frac{1}{3\pi}\int_0^{\pi/2}\frac{136}{15}\cos\theta\,d\theta = \frac{136}{45\pi}$, so that the center of mass is $\left(\frac{136}{45\pi},\ \frac{136}{45\pi}\right)$. The density of the plate increases as you move away from the origin.

13.6.27 Assuming density 1, the mass is the volume of a half-sphere of radius 4, which is $\frac{1}{2}\cdot\frac{4}{3}\pi\cdot 4^3 = \frac{128\pi}{3}$. By symmetry, $\overline{x}=\overline{y}=0$. Also, $\overline{z}=\frac{1}{(128\pi)/3}\int_0^4\int_0^{2\pi}\int_0^{\pi/2}\rho\cos\varphi\cdot\rho^2\sin\varphi\,d\varphi\,d\theta\,d\rho =$
$\frac{3}{128\pi}\int_0^4\int_0^{2\pi}\int_0^{\pi/2}\rho^3\cos\varphi\sin\varphi\,d\varphi\,d\theta\,d\rho = \frac{3}{128}\int_0^4\rho^3\,d\rho = \frac{3}{2}$, so the center of mass is at $\left(0,\ 0,\ \frac{3}{2}\right)$.

13.6.28 Assuming density 1, the mass is $\int_0^{2\pi}\int_0^5\int_{r^2}^{25} r\,dz\,dr\,d\theta = 2500\pi$. By symmetry, $\overline{x}=\overline{y}=0$, and $\overline{z}=\frac{1}{2500\pi}\int_0^{2\pi}\int_0^5\int_{r^2}^{25} r\,z\,dz\,dr\,d\theta = \frac{1}{2500\pi}\cdot\frac{312500\pi}{7} = \frac{125}{7}$, so that the center of mass is at $\left(0,0,\frac{125}{7}\right)$.

13.6.29 Assuming density 1, the mass is the volume of a pyramid with height 1 and base area $\frac{1}{2}$, so the volume is $\frac{1}{6}$. The region is symmetric with respect to the line $x=y=z$, $\overline{x}=\overline{y}=\overline{z}$, and $\overline{z}=\frac{1}{1/6}\int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx = 3\int_0^1\int_0^{1-x}(1-x-y)^2\,dy\,dx = \frac{1}{4}$, so the center of mass is $\left(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)$.

13.6.30 Assuming density 1, the mass is the volume of a cone with base area 256π and height 16, so is $\frac{4096\pi}{3}$. By symmetry, $\overline{x}=\overline{y}=0$, and $\overline{z}=\frac{1}{(4096\pi)/3}\int_0^{16}\int_0^{16-r}\int_0^{2\pi} rz\,d\theta\,dz\,dr = \frac{3}{4096}\int_0^{16} r(16-r)^2\,dr = 4$, so the center of mass is $(0,\ 0,\ 4)$.

13.6.31 Assuming density 1, the mass is $\int_0^1\int_0^{2\pi}\int_0^{1-r\sin\theta} r\,dz\,d\theta\,dr = \int_0^1\int_0^{2\pi} r(1-r\sin\theta)\,d\theta\,dr = \pi$. The region is symmetric around the yz-plane, so $\overline{x}=0$. We have $\overline{y}=\frac{1}{\pi}\int_0^1\int_0^{2\pi}\int_0^{1-r\sin\theta} r^2\sin\theta\,dz\,d\theta\,dr = \frac{1}{\pi}\int_0^1\int_0^{2\pi} r^2(1-r\sin\theta)\sin\theta\,d\theta\,dr = -\frac{1}{4}$. $\overline{z}=\frac{1}{\pi}\int_0^1\int_0^{2\pi}\int_0^{1-r\sin\theta} rz\,dz\,d\theta\,dr = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} r(1-r\sin\theta)^2\,d\theta\,dr = \frac{5}{8}$, so that the center of mass is $\left(0,\ -\frac{1}{4},\ \frac{5}{8}\right)$.

13.6.32 Assuming density 1, the mass is $\int_0^2\int_0^{2\pi}\int_0^{2\sqrt{4-r^2}} r\,dz\,d\theta\,dr = 4\pi\int_0^2 r\sqrt{4-r^2}\,dr = \frac{32\pi}{3}$. By symmetry, $\overline{x}=\overline{y}=0$, and $\overline{z}=\frac{1}{(32\pi/3)}\int_0^2\int_0^{2\pi}\int_0^{2\sqrt{4-r^2}} rz\,dz\,d\theta\,dr = \frac{3}{8}\int_0^2 r\left(4-r^2\right)dr = \frac{3}{2}$, so that the center of mass is $\left(0,\ 0,\ \frac{3}{2}\right)$.

13.6.33 The mass is $\int_0^4\int_0^1\int_0^1\left(1+\frac{x}{2}\right)dz\,dy\,dx = \int_0^4\left(1+\frac{x}{2}\right)dx = 8$. By symmetry, Because the density depends only on x, $\overline{y}=\overline{z}=\frac{1}{2}$, while $\overline{x}=\frac{1}{8}\int_0^4\int_0^1\int_0^1\left(x+\frac{x^2}{2}\right)dz\,dy\,dx = \frac{1}{8}\int_0^4\left(x+\frac{x^2}{2}\right)dx = \frac{7}{3}$, so the center of mass is $\left(\frac{7}{3},\ \frac{1}{2},\ \frac{1}{2}\right)$.

13.6.34 The mass is $\int_0^4\int_0^{\sqrt{4-z}}\int_0^{2\pi} r\,(5-z)\;d\theta\,dr\,dz = 2\pi\int_0^4\int_0^{\sqrt{4-z}} r\,(5-z)\;dr\,dz = \pi\int_0^4 (5-z)\,(4-z)\;dz$ $=\frac{88\pi}{3}$. By symmetry, because the density depends only on z, $\overline{x}=\overline{y}=0$, while
$\overline{z}=\frac{1}{(88\pi)/3}\int_0^4\int_0^{\sqrt{4-z}}\int_0^{2\pi} rz\,(5-z)\;d\theta\,dr\,dz = \frac{3}{44}\int_0^4\int_0^{\sqrt{4-z}} rz\,(5-z)\;\,dr\,dz = \frac{3}{88}\int_0^4 (5-z)\,(4-z)\;dz = \frac{12}{11}$, so the center of mass is $\left(0,\,0,\,\frac{12}{11}\right)$.

13.6.35 The mass of the sphere is given by $\int_0^6\int_0^{\pi/2}\int_0^{2\pi}\left(1+\frac{\rho}{4}\right)\rho^2\,\sin\varphi\,d\theta\,d\varphi\,d\rho = 2\pi\int_0^6\left(1+\frac{\rho}{4}\right)\rho^2\,d\rho =$ 306π. By symmetry, because the density function depends only on ρ, $\overline{x}=\overline{y}=0$. Also,
$\overline{z}=\frac{1}{306\pi}\int_0^6\int_0^{\pi/2}\int_0^{2\pi}\left(1+\frac{\rho}{4}\right)\rho^2\,\sin\varphi\cdot\rho\,\cos\varphi\,d\theta\,d\varphi\,d\rho = \frac{1}{153}\int_0^6\int_0^{\pi/2}\rho^3\,\left(1+\frac{\rho}{4}\right)\,\sin\varphi\,\cos\varphi\,d\varphi\,d\rho =$
$\frac{1}{306}\int_0^6\rho^3\left(1+\frac{\rho}{4}\right)\;d\rho = \frac{198}{85}$, so the center of mass is $\left(0,\,0,\,\frac{198}{85}\right)$.

13.6.36 The mass of the cube is given by $\int_0^1\int_0^1\int_0^1 (2+x+y+z)\;dz\,dy\,dx = \int_0^1\int_0^1\left(\frac{5}{2}+x+y\right)\,dy\,dx =$ $\int_0^1 (3+x)\;dx = \frac{7}{2}$. Both the region and the density function are symmetric with respect to the line $x=y=z$, so $\overline{x}=\overline{y}=\overline{z}$, and $\overline{x}=\frac{1}{7/2}\int_0^1\int_0^1\int_0^1\left(2x+x^2+xy+xz\right)\,dz\,dy\,dx = \frac{2}{7}\int_0^1\int_0^1\left(\frac{5}{2}x+x^2+xy\right)\,dy\,dx =$ $\frac{2}{7}\int_0^1\left(3x+x^2\right)\;dx = \frac{11}{21}$, so that the center of mass is $\left(\frac{11}{21},\,\frac{11}{21},\,\frac{11}{21}\right)$.

13.6.37 The mass is $\int_0^1\int_0^4\int_0^x (2+y)\;dz\,dy\,dx = \int_0^1\int_0^4 (2x+xy)\;dy\,dx = \int_0^1 16x\,dx = 8$. Then $\overline{x} =$ $\frac{1}{8}\int_0^1\int_0^4\int_0^x (2x+xy)\;dz\,dy\,dx = \frac{1}{8}\int_0^1\int_0^4 x^2\,(y+2)\,dy\,dx = 2\int_0^1 x^2\;dx = \frac{2}{3}$. $\overline{y}=\frac{1}{8}\int_0^1\int_0^4\int_0^x\left(2y+y^2\right)\,dz\,dy\,dx$ $=\frac{1}{8}\int_0^1\int_0^4 x\left(y+2y^2\right)dy\,dx = \frac{14}{3}\int_0^1 x\,dx = \frac{7}{3}$. $\overline{z}=\frac{1}{8}\int_0^1\int_0^4\int_0^x (2z+yz)\;dz\,dy\,dx = \frac{1}{8}\int_0^1\int_0^4\left(x^2+\frac{x^2y}{2}\right)dy\,dx$ $=\frac{1}{8}\int_0^1 8x^2\,dx = \frac{1}{3}$, so that the center of mass is $\left(\frac{2}{3},\,\frac{7}{3},\,\frac{1}{3}\right)$.

13.6.38 The mass is $\int_0^9\int_0^{9-r}\int_0^{2\pi} r\,(1+z)\;d\theta\,dz\,dr = 2\pi\int_0^9 r\left(9-r+\frac{(9-r)^2}{2}\right)\,dr = \frac{3159\pi}{4}$. By symmetry and the fact that the density depends only on z, we have $\overline{x}=\overline{y}=0$. We also have
$\overline{z}=\frac{1}{(3159\pi)/4}\int_0^9\int_0^{9-r}\int_0^{2\pi} r\left(z+z^2\right)\,d\theta\,dz\,dr = \frac{8}{3159}\int_0^9 r\left(\frac{(9-r)^2}{2}+\frac{(9-r)^3}{3}\right)\,dr = \frac{207}{65}$, so that the center of mass is $\left(0,0,\frac{207}{65}\right)$.

13.6.39

a. False. It has a center of mass with a y-coordinate of zero.

b. True, Because every point is balanced by the corresponding point on the other side of the origin.

c. False. For example, the annulus $1\le r\le 3$ has center of mass at the origin.

d. False. For example, the solid resulting from revolving the annulus in part (c) about the x-axis is connected, but its center of mass is at the origin.

13.6.40 The mass of the rod is $\int_0^L 2e^{-x/3}\,dx = 6\left(1-e^{-L/3}\right)$, so its center of mass is given by
$\frac{1}{6\left(1-e^{-L/3}\right)}\int_0^L 2x\,e^{-x/3}\,dx = \frac{18-18e^{-L/3}-6Le^{-L/3}}{6\left(1-e^{-L/3}\right)}$, so that as $L\to\infty$, the center of mass approaches
$\lim_{L\to\infty}\frac{18-18e^{-L/3}-6Le^{-L/3}}{6\left(1-e^{-L/3}\right)} = 3$, because all terms involving L approach 0 in the limit.

13.6.41 The mass of the rod is $\int_0^L\frac{10}{1+x^2}\,dx = 10\tan^{-1}(L)$, so its center of mass is $\frac{1}{\tan^{-1}(L)}\int_0^L\frac{10x}{1+x^2}\,dx =$ $\frac{\ln\left(1+L^2\right)}{\tan^{-1}(L)}$. As $L\to\infty$, the numerator grows without bound while the denominator approaches $\frac{\pi}{2}$, so $\overline{x}\to\infty$ as $L\to\infty$.

13.6.42 The mass of the plate is $\int_0^L\int_{-e^{-x}}^{e^{-x}}\,dy\,dx = \int_0^L 2e^{-x}\,dx = 2-2e^{-L}$, so that
$\lim_{L\to\infty}\overline{x} = \lim_{L\to\infty}\frac{1}{2-2e^{-L}}\int_0^L\int_{-e^{-x}}^{e^{-x}} x\,dy\,dx = \lim_{L\to\infty}\frac{1}{2-2e^{-L}}\int_0^L 2x\,e^{-x}\,dx = \lim_{L\to\infty}\frac{2-2e^{-L}(L+1)}{2-2e^{-L}} = 1$.
Also, $\lim_{L\to\infty}\overline{y} = \lim_{L\to\infty}\frac{1}{2-2e^{-L}}\int_0^L\int_{-e^{-x}}^{e^{-x}} y\,dy\,dx = \lim_{L\to\infty} 0 = 0$. Thus as $L\to\infty$, the center of mass approaches (1, 0).

13.6.43 The mass of the plate, assuming density 1, is $8+4=12$ Because the area of the rectangle is 8 and the two triangles together to form a 2×2 square. By symmetry, $\overline{x}=0$. Compute $\overline{y}$ by computing M_x for each of the three pieces. The moments around the x-axis are equal for the two triangles. So we need only compute M_x for one of the triangles. This is $M_x=\int_2^4\int_0^{4-x} y\,dy\,dx=\int_2^4 \frac{1}{2}(4-x)^2\,dx=\frac{4}{3}$

The moment around the x-axis for the rectangle is $M_x=\int_{-2}^2\int_0^2 y\,dy\,dx=8$, so that $\overline{y}=\frac{1}{12}\left(\frac{4}{3}+8+\frac{4}{3}\right)=\frac{8}{9}$. Thus the center of mass is at $\left(0,\ \frac{8}{9}\right)$.

13.6.44 Assuming density 1, the mass of the plate is the area of the outer rectangle (48) less the area of the inner, missing, rectangle (4), so the mass is 44. By symmetry, $\overline{x}=0$. To compute $\overline{y}$, compute the moment around the x-axis by computing the moment for the outer rectangle and that for the missing rectangle, and subtract one from the other. Thus $\overline{y}=\frac{1}{44}\left(\int_{-4}^4\int_{-4}^2 y\,dy\,dx-\int_{-2}^2\int_{-1}^0 y\,dy\,dx\right)=\frac{1}{44}(-48+2)=-\frac{23}{22}$. Thus the center of mass is at $\left(0,\ -\frac{23}{22}\right)$.

13.6.45 The mass of the region (assuming density 1) is $\frac{1}{2}\pi r^2=2\pi$, by symmetry, $\overline{x}=0$, and $\overline{y}=\frac{1}{2\pi}\int_0^\pi\int_0^2 r^2\sin\theta\,dr\,d\theta=\frac{1}{2\pi}\int_0^\pi \frac{8}{3}\sin\theta\,d\theta=\frac{8}{3\pi}$, so the center of mass is at $\left(0,\ \frac{8}{3\pi}\right)$.

13.6.46 The mass of the region is π; by symmetry, $\overline{x}=\overline{y}$, and $\overline{x}=\frac{1}{\pi}\int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta=\frac{8}{3\pi}\int_0^{\pi/2}\cos\theta\,d\theta$ $=\frac{8}{3\pi}$, thus the center of mass is at $\left(\frac{8}{3\pi},\ \frac{8}{3\pi}\right)$.

13.6.47 The mass of the cardioid is $\int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta=\frac{1}{2}\int_0^{2\pi}(1+\cos\theta)^2\ d\theta=\frac{3\pi}{2}$. By symmetry, $\overline{y}=0$, and $\overline{x}=\frac{1}{(3\pi)/2}\int_0^{2\pi}\int_0^{1+\cos\theta} r^2\cos\theta\,dr\,d\theta=\frac{2}{9\pi}\int_0^{2\pi}(1+\cos\theta)^3\cos\theta\,d\theta=\frac{5}{6}$, so that the center of mass is at $\left(\frac{5}{6},\,0\right)$.

13.6.48 The mass of the cardioid is $\int_0^{2\pi}\int_0^{3-3\cos\theta} r\,dr\,d\theta=\frac{9}{2}\int_0^{2\pi}(1-\cos\theta)^2\,d\theta=\frac{27\pi}{2}$. By symmetry, $\overline{y}=0$, and $\overline{x}=\frac{1}{(27\pi)/2}\int_0^{2\pi}\int_0^{3-3\cos\theta} r^2\cos\theta\,dr\,d\theta=\frac{2}{3\pi}\int_0^{2\pi}(1-\cos\theta)^3\cos\theta\,d\theta=-\frac{5}{2}$, so that the center of mass is at $\left(-\frac{5}{2},\,0\right)$.

13.6.49 The mass of the leaf is $\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta=\frac{1}{2}\int_0^{\pi/2}\sin^2 2\theta\,d\theta=\frac{\pi}{8}$. By symmetry, $\overline{x}=\overline{y}$, and $\overline{x}=\frac{1}{\pi/8}\int_0^{\pi/2}\int_0^{\sin 2\theta} r^2\cos\theta\,dr\,d\theta=\frac{8}{3\pi}\int_0^{\pi/2}\sin^3 2\theta\,\cos\theta\,d\theta=\frac{128}{105\pi}$, so that the center of mass is at $\left(\frac{128}{105\pi},\ \frac{128}{105\pi}\right)$.

13.6.50 The mass of the limacon is $\int_0^{2\pi}\int_0^{2+\cos\theta} r\,dr\,d\theta=\frac{1}{2}\int_0^{2\pi}(2+\cos\theta)^2\,d\theta=\frac{9\pi}{2}$. By symmetry, $\overline{y}=0$, and $\overline{x}=\frac{1}{(9\pi)/2}\int_0^{2\pi}\int_0^{2+\cos\theta} r^2\cos\theta\,dr\,d\theta=\frac{1}{2}\int_0^{2\pi}(2+\cos\theta)^3\cos\theta\,d\theta=\frac{17}{18}$, so that the center of mass is at $\left(\frac{17}{18},\,0\right)$.

13.6.51 Assume the origin is at the midpoint of the diameter with the y-axis pointing up. Assume the density is one. The mass is πa (the length of the wire). The moment is $\int_0^\pi (a\sin\theta)\,d\theta=2a$. Therefore $\overline{y}=\frac{2a}{\pi}$, while $\overline{x}=0$ by symmetry.

13.6.52 The line $y=b$ intersects the parabola at $x=\pm\sqrt{\frac{b}{a}}$, so the mass of the plate (assuming density 1) is $\int_{-\sqrt{\frac{b}{a}}}^{\sqrt{\frac{b}{a}}}\int_{ax^2}^b 1\,dy\,dx=\int_{-\sqrt{\frac{b}{a}}}^{\sqrt{\frac{b}{a}}}\left(b-ax^2\right)\,dx=\frac{4b}{3}\sqrt{\frac{b}{a}}$. By symmetry, $\overline{x}=0$, and $\overline{y}=\frac{1}{\left(\frac{4b}{3}\sqrt{\frac{b}{a}}\right)}\int_{-\sqrt{\frac{b}{a}}}^{\sqrt{\frac{b}{a}}}\int_{ax^2}^b y\,dy\,dx$ $=\frac{3}{8b}\sqrt{\frac{a}{b}}\int_{-\sqrt{\frac{b}{a}}}^{\sqrt{\frac{b}{a}}}\left(b^2-a^2x^4\right)\,dx=\frac{3b}{5}$, so that the center of mass is at $\left(0,\ \frac{3b}{5}\right)$ and is independent of a.

13.6.53 The mass of the region, assuming density 1, is $a^2-\frac{\pi}{4}a^2=\frac{4-\pi}{4}a^2$. By symmetry, $\overline{x}=\overline{y}$, and $\overline{x}=\frac{1}{a^2(4-\pi)/4}\int_0^a\int_{\sqrt{a^2-x^2}}^a x\,dy\,dx=\frac{4}{a^2(4-\pi)}\int_0^a x\left(a-\sqrt{a^2-x^2}\right)\,dx=\frac{2a}{3(4-\pi)}$, so that the center of mass is at $\left(\frac{2a}{3(4-\pi)},\ \frac{2a}{3(4-\pi)}\right)$.

13.6.54 Clearly the center of mass is at the center of the box - halfway between each opposite pair of faces. The easiest way to see this is to place the origin at the center of the box; then (for example) $\overline{x}=\frac{1}{m}\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} x\,dx\,dy\,dz=\frac{1}{m}\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\frac{1}{2}x^2\Big|_{-a/2}^{a/2}\,dy\,dz=0$.

13.6.55 Place the origin at the vertex of the cone and let the z-axis be the axis of the cone. Then the cone has equation (in cylindrical coordinates (a, θ, z)) $z = \frac{h}{r}a$. The mass of the cone is $\frac{1}{3}\pi r^2 h$, and by symmetry $\overline{x} = \overline{y} = 0$. Now $\overline{z} = \frac{1}{\pi r^2 h/3}\int_0^r \int_0^{(h/r)a} \int_0^{2\pi} az\, d\theta\, dz\, da = \frac{3h}{r^4}\int_0^r a^3\, da = \frac{3h}{4}$, so that the center of mass is one quarter of the way from the base to the vertex.

13.6.56 Place the origin at the center of the sphere. The sphere has mass $\frac{2}{3}\pi a^3$, so (in spherical coordinates) $\overline{z} = \frac{1}{(2\pi a^3)/3}\int_0^a \int_0^{\pi/2} \int_0^{2\pi} \rho^3 \sin\varphi \cos\varphi\, d\theta\, d\varphi\, d\rho = \frac{3}{a^3}\int_0^a \int_0^{\pi/2} \rho^3 \sin\varphi \cos\varphi\, d\varphi\, d\rho = \frac{3}{8}a$, so that the center of mass is $\frac{3}{8}$ of the way from the origin to the top of the sphere.

13.6.57 Place the origin at the middle of the base of the triangle. Then the y-coordinate of the center of mass can be determined. If h is the height of the triangle, its area is $\frac{bh}{2}$, $\overline{x} = 0$ by symmetry, and $\overline{y} = \frac{1}{(bh)/2}\int_0^h \int_{b(y-h)/(2h)}^{-b(y-h)/(2h)} y\, dx\, dy = \frac{2}{bh}\int_0^h \frac{-b(y-h)}{h}\, dy = -\frac{2}{h^2}\int_0^h \left(y^2 - hy\right) dy = \frac{h}{3}$, so that the center of mass is $\frac{1}{3}$ of the way from the base to the vertex.

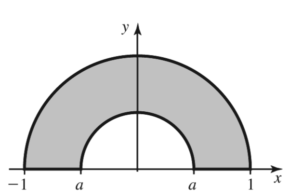
13.6.58 The tetrahedron has height a, and the base has area $\frac{a^2}{2}$, so the volume of the tetrahedron is $\frac{a^3}{6}$. By symmetry, $\overline{x} = \overline{y} = \overline{z}$, and $\overline{x} = \frac{1}{a^3/6}\int_0^a \int_0^{a(1-x/a)} \int_0^{a(1-x/a-y/a)} x\, dz\, dy\, dx$. Make the change of variable $u = \frac{x}{a}$., $v = \frac{y}{a}$, $w = \frac{z}{a}$; then $dx = a\, du$, $du = a\, dv$, $dz = a\, dw$ and then $\overline{x} = \frac{1}{a^3/6}\int_0^a \int_0^{1-u} \int_0^{1-u-v} a^4 u\, dw\, dv\, du = 6a\int_0^1 \int_0^{1-u} u\,(1-u-v)\, dv\, du = \frac{a}{4}$, so the center of mass is $\left(\frac{a}{4}, \frac{a}{4}, \frac{a}{4}\right)$.

13.6.59 Place the origin at the center of the ellipse with the circular base of radius r in the xy-plane, so that the top of the ellipsoid is on the positive z-axis, at $(0, 0, a)$. Use cylindrical coordinates (ρ, θ, z); then the equation for the top half of the ellipsoid is $z = a\sqrt{1 - \frac{\rho^2}{r^2}}$. We know (Problem 80, Section 13.5) that the volume of the top half of this ellipsoid is $\frac{2\pi r^2 a}{3}$, so $\overline{z} = \frac{1}{(2\pi r^2 a)/3}\int_0^r \int_0^{a\sqrt{1-\rho^2/r^2}} \int_0^{2\pi} \rho z\, d\theta\, dz\, d\rho = \frac{3a}{2r^2}\int_0^r \rho\left(1 - \frac{\rho^2}{r^2}\right) d\rho = \frac{3}{8}a$, so the center of mass is $\frac{3}{8}$ of the way from the base to the top of the ellipsoid.

13.6.60 The area of the country is (adding up the large rectangle and the two small squares) $48 + 2\cdot 4 = 56$. $\overline{x} = 0$ by symmetry. To compute $\overline{y}$, we subtract the moment of the missing small rectangle around the x-axis from that of the large rectangle. But the large rectangle is symmetric around the x-axis, so its moment is zero and thus $\overline{y} = \frac{1}{56}\left(0 - \int_{-4}^{-2}\int_{-2}^{2} y\, dx\, dy\right) = -\frac{1}{14}\int_{-4}^{-2} y\, dy = \frac{3}{7}$, so that the geographical center is at $\left(0, \frac{3}{7}\right)$. The population center is a discrete computation. Count population in thousands: $\overline{x} = \frac{10\cdot(-2)+15\cdot 2+20\cdot 2+5\cdot 4+15\cdot(-2)}{10+15+20+5+15} = \frac{40}{65} = \frac{8}{13}\overline{y} = \frac{10\cdot 2+15\cdot 3+20\cdot 0+5\cdot(-4)+15\cdot(-2)}{10+15+20+5+15} = \frac{15}{65} = \frac{3}{13}$, so that the population center is at $\left(\frac{8}{13}, \frac{3}{13}\right)$.

13.6.61

a. The mass of the plate is the difference of the area of the two semicircles, so is $\frac{1}{2}\pi\left(1 - a^2\right)$. The y-coordinate of the center of mass is then $\overline{y} = \frac{1}{\pi(1-a^2)/2}\int_a^1 \int_0^{\pi} r^2 \sin\theta\, d\theta\, dr = \frac{4}{3\pi(1-a^2)}\int_a^1 r^2\, dr = \frac{4\left(1-a^3\right)}{3\pi(1-a^2)} = \frac{4(a^2+a+1)}{3(a+1)\pi}$.



b. The center of mass always has x coordinate 0, so it lies on the edge of the plate exactly when $\frac{4(a^2+a+1)}{3(a+1)\pi} = a$ or 1. Solving for equality with 1 gives no solutions in the range $0 \le a \le 1$. Solving for equality with a gives $a = -\frac{1}{2}\left(1 - \sqrt{\frac{3(\pi+4)}{3\pi-4}}\right) \approx 0.49366$ while the other solution is outside of the range $0 \le a \le 1$.

13.6.62

a. The mass of the solid is the difference of the volumes of the two hemispheres, so is $\frac{2}{3}\pi\left(1-a^3\right)$. The z-coordinate of the center of mass is then $\overline{z} = \frac{1}{2\pi(1-a^3)/3}\int_0^1\int_0^{\pi/2}\int_0^{2\pi} r^3\cos\varphi\,\sin\varphi\,d\theta\,d\varphi\,dr = \frac{3}{2(1-a^3)}\int_0^1 r^3\,dr = \frac{3\left(1-a^4\right)}{8\left(1-a^3\right)}$.

b. The center of mass always has x and y-coordinates 0, so it lies on the edge of the plate exactly when $\frac{3\left(1-a^4\right)}{8\left(1-a^3\right)} = a$ or 1. Solving for equality with 1 gives no solutions in the range $0 \le a \le 1$; solving for equality with a gives $a = \frac{\left(1450+450\sqrt{11}\right)^{2/3}-5\left(1450+450\sqrt{11}\right)^{1/3}-50}{15\left(1450+450\sqrt{11}\right)^{1/3}} \approx 0.38936$.

13.6.63 Place the origin at the center of the bottom of the soda can. If the height of soda in the can is h, for $0 \le h \le 12$, the mass of the can is $16\pi h + \frac{16}{1000}\pi\left(12-h\right) = \frac{6}{125}\pi\left(333h+4\right)$. To compute $\overline{z}$, compute the moments around the x-axis separately for the soda and the air:
$\overline{z} = \frac{1}{6\pi(333h+4)/125}\int_0^4\int_0^{2\pi}\left(\frac{1}{1000}\int_h^{12} zr\,dz + \int_0^h zr\,dz\right)d\theta\,dr =$
$\frac{125}{6\pi(333h+4)}\int_0^4\int_0^{2\pi}\left(\frac{1}{2000}\left(144-h^2\right)r + \frac{1}{2}h^2 r\right)d\theta\,dr = \frac{125}{6\pi(333h+4)}\cdot\frac{\pi\left(144+999h^2\right)}{125} = \frac{3}{2}\cdot\frac{111h^2+16}{333h+4}$. The center of mass is at its lowest point when the derivative of this function is zero, i.e. when $\frac{333h}{333h+4} - \frac{999\left(16+111h^2\right)}{2(333h+4)^2} = 0$. Placing over a common denominator, setting the numerator to zero and solving gives $h = \frac{40\sqrt{10}-4}{333} \approx 0.3678$ cm.

13.6.64

a. There are two cases: either $0 < a \le b$ or $0 < b \le a$. In either case, the area of the triangle is $\frac{1}{2}bh$.

Case 1: $0 < a \le b$.

$\overline{x} = \frac{1}{bh/2}\left(\int_0^a\int_0^{(h/a)x} x\,dy\,dx + \int_a^b\int_0^{(h/(a-b))(x-b)} x\,dy\,dx\right) =$
$\frac{2}{bh}\left(\frac{h}{a}\int_0^a x^2\,dx + \frac{h}{a-b}\int_a^b\left(x^2-bx\right)\,dx\right) = \frac{2}{bh}\left(\frac{h}{3a}a^3 + \frac{h\left(b^2+ab-2a^2\right)}{6}\right) = \frac{a+b}{3}$.
$\overline{y} = \frac{1}{bh/2}\left(\int_0^a\int_0^{(h/a)x} y\,dy\,dx + \int_a^b\int_0^{(h/(a-b))(x-b)} y\,dy\,dx\right) =$
$\frac{2}{bh}\left(\frac{h^2}{2a^2}\int_0^a x^2\,dx + \frac{h^2}{2(a-b)^2}\int_a^b\left(x-b\right)^2\,dx\right) = \frac{2}{bh}\left(\frac{ah^2}{6} - \frac{h^2(a-b)}{6}\right) = \frac{2}{bh}\cdot\frac{bh^2}{6} = \frac{h}{3}$.

Case 2: $0 < b \le a$.

$\overline{x} = \frac{1}{bh/2}\left(\int_0^b\int_0^{(h/a)x} x\,dy\,dx + \int_b^a\int_{(h/(a-b))(x-b)}^{(h/a)x} x\,dy\,dx\right) =$
$\frac{2}{bh}\left(\frac{h}{a}\int_0^b x^2\,dx + \int_b^a x\left(\frac{h}{a}x - \frac{h}{a-b}\left(x-b\right)\right)\,dx\right) = \frac{2}{bh}\left(\frac{hb^3}{3a} + \frac{\left(a^2+ab-2b^2\right)bh}{6a}\right) = \frac{a+b}{3}$.
$\overline{y} = \frac{1}{bh/2}\left(\int_0^b\int_0^{(h/a)x} y\,dy\,dx + \int_a^b\int_{(h/(a-b))(x-b)}^{(h/a)x} y\,dy\,dx\right) =$
$\frac{1}{bh}\left(\frac{h^2}{a^2}\int_0^b x^2\,dx + \int_b^a\frac{h^2}{a^2}x^2 - \frac{h^2}{(a-b)^2}\left(x-b\right)^2\,dx\right) = \frac{1}{bh}\left(\frac{h^2b^3}{3a^2} + \frac{h^2b\left(a^2-b^2\right)}{3a^2}\right) = \frac{h}{3}$.

In either case, the centroid is at $\left(\frac{a+b}{3},\ \frac{h}{3}\right)$.

b. Because each median bisects the triangle, the centroid must lie on each median Because the triangle will be balanced with respect to the axis determined by the median. Because this is true of each median, the centroid must be at the intersection of the medians.

13.6.65

a. Place the origin at Q, with the circles to the right of the y-axis. Then using polar coordinates $(a,\,\theta)$, the equation of the circles are $a = 2R\cos\theta$ and $a = 2r\cos\theta$ for $0 \le \theta \le \pi$. The mass of the earring is $\pi\left(R^2-r^2\right)$, and $\overline{y} = 0$.

$\overline{x} = \frac{1}{\pi(R^2-r^2)} \int_0^\pi \int_{2r\cos\theta}^{2R\cos\theta} a^2 \cos\theta \, da \, d\theta = \frac{8\left(R^3-r^3\right)}{3\pi(R^2-r^2)} \int_0^\pi \cos^4\theta \, d\theta = \frac{R^2+Rr+r^2}{R+r}$.

With the origin instead at the center of the large circle, the equations of the circles are $x^2+y^2=R^2$ and $(x-(R-r))^2+y^2=r^2$. Then the moment around the y-axis of the large circle is zero, so to compute $\overline{x}$ for the earring, we compute $\overline{x} = \frac{1}{\pi(R^2-r^2)}\left(0 - \int_{R-2r}^{R} \int_{-\sqrt{r^2-(x-(R-r))^2}}^{\sqrt{r^2-(x-(R-r))^2}} x \, dy \, dx\right) = -\frac{1}{\pi(R^2-r^2)} \int_{R-2r}^{R} x\sqrt{r^2-(x-(R-r))^2} \, dx = -\frac{r^2}{R+r}$.

b. With the origin at the center of the large circle, point P is $(R-2r, 0)$, so we want $R-2r = -\frac{r^2}{R+r}$. Multiplying both sides by $R+r$, dividing by r^2 and letting $x = \frac{R}{r}$, we find that $x = \frac{1}{x-1}$ or $x^2-x-1=0$, which has roots $\frac{1\pm\sqrt{5}}{2}$. Because $R, r > 0$, it must be the positive value, so $x = \frac{1+\sqrt{5}}{2}$ satisfies the condition.

13.7 Change of Variables in Multiple Integrals

13.7.1 It is the square with vertices (0, 0), (2, 0), (2, 2), and (0, 2).

13.7.2 The Jacobian is $J(u, v) = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix}$.

13.7.3 The Jacobian is $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$, so $\iint\limits_R f(x, y) \, dA = \iint\limits_S f(u+v, u-v) \left|J(u, v)\right| dA = \iint\limits_S 2f(u+v, u-v) \, dA = \int_0^1 \int_0^1 2f(u+v, u-v) \, dv \, du$.

13.7.4 It is the cube of side length $\frac{1}{2}$ in the first octant with one vertex at the origin.

13.7.5 $u = \frac{x}{2}$, so $0 \le u \le 1$ means $0 \le \frac{x}{2} \le 1$ or $0 \le x \le 2$. $v = 2y$, so $0 \le v \le 1$ means $0 \le 2y \le 1$, or $0 \le y \le \frac{1}{2}$. Thus the image is the region $\left\{0 \le x \le 2, \, 0 \le y \le \frac{1}{2}\right\}$, which is a rectangle in the first quadrant with vertices $(0, 0)$, $(2, 0)$, $\left(2, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}\right)$.

13.7.6 $0 \le u \le 1 \Rightarrow 0 \le -x \le 1 \Rightarrow 0 \ge x \ge -1$, and $0 \le v \le 1 \Rightarrow 0 \le -y \le 1 \Rightarrow 0 \ge y \ge -1$, so we get a unit square in the third quadrant with one vertex at the origin.

13.7.7 Solving for u and v gives $u = x+y$ and $v = x-y$. The region is thus the square bounded by the lines $x+y=0$, $x+y=1$, $x-y=0$, and $x-y=1$.

13.7.8 Solving for u and v gives $u = \frac{y}{2}$ and $v = x-y$. The region is thus the parallelogram bounded by the lines $y=0$, $y=2$, $x-y=0$, and $x-y=1$.

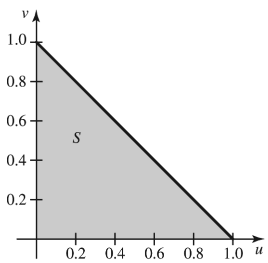
13.7.9 From (0, 0) to (1, 0), $x = u^2$ and $y = 0$, so this traces out the segment from (0, 0) to (1, 0). Similarly, from (0, 1) to (0, 0), $x = -v^2$ and $y = 0$, so this traces out the segment from (−1, 0) to (0, 0). From (1, 0) to (1, 1), $(x, y) = \left(1-v^2, 2v\right)$, so that (x, y) satisfies the equation $y^2 = 4-4x$. From (1, 1) to (0, 1), $(x, y) = \left(u^2-1, 2u\right)$, so that (x, y) satisfies the equation $y^2 = 4+4x$. So the result is the region enclosed by the x-axis and the parabolas $y^2 = 4-4x$ and $y^2 = 4+4x$.

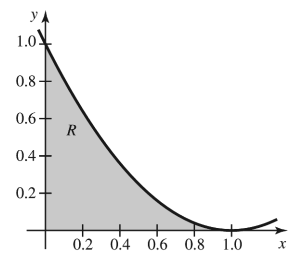
13.7.10 This is the result of interchanging x and y in the previous exercise, so the result is the region enclosed by the y-axis and the parabolas $x^2 = 4-4y$ and $x^2 = 4+4y$.

13.7.11 As (u, v) goes from (0, 0) to (1, 0), (x, y) goes from (0, 0) to (1, 0) along the x-axis. As (u, v) goes from (1, 0) to (1, 1), (x, y) traces out the upper half of the unit circle. As (u, v) goes from (1, 1) to (0, 1), (x, y) goes from (−1, 0) to (0, 0) along the x-axis. Finally, as (u, v) goes from (0, 1) to (0, 0), (x, y) is stationary at the origin. Thus the region swept out is the upper half of the unit circle.

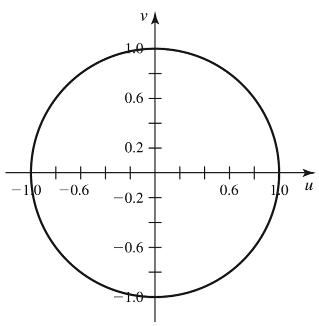
13.7.12 As (u, v) goes from $(0, 0)$ to $(1, 0)$, (x, y) is stationary at the origin. As (u, v) goes from $(1, 0)$ to $(1, 1)$, (x, y) traces out the segment from $(0, 0)$ to $(0, -1)$. As (u, v) goes from $(1, 1)$ to $(0, 1)$, (x, y) traces out the right half of the unit circle. Finally, as (u, v) goes from $(0, 1)$ to $(0, 0)$, (x, y) traces out the segment from $(1, 0)$ to $(0, 0)$. The result is the right half of the unit circle.

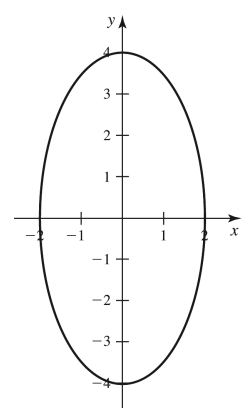
13.7.13 $R = \{(x, y) : \sqrt{y} \le 1 - x,\, x \ge 0,\, y \ge 0\} = \left\{(x, y) : y \le (1 - x)^2,\, x \ge 0,\, y \ge 0\right\}$.



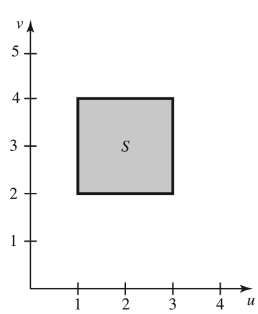


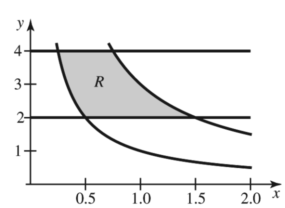
13.7.14 $R = \left\{(x, y) : \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 \le 1\right\} = \{(x, y) : 4x^2 + y^2 \le 16\}$.



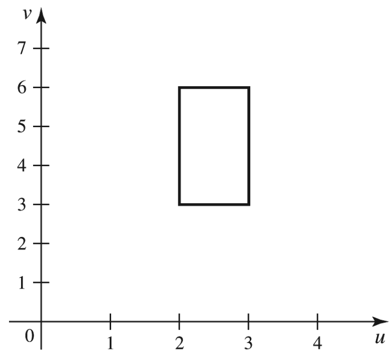


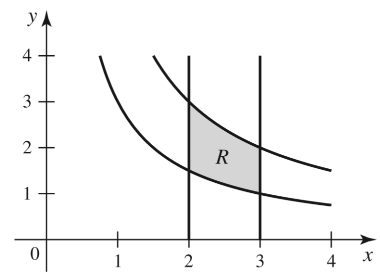
13.7.15 $R = \{(x, y) : 1 \le xy \le 3,\, 2 \le y \le 4\} = \left\{(x, y) : 2 \le y \le 4,\, \frac{1}{y} \le x \le \frac{3}{y}\right\}$.





13.7.16 $R = \{(x, y) : 2 \le x \le 3,\, 3 \le xy \le 6\} = \left\{(x, y) : 2 \le x \le 3,\, \frac{3}{x} \le y \le \frac{6}{x}\right\}$.





13.7.17 $J = \begin{vmatrix} 3 & 0 \\ 0 & -3 \end{vmatrix} = -9.$

13.7.18 $J = \begin{vmatrix} 0 & 4 \\ -2 & 0 \end{vmatrix} = 8.$

13.7.19 $J = \begin{vmatrix} 2v & 2u \\ 2u & -2v \end{vmatrix} = -4\left(u^2 + v^2\right).$

13.7.20 $J = \begin{vmatrix} \cos(\pi v) & -u\pi\sin(\pi v) \\ \sin(\pi v) & u\pi\cos(\pi v) \end{vmatrix} = u\pi.$

13.7.21 $J = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} = -1.$

13.7.22 $J = \begin{vmatrix} v^{-1} & -uv^{-2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}.$

13.7.23 Add the two equations to get $3x = u + v$, so $x = \frac{u+v}{3}$, and $y = u - \frac{u+v}{3} = \frac{2u-v}{3}$. The Jacobian is then $J = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}$

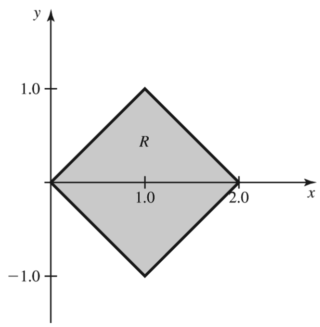
13.7.24 $x = v$, so $y = \frac{u}{v}$. The Jacobian is then $J = \begin{vmatrix} 0 & 1 \\ v^{-1} & -uv^{-2} \end{vmatrix} = -\frac{1}{v}.$

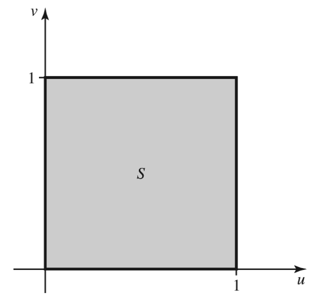
13.7.25 Add twice the second equation to the first to get $-y = u + 2v$ so that $y = -u - 2v$, and then $x = -u - 3v$. The Jacobian is $J = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = -1.$

13.7.26 Subtract twice the second equation from the first to get $u - 2v = -5x$ so that $x = \frac{2v-u}{5}$, and then $2y = v - 3x = v - 3\frac{2v-u}{5} = \frac{3u-v}{5}$, so that $y = \frac{3u-v}{10}$. The Jacobian is $J = \begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{10} \end{vmatrix} = -\frac{1}{10}.$

13.7.27

a.





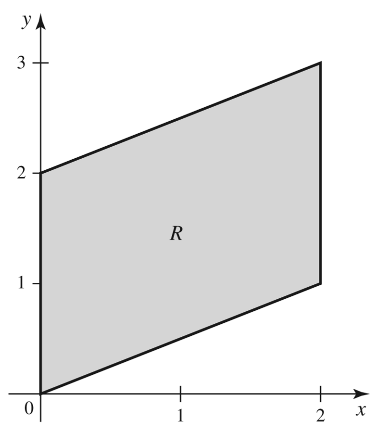
b. The region S is the first-quadrant unit square with one vertex at the origin, so the limits of integration are $0 \le u,\, v \le 1$.

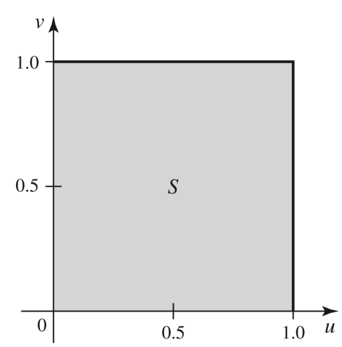
c. The Jacobian of the transformation is $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$.

d. $\iint\limits_R xy\; dA = \int_0^1 \int_0^1 \frac{u+v}{2} \cdot \frac{u-v}{2}\, |-2|\; dv\, du = \frac{1}{2} \int_0^1 \int_0^1 \left(u^2 - v^2\right)\, dv\, du = \frac{1}{2} \int_0^1 \left(u^2 - \frac{1}{3}\right) du = 0.$

13.7.28

a. $S = \{(u,\, v) : 0 \le 2u \le 2,\ 2u \le 4v + 2u \le 2u + 4\} = \{(u,\, v) : 0 \le u \le 1,\ u \le 2v + u \le u + 2\} = \{(u,\, v) : 0 \le u \le 1,\ 0 \le v \le 1\}$.





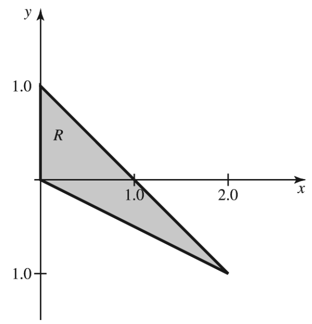
b. From the above computation, the new integration limits are $0 \le u,\, v \le 1$.

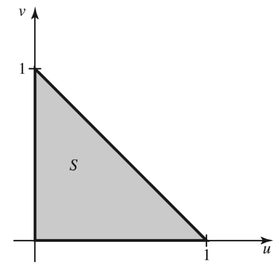
c. The Jacobian is $\begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8$.

d. $\iint\limits_R x^2 y\; dA = 64 \int_0^1 \int_0^1 u^2\,(2v + u)\; dv\, du = 64 \int_0^1 \left(u^2 + u^3\right) du = \frac{112}{3}.$

13.7.29

a. $S = \{(u,\, v) : 0 \le 2u \le 2,\ -u \le v - u \le 1 - 2u\} = \{(u,\, v) : 0 \le u \le 1,\ 0 \le v \le 1 - u\}$





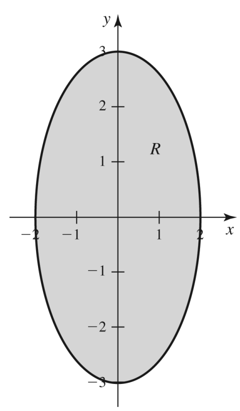
b. From the above, the new limits of integration are $0 \le u \le 1$, $0 \le v \le 1 - u$.

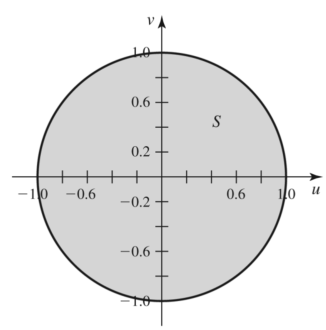
c. The Jacobian is $\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = 2$.

d. $\iint\limits_R x^2\sqrt{x+2y}\;dA = 8\int_0^1\int_0^{1-u} u^2\sqrt{2v}\,dv\,du = \frac{16\sqrt{2}}{3}\int_0^1 u^2\,(1-u)^{3/2}\;du = \frac{256\sqrt{2}}{945}$.

13.7.30

a. Under the given transformation, $9x^2 + 4y^2 = 36$ becomes $9\,(2u)^2 + 4\,(3v)^2 = 36u^2 + 36v^2 = 36$, which is the unit circle $u^2 + v^2 = 1$.





b. The limits of integration over the unit circle are $-1 \le u \le 1$, $-\sqrt{1-u^2} \le v \le \sqrt{1-u^2}$.

c. The Jacobian is $\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$.

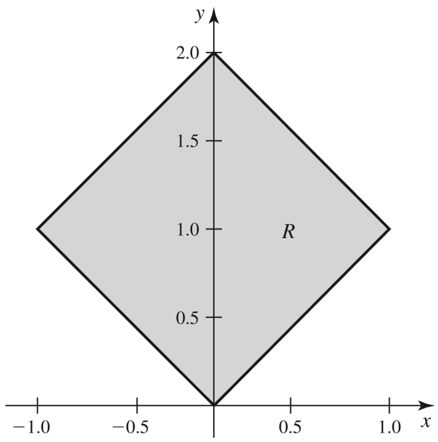
d. $\iint\limits_R xy\;dA = 36\int_{-1}^1\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} uv\,dv\,du = 0$.

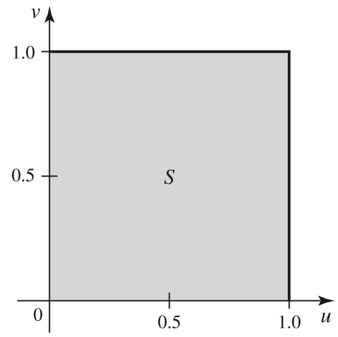
13.7.31 Use the substitution $y = v$, $x = u + v$. Then $S = \{(u,\,v) : 0 \le v \le 1,\, v \le u+v \le v+2\} = \{(u,\,v) : 0 \le v \le 1,\, 0 \le u \le 2\}$.

The Jacobian of this transformation is $J\,(u,\,v) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$ so that

$$\int_0^1\int_y^{y+2}\sqrt{x-y}\,dx\,dy = \int_0^1\int_0^2\sqrt{u}\,du\,dv = \frac{2}{3}2^{3/2} = \frac{4\sqrt{2}}{3}.$$

13.7.32 Use the substitution $x = u - v$, $y = u + v$. Then S becomes the unit square in the first quadrant with one vertex at the origin, as can be seen by tracing each edge.

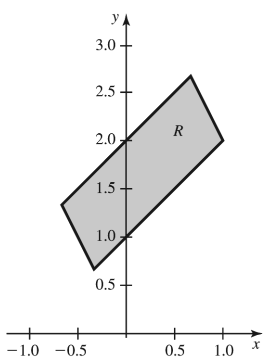


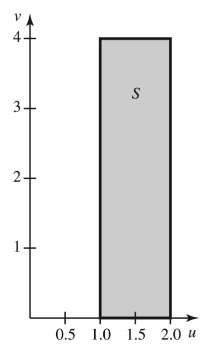


The Jacobian of this transformation is $J(u, v) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$ so that

$\iint\limits_R \sqrt{y^2 - x^2}\, dA = 2\int_0^1 \int_0^1 \sqrt{(u+v)^2 - (u-v)^2}\, dv\, du = 2\int_0^1 \int_0^1 \sqrt{4uv}\, dv\, du = \frac{16}{9}$.

13.7.33 The points of intersection of the given lines are $\left(-\frac{1}{3}, \frac{2}{3}\right)$, $\left(-\frac{2}{3}, \frac{4}{3}\right)$, $\left(\frac{2}{3}, \frac{8}{3}\right)$, and $(1, 2)$. Setting $u = y - x$, $v = y + 2x$ sends these points into the rectangle with vertices $(1, 0)$, $(2, 0)$, $(2, 4)$, and $(1, 4)$.



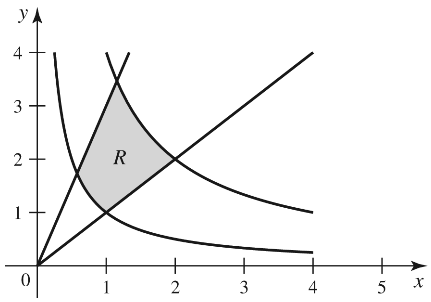


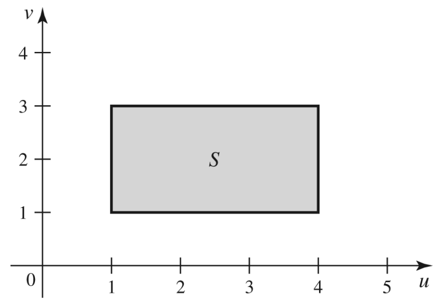
So use the transformation (solving for x, y) $x = \frac{v-u}{3}$, $y = \frac{v+2u}{3}$, $J(u, v) = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = -\frac{1}{3}$, so that

$\iint\limits_R \left(\frac{y-x}{y+2x+1}\right)^4 dA = \frac{1}{3}\int_1^2 \int_0^4 \left(\frac{u}{v+1}\right)^4 dv\, du = \frac{1}{3}\int_1^2 \frac{124}{375} u^4\, du = \frac{3844}{5625}$.

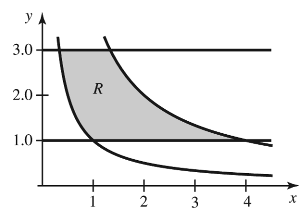
13.7.34 Use the transformation $u = xy$, $v = \frac{y}{x}$, so that $x = \sqrt{\frac{u}{v}}$, $y = \sqrt{uv}$. The Jacobian is

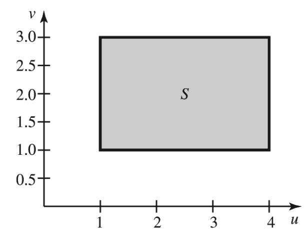
$\begin{vmatrix} \frac{u^{-1/2}v^{-1/2}}{2} & -\frac{u^{-1/2}v^{-3/2}}{2} \\ \frac{u^{-1/2}v^{1/2}}{2} & \frac{u^{1/2}v^{-1/2}}{2} \end{vmatrix} = \frac{1}{2v}$, and the new region is the square with vertices $(1, 1)$, $(1, 3)$, $(4, 1)$, and $(4, 3)$, so that $\iint\limits_R e^{xy}\, dA = \int_1^4 \int_1^3 e^u\, \frac{1}{2v}\, dv\, du = \frac{\ln 3}{2} \int_1^4 e^u\, du = \frac{\ln 3}{2}\left(e^4 - e\right)$.





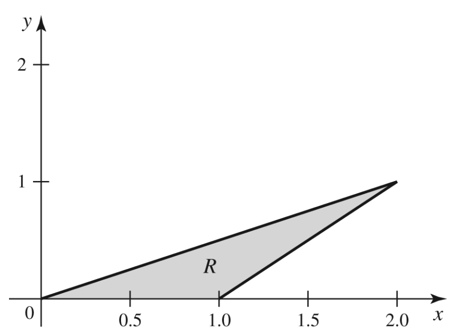
13.7.35 Use the transformation $u = xy$, $v = y$, so that $x = \frac{u}{v}$, $y = v$. The new region is the square with vertices (1, 1), (4, 1), (4, 3), and (1, 3). The Jacobian of the transformation is $\begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$.

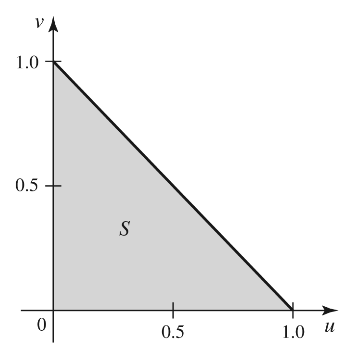




Thus $\iint\limits_R xy\,dA = \int_1^4 \int_1^3 \frac{u}{v}\,dv\,du = \int_1^4 u \ln 3\,du = \frac{15 \ln 3}{2}$.

13.7.36 R is the interior of the triangle with vertices (0, 0), (1, 0), and (2, 1). Looking at the form of the integrand, try $u = x - 2y$, $v = y$ so that $x = u + 2v$, $y = v$. Then S is the triangle bounded by (0, 0), (1, 0), and (0, 1).





The Jacobian of this transformation is $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$, and the integral is equal to $\iint\limits_R (x - y)\sqrt{x - 2y}\,dA = \int_0^1 \int_0^{1-u} (u + v)\sqrt{u}\,dv\,du = \frac{4}{21}$.

13.7.37 $J(u, v, w) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2.$

13.7.38 $J(u, v, w) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -4.$

13.7.39 $J(u, v, w) = \begin{vmatrix} 0 & w & v \\ w & 0 & u \\ 2u & -2v & 0 \end{vmatrix} = 2w\left(u^2 - v^2\right)$.

13.7.40 Solving for x, y, and z gives $x = \frac{u+v+w}{2}$, $y = \frac{-u+v+w}{2}$, $z = \frac{u-v+w}{2}$, so that
$J(u, v, w) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$.

13.7.41 Let $u = y - x$, $v = z - y$, $w = z$; then $x = w - v - u$, $y = w - v$, $z = w$ and the Jacobian is $J(u, v, w) = \begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$. The new region is clearly $0 \le u \le 2$, $0 \le v \le 1$, $0 \le w \le 3$, so we obtain $\iiint\limits_D xy\, dV = \int_0^2 \int_0^1 \int_0^3 (w - v - u)(w - v)\, dw\, dv\, du = 5$.

13.7.42 Let $u = y - 2x$, $v = z - 3y$, $w = z - 4x$. Then $x = -\frac{3}{2}u - \frac{1}{2}v + \frac{1}{2}w$, $y = -2u - v + w$, $z = -6u - 2v + 3w$, so the Jacobian is $J(u, v, w) = \begin{vmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3 \end{vmatrix} = \frac{1}{2}$, and the new region of integration is $0 \le u \le 1$, $0 \le v \le 1$, $0 \le w \le 3$. Thus $\iiint\limits_D dV = \int_0^3 \int_0^1 \int_0^1 \frac{1}{2}\, du\, dv\, dw = \frac{3}{2}$.

13.7.43 Using the given change of variables, the Jacobian is $\begin{vmatrix} 4\cos v - 4u & \sin v & 0 \\ 2\sin v 2u & \cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = 8u$, and the new range of integration is $0 \le u \le 1$, $0 \le v \le 2\pi$, $0 \le w \le 16 - 16u^2$. Thus $\iiint\limits_D z\, dV = \int_0^1 \int_0^{2\pi} \int_0^{16-16u^2} 8uw\, dw\, dv\, du = \int_0^1 2048\pi\, u\left(u^2 - 1\right)^2 du = \frac{1024\pi}{3}$.

(Note that this change of variables is essentially cylindrical coordinates, with an adjustment for the fact that we are integrating over a paraboloid with differently sized axes.)

13.7.44 Using the given change of variables, the Jacobian is $\begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$. In uvw-coordinates, the equation becomes $u^2 + v^2 + w^2 = 1$, so the integral is half the volume of the unit sphere, or $\frac{2\pi}{3}$. Multiplying by the Jacobian gives 4π.

13.7.45

a. True. This is because $g(u, v)$ and $h(u, v)$ are of the form $au + bv$ so their partial derivatives are constants.

b. True. This is because the transformation maps lines to lines.

c. True. It simply halves lengths and reflects in the x axis.

13.7.46 Expand the determinant about the third row. $J(r, \theta, \mathrm{Z}) = \begin{vmatrix} \cos\theta & r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$.

13.7.47 Expand the determinant about the third row. $J(\rho, \varphi, \theta) = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$
$= \cos\varphi\left(\rho^2\sin\varphi\,\cos\varphi\cos^2\theta + \rho^2\sin\varphi\,\cos\varphi\sin^2\theta\right) + \rho\,\sin\varphi\left(\rho\,\sin^2\varphi\,\cos^2\theta + \rho\,\sin^2\varphi\,\sin^2\theta\right) =$
$\rho^2\sin\varphi\,\cos^2\varphi + \rho^2\sin^3\varphi = \rho^2\sin\varphi$.

13.7.48 Under T, the ellipse becomes $u^2+v^2=1$, the unit circle, with area π. The Jacobian of the transformation is ab, so the ellipse area is πab.

13.7.49 This integral is four times the integral over the first quadrant. This is $4ab\int_0^1\int_0^{\sqrt{1-u^2}} abuv\,dv\,du = 2a^2b^2\int_0^1 u\left(1-u^2\right)du = \frac{a^2b^2}{2}$.

13.7.50 The area of R is given by Exercise 48 ($\frac{\pi ab}{2}$), and $\overline{x}=0$ by symmetry. To find $\overline{y}$, we compute using the same change of variables (where R^+ is the upper half) $\frac{1}{(\pi ab)/2}\int_{-1}^1\int_0^{\sqrt{1-u^2}} ab^2v\,dv\,du = \frac{b}{\pi}\int_{-1}^1\left(1-u^2\right)du = \frac{4b}{3\pi}$.

13.7.51 The distance of a point from the origin is $\sqrt{x^2+y^2}=\sqrt{a^2u^2+b^2v^2}$, so the average squared distance is $\frac{1}{\pi ab}\int_{-1}^1\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\left(a^2u^2+b^2v^2\right)dv\,du = \frac{a^2+b^2}{4}$.

13.7.52 The distance of a point in the upper half of R and the x-axis is simply the point's y-coordinate, so the average distance is $\frac{1}{(\pi ab)/2}\int_{-1}^1\int_0^{\sqrt{1-u^2}} ab^2v\,dv\,du = \frac{4b}{3\pi}$.

13.7.53 Under the given transformation, the equation becomes $u^2+v^2+w^2=1$, the unit sphere. The Jacobian of the transformation is abc, so the volume of D is abc times the volume of the unit sphere, or $\frac{4}{3}\pi abc$.

13.7.54 This integral is eight times the integral over the first octant, so it is
$8abc\int_0^1\int_0^{\sqrt{1-u^2}}\int_0^{\sqrt{1-u^2-v^2}} abcuvw\,dw\,dv\,du = \frac{a^2b^2c^2}{6}$.

13.7.55 The mass of the upper half is $\frac{2\pi abc}{3}$ by Problem 53, and $\overline{x}=\overline{y}=0$ by symmetry.
$\overline{z} = \frac{1}{(2\pi abc)/3}\int_{-1}^1\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_0^{\sqrt{1-u^2-v^2}} abc^2w\;dw\,dv\,du = \frac{3c}{2\pi}\int_{-1}^1\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\left(1-u^2-v^2\right)dv\,du = \frac{3c}{8}$.

13.7.56 The distance of a point from the origin is $\sqrt{x^2+y^2+z^2}=\sqrt{a^2u^2+b^2v^2+c^2w^2}$, so the average squared distance is $\frac{3}{4\pi abc}\int_{-1}^1\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}}\left(a^2u^2+b^2v^2+c^2w^2\right)abc\,dw\,dv\,du$

To evaluate this integral, switch to cylindrical coordinates; we obtain
$\frac{3}{2\pi}\int_0^1\int_0^{2\pi}\int_0^{\sqrt{1-r^2}} r\left(a^2r^2\cos^2\theta+b^2r^2\sin^2\theta+c^2z^2\right)dz\,d\theta\,dr =$
$\frac{1}{2\pi}\int_0^1\int_0^{2\pi} r\sqrt{1-r^2}\left(c^2-c^2r^2+3b^2r^2+3a^2r^2\cos^2\theta-3b^2r^2\cos^2\theta\right)d\theta\,dr = \frac{a^2+b^2+c^2}{5}$.

13.7.57

a. The line $u=a$ is the set $\{(a,\,v)\}$, which maps under T to $\left\{\left(a^2-v^2,\,2av\right)\right\}$. But points of this form satisfy the equation $a^2-x=\frac{1}{4a^2}y^2$, or $x=-\frac{1}{4a^2}y^2+a^2$, which is a parabola opening in the negative x direction. The vertex of the parabola $x=Ay^2+By+C$ is at $\left(C-\frac{B^2}{4A},\,-\frac{B}{2A}\right)$, which for this parabola is $\left(a^2,\,0\right)$, which lies on the positive x-axis.

b. The line $v=b$ is the set $\{(u,\,b)\}$, which maps under T to $\left\{\left(u^2-b^2,\,2ub\right)\right\}$. But points of this form satisfy the equation $b^2+x=\frac{1}{4b^2}y^2$, or $x=\frac{1}{4b^2}y^2-b^2$, which is a parabola opening in the positive x direction. The vertex of the parabola $x=Ay^2+By+C$ is at $\left(C-\frac{B^2}{4A},\,-\frac{B}{2A}\right)$, which for this parabola is $\left(-b^2,\,0\right)$, which lies on the negative x-axis.

c. $J(u,\,v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4\left(u^2 + v^2\right)$.

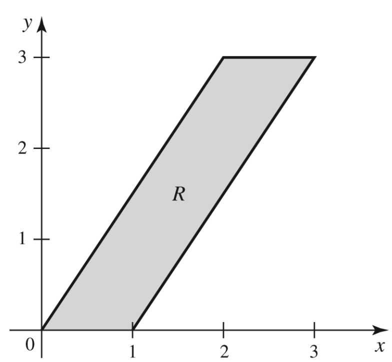
d. Use the transformation $x = v^2 - u^2$, $y = 2uv$. The Jacobian of this transformation is $\begin{vmatrix} -2u & 2v \\ 2v & 2u \end{vmatrix} = -4\left(u^2 + v^2\right)$. $x = 4 - \frac{1}{16}y^2$ corresponds to the lines $u = \pm 2$ (by an analysis similar to part (a)), while $x = \frac{1}{4}y^2 - 1$ corresponds to the line $v = \pm 1$, so that the rectangle with vertices $(-2,\,-1)$, $(-2,\,1)$, $(2,\,1)$, $(2,\,-1)$ is mapped to the region bounded by the parabolas. However, note that the area of that rectangle to the left of the v-axis and the area to the right of the v-axis are each mapped onto that region (that is, the map is $2:1$; it is a simple computation to see that $(a,\,b)$ and $(-a,\,-b)$ map to the same xy point). Thus to determine the area of the original region, we want to integrate over only (say) the right half of the rectangle. So the area is $\int_0^2 \int_{-1}^1 4\left(u^2 + v^2\right)\,dv\,du = \frac{80}{3}$.

e. As in part (d), use the transformation $x = v^2 - u^2$, $y = 2uv$, with Jacobian $-4\left(u^2 + v^2\right)$. The correspondences are: $x = \frac{y^2}{4} - 1 \Leftrightarrow u = 1x = \frac{y^2}{64} - 16 \Leftrightarrow u = 4x = 9 - \frac{y^2}{36} \Leftrightarrow v = \pm 3x = 4 - \frac{y^2}{16} \Leftrightarrow v = \pm 2$. Because we are looking at the positive portion of the bounded piece, the new range of integration is $1 \le u \le 4$, $2 \le v \le 3$. Thus the area is $\int_1^4 \int_2^3 4\left(u^2 + v^2\right)\,dv\,du = 160$.

f. This simply reverses the roles of x and y in parts (a) and (b). Thus lines $u = a$ in the uv plane map to parabolas in the xy plane that open in the negative y direction with vertices on the positive y-axis, while lines $v = b$ in the uv plane map to parabolas in the xy plane that open in the positive y direction with vertices on the positive y-axis.

13.7.58

a. Here is the effect of the shear transformation with $a = 1$, $b = 2$, $c = 3$ ($x = u + 2v$; $y = 3v$):



The y coordinates are all multiplied by c, while the x coordinates are gotten by expanding by a and then adding b times the y coordinate. This has the effect of pushing the square further and further to the right as the y coordinate increases.

b. $J(u,\,v) = \begin{vmatrix} a & b \\ 0 & c \end{vmatrix} = ac$

c. R is a parallelogram with vertices $(0,\,0)$, $(a,\,0)$, $(a+b,\,c)$, and $(b,\,c)$, so it has base a and height c, so has area ac (which is exactly what we would expect because the Jacobian is ac).

d. By symmetry, $\overline{y} = \frac{c}{2}$ and $\overline{x} = \frac{a+b}{2}$.

e. The analogous shear transformation is $x = au$, $y = bu + cv$.

13.7.59

a. The z coordinate remains constant, while the other coordinates are stretched by an amount depending on all three coordinates. The result is thus a parallelepiped with its base in the xy plane.

b. $J(u, v, w) = \begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{vmatrix} = ad$

c. D has a height of 1 Because S and the z coordinate remains unchanged. Its base is in the xy plane, and is the parallelogram that is the result of the shear transformation $x = au + by$, $y = dv$ (set $w = 0$ in the original equations to see this). The area of this parallelogram, from Problem 58, is ad, so the total volume is ad (again what we would expect from the Jacobian).

d. By symmetry, the center of mass is $\left(\frac{a+b+c}{2}, \frac{d+e}{2}, \frac{1}{2}\right)$.

13.7.60

a. $J(u, v) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

b. R is the parallelogram with vertices $(0, 0)$, (a, c), (b, d), and $(a + b, c + d)$. Compute its area by thinking of (a, c) and (b, d) as vectors in 3-space with zero z coordinate; then the area of the parallelogram is the cross-product $\left|\left\langle a, c, 0\right\rangle \times \left\langle b, d, 0\right\rangle\right| = \left|\left\langle 0, 0, ad - bc\right\rangle\right| = |ad - bc| = |J(u, v)|$

c. Let $P = (q, r)$ and let ℓ be the line given by $(q + et, r + ft)$; $0 \le t \le 1$ so that $Q = (q + e, r + f)$. Then $T(P) = (aq + br, cq + dr)$, while $T(Q) = (aq + ae + br + bf, cq + ce + dr + df)$. Then $T((q + et, r + ft)) = (aq + br, (ae + bf)t, cq + dr + (ce + df)t) =$ $(aq + br, cq + dr) + (ae + bf, ce + df)t = T(P) + (ae + bf, ce + df)t$. Now, when $t = 1$, this is $T(Q)$, so that ℓ maps to a line from $T(P)$ to $T(Q)$.

d. By moving and reflecting the x and y axes if necessary, we may assume that the vertices of the parallelogram are $(0, 0)$, $(r, 0)$, (s, t), and $(r + s, t)$ for $r, s, t > 0$. The area of this parallelogram is then rt (base times height). Then the vertices of S are $(0, 0)$, (ar, cr), $(as + bt, cs + dt)$, and $(ar + as + bt, cr + cs + dt)$. Again regarding (ar, cr) and $(as + bt, cs + dt)$ as vectors in 3-space, the area of the parallelogram is $\left|\left\langle ar, cr, 0\right\rangle \times \left\langle as + bt, cs + dt, 0\right\rangle\right| = \left|\left\langle 0, 0, ar(cs + dt) - cr(as + bt)\right\rangle\right| =$ $\left|\left\langle 0, 0, (ad - bc)rt\right\rangle\right| = |J(u, v)|\,rt = |J(u, v)|\,\text{area}(S)$.

13.7.61

a. This is just the definition of the transformation T, which is given by $x = g(u, v)$, $y = h(u, v)$, so that the image of (x, y) is $T((x, y)) = (g(u, v), h(u, v))$. Apply this to the coordinates of the points O, P, Q.

b. The Taylor expansions of g and h at $(0, 0)$ are

$$g(u, v) = g(0, 0) + u\,g_u(0, 0) + v\,g_v(0, 0) + \text{ terms involving higher derivatives}$$

$$rh(u, v) = h(0, 0) + u\,h_u(0, 0) + v\,h_v(0, 0) + \text{terms involving higher derivatives.}$$

Substituting the two points $(\triangle u, 0)$ and $(0, \triangle v)$ into these equations and considering only the terms up through the first derivative gives the desired result.

c. From part (b),

$$P' \approx (g(0, 0) + g_u(0, 0)\triangle u,\ h(0, 0) + h_u(0, 0)\triangle u)$$
$$Q' \approx (g(0, 0) + g_v(0, 0)\triangle v,\ h(0, 0) + h_v(0, 0)\triangle v),$$

so that $\overrightarrow{O'P'} \approx \triangle u \left\langle g_u(0, 0), h_u(0, 0) \right\rangle$ and $\overrightarrow{O'Q'} \approx \triangle v \left\langle g_v(0, 0), h_v(0, 0) \right\rangle$. The area of the parallelogram determined by $\overrightarrow{O'P'}$ and $\overrightarrow{O'Q'}$ is the magnitude of the cross product of these vectors (considered as vectors in 3-space with zero z coordinate), so the area of the resulting region is approximately

$$\begin{aligned}
&\left| \triangle u \left\langle g_u(0, 0), h_u(0, 0), 0 \right\rangle \times \triangle v \left\langle g_v(0, 0), h_v(0, 0), 0 \right\rangle \right| \\
&= \triangle u \triangle v \left| \left\langle g_u(0, 0), h_u(0, 0), 0 \right\rangle \times \left\langle g_v(0, 0), h_v(0, 0), 0 \right\rangle \right| \\
&= \triangle u \triangle v \left| g_u(0, 0)\, h_v(0, 0) - g_v(0, 0)\, h_u(0, 0) \right| \\
&= \left| J(u, v) \right| \triangle u \triangle v.
\end{aligned}$$

d. Because the area of R is $\triangle u \triangle v$, the ratio is approximately $|J(u, v)|$.

13.7.62

a. Let $n_1 = \langle at, bt, 0 \rangle$, $n_2 = \langle ct, 0, dt \rangle$, $n_3 = \langle 0, et, ft \rangle$ for $t \in R$. Then n_1 is normal to the first pair, n_2 to the second pair, and n_3 to the third pair.

b. The triple scalar product of any three vectors is the volume of the parallelepiped that they determine. This volume is zero if and only if they are coplanar.

c. The triple scalar product is $\langle at, bt, 0 \rangle \cdot (\langle ct, 0, dt \rangle \times \langle 0, et, ft \rangle) = \langle at, bt, 0 \rangle \cdot t^2 \langle -de, cf, ce \rangle = t^3(-ade - bcf)$, so the normal vectors are coplanar if and only if $ade + bcf = 0$.

d. If the three vectors are coplanar, then the cross product of any two of them is perpendicular to the plane they determine. Thus, for example, from part (c), $N = t(de, cf, -ce)$ is normal to the plane. So any line in the direction of N is parallel to all six planes. By choosing the line appropriately, we can ensure that it does not lie in any of the six planes, and thus it does not intersect any of them. Thus the six planes do not form a bounded region if $ade + bcf = 0$.

e. The Jacobian is $\begin{vmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & f \end{vmatrix} = -ade - bcf$. The Jacobian is zero if R is unbounded.

Chapter Thirteen Review

1

a. False. For example, if $g(x, y) = 2$, then $\int_c^d \int_a^b 2\, dx\, dy = 2(b - a)(d - c)$ while $\left(\int_c^d 2\, dy\right)\left(\int_a^b 2\, dx\right) = 4(b - a)(d - c)$.

b. True. The first set is the set whose φ coordinate is $\frac{\pi}{2}$; φ is the angle the line to the point makes with the z-axis, so this is the set of points in the xy-plane.

c. False. The integrand doesn't change.

d. False. For example, it maps the standard unit square into the square with vertices $(0, 0)$, $(0, -1)$, $(1, -1)$, $(1, 0)$.

2 $\int_1^2 \int_1^4 \frac{xy}{(x^2+y^2)^2}\, dx\, dy = \frac{1}{2} \int_1^2 \int_1^4 \frac{(2x)y}{(x^2+y^2)^2}\, dx\, dy = -\frac{1}{2} \int_1^2 \left. \frac{y}{x^2+y^2} \right|_1^4 dy = -\frac{1}{2} \int_1^2 \left(\frac{y}{y^2+16} - \frac{y}{y^2+1} \right) dy = \frac{1}{4} \ln 17 - \frac{3}{4} \ln 2.$

3 $\int_1^3 \int_1^{e^x} \frac{x}{y}\, dy\, dx = \int_1^3 x \ln y \Big|_1^{e^x} dx = \int_1^3 x^2 dx = \frac{26}{3}.$

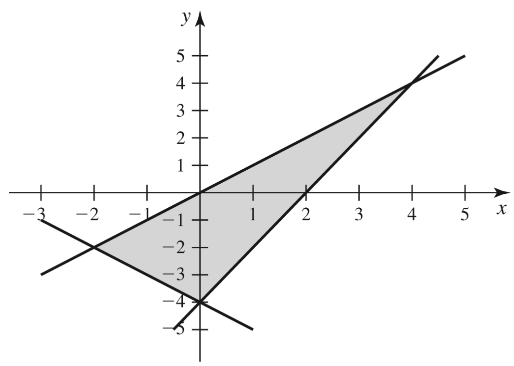
4 $\int_1^2 \int_0^{\ln x} x^3 e^y \, dy \, dx = \int_1^2 x^3 e^y \Big|_0^{\ln x} dx = \int_1^2 \left(x^4 - x^3\right) dx = \frac{49}{20}.$

5 $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) \, dx \, dy.$

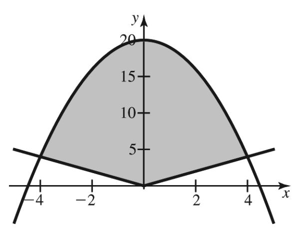
6 $\int_{-1}^1 \int_0^{x+1} f(x, y) \, dy \, dx.$

7 $\int_0^1 \int_0^{\sqrt{1-x^2}} f(x, y) \, dy \, dx.$

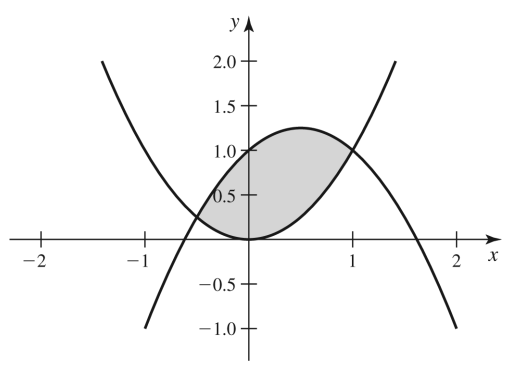
8 $\int_{-2}^0 \int_{-x-4}^{x} 1 \, dy \, dx + \int_0^4 \int_{2x-4}^{x} 1 \, dy \, dx = \int_{-2}^0 (2x+4) \, dx + \int_0^4 (4-x) \, dx = 12.$



9 $2 \int_0^4 \int_x^{20-x^2} 1 \, dy \, dx = 2 \int_0^4 \left(20 - x^2 - x\right) dx = \frac{304}{3}.$



10 $\int_{-1/2}^1 \int_{x^2}^{1+x-x^2} 1 \, dy \, dx = \int_{-1/2}^1 \left(1 + x - 2x^2\right) dx = \frac{9}{8}.$



11 $\int_1^2 \int_0^{x^{3/2}} \frac{2y}{\sqrt{x^4+1}} \, dy \, dx = \int_1^2 \frac{x^3}{\sqrt{x^4+1}} \, dx = \frac{\sqrt{17}-\sqrt{2}}{2}.$

12 $\int_1^4 \int_0^{\sqrt{x}} x^{-1/2} e^y dy \, dx = \int_1^4 x^{-1/2} \left(e^{x^{1/2}} - 1\right) dx = 2e^2 - 2e - 2.$

13 $\int_0^\pi \int_0^{4 \sin\theta} r^2 (\cos\theta + \sin\theta) \, dr \, d\theta = \frac{64}{3} \int_0^\pi \sin^3\theta \, (\cos\theta + \sin\theta) \, d\theta = 8\pi.$

14 $\int_0^2 \int_0^x \left(x^2+y^2\right) dy\,dx = \int_0^2 \frac{4}{3}x^3\,dx = \frac{16}{3}$.

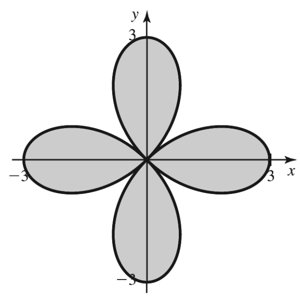
15 $\int_0^1 \int_{y^{1/3}}^1 x^{10}\cos\left(\pi x^4 y\right)\,dx\,dy = \int_0^1 \int_0^{x^3} x^{10}\cos\left(\pi x^4 y\right)\,dy\,dx = \int_0^1 \frac{x^{10}}{\pi x^4}\,\sin\left(\pi x^7\right)\,dx =$
$\frac{1}{\pi}\int_0^1 x^6\,\sin\left(\pi x^7\right)\,dx = \frac{2}{7\pi^2}$.

16 $\int_0^2 \int_{y^2}^4 x^8 y\sqrt{1+x^4y^2}\,dx\,dy = \int_0^4 \int_0^{\sqrt{x}} x^8 y\sqrt{1+x^4y^2}\,dy\,dx = \frac{1}{3}\int_0^4 x^4\left(-1+\left(1+x^5\right)^{3/2}\right)dx =$
$\frac{420250}{3}\sqrt{41} - \frac{5122}{25}$.

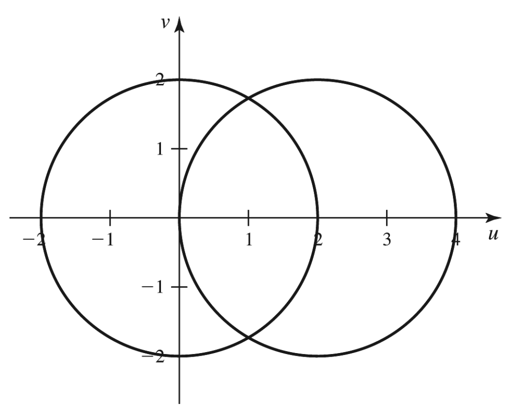
17 $\int_0^1 \int_0^{\pi/2} 3\left(r\,\cos\theta\right)^2\,r\,\sin\theta\,r\,d\theta\,dr = \int_0^1 \int_0^{\pi/2} 3r^4\cos^2\theta\,\sin\theta\,d\theta\,dr = \int_0^1 r^4\,dr = \frac{1}{5}$.

18 $\int_1^4 \int_0^\pi \frac{1}{(1+r^2)^2}\,r\,d\theta\,dr = \pi\int_1^4 \frac{r}{(1+r^2)^2}\,dr = \frac{15}{68}\pi$.

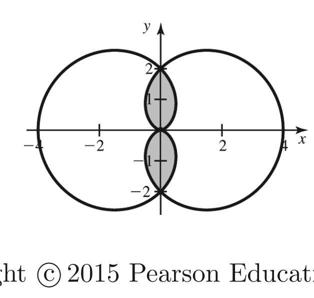
19 The area is four times the area of one leaf, so is $4\int_{-\pi/4}^{\pi/4}\int_0^{3\,\cos 2\theta} r\,dr\,d\theta = 18\int_{-\pi/4}^{\pi/4}\cos^2 2\theta\,d\theta = \frac{9\pi}{2}$.



20 The area is twice the area of the region above the x-axis. The two circles intersect above the x axis at $\cos\theta = \frac{1}{2}$, or $\theta = \frac{\pi}{3}$. Thus the area of the region is $2\left(\int_0^{\pi/3}\int_0^2 r\,dr\,d\theta + \int_{\pi/3}^{\pi/2}\int_0^{4\,\cos\theta} r\,dr\,d\theta\right) =$
$2\left(\int_0^{\pi/3} 2\,d\theta + \int_{\pi/3}^{\pi/2} 8\cos^2\theta\,d\theta\right) = \frac{4\pi}{3} + \frac{4\pi}{3} - 2\sqrt{3} = \frac{8\pi}{3} - 2\sqrt{3}$.



21 The area is four times the area bounded by the cardioid $2-2\cos\theta$ and the y-axis for positive x (i.e. for $0 \le \theta \le \frac{\pi}{2}$), Because all four portions of the area are congruent. Thus it is $4\int_0^{\pi/2}\int_0^{2-2\,\cos\theta} r\,dr\,d\theta =$
$8\int_0^{\pi/2}\left(1-\cos\theta\right)^2\,d\theta = 6\pi - 16$.



22 The area of the disk is $\pi \cdot 4^2 = 16\pi$. Using polar coordinates, the average value is then $\frac{1}{16\pi}\int_0^4\int_0^{2\pi} r\sqrt{16-r^2}\,d\theta\,dr = \frac{1}{8}\int_0^4 r\sqrt{16-r^2}\,dr = \frac{8}{3}$.

23 The volume of the cone is $\frac{1}{3}$ times the area of the base times the height. The base (at $z=8$) is a circle of radius 4, so the volume is $\frac{1}{3}\pi\cdot 4^2\cdot 8 = \frac{128\pi}{3}$. Use cylindrical coordinates to integrate; then the distance to the z-axis is r and the average is $\frac{1}{(128\pi)/3}\int_0^8\int_0^{z/2}\int_0^{2\pi} r^2\,d\theta\,dr\,dz = \frac{1}{64}\int_0^8 \frac{z^3}{8}\,dz = 2$.

24 $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} f(x,\,y,\,z)\;dz\,dy\,dx$.

25 $\int_0^4\int_0^{\sqrt{16-z^2}}\int_0^{\sqrt{16-y^2-z^2}} f(x,\,y,\,z)\;dx\,dy\,dz$.

26 $\int_0^2\int_y^2\int_0^{9-x^2} f(x,\,y,\,z)\;dz\,dx\,dy$.

27 $\int_0^1\int_{-z}^{z}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx\,dz = 2\int_0^1\int_{-z}^{z}\sqrt{1-x^2}\,dx\,dz = 2\int_0^1\left(z\sqrt{1-z^2}+\arcsin(z)\right)dz = \pi - \frac{4}{3}$.

28 $\int_0^\pi\int_0^y\int_0^{\sin x} dz\,dx\,dy = \int_0^\pi\int_0^y \sin x\,dx\,dy = \int_0^\pi (1-\cos y)\;dy = \pi$.

29 The region in the xy-plane can be restated as $0 \le x \le 2$, $0 \le y \le \frac{x}{2}$. Reordering the integral gives $\int_1^9\int_0^2\int_0^{x/2} \frac{4\sin\left(x^2\right)}{\sqrt{z}}\,dy\,dx\,dz = 2\int_1^9\int_0^2 \frac{x\sin\left(x^2\right)}{\sqrt{z}}\,dx\,dz = (1-\cos 4)\int_1^9 z^{-1/2}\,dz = 4-4\cos 4$.

30 This is the integral of a function over the right half of the ellipse $\frac{x^2}{2}+y^2=2$. Change the order of integration of x and y so that $-\sqrt{2}\le y\le\sqrt{2}$, $0\le x\le\sqrt{4-2y^2}$. We have $\int_{-\sqrt{2}}^{\sqrt{2}}\int_0^{\sqrt{4-2y^2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy = \int_{-\sqrt{2}}^{\sqrt{2}}\int_0^{\sqrt{4-2y^2}}\left(8-2x^2-4y^2\right)\,dx\,dy = \int_{-\sqrt{2}}^{\sqrt{2}}\left(\left(8-4y^2\right)\sqrt{4-2y^2}-\frac{2}{3}\left(4-2y^2\right)^{3/2}\right)dy = 4\pi\sqrt{2}$.

31 Reorder to integrate with respect to x last. Because $0\le y\le 2$, we have $0\le x\le 4$. The integral is $\int_0^4\int_{\sqrt{x}}^2\int_0^{y^{1/3}} yz^5\left(1+x+y^2+z^6\right)^2\,dz\,dy\,dx = \frac{1}{18}\int_0^4\int_{\sqrt{x}}^2\left(7y^7+9y^5+9xy^5+3y^3+6xy^3+3x^2y^3\right)\,dy\,dx = \frac{1}{18}\int_0^4\left(332-\frac{25}{8}x^4-3x^3+\frac{45}{4}x^2+120x\right)\,dx = \frac{848}{9}$.

32 $\int_0^1\int_0^{3-3x}\int_0^2 1\,dz\,dy\,dx = \int_0^1 (6-6x)\;dx = 3$.

33 $\int_0^2\int_0^\pi\int_0^{r\sin\theta} r\,dz\,d\theta\,dr = \int_0^2\int_0^\pi r^2\sin\theta\,d\theta\,dr = \int_0^2 2r^2\,dr = \frac{16}{3}$.

34 Note that $2-x^2-(1+y^2) = 1-(x^2+y^2)$. Note also that the projection of the intersection of the surfaces to the plane $z=0$ is the unit circle. We use cylindrical coordinates to obtain $\int_0^{2\pi}\int_0^1(1-r^2)r\,dr\,d\theta = \int_0^{2\pi}\int_0^1(r-r^3)\,dr\,d\theta = 2\pi\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}$.

35 Look at the intersection of the cylinders from the positive z-axis. The vertical sides of the region lie on the cylinder $x^2+y^2=4$, and the top and bottom lie on $x^2+z^2=4$. Thus the region is $\left\{(x,\,y,\,z): x^2+y^2\le 4,\ -\sqrt{4-x^2}\le z\le\sqrt{4-x^2}\right\}$. Thus the volume is $\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1\,dz\,dy\,dx = \int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2\sqrt{4-x^2}\,dy\,dx = \int_{-2}^2 4\left(4-x^2\right)\,dx = \frac{128}{3}$.

36 $\int_0^1\int_0^x\int_0^y 1\,dz\,dy\,dx = \int_0^1\int_0^x y\,dy\,dx = \int_0^1 \frac{x^2}{2}\,dx = \frac{1}{6}$.

37 Rewrite the integral as $\int_0^{1/2}\int_{\sin^{-1}x}^{\sin^{-1}2x} 1\,dy\,dx$ and then change the order of integration. This results in the integral breaking up into two integrals, and we obtain

$$\int_0^{\pi/6}\int_{(\sin y)/2}^{\sin y} 1\,dx\,dy + \int_{\pi/6}^{\pi/2}\int_{(\sin y)/2}^{1/2} 1\,dx\,dy = \int_0^{\pi/6}\frac{\sin y}{2}\,dy + \int_{\pi/6}^{\pi/2}\left(\frac{1}{2}-\frac{\sin y}{2}\right)dy = \frac{1-\sqrt{3}}{2}+\frac{\pi}{6}.$$

38 The plane of the slanted surface of the tetrahedron is $6x+3y+2z=6$, so the possibilities are

$$\begin{aligned}\int_0^1\int_0^{2-2x}\int_0^{(6-6x-3y)/2} dz\,dy\,dx &= \int_0^1\int_0^{3-3x}\int_0^{(6-6x-2z)/3} dy\,dz\,dx\\ &= \int_0^2\int_0^{(2-y)/2}\int_0^{(6-6x-3y)/2} dz\,dx\,dy\\ &= \int_0^2\int_0^{2-2x}\int_0^{(6-3y-2z)/6} dx\,dz\,dy\\ &= \int_0^3\int_0^{(3-z)/3}\int_0^{(6-6x-2z)/3} dy\,dx\,dz\\ &= \int_0^3\int_0^{2(3-z)/3}\int_0^{(6-3y-2z)/6} dx\,dy\,dz.\end{aligned}$$

39

a. $\int_0^2\int_0^{z^3}\int_0^{y^2} 1\,dx\,dy\,dz = \int_0^2\int_0^{z^3} y^2\,dy\,dz = \frac{1}{3}\int_0^2 z^9\,dz = \frac{512}{15}$.

b. In theory, there are a total of six arrangements of dx, dy and dz. Thus there are five possible integration orders other than this one. For example, using $dx\,dz\,dy$: $\int_0^8\int_{\sqrt[3]{y}}^2\int_0^{y^2} 1\,dx\,dz\,dy = \int_0^8\int_{\sqrt[3]{y}}^2 y^2\,dz\,dy = \int_0^8 y^2\left(2-\sqrt[3]{y}\right)\,dy = \frac{512}{15}$.

c. $\int_0^2\int_0^{z^q}\int_0^{y^p} 1\,dx\,dy\,dz = \int_0^2\int_0^{z^q} y^p\,dy\,dz = \frac{1}{p+1}\int_0^2 z^{q(p+1)}\,dz = \frac{2^{q(p+1)+1}}{(p+1)(q(p+1)+1)}$.

40 Use cylindrical coordinates. The volume of the paraboloid is $\int_0^2\int_0^{2\pi}\int_0^{4-r^2} r\,dz\,d\theta\,dr = 2\pi\int_0^2\left(4r-r^3\right)\,dr = 8\pi$. The distance of a point from the origin is $\sqrt{r^2+z^2}$, so the average squared distance is $\frac{1}{8\pi}\int_0^2\int_0^{2\pi}\int_0^{4-r^2} r\left(r^2+z^2\right)\,dz\,d\theta\,dr = \frac{1}{4}\int_0^2\left(\frac{1}{3}r\left(4-r^2\right)^3 + r^3\left(4-r^2\right)\right)\,dr = 4$.

41 The volume of the prism is $\int_0^1\int_0^{3-3x}\int_0^2 1\,dz\,dy\,dx = \int_0^1(6-6x)\,dx = 3$. Thus the average x coordinate is $\frac{1}{3}\int_0^1\int_0^{3-3x}\int_0^2 x\,dz\,dy\,dx = \frac{1}{3}\int_0^1 2x(3-3x)\,dx = \frac{1}{3}$.

42 This is a quarter of a cylinder of radius 3 and length 3 oriented along the z-axis, so$\int_0^3\int_0^{\pi/2}\int_0^3 r^3\cdot r\,dz\,d\theta\,dr = \frac{3\pi}{2}\int_0^3 r^4\,dr = \frac{729\pi}{10}$.

43 Use cylindrical coordinates. We have $\int_{-2}^2\int_{-\pi/2}^{\pi/2}\int_0^1 \frac{1}{(1+r^2)^2} r\,dr\,d\theta\,dz$. This can be written as $(2-(-2))\cdot\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)\int_0^1\frac{r}{(1+r^2)^2}\,dr = 4\pi\int_1^2\frac{1}{2}\left(\frac{1}{u^2}\right)\,du = 2\pi\left(-\frac{1}{u}\right)\Big|_1^2 = \pi$.

44 $z=\sqrt{29}$ when $x^2+y^2=25$, so $0\le r\le 5$. The volume is $\int_0^5\int_0^{2\pi}\int_{\sqrt{4+r^2}}^{\sqrt{29}} r\,dz\,d\theta\,dr = 2\pi\int_0^5 r\left(\sqrt{29}-\sqrt{4+r^2}\right)\,dr = \frac{1}{3}\pi\left(17\sqrt{29}+16\right)$.

45 $\int_0^4\int_0^\pi\int_0^{2\cos\theta} r\,dr\,d\theta\,dz = 2\int_0^4\int_0^\pi\cos^2\theta\,d\theta\,dz = 4\pi$.

46 $\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/2}\cos^3\varphi\,\sin\varphi\,d\varphi\,d\theta = \frac{4\pi}{3}$.

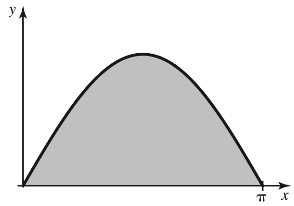
47 $\int_0^\pi\int_0^{\pi/4}\int_{2\sec\varphi}^{4\sec\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{56}{3}\int_0^\pi\int_0^{\pi/4}\sec^3\varphi\,\sin\varphi\,d\varphi\,d\theta = \frac{28\pi}{3}$.

48 $\int_0^\pi\int_0^{(1-\cos\varphi)/2}\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{\pi}{12}\int_0^\pi(1-\cos\varphi)^3\,\sin\varphi\,d\rho\,d\varphi = \frac{\pi}{3}$.

49 $\int_0^{\pi/2}\int_0^{4\sin2\varphi}\int_0^{2\pi}\rho^2\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{128\pi}{3}\int_0^{\pi/2}\sin^3 2\varphi\,\sin\varphi\,d\varphi = \frac{2048\pi}{105}$.

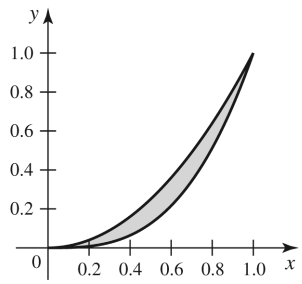
50 $\int_0^{\pi/4}\int_0^{4\cos\varphi}\int_0^{2\pi}\rho^2\,\sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{128\pi}{3}\int_0^{\pi/4}\cos^3\varphi\,\sin\varphi\,d\varphi = 8\pi.$

51



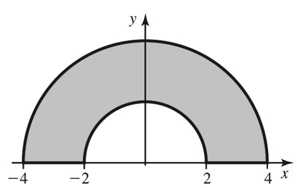
The mass of the plate is $\int_0^{\pi}\sin x\,dx = -\cos\pi + \cos 0 = 2$. By symmetry, $\overline{x} = \frac{\pi}{2}$, $\overline{y} = \frac{1}{2}\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx = \frac{1}{4}\int_0^{4}\sin^2 x\,dx = \frac{\pi}{8}$. The center of mass is $\left(\frac{\pi}{2},\ \frac{\pi}{8}\right)$.

52



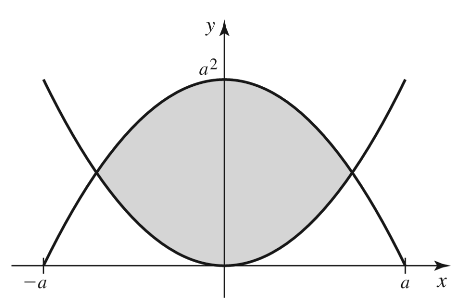
The mass of the plate is $\int_0^1\int_{x^3}^{x^2} 1\,dy\,dx = \int_0^1\left(x^2 - x^3\right)dx = \frac{1}{12}$, so that $\overline{x} = \frac{1}{1/12}\int_0^1\int_{x^3}^{x^2} x\,dy\,dx = 12\int_0^1\left(x^3 - x^4\right)dx = \frac{3}{5}\overline{y} = \frac{1}{1/12}\int_0^1\int_{x^3}^{x^2} y\,dy\,dx = 6\int_0^1\left(x^4 - x^6\right)dx = \frac{12}{35}$. The center of mass is at $\left(\frac{3}{5},\ \frac{12}{35}\right)$.

53



By symmetry, $\overline{x} = 0$. The mass of the region is $8\pi - 2\pi = 6\pi$, so $\overline{y} = \frac{1}{6\pi}\int_0^{\pi}\int_2^4 r^2\sin\theta\,dr\,d\theta = \frac{28}{9\pi}\int_0^{\pi}\sin\theta\,d\theta = \frac{56}{9\pi}$. The center of mass is at $\left(0,\ \frac{56}{9\pi}\right)$.

54



By symmetry, $\overline{x} = 0$. The two parabolas intersect where $x^2 = a^2 - x^2$, so where $x = \pm\frac{a}{\sqrt{2}}$. The mass is then $\int_{-a/\sqrt{2}}^{a/\sqrt{2}}\int_{x^2}^{a^2-x^2} 1\,dy\,dx = \int_{-a/\sqrt{2}}^{a/\sqrt{2}}\left(a^2 - 2x^2\right)dx = \frac{2}{3}a^3\sqrt{2}$ so that $\overline{y} = \frac{1}{2a^3\sqrt{2}/3}\int_{-a/\sqrt{2}}^{a/\sqrt{2}}\int_{x^2}^{a^2-x^2} y\,dy\,dx = \frac{3}{4a^3\sqrt{2}}\int_{-a/\sqrt{2}}^{a/\sqrt{2}}\left(\left(a^2 - x^2\right)^2 - x^4\right)dx = \frac{1}{2}a^2$. The center of mass is at $\left(0,\ \frac{1}{2}a^2\right)$.

55 By symmetry, $\overline{x} = \overline{y} = 0$. The mass (using cylindrical coordinates) is

$$\int_0^6 \int_{r^2}^{36} \int_0^{2\pi} r\, d\theta\, dz\, dr = 2\pi \int_0^6 r\left(36 - r^2\right)\, dr = 648\pi,$$

so $\overline{z} = \frac{1}{648\pi}\int_0^6 \int_{r^2}^{36} \int_0^{2\pi} rz\, d\theta\, dz\, dr = \frac{1}{648}\int_0^6 r\left(1296 - r^4\right)\, dr = 24$. The center of mass is (0, 0, 24).

56 The mass is $\frac{1}{3}$ times the area of the base (4) times the height (4), so is $\frac{16}{3}$. The plane intersects the x-axis at $x = 4$, the y-axis at $y = 2$, and the z-axis at $z = 4$, so by symmetry, $\overline{x} = \overline{z}$. Then $\overline{x} = \frac{1}{16/3}\int_0^2 \int_0^{4-2y} \int_0^{4-x-2y} x\, dz\, dx\, dy = \frac{3}{16}\int_0^2 \int_0^{4-2y} \left(4x - x^2 - 2xy\right)\, dx\, dy = \frac{1}{32}\int_0^2 (4-2y)^3\, dy = 1$. $\overline{y} = \frac{1}{16/3}\int_0^2 \int_0^{4-2y} \int_0^{4-x-2y} y\, dz\, dx\, dy = \frac{3}{16}\int_0^2 \int_0^{4-2y} \left(4y - xy - 2y^2\right)\, dx\, dy = \frac{3}{16}\int_0^2 \left(8y - 8y^2 + 2y^3\right) dy = \frac{1}{2}$, so the center of mass is $\left(1, \frac{1}{2}, 1\right)$.

57 Use spherical coordinates. The mass is $\int_0^{\pi/2} \int_0^{16} \int_0^{2\pi} \left(1 + \frac{\rho}{4}\right) \rho^2 \sin\varphi\, d\theta\, d\rho\, d\varphi =$ $2\pi \int_0^{\pi/2} \int_0^{16} \left(\rho^2 + \frac{\rho^3}{4}\right) \sin\varphi\, d\rho\, d\varphi = \frac{32768\pi}{3}\int_0^{\pi/2} \sin\varphi\, d\varphi = \frac{32768\pi}{3}$. By symmetry of the region and of the density function around the z axis, we have $\overline{x} = \overline{y} = 0$. $\overline{z} = \frac{1}{32768\pi/3}\int_0^{\pi/2} \int_0^{16} \int_0^{2\pi} \rho\cos\varphi\left(1 + \frac{\rho}{4}\right)\rho^2 \sin\varphi\, d\theta\, d\rho\, d\varphi$ $= \frac{3}{32768}\int_0^{\pi/2} \int_0^{16} \left(\rho^3 + \frac{\rho^4}{4}\right) \cos\varphi\, \sin\varphi\, d\rho\, d\varphi = \frac{3}{32768}\int_0^{\pi/2} \frac{16384 \cdot 21}{5} \cos\varphi\, \sin\varphi\, d\varphi = \frac{63}{10}$, so the center of mass is $\left(0, 0, \frac{63}{10}\right)$.

58 The mass is $\int_0^2 \int_0^2 \int_0^2 (1 + x + y + z)\, dz\, dy\, dx = 32$. By symmetry, $\overline{x} = \overline{y} = \overline{z}$, and $\overline{x} = \frac{1}{32}\int_0^2 \int_0^2 \int_0^2 x\,(1 + x + y + z)\, dz\, dy\, dx = \frac{13}{12}$, so the center of mass is $\left(\frac{13}{12}, \frac{13}{12}, \frac{13}{12}\right)$.

59 Place the vertex at the origin with the paraboloid opening upwards along the positive z-axis. Then the equation of the paraboloid is $z = \frac{h}{R^2}r^2$. Its mass is $\int_0^R \int_0^{2\pi} \int_{hr^2/R^2}^{h} r\, dz\, d\theta\, dr = 2\pi \int_0^R \left(r - \frac{h}{R^2}r^2\right) dr =$ $\frac{1}{2}\pi h R^2$. Then $\overline{z} = \frac{1}{\pi h R^2/2}\int_0^R \int_0^{2\pi} \int_{hr^2/R^2}^{h} rz\, dz\, d\theta\, dr = \frac{2}{hR^2}\int_0^R rh^2 \frac{R^4 - r^4}{R^4} dr = \frac{2}{3}h$, so that the center of mass is $\frac{1}{3}$ of the way from the base to the vertex.

60 Place one vertex at the origin and another at $(s, 0)$ so that the third is at $\left(\frac{s}{2}, \frac{s\sqrt{3}}{2}\right)$. Then the equations of the two sides of the triangle to that third vertex are $y = x\sqrt{3}$ and $y = -\sqrt{3}\,(x - s)$. The area of the triangle is $\frac{s^2\sqrt{3}}{4}$ (one half the base times the height), so $\overline{y} = \frac{1}{s^2\sqrt{3}/4}\int_0^{s/2} \int_0^{\sqrt{3}x} y\, dy\, dx + \int_{s/2}^{s} \int_0^{-\sqrt{3}(x-s)} y\, dy\, dx =$ $\frac{2}{s^2\sqrt{3}}\left(\int_0^{s/2} 3x^2\, dx + \int_{s/2}^{s} 3\,(x - s)^2\, dx\right) = \frac{6}{s^2\sqrt{3}}\left(\frac{s^3}{24} + \frac{s^3}{24}\right) = \frac{\sqrt{3}}{6}s$, which is one third the height of the triangle.

61 Place one of the vertices of the base at the origin and the other at $(b, 0)$. Some simple right triangle analysis shows that the third vertex is at $\left(\frac{b}{2}, \sqrt{s^2 - \frac{b^2}{4}}\right)$. The area of the triangle is half the base times the height, or $\frac{b}{2} \cdot \frac{1}{2}\sqrt{4s^2 - b^2} = \frac{b\sqrt{4s^2 - b^2}}{4}$.

Then $\overline{y} = \frac{1}{b\sqrt{4s^2 - b^2}/4}\left(\int_0^{b/2} \int_0^{x\sqrt{4s^2 - b^2}/b} y\, dy\, dx + \int_{b/2}^{b} \int_0^{-(x-b)\sqrt{4s^2 - b^2}/b} y\, dy\, dx\right) =$ $\frac{2}{b\sqrt{4s^2 - b^2}}\left(\int_0^{b/2} \frac{4s^2 - b^2}{b^2}x^2\, dx + \int_{b/2}^{b} \frac{4s^2 - b^2}{b^2}\,(x - b)^2\, dx\right) = \frac{2\sqrt{4s^2 - b^2}}{b^3}\left(\frac{b^3}{24} + \frac{b^3}{24}\right) = \frac{\sqrt{4s^2 - b^2}}{6}$, which is one third the height of the triangle.

62 The projection of the tetrahedron on the xy-plane is the triangle with vertices (0, 0), (1, 0), and (0, 2); the height of the tetrahedron is 3. Thus its volume is $\frac{1}{3} \cdot 3 \cdot 1 = 1$.

Then $\overline{x} = \int_0^1 \int_0^{2-2x} \int_0^{3(1-x-y/2)} x\, dz\, dy\, dx = \int_0^1 \int_0^{2-2x} 3x\left(1 - x - \frac{y}{2}\right) dy\, dx = \int_0^1 \left(3x - 6x^2 + 3x^3\right) dx =$ $\frac{1}{4}$. $\overline{y} = \int_0^1 \int_0^{2-2x} \int_0^{3(1-x-y/2)} y\, dz\, dy\, dx = \int_0^1 \int_0^{2-2x} 3y\left(1 - x - \frac{y}{2}\right) dy\, dx = \int_0^1 \left(2 - 6x + 6x^2 - 2x^3\right) dx = \frac{1}{2}$. $\overline{z} = \int_0^1 \int_0^{2-2x} \int_0^{3(1-x-y/2)} z\, dz\, dy\, dx = \frac{9}{2}\int_0^1 \int_0^{2-2x} \left(1 - x - \frac{y}{2}\right)^2 dy\, dx = 3\int_0^1 (1 - x)^3\, dx = \frac{3}{4}$, so the center of mass is at $\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right)$.

63 The equation of the cone in cylindrical coordinates is $z = 2 - \frac{1}{2}r$. The volume of the cone is one third the area of the base times the height, or $\frac{32\pi}{3}$.

a. The volume of a slice of $\frac{\pi}{4}$ radians is $\int_0^4 \int_0^{\pi/4} \int_0^{2-r/2} r\,dz\,d\theta\,dr = \frac{\pi}{4} \int_0^4 r\left(2 - \frac{r}{2}\right)\,dr = \frac{4\pi}{3}$. The volume of the wedge is in fact one eighth the volume of the entire cone ($\frac{\pi}{4}$ radians is an eighth-circle).

b. The volume of a slice of Q radians is $\int_0^4 \int_0^Q \int_0^{2-r/2} r\,dz\,d\theta\,dr = Q \int_0^4 r\left(2 - \frac{r}{2}\right)\,dr = \frac{16}{3}Q$. Geometrically, Q radians is $\frac{Q}{2\pi}$ of a circle, and indeed the volume of the slice is $\frac{Q}{2\pi}$ times the volume of the cone.

64 Compute the volume of empty space, using spherical coordinates with the center of the tank at the origin. The water level is at $\varphi = \frac{\pi}{3}$ (Pythagorean theorem), so the volume of the empty spherical cap is $\int_0^{\pi/3} \int_{\sec\varphi/2}^{1} \int_0^{2\pi} \rho^2 \sin\varphi\,d\theta\,d\rho\,d\varphi = \frac{2\pi}{3} \int_0^{\pi/3} \left(1 - \frac{\sec^3\varphi}{8}\right) \sin\varphi\,d\varphi = \frac{5\pi}{24}$ cubic feet. The total volume of the sphere is $\frac{4\pi}{3}$ cubic feet.

a. The volume of water is $\frac{4\pi}{3} - \frac{5\pi}{24} = \frac{27\pi}{24} = \frac{9\pi}{8}$ cubic feet and it weighs $1.125 \cdot 62.5 \cdot \pi \approx 220.893$ pounds.

b. The amount of water that must be added to fill the tank is the volume of empty space, or $\frac{5\pi}{24}$ cubic feet.

65 The transformation just switches the coordinates, so the image is again the unit square.

66 $T = \{(x, y) : 0 \le v \le 1, 0 \le u \le 1\} = \{(x, y) : 0 \le -x \le 1, 0 \le y \le 1\} = \{(x, y) : -1 \le x \le 0, 0 \le y \le 1\}$ so that T is the unit square in the second quadrant with one vertex at the origin.

67 As (u, v) goes from $(0, 0)$ to $(1, 0)$, (x, y) goes from $(0, 0)$ to $\left(\frac{1}{2}, \frac{1}{2}\right)$; as (u, v) goes from $(1, 0)$ to $(1, 1)$, (x, y) goes from $\left(\frac{1}{2}, -\frac{1}{2}\right)$ to $(1, 0)$; as (u, v) goes from $(1, 1)$ to $(0, 1)$, (x, y) goes from $(1, 0)$ to $\left(\frac{1}{2}, \frac{1}{2}\right)$. Thus the image of S is the diamond with vertices $(0, 0)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, $(1, 0)$, and $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

68 The transformation leaves the x coordinate alone and stretches and translates the y coordinate. So the new region is the rectangle with vertices $(0, 2)$, $(1, 2)$, $(1, 4)$, $(0, 4)$.

69 $J(u, v) = \begin{vmatrix} 4 & -1 \\ -2 & 3 \end{vmatrix} = 10.$

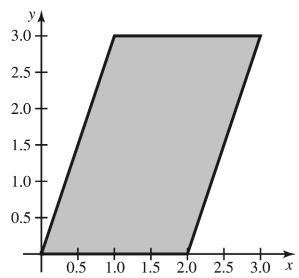
70 $J(u, v) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$

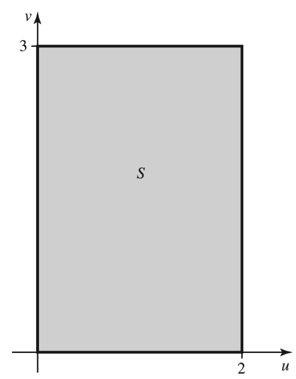
71 $J(u, v) = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6.$

72 $J(u, v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4\left(u^2 + v^2\right).$

73

a.





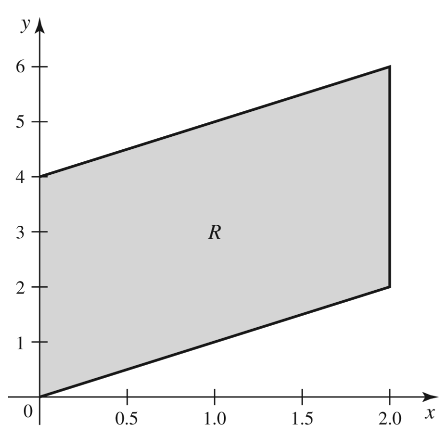
b. $S = \left\{(u, v) : \frac{v}{3} \leq u + \frac{v}{3} \leq \frac{v+6}{3}, 0 \leq v \leq 3\right\} = \{(u, v) : 0 \leq u \leq 2, 0 \leq v \leq 3\}$.

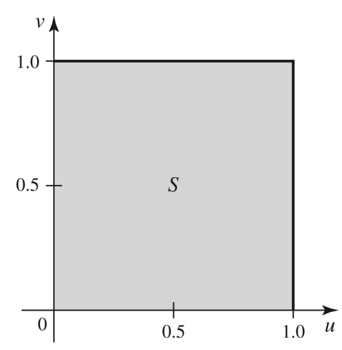
c. $J(u, v) = \begin{vmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{vmatrix} = 1$

d. $\iint\limits_R xy^2 \, dA = \int_0^2 \int_0^3 \left(u + \frac{v}{3}\right) v^2 \, dv \, du = \frac{63}{2}$.

74

a.





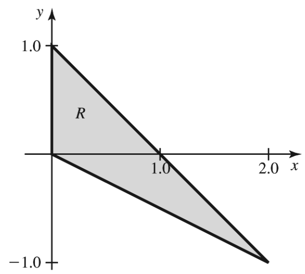
b. $S = \{(u, v) : 0 \leq 2u \leq 2, 2u \leq 4v + 2u \leq 2u + 4\} = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}$.

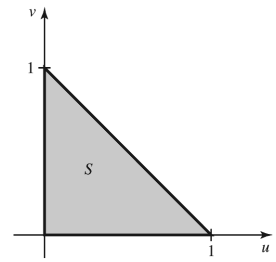
c. $J(u, v) = \begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8$.

d. $\iint\limits_R 3xy^2 \, dA = \int_0^1 \int_0^1 24\,(2u)\,(4v + 2u)^2 \, dv \, du = 192 \int_0^1 \int_0^1 u\,(2v + u)^2 \, dv \, du = 304$.

75

a.





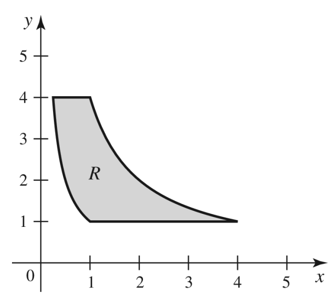
b. $S = \left\{(u, v) : 0 \le 2u \le 2, -\frac{2u}{2} \le v - u \le 1 - 2u\right\} = \{(u, v) : 0 \le u \le 1, 0 \le v \le 1 - u\}$.

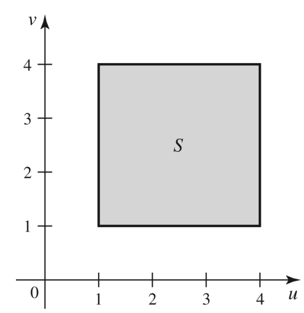
c. $J(u, v) = \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = 2$.

d. Switch the order of integration:
$\iint\limits_R x^2 \sqrt{x + 2y}\, dA = 2 \int_0^1 \int_0^{1-v} (2u)^2 \sqrt{2v}\, du\, dv = \frac{8\sqrt{2}}{3} \int_0^1 (1 - v)^3 \sqrt{v}\, dv = \frac{256\sqrt{2}}{945}$.

76

a.



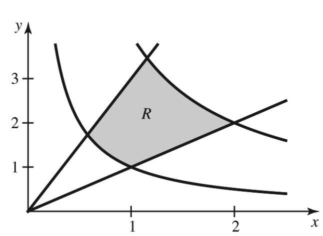


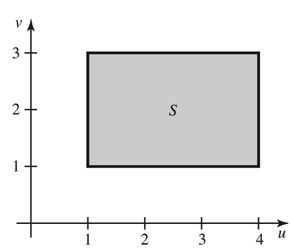
b. $xy = 1$ becomes $u = 1$ while $xy = 4$ becomes $u = 4$; $y = 1$ becomes $v = 1$; $y = 4$ becomes $v = 4$. So $S = \{(u, v) : 1 \le u \le 4, 1 \le v \le 4\}$.

c. $J(u, v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ 0 & 1 \end{vmatrix} = v^{-1}$.

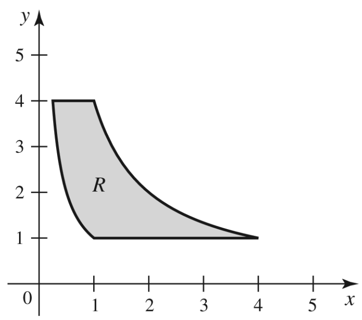
d. $\iint\limits_R xy^2\, dA = \int_1^4 \int_1^4 \frac{u}{v}\, v^2\, v^{-1}\, dv\, du = \int_1^4 \int_1^4 u\, dv\, du = \frac{45}{2}$.

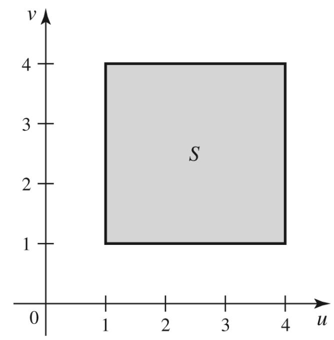
77 Use the transformation $u = xy$; $v = \frac{y}{x}$ so that $x = \sqrt{\frac{u}{v}}$, $y = \sqrt{uv}$. Then $xy = u$ and $\frac{y}{x} = v$, so the new region S is $\{(u, v) : 1 \le u \le 4, 1 \le v \le 3\}$. The Jacobian of this transformation is $J(u, v) = \begin{vmatrix} \frac{u^{-1/2}v^{-1/2}}{2} & -\frac{u^{1/2}v^{-3/2}}{2} \\ \frac{v^{1/2}u^{-1/2}}{2} & \frac{u^{1/2}v^{-1/2}}{2} \end{vmatrix} = \frac{1}{2v}$ so that $\iint\limits_R y^4\, dA = \frac{1}{2} \int_1^4 \int_1^3 u^2 v\, dv\, du = 42$.





78 Let $x = u - v$, $y = u + 2v$; then $y = x$ becomes $v = 0$, $y = x - 3$ becomes $v = -1$, $y = -2x + 3$ becomes $u = 1$, and $y = -2x - 3$ becomes $u = -1$. So the new region of integration is $S = \{(u, v) : -1 \le u \le 1, -1 \le v \le 0\}$. The Jacobian is $J(u, v) = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3$, so that $\iint_R \left(y^2 + xy - 2x^2\right) dA$ $= 3\int_{-1}^{1}\int_{-1}^{0}\left((u+2v)^2 + (u-v)(u+2v) - 2(u-v)^2\right) dv\,du = 3\int_{-1}^{1}\int_{-1}^{0} 9uv\,dv\,du = 0.$





79 Use the transformation $u = x + 2y$, $v = x - z$, $w = 2y - z$; solving for x, y, z gives $x = \frac{u+v-w}{2}$, $y = \frac{u-v+w}{4}$, $z = \frac{u-v-w}{2}$.

The new range of integration becomes $1 \le u \le 2$, $0 \le v \le 2$, $0 \le w \le 3$. The Jacobian is $J(u, v, w) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{4}$ so that $\iiint_D yz\,dV = \frac{1}{32}\int_1^2\int_0^2\int_0^3 (u-v+w)(u-v-w)\,dw\,dv\,du = -\frac{7}{16}$.

80 Use the transformation $u = y - 2x$, $v = z - 3y$, $w = z - 4x$; solving for x, y, z gives $x = \frac{-3u-v+w}{2}$, $y = -2u - v + w$, $z = -6u - 2v + 3w$. The new range of integration becomes $0 \le u \le 1$, $0 \le v \le 1$, $0 \le w \le 3$.

The Jacobian is $J(u, v, w) = \begin{vmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3 \end{vmatrix} = \frac{1}{2}$ so that

$$\iiint_D x\,dV = \frac{1}{4}\int_0^1\int_0^1\int_0^3 (-3u - v + w)\,dw\,dv\,du = -\frac{3}{8}.$$

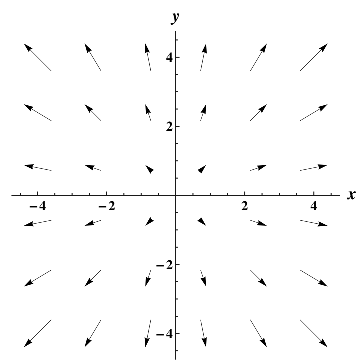
Chapter 14

Vector Calculus

14.1 Vector Fields

14.1.1 A vector field describes the motion of the air as a vector at each point in the room.

14.1.2

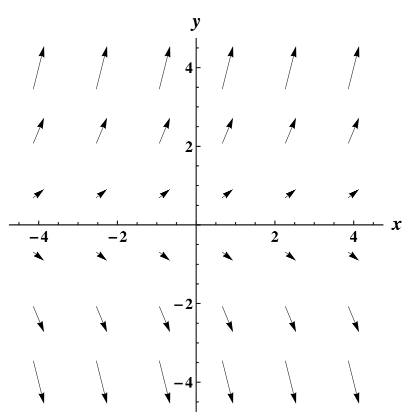


14.1.3 At selected points (a, b), plot the vector $\langle f(a,b), g(a,b)\rangle$.

14.1.4 The gradient of a function at a point is a vector describing the direction in which the value of the function is increasing most rapidly. The collection of these vectors over all points is a vector field.

14.1.5 The gradient field gives, at each point, the direction in which the temperature is increasing most rapidly and the amount of increase.

14.1.6



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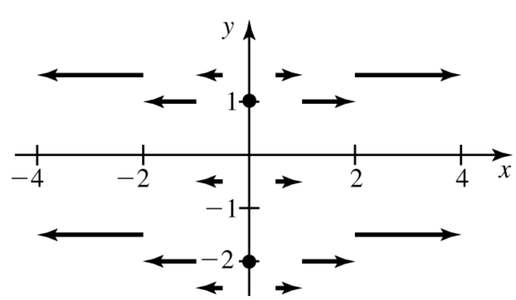

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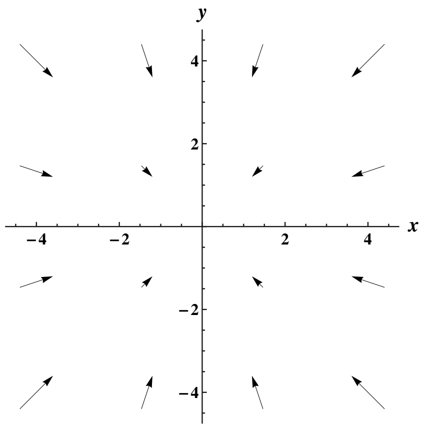

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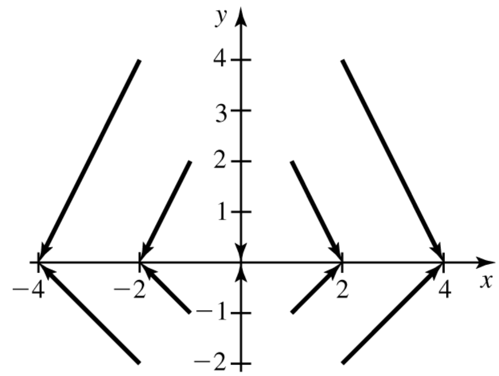
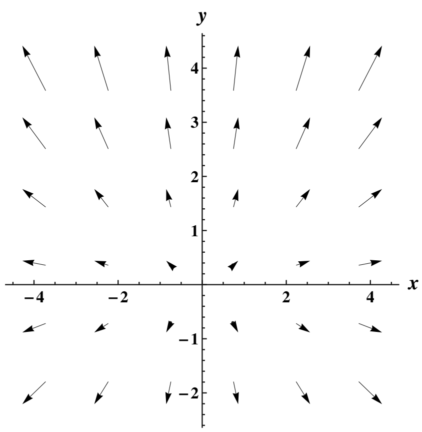


14.1.10



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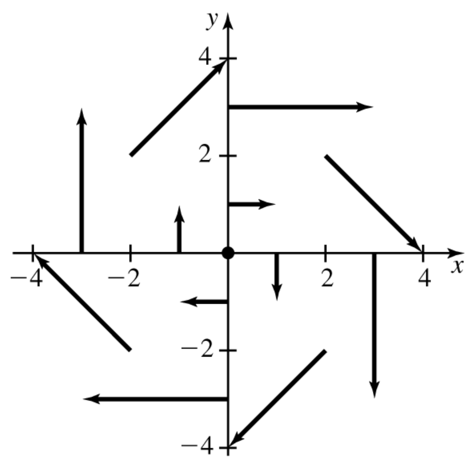

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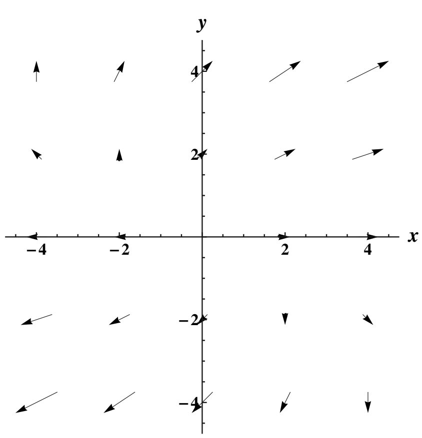

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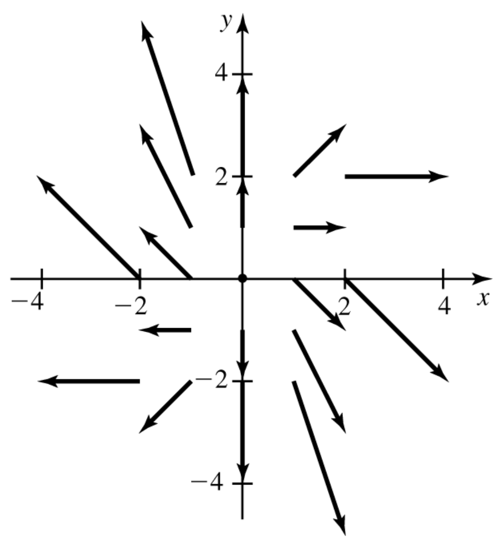
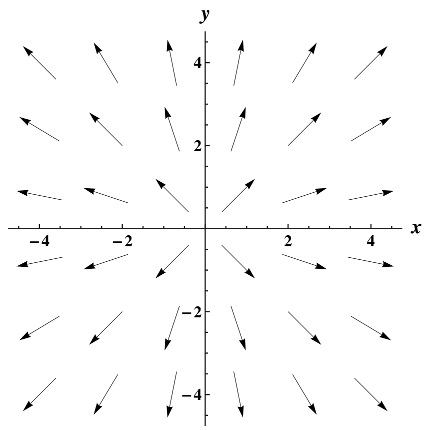
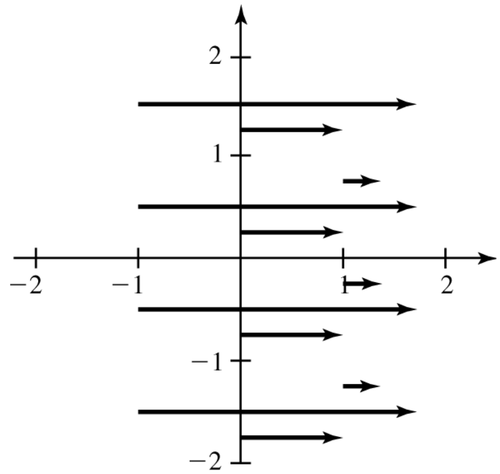


14.1.14

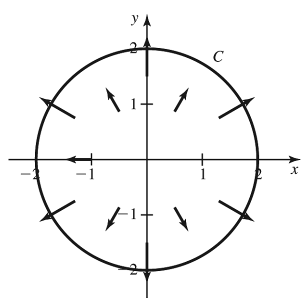


14.1.15

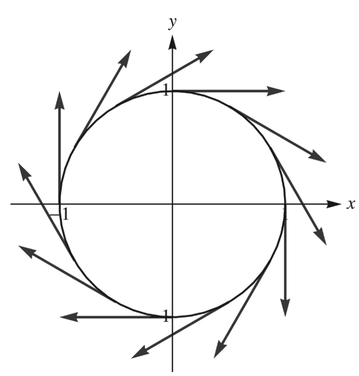


14.1.16 (a) corresponds to (D), since (D) has zero x component, and the y component increases as y does. (b) corresponds to (C), since the x component appears to be zero along the line $y = x$. (c) corresponds to (B) since the y component is zero on the x-axis and the x component is zero on the y axis. Finally, (d) corresponds to (A) since the x component is zero on the x-axis and the y component is zero on the y-axis.

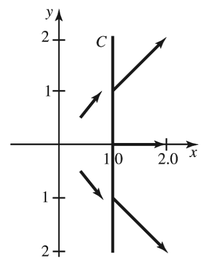
14.1.17 Here C is the circle of radius 2, so a vector tangent to the circle is $\langle -y, x \rangle$. So F is normal to C at (x, y) since $\langle -y, x \rangle \cdot \langle x, y \rangle = 0$.



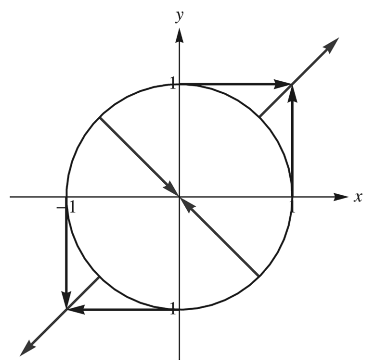
14.1.18 C is the circle of radius 1, so a vector tangent at (x, y) is $\langle -y, x \rangle$. Then F is a scalar multiple of the tangent vector, so it is tangent to C at all points.



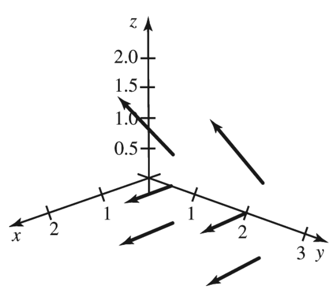
14.1.19 C is the vertical line at $x = 1$, so the tangent vector is a multiple of $\langle 0, y \rangle$ at all points. Then $\langle x, y \rangle$ is never a multiple of $(0, y)$ for any point on C (since $x = 1$ there), and $\langle x, y \rangle \cdot \langle 0, y \rangle = y^2$ is zero for $y = 0$, so that F is normal to C at $(1, 0)$.



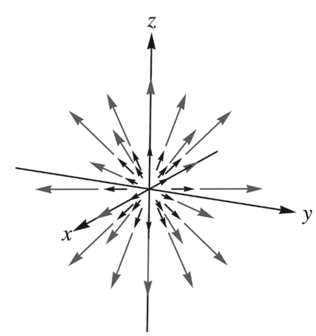
14.1.20 C is the circle of radius 1, so a vector tangent at (x, y) is $\langle -y, x\rangle$. F is a multiple of the tangent vector for $y = 0$ or for $x = 0$, so that F is tangent to C at $(\pm 1, 0)$ and at $(0, \pm 1)$. $\mathrm{F}\cdot\langle -y, x\rangle = x^2 - y^2$, so that F is normal to C at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$.



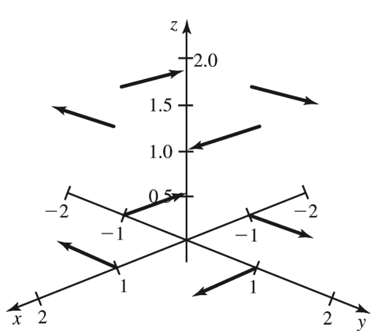
14.1.21



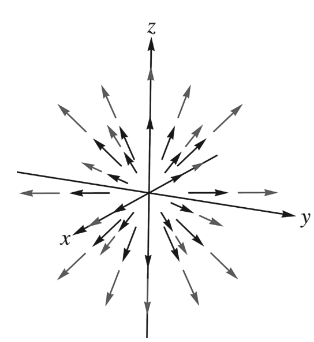
14.1.22



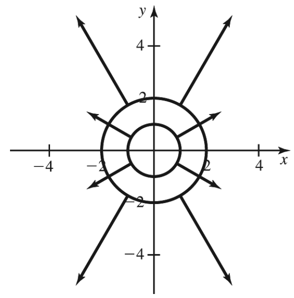
14.1.23



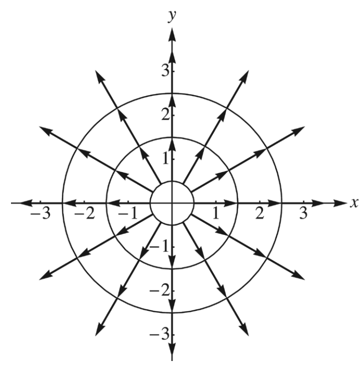
14.1.24



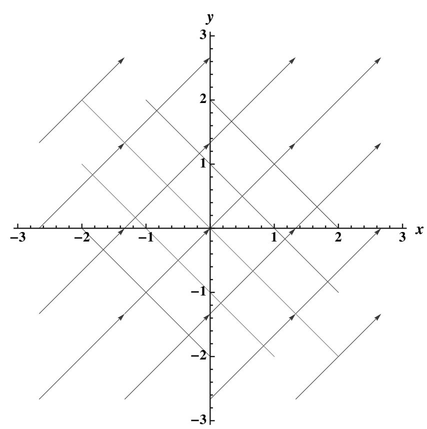
14.1.25 The gradient field is $\langle \varphi_x, \varphi_y \rangle = \langle 2x, 2y \rangle$.



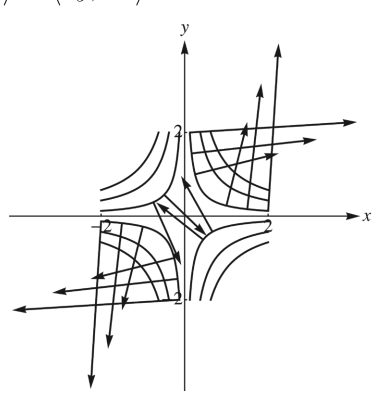
14.1.26 The gradient field is $\langle \varphi_x, \varphi_y \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$.



14.1.27 The gradient field is $\langle \varphi_x, \varphi_y \rangle = \langle 1, 1 \rangle$.



14.1.28 The gradient field is $\langle \varphi_x, \varphi_y \rangle = \langle 2y, 2x \rangle$.



14.1.29 $\nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \langle 2xy - y^2, x^2 - 2xy \rangle$.

14.1.30 $\nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \left\langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \right\rangle = \frac{1}{2\sqrt{xy}} \langle y, x \rangle$.

14.1.31 $\nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \langle 1/y, -x/y^2 \rangle$.

14.1.32 $\nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \frac{1}{x^2+y^2} \langle -y, x \rangle = \frac{1}{|\mathbf{r}|^2} \langle -y, x \rangle$.

14.1.33 $\nabla\varphi = \langle x, y, z \rangle = \mathbf{r}$.

14.1.34 $\nabla\varphi = \left\langle \frac{2x}{1+x^2+y^2+z^2}, \frac{2y}{1+x^2+y^2+z^2}, \frac{2z}{1+x^2+y^2+z^2} \right\rangle = \frac{2\,\mathbf{r}}{|\mathbf{r}|^2+1}$.

14.1.35 $\nabla\varphi = -\left(x^2 + y^2 + z^2\right)^{-3/2} \langle x,\, y,\, z \rangle = -\frac{\mathbf{r}}{|\mathbf{r}|^3}$.

14.1.36 $\nabla\varphi = \left\langle e^{-z} \cos(x+y), e^{-z} \cos(x+y), -e^{-z} \sin(x+y) \right\rangle$.

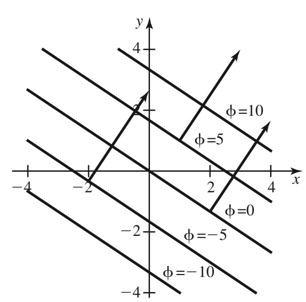
14.1.37

a. The gradient field is $\langle 2, 3 \rangle$.

b. The equipotential curve at $(1, 1)$ is $2x + 3y = 5$, which is a line of slope $-\frac{2}{3}$ so has a tangent vector at (x, y) parallel to $\langle 1, -\frac{2}{3} \rangle$. But $\langle 2, 3 \rangle \cdot \langle 1, -\frac{2}{3} \rangle = 0$, so the gradient field is normal to the equipotential line through $(1, 1)$.

c. The equipotential curve at any point is a line of slope $-\frac{2}{3}$ and thus has a tangent vector at (x, y) parallel to $\langle x, -\frac{2}{3}y \rangle$. The same argument as in part (b) shows that this is normal to the gradient field.

d.



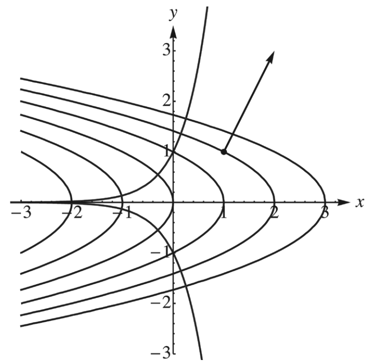
14.1.38

a. The gradient field is $\langle 1, 2y \rangle$.

b. At $(1, 1)$, the tangent vector is parallel to $\langle -\varphi_y(1, 1), \varphi_x(1, 1) \rangle = \langle -2, 1 \rangle$, which is normal to the gradient at $(1, 1)$ (which is $\langle 1, 2 \rangle$).

c. At (x, y), the tangent vector is parallel to $\langle -2y, 1 \rangle$, and $\langle 1, 2y \rangle \cdot \langle -2y, 1 \rangle = 0$, so the gradient is everywhere normal to the equipotential curves.

d.



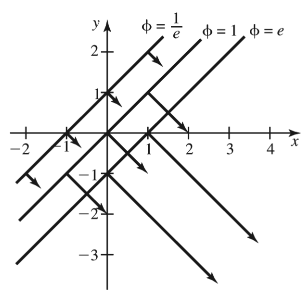
14.1.39

a. The gradient field is $\left\langle e^{x-y},\, -e^{x-y}\right\rangle$.

b. At (1, 1), the tangent vector is parallel to $\left\langle -\varphi_y(1,\,1),\, \varphi_x(1,\,1)\right\rangle = \left\langle 1,\, 1\right\rangle$, which is normal to the gradient $\left\langle 1,\, -1\right\rangle$ at (1, 1).

c. At $(x,\, y)$, the tangent vector is parallel to $\left\langle e^{x-y},\, e^{x-y}\right\rangle$, and $\left\langle e^{x-y},\, -e^{x-y}\right\rangle \cdot \left\langle e^{x-y},\, e^{x-y}\right\rangle = 0$, so the gradient is everywhere normal to the equipotential curves.

d.



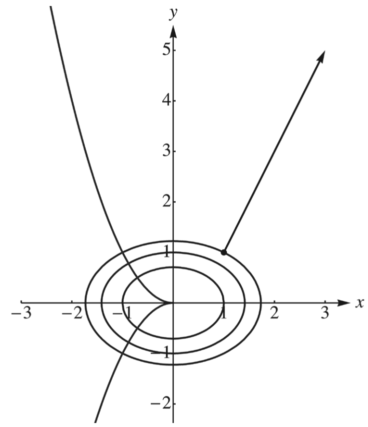
14.1.40

a. The gradient field is $\left\langle 2x,\, 4y\right\rangle$.

b. At (1, 1), the tangent vector is parallel to $\left\langle -\varphi_y(1,\,1),\, \varphi_x(1,\,1)\right\rangle = \left\langle -4,\, 2\right\rangle$, which is normal to the gradient $\left\langle 2,\, 4\right\rangle$ at (1, 1).

c. At $(x,\, y)$, the tangent vector is parallel to $\left\langle -4y,\, 2x\right\rangle$, which is normal to the gradient field.

d.



14.1.41

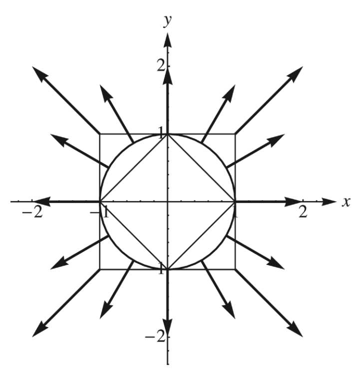
a. True. $(\varphi_1)_x = (\varphi_2)_x = 3x^2$, and $(\varphi_1)_y = (\varphi_2)_y = 1$.

b. False. It is constant in magnitude (magnitude 1) but not direction.

c. True. For example, it points outwards along the line $y = x$ but horizontally along the line $x = 0$.

14.1.42

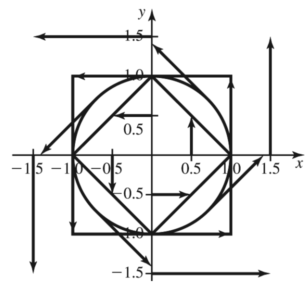
a. The magnitude of the vector field is $\sqrt{x^2 + y^2}$. Thus on C, the magnitude is 1 on the boundary of C and less than 1 elsewhere. On S, the magnitude is a maximum at the corners of S, where it is $\sqrt{2}$. Finally, on D, the magnitude is a maximum at the corners, where it is 1.



b. The vector field is everywhere directed outwards.

14.1.43

a. This is a rotational field with magnitude $\sqrt{x^2 + y^2}$ at (x, y). Thus the answer to this question is the same as for the previous question: on C, the magnitude is 1 on the boundary of C and less than 1 elsewhere. On S, the magnitude is a maximum at the corners of S, where it is $\sqrt{2}$. Finally, on D, the magnitude is a maximum at the corners, where it is 1.



b. For S and D the field is directed out of the region on line segments between any vertex and the midpoint of the boundary line when proceeding in a counterclockwise direction; on C the vector field is tangent to the boundary curve everywhere.

14.1.44 For example, $\mathbf{F} = \langle y, 0 \rangle$.

14.1.45 For example, $\mathbf{F} = \langle -y, x \rangle$ or $\mathbf{F} = \langle -1, 1 \rangle$.

14.1.46 For example, $\mathbf{F} = \langle -y, x \rangle$. The magnitude is $\sqrt{x^2+y^2}$.

14.1.47 For example, $\mathbf{F} = \frac{1}{\sqrt{x^2+y^2}}\langle x, y \rangle = \frac{\mathbf{r}}{|\mathbf{r}|}$, $\mathbf{F}(0,0) = \mathbf{0}$.

14.1.48

a. $V(x, y) = k\left(x^2+y^2\right)^{-1/2}$, so $\mathbf{E} = -\nabla V = -\langle V_x, V_y \rangle = k\left(x^2+y^2\right)^{-3/2}\langle x, y \rangle$.

b. From the above formula, the field is a varying multiple of $\langle x, y \rangle$, which is a radial field pointing away from the origin. The radial component of $\mathbf{E}$ is thus $|\mathbf{E}| = k\left(x^2+y^2\right)^{-3/2}\left|\langle x, y \rangle\right| = k\left(x^2+y^2\right)^{-3/2}\left(x^2+y^2\right)^{1/2} = \frac{k}{r^2}$.

c. The equipotential curves are curves of the form $\dfrac{k}{\sqrt{x^2+y^2}} = C$ so that $\sqrt{x^2+y^2} = \frac{k}{C}$ and the equipotential curves are circles. Thus the tangent vectors to the equipotential curves are proportional to $\left|\langle -y, x \rangle\right|$ and thus are normal to $\mathbf{E}$, which is proportional to $\left|\langle x, y \rangle\right|$.

14.1.49

a. $V_x = \frac{c\sqrt{x^2+y^2}}{r_0} \cdot \left(-\frac{1}{2}r_0\left(x^2+y^2\right)\right)$ and similarly, $V_y = -\frac{cy}{x^2+y^2}$, so that $\mathbf{E} = -\nabla V = \frac{c}{x^2+y^2}\langle x, y \rangle = \frac{c}{|\mathbf{r}|^2}\mathbf{r} = \frac{c}{|\mathbf{r}|} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}$.

b. From the above formula, the field is a varying multiple of $\langle x, y \rangle$, which is a radial field pointing away from the origin. The radial component of $\mathbf{E}$ is thus $|\mathbf{E}| = \dfrac{c}{x^2+y^2}\sqrt{x^2+y^2} = \dfrac{c}{\sqrt{x^2+y^2}}$.

c. The equipotential curves are curves of the form $c \ln\left(\frac{r_0}{\sqrt{x^2+y^2}}\right) = K$, so are solutions to $\frac{r_0}{\sqrt{x^2+y^2}} = e^{cK} = C$ and thus of the form $\sqrt{x^2+y^2} = K_0$ for some constant K_0. Hence the equipotential curves are circles, so have tangent vectors proportional to $\langle -y, x \rangle$; these are clearly normal to $\mathbf{E}$, which is proportional to $\langle x, y \rangle$.

14.1.50

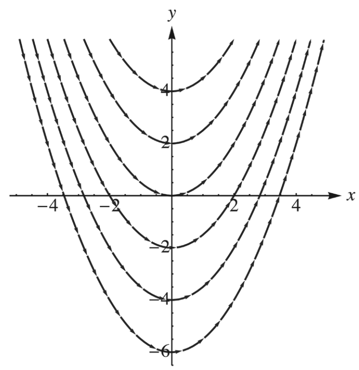
a. $U_x = \left(GMm\left(x^2+y^2+z^2\right)^{-1/2}\right)_x = -\frac{1}{2}GMm\left(x^2+y^2+z^2\right)^{-3/2}(2x) = -GMmx\left(x^2+y^2+z^2\right)^{-3/2}$ and similarly for U_y and U_z. Thus $\mathbf{F} = -\nabla U = GMm\left(x^2+y^2+z^2\right)^{-3/2}\langle x, y, z \rangle$.

b. From the above formula the field is a varying multiple of $\langle x, y, z\rangle$, which is a radial field pointing away from the origin. The radial component of $\mathbf{F}$ is thus $|\mathbf{F}| = GMm\left(x^2+y^2+z^2\right)^{-3/2}\sqrt{x^2+y^2+z^2} = \frac{GMm}{r^2}$.

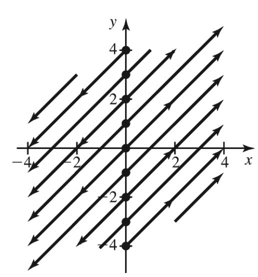
c. The equipotential surfaces are solutions to $GMm\sqrt{x^2+y^2+z^2} = K$ and so are spheres. The tangent plane at (x_0, y_0, z_0) is $U_x(x_0, y_0, z_0)(x - x_0) + U_y(x_0, y_0, z_0)(y - y_0) + U_z(x_0, y_0, z_0)(z - z_0)$ and so a normal to the plane is $\langle U_x, U_y, U_z\rangle$, which is proportional to $\mathbf{F}$.

14.1.51 The flow curve $y(x)$ of the vector field F at (x, y) is defined to be a continuous curve through (x, y) that is aligned with the vector field, i.e. whose tangent at (x, y) is given by $\mathbf{F}(x, y) = \langle f(x, y), g(x, y)\rangle$. The slope of the tangent line is then $\frac{g(x,y)}{f(x,y)}$, so this is $y'(x)$.

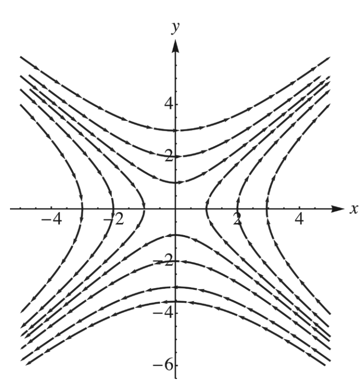
14.1.52 The streamlines satisfy $y'(x) = x$, so that $y(x) = \frac{1}{2}x^2 + C$.



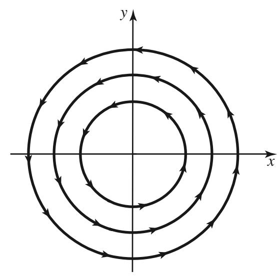
14.1.53 The streamlines satisfy $y'(x) = 1$, so that $y(x) = x + C$.



14.1.54 The streamlines satisfy $y'(x) = \frac{x}{y}$. But also $\frac{d}{dx}\left(y^2\right) = 2yy'(x)$, so that $y'(x) = \frac{\frac{d}{dx}\left(y^2\right)}{2y}$ and thus $\frac{d}{dx}\left(y^2\right) = 2x$. Thus $y^2 = x^2 + C$ and the streamlines are the hyperbolas $x^2 - y^2 = K$.



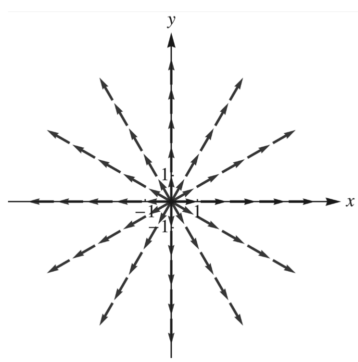
14.1.55 The streamlines satisfy $y'(x) = -\frac{x}{y}$. Because $\frac{d}{dx}\left(y^2\right) = 2yy'(x)$, we have $y'(x) = \frac{\frac{d}{dx}\left(y^2\right)}{2y}$ and thus $\frac{d}{dx}\left(y^2\right) = -2x$. Thus $y^2 = -x^2 + C$ and the streamlines are the circles $x^2 + y^2 = C$.



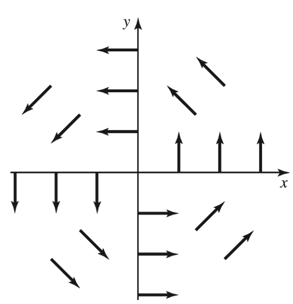
14.1.56 $\mathbf{u}_r$ is the unit vector forming an angle of θ with $\mathbf{i}$, so it is of unit length and proportional to $\langle \cos\theta, \sin\theta \rangle$. But $\cos^2(\theta) + \sin^2(\theta) = 1$, so this is in fact the unit vector. Thus $\mathbf{u}_r = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j}$. Similarly, $\mathbf{u}_\theta$ forms an angle of $\theta + \frac{\pi}{2}$ with $\mathbf{i}$, so it is the unit vector $\left\langle \cos\left(\theta + \frac{\pi}{2}\right), \sin\left(\theta + \frac{\pi}{2}\right)\right\rangle = \langle -\sin(\theta), \cos(\theta)\rangle$. The other two formulas can be found by solving these for $\mathbf{i}$, $\mathbf{j}$ as linear equations. For example, $\mathbf{u}_r = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j}$, $\mathbf{u}_\theta = -\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j}$. Multiply the first equation by $\sin(\theta)$ and the second by $\cos(\theta)$ and add to obtain $\mathbf{u}_r \sin(\theta) + \mathbf{u}_\theta \cos(\theta) = \sin^2(\theta)\,\mathbf{j} + \cos^2(\theta)\,\mathbf{j} = \mathbf{j}$.

14.1.57 For $\theta = 0$, $\mathbf{u}_r$ is coincident with $\mathbf{i}$, and $\mathbf{u}_\theta$ with $\mathbf{j}$. From the formula $\mathbf{u}_r = \cos(0)\,\mathbf{i} + \sin(0)\,\mathbf{j} = \mathbf{i}$, $\mathbf{u}_\theta = -\sin(0)\,\mathbf{i} + \cos(0)\,\mathbf{j} = \mathbf{j}$. For $\theta = \frac{\pi}{2}$, the picture implies that we should have $\mathbf{u}_r = \mathbf{j}$, $\mathbf{u}_\theta = -\mathbf{i}$. From the formulas, $\mathbf{u}_r = \cos\left(\frac{\pi}{2}\right)\mathbf{i} + \sin\left(\frac{\pi}{2}\right)\mathbf{j} = \mathbf{j}$, $\mathbf{u}_\theta = -\sin\left(\frac{\pi}{2}\right)\mathbf{i} + \cos\left(\frac{\pi}{2}\right)\mathbf{j} = -\mathbf{i}$. For $\theta = \pi$, $\mathbf{u}_r$ is coincident with $-\mathbf{i}$, and $\mathbf{u}_\theta$ with $-\mathbf{j}$. From the formula $\mathbf{u}_r = \cos(\pi)\,\mathbf{i} + \sin(\pi)\,\mathbf{j} = -\mathbf{i}$, $\mathbf{u}_\theta = -\sin(\pi)\,\mathbf{i} + \cos(\pi)\,\mathbf{j} = -\mathbf{j}$. For $\theta = \frac{3\pi}{2}$, the picture implies that we should have $\mathbf{u}_r = -\mathbf{j}$, $\mathbf{u}_\theta = \mathbf{i}$. From the formulas, $\mathbf{u}_r = \cos\left(\frac{3\pi}{2}\right)\mathbf{i} + \sin\left(\frac{3\pi}{2}\right)\mathbf{j} = -\mathbf{j}$, $\mathbf{u}_\theta = -\sin\left(\frac{3\pi}{2}\right)\mathbf{i} + \cos\left(\frac{3\pi}{2}\right)\mathbf{j} = \mathbf{i}$.

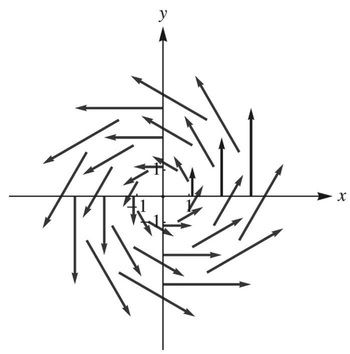
14.1.58 $\mathbf{F}(r, \theta) = \mathbf{u}_r = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j} = \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\mathbf{j} = \dfrac{1}{\sqrt{x^2+y^2}}(x\mathbf{i} + y\mathbf{j})$.



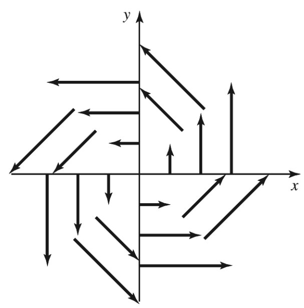
14.1.59 $\mathbf{F}(r, \theta) = \mathbf{u}_\theta = -\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j} = -\dfrac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{j} = \dfrac{1}{\sqrt{x^2+y^2}}(-y\mathbf{i} + x\mathbf{j})$.



14.1.60 $\mathbf{F}(r,\theta) = r\,\mathbf{u}_\theta = \sqrt{x^2+y^2}\,(-\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j}) = \sqrt{x^2+y^2}\frac{1}{\sqrt{x^2+y^2}}(-y\mathbf{i}+x\mathbf{j}) = \langle -y, x\rangle$.



14.1.61 $\mathbf{F}(x,y) = -y\,(\mathbf{u}_r\cos(\theta) - \mathbf{u}_\theta\sin(\theta)) + x\,(\mathbf{u}_r\sin(\theta) + \mathbf{u}_\theta\cos(\theta)) =$
$-r\sin(\theta)\,(\cos(\theta)\,\mathbf{u}_r - \sin(\theta)\,\mathbf{u}_\theta) + r\cos(\theta)\,(\sin(\theta)\cos(\theta)\,\mathbf{u}_r + \cos(\theta)\,\mathbf{u}_\theta) = r\left(\sin^2(\theta) + \cos^2(\theta)\right)\mathbf{u}_\theta = r\mathbf{u}_\theta.$



14.2 Line Integrals

14.2.1 A single-variable integral integrates along a segment while a line integral integrates along an arbitrary curve.

14.2.2 The integral is evaluated by evaluating the integral of$f \cdot |\mathbf{r}'(t)|$ where$\mathbf{r}'(t)$ expresses the velocity of the parameterization with respect to arc length.

14.2.3 $|\mathbf{r}'(t)| = \sqrt{(\mathbf{r}'_x)^2 + (\mathbf{r}'_y)^2} = \sqrt{1+4t^2}$

14.2.4 Choose a parameterization $\mathbf{r}(t)$ for C; then $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ becomes $\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt$.

14.2.5 Because T= $\langle x'(t), y'(t), z'(t)\rangle$, $\int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t))\,dt$ is simply a rewriting of the dot product.

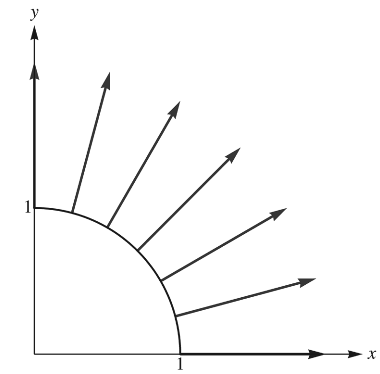
14.2.6 The circulation measures the degree to which the vector field is positively aligned with (has a positive dot product with) the curve C as C is traversed with a particular orientation.

14.2.7 Take the line integral of $\mathbf{F}\cdot\mathbf{T}$ along the curve using arc length parameterization.

14.2.8 The flux measures the degree to which the vector field is outwards normal to the curve C as C is traversed with a particular orientation.

14.2.9 Take the line integral of $\mathbf{F}\cdot\mathbf{n}$ along the curve using arc length parameterization.

14.2.10 One parameterization for the curve is $\langle \cos t, \sin t\rangle$, $0 \le t \le \frac{\pi}{2}$.



14.2.11 $\mathbf{r}(s)$ is an arc length parameterization, so we have $\int_C xy\,ds = \int_0^{2\pi} \cos(s)\sin(s)\,ds = 0$.

14.2.12 Choose the arc length parameterization $\mathbf{r}(s) = \langle \cos(s), \sin(s)\rangle$ then we have $\int_C (x+y)\,ds = \int_0^{2\pi} (\cos(s)+\sin(s))\,ds = (\sin(s)-\cos(s))\big|_0^{2\pi} = 0$.

14.2.13 With $\mathbf{r}(s) = \langle \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\rangle$, $|\mathbf{r}'(t)| = 1$, so that we have $\int_C \left(x^2 - 2y^2\right) ds = \int_0^4 \left(\frac{s^2}{2} - 2\frac{s^2}{2}\right) ds = -\int_0^4 \frac{s^2}{2}\,ds = -\frac{s^3}{6}\big|_0^4 = -\frac{32}{3}$.

14.2.14 With $\mathbf{r}(s) = \langle \frac{s}{\sqrt{2}}, 1 - \frac{s}{\sqrt{2}}\rangle$, $|\mathbf{r}'(t)| = 1$, so that $\int_C x^2 y\,ds = \int_0^4 \frac{s^2}{2}\left(1 - \frac{s}{\sqrt{2}}\right) ds = \left(\frac{s^3}{6} - \frac{s^4}{8\sqrt{2}}\right)\Big|_0^4 = \frac{32}{3} - 16\sqrt{2}$.

14.2.15

a. $\mathbf{r}(t) = \langle 4\cos t, 4\sin t\rangle$, $0 \le t \le 2\pi$.

b. $|\mathbf{r}'(t)| = \sqrt{(-4\sin t)^2 + (4\cos t)^2} = 4$.

c. $\int_C \left(x^2+y^2\right) ds = \int_0^{2\pi} 4\left(16\cos^2 t + 16\sin^2 t\right) dt = \int_0^{2\pi} 64\,dt = 128\pi$.

14.2.16

a. $\mathbf{r}(t) = \langle 5t, 5t\rangle$, $0 \le t \le 1$.

b. $|\mathbf{r}'(t)| = \sqrt{(5)^2 + (5)^2} = 5\sqrt{2}$.

c. $\int_C \left(x^2+y^2\right) ds = \int_0^1 50t^2 \cdot 5\sqrt{2}\,dt = 250\sqrt{2}\int_0^1 t^2\,dt = \frac{250\sqrt{2}}{3}$.

14.2.17

a. $\mathbf{r}(t) = \langle t, t\rangle$, $1 \le t \le 10$.

b. $|\mathbf{r}'(t)| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$.

c. $\int_C \frac{x}{x^2+y^2}\,ds = \int_1^{10} \frac{t}{t^2+t^2} \cdot \sqrt{2}\,dt = \frac{\sqrt{2}}{2}\int_1^{10} \frac{1}{t}\,dt = \frac{\sqrt{2}}{2}\ln 10$.

14.2.18

a. $\mathbf{r}(t) = \langle t, t^2\rangle$, $0 \le t \le 1$.

b. $|\mathbf{r}'(t)| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1+4t^2}$.

c. $\int_C (xy)^{1/3}\,ds = \int_0^1 \left(t^3\right)^{1/3} \sqrt{1+4t^2}\,dt = \int_0^1 t\sqrt{1+4t^2}\,dt = \frac{5\sqrt{5}-1}{12}$.

14.2.19

a. $\mathbf{r}(t) = \langle 2\cos t, 4\sin t\rangle$, $0 \le t \le \pi/2$.

b. $|\mathbf{r}'(t)| = \sqrt{4\sin^2 t + 16\cos^2 t} = \sqrt{4 + 12\cos^2 t} = 2\sqrt{1 + 3\cos^2 t}$.

c. $\int_C xy\,ds = \int_0^{\pi/2} 16\sin t\cos t\sqrt{1+3\cos^2 t}\,dt$. Let $u = (1 + 3\cos^2 t)\,dt$ so that $du = -6\cos t\sin t\,dt$. Substituting gives $\frac{8}{3}\int_1^4 u^{1/2}\,du = \frac{16}{9}\left(u^{3/2}\right)\Big|_1^4 = \frac{16}{9}(8-1) = \frac{112}{9}$.

14.2.20

a. $\mathrm{r}_1(t) = \langle t-1, t\rangle$, $\mathrm{r}_2(t) = \langle t, 1-t\rangle$, $0 \le t \le 1$.

b. $|\mathbf{r}_1'(t)| = \sqrt{2}$,$|\mathbf{r}_2'(t)| = \sqrt{2}$.

c. $\int_C (2x - 3y)\ ds = \int_0^1 (2(t-1) - 3t)\ \sqrt{2}\,dt + \int_0^1 (2t - 3(1-t))\ \sqrt{2}\,dt = \sqrt{2}\int_0^1 (4t-5)\ dt = -3\sqrt{2}$.

14.2.21 Let $\mathbf{r}(t) = \langle t+1, 4t+1\rangle$, $0 \le t \le 1$. Then $|\mathbf{r}'(t)| = \sqrt{17}$ and $\int_C (x+2y)\ ds =$ $\int_0^1 ((t+1) + 2(4t+1))\cdot\sqrt{17}\,dt = \sqrt{17}\int_0^1 (9t+3)\ dt = \frac{15}{2}\sqrt{17}$. The length of the line segment is $\sqrt{17}$, so the average value is $\frac{15}{2}$.

14.2.22 Let $\mathbf{r}(t) = \langle 9\cos t, 9\sin t\rangle$, $0 \le t \le 2\pi$. Then $|\mathbf{r}'(t)| = 9$, and

$$\int_C \left(x^2 + 4y^2\right)\ ds = \int_0^{2\pi} \left(81\cos^2 t + 4\cdot 81\sin^2 t\right) 9\,dt = 3645\pi.$$

The circumference of the circle is $2\pi\cdot 9 = 18\pi$, so the average value is $\frac{3645}{18} = \frac{405}{2}$.

14.2.23 Let $\mathbf{r}(t) = \langle t, t^{3/2}\rangle$ for $0 \le t \le 5$. Then $|\mathbf{r}'(t)| = \sqrt{1 + 9t/4}$ and

$$\int_C \sqrt{4 + 9y^{2/3}}ds = \int_0^5 \sqrt{4+9t}\sqrt{1+9t/4}\,dt = \frac{1}{2}\int_0^5 (4+9t)\,dt = \frac{1}{2}\left(4t + 9t^2/2\right)\Big|_0^5 = \frac{265}{4}.$$

The length of the curve is $\int_0^5 \sqrt{1+9x/4}\,dx = \frac{335}{27}$ so the average value is $\frac{1431}{268}$.

14.2.24 Let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. Then $|\mathbf{r}'(t)| = 1$ and $\int_C (x\,e^y)\ ds = \int_0^{2\pi} \cos t\,e^{\sin t}\,dt = 0$. Thus, the average value is 0.

14.2.25 $|\mathbf{r}'(t)| = \sqrt{4\sin^2 t + 0 + 4\cos^2 t} = 2$, so $\int_C (x+y+z)\ ds = 2\int_0^{2\pi} (2\cos(t) + 2\sin t)\ dt = 0$.

14.2.26 $|\mathbf{r}'(t)| = \sqrt{0 + 9\sin^2 t + 9\cos^2 t} = 3$, so $\int_C (x - y + 2z)\ ds = 3\int_0^{2\pi} (1 - 3\cos(t) + 6\sin t)\ dt = 6\pi$.

14.2.27 Let $\mathbf{r}(t) = \langle t, 2t, 3t\rangle$, $0 \le t \le 1$. Then $|\mathbf{r}'(t)| = \sqrt{14}$, so $\int_C (xyz)\ ds = \sqrt{14}\int_0^1 6t^3\,dt =$ $\sqrt{14}\left(\frac{3t^4}{2}\right)\Big|_0^1 = \frac{3}{2}\sqrt{14}$.

14.2.28 Let $\mathbf{r}(t) = \langle 2t+1, 2t+4, 2t+1\rangle$, $0 \le t \le 1$. Then $|\mathbf{r}'(t)| = \sqrt{12} = 2\sqrt{3}$, so $\int_C \frac{xy}{z}\,ds =$ $2\sqrt{3}\int_0^1 \frac{(2t+1)(2t+4)}{(2t+1)}\,dt = 2\sqrt{3}\int_0^1 (2t+4)\ dt = 10\sqrt{3}$.

14.2.29 $|\mathbf{r}'(t)| = \sqrt{10}$, so $\int_C\ (y - z)\ ds = \sqrt{10}\int_0^{2\pi} (3\sin t - t)\ dt = -2\sqrt{10}\,\pi^2$.

14.2.30 $|\mathbf{r}'(t)| = \sqrt{21}$, so $\int_C xe^{yz}\,ds = \sqrt{21}\int_1^2 t\,e^{-8t^2}\,dt = \frac{\sqrt{21}\left(e^{24}-1\right)}{16e^{32}}$.

14.2.31 The length of the curve is the line integral of 1 along the curve. Then $\mathbf{r}'(t)$ simplifies to $\langle 5\cos(t/4), -5\sin(t/4), \frac{1}{2}\rangle$, and then

$$|\mathbf{r}'(t)| = \sqrt{25 + \frac{1}{4}} = \frac{1}{2}\sqrt{101}$$

so that the arc length is $\frac{1}{2}\int_0^2 \sqrt{101}\, dt = \sqrt{101}$.

14.2.32 The length of the curve is the line integral of 1 along the curve.
$|\mathbf{r}'(t)| = \sqrt{900\cos^2(t) + 1600\cos^2 t + 2500\sin^2 t} = 50$ so that $\int_C 1\, ds = 50\int_0^{2\pi} dt = 100\pi$.

14.2.33 $\mathbf{r}'(t) = \langle 4, 2t\rangle$, and $\mathbf{F}\cdot\mathbf{r}'(t) = \langle 4t, t^2\rangle \cdot \langle 4, 2t\rangle = 16t + 2t^3$, so that

$$\int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_0^1 \mathbf{F}\cdot\mathbf{r}'(t)\, dt = \int_0^1 \left(16t + 2t^3\right)\, dt = \frac{17}{2}.$$

14.2.34 $\mathbf{r}'(t) = \langle -4\sin t, 4\cos t\rangle$, so $\mathbf{F}\cdot\mathbf{r}'(t) = \langle -4\sin t, 4\cos t\rangle \cdot \langle -4\sin t, 4\cos t\rangle = 16$. Then $\int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_0^\pi \mathbf{F}\cdot\mathbf{r}'(t)\, dt = \int_0^\pi 16\, dt = 16\pi$.

14.2.35 Let $\mathbf{r}(t) = \langle 4t+1, 9t+1\rangle$, $0 \le t \le 1$; then $\mathbf{r}'(t) = \langle 4, 9\rangle$ and $\mathbf{F}\cdot\mathbf{r}'(t) = \langle 9t+1, 4t+1\rangle\cdot\langle 4, 9\rangle = 72t+13$. Then $\int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_0^1 (72t+13)\, dt = 49$.

14.2.36 Let $\mathbf{r}(t) = \langle t, t^2\rangle$, $0 \le t \le 1$; then $\mathbf{r}'(t) = \langle 1, 2t\rangle$ and $\mathbf{F}\cdot\mathbf{r}'(t) = \langle -t^2, t\rangle \cdot \langle 1, 2t\rangle = t^2$. Then $\int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_0^1 t^2\, dt = \frac{1}{3}$.

14.2.37 $\mathbf{r}'(t) = \langle 2t, 6t\rangle$; then $\mathbf{F}\cdot\mathbf{r}'(t) = \left(10t^4\right)^{-3/2}\langle t^2, 3t^2\rangle \cdot \langle 2t, 6t\rangle = \frac{20t^3}{10\sqrt{10}\, t^6} = \frac{2}{\sqrt{10}}\, t^{-3}$ and $\int_C \mathbf{F}\cdot\mathbf{T}\, ds = \frac{2}{\sqrt{10}}\int_1^2 t^{-3}\, dt = \frac{3}{40}\sqrt{10}$.

14.2.38 $\mathbf{r}'(t) = \langle 1, 4\rangle$, so that $\mathbf{F}\cdot\mathbf{r}'(t) = \frac{1}{17t^2}\langle t, 4t\rangle \cdot \langle 1, 4\rangle = \frac{1}{t}$. Then $\int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_1^{10} \frac{1}{t}\, dt = \ln(10)$.

14.2.39 $\mathbf{r}_1(t) = \langle 1-t, 2-2t\rangle$, $\mathbf{r}_2(t) = \langle 0, 4t\rangle$, $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle -1, -2\rangle$, $\mathbf{r}_2'(t) = \langle 0, 4\rangle$, so that $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\, dt = \int_0^1 \langle 2-2t, 1-t\rangle \cdot \langle -1, -2\rangle\, dt + \int_0^1 \langle 4t, 0\rangle \cdot \langle 0, 4\rangle\, dt = \int_0^1 0\, dt = 0$.

14.2.40 $\mathbf{r}_1(t) = \langle t-1, 8t\rangle$, $\mathbf{r}_2(t) = \langle 2t, 8\rangle$, $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle 1, 8\rangle$, $\mathbf{r}_2'(t) = \langle 2, 0\rangle$, so that $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\, dt = \int_0^1 \langle t-1, 8t\rangle \cdot \langle 1, 8\rangle\, dt + \int_0^1 \langle 2t, 8\rangle \cdot \langle 2, 0\rangle\, dt = \int_0^1 (69t-1)\, dt = \frac{67}{2}$.

14.2.41 $\mathbf{r}(t) = \langle 2t, 8t^2\rangle$, $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 2, 16t\rangle$, so $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\, dt = \int_0^1 \langle 8t^2, 2t\rangle \cdot \langle 2, 16t\rangle\, dt = \int_0^1 48t^2\, dt = 16$.

14.2.42 $\mathbf{r}(t) = \langle 2t+1, 8-4t\rangle$, $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 2, -4\rangle$, so $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\, dt = \int_0^1 \langle 8-4t, -1-2t\rangle \cdot \langle 2, -4\rangle\, dt = \int_0^1 20\, dt = 20$.

14.2.43 $\mathbf{r}'(t) = \langle -4\sin t, 4\cos t, -4\sin t\rangle$. Thus we have $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\, dt =$
$\int_0^{2\pi} \langle 4\cos t, 4\sin t, 4\cos t\rangle \cdot \langle -4\sin t, 4\cos t, -4\sin t\rangle\, dt = \int_0^{2\pi} -16\sin t\cos(t)\, dt = 0$.

14.2.44 $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, \frac{1}{2\pi}\rangle$, so $\int_C \mathbf{F}\cdot\mathbf{r}'t\, dt = \int_0^{2\pi} \langle -2\sin t, 2\cos t, \frac{t}{2\pi}\rangle \cdot \langle -2\sin t, 2\cos t, \frac{1}{2\pi}\rangle\, dt = \int_0^{2\pi} \left(4 + \frac{t}{4\pi^2}\right)\, dt = \frac{1}{2} + 8\pi$.

14.2.45 Let $\mathbf{r}(t) = \langle t+1, t+1, t+1\rangle$, $0 \le t \le 9$, so that $\mathbf{r}'(t) = \langle 1, 1, 1\rangle$. Then $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\, dt = \int_0^9 \frac{1}{\left(3(t+1)^2\right)^{3/2}}\langle t+1, t+1, t+1\rangle \cdot \langle 1, 1, 1\rangle\, dt = \frac{1}{\sqrt{3}}\int_0^9 \frac{1}{(t+1)^3}(t+1)\, dt = \frac{1}{\sqrt{3}}\int_0^9 \frac{1}{(t+1)^2}\, dt = \frac{3}{10}\sqrt{3}$.

14.2.46 Let $\mathbf{r}(t) = \langle 7t+1, 3t+1, t+1\rangle$, $0 \le t \le 1$, so that $\mathbf{r}'(t) = \langle 7, 3, 1\rangle$. Then $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\, dt = \int_0^1 \frac{1}{(7t+1)^2+(3t+1)^2+(t+1)^2}\langle 7t+1, 3t+1, t+1\rangle \cdot \langle 7, 3, 1\rangle\, dt = \int_0^1 \frac{59t+11}{(7t+1)^2+(3t+1)^2+(t+1)^2}\, dt = \ln(2) + \frac{1}{2}\ln(7) = \ln\left(2\sqrt{7}\right)$.

14.2.47

a. Looking at the vector field, it appears that the vector field points counterclockwise just as much as it points clockwise at the boundary of the region, so we would expect the circulation to be zero.

b. $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, so $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}\langle 2(\sin t - \cos t), 2\cos t\rangle \cdot \langle -2\sin(t), 2\cos t\rangle\,dt = -4\int_0^{2\pi}\left(\cos^2 t + \sin t\cos t - \sin^2 t\right)\,dt = 0$.

14.2.48

a. Looking at the vector field, it appears that the vector field points counterclockwise just as much as it points clockwise at the boundary of the region, so we would expect the circulation to be zero.

b. Parameterize the boundary by four paths: $\mathbf{r}_1(t) = \langle 2, -2+4t\rangle$, $\mathbf{r}_2(t) = \langle 2-4t, 2\rangle$, $\mathbf{r}_3(t) = \langle -2, 2-4t\rangle$, $\mathbf{r}_4(t) = \langle -2+4t, -2\rangle$, all with $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle 0, 4\rangle$, $\mathbf{r}_2'(t) = \langle -4, 0\rangle$, $\mathbf{r}_3'(t) = \langle 0, -4\rangle$, $\mathbf{r}_4'(t) = \langle 4, 0\rangle$. Then $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}\cdot\langle 2, -2+4t\rangle\cdot\langle 0,4\rangle\,dt + \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}\cdot\langle 2-4t, 2\rangle\cdot\langle -4, 0\rangle\,dt + \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}\cdot\langle -2, 2-4t\rangle\cdot\langle 0,-4\rangle\,dt + \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}\cdot\langle -2+4t, -2\rangle\cdot\langle 4,0\rangle\,dt = \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}(-8+16t-8+16t-8+16t-8+16t)\,dt = \int_0^1 \frac{64t-32}{\sqrt{8-16t+16t^2}}\,dt = 0$.

14.2.49

a. Looking at the vector field, the inward-pointing vectors (in quadrants II and IV) appear larger than the outward-pointing vectors (in quadrants I and III). Thus we would expect the flux to be negative.

b. $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, so that $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi}(2(\sin t - \cos t)\cdot 2\cos t - 2\cos t\cdot(-2\sin t))\,dt = -4\int_0^{2\pi}\left(\cos^2 t - 2\sin t\cos t\right)\,dt = -4\pi$.

14.2.50

a. The vector field points outwards everywhere, so we would expect the flux to be positive.

b. Parameterize the boundary by four paths: $\mathbf{r}_1(t) = \langle 2, -2+4t\rangle$, $\mathbf{r}_2(t) = \langle 2-4t, 2\rangle$, $\mathbf{r}_3(t) = \langle -2, 2-4t\rangle$, $\mathbf{r}_4(t) = \langle -2+4t, -2\rangle$, all with $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle 0, 4\rangle$, $\mathbf{r}_2'(t) = \langle -4, 0\rangle$, $\mathbf{r}_3'(t) = \langle 0, -4\rangle$, $\mathbf{r}_4'(t) = \langle 4, 0\rangle$. Then $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}(2\cdot 4 - 0)\,dt + \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}(0+2\cdot 4)\,dt + \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}((-2)(-4)-0)\,dt + \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}(0+2\cdot 4)\,dt = \int_0^1 \frac{32}{\sqrt{8-16t+16t^2}}\,dt = 8\ln\left(2\sqrt{2}+3\right) \approx 14.1$.

14.2.51

a. True. This is the definition of an arc length parameterization.

b. True. Let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$. Then $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$, and $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}\langle \sin t, \cos t\rangle\cdot\langle -\sin t, \cos t\rangle\,dt = \int_0^{2\pi}\left(\cos^2 t - \sin^2 t\right)\,dt = 0$. $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi}(\sin t\cdot\cos t - \cos t(-\sin t))\,dt = 2\int_0^{2\pi}\sin t\cos t\,dt = 0$.

c. True.

d. True. It is the line integral $\int_C \mathbf{F}\cdot\mathbf{n}ds$.

14.2.52 The work done on either path is simply $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt$.

a. Here $\mathbf{r}(t) = \langle -t, 50\rangle$, for $-100 \le t \le 100$, so $\mathbf{r}'(t) = \langle -1, 0\rangle$ and $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_{-100}^{100} -150\,dt = 30000$

b. Here $\mathbf{r}(t) = \langle 100\cos t, 100\sin t\rangle$ and $\mathbf{r}'(t) = \langle -100\sin t, 100\cos t\rangle$, so $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{\pi} -15000\sin t\,dt = 30000$. The same amount of work is done along each path.

14.2.53

a. For the first path, the work done is $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_{-100}^{100} -141\,dt = 28200$, and for the second path $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{\pi}(-14100\sin t - 5000\cos(t))\,dt = 28200$,
so again they are equal.

b. For the first path, the work done is $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_{-100}^{100} -141\,dt = 28200$, while for the second path, $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{\pi}(-14100\sin t - 5000\cos t)\,dt = 28200$, so the amount of work is still equal along the two paths.

14.2.54

a. Let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \le t \le 2\pi$ so that $|\mathbf{r}'(t)| = 1$. Then $\int_C f\,ds = \int_0^{2\pi}(\cos t + 2\sin(t))\,dt = 0$.

b. Let $\mathbf{r}(t) = \langle \cos t, -\sin t\rangle$ for $0 \le t \le 2\pi$ so that $|\mathbf{r}'(t)| = 1$. Note that this parameterization traces out the unit circle, but clockwise. Then $\int_C f\,ds = \int_0^{2\pi}(\cos t - 2\sin(t))\,dt = 0$.

c. The two integrals are equal.

14.2.55

a. Let $\mathbf{r}(t) = \langle t, t^2\rangle$ for $0 \le t \le 1$ so that $|\mathbf{r}'(t)| = \sqrt{4t^2+1}$, and $\int_C f\,ds = \int_0^1 t\sqrt{4t^2+1}\,dt = \frac{5\sqrt{5}-1}{12}$.

b. $\mathbf{r}(t) = \langle 1-t, (1-t)^2\rangle$ for $0 \le t \le 1$. Then $|\mathbf{r}'(t)| = \sqrt{(-1)^2 + (-2(1-t))^2} = \sqrt{4t^2-8t+5}$ and $\int_C f\,ds = \int_0^1 (1-t)\sqrt{4t^2-8t+5}\,dt = \frac{5\sqrt{5}-1}{12}$.

c. The two integrals are equal.

14.2.56 Letting $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ and $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$ as usual, we have $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}\left(-b\sin^2 t + c\cos^2 t\right)dt = \pi(c-b)$, so that the circulation is zero only for $b = c$.

14.2.57 Let $\mathbf{r}(t) = \langle r\cos t, r\sin t\rangle$ for a circle of radius r, so that $\mathbf{r}'(t) = \langle -r\sin t, r\cos(t)\rangle$. Then $\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}((ar\cos t + br\sin t)(-r\sin t) + (cr\cos t + dr\sin t)(r\cos t))\,dt = r^2\int_0^{2\pi}\left(-a\sin t\cos t - b\sin^2 t + c\cos^2 t + d\sin t\cos t\right)dt = \pi(c-b)\,r^2$, so that the circulation is zero provided $b = c$.

14.2.58 Let $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$; then the flux is $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi}(a\cos t\cos t - d\sin t(-\sin t))\,dt = (a+d)\pi$, so the flux is zero if $a = -d$.

14.2.59 Let $\mathbf{r}(t) = \langle r\cos t, r\sin t\rangle$ for a circle of radius r; then the flux is $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi}((ar\cos t + br\sin t)(r\cos t) - (cr\cos t + dr\sin t)(-r\sin t))\,dt = r^2\int_0^{2\pi}\left(a\cos^2 t + b\sin t\cos t + c\sin t\cos t + d\sin^2 t\right)dt = r^2(a+d)\pi$, so the flux is zero provided that $a = -d$.

14.2.60 Parameterize C_1 by $\mathbf{r}(t) = \langle 1-t, t\rangle$ for $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle -1, 1\rangle$, and $\int_{C_1}\mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1((-t)(-1) + (1-t)(1))\,dt = \int_0^1 1\,dt = 1$.

Parameterize C_2 by $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \le t \le \frac{\pi}{2}$. Then $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$, and $\int_{C_2}\mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{\pi/2}((-\sin t)(-\sin t) + (\cos t)(\cos t))\,dt = \int_0^{\pi/2} 1\,dt = \frac{\pi}{2}$.

Finally, parameterize C_3 by the two paths $\mathrm{r}_1(t) = \langle 1-t, 0\rangle$ and $\mathrm{r}_2(t) = \langle 0, t\rangle$ for $0 \le t \le 1$, so that $\mathrm{r}_1'(t) = \langle -1, 0\rangle$ and $\mathrm{r}_2'(t) = \langle 0, 1\rangle$. Then $\int_{C_3}\mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1((0)(-1) + (1-t)(0))\,dt + \int_0^1((-t)(0) + (0)(1))\,dt = 0$. None of the three path integrals are equal.

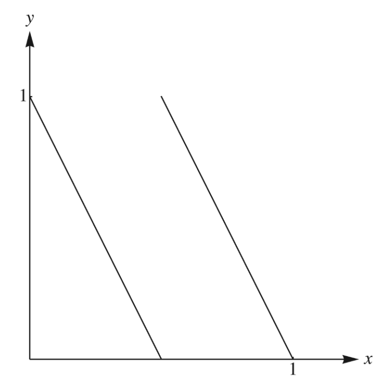
14.2.61 Using the same parameterizations as for the previous problem, we have $\int_{C_1}\mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1((t)(-1) + (1-t)(1))\,dt = \int_0^1(1-2t)\,dt = 0$, $\int_{C_2}\mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{\pi/2}((\sin t)(-\sin t) + (\cos t)(\cos t))\,dt = \int_0^{\pi/2}\left(\cos^2 t - \sin^2 t\right)dt = 0$, $\int_{C_3}\mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1((0)(-1) + (1-t)(0))\,dt + \int_0^1(t\cdot 0 + 0\cdot 1)\,dt = 0$. All three are equal to zero.

14.2.62 $\mathbf{r}'(\theta) = \langle -\sin\theta, \cos\theta\rangle$, so that $|\mathbf{r}'(\theta)| = 1$ and $\int_C \rho\, ds = \int_0^\pi \left(\frac{2\theta}{\pi} + 1\right) d\theta = 2\pi$.

14.2.63 Parameterize C by $\mathbf{r}(t) = \langle t, 2t^2\rangle$ for $0 \le t \le 3$. Then $|\mathbf{r}'(t)| = \sqrt{1+16t^2}$ and $\int_C \rho\, ds = \int_0^3 \left(1+2t^3\right)\sqrt{1+16t^2}\, dt \approx 409.5$.

14.2.64

a.



b. The gradient is $-50\mathbf{i} - 25\mathbf{j}$.

c. $\mathbf{F} = 50\mathbf{i} + 25\mathbf{j}$.

d. Parameterize the boundary C by $\mathbf{r}(t) = \langle 1, t\rangle$ for $0 \le t \le 1$. Then $\mathbf{r}'(t) = \langle 0, 1\rangle$ and $\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_0^1 (50\cdot 1 - 25\cdot 0)\, dt = 50$.

e. Parameterize the boundary C by $\mathbf{r}(t) = \langle 1-t, 1\rangle$ for $0 \le t \le 1$ (note: we do not use $\langle t, 1\rangle$ because we need counterclockwise orientation). Then $\mathbf{r}'(t) = \langle -1, 0\rangle$ and $\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_0^1 (50\cdot 0 - 25\cdot(-1))\, dt = 25$.

14.2.65 $\mathbf{r}'(t) = \langle 1,1,1\rangle$ and $|\mathbf{r}(t)| = t\sqrt{3}$, so the work is $\int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_1^a \frac{3t}{(t\sqrt{3})^p} dt = 3^{1-p/2}\int_1^a t^{1-p}\, dt$

a. For $p = 2$, we have $\int_1^a \frac{1}{t} dt = \ln a$.

b. The work is not finite.

c. For $p = 4$, we have $\frac{1}{3}\int_1^a t^{-3}\, dt = -\frac{1}{6t^2}\Big|_1^a = \frac{1}{6} - \frac{1}{6a^2}$.

d. As $a \to \infty$, the work approaches $\frac{1}{6}$.

e. For the general $p > 1$, the analysis above shows that the integral is (for $p \neq 2$) $3^{1-p/2}\frac{1}{2-p}t^{2-p}\Big|_1^a = \frac{3^{1-p/2}}{2-p}\left(-1 + a^{2-p}\right)$ while for $p = 2$ the integral is (from part (a)) $\ln a$.

e. This approaches a limit only for $p > 2$ (when $2 - p < 0$). This limit is $\frac{3^{1-p/2}}{p-2}$.

14.2.66

a. $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, so the flux along the quarter circle is $\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_0^{\pi/2} ((2\sin t)(2\cos t) - (2\cos t)(-2\sin t))\, dt = \int_0^{\pi/2} 8\sin t\cos t\, dt = 4$.

b. With $\mathbf{r}'(t)$ as above, the flux is $\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_{\pi/2}^{\pi} (8\sin t\, \cos t)\, dt = -4$.

c. Both the normal vectors and the vector field F in the third quadrant are the negatives of their values in the first quadrant, so their dot product is the same. Thus the flux is identical.

d. Both the normal vectors and the vector field F in the fourth quadrant are the negatives of their values in the second quadrant, so their dot product is the same. Thus the flux is identical.

e. The total flux is $4 - 4 + 4 - 4 = 0$.

14.2.67 We use four line segment parameterizations for the rectangle, all for $0 \le t \le 1$: $\mathbf{r}_1(t) = \langle at, 0\rangle$, so $\mathbf{r}_1'(t) = \langle a, 0\rangle$. $\mathbf{r}_2(t) = \langle a, bt\rangle$, so $\mathbf{r}_2'(t) = \langle 0, b\rangle$. $\mathbf{r}_3(t) = \langle a - at, b\rangle$, so $\mathbf{r}_3'(t) = \langle -a, 0\rangle$. $\mathbf{r}_4(t) = \langle 0, b - bt\rangle$, so $\mathbf{r}_4'(t) = \langle 0, -b\rangle$. Then $\int_C x\,dy = \int_0^1 (at \cdot 0 + a \cdot b + (a - at) \cdot 0 + 0 \cdot (-b))\,dt = \int_0^1 ab\,dt = ab$.

14.2.68 Parameterize C by $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$, so that $\mathbf{r}'(t) = \langle -a\sin t, a\cos t\rangle$. Then $-\int_C y\,dx = -\int_0^{2\pi} (a\sin t \cdot (-a\sin t))\,dt = a^2 \int_0^{2\pi} \sin^2 t\,dt = \pi a^2$.

14.3 Conservative Vector Fields

14.3.1 A simple curve has no self-intersections; the initial and terminal points of a closed curve are identical.

14.3.2 A region is connected, roughly speaking, if it consists of one piece. A simply connected region has the property that every closed loop can be contracted to a point.

14.3.3 If $\mathbf{F} = \langle f, g\rangle$ is a vector field in $\mathbb{R}^2$ and if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, then $\mathbf{F}$ is conservative.

14.3.4 If $\mathbf{F} = \langle f, g, h\rangle$ is a vector field in $\mathbb{R}^3$ and if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}$ and $\frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$, then $\mathbf{F}$ is conservative.

14.3.5 Integrate f with respect to x to get an answer where the "constant" is actually a function of y. Take the partial with respect to y and equate with g to compute the constant.

14.3.6 The integral is $\varphi(B) - \varphi(A)$ where $\nabla\varphi =$F.

14.3.7 The integral is zero.

14.3.8

- There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$.
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points A and B in R and all smooth oriented curves C from A to B (path independence).
- $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple smooth closed oriented curves C in R.

14.3.9 Yes, because $1_y = 1_x = 0$.

14.3.10 Yes, because $x_y = y_x = 0$.

14.3.11 Yes, because $-y_y = -x_x = -1$.

14.3.12 No. $-y_y = -1$, but $(x + y)_x = 1$.

14.3.13 Yes. $\frac{\partial}{\partial y} e^{-x}\cos(y) = -e^{-x}\sin(y) = \frac{\partial}{\partial x} e^{-x}\sin(y)$.

14.3.14 Yes. $\frac{\partial}{\partial y}\left(2x^3 + xy^2\right) = 2xy$, and $\frac{\partial}{\partial x}\left(2y^3 + x^2 y\right) = 2xy$.

14.3.15 Yes. $\frac{\partial}{\partial y}(x) = \frac{\partial}{\partial x}(y) = 0$. A potential function is $\frac{x^2+y^2}{2}$.

14.3.16 Yes. $\frac{\partial}{\partial y}(-y) = \frac{\partial}{\partial x}(-x) = -1$. A potential function is $-xy$.

14.3.17 No. $\frac{\partial}{\partial y}\left(x^3 - xy\right) = -x$, and $\frac{\partial}{\partial x}\left(\frac{x^2}{2} + y\right) = x$.

14.3.18 Yes. $\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-2xy}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)$. Integrating $\frac{x}{x^2+y^2}$ with respect to x, we see that a potential function is $\frac{1}{2}\ln\left(x^2+y^2\right)$.

14.3.19 Yes. $\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{-xy}{(x^2+y^2)^{3/2}} = \frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$. Integrating $\frac{x}{\sqrt{x^2+y^2}}$ with respect to x, we obtain a potential function of $\sqrt{x^2+y^2}$.

14.3.20 Yes. $\frac{\partial}{\partial y}(y) = \frac{\partial}{\partial x}(x) = 1$, $\frac{\partial}{\partial z}(y) = \frac{\partial}{\partial x}(1) = 0$ and $\frac{\partial}{\partial z}(y) = \frac{\partial}{\partial y}(1) = 0$. A potential function is $xy+z$. To see this, integrate y with respect to x to get $xy + f(y,z)$; differentiating with respect to y gives $x + f_y(y,z) = x$, so that $f(y,z)$ is actually a function of z, say $g(z)$. But then differentiating $xy + g(z)$ with respect to z gives $g'(z) = 1$, so $g(z) = z$.

14.3.21 Yes, because the mixed partials are pairwise equal. A potential function is $xz + y$.

14.3.22 Yes, because the mixed partials are pairwise equal. A potential function is xyz.

14.3.23 Yes, because the mixed partials are pairwise equal. To find the potential function, integrate $y+z$ with respect to x to get $x(y+z) + f(y,z)$; differentiating with respect to y gives $x+z = x + f_y(y,z)$ so that $f_y(y,z) = z$ and$f(y,z) = yz$. Thus a potential function is $xy + yz + xz$.

14.3.24 Yes, because the mixed partials are pairwise equal. As in problem 18, a potential function is $\frac{1}{2}\ln\left(x^2+y^2+z^2\right)$.

14.3.25 Yes, because the mixed partials are pairwise equal. As in problem 19, a potential function is $\sqrt{x^2+y^2+z^2}$.

14.3.26 Yes, because the mixed partials are pairwise equal (and zero). A potential function is $\frac{1}{4}x^4+y^2-\frac{1}{4}z^4$.

14.3.27

a. $\nabla\varphi = \langle y, x\rangle$, so $\int_C \nabla\varphi\cdot\mathbf{r}'(t)\,dt = \int_0^\pi \langle \sin t, \cos t\rangle\cdot\langle -\sin t, \cos t\rangle\,dt = \int_0^\pi \cos(2t)\,dt = 0$.

b. Since $\nabla\varphi$ is obviously conservative, the integral is simply $\varphi(\cos(\pi), \sin(\pi)) - \varphi(\cos(0), \sin(0)) = 0$.

14.3.28

a. $\nabla\varphi = \langle x, y\rangle$, so $\int_C \nabla\varphi\cdot\mathbf{r}'(t)\,dt = \int_0^\pi \langle \sin t, \cos t\rangle\cdot\langle \cos t, -\sin t\rangle\,dt = \int_0^\pi 0\,dt = 0$.

b. The integral is $\varphi(\sin(\pi), \cos(\pi)) - \varphi(\sin(0), \cos(0)) = \frac{1}{2} - \frac{1}{2} = 0$.

14.3.29

a. $\nabla\varphi = \langle 1, 3\rangle$, so $\int_C \nabla\varphi\cdot\mathbf{r}'(t)\,dt = \int_0^2 \langle 1, 3\rangle\cdot\langle -1, 1\rangle\,dt = \int_0^2 2\,dt = 4$.

b. The integral is $\varphi(0,2) - \varphi(2,0) = 6 - 2 = 4$. (Note that$\varphi(0,2)$ is φ evaluated at the point where $t = 2$).

14.3.30

a. $\nabla\varphi = \langle 1,1,1\rangle$, so $\int_C \nabla\varphi\cdot\mathbf{r}'(t)\,dt = \int_0^\pi \langle 1,1,1\rangle\cdot\langle \cos t, -\sin(t), \frac{1}{\pi}\rangle\,dt = \int_0^\pi \left(\cos t - \sin t + \frac{1}{\pi}\right)\,dt = -1$.

b. The integral is $\varphi(0,-1,1) - \varphi(0,1,0) = 0 - 1 = -1$.

14.3.31

a. $\nabla\varphi = \langle x, y, z\rangle$, so $\int_C \nabla\varphi \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} \langle \cos t, \sin t, \frac{t}{\pi}\rangle \cdot \langle -\sin t, \cos t, \frac{1}{\pi}\rangle\, dt = \int_0^{2\pi} \left(\frac{t}{\pi^2}\right)\, dt = 2$.

b. The integral is $\varphi(1,0,2) - \varphi(1,0,0) = \frac{5}{2} - \frac{1}{2} = 2$.

14.3.32

a. $\nabla\varphi = \langle y+z, x+z, x+y\rangle$, so $\int_C \nabla\varphi \cdot \mathbf{r}'(t)\, dt = \int_0^4 \langle 5t, 4t, 3t\rangle \cdot \langle 1,2,3\rangle\, dt = \int_0^4 22t\, dt = 176$.

b. The integral is $\varphi(4,8,12) - \varphi(0,0,0) = 176 - 0 = 176$.

14.3.33 Parameterize C by $\mathbf{r}(t) = \langle 4\cos t, 4\sin t\rangle$, $0 \le t \le 2\pi$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 4\cos t, 4\sin t\rangle \cdot \langle -4\sin t, 4\cos t\rangle\, dt = \int_0^{2\pi} 0\, dt = 0$.

14.3.34 Parameterize C by $\mathbf{r}(t) = \langle 8\cos t, 8\sin t\rangle$, $0 \le t \le 2\pi$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 8\sin t, 8\cos t\rangle \cdot \langle -8\sin t, 8\cos t\rangle\, dt = 8\int_0^{2\pi} \left(\cos^2 t - \sin^2(t)\right)\, dt = 0$.

14.3.35 Parameterize C by three paths, all for $0 \le t \le 1$: $\mathbf{r}_1(t) = \langle t, t-1\rangle$, so $\mathbf{r}_1'(t) = \langle 1,1\rangle$. $\mathbf{r}_2(t) = \langle 1-t, t\rangle$, so $\mathbf{r}_2'(t) = \langle -1, 1\rangle$. $\mathbf{r}_3(t) = \langle 0, 1-2t\rangle$, so $\mathbf{r}_3'(t) = \langle 0, -2\rangle$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \langle t, t-1\rangle \cdot \langle 1,1\rangle\, dt + \int_0^1 \langle 1-t, t\rangle \cdot \langle -1,1\rangle\, dt + \int_0^1 \langle 0, 1-2t\rangle \cdot \langle 0,-2\rangle\, dt = \int_0^1 (t + (t-1) + (t-1) + t - 2(1-2t))\, dt = \int_0^1 (8t-4)\, dt = 0$.

14.3.36 Parameterize C by $\mathbf{r}(t) = \langle 3\cos t, 3\sin t\rangle$, $0 \le t \le 2\pi$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 3\sin t, -3\cos t\rangle \cdot \langle -3\sin t, 3\cos t\rangle\, dt = -9\int_0^{2\pi} 1\, dt = -18\pi$. This integral is not zero because the vector field $\langle y, -x\rangle$ is not conservative: $\frac{\partial}{\partial y}(y) = 1$, while $\frac{\partial}{\partial x}(-x) = -1$.

14.3.37 Using the given parameterization,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle \cos t, \sin t, 2\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt = \int_0^{2\pi} 0\, dt = 0.$$

14.3.38 Using the given parameterization,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle \sin t - \cos t, 0, \cos t - \sin t\rangle \cdot \langle -\sin t, \cos t, -\sin t\rangle\, dt = \int_0^{2\pi} 0\, dt = 0.$$

14.3.39

a. False. Parametrize the curve by $x = 4\cos t + 1$, $y = 4\sin t$, for $0 \le t \le 2\pi$. Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle -4\sin t, 4\cos t + 1\rangle \cdot \langle -4\sin t, 4\cos t\rangle\, dt = \int_0^{2\pi} (16\sin^2 t + 16\cos^2 t + 4\cos t)\, dt = 32\pi \ne 0$.

b. True. This is because $\mathbf{F}$ is conservative.

c. True. If the vector field is $\langle a, b\rangle$, then a potential function is $ax + by$.

d. True. This is because $\frac{\partial}{\partial y} f(x) = \frac{\partial}{\partial x} g(y) = 0$.

e. True. This follows from the definitions.

14.3.40 Write $\varphi(x,y,z) = 1 + x^2yz$. Then using the Fundamental Theorem, this integral is equal to $\varphi(\cos(8\pi), \sin(8\pi), 4\pi) - \varphi(\cos(0), \sin(0), 0) = \varphi(1,0,4\pi) - \varphi(1,0,0) = 1 - 1 = 0$.

14.3.41 Write $\varphi(x,y) = e^{-x}\cos(y)$. Then using the Fundamental Theorem, this integral is equal to $\varphi(\ln 2, 2\pi) - \varphi(0,0) = e^{-\ln 2}\cos(2\pi) - e^0\cos(0) = \frac{1}{2} - 1 = -\frac{1}{2}$.

14.3.42 If $x = x(t)$, $y = y(t)$, then $\oint_C e^{-x}(\cos y\, dx + \sin y\, dy) = \oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle e^{-x}\cos y, e^{-x}\sin y\rangle$. This is a conservative vector field, because $\frac{\partial}{\partial y}(e^{-x}\cos y) = -e^{-x}\sin y = \frac{\partial}{\partial x}(e^{-x}\sin y)$, so that the integral around the closed curve is zero.

14.3.43 **F** is a conservative vector field; a potential function can be found by integrating x^2 with respect to y to obtain $x^2y+f(x,z)$; differentiate with respect to z to get $f_z(x,z)=2xz$, so that $f(x,z)=xz^2+g(x)$. Thus the potential function is $x^2y+xz^2+g(x)$; differentiating with respect to x gives $2xy+z^2+g_x(x)=2xy+z^2$, so that $g_x(x)=0$ and we may take $g(x)=0$. So if $\varphi=x^2y+xz^2$, then $\nabla\varphi=$F and thus the integral is zero because both sine and cosine, and thus φ, have the same values at the two endpoints of C.

14.3.44 $\oint_C ds$ is the length of the curve C, which is 2π. The other two integrals are zero, because they are the same as integrating the conservative vector fields $\langle 1,0,0\rangle$, and $\langle 0,1,0\rangle$ respectively around a closed curve.

14.3.45 This is a conservative vector field with potential function $\varphi(x,y)=\frac{1}{2}x^2+2y$, so the work is $\varphi(2,4)-\varphi(0,0)=10$.

14.3.46 This is a conservative vector field with potential function $\varphi(x,y)=\frac{1}{2}\left(x^2+y^2\right)$, so the work is $\varphi(3,-6)-\varphi(1,1)=\frac{45}{2}-1=\frac{43}{2}$.

14.3.47 This is a conservative vector field with potential function $\varphi(x,y,z)=\frac{1}{2}\left(x^2+y^2+z^2\right)$, so the work is $\varphi(2,4,6)-\varphi(1,2,1)=28-3=25$.

14.3.48 This is not a conservative vector field, because for example $\frac{\partial}{\partial z}\left(e^{x+y}\right)=0$, while $\frac{\partial}{\partial x}\left(ze^{x+y}\right)=ze^{x+y}$. Parameterize C by $\langle -t,2t,-4t\rangle$, $0\le t\le 1$; then $\int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^1\langle e^t,e^t,-4te^t\rangle\cdot\langle -1,2,-4\rangle\,dt=\int_0^1\left(e^t+16te^t\right)dt=e+15$.

14.3.49 For C_1, the vector field points "against" the curve for most of its length, and with larger magnitude, so the integral is negative. For C_2, the vector field points with the curve for its entire length, so the integral is positive.

14.3.50 For C_1, the vector field points against the curve for its entire length; for C_2, the vector field points with the curve, so the integral over C_1 is negative and the integral over C_2 is positive.

14.3.51 $\mathbf{F}=\langle a,b,c\rangle$ is a conservative force field with potential function $\varphi(x,y,z)=ax+by+cz$, so the work done is $\varphi(B)-\varphi(A)=\mathbf{F}\cdot B-\mathbf{F}\cdot A=\mathbf{F}\cdot(B-A)=\mathbf{F}\cdot\overrightarrow{AB}$.

14.3.52

a. The acceleration is the time derivative of velocity, so Newton's second law says that $m\frac{d\mathbf{v}}{dt}=m\mathbf{a}=\mathbf{F}=-\nabla\varphi$

b. By the product rule, $\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v})=\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}+\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}=2\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}$, and the desired equation follows.

c. Multiplying part (a) by $\mathbf{v}=\mathbf{r}'$ and using (b), we have (letting C be a path from A to B) $m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}=-\nabla\varphi\cdot\mathbf{r}'=\frac{1}{2}m\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v})$.

   Thus, $\int_C -\nabla\varphi\cdot\mathbf{r}'=\frac{1}{2}m\int_A^B\frac{d}{dt}\left(\mathbf{v}(t)\cdot\mathbf{v}(t)\right)dt$, so $\varphi(A)-\varphi(B)=\frac{1}{2}m\left(|\mathbf{v}|\right)^2\Big|_A^B$. The last equality follows because the integrand is a conservative vector field. Thus $\frac{1}{2}m\left|\mathbf{v}(B)\right|^2-\frac{1}{2}m\left|\mathbf{v}(A)\right|^2=\varphi(A)-\varphi(B)$, so $\frac{1}{2}m\left|\mathbf{v}(B)\right|^2+\varphi(B)=\frac{1}{2}m\left|\mathbf{v}(A)\right|^2+\varphi(A)$.

14.3.53

a. Away from the origin (where the denominator of the force field equation is undefined), the force field is conservative because, for example,

$$\frac{\partial}{\partial y}GMm\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}=GMm\frac{2xy}{\left(x^2+y^2+z^2\right)^{5/2}}=\frac{\partial}{\partial x}GMm\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}.$$

b. A potential function for the force field is $\varphi(x,y,z)=GMm\left(x^2+y^2+z^2\right)^{-1/2}=GMm\frac{1}{|\mathbf{r}|}$.

c. The work done in moving the point from A to B, because the force field is conservative, is $\varphi(B) - \varphi(A) = GMm\left(\frac{1}{|B|} - \frac{1}{|A|}\right) = GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$.

d. Because the field is conservative, the work done does not depend on the path.

14.3.54 This vector field is $\mathbf{F} = \langle x, y, z\rangle\left(x^2+y^2+z^2\right)^{-p/2}$, so away from the origin is conservative with potential function $\varphi(x,y,z) = \frac{1}{2-p}\left(x^2+y^2+z^2\right)^{1-p/2}$ as long as $p \neq 2$. When $p = 2$, the potential function is $\varphi = \frac{1}{2}\ln(x^2+y^2+z^2)$. The field is conservative at the origin if it is defined and if its potential function is defined, i.e. if both $-\frac{p}{2}$ and $1-\frac{p}{2}$ are nonnegative, which happens only if $p \le 0$.

14.3.55

a. This field is $\mathbf{F} = \langle -y, x\rangle\left(x^2+y^2\right)^{-p/2}$, and we have $\frac{\partial}{\partial y}\left(-y\left(x^2+y^2\right)^{-p/2}\right) = -\left(x^2+y^2\right)^{-p/2} + py^2\frac{\left(x^2+y^2\right)^{-p/2}}{x^2+y^2} = -\left(x^2+y^2\right)^{-p/2} + py^2\left(x^2+y^2\right)^{-1-p/2} = \frac{-x^2+(p-1)y^2}{(x^2+y^2)^{1+p/2}}$. $\frac{\partial}{\partial x}\left(x\left(x^2+y^2\right)^{-p/2}\right) = \left(x^2+y^2\right)^{-p/2} - px^2\frac{\left(x^2+y^2\right)^{-p/2}}{x^2+y^2} = \left(x^2+y^2\right)^{-p/2} - px^2\left(x^2+y^2\right)^{-1-p/2} = \frac{-(p-1)x^2+y^2}{(x^2+y^2)^{1+p/2}}$.

For the force field to be conservative, these two would have to be equal. However, their difference is $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2\left(x^2+y^2\right)^{-p/2} - p\left(x^2+y^2\right)\left(x^2+y^2\right)^{-1-p/2} = 2\left(x^2+y^2\right)^{-p/2} - p\left(x^2+y^2\right)^{-p/2} = (2-p)\left(x^2+y^2\right)^{-p/2}$ which is in general nonzero.

b. From the above formula, if $p = 2$, then the mixed partials are equal, so that $\mathbf{F}$ is conservative.

c. For $p = 2$, $\mathbf{F} = \frac{1}{x^2+y^2}\langle -y, x\rangle$. Integrating the x component of F with respect to x gives $\varphi = \tan^{-1}\left(\frac{y}{x}\right)$.

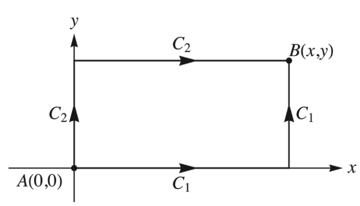
14.3.56

a. Because $\frac{\partial}{\partial y}(ax+by) = b$ and $\frac{\partial}{\partial x}(cx+dy) = c$, the field is conservative when $b = c$.

b. Because $\frac{\partial}{\partial y}\left(ax^2 - by^2\right) = -2by$ and $\frac{\partial}{\partial x}(cxy) = cy$, the field is conservative when $c = -2b$.

14.3.57

a.



b. Parameterize C_1 by two paths: $r_1(t) = \langle t, 0\rangle$, $0 \le t \le x$, and $r_2(t) = \langle x, t\rangle$, $0 \le t \le y$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle 2t, -t\rangle\cdot\langle 1,0\rangle\,dt + \int_0^y \langle 2x-t, -x+2t\rangle\cdot\langle 0,1\rangle\,dt = \int_0^x 2t\,dt + \int_0^y (2t-x)\,dt = x^2+y^2-xy$.

c. Parameterize C_2 by the paths $r_1(t) = \langle 0, t\rangle$, $0 \le t \le y$, and $r_2(t) = \langle t, y\rangle$, $0 \le t \le x$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^y \langle -t, 2t\rangle\cdot\langle 0,1\rangle\,dt + \int_0^x \langle 2t-y, -t+2y\rangle\cdot\langle 1,0\rangle\,dt = \int_0^y 2t\,dt + \int_0^x (2t-y)\,dt = x^2+y^2-xy$.

14.3.58 Using problem 57, we have the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, and $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle 0, -t\rangle\cdot\langle 1,0\rangle\,dt + \int_0^y \langle -t, -x\rangle\cdot\langle 0,1\rangle\,dt = \int_0^x 0dt + \int_0^y (-x)\,dt = -xy$.

14.3.59 Using problem 57 and the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, we have $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle t, 0\rangle\cdot\langle 1,0\rangle\,dt + \int_0^y \langle x, t\rangle\cdot\langle 0,1\rangle\,dt = \int_0^x t\,dt + \int_0^y t\,dt = \frac{1}{2}\left(x^2+y^2\right)$.

14.3.60 Using problem 57 and the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, note that $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \langle x, y\rangle\left(x^2+y^2\right)^{-1/2}$, and $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle 1,0\rangle\cdot\langle 1,0\rangle\,dt + \int_0^y \left\langle \frac{x}{\sqrt{x^2+t^2}}, \frac{t}{\sqrt{x^2+t^2}}\right\rangle\cdot\langle 0,1\rangle\,dt = \int_0^x 1\,dt + \int_0^y \frac{t}{\sqrt{x^2+t^2}}\,dt = x + \sqrt{x^2+y^2} - x = \sqrt{x^2+y^2}$.

14.3.61 Using problem 57 and the same paths $\mathbf{r}_1$, $\mathbf{r}_2$, we have $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^x \langle 2t^3, 0\rangle\cdot\langle 1,0\rangle\,dt + \int_0^y \langle 2x^3 + xt^2, 2t^3 + x^2t\rangle\cdot\langle 0,1\rangle\,dt = \int_0^x 2t^3\,dt + \int_0^y \left(2t^3 + x^2t\right)\,dt = \frac{1}{2}\left(x^4 + x^2y^2 + y^4\right)$.

14.4 Green's Theorem

14.4.1 As with the Fundamental Theorem of Calculus, it allows evaluation of the integral of a derivative by looking at the value of the underlying function on the boundary of a region (or, in the case of the Fundamental Theorem, an interval).

14.4.2 The line integral for flux corresponds to the double integral of the divergence; the line integral for circulation to the double integral of the curl.

14.4.3 The curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = y^2 + 4x^3 - 4x^3 = y^2$.

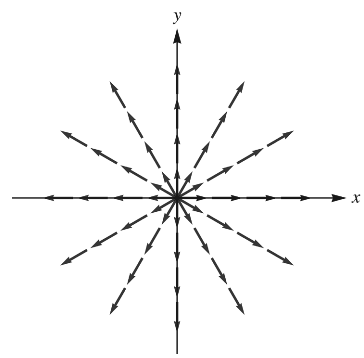
14.4.4 The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 12x^2y + 2xy$.

14.4.5 The area is $\frac{1}{2}\oint_C (x\,dy - y\,dx)$ where C is the boundary of the region.

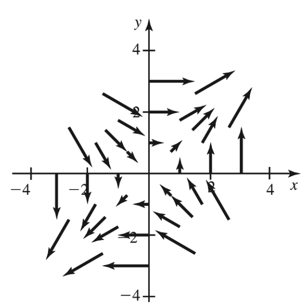
14.4.6 Because the curl being zero is an equivalent condition to the field being conservative.

14.4.7 Because the flux is the integrand in Green's theorem, so the integral vanishes.

14.4.8 A conservative vector field such as $\langle x, y\rangle$ will have zero curl:



14.4.9



14.4.10 A conservative and a source-free field each have functions (a potential function in the case of a conservative field; a stream function in the case of a source-free field) that closely reflect the vector field. The properties of the partials of these functions are such that the curl (or divergence, for a source-free field) vanish.

14.4.11

a. The curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0 - 0 = 0$.

b. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle \cos t, \sin t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt = 0$. $\iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint\limits_R 0\, dA = 0$.

c. The vector field is conservative because its curl is zero.

14.4.12

a. The curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1 - 1 = 0$.

b. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 0, t\rangle \cdot \langle 1, 0\rangle\, dt + \int_0^1 \langle t, 1\rangle \cdot \langle 0, 1\rangle\, dt + \int_0^1 \langle 1, 1-t\rangle \cdot \langle -1, 0\rangle\, dt + \int_0^1 \langle 1-t, 0\rangle \cdot \langle 0, -1\rangle\, dt = 0$.
$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R 0\, dA = 0$.

c. The vector field is conservative because its curl is zero.

14.4.13

a. The curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = -2 - 2 = -4$.

b. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \langle 0, -2t\rangle \cdot \langle 1, 0\rangle\, dt + \int_\pi^0 \langle 2\sin t, -2t\rangle \cdot \langle 1, \cos t\rangle\, dt = \int_\pi^0 (2\sin t - 2t\cos t)\, dt = -8$.
$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R (-4)\, dA = -4\int_0^\pi \sin x\, dx = -8$.

c. It is not conservative because the curl is nonzero.

14.4.14

a. The curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 3 + 3 = 6$.

b. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 0, 3t\rangle \cdot \langle 1, 0\rangle\, dt + \int_0^1 \langle -6t, 3-3t\rangle \cdot \langle -1, 2\rangle\, dt + \int_0^1 \langle -3(2-2t), 0\rangle \cdot \langle 0, -2\rangle\, dt =$
$\int_0^1 (6t + 6 - 6t)\, dt = 6$.
$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R 6\, dA = 6\int_0^1 (2 - 2x)\, dx = 6$.

c. No, because the curl is nonzero.

14.4.15

a. The curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2x - 2x = 0$

b. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \langle 0, t^2\rangle \cdot \langle 1, 0\rangle\, dt + \int_2^0 \langle 2t^2(2-t), t^2 - t^2(2-t)^2\rangle \cdot \langle 1, 2-2t\rangle\, dt =$
$\int_2^0 \left(2t^2(2-t) + t^2\left(1 - (2-t)^2\right)(2-2t)\right) dt = 0$.
$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R 0\, dA = 0$.

c. Yes, because the curl is zero.

14.4.16

a. The curl is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2x$.

b. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 0, \sin^2 t + \cos^2 t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt = \int_0^{2\pi} \cos t\, dt = 0$.
$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_R (2x)\, dA = \int_0^{2\pi}\int_0^1 2r\cos\theta\, r\, dr\, d\theta = 0$.

c. No, because the curl is nonzero.

14.4.17 Parameterize the boundary by $x = 5\cos t$, $y = 5\sin t$, $0 \le t \le 2\pi$. Then $dx = -5\sin t\, dt$, $dy = 5\cos t\, dt$, and the area is $\frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} ((5\cos t)\cdot(5\cos t) - (5\sin t)\cdot(-5\sin t))\, dt = \frac{25}{2}\int_0^{2\pi} 1\, dt = 25\pi$.

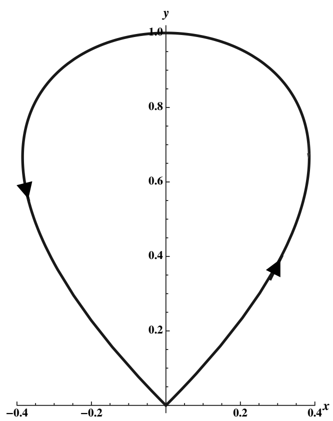
14.4.18 Parameterize the boundary by $x = 6\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$. Then $dx = -6\sin t\,dt$, $dy = 4\cos t\,dt$, and the area is $\frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}((6\,\cos t)\cdot(4\cos t) - (4\sin t)\cdot(-6\sin t))\,dt = 12\int_0^{2\pi} 1\,dt = 24\pi$.

14.4.19 Parameterize the boundary by $x = 4\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$. Then $dx = -4\sin t\,dt$, $dy = 4\cos t\,dt$, and the area is $\frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}((4\,\cos t)\cdot(4\cos t) - (4\sin t)\cdot(-4\sin t))\,dt = 8\int_0^{2\pi} 1\,dt = 16\pi$.

14.4.20 Note that C_1 can be parameterized by $x = -\frac{\sqrt{2}}{2}(1-t) + \frac{\sqrt{2}}{2}t$ and $y = \frac{\sqrt{2}}{2}$, $0 \le t \le 1$ while C_2 can be parametrized by $x = \cos t$ and $y = \sin t$ for $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$. For C_1 we have $dy = 0$ and $dx = \sqrt{2}\,dt$. Thus $\frac{1}{2}\oint_{C_1} x\,dy - y\,dx = \frac{1}{2}\int_0^1 -\frac{\sqrt{2}}{2}\cdot\sqrt{2}\,dt = -\frac{1}{2}$. For C_2, we have $dx = -\sin t\,dt$ and $dy = \cos t\,dt$, and we have $\frac{1}{2}\oint_{C_2} x\,dy - y\,dx = \frac{1}{2}\int_{-\pi/4}^{\pi/4} dt = \frac{\pi}{4}$. Thus the area is $\frac{\pi}{4} - \frac{1}{2}$. As a quick check, note that the region could be thought of as a quarter circle of radius one minus a triangle with area $\frac{1}{2}$.

14.4.21 Traverse the first path from -2 to 2, then the second path back from 2 to -2. The area is then $\frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_{-2}^{2}\left(t\cdot(4\,t) - \left(2t^2\right)\cdot 1\right)\,dt + \frac{1}{2}\int_{2}^{-2}\left(t\cdot(-2\,t) - \left(12 - t^2\right)\cdot 1\right)\,dt = \frac{1}{2}\int_{-2}^{2} 2t^2\,dt + \frac{1}{2}\int_{2}^{-2}\left(-2t^2 - 12 + t^2\right)\,dt = 32$.

14.4.22



We have $dx = \left(1 - t^2 + t\cdot(-2t)\right)\,dt = \left(1 - 3t^2\right)\,dt$ and $dy = -2t\,dt$. We parameterize the curve from $t = 1$ to $t = -1$ so that we traverse the region counterclockwise; then the area is $\frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_1^{-1}\left(t\left(1-t^2\right)(-2t) - \left(1-t^2\right)\left(1-3t^2\right)\right)\,dt = \frac{8}{15}$.

14.4.23

a. The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 1 + 1 = 2$.

b. $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = 4\int_0^{2\pi}(\cos t\,(\cos t) - \sin t\,(-\sin t))\,dt = 8\pi$.
$\iint\limits_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\,dA = \iint\limits_R (2)\,dA = 2\cdot 4\pi = 8\pi$.

c. It is not source-free because its divergence is nonzero.

14.4.24

a. The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 + 0 = 0$.

b. $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 (0\,(0) + t\,(1))\,dt + \int_0^1 (t\,(1) + 1\,(0))\,dt + \int_0^1 ((1)\,(0) + (1-t)\,(-1))\,dt + \int_0^1 ((1-t)\,(-1) + 0\,(0))\,dt = \int_0^1 (2t + t - 1 + t - 1)\,dt = \int_0^1 (4t - 2)\,dt = 0$.
$\iint\limits_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\,dA = \iint\limits_R 0\,dA = 0$.

c. Yes, because its divergence is zero.

14.4.25

a. The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$.

b. $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{-2}^{2} \left(0\,(0) + 3t\,(1)\right)\, dt + \int_{-2}^{2} \left(\left(4 - t^2\right)(-2t) + 3t\,(1)\right)\, dt = 0.$

$\iint\limits_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\, dA = \iint\limits_R 0\, dA = 0.$

c. Yes, because its divergence is zero.

14.4.26

a. The divergence is $0 + 0 = 0$.

b. Parameterize the triangle by the three paths: $\mathbf{r}_1(t) = \langle 3t, 0\rangle$, $\mathbf{r}_2(t) = \langle 3 - 3t, t\rangle$, and $\mathbf{r}_3(t) = \langle 0, 1 - t\rangle$, all for $0 \le t \le 1$. Then $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^1 \left(0\,(0) + 9t\,(3)\right)\, dt + \int_0^1 \left(-3t\,(1) + (9 - 9t)(-3)\right)\, dt + \int_0^1 \left((3t - 3)(-1) - 0\,(0)\right)\, dt = \int_0^1 (27t - 3t - 27 + 27t - 3t + 3)\, dt = \int_0^1 (48t - 24)\, dt = 0.$

$\iint\limits_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\, dA = \iint\limits_R 0\, dA = 0.$

c. Yes, because its divergence is zero.

14.4.27

a. The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 2y - 2y = 0$.

b. Parameterize the region by $\mathbf{r}_1(t) = \langle t,\ 0\rangle$, and $\mathbf{r}_1(t) = \langle t, t\,(2 - t)\rangle$ for $0 \le t \le 2$; traverse the second path from $t = 2$ to $t = 0$ to make a counterclockwise closed curve. Then $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^2 \left(0\,(0) - t^2\,(1)\right)\, dt + \int_2^0 \left(2t^2\,(2 - t)(2 - 2t) - t^2\left(1 - (2 - t)^2\right)(1)\right)\, dt = 0.$

$\iint\limits_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\, dA = \iint\limits_R 0\, dA = 0$

c. Yes, because its divergence is zero.

14.4.28

a. The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 2x$.

b. $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \left(\left(\sin^2 t + \cos^2 t\right) \cdot \cos t + 0 \cdot \sin t\right)\, dt = \int_0^{2\pi} \cos t\, dt = 0.$

$\iint\limits_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\, dA = \iint\limits_R 2x\, dA = \int_0^{2\pi} \int_0^1 2\cos\theta \cdot r\, dr\, d\theta = 0.$

c. No, because its divergence is nonzero.

14.4.29 The line integral, using the flux form of Green's theorem, is equal to
$\iint\limits_R \left(\frac{\partial}{\partial x}\left(2x + e^{y^2}\right) + \frac{\partial}{\partial y}\left(4y^2 + e^{x^2}\right)\right)\, dA = \iint\limits_R (2 + 8y)\, dA = \int_0^1 \int_0^1 (2 + 8y)\, dx\, dy = \int_0^1 (2 + 8y)\, dy = 6.$

14.4.30 Using the flux form of Green's theorem, the integral is equal to $\iint\limits_R \left(\frac{\partial}{\partial x}(2x - 3y) + \frac{\partial}{\partial y}(3x + 4y)\right)\, dA$ $= \iint\limits_R 6\, dA = 6 \times \text{area of } R = 6\pi.$

14.4.31 Using the flux form of Green's theorem, the integral is equal to $\iint\limits_R \left(\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(xy)\right)\, dA =$ $\int_0^2 \int_0^{4-2x} x\, dy\, dx = \int_0^2 \left(4x - 2x^2\right)\, dx = \frac{8}{3}.$

14.4.32 Using the flux form of Green's theorem, the integral is equal to (note the leading minus sign to correct for the orientation)

$$-\iint_R \left(\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}\left(2y^2\right)\right) dA = -\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (2x+4y)\; dy\, dx$$
$$= -\int_{-1}^{1} \left(2xy+2y^2\right)\Big|_{y=0}^{y=\sqrt{1-x^2}} dx = -\int_{-1}^{1}\left(2x\sqrt{1-x^2}+2-2x^2\right) dx = -\frac{8}{3}.$$

14.4.33 Using the circulation form of Green's theorem, the integral is equal to

$$\iint_R \left(\frac{\partial}{\partial x}\left(4x+y^3\right)-\frac{\partial}{\partial y}\left(x^2+y^2\right)\right) dA = \iint_R (4-2y)\; dA = \int_0^{\pi}\int_0^{\sin(x)} (4-2y)\; dy\, dx = 8-\frac{\pi}{2}.$$

14.4.34 Using the flux form of Green's theorem, the integral is equal to $\iint_R \left(\frac{\partial}{\partial x}\left(e^{x-y}\right)+\frac{\partial}{\partial y}\left(e^{y-x}\right)\right) dA = \iint_R \left(e^{x-y}+e^{y-x}\right) dA = \int_0^1\int_0^x \left(e^{x-y}+e^{y-x}\right) dy\, dx = \int_0^1 \left(-e^{x-y}+e^{y-x}\right)\Big|_{y=0}^{y=x} dx = \int_0^1 \left(e^x-e^{-x}\right) dx = e+e^{-1}-2.$

14.4.35

a. Using Green's theorem, the circulation is $\iint_R \left(\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial y}(x)\right) dA = \iint_R 0\, dA = 0,$

b. Using Green's theorem, the flux is $\iint_R \left(\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)\right) dA = \iint_R 2\, dA = 2\cdot \text{area of } R = 2\cdot\frac{1}{2}(4\pi-\pi) = 3\pi.$

14.4.36

a. Using Green's theorem, the circulation is $\iint_R \left(\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(-y)\right) dA = \iint_R 2\, dA = 2\cdot \text{area of} R = 2\cdot(9\pi-\pi) = 16\pi.$

b. Using Green's theorem, the flux is $\iint_R \left(\frac{\partial}{\partial x}(-y)+\frac{\partial}{\partial y}(x)\right) dA = 0.$

14.4.37

a. Using Green's theorem, the circulation is $\iint_R \left(\frac{\partial}{\partial x}(x-4y)-\frac{\partial}{\partial y}(2x+y)\right) dA = \iint_R (1-1)\; dA = 0.$

b. Using Green's theorem, the flux is $\iint_R \left(\frac{\partial}{\partial x}(2x+y)+\frac{\partial}{\partial y}(x-4y)\right) dA = \iint_R (-2)\; dA = -2\cdot\text{area of } R = -2\cdot\frac{1}{4}(16\pi-\pi) = -\frac{15}{2}\pi.$

14.4.38

a. Using Green's theorem, the circulation is $\iint_R \left(\frac{\partial}{\partial x}(-x+2y)-\frac{\partial}{\partial y}(x-y)\right) dA = \iint_R (0)\; dA = 0.$

b. Using Green's theorem, the flux is $\iint_R \left(\frac{\partial}{\partial x}(x-y)+\frac{\partial}{\partial y}(-x+2y)\right) dA = \iint_R 3\, dA = 3\cdot\text{area of } R = 6.$

14.4.39

a. True. This is the definition of work along a path.

b. False. Divergence corresponds to flux, so if the divergence is zero throughout a region, the flux is zero across the boundary.

c. True. This follows from Green's theorem.

14.4.40

a. The circulation is $\iint\limits_R \left(\frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) - \frac{\partial}{\partial y}\left(\ln\left(x^2+y^2\right)\right)\right) dA = \iint\limits_R \left(\frac{-y}{x^2+y^2} - \frac{2y}{x^2+y^2}\right) dA =$ $-3\iint\limits_R \frac{y}{x^2+y^2}\, dA = -3\int_0^{2\pi}\int_1^2 \frac{r\sin\theta}{r^2} r\, dr\, d\theta = -3\int_0^{2\pi}\int_1^2 2\sin\theta\, dr\, d\theta = -3\int_0^{2\pi}\sin\theta\, d\theta = 0.$

b. The flux is $\iint\limits_R \left(\frac{\partial}{\partial x}\left(\ln\left(x^2+y^2\right)\right) + \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)\right) dA = \iint\limits_R \frac{3x}{x^2+y^2}\, dA =$ $\int_0^{2\pi}\int_1^2 \frac{3r\cos\theta}{r^2} r\, dr\, d\theta = \int_0^{2\pi}\int_1^2 3\cos\theta\, dr\, d\theta = \int_0^{2\pi} 3\cos(\theta)\, d\theta = 0.$

14.4.41

a. Because **F** is conservative, the circulation on the boundary of R is zero.

b. $\mathbf{F} = \left(x^2+y^2\right)^{-1/2}\langle x, y\rangle$, so the flux is $\iint\limits_R \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right) dA =$ $\iint\limits_R \left(x^2+y^2\right)^{-1/2} dA = \int_0^{\pi}\int_1^3 \frac{1}{r} r\, dr\, d\theta = \int_0^{\pi}\int_1^3 1\, dr\, d\theta = 2\pi.$

14.4.42

a. The circulation is

$$\iint\limits_R \left(\frac{\partial}{\partial x}(-\sin x) - \frac{\partial}{\partial y}(y\cos x)\right) dA = 2\iint\limits_R (-\cos x)\, dA$$
$$= 2\int_0^{\pi/2}\int_0^{\pi/2}(-\cos x)\, dy\, dx = \int_0^{\pi/2}(-\pi\cos x)\, dx = -\pi.$$

b. The flux is

$$\iint\limits_R \left(\frac{\partial}{\partial x}(y\cos x) + \frac{\partial}{\partial y}(-\sin x)\right) dA = \iint\limits_R (-y\sin x)\, dA$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}(-y\sin x)\, dy\, dx = -\frac{1}{8}\pi^2.$$

14.4.43 Note that the region is the area between $x = 3y^2$ and$x = 36 - y^2$, which intersect at $y = 3$.

a. The circulation is $\iint\limits_R \left(\frac{\partial}{\partial x}\left(x^2-y\right) - \frac{\partial}{\partial y}\left(x+y^2\right)\right) dA = \iint\limits_R (2x-2y)\, dA = \int_{-3}^{3}\int_{3y^2}^{36-y^2}(2x-2y)\, dx\, dy$ $= \int_{-3}^{3}\left(1296 - 72y - 72y^2 + 8y^3 - 8y^4\right) dy = \frac{28512}{5}.$

b. The flux is $\iint\limits_R \left(\frac{\partial}{\partial x}\left(x+y^2\right) + \frac{\partial}{\partial y}\left(x^2-y\right)\right) dA = \iint\limits_R (0)\, dA = 0.$

14.4.44 By Green's theorem, $\oint_C 1\, dx = \oint_C 1\, dx + 0\, dy = \iint\limits_R \left(\frac{\partial 0}{\partial x} - \frac{\partial 1}{\partial y}\right) dA = \iint\limits_R 0\, dA = 0.$ Similarly, $\oint_C 1\, dy = \oint_C 0\, dx + 1\, dy = \iint\limits_R \left(\frac{\partial 1}{\partial x} - \frac{\partial 0}{\partial y}\right) dA = \iint\limits_R 0\, dA = 0.$

14.4.45 Because $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 0$, the integral is zero (because **F** is conservative.)

14.4.46 Let $f(x,y)=0$, $g(x,y)=xy^2+y^4$; then

$$\iint_R \left(2xy+4y^3\right)\,dA=\iint_R \left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)dA=\oint_C (0\,dy-g\,dx)=-\oint_C \left(xy^2+y^4\right)\,dx$$

by Green's theorem, where C is the boundary of the triangle. To evaluate this line integral, parameterize C by three paths, all for $0\le t\le 1$: $\mathbf{r}_1(t)=\langle t,0\rangle$, so that $\mathbf{r}_1'(t)=\langle 1,0\rangle$. $\mathbf{r}_2(t)=\langle 1-t,t\rangle$, so that $\mathbf{r}_2'(t)=\langle -1,1\rangle$. $\mathbf{r}_3(t)=\langle 0,1-t\rangle$, so that $\mathbf{r}_3'(t)=\langle 0,-1\rangle$. Then $-\oint_C\left(xy^2+y^4\right)dx=-\int_0^1\left(t\left(0^2\right)-0^4\right)dt-\int_0^1 t^2(1-t)(-1)\,dt-\int_0^1\left(0\,(1-t)^2+t^4\cdot 0\right)dt=\int_0^1\left(t^4-t^3+t^2\right)dt=\frac{17}{60}$.

14.4.47 By Green's theorem,

$$\oint_C xy^2\,dx+\left(x^2y+2x\right)\,dy=\iint_R\left(\frac{\partial}{\partial x}\left(x^2y+2x\right)-\frac{\partial}{\partial y}\left(xy^2\right)\right)dA$$
$$=\iint_R (2xy+2-2xy)\,dA=\iint_R 2\,dA=2\cdot\text{area of }A.$$

14.4.48 Using the circulation form of Green's theorem, the integral is

$$\oint_C ay\,dx+bx\,dy=\iint_R (b-a)\,dA=(b-a)\cdot\text{area of }A.$$

14.4.49

a. The divergence is $\frac{\partial}{\partial x}(4)+\frac{\partial}{\partial y}(2)=0$.

b. $\psi=4y-2x$.

14.4.50

a. The divergence is $\frac{\partial}{\partial x}\left(y^2\right)+\frac{\partial}{\partial y}\left(x^2\right)=0+0=0$.

b. $\psi=\frac{1}{3}\left(y^3-x^3\right)$.

14.4.51

a. The divergence is $\frac{\partial}{\partial x}\left(-e^{-x}\sin y\right)+\frac{\partial}{\partial y}\left(e^{-x}\cos y\right)=e^{-x}\sin y-e^{-x}\sin y=0$.

b. $\psi=e^{-x}\cos y$.

14.4.52

a. The divergence is $\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}(-2xy)=2x-2x=0$.

b. $\psi=x^2y$.

14.4.53

a. The curl and divergence are $\operatorname{curl}\mathbf{F}=\frac{\partial}{\partial x}\left(-e^x\sin y\right)-\frac{\partial}{\partial y}\left(e^x\cos y\right)=-e^x\sin y+e^x\sin y=0$. $\operatorname{div}\mathbf{F}=\frac{\partial}{\partial x}\left(e^x\cos y\right)+\frac{\partial}{\partial y}\left(-e^x\sin y\right)=e^x\cos y-e^x\cos y=0$.

b. $\varphi(x,y)=e^x\cos(y)$. $\psi(x,y)=e^x\sin(y)$.

c. $\varphi_{xx}+\varphi_{yy}=\frac{\partial^2}{\partial x^2}\left(e^x\cos y\right)+\frac{\partial^2}{\partial y^2}\left(e^x\cos y\right)=e^x\cos y-e^x\cos y=0$. $\psi_{xx}+\psi_{yy}=\frac{\partial^2}{\partial x^2}\left(e^x\sin y\right)+\frac{\partial^2}{\partial y^2}\left(e^x\sin y\right)=e^x\sin y-e^x\sin y=0$.

14.4.54

a. The curl and divergence are $\operatorname{curl}\mathbf{F} = \frac{\partial}{\partial x}\left(y^3 - 3x^2y\right) - \frac{\partial}{\partial y}\left(x^3 - 3xy^2\right) = -6xy + 6xy = 0$. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}\left(x^3 - 3xy^2\right) + \frac{\partial}{\partial y}\left(y^3 - 3x^2y\right) = 3x^2 - 3y^2 + 3y^2 - 3x^2 = 0$

b. $\varphi(x,y) = \frac{1}{4}\left(x^4 + y^4\right) - \frac{3}{2}x^2y^2$. $\psi(x,y) = x^3y - xy^3$.

c. $\varphi_{xx} + \varphi_{yy} = \frac{\partial^2}{\partial x^2}\left(\frac{1}{4}\left(x^4+y^4\right) - \frac{3}{2}x^2y^2\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{4}\left(x^4+y^4\right) - \frac{3}{2}x^2y^2\right) = \frac{\partial}{\partial x}\left(x^3 - 3xy^2\right) + \frac{\partial}{\partial y}\left(y^3 - 3x^2y\right) = 3x^2 - 3y^2 + 3y^2 - 3x^2 = 0$. $\psi_{xx} + \psi_{yy} = \frac{\partial^2}{\partial x^2}\left(x^3y - xy^3\right) + \frac{\partial^2}{\partial y^2}\left(x^3y - xy^3\right) = \frac{\partial}{\partial x}\left(3x^2y - y^3\right) + \frac{\partial}{\partial y}\left(x^3 - 3xy^2\right) = 6xy - 6xy = 0$.

14.4.55

a. The curl and divergence are $\operatorname{curl}\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right) - \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0$.
$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right) = 0$.

b. $\varphi(x,y) = x\tan^{-1}\left(\frac{y}{x}\right) - y + \frac{1}{2}y\ln\left(x^2+y^2\right)$. $\psi(x,y) = \int -\frac{1}{2}\ln\left(x^2+y^2\right)\,dy = y\tan^{-1}\left(\frac{y}{x}\right) - \frac{x}{2}\ln\left(x^2+y^2\right) + x$.

c. $\varphi_{xx} + \varphi_{yy} = \frac{\partial^2}{\partial x^2}\left(x\tan^{-1}\left(\frac{y}{x}\right) - y + \frac{1}{2}y\ln\left(x^2+y^2\right)\right) + \frac{\partial^2}{\partial y^2}\left(x\tan^{-1}\left(\frac{y}{x}\right) - y + \frac{1}{2}y\ln\left(x^2+y^2\right)\right) = \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right) - \frac{xy}{x^2+y^2} + \frac{xy}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x^2}{x^2+y^2} - 1 + \frac{1}{2}\ln\left(x^2+y^2\right) + \frac{y^2}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right) = 0$.

$\psi_{xx} + \psi_{yy} = \frac{\partial^2}{\partial x^2}\left(y\tan^{-1}\left(\frac{y}{x}\right) - \frac{x}{2}\ln\left(x^2+y^2\right) + x\right) + \frac{\partial^2}{\partial y^2}\left(y\tan^{-1}\left(\frac{y}{x}\right) - \frac{x}{2}\ln\left(x^2+y^2\right) + x\right) = \frac{\partial}{\partial x}\left(-\frac{y^2}{x^2+y^2} - \frac{x^2}{x^2+y^2} - \frac{1}{2}\ln\left(x^2+y^2\right) + 1\right) + \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right) + \frac{xy}{x^2+y^2} - \frac{xy}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right) + \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0$.

14.4.56

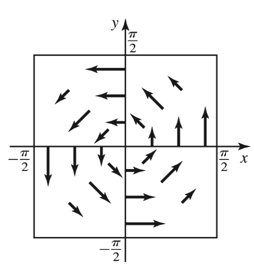
a. The curl and divergence are $\operatorname{curl}\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = 0$, and
$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = 0$.

b. $\varphi(x,y) = \frac{1}{2}\ln\left(x^2+y^2\right)$. $\psi(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$.

c. $\varphi_{xx} + \varphi_{yy} = \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$. $\psi_{xx} + \psi_{yy} = \frac{\partial^2}{\partial x^2}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + \frac{\partial^2}{\partial y^2}\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$.

14.4.57

a. The velocity field is $\langle -4\cos x\sin y, 4\sin x\cos y\rangle$.



b. The field is source-free if its divergence is zero.

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(-4\cos x\sin y) + \frac{\partial}{\partial y}(4\sin x\cos y) = 4\sin x\sin y - 4\sin x\sin y = 0,$$

so the field is source-free.

c. The field is irrotational if its curl is zero.

$$\operatorname{curl}\mathbf{F} = \frac{\partial}{\partial x}(4\sin x\cos y) - \frac{\partial}{\partial y}(-4\cos x\sin y) = 4\cos x\cos y + 4\cos x\cos y = 8\cos x\cos y,$$

so the field is not irrotational.

d. Since the field is source-free, it has zero flux across the boundary.

e. The circulation around the boundary of the rectangle is (by Green's theorem) given by

$$\iint\limits_R 8\cos x\cos y\,dA = \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} 8\cos x\cos y\,dy\,dx = \int_{-\pi/2}^{\pi/2} 16\cos x\,dx = 32.$$

14.4.58 If $f(x)$ is continuous, then the circulation form of Green's theorem says that $\oint_C \frac{f(x)}{c}dy = \frac{1}{c}\iint\limits_R \frac{df}{dx}\,dA$. The right side of this equation evaluates to $\frac{1}{c}\iint\limits_R \frac{df}{dx}\,dA = \frac{1}{c}\int_a^b\int_0^c \frac{df}{dx}dy\,dx = \int_a^b \frac{df}{dx}\,dx$. To evaluate the left side, parameterize the boundary of R with four paths, each for $0 \le t \le 1$: $\mathbf{r}_1(t) = \langle a + (b-a)t, 0\rangle$, so $\mathbf{r}_1'(t) = \langle b-a, 0\rangle$. $\mathbf{r}_2(t) = \langle b, ct\rangle$, so $\mathbf{r}_2'(t) = \langle 0, c\rangle$. $\mathbf{r}_3(t) = \langle b + (a-b)t, c\rangle$, so $\mathbf{r}_3'(t) = \langle a-b, 0\rangle$. $\mathbf{r}_4(t) = \langle a, c - ct\rangle$, so $\mathbf{r}_4'(t) = \langle 0, -c\rangle$. Then we evaluate $\mathbf{F}\cdot\mathbf{r}_i$ for each i and add: $\frac{1}{c}\oint_C f(x)\,dy = \frac{1}{c}\int_0^1 (f(a+(b-a)t)\cdot 0 + f(b)\cdot c + f(b+(a-b)t)\cdot 0 + f(a)\cdot(-c))\,dt = f(b) - f(a)$.

14.4.59 If $f(x)$ is continuous, then the flux form of Green's theorem says that $\oint_C \frac{f(x)}{c}dx = \frac{1}{c}\iint\limits_R \frac{df}{dx}\,dA$. The right side of this equation evaluates to $\frac{1}{c}\iint\limits_R \frac{df}{dx}\,dA = \frac{1}{c}\int_a^b\int_0^c \frac{df}{dx}dy\,dx = \int_a^b \frac{df}{dx}\,dx$. To evaluate the left side, parameterize the boundary of R with four paths, each for $0 \le t \le 1$: $\mathbf{r}_1(t) = \langle a + (b-a)t, 0\rangle$, so $\mathbf{r}_1'(t) = \langle b-a, 0\rangle$. $\mathbf{r}_2(t) = \langle b, ct\rangle$, so $\mathbf{r}_2'(t) = \langle 0, c\rangle$. $\mathbf{r}_3(t) = \langle b + (a-b)t, c\rangle$, so $\mathbf{r}_3'(t) = \langle a-b, 0\rangle$. $\mathbf{r}_4(t) = \langle a, c - ct\rangle$, so $\mathbf{r}_4'(t) = \langle 0, -c\rangle$. Then we evaluate $\mathbf{F}\cdot\mathbf{r}_i$ for each i and add: $\oint_C \frac{f(x)}{c}\,dy = \frac{1}{c}\int_0^1 (0 + f(b)\cdot c + 0 + f(a)\cdot(-c))\,dt = f(b) - f(a)$ so that $\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$.

14.4.60

a. The curl is $\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = 0$.

b. Take a line integral around the unit circle, parameterized as $\langle\cos t, \sin t\rangle$. The circulation is then $\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_0^{2\pi} (-\sin t(-\sin t) + \cos t\cos t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi$.

c. The vector field is not defined everywhere in R; specifically, it is undefined at the origin.

14.4.61

a. The divergence is $\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=0.$

b. Take a line integral around the unit circle, parameterized as $\langle \cos t, \sin t\rangle$. The flux is then $\oint_C \frac{x}{x^2+y^2}\,dy + \frac{y}{x^2+y^2}\,dx = \int_0^{2\pi}(\cos t\cos t - \sin t\,(-\sin t))\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$

c. The vector field is not defined everywhere in R; specifically, it is undefined at the origin.

14.4.62

a. Green's theorem does not apply to a region including the origin because $\mathbf{F}$ is not defined at the origin.

b. $\iint\limits_R \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right)dA = \iint\limits_R \left(x^2+y^2\right)^{-1/2}\,dA = \int_0^{2\pi}\int_0^1 \frac{1}{r} r\,dr\,d\theta = 2\pi.$

c. $\oint_C \frac{x}{\sqrt{x^2+y^2}}dy - \frac{y}{\sqrt{x^2+y^2}}dx = \int_0^{2\pi}(\cos t\,(\cos t) - \sin t\,(-\sin t))\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$

d. They do agree. Because Green's theorem does not apply, there is no particular reason why they should.

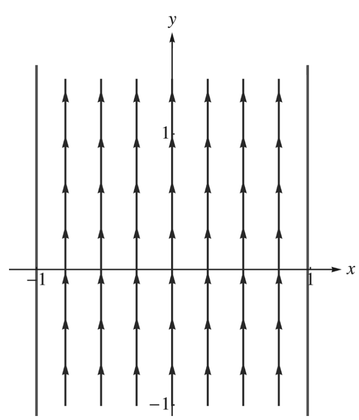
14.4.63 Because ψ is a stream function, $d\psi = \psi_x\,dx + \psi_y\,dy$, so the flux integral is $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C f\,dy - g\,dx = \int_C \psi_y\,dy - \psi_x\,dx = \int_C d\psi = \psi(B) - \psi(A)$, so that the integral is independent of the path.

14.4.64 Showing that $\mathbf{F}$ is tangent to the level curves of the stream function is the same as showing that $\mathbf{F}$ is normal to the gradient of the stream function. But that gradient is $\langle \psi_x, \psi_y\rangle$, and $\mathbf{F}\cdot\langle \psi_x\,,\,\psi_y\rangle = \langle -\psi_y, \psi_x\rangle\cdot\langle \psi_x\,,\psi_y\rangle = 0.$

14.4.65 Showing that the level curves of φ and ψ are orthogonal is equivalent to showing that the gradients of φ andψ are orthogonal. But $\nabla\varphi\cdot\nabla\psi = \langle f, g\rangle\cdot\langle -g, f\rangle = 0.$

14.4.66

a. The stream function is found by taking $-\int\left(1-x^2\right)\,dx = \frac{1}{3}x^3 - x$. A plot together with some streamlines is



b. The curl of F is $\frac{\partial}{\partial x}\left(1-x^2\right) = -2x$, so the curl on $x=0$ is 0; on $x=\frac{1}{4}$ it is $-\frac{1}{2}$; on $x=\frac{1}{2}$, it is -1, and on $x=1$, it is -2.

c. The circulation is (by Green's theorem) $\iint\limits_R (-2x)\,dA = \int_{-5}^{5}\int_{-1}^{1}(-2x)\,dx\,dy = \int_{-5}^{5} 0\,dy = 0.$

d. The curl is positive for negative x and negative for positive x. These cancel, giving a net circulation of zero. This can easily be seen from the picture - any circulation resulting from the top boundary $(y=1)$ is cancelled by the circulation in the opposite direction resulting from the bottom boundary.

14.5 Divergence and Curl

14.5.1 The divergence is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$.

14.5.2 The divergence measures the expansion or contraction of the vector field at each point.

14.5.3 It means that the field has no sources or sinks.

14.5.4 The curl is $\nabla \times \text{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}$.

14.5.5 The curl indicates the axis and speed of rotation of a vector field at each point.

14.5.6 It means that the vector field is irrotational.

14.5.7 $\nabla \cdot (\nabla \times \mathbf{F}) = 0$; see Theorem 14.10.

14.5.8 Here $\mathbf{u}$ is a potential function, so $\nabla \times \nabla\mathbf{u}$ is the curl of a conservative vector field, which is 0.

14.5.9 $\frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(-3z) = 3$.

14.5.10 $\frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(3x) + \frac{\partial}{\partial z}(z) = 1$.

14.5.11 $\frac{\partial}{\partial x}(12x) + \frac{\partial}{\partial y}(-6y) + \frac{\partial}{\partial z}(-6z) = 0$.

14.5.12 $\frac{\partial}{\partial x}\left(x^2yz\right) + \frac{\partial}{\partial y}\left(-xy^2z\right) + \frac{\partial}{\partial z}\left(-xyz^2\right) = 2xyz - 2xyz - 2xyz = -2xyz$.

14.5.13 $\frac{\partial}{\partial x}\left(x^2 - y^2\right) + \frac{\partial}{\partial y}\left(y^2 - z^2\right) + \frac{\partial}{\partial z}\left(z^2 - x^2\right) = 2x + 2y + 2z$.

14.5.14 $\frac{\partial}{\partial x}\left(e^{y-x}\right) + \frac{\partial}{\partial y}\left(e^{z-y}\right) + \frac{\partial}{\partial z}\left(e^{x-z}\right) = -\left(e^{y-x} + e^{z-y} + e^{x-z}\right)$.

14.5.15 $\frac{\partial}{\partial x}\left(\frac{x}{1+x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{1+x^2+y^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{1+x^2+y^2}\right) = \frac{x^2+y^2+3}{(1+x^2+y^2)^2}$.

14.5.16 $\frac{\partial}{\partial x}(yz\sin(x)) + \frac{\partial}{\partial y}(xz\cos(y)) + \frac{\partial}{\partial z}(xy\cos(z)) = yz\cos(x) - xz\sin(y) - xy\sin(z)$.

14.5.17 $\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right) =$
$\frac{1}{(x^2+y^2+z^2)^2}\left(\left(z^2 + y^2 - x^2\right) + \left(x^2 + z^2 - y^2\right) + \left(x^2 + y^2 - z^2\right)\right) = \frac{1}{(x^2+y^2+z^2)^2}\left(x^2 + y^2 + z^2\right) = \frac{1}{|\mathbf{r}|^2}$.

14.5.18 $\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) =$
$\frac{1}{(x^2+y^2+z^2)^{5/2}}\left(\left(z^2 + y^2 - 2x^2\right) + \left(x^2 + z^2 - 2y^2\right) + \left(x^2 + y^2 - 2z^2\right)\right) = 0$.

14.5.19 $\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^2}\right) =$
$\frac{1}{(x^2+y^2+z^2)^3}\left(\left(z^2 + y^2 - 3x^2\right) + \left(x^2 + z^2 - 3y^2\right) + \left(x^2 + y^2 - 3z^2\right)\right) = \frac{-1}{(x^2+y^2+z^2)^3}\left(x^2 + y^2 + z^2\right) = \frac{-1}{|\mathbf{r}|^4}$

14.5.20 $\frac{\partial}{\partial x}\left(x\left(x^2 + y^2 + z^2\right)\right) + \frac{\partial}{\partial y}\left(y\left(x^2 + y^2 + z^2\right)\right) + \frac{\partial}{\partial z}\left(z\left(x^2 + y^2 + z^2\right)\right) =$
$\left(3x^2 + y^2 + z^2\right) + \left(x^2 + 3y^2 + z^2\right) + \left(x^2 + y^2 + 3z^2\right) = 5\,|\mathbf{r}|^2$

14.5.21

a. At both P and Q, the arrows going away from the point are larger in both number and magnitude than those going in, so we would expect the divergence to be positive at both points.

b. The divergence is $\frac{\partial}{\partial \boldsymbol{x}}(x) + \frac{\partial}{\partial y}(x + y) = 1 + 1 = 2$, so is positive everywhere.

c. The arrows all point roughly away from the origin, so we the flux is outward everywhere.

d. The net flux across C should be positive.

14.5.22

a. At P, the divergence should be positive, while at Q, the larger arrows point in towards Q, so the divergence should be negative.

b. The divergence is $\frac{\partial}{\partial \boldsymbol{x}}(x) + \frac{\partial}{\partial y}\left(y^2\right) = 1 + 2y$; at $P = (-1, 1)$, this is 3, while at $Q = (-1, -1)$, it is -1.

c. The flux is outward above the line $y = -1$ (approximately); below this line, the flux is inward across C.

d. The size of the arrows pointing outward at the top of the circle seems to roughly equal those pointing inward at the bottom, so the remaining outward-pointing arrows result in a net positive flux across C.

14.5.23

a. The axis of rotation is $\langle 1, 0, 0\rangle$, the x-axis. $\nabla \times \mathbf{F} = \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ x & y & z \end{vmatrix} = \nabla \times (-z\mathbf{j} + y\mathbf{k}) = (1+1)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = 2\mathbf{i}$. It is in the same direction as the axis of rotation.

b. The magnitude of the curl is $|2\mathbf{i}| = 2$.

14.5.24

a. The axis of rotation is $\langle 1, -1, 0\rangle$. $\nabla \times \mathbf{F} = \nabla \times \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 1 & -1 & 0 \\ x & y & z \end{vmatrix} = \nabla \times (-z\mathrm{i} - z\mathrm{j} + (x+y)\,\mathrm{k}) = (1+1)\,\mathrm{i} + (-1-1)\,\mathrm{j} + (0-0)\,\mathrm{k} = 2\langle 1, -1, 0\rangle$

and the curl is in the same direction as the axis of rotation.

b. The magnitude of the curl is $2\,|\langle 1, -1, 0\rangle| = 2\sqrt{2}$

14.5.25

a. The axis of rotation is $\langle 1, -1, 1\rangle$.

$\nabla \times \mathbf{F} = \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix} = \nabla \times (-(y+z)\,\mathbf{i} + (x-z)\,\mathbf{j} + (x+y)\,\mathbf{k}) = (1+1)\,\mathbf{i} + (-1-1)\,\mathbf{j} + (1+1)\,\mathbf{k} = 2\langle 1, -1, 1\rangle$, and the curl is in the same direction as the axis of rotation.

b. The magnitude of the curl is $2\,|\langle 1, -1, 1\rangle| = 2\sqrt{3}$.

14.5.26

a. The axis of rotation is $\langle 1, -2, -3\rangle$.

$\nabla \times \mathbf{F} = \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -3 \\ x & y & z \end{vmatrix} = \nabla \times ((3y-2z)\,\mathbf{i} + (-3x-z)\,\mathbf{j} + (2x+y)\,\mathbf{k}) = (1+1)\,\mathbf{i} + (-2-2)\,\mathbf{j} + (-3-3)\,\mathbf{k} = 2\langle 1, -2, -3\rangle$, and the curl is in the same direction as the axis of rotation.

b. The magnitude of the curl is $2\,|\langle 1, -2, -3\rangle| = 2\sqrt{14}$.

14.5.27 $\nabla \times \langle x^2 - y^2, xy, z\rangle = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (y+2y)\,\mathbf{k} = 3y\mathbf{k}$.

14.5.28 $\nabla \times \langle 0, z^2 - y^2, -yz \rangle = (-z - 2z)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = -3z\mathbf{i}$.

14.5.29 $\nabla \times \langle x^2 - z^2, 1, 2xz \rangle = (0-0)\,\mathbf{i} + (-2z - 2z)\,\mathbf{j} + (0-0)\,\mathbf{k} = -4z\mathbf{j}$.

14.5.30 $\nabla \times \langle x, y, z \rangle = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$.

14.5.31 $\nabla \times \frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle = \frac{1}{(x^2+y^2+z^2)^{5/2}} \left((-3yz + 3yz)\,\mathbf{i} + (-3xz + 3xz)\,\mathbf{j} + (-3xy + 3xy)\,\mathbf{k} \right) = \mathbf{0}$.

14.5.32 $\nabla \times \frac{1}{(x^2+y^2+z^2)^{1/2}} \langle x, y, z \rangle = \frac{1}{(x^2+y^2+z^2)^{3/2}} \left((-yz + yz)\,\mathbf{i} + (-xz + xz)\,\mathbf{j} + (-xy + xy)\,\mathbf{k} \right) = \mathbf{0}$.

14.5.33 $\nabla \times \langle \mathbf{z}^2 \sin(y), xz^2 \cos(y), 2xz \sin(y) \rangle = (2xz\cos(y) - 2xz\cos(y))\,\mathbf{i} + (2z\sin(y) - 2z\sin(y))\,\mathbf{j} + \left(z^2\cos(y) - z^2\cos(y)\right)\mathbf{k} = \mathbf{0}$.

14.5.34 $\nabla \times \langle 3xz^3 e^{y2}, 2xz^3 e^{y2}, 3xz^2 e^{y2} \rangle = \left(6xyz^2 e^{y2} - 6xz^2 e^{y2}\right)\mathbf{i} + \left(9xz^2 e^{y2} - 3z^2 e^{y2}\right)\mathbf{j} + \left(2z^3 e^{y2} - 6xyz^3 e^{y2}\right)\mathbf{k} = z^2 e^{y2} \left((6xy - 6x)\,\mathbf{i} + (9x - 3)\,\mathbf{j} + (2z - 6xyz)\,\mathbf{k} \right)$.

14.5.35 Simply compute it: $\left\langle \frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2+z^2)^{3/2}}\right), \frac{\partial}{\partial y}\left(\frac{1}{(x^2+y^2+z^2)^{3/2}}\right), \frac{\partial}{\partial z}\left(\frac{1}{(x^2+y^2+z^2)^{3/2}}\right) \right\rangle = \left\langle \frac{-3x}{(x^2+y^2+z^2)^{5/2}}, \frac{-3y}{(x^2+y^2+z^2)^{5/2}}, \frac{-3z}{(x^2+y^2+z^2)^{5/2}} \right\rangle = \frac{-3\mathbf{r}}{|\mathbf{r}|^5}$.

14.5.36 $\left\langle \frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2+z^2}\right), \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2+z^2}\right), \frac{\partial}{\partial z}\left(\frac{1}{x^2+y^2+z^2}\right) \right\rangle = \left\langle \frac{-2x}{(x^2+y^2+z^2)^2}, \frac{-2y}{(x^2+y^2+z^2)^2}, \frac{-2z}{(x^2+y^2+z^2)^2} \right\rangle = \frac{-2\mathbf{r}}{|\mathbf{r}|^4}$.

14.5.37 $\nabla\left(\frac{1}{|\mathbf{r}|^2}\right) = \frac{-2\mathbf{r}}{|\mathbf{r}|^4}$, from Problem 36; applying Theorem 14.8 we have $\nabla \cdot \nabla\left(\frac{1}{|\mathbf{r}|^2}\right) = -2\nabla \cdot \frac{\mathbf{r}}{|\mathbf{r}|^4} = -2\frac{3-4}{|\mathbf{r}|^4} = \frac{2}{|\mathbf{r}|^4}$.

14.5.38 $\nabla(\ln|\mathbf{r}|) = \nabla\left(\ln\left(\sqrt{x^2+y^2+z^2}\right)\right) = \frac{1}{2}\nabla\left(\ln\left(x^2+y^2+z^2\right)\right) = \frac{1}{2(x^2+y^2+z^2)}\langle 2x, 2y, 2z \rangle = \frac{\mathbf{r}}{|\mathbf{r}|^2}$.

14.5.39

a. False. For example, F= $\langle y, z, x \rangle$ has zero divergence yet is not constant.

b. False. For example, F= $\langle x, y, z \rangle$ is a counterexample.

c. False. For example, consider the vector field $\langle 0, 1 - x^2 \rangle$ from problem 66 in the previous section.

d. False. For example, F= $\langle x, 0, 0 \rangle$ has divergence 1.

e. False. For example, the curl of $\langle z, -z, y \rangle$ is $\langle 2, 1, 0 \rangle$.

14.5.40

a. $(\mathbf{F} \cdot \nabla)\,u = \left(\langle f, g, h \rangle \cdot \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle\right) u = \left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}\right) u = f\frac{\partial u}{\partial x} + g\frac{\partial u}{\partial y} + h\frac{\partial u}{\partial z}$.

b. Because $\mathbf{F} = \langle 1, 1, 1 \rangle$, $\mathbf{F} \cdot \nabla\left(xy^2z^3\right) = \frac{\partial}{\partial x}\left(xy^2z^3\right) + \frac{\partial}{\partial y}\left(xy^2z^3\right) + \frac{\partial}{\partial z}\left(xy^2z^3\right) = y^2z^3 + 2xyz^3 + 3xy^2z^2$.

14.5.41

a. No; divergence is a concept that applies to vector fields.

b. No; the gradient applies to functions.

c. Yes; this is the divergence of the gradient and is thus a scalar function.

d. No, since $\nabla \cdot \varphi$ does not make sense (part (a)).

e. No; curl applies to vector fields, $\nabla \times \varphi$ does not make sense.

f. No, since $\nabla \cdot$F is a function, so that applying $\nabla \cdot$ to it does not make sense.

g. Yes, this is the curl of a vector field and is thus a vector field.

h. No, since $\nabla \cdot F$ is a function, not a vector field.

i. Yes; this is the curl of the curl of a vector field and is thus a vector field.

14.5.42 Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$; then $\mathbf{F} = \mathbf{a} \times \mathbf{r} = (a_2 z - a_3 y)\,\mathbf{i} + (a_3 x - a_1 z)\,\mathbf{j} + (a_1 y - a_2 x)\,\mathbf{k}$, so that $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x) = 0$.

14.5.43

a. $\langle 0,1,0 \rangle \times \langle 0,1,1 \rangle = \langle 1,0,0 \rangle$, so $\mathbf{F}$ points in the positive x-direction. $\langle 0,1,0 \rangle \times \langle 1,1,0 \rangle = \langle 0,0,-1 \rangle$, so $\mathbf{F}$ points in the negative z-direction. $\langle 0,1,0 \rangle \times \langle 0,1,-1 \rangle = \langle -1,0,0 \rangle$, so $\mathbf{F}$ points in the negative x-direction. $\langle 0,1,0 \rangle \times \langle -1,1,0 \rangle = \langle 0,0,1 \rangle$, so $\mathbf{F}$ points in the positive z-direction.

b. Note that these vectors circle the y-axis in the counterclockwise direction looking along $\mathbf{a}$ from head to tail.

14.5.44 Note that $\mathbf{a} \times \mathbf{r} = \langle 0,1,0 \rangle \times \langle x,y,z \rangle = \langle z,0,-x \rangle$ is a rotational field whose vectors circle the y-axis in the counterclockwise direction looking along $\mathbf{a}$ from head to tail.

14.5.45 Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$; then $\mathbf{F} = \mathbf{a} \times \mathbf{r} = (a_2 z - a_3 y)\,\mathbf{i} + (a_3 x - a_1 z)\,\mathbf{j} + (a_1 y - a_2 x)\,\mathbf{k}$, so that $\nabla \times \mathbf{F} = \frac{\partial}{\partial x}(a_1 + a_1)\,\mathbf{i} + \frac{\partial}{\partial y}(a_2 + a_2)\,\mathbf{j} + \frac{\partial}{\partial z}(a_3 + a_3)\,\mathbf{k} = 2\mathbf{a}$.

14.5.46 The field switches from inward-pointing to outward-pointing at points where it is tangent to the circle $x^2 + y^2 = 2$, i.e. where it is orthogonal to the normal to the circle. The normal to the circle at (x,y) is a multiple of (x,y), so we want to find x, y so that $\langle x,y \rangle \cdot \langle x^2, y \rangle = x^3 + y^2 = 0$ with $x^2 + y^2 = 2$. Thus $x^3 - x^2 + 2 = 0$. The solutions are $x = -1$ and $y = \pm 1$.

14.5.47 $\operatorname{div} \mathbf{F} = 2x + 2xyz + 2x = 2x\,(yz + 2)$; this function clearly achieves its maximum magnitude at $(-1,1,1)$, $(-1,-1,-1)$, $(1,1,1)$, and $(1,-1,-1)$, where its magnitude is 6.

14.5.48 For $\mathbf{F} = \langle z, 0, -y \rangle$, $\operatorname{curl} \mathbf{F} = \langle -1, 1, 0 \rangle$.

a. The component of $\operatorname{curl} \mathbf{F}$ in the direction $\langle 1,0,0 \rangle$ is -1.

b. The component of $\operatorname{curl} \mathbf{F}$ in the direction $\langle 1,-1,1 \rangle$ is $\frac{1}{\sqrt{2}} \langle -1,1,0 \rangle \cdot \langle 1,-1,1 \rangle = -\sqrt{2}$.

c. The component of $\operatorname{curl} \mathbf{F}$ in the direction $\langle a,b,c \rangle$ is $\frac{b-a}{a^2+b^2+c^2}$; this has its maximum in the direction $\langle 1,-1,0 \rangle$.

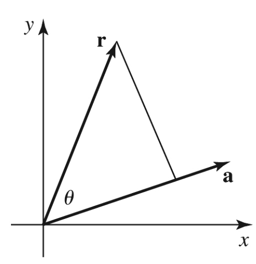
14.5.49 $\operatorname{curl} \mathbf{F} = \langle 0+2, 0+1, 0-1 \rangle = \langle 2,1,-1 \rangle$. If $\mathbf{n} = \langle a,b,c \rangle$, then $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = 0$ when $2a + b - c = 0$ so that $c = 2a + b$; thus all such vectors are of the form $\langle a, b, 2a+b \rangle$, where a, b are real numbers.

14.5.50 $\mathbf{F} = \langle z, 0, 0 \rangle$, or $\mathbf{F} = \langle 0, 0, -x \rangle$, so it is not unique.

14.5.51 $\mathbf{F} = \frac{1}{2} \langle y^2 + z^2, 0, 0 \rangle$ or $\mathbf{F} = \langle 0, -xy, -xz \rangle$, so it is not unique.

14.5.52

a. Looking at the picture, it is clear that the distance from P to $\mathbf{a}$ is $|\mathbf{r}|$.



b. The velocity field is $\mathbf{a} \times \mathbf{r}$, so the speed, which is the magnitude of velocity, is $|\mathbf{a} \times \mathbf{r}|$. Now $|\mathbf{a} \times \mathbf{r}| = |\mathbf{a}| \cdot |\mathbf{r}| \cdot \sin(\theta) \cdot \mathbf{n} = R\,|\mathbf{a}|\,\mathbf{n}$, where $\mathbf{n}$ is a vector normal to the plane determined by $\mathbf{a}$ and $\mathbf{r}$. Thus the motion of the particle is always perpendicular to this plane, so it rotates about the axis $\mathbf{a}$. It is moving at a speed $R\,|\mathbf{a}|$ around a circle of radius R, so its angular speed is $\frac{R|\mathbf{a}|}{R} = |\mathbf{a}|$.

c. Because $|\nabla \times \mathbf{v}| = 2\,|\mathbf{a}|$, it follows from part (b) that $\omega = |\mathbf{a}| = \frac{1}{2}\,|\nabla \times \mathbf{v}|$.

14.5.53 The curl of this vector field is $\langle 0,1,0\rangle$. The component of the curl along some unit vector $\mathbf{n}$ is $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$.

a. $\langle 0,1,0\rangle \cdot \langle 1,0,0\rangle = 0$, so the wheel does not spin.

b. $\langle 0,1,0\rangle \cdot \langle 0,1,0\rangle = 1$, so the wheel spins clockwise (looking towards positive y).

c. $\langle 0,1,0\rangle \cdot \langle 0,0,1\rangle = 0$, so the wheel does not spin.

14.5.54 The curl of the vector field is $\nabla \times \mathbf{v} = \langle -2,0,2\rangle$.

a. The wheel is placed with its axis in the direction $\langle 0,0,1\rangle$, so the component of velocity in that direction is $\langle -2,0,2\rangle \cdot \langle 0,0,1\rangle = 2$, and $\omega = \frac{1}{2} \cdot 2 = 1$.

b. The wheel is placed with its axis in the direction $\langle 0,1,0\rangle$, so the component of velocity in that direction is $\langle -2,0,2\rangle \cdot \langle 0,1,0\rangle = 0$, and the wheel does not turn.

c. The wheel is placed with its axis in the direction $\langle 1,0,0\rangle$, so the component of velocity in that direction is $\langle -2,0,2\rangle \cdot \langle 1,0,0\rangle = -2$, and $\omega = \frac{1}{2} \cdot |-2| = 1$.

14.5.55 The curl of the vector field is $\nabla\times$v$=\langle -20,0,0\rangle$. Because the wheel is placed with its axis normal to the plane $x+y+z=1$, its axis must point in the direction $\langle 1,1,1\rangle$ (with unit vector $\frac{1}{\sqrt{3}}\langle 1,1,1\rangle$). Thus, the component of velocity along that direction is $\frac{1}{\sqrt{3}}\langle -20,0,0\rangle \cdot \langle 1,1,1\rangle = \frac{-20}{\sqrt{3}}$ and then ω is the absolute value of one half of that amount, or $\omega = \frac{10}{\sqrt{3}}$ or $\frac{5}{\pi\sqrt{3}} \approx 0.9189$ revolutions per time unit.

14.5.56 $\mathbf{F} = -100k\nabla e^{-\sqrt{x^2+y^2+z^2}} = \frac{100ke^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\langle x,y,z\rangle$. Looking at the x component, its contribution to the divergence is $100k\frac{\partial}{\partial x}\left[\frac{xe^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right] = -100k\frac{\left(x^2\sqrt{x^2+y^2+z^2}-y^2-z^2\right)e^{-\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)^{3/2}}$ and similarly for the y and z components. Thus the divergence is the sum of these three terms, which is $-100k\frac{e^{-\sqrt{x^2+y^2+z^2}}}{(x^2+y^2+z^2)^{3/2}}\left(\left(x^2+y^2+z^2\right)^{3/2} - 2\left(x^2+y^2+z^2\right)\right) = 100k\frac{e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\left(2-\sqrt{x^2+y^2+z^2}\right)$.

14.5.57 $\mathbf{F} = -100k\nabla e^{-x^2+y^2+z^2} = -200ke^{-x^2+y^2+z^2}\langle -x,y,z\rangle$, so the divergence is

$$\begin{aligned}
&-200k\left(\frac{\partial}{\partial x}\left(-xe^{-x^2+y^2+z^2}\right) + \frac{\partial}{\partial y}\left(ye^{-x^2+y^2+z^2}\right) + \frac{\partial}{\partial z}\left(ze^{-x^2+y^2+z^2}\right)\right)\\
&= -200k\Big(-e^{-x^2+y^2+z^2} + 2x^2e^{-x^2+y^2+z^2} + e^{-x^2+y^2+z^2} + 2y^2e^{-x^2+y^2+z^2} + e^{-x^2+y^2+z^2} + 2z^2e^{-x^2+y^2+z^2}\Big)\\
&= -200k\left(e^{-x^2+y^2+z^2} + 2\left(x^2+y^2+z^2\right)e^{-x^2+y^2+z^2}\right)\\
&= -200k\left(2x^2+2y^2+2z^2+1\right)e^{-x^2+y^2+z^2}.
\end{aligned}$$

14.5.58 $\mathbf{F} = -100k\nabla\left(1+\sqrt{1+x^2+y^2+z^2}\right) = -100k\left(x^2+y^2+z^2\right)^{-1/2}\langle x,y,z\rangle$, and thus the divergence is $\nabla \cdot \mathbf{F} = \frac{-200k}{\sqrt{x^2+y^2+z^2}}$

14.5.59

a. $\mathbf{F} = -\boldsymbol{\nabla}\varphi = -GMm\left\langle \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right], \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right], \frac{\partial}{\partial z}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right]\right\rangle =$
$-GMm\left(x^2+y^2+z^2\right)^{-3/2}\langle -x,-y,-z\rangle = GMm\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle = GMm\frac{\mathbf{r}}{|\mathbf{r}|^3}$.

b. $\frac{\partial}{\partial y}x\left(x^2+y^2+z^2\right)^{-3/2} = -3xy\left(x^2+y^2+z^2\right)^{-5/2}$. Applying this pattern in computing the curl gives $\boldsymbol{\nabla}\times\mathbf{F} = GMm\left(x^2+y^2+z^2\right)^{-5/2}\left((-3yz+3yz)\,\mathbf{i}+(-3xz+3xz)\,\mathbf{j}+(-3xy+3xy)\,\mathbf{k}\right) = \mathbf{0}$, so the field is irrotational.

14.5.60 Note: this is identical to the previous problem except for the constant.

a. $\mathbf{F} = -\boldsymbol{\nabla}\varphi = -\frac{q}{4\pi\epsilon_0}\left\langle \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right], \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right], \frac{\partial}{\partial z}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right]\right\rangle =$
$-\frac{q}{4\pi\epsilon_0}\left(x^2+y^2+z^2\right)^{-3/2}\langle -x,-y,-z\rangle = \frac{q}{4\pi\epsilon_0}\left(x^2+y^2+z^2\right)^{-3/2}\langle x,y,z\rangle = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3}$.

b. $\frac{\partial}{\partial y}x\left(x^2+y^2+z^2\right)^{-3/2} = -3xy\left(x^2+y^2+z^2\right)^{-5/2}$. Applying this pattern in computing the curl gives $\boldsymbol{\nabla}\times\mathbf{F} = \frac{q}{4\pi\epsilon_0}\left(x^2+y^2+z^2\right)^{-5/2}\left((-3yz+3yz)\,\mathbf{i}+(-3xz+3xz)\,\mathbf{j}+(-3xy+3xy)\,\mathbf{k}\right) = \mathbf{0}$, so the field is irrotational.

14.5.61 Using Exercise 40, we have

$$\rho\left(\langle\frac{\partial u}{\partial t},\frac{\partial v}{\partial t},\frac{\partial w}{\partial t}\rangle + \left(u\frac{\partial}{\partial t}+v\frac{\partial}{\partial t}+w\frac{\partial}{\partial t}\right)\langle u,v,w\rangle\right) = -\langle\frac{\partial p}{\partial t},\frac{\partial p}{\partial t},\frac{\partial p}{\partial t}\rangle + \mu\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial t^2}\right)\langle u,v,w\rangle,$$

so that

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$
$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$
$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right).$$

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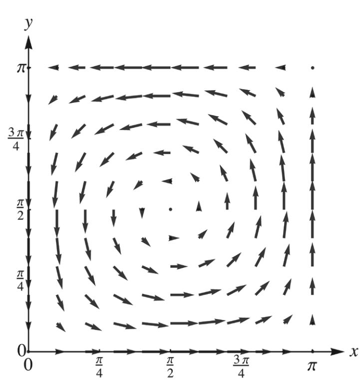
14.5.62

a. $\boldsymbol{\nabla}\times\langle 2,-3y,5z\rangle = \mathbf{0}$ and $\boldsymbol{\nabla}\times\langle y,x-z,-y\rangle = \mathbf{0}$ so they are both irrotational.

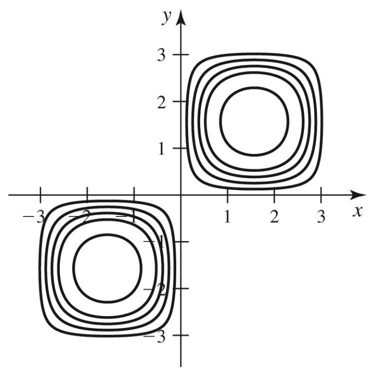
b. If ψ is defined as stated, then $\boldsymbol{\nabla}^2\psi = \frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi + \frac{\partial^2}{\partial z^2}\psi = -\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}$ while $\zeta = \mathrm{k}\cdot\boldsymbol{\nabla}\times\langle u,v,0\rangle = \mathrm{k}\cdot\left(-\frac{\partial v}{\partial z}\mathrm{i}+\frac{\partial u}{\partial z}\mathrm{j}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\mathrm{k}\right) = \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$ so that $\boldsymbol{\nabla}^2\psi = -\zeta$ as desired.

c. $u = \frac{\partial\psi}{\partial y} = \sin(x)\cos(y)$ and $v = -\frac{\partial\psi}{\partial x} = -\cos(x)\sin(y)$. The velocity field looks like



d. The vorticity function is $\zeta = -\nabla^2\psi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \sin(x)\sin(y) + \sin(x)\sin(y) = 2\sin(x)\sin(y)$

The diagram shows level curves for ζ at $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1 and $\frac{3}{2}$ (from outer to inner).

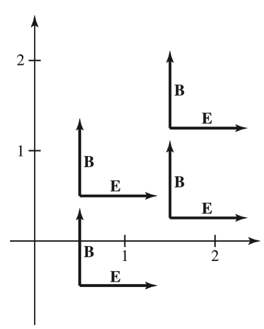


Using implicit differentiation (or from looking at the diagram), ζ achieves its maximum at $x = y = \frac{\pi}{2}$, where it has value 2, and its minimum on the boundary, where it is zero.

14.5.63

a. We have $\nabla \times \mathbf{B} = -\frac{\partial}{\partial z}(A\sin(kz - \omega t))\,\mathbf{i} + 0\mathbf{j} + \frac{\partial}{\partial x}(A\sin(kz-\omega t))\,\mathbf{k} = -Ak\cos(kz-\omega t)\,\mathbf{i}$.
$C\frac{\partial \mathbf{E}}{\partial t} = C\frac{\partial}{\partial t}(A\sin(kz-\omega t)\,\mathbf{i}) = -\omega CA\cos(kz-\omega t)\,\mathbf{i}$ so that the two are equal when $k = \omega C$, or $\omega = \frac{k}{C}$.

b.



14.5.64 Let $\mathbf{V} = \langle xy, -\frac{1}{2}y^2, 0\rangle$ and $\mathbf{W} = \langle 0, \frac{1}{2}y^2, 0\rangle$. Then $\nabla \cdot \mathbf{V} = 0$ and $\nabla \times \mathbf{W} = \mathbf{0}$.

14.5.65 Let $\mathbf{F} = \langle f, g, h\rangle$ and $\mathbf{G} = \langle k, m, n\rangle$. Then

a. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \langle f+k, g+m, h+n\rangle = \frac{\partial}{\partial x}(f+k) + \frac{\partial}{\partial y}(g+m) + \frac{\partial}{\partial z}(h+n) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} + \frac{\partial k}{\partial x} + \frac{\partial m}{\partial y} + \frac{\partial n}{\partial z} = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$.

b. $\nabla \times (\mathbf{F}+\mathbf{G}) = \left(\frac{\partial}{\partial y}(h+n) - \frac{\partial}{\partial z}(g+m)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(f+k) - \frac{\partial}{\partial x}(h+n)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(g+m) - \frac{\partial}{\partial y}(f+k)\right)\mathbf{k} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} + \left(\frac{\partial n}{\partial y} - \frac{\partial m}{\partial z}\right)\mathbf{i} + \left(\frac{\partial k}{\partial z} - \frac{\partial n}{\partial x}\right)\mathbf{j} + \left(\frac{\partial m}{\partial x} - \frac{\partial k}{\partial y}\right)\mathbf{k} = \nabla\times\mathbf{F} + \nabla\times\mathbf{G}$.

c. $\nabla \cdot (c\mathbf{F}) = \frac{\partial}{\partial x}(cf) + \frac{\partial}{\partial y}(cg) + \frac{\partial}{\partial z}(ch) = c\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) = c(\nabla \cdot \mathbf{F})$.

d. $\nabla \times (c\mathbf{F}) = \left(\frac{\partial}{\partial y}(ch) - \frac{\partial}{\partial z}(cg)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(cf) - \frac{\partial}{\partial x}(ch)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(cg) - \frac{\partial}{\partial y}(cf)\right)\mathbf{k} = c\left[\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}\right] = c(\nabla \times \mathbf{F})$.

14.5.66 The statement is not true. The conditions imply that $\mathbf{F}-\mathbf{G}$ is irrotational and source-free, but this can happen with nonconstant vector fields. For example, if $\mathbf{F}=\langle x^2-y^2,-2xy\rangle$ and $\mathbf{G}=\langle 2xy,x^2-y^2\rangle$, then $\mathbf{F}-\mathbf{G}$ is irrotational and source-free (i.e. has zero curl and zero divergence); in fact, both $\mathbf{F}$ and $\mathbf{G}$ do, but clearly the two vector fields do not differ by a constant.

14.5.67 $\nabla\cdot(\varphi\mathbf{F})=\nabla\cdot\langle\varphi f,\varphi g,\varphi h\rangle=\frac{\partial}{\partial x}(\varphi f)+\frac{\partial}{\partial y}(\varphi g)+\frac{\partial}{\partial z}(\varphi h)=\varphi\frac{\partial f}{\partial x}+f\frac{\partial\varphi}{\partial x}+\varphi\frac{\partial g}{\partial y}+g\frac{\partial\varphi}{\partial y}+\varphi\frac{\partial h}{\partial z}+h\frac{\partial\varphi}{\partial z}=\varphi\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)+\langle f,g,h\rangle\langle\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}\rangle=\varphi\nabla\cdot\mathbf{F}+\nabla\boldsymbol{\varphi}\cdot\mathbf{F}$

14.5.68 $\nabla\times(\varphi\mathbf{F})=\nabla\times\langle\varphi f,\varphi g,\varphi h\rangle=\langle\frac{\partial}{\partial y}(\varphi h)-\frac{\partial}{\partial z}(\varphi g),\frac{\partial}{\partial z}(\varphi f)-\frac{\partial}{\partial x}(\varphi h),\frac{\partial}{\partial x}(\varphi g)-\frac{\partial}{\partial y}(\varphi f)\rangle=\langle\varphi\frac{\partial h}{\partial y}+h\frac{\partial\varphi}{\partial y}-\varphi\frac{\partial g}{\partial z}-g\frac{\partial\varphi}{\partial z},\varphi\frac{\partial f}{\partial z}+f\frac{\partial\varphi}{\partial z}-\varphi\frac{\partial h}{\partial x}-h\frac{\partial\varphi}{\partial x},\varphi\frac{\partial g}{\partial x}+g\frac{\partial\varphi}{\partial x}-\varphi\frac{\partial f}{\partial y}-f\frac{\partial\varphi}{\partial y}\rangle=\langle\varphi\frac{\partial h}{\partial y}-\varphi\frac{\partial g}{\partial zy},\varphi\frac{\partial f}{\partial z}-\varphi\frac{\partial h}{\partial x},\varphi\frac{\partial g}{\partial x}-\varphi\frac{\partial f}{\partial y}\rangle+\langle h\frac{\partial\varphi}{\partial y}-g\frac{\partial\varphi}{\partial z},f\frac{\partial\varphi}{\partial z}-h\frac{\partial\varphi}{\partial x},g\frac{\partial\varphi}{\partial x}-f\frac{\partial\varphi}{\partial y}\rangle=\varphi\nabla\times\mathbf{F}+\nabla\boldsymbol{\varphi}\times\mathbf{F}$

14.5.69 If $\mathbf{F}=\langle f,g,h\rangle$ and $\mathbf{G}=\langle k,m,n\rangle$, then

$$\begin{aligned}\mathbf{G}\cdot(\nabla\times\mathbf{F})-\mathbf{F}\cdot(\nabla\times\mathbf{G})&=\langle k,m,n\rangle\cdot\langle\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z},\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x},\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\rangle\\&\quad-\langle f,g,h\rangle\cdot\langle\frac{\partial n}{\partial y}-\frac{\partial m}{\partial z},\frac{\partial k}{\partial z}-\frac{\partial n}{\partial x},\frac{\partial m}{\partial x}-\frac{\partial k}{\partial y}\rangle\\&=k\frac{\partial h}{\partial y}-k\frac{\partial g}{\partial z}+m\frac{\partial f}{\partial z}-m\frac{\partial h}{\partial x}+n\frac{\partial g}{\partial x}-n\frac{\partial f}{\partial y}\\&\quad-f\frac{\partial n}{\partial y}+f\frac{\partial m}{\partial z}-g\frac{\partial k}{\partial z}+g\frac{\partial n}{\partial x}-h\frac{\partial m}{\partial x}+h\frac{\partial k}{\partial y}\\&=n\frac{\partial g}{\partial x}+g\frac{\partial n}{\partial x}-h\frac{\partial m}{\partial x}-m\frac{\partial h}{\partial x}+h\frac{\partial k}{\partial y}+k\frac{\partial h}{\partial y}-f\frac{\partial n}{\partial y}-n\frac{\partial f}{\partial y}\\&\quad+f\frac{\partial m}{\partial z}+m\frac{\partial f}{\partial z}-g\frac{\partial k}{\partial z}-k\frac{\partial g}{\partial z}\\&=\frac{\partial}{\partial x}(gn-hm)+\frac{\partial}{\partial y}(hk-fn)+\frac{\partial}{\partial z}(fm-gk)=\nabla\cdot(\mathbf{F}\times\mathbf{G}).\end{aligned}$$

14.5.70 First, $(\mathbf{G}\cdot\nabla)\mathbf{F}=\left(k\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)\langle f,g,h\rangle=\langle k\frac{\partial f}{\partial x}+m\frac{\partial f}{\partial y}+n\frac{\partial f}{\partial z},k\frac{\partial g}{\partial x}+m\frac{\partial g}{\partial y}+n\frac{\partial g}{\partial z},k\frac{\partial h}{\partial x}+m\frac{\partial h}{\partial y}+n\frac{\partial h}{\partial z}\rangle$ and similarly for $(\mathbf{F}\cdot\nabla)\mathbf{G}$. Next, $\mathbf{G}(\nabla\cdot\mathbf{F})=\langle k,m,n\rangle\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)=\langle k\frac{\partial f}{\partial x}+k\frac{\partial g}{\partial y}+k\frac{\partial h}{\partial z},m\frac{\partial f}{\partial x}+m\frac{\partial g}{\partial y}+m\frac{\partial h}{\partial z},n\frac{\partial f}{\partial x}+n\frac{\partial g}{\partial y}+n\frac{\partial h}{\partial z}\rangle$ and similarly for $\mathbf{F}(\nabla\cdot\mathbf{G})$. Thus

$$\begin{aligned}&(\mathbf{G}\cdot\nabla)-\mathbf{G}(\nabla\cdot\mathbf{F})-(\mathbf{F}\cdot\nabla)\mathbf{G}+\mathbf{F}(\nabla\cdot\mathbf{G})\\&=\langle k\frac{\partial f}{\partial x}+m\frac{\partial f}{\partial y}+n\frac{\partial f}{\partial z},k\frac{\partial g}{\partial x}+m\frac{\partial g}{\partial y}+n\frac{\partial g}{\partial z},k\frac{\partial h}{\partial x}+m\frac{\partial h}{\partial y}+n\frac{\partial h}{\partial z}\rangle\\&\quad-\langle k\frac{\partial f}{\partial x}+k\frac{\partial g}{\partial y}+k\frac{\partial h}{\partial z},m\frac{\partial f}{\partial x}+m\frac{\partial g}{\partial y}+m\frac{\partial h}{\partial z},n\frac{\partial f}{\partial x}+n\frac{\partial g}{\partial y}+n\frac{\partial h}{\partial z}\rangle\\&\quad-\langle f\frac{\partial k}{\partial x}+g\frac{\partial k}{\partial y}+h\frac{\partial k}{\partial z},f\frac{\partial m}{\partial x}+g\frac{\partial m}{\partial y}+h\frac{\partial m}{\partial z},f\frac{\partial n}{\partial x}+g\frac{\partial n}{\partial y}+h\frac{\partial n}{\partial z}\rangle\\&\quad+\langle f\frac{\partial k}{\partial x}+f\frac{\partial m}{\partial y}+f\frac{\partial n}{\partial z},g\frac{\partial k}{\partial x}+g\frac{\partial m}{\partial y}+g\frac{\partial n}{\partial z},h\frac{\partial k}{\partial x}+h\frac{\partial m}{\partial y}+h\frac{\partial n}{\partial z}\rangle\\&=\langle\frac{\partial}{\partial y}(fm-gk)-\frac{\partial}{\partial z}(hk-fn),\frac{\partial}{\partial z}(gn-hm)-\frac{\partial}{\partial x}(fm-gk),\frac{\partial}{\partial x}(hk-fn)-\frac{\partial}{\partial y}(gn-hm).\end{aligned}$$

But $\mathbf{F}\times\mathbf{G}=\langle gn-hm,hk-fn,fm-gk\rangle$, so the above expression is indeed equal to $\nabla\times(\mathbf{F}\times\mathbf{G})$.

14.5.71 Use the values of$(\mathbf{G}\cdot\nabla)\mathbf{F}$ and $(\mathbf{F}\cdot\nabla)\mathbf{G}$ from the previous problem. Then $\mathbf{G}\times(\nabla\times\mathbf{F})=\langle k,m,n\rangle\times\langle\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z},\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x},\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\rangle=\langle m\frac{\partial g}{\partial x}-m\frac{\partial f}{\partial y}-n\frac{\partial f}{\partial z}+n\frac{\partial h}{\partial x},n\frac{\partial h}{\partial y}-n\frac{\partial g}{\partial z}-k\frac{\partial g}{\partial x}+k\frac{\partial f}{\partial y},k\frac{\partial f}{\partial z}-k\frac{\partial h}{\partial x}-m\frac{\partial h}{\partial y}+m\frac{\partial g}{\partial z}\rangle$ and similarly for $\mathbf{F}\times(\nabla\times\mathbf{G})$.

Thus

$$
\begin{aligned}
&(\mathbf{G}\cdot\boldsymbol{\nabla})\,\mathbf{F}+(\mathbf{F}\cdot\boldsymbol{\nabla})\,\mathbf{G}+\mathbf{G}\times(\boldsymbol{\nabla}\times\mathbf{F})+\mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{G})\\
&=\langle k\frac{\partial f}{\partial x}+m\frac{\partial f}{\partial y}+n\frac{\partial f}{\partial z},k\frac{\partial g}{\partial x}+m\frac{\partial g}{\partial y}+n\frac{\partial g}{\partial z},k\frac{\partial h}{\partial x}+m\frac{\partial h}{\partial y}+n\frac{\partial h}{\partial z}\rangle\\
&\quad+\langle f\frac{\partial k}{\partial x}+g\frac{\partial k}{\partial y}+h\frac{\partial k}{\partial z},f\frac{\partial m}{\partial x}+g\frac{\partial m}{\partial y}+h\frac{\partial m}{\partial z},f\frac{\partial n}{\partial x}+g\frac{\partial n}{\partial y}+h\frac{\partial n}{\partial z}\rangle\\
&\quad+\langle m\frac{\partial g}{\partial x}-m\frac{\partial f}{\partial y}-n\frac{\partial f}{\partial z}+n\frac{\partial h}{\partial x},n\frac{\partial h}{\partial y}-n\frac{\partial g}{\partial z}-k\frac{\partial g}{\partial x}+k\frac{\partial f}{\partial y},k\frac{\partial f}{\partial z}-k\frac{\partial h}{\partial x}-m\frac{\partial h}{\partial y}+m\frac{\partial g}{\partial z}\rangle\\
&\quad+\langle g\frac{\partial m}{\partial x}-g\frac{\partial k}{\partial y}-h\frac{\partial k}{\partial z}+h\frac{\partial n}{\partial x},h\frac{\partial n}{\partial y}-h\frac{\partial m}{\partial z}-f\frac{\partial m}{\partial x}+f\frac{\partial k}{\partial y},f\frac{\partial k}{\partial z}-f\frac{\partial n}{\partial x}-g\frac{\partial n}{\partial y}+g\frac{\partial m}{\partial z}\rangle\\
&=\langle\frac{\partial}{\partial x}(fk+gm+hn),\frac{\partial}{\partial y}(fk+gm+hn),\frac{\partial}{\partial z}(fk+gm+hn)\rangle=\nabla(\mathbf{F}\cdot\mathbf{G}).
\end{aligned}
$$

14.5.72

$$
\begin{aligned}
&\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F})=\boldsymbol{\nabla}\times\langle\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z},\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x},\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\rangle\\
&\quad=\langle\frac{\partial^2 g}{\partial y\partial x}-\frac{\partial^2 f}{\partial y^2}-\frac{\partial^2 f}{\partial z^2}+\frac{\partial^2 h}{\partial z\partial x},\frac{\partial^2 h}{\partial z\partial y}-\frac{\partial^2 g}{\partial z^2}-\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 f}{\partial x\partial y},\frac{\partial^2 f}{\partial x\partial z}-\frac{\partial^2 h}{\partial x^2}-\frac{\partial^2 h}{\partial^2 y}-\frac{\partial^2 g}{\partial y\partial z}\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
&\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F})-(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\,\mathbf{F}=\boldsymbol{\nabla}\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\langle f,g,h\rangle\\
&=\langle\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 g}{\partial x\partial y}+\frac{\partial^2 h}{\partial x\partial z},\frac{\partial^2 f}{\partial y\partial x}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 h}{\partial y\partial z},\frac{\partial^2 f}{\partial z\partial x}+\frac{\partial^2 g}{\partial z\partial y}+\frac{\partial^2 h}{\partial z^2}\rangle\\
&\quad-\langle\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2},\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2},\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}\rangle.
\end{aligned}
$$

The two expressions are equal after cancellations and noting that mixed partials are equal.

14.5.73

$$
\begin{aligned}
&\nabla\cdot\frac{\langle x,y,z\rangle}{(x^2+y^2+z^2)^{p/2}}\\
&=\frac{(1-p)\,x^2+y^2+z^2}{(x^2+y^2+z^2)^{1+p/2}}+\frac{x^2+(1-p)\,y^2+z^2}{(x^2+y^2+z^2)^{1+p/2}}+\frac{x^2+y^2+(1-p)\,z^2}{(x^2+y^2+z^2)^{1+p/2}}\\
&=\frac{(3-p)\left(x^2+y^2+z^2\right)}{(x^2+y^2+z^2)^{1+p/2}}=\frac{3-p}{(x^2+y^2+z^2)^{p/2}}=\frac{3-p}{|\mathbf{r}|^p}
\end{aligned}
$$

14.5.74 $\boldsymbol{\nabla}\left(\frac{1}{|r|^p}\right)=\boldsymbol{\nabla}\left(\frac{1}{(x^2+y^2+z^2)^{p/2}}\right)=-\frac{p}{(x^2+y^2+z^2)^{1+p/2}}\langle x,y,z\rangle=-\frac{p\,\mathbf{r}}{|\mathbf{r}|^{p+2}}$.

14.5.75 $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\left(\frac{1}{|r|^p}\right)=\boldsymbol{\nabla}\cdot\left(-\frac{p\,\mathbf{r}}{|\mathbf{r}|^{p+2}}\right)$ by Exercise 72, and then by Exercise 71, $\boldsymbol{\nabla}\cdot\left(-\frac{p\,\mathbf{r}}{|\mathbf{r}|^{p+2}}\right)=-p\,\boldsymbol{\nabla}\cdot\frac{\mathbf{r}}{|\mathbf{r}|^{p+2}}=\frac{-p(3-(p+2))}{|\mathbf{r}|^{p+2}}=\frac{p(p-1)}{|\mathbf{r}|^{p+2}}$.

14.6 Surface Integrals

14.6.1 $\mathbf{r}(u,v)=\langle a\cos u,a\sin u,v\rangle$ where $0\le u\le 2\pi$; $0\le v\le h$.

14.6.2 $\mathbf{r}(u,v)=\langle\frac{av}{h}\cos u,\frac{av}{h}\sin u,v\rangle$ where $0\le u\le 2\pi$; $0\le v\le h$.

14.6.3 $\mathbf{r}(u,v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$ where $0 \le u \le \pi$; $0 \le v \le 2\pi$.

14.6.4 A cone of height h and radius a has equation$a^2z^2 = h^2\left(x^2+y^2\right)$; thus $z_x = \frac{h^2x}{a^2z}$ and similarly for z_y, so compute

$$\iint_S f(x,y,z)\, dS = \iint_R f\left(x, y, \frac{h^2}{a^2}\sqrt{x^2+y^2}\right)\sqrt{\left(\frac{h^2x}{a^2z}\right)^2 + \left(\frac{h^2y}{a^2z}\right)^2 + 1}\, dA$$
$$= \iint_R f\left(x, y, \frac{h^2}{a^2}\sqrt{x^2+y^2}\right)\sqrt{1+\frac{h^2}{a^2}}\, dA.$$

14.6.5 Use the parametric description from problem 3 and compute
$\int_0^\pi \int_0^{2\pi} a^2 f(a\sin u\cos v, a\sin u\sin v, a\cos u)\sin u\, du\, dv$.

14.6.6 For $\mathbf{F} = \langle f, g, h\rangle$, evaluate the integral $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R (-f\, z_x - g\, z_y + h)\, dA$ using the explicit form from problem 4.

14.6.7 Using the parameterization from the text, and the fact that for the sphere (see Example 2(b)), $\mathbf{t}_u \times \mathbf{t}_v = \langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\sin u\cos u\rangle$, compute

$$\iint_S \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\, dS = \iint_R a^2\sin u\,(f\sin u\cos v + g\sin u\sin v + h\cos u)\, dA$$
$$= \int_0^\pi \int_0^{2\pi} a^2\sin u\,(f\sin u\cos v + g\sin u\sin v + h\cos u)\, dv\, du.$$

14.6.8 It means that we can make a consistent choice of normal vectors such that when you walk along the surface, the direction of the normal vectors does not change discontinuously.

14.6.9 The usual orientation of a closed surface is that the normal vectors point outwards.

14.6.10 Because the vector field is vertical, the same amount of materials goes through the surface as through its projection on the xy-plane.

14.6.11 $\langle u, v, \frac{1}{3}(16-2u+4v)\rangle$, $|u| < \infty$, $|v| < \infty$.

14.6.12 $\langle 4\sin u\cos v, 4\sin u\sin v, 4\cos v\rangle$, $0 \le u \le \frac{\pi}{4}$, $0 \le v \le 2\pi$.

14.6.13 $\langle v\cos u, v\sin u, v\rangle$, $0 \le u \le 2\pi$, $2 \le v \le 8$.

14.6.14 $\left\langle \frac{v}{2}\cos u, \frac{v}{2}\sin u, v\right\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 4$.

14.6.15 $\langle 3\cos u, 3\sin u, v\rangle$, $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 3$.

14.6.16 $\langle v, 6\cos u, 6\sin u\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 9$.

14.6.17 The segment of the plane $z = 2x+3y-1$ above $[1,3]\times[2,4]$.

14.6.18 The segment of the plane $z = 2-y$ above $[0,2]\times[0,4]$.

14.6.19 The portion of the cone $z^2 = 16x^2+16y^2$ of height 12 and radius 3, where $y \ge 0$.

14.6.20 The cylinder $y^2+z^2 = 36$ of radius 6 whose axis is the x-axis, for $0 \le x \le 2$.

14.6.21 Using the standard parametric description of the cylinder, we have $\mathbf{r}(u,v) = \langle 4\cos u, 4\sin u, v\rangle$ for $0 \le v \le 7$, $0 \le u \le \pi$. Then $|\mathbf{t}_u \times \mathbf{t}_v| = 4$ and the area is $\iint_S 1\, dS = \iint_R 4\, dA = \int_0^\pi \int_0^7 4\, dv\, du = 28\pi$.

14.6.22 The plane has the parametric description $\mathbf{r}(u,v) = \langle u, v, 3-u-3v\rangle$, $0 \le v \le 1$, $0 \le u \le 3-3v$. Then $\mathbf{t}_u \times \mathbf{t}_v = \langle 1,0,-1\rangle \times \langle 0,1,-3\rangle = \langle 1,3,1\rangle$, so that $|\mathbf{t}_u \times \mathbf{t}_v| = \sqrt{11}$. Then $\iint\limits_S 1\,dS = \sqrt{11}\iint\limits_R 1\,dA = \sqrt{11}\int_0^1\int_0^{3-3v} 1\,du\,dv = \sqrt{11}\int_0^1 (3-3v)\,dv = \frac{3\sqrt{11}}{2}$.

14.6.23 The plane has parametric description $\mathbf{r}(u,v) = \langle u, v, 10-u-v\rangle$, for $-2 \le u \le 2$, $-2 \le v \le 2$. Then $\mathbf{t}_u \times \mathbf{t}_v = \langle 1,0,-1\rangle \times \langle 0,1,-1\rangle = \langle 1,1,1\rangle$, so that $\iint\limits_S 1\,dS = \sqrt{3}\iint\limits_R 1\,dA = 16\sqrt{3}$.

14.6.24 Using the standard parametric description of the sphere with $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 2\pi$, we have $|\mathrm{t}_u \times \mathrm{t}_v| = a^2 \sin u$ so that $\iint\limits_S 1\,dS = \iint\limits_R 100\,\sin u\,dA = \int_0^{\pi/2}\int_0^{2\pi} 100\sin u\,dv\,du = 200\pi\int_0^{\pi/2}\sin u\,du = 200\pi$.

14.6.25 Parameterize the cone by $\mathbf{r}(u,v) = \left\langle \frac{r}{h}v\cos u, \frac{r}{h}v\sin u, v\right\rangle$, for $0 \le v \le h$, $0 \le u \le 2\pi$; then $\mathbf{t}_u \times \mathbf{t}_v = \left\langle -\frac{r}{h}v\sin u, \frac{r}{h}v\cos u, 0\right\rangle \times \left\langle \frac{r}{h}\cos u, \frac{r}{h}\sin u, 1\right\rangle$ and $|\mathrm{t}_u \times \mathrm{t}_v| = \frac{r}{h^2}v\sqrt{h^2+r^2}$. Then $\iint\limits_S 1\,dS = \frac{r\sqrt{h^2+r^2}}{h^2}\iint\limits_R v\,dA = \frac{r\sqrt{h^2+r^2}}{h^2}\int_0^{2\pi}\int_0^h v\,dv\,du = \frac{r\sqrt{h^2+r^2}}{h^2}\int_0^{2\pi}\frac{1}{2}h^2\,du = \frac{2\pi r\sqrt{h^2+r^2}}{2} = \pi r\sqrt{h^2+r^2}$.

14.6.26 Using the standard parameterization of the sphere for $1 \le 2\cos u \le 2$, or $0 \le u \le \frac{\pi}{3}$, and $0 \le v \le 2\pi$, we obtain $\iint\limits_S 1\,dS = \iint\limits_R 4\,\sin u\,dA = 4\int_0^{2\pi}\int_0^{\pi/3}\sin u\,du\,dv = 4\int_0^{2\pi}(-\cos u)\big|_{u=0}^{u=\pi/3}\,dv = 4\int_0^{2\pi}\frac{1}{2}\,dv = 4\pi$.

14.6.27 Using the standard parameterization of the sphere for $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 2\pi$, we obtain $\iint\limits_S (x^2+y^2)\,dS = \int_0^{2\pi}\int_0^{\pi/2} 36\sin^2 u \cdot 36\sin u\,du\,dv = 1296\int_0^{2\pi}\int_0^{\pi/2}\sin^3 u\,du\,dv = 1296\int_0^{2\pi}\frac{2}{3}\,dv = 1728\pi$.

14.6.28 Use the standard parameterization of the cylinder, for $0 \le u \le 2\pi$, $0 \le v \le 3$; then $\iint\limits_S y\,dS = \int_0^{2\pi}\int_0^3 3\sin u \cdot 3\,dv\,du = 27\int_0^{2\pi}\sin u\,du = 0$.

14.6.29 Use the standard parameterization, for $0 \le u \le 2\pi$, $0 \le v \le 3$; then $\iint\limits_S x\,dS = \int_0^{2\pi}\int_0^3 \cos u \cdot 1\,dv\,du = 3\int_0^{2\pi}\cos u\,du = 0$.

14.6.30 Using the standard parameterization (with $u = \varphi$ and $v = \theta$) for $0 \le u \le \frac{\pi}{2}$ and $0 \le v \le \frac{\pi}{2}$, we have $\iint\limits_R \cos u\,dA = \int_0^{\pi/2}\int_0^{\pi/2}\cos u \cdot \sin u\,du\,dv = \frac{\pi}{4}$.

14.6.31 $2z\,dz = 8x\,dx$, so $z_x = \frac{4x}{z}$; similarly, $z_y = \frac{4y}{z}$. Thus $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{\frac{16x^2+16y^2+z^2}{z^2}} = \sqrt{\frac{20(x^2+y^2)}{4(x^2+y^2)}} = \sqrt{5}$. Further, this cone sits over $x^2+y^2 = 4$. Then $\iint\limits_S 1\,dS = \iint\limits_R \sqrt{5}\,dA = 4\pi\sqrt{5}$.

14.6.32 $dz = 4x\,dx$ so that $z_x = 4x$ and similarly $z_y = 4y$. The paraboloid sits over $x^2+y^2 = 4$ Thus $\iint\limits_S 1\,dS = \iint\limits_R \sqrt{16x^2+16y^2+1}\,dA = \int_0^{2\pi}\int_0^2 r\sqrt{16r^2+1}\,dr\,d\theta = \frac{(65\sqrt{65}-1)\pi}{24}$.

14.6.33 $z_x = 2x$ and $z_y = 0$, so that $\iint\limits_S 1\,dS = \int_0^4\int_{-2}^2 \sqrt{4x^2+1}\,dx\,dy = 8\sqrt{17} + 2\,\ln\left(\sqrt{17}+4\right)$.

14.6.34 We have $z_x = 2x$, $z_y = -2y$, so $\iint\limits_S 1\,dS = \iint\limits_R \sqrt{4x^2+4y^2+1}\,dA = \int_{-\pi/4}^{\pi/4}\int_0^4 r\sqrt{4r^2+1}\,dr\,d\theta = \frac{(65\sqrt{65}-1)\pi}{24}$.

14.6.35 $z_x = z_y = -1$, so $\iint\limits_S xy\,dS = \sqrt{3}\iint\limits_R xy\,dA = \sqrt{3}\int_0^2\int_0^{2-x} xy\,dy\,dx = \frac{2\sqrt{3}}{3}$.

14.6.36 $z_x = 2x$, $z_y = 2y$, and the paraboloid sits over $x^2+y^2=4$, so $\iint_S \left(x^2+y^2\right) dS =$ $\iint_R \left(x^2+y^2\right)\sqrt{4x^2+4y^2+1}\,dA = \int_0^{2\pi}\int_0^2 r^3\sqrt{4r^2+1}\,dr\,d\theta = \frac{\left(391\sqrt{17}+1\right)\pi}{60}$.

14.6.37 $x^2+y^2+z^2=25$, so $z_x = --\frac{x}{z}$ and $z_y = -\frac{y}{z}$. Then $\iint_S \left(25-x^2-y^2\right) dS =$ $\iint_R \left(25-x^2-y^2\right)\sqrt{\frac{x^2+y^2+z^2}{z^2}}\,dA = 5\iint_R \sqrt{25-x^2-y^2}\,dA = 5\int_0^{2\pi}\int_0^5 r\sqrt{25-r^2}\,dr\,d\theta = \frac{1250\pi}{3}$.

14.6.38 $z_x=-1$, $z_y=-2$, and the limits of integration are $0\le y\le 4$, $0\le x\le 8-2y$. Then $\iint_S e^z\,dS =$ $\iint_R e^{8-x-2y}\sqrt{6}\,dA = \sqrt{6}\int_0^4\int_0^{8-2y} e^{8-x-2y}\,dx\,dy = \frac{\sqrt{6}\left(e^8-9\right)}{2}$.

14.6.39 $z_x=-3$, $z_y=-4$, so $\sqrt{z_x^2+z_y^2+1}=\sqrt{26}$. The area of the part of the plane is $\iint_S 1\,dS =$ $\sqrt{26}\iint_R 1\,dA = 4\sqrt{26}$ and the surface integral of temperature is $\iint_S e^{3x+4y-6}\,dS = \sqrt{26}\int_{-1}^1\int_{-1}^1 e^{3x+4y-6}\,dx\,dy$ $= \frac{\sqrt{26}}{12}\left(e-e^{-5}-e^{-7}+e^{-13}\right)$. Thus the average temperature is the ratio of the two, or $\frac{1}{48}\left(e-e^{-5}-e^{-7}+e^{-13}\right)$.

14.6.40 $z_x=-2x$, $z_y=-2y$. The paraboloid sits over $x^2+y^2=4$. Thus the area of the paraboloid is $\iint_S 1\,dS = \iint_R \sqrt{4x^2+4y^2+1}\,dA = \int_0^{2\pi}\int_0^2 r\sqrt{4r^2+1}\,dr\,d\theta = \frac{\left(17\sqrt{17}-1\right)\pi}{6}$ and the integral of the square of the distance from the origin is $\iint_S \left(x^2+y^2+z^2\right) dS = \iint_R \left(x^2+y^2+\left(4-x^2-y^2\right)^2\right)\sqrt{4x^2+4y^2+1}\,dA =$ $\int_0^{2\pi}\int_0^2 r\left(r^2+\left(4-r^2\right)^2\right)\sqrt{4r^2+1}\,dr\,d\theta = \frac{\left(255\sqrt{17}-39\right)\pi}{14}$. Thus the average squared distance is $\frac{\left(255\sqrt{17}-39\right)\pi}{14}\cdot\frac{6}{\left(17\sqrt{17}-1\right)\pi} = \frac{9\left(85\sqrt{17}-13\right)}{7\left(17\sqrt{17}-1\right)}$.

14.6.41 $z_x=-\frac{x}{z}$ and $z_y=-\frac{y}{z}$, so that $\sqrt{z_x^2+z_y^2+1}=\sqrt{\frac{x^2+y^2+z^2}{z^2}} = \left(1-x^2-y^2\right)^{-1/2}$. The area of the sphere is $\frac{1}{8}\cdot 4\pi = \frac{\pi}{2}$, and the integral of the function is

$$\iint_S xyz\,dS = \iint_R xy\left(1-x^2-y^2\right)^{1/2}\left(1-x^2-y^2\right)^{-1/2}\,dA = \int_0^{\pi/2}\int_0^1 r^3\sin(\theta)\cos(\theta)\,dr\,d\theta = \frac{1}{8},$$

so that the average value is $\frac{1}{4\pi}$.

14.6.42 $z_x=\frac{x}{z}$ and $z_y=\frac{y}{z}$ so that $\sqrt{z_x^2+z_y^2+1}=\sqrt{2}$. The cone sits over $x^2+y^2=4$. The area of the cone is $\iint_S 1\,dS = \sqrt{2}\iint_R dA = 4\pi\sqrt{2}$, and the integral of the temperature function is $\iint_S (100-25z)\,dS =$ $\sqrt{2}\iint_R\left(100-25\sqrt{x^2+y^2}\right)dA = \sqrt{2}\int_0^{2\pi}\int_0^2 r\,(100-25r)\,dr\,d\theta = \frac{800\pi\sqrt{2}}{3}$ so that the average temperature is $\frac{200}{3}$.

14.6.43 $z_x=z_y=-1$, so the normal vector is $\langle 1,1,1\rangle$, which points in the positive z-direction. Then $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R (0\cdot 1+0\cdot 1-1\cdot 1)\,dA = \int_0^4\int_0^{4-x}(-1)\,dy\,dx = -8$.

14.6.44 $z_x=-2$, $z_y=-5$, so the normal vector is $\langle 2,5,1\rangle$, which points in the positive z-direction. $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R (x\cdot 2+y\cdot 5+z\cdot 1)\,dA = \int_0^2\int_0^{(10-5y)/2}(2x+5y+(10-2x-5y))\,dx\,dy = \int_0^2\int_0^{(10-5y)/2} 10\,dx\,dy$ $= 50$.

14.6.45 We have $z = \sqrt{x^2+y^2}$; then $\mathbf{n} = \left\langle -\frac{x}{z}, -\frac{y}{z}, 1 \right\rangle$, which points upwards.
$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \left(\left(-\frac{x}{z}\right)\cdot x + \left(-\frac{y}{z}\right)\cdot y + 1\cdot z\right)\,dA = \iint_R \left(z - \frac{x^2+y^2}{z}\right)\,dA = \iint_R (z-z)\,dA = 0.$

14.6.46 $z_x = 0$, $z_y = -\sin(y)$, so an upward-pointing normal is $\langle 0, \sin y, 1\rangle$. We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R (2\sin y\cos y + xy)\,dA = \int_0^4 \int_{-\pi}^{\pi} (2\sin y\,\cos y + xy)\,dy\,dx = 0.$

14.6.47 An outward-pointing normal is $\frac{\mathbf{r}}{|\mathbf{r}|}$. The sphere has radius a, so the vector field is in fact $\frac{\mathbf{r}}{|\mathbf{r}|^3}$.
$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\frac{\mathbf{r}}{|\mathbf{r}|}\,dS = \iint_S \frac{1}{|\mathbf{r}|^2}dS = \iint_S \frac{1}{a^2}dS = \frac{1}{a^2}\iint_S 1\,dS = \frac{1}{a^2}4\pi\,a^2 = 4\pi.$

14.6.48 The parametric form is $\langle u, u^2, v\rangle$ for $0 \le u \le 1$, $0 \le v \le 4$. We have $\mathbf{t}_u = \langle 1, 2u, 0\rangle$ and $\mathbf{t}_v = \langle 0,0,1\rangle$, so that $\mathbf{t}_u \times \mathbf{t}_v = \langle 2u, -1, 0\rangle$; since we want normal vectors to point in the positive y direction, we choose$\langle -2u, 1, 0\rangle$ for the normal vector. Then $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \langle -u^2, u, 1\rangle \cdot \langle -2u, 1, 0\rangle\,dA = \iint_R \left(u + 2u^3\right)\,dA = \int_0^4\int_0^1 \left(u+2u^3\right)\,du\,dv = 4.$

14.6.49

a. True. The formula in Theorem 14.12 gives $\iint_S f(x,y,z)\,dS = \iint_R f(x,y,10)\sqrt{0+0+1}\,dA.$

b. False. The formula in Theorem 14.12 gives

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,x)\sqrt{1+0+1}\,dA = \sqrt{2}\iint_R f(x,y,x)\,dA.$$

c. True. Substituting $2u$ for u and $\sqrt{v}$ for v in the first parameterization gives $\langle \sqrt{v}\cos 2u, \sqrt{v}\sin 2u, v\rangle$, $0 \le 2u \le \pi$, $0 \le \sqrt{v} \le 2$. Simplifying the bounds conditions gives the second parameterization.

d. True. The standard parameterization is $\langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$ for $0 \le u \le \pi$, $0 \le v \le 2\pi$. Then $\mathbf{t}_u \times \mathbf{t}_v = \langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\rangle$, and it is easily seen that these are outward-pointing vectors, by considering various ranges for u and v.

14.6.50 $\nabla \ln|\mathbf{r}| = \nabla \ln\sqrt{x^2+y^2+z^2} = \frac{1}{|\mathbf{r}|^2}\langle x,y,z\rangle = \frac{1}{a^2}\langle x,y,z\rangle$ on the sphere of radius a; using the explicit description for the sphere, we have $\mathbf{n} = \left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$, so that

$$\iint_S \nabla \ln|\mathbf{r}|\cdot\mathbf{n}\,dS = \iint_R \frac{1}{a^2}\langle x,y,z\rangle\cdot\left\langle\frac{x}{z},\frac{y}{z},1\right\rangle dA = \frac{1}{a^2}\iint_R \left(\frac{x^2+y^2}{z}+z\right)\,dA =$$

$$\frac{1}{a^2}\iint_R\left(\frac{x^2+y^2+z^2}{z}\right)\,dA = \frac{1}{a^2}\iint_R \frac{a^2}{z}\,dA = \iint_R \frac{1}{z}\,dA = \int_0^{2\pi}\int_0^a \frac{1}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = 2\pi a.$$

14.6.51 Parameterize the surface by $\langle 2\cos u, 2\sin u, v\rangle$ for $0 \le u \le 2\pi$, $0 \le v \le 8$. Then the normal vector has magnitude 2, and

$$\iint_S |\mathbf{r}|\,dS = \iint_R \sqrt{x^2+y^2+z^2}\,dA = 2\int_0^{2\pi}\int_0^8 \sqrt{4+v^2}\,dv\,du = 8\pi\left(4\sqrt{17}+\ln\left(4+\sqrt{17}\right)\right).$$

14.6.52 $z_x = 0, z_y = -1$, so

$$\iint_S xyz\,dS = \iint_R xyz\sqrt{2}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^2 r\cdot r\cos\theta\cdot r\sin\theta\cdot(6-r\sin\theta)\,dr\,d\theta =$$

$$\sqrt{2}\int_0^{2\pi}\int_0^2 r^3\cos\theta\,\sin\theta\,(6-r\sin\theta)\,dr\,d\theta = 0.$$

14.6.53 The normal vector is $\langle x, 0, z\rangle$, so

$$\iint\limits_S \frac{1}{\sqrt{x^2+z^2}}\langle x,0,z\rangle \cdot \langle x,0,z\rangle \, dS = \iint\limits_R \sqrt{x^2+z^2}\, dA = \int_{-2}^{2}\int_0^{2\pi} a\, dA = 8\pi a.$$

14.6.54 The two curves intersect where $x^2+y^2 = 16-x^2-y^2$, so on the plane $z = 2\sqrt{2}$. The projection on the xy-plane of the circle of intersection is $x^2+y^2=8$. The total surface area of the sphere, of radius 4, is 64π. Finally, the outward normals to the sphere are $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$.

a. From part (b) below and the fact that the total surface area is 64π, we get an answer of $64\pi - 32\pi + 16\pi\sqrt{2} = 16\pi\left(2+\sqrt{2}\right)$.

b. $\iint\limits_S 1\, dS = \iint\limits_R \sqrt{\frac{x^2}{z^2}+\frac{y^2}{z^2}+1}\, dA = \iint\limits_R \frac{4}{z}\, dA = \iint\limits_R \frac{4}{\sqrt{16-x^2-y^2}}\, dA = \int_0^{2\pi}\int_0^{\sqrt{8}} \frac{4r}{\sqrt{16-r^2}}\, dr\, d\theta =$ $16\pi\left(2-\sqrt{2}\right)$.

c. For the cone $z^2 = x^2+y^2$, we have $\sqrt{z_x^2+z_y^2+1} = \sqrt{2}$, so $\iint\limits_S 1\, dS = \iint\limits_R \sqrt{2}\, dA = \sqrt{2}\cdot\pi\cdot\left(\sqrt{8}\right)^2 = 8\pi\sqrt{2}$.

14.6.55

a. The surface of the cylinder inside the sphere is defined parametrically by $\langle 1+\cos u, \sin u, v\rangle$ where $0 \le u \le 2\pi$ and also (because the z-coordinate must stay inside the sphere of radius 2), $0 \le v \le \sqrt{4-(1+\sin u)^2-\cos^2 u}$, or $0 \le v \le \sqrt{2-2\sin u}$. The normal is $\langle \cos u, \sin u, 0\rangle$, which has magnitude 1, so we have $\iint\limits_S 1\, dS = \iint\limits_R 1\, dA = \int_0^{2\pi}\int_0^{\sqrt{2-2\sin u}} 1\, dv\, du = \int_0^{2\pi}\sqrt{2-2\sin u}\, du = 8$.

b. To find a parameterization of the portion of the sphere cut by the cylinder above the z-axis, first note that it is sufficient to do this for the portion of the sphere in the first octant, and then double the result. Now, the first octant is determined by $0 \le u \le \frac{\pi}{2}$, $0 \le v \le \frac{\pi}{2}$ in the standard parameterization $\langle 2\sin u\cos v, 2\sin u \sin v, 2\cos u\rangle$. For each point on the boundary of the intersection, we must have$(x-1)^2+y^2=1$ or, substituting from the parameterization, $1 = (2\sin u\cos v - 1)^2 + (2\sin u \sin v)^2 = 4\sin^2 u\cos^2 v + 4\sin^2 u \sin^2 v - 4\sin u\cos v + 1$, so that $\sin^2 u = \sin u \cos v$, and we must have $v = \cos^{-1}(\sin(u)) = \frac{\pi}{2} - u$. Thus, the surface is determined by $0 \le u \le \frac{\pi}{2}$, $0 \le v \le \frac{\pi}{2} - u$, and the surface area is $\iint\limits_S 1\, dS = \iint\limits_R |\mathbf{t}_u \times \mathbf{t}_v|\, dA = \int_0^{\pi/2}\int_0^{\pi/2-u} 4\sin u\, dv\, du = 2\pi - 4$. Doubling this to account for the other quadrant, we have $4\pi - 8$.

14.6.56 We have $z = -\frac{c}{a}x + \left(-\frac{c}{b}\right)y + c$, so that $z_x = -\frac{c}{a}$, $z_y = -\frac{c}{b}$, and an upward-pointing normal is $\left\langle \frac{c}{a}, \frac{c}{b}, 1\right\rangle$. Then $\iint\limits_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint\limits_R \left(\frac{c}{a}x + \frac{c}{b}y + z\right) dA = \iint\limits_R \left(\frac{c}{a}x + \frac{c}{b}y + \left(-\frac{c}{a}x + \left(-\frac{c}{b}\right)y + c\right)\right) dA = \iint\limits_R c\, dA$, which is c times the area of A. Recall that if the vector field is vertical, then the flux is equal to the area of the base. As c increases, the slope of the plane gets closer to vertical, so that the x and y components of the vector field $\langle x, y, z\rangle$ contribute more to the flux; also, the values of z get larger. Thus the flux increases as c does.

14.6.57

a. Using the standard parameterization, $\iint\limits_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint\limits_R \left(\frac{-x^2}{z} + \frac{-y^2}{z} + z\right) dA = \iint\limits_R 0\, dA = 0$, because $x^2+y^2=z^2$, so that the flux is zero. This is due to the fact that the field $\mathbf{F}$ is aligned with the cone at all points on the cone.

b. $2z\, dz = \left(\frac{2x}{a^2}\right) dx$ so that $z_x = \frac{x}{a^2 z}$. Then $\iint\limits_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint\limits_R \left(\frac{-x^2}{a^2 z} + \frac{-y^2}{a^2 z} + z\right) dA = \iint\limits_R \left(\frac{-a^2 z^2}{a^2 z} + z\right) dA =$ 0, so the flux is again zero. This is because the flow is a radial flow, so is always tangent to this surface.

14.6.58 Parameterize the cone by $\mathbf{r}(u,v) = \left\langle \frac{a}{h}v\cos u, \frac{a}{h}v\sin u, v\right\rangle$for $0 \le v \le h$, $0 \le u \le 2\pi$; then $\mathbf{t}_u \times \mathbf{t}_v = \left\langle -\frac{a}{h}v\sin u, \frac{a}{h}v\cos u, 0\right\rangle \times \left\langle \frac{a}{h}\cos u, \frac{a}{h}\sin u, 1\right\rangle$ and $|\mathbf{t}_u \times \mathbf{t}_v| = \frac{a}{h^2}v\sqrt{h^2+a^2}$. Then $\iint_S 1\,dS = \frac{a\sqrt{h^2+a^2}}{h^2}\iint_R v\,dA = \frac{a\sqrt{h^2+a^2}}{h^2}\int_0^{2\pi}\int_0^h v\,dv\,du = \frac{a\sqrt{h^2+a^2}}{h^2}\int_0^{2\pi}\frac{1}{2}h^2\,du = \frac{2\pi a\sqrt{h^2+a^2}}{2} = \pi a\sqrt{h^2+a^2}$.

14.6.59 Because the cap has height h, the circle at the boundary of the cap has radius $\sqrt{a^2-(a-h)^2} = \sqrt{2ah-h^2}$, so that the equation of the base of the region is $x^2+y^2 = 2ah-h^2$. The outward normals to the sphere are $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$. Thus $\iint_S 1\,dS = \iint_R \sqrt{\frac{x^2}{z^2}+\frac{y^2}{z^2}+1}\,dA = \iint_R \sqrt{\frac{a^2-z^2}{z^2}+1}\,dA = \iint_R \frac{a}{z}\,dA = \iint_R \frac{a}{\sqrt{a^2-x^2-y^2}}\,dA = \int_0^{2\pi}\int_0^{\sqrt{2ah-h^2}} \frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta = 2\pi ah$. (See also problem 54(b)).

14.6.60 Using a parametric description, we have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \frac{1}{(x^2+y^2+z^2)^{p/2}}\langle x,y,z\rangle\cdot(\mathbf{t}_u\times\mathbf{t}_v)\,dA = \frac{1}{a^p}\iint_R \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle\cdot\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\rangle\,dA =$
$\frac{1}{a^p}\iint_R \left(a^3\sin^3 u\cos^2 v + a^3\sin^3 u\sin^2 v + a^3\cos^2 u\sin u\right)dA = a^{3-p}\int_0^{\pi}\int_0^{2\pi}\left(\sin^3 u + \cos^2 u\sin u\right)dv\,du =$
$a^{3-p}\int_0^{\pi}\int_0^{2\pi}\sin u\,dv\,du = \frac{4\pi}{a^{p-3}}$. Using an explicit description, compute the flux on the upper half hemisphere and double it. There, for $z \ge 0$, we have $z_x = -\frac{x}{z}$, $z_y = -\frac{y}{z}$, so that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \frac{1}{(x^2+y^2+z^2)^{p/2}}\left(\frac{x^2}{z}+\frac{y^2}{z}+z\right)dA$$
$$= \frac{1}{a^p}\iint_R \frac{a^2}{z}\,dA = a^{2-p}\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = \frac{2\pi}{a^{p-3}}.$$

After doubling, we get the same answer.

14.6.61 F$= -\nabla T = -\langle T_x, T_y, T_z\rangle = \langle 100e^{-x-y}, 100e^{-x-y}, 0\rangle$. Thus the flow is parallel to the two sides where $z = \pm 1$ so that the flux is zero there. We thus need only compute the flux on the remaining four sides. Parameterize the sides as

$S_1 : \langle -1, y, z\rangle$	$\mathbf{t}_y\times\mathbf{t}_z = \langle 1,0,0\rangle$
$S_2 : \langle 1, y, z\rangle$	$\mathbf{t}_y\times\mathbf{t}_z = \langle 1,0,0\rangle$
$S_3 : \langle x, -1, z\rangle$	$\mathbf{t}_x\times\mathbf{t}_z = \langle 0,-1,0\rangle$
$S_4 : \langle x, 1, z\rangle$	$\mathbf{t}_x\times\mathbf{t}_z = \langle 0,-1,0\rangle$

for $-1 \le x,y,z \le 1$.

We are looking for the outward flux, so we must choose outward normals, which are (respectively) $\langle -1,0,0\rangle$, $\langle 1,0,0\rangle$,$\langle 0,-1,0\rangle$, and $\langle 0,1,0\rangle$. Then

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS_1 = \iint_R -100e^{-x-y}\,dA = -100\int_{-1}^1\int_{-1}^1 e^{1-y}\,dz\,dy = -200e^2+200$$
$$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS_2 = \iint_R -100e^{-x-y}\,dA = 100\int_{-1}^1\int_{-1}^1 e^{-1-y}\,dz\,dy = -200e^{-2}+200$$
$$\iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS_3 = \iint_R -100e^{-x-y}\,dA = -100\int_{-1}^1\int_{-1}^1 e^{-x+1}dz\,dx = -200e^2+200$$
$$\iint_{S_4} \mathbf{F}\cdot\mathbf{n}\,dS_4 = \iint_R -100e^{-x-y}\,dA = 100\int_{-1}^1\int_{-1}^1 e^{-x-1}dz\,dx = -200e^{-2}+200$$

so that the total flux is $-400\left(e^2+e^{-2}-2\right) = -400\left(e-\frac{1}{e}\right)^2$.

14.6.62 F$= -\nabla T = -\langle T_x, T_y, T_z\rangle = \langle 200e^{-x^2-y^2-z^2}, 200e^{-x^2-y^2-z^2}, 200e^{-x^2-y^2-z^2}\rangle$. Thus integrating over the top half of the sphere gives $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \langle 200xe^{-x^2-y^2-z^2}, 200ye^{-x^2-y^2-z^2}, 200ze^{-x^2-y^2-z^2}\rangle \cdot \left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle dA = 200e^{-a^2}\iint_R \left(\frac{x^2}{z}+\frac{y^2}{z}+z\right) dA = 200a^2e^{-a^2}\iint_R \frac{1}{z}dA = 200a^2e^{-a^2}\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = 400\pi a^3e^{-a^2}$, and because the vector field is symmetric, the answer is $800\pi a^3e^{-a^2}$.

14.6.63 $\mathbf{F} = -\nabla T = \frac{2}{x^2+y^2+z^2}\langle x, y, z\rangle$. Thus integrating on the top half of the sphere gives $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \left(\frac{x^2}{z}+\frac{y^2}{z}+z\right) dA = 2\iint_R \frac{1}{z}dA = 2\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = 4\pi a$, and because the vector field is symmetric, the answer is $2\cdot 4\pi a = 8\pi a$.

14.6.64

a. $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \langle x, y, 0\rangle \cdot \left\langle x, y, 0\right\rangle dA = \iint_R \left(x^2+y^2\right) dA = a^2\int_0^{2\pi}\int_0^a r\,dr\,d\theta = \pi a^4$.

b. On the cylinder, this field is $\left(x^2+y^2\right)^{-p/2} = a^{-p}$ times as large as the field in part (a), so the flux is $4\pi La^{2-p}$.

c. As $a\to\infty$, this converges for $2-p\le 0$, or $p\ge 2$.

d. As $L\to\infty$, the flux never converges.

14.6.65

a. From problem 60, the outward flux across a sphere of radius b is $\frac{4\pi}{b^{p-3}}$, so the total flux across the concentric spheres when $p=0$ is $4\pi b^3 - 4\pi a^3 = 4\pi\left(b^3-a^3\right)$.

b. For $p=3$, the flux across the sphere of radius b is 4π, so the net flux is zero across S.

14.6.66 By symmetry, $\overline{x} = \overline{y} = 0$. Since the shell has constant density, we assume the density is 1; then its mass is $2\pi a^2$, and $M_{xy} = \iint_S z\,dS = \iint_R z\sqrt{\left(\frac{-x}{z}\right)^2 + \left(\frac{-y}{z}\right)^2 + 1}\,dA = \iint_R z\cdot\frac{a}{z}\,dA = a\int_0^{2\pi}\int_0^a r\,dr\,d\theta = \pi\,a^3$ so that $\overline{z} = \frac{a}{2}$.

14.6.67 The cone is rotationally symmetric around the z axis, so $\overline{x} = \overline{y} = 0$. Parameterize the cone by $\left\langle \frac{r}{h}v\cos u, \frac{r}{h}v\sin u, v\right\rangle$. Then from problem 25, the surface area of the cone is $\pi r\sqrt{h^2+r^2}$, so its mass is $\rho\pi r\sqrt{h^2+r^2}$. Using the parameterization from that problem, $|\mathbf{t}_u\times\mathbf{t}_v| = \frac{r}{h^2}v\sqrt{h^2+r^2}$, so that $M_{xy} = \rho\iint_S z\,dS = \rho\frac{r\sqrt{h^2+r^2}}{h^2}\iint_R vz dA = \rho\frac{r\sqrt{h^2+r^2}}{h^2}\int_0^{2\pi}\int_0^h v^2\,dv\,du = \rho\frac{r\sqrt{h^2+r^2}}{h^2}\int_0^{2\pi}\frac{1}{3}h^3\,du = \rho\frac{2\pi rh\sqrt{h^2+r^2}}{3}$ so that $\overline{z} = \frac{M_{xy}}{m} = \rho\frac{2\pi rh\sqrt{h^2+r^2}}{3}\cdot\frac{1}{\rho\pi r\sqrt{h^2+r^2}} = \frac{2h}{3}$.

14.6.68 Assume the shell has density 1. Then the mass of the shell, which is half the area of the entire cylinder, is ah. Further, by symmetry, $\overline{x} = \overline{y} = 0$. Use the parameterization $\langle a\cos u, v, a\sin u\rangle$; then $|\mathbf{t}_u\times\mathbf{t}_v| = a$ and $M_{xy} = \iint_S z\,dS = a\iint_R z\,dA = a\int_{-h/2}^{h/2}\int_0^{\pi} a\sin u\,du\,dv = 2a^2h$ and $\overline{z} = \frac{2}{\pi}a$.

14.6.69 Using the standard parameterization, the mass of the shell is $m = \iint_S (1+z)\,dS = a\iint_R (1+z)\,dA = a\int_0^{2\pi}\int_0^2 (1+v)\,dv\,du = 8\pi a$. The density does not depend on either x or y, and the cylinder is symmetric about the z axis, so $\overline{x} = \overline{y} = 0$. Then

$$\mathrm{M}_{xy} = \iint_S z\,(1+z)\,dS = a\iint_R z\,(1+z)\,dA = a\int_0^{2\pi}\int_0^2 v\,(1+v)\,dv\,du = \frac{28\pi a}{3}.$$

Then $\overline{z} = \frac{7}{6}$.

14.6.70 $\mathbf{t}_u = \langle a\cos u\cos v, a\cos u\sin v, -a\sin u\rangle$ and $\mathbf{t}_v = \langle -a\sin u\sin v, a\sin u\cos v, 0\rangle$, and then $\mathbf{t}_u\times\mathbf{t}_v = \langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\cos u\sin u\rangle$ so that $|\mathbf{t}_u\times\mathbf{t}_v| = a^2\left|\langle\sin^2 u\cos v, \sin^2 u\sin v, \cos u\sin u\rangle\right| = a^2\sqrt{\sin^4 u\cos^2 v+\sin^4 u\sin^2 v+\cos^2 u\sin^2 u} = a^2\sqrt{\sin^4 u+\sin^2 u\cos^2 u} = a^2\sin u$.

14.6.71 The explicit formula $z = g(x,y)$ becomes, on regarding x and y as parameters, the parametric form $\langle x, y, g(x,y)\rangle$, and now $\mathbf{t}_x = \langle 1,0,z_x\rangle$ and $\mathbf{t}_y = \langle 0,1,z_y\rangle$. Then $\mathbf{t}_x\times\mathbf{t}_y = \langle -z_x, -z_y, 1\rangle$, so that $|\mathbf{t}_x\times\mathbf{t}_y| = \sqrt{z_x^2+z_y^2+1}$. Now the formula $\iint\limits_S f(x,y,z)\,dS = \iint\limits_R f(x,y,g(x,y))\sqrt{z_x^2+z_y^2+1}\,dA$ follows from the definition of the surface integral for parameterized surfaces.

14.6.72

a. Each point on the graph of f on $[a,b]$, say $f(x)$ becomes, in the surface of revolution, a circle of radius $f(x)$ with center on the x-axis. Letting $u = x$, that circle is then parameterized by $\langle f(u)\cos v, f(u)\sin v\rangle$ for $0\le v\le 2\pi$, so the entire surface is parameterized by $\langle u, f(u)\cos v, f(u)\sin v\rangle$, $a\le u\le b$, $0\le v\le 2\pi$.

b. $\mathbf{t}_u = \langle 1, f'(u)\cos v, f'(u)\sin v\rangle$ and $\mathbf{t}_v = \langle 0, -f(u)\sin v, f(u)\cos v\rangle$, and then

$$\mathbf{t}_u\times\mathbf{t}_v = \langle f'(u)f(u), -f(u)\cos v, -f(u)\sin v\rangle$$

and

$$|\mathbf{t}_u\times\mathbf{t}_v| = \sqrt{f'(u)^2 f(u)^2 + f(u)^2\left(\sin^2 v+\cos^2 v\right)} = f(u)\sqrt{f'(u)^2+1}.$$

We have

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_R f(u)\sqrt{f'(u)^2+1}\,dA = \int_a^b\int_0^{2\pi} f(u)\sqrt{f'(u)^2+1}\,dv\,du$$
$$= 2\pi\int_a^b f(u)\sqrt{f'(u)^2+1}\,du.$$

c. The area of the surface is $2\pi\int_1^2 x^3\sqrt{9x^4+1}\,dx = \frac{\pi}{27}\left(145^{3/2}-10^{3/2}\right)$.

d. The area of the surface is

$$2\pi\int_3^4\left(25-x^2\right)^{1/2}\sqrt{\left(-x\left(25-x^2\right)^{-1/2}\right)^2+1}dx = 2\pi\int_3^4\left(25-x^2\right)^{1/2}\sqrt{\frac{x^2}{25-x^2}+1}\,dx$$
$$= 2\pi\int_3^4\left(25-x^2\right)^{1/2}\sqrt{\frac{25}{25-x^2}}\,dx = 2\pi\int_3^4 5\,dx = 10\pi.$$

14.6.73 We have $z = s(x,y)$, so a normal vector is $\langle -z_x, -z_y, 1\rangle$. Since we are interested in the downward flux, we choose a downward-pointing normal, which is $\langle \mathrm{s}_x(x,y), s_y(x,y), -1\rangle$. Then $\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_R \langle 0,0,-1\rangle\cdot\langle \mathrm{s}_x(x,y), s_y(x,y), -1\rangle\,dA = \iint\limits_R 1\,dA$, which is the area of R. Since the vector field is constant and pointed downwards vertically, everything that goes through the surface is matched by something going through R.

14.6.74

a. Imagine the torus as built by starting with a circle of radius R in the xy plane, centered at the origin. From each point on this circle, parameterize as $\langle R\cos v, R\sin v, 0\rangle$, we can reach a circle of points on the surface by making it the center of a circle of radius r, parameterized by u. This second circle is drawn in a vertical plane that includes the z-axis. Each point of this second circle is thus in the plane determined by $\langle R\cos v, R\sin v, 0\rangle$ and the z-axis; its z-coordinate will be $r\sin u$ and its x and y-coordinates will then be (from its center) $r\cos v\cos u$ and $r\sin v\cos u$. Thus the set of points on the torus can be parameterized by the sum of these vectors, which is $\langle R\cos v, R\sin v, 0\rangle + \langle r\cos v\cos u, r\sin v\cos u, r\sin u\rangle = \langle (R+r\cos u)\cos v, (R+r\cos u)\sin v, r\sin u\rangle$

b. From the parameterization above, we have

$$\mathbf{t}_u = \langle -r\sin u\cos v, -r\sin u\,\sin v, r\cos u\rangle$$
$$\mathbf{t}_v = \langle -(R+r\cos u)\sin v, (R+r\cos u)\cos v, 0\rangle$$

so that

$$\mathbf{t}_u \times \mathbf{t}_v = (R+r\cos u)\langle -r\,\cos u\cos v, -r\cos u\sin v, -r\,\sin u\rangle$$

and $|\mathbf{t}_u \times \mathbf{t}_v| = (R+r\cos u)\sqrt{r^2\cos^2 u\cos^2 v + r^2\cos^2 u\sin^2 v + r^2\sin^2 u} = r(R+r\cos u)$, so that the area of the torus is $\iint\limits_S 1\,dS = \int_0^{2\pi}\int_0^{2\pi} r(R+r\,\cos u)\,du\,dv = 4\pi^2 Rr$.

14.6.75 The goal is to start with the surface area formula $A = \iint\limits_R \sqrt{z_x^2+z_y^2+1}\,dA$ of Theorem 14.12 (where we have set $f(x,y,g(x,y)) = 1$) and derive the surface area formula $A = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$ of section 6.6. Because S is generated by revolving the graph of f about the x-axis, we can use symmetry and take $R = \{(x,y) : a \le x \le b, 0 \le y \le f(x)\}$. The resulting surface area is then one-quarter of the desired surface area. The key observation is that the surface generated is given by $z^2 = f(x)^2 - y^2$ over the region R. It follows that $2zz_x = 2ff'$, or $z_x = \frac{ff'}{2}$ and $2zz_y = -2y$, or $z_y = \frac{-y}{z}$. Substituting, we have that the surface area is $A = \iint\limits_R \sqrt{z_x^2+z_y^2+1}\,dA = 4\int_a^b\int_0^{f(x)} \sqrt{\left(\frac{ff'}{z}\right)^2 + \left(\frac{-y}{z}\right)^2 + 1}\,dy\,dx = 4\int_a^b\int_0^{f(x)} \sqrt{\frac{f^2f'2+y^2+z^2}{z^2}}\,dy\,dx = 4\int_a^b\int_0^{f(x)}\sqrt{\frac{f^2f'^2+f^2}{f^2-y^2}}\,dy\,dx$, because $z^2 = f^2 - y^2$.

Continuing, we have $A = 4\int_a^b\int_0^{f(x)} \frac{f(x)\sqrt{1+f'(x)^2}}{\sqrt{f(x)^2-y}}\,dy\,dx = 4\int_a^b f(x)\sqrt{1+f'(x)^2}\int_0^{f(x)}\frac{1}{\sqrt{f(x)^2-y}}\,dy\,dx = 4\int_a^b f(x)\sqrt{1+f'(x)^2}\left(\sin^{-1}\frac{y}{f(x)}\Big|_0^{f(x)}\right)dx = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$.

14.7 Stokes' Theorem

14.7.1 It measures the circulation of the vector field $\mathbf{F}$ along the closed curve C.

14.7.2 It measures the accumulated rotation of the vector field $\mathbf{F}$ over the surface S.

14.7.3 It says that the circulation of a vector field along a closed curve is equal to the net circulation of the field over a surface whose boundary is that curve, so that either can be calculated from the other.

14.7.4 This is the fundamental theorem of line integrals - the integral of any conservative vector field around a closed curve is zero.

14.7.5 The line integral is $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_R \langle \sin t, -\cos t, 10\rangle\cdot\langle -\sin t, \cos t, 0\rangle\,dA = \int_0^{2\pi}(-1)\,dt = -2\pi$. For the surface integral, use the standard parameterization of the sphere; then n= $\langle \sin^2 u\cos v, \sin^2 u\sin v, \cos u\sin u\rangle$ and $\nabla\times\mathbf{F} = \langle 0,0,-2\rangle$ so that $\iint\limits_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^{\pi/2}(-2\cos u\,\sin u)\,du\,dv = -2\pi$.

14.7.6 The line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_R \langle 0, -2\cos t, 2\sin t\rangle\cdot\langle -2\,\sin t, 2\cos t, 0\rangle\,dA = \int_0^{2\pi}\left(-4\,\cos^2 t\right)\,dt = -4\pi.$$

$\nabla\times\mathbf{F} = \langle 1,0,-1\rangle$. The outward normal to the sphere is $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$, so the surface integral is $\iint\limits_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \iint\limits_R \langle 1,0,-1\rangle\cdot\left\langle\frac{x}{z},\frac{y}{z},1\right\rangle dA = \int_0^{2\pi}\int_0^2\left(\frac{r^2\cos\theta}{\sqrt{4-r^2}} - r\right)dr\,d\theta = -4\pi$.

14.7.7 The line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle 2\sqrt{2}\cos t, 2\sqrt{2}\sin t, 0\rangle \cdot \langle -2\sqrt{2}\sin t, 2\sqrt{2}\cos t, 0\rangle\, dt = \int_0^{2\pi} 0\, dt = 0.$$

For the surface integral, we have $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{0}$ so that the surface integral is also zero.

14.7.8 The boundary of the region is the intersection of the sphere with the plane $z = 12$, which has the equation $x^2 + y^2 = 25$ and $z = 12$. Thus the line integral is $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle 24, -4\cdot 5\cos t, 3\cdot 5\sin t\rangle \cdot \langle -5\sin t, 5\,\cos t, 0\rangle\, dt = \int_0^{2\pi}\left(-120\sin t - 100\cos^2 t\right)\, dt = -100\pi$.

The surface sits over $x^2 + y^2 = 25$ and the normal to the sphere is $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$. $\boldsymbol{\nabla}\times\mathbf{F} = \langle 3, 2, -4\rangle$, so the surface integral is $\iint\limits_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = \int_0^{2\pi}\int_0^5 r\left(\frac{3r\cos\theta}{\sqrt{169-r^2}} + \frac{2r\sin\theta}{\sqrt{169-r^2}} - 4\right) dr\, d\theta = -100\pi$.

14.7.9 The boundary of the region is the intersection of the sphere with the plane $z = \sqrt{7}$, which has the equation $x^2 + y^2 = 9$ and $z = \sqrt{7}$. Then the line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle 3\sin t - \sqrt{7}, \sqrt{7} - 3\cos t, 3\cos t - 3\,\sin t\rangle \cdot \langle -3\sin t, 3\cos t, 0\rangle\, dt$$

$$= \int_0^{2\pi}\left(-9\sin^2 t + 3\sqrt{7}\sin t + 3\sqrt{7}\cos t - 9\cos^2 t\right)\, dt = \int_0^{2\pi}\left(-9 + 3\sqrt{7}\sin t + 3\sqrt{7}\,\cos t\right)\, dt = -18\pi.$$

The surface sits over $x^2 + y^2 = 9$ and the normal to the sphere is $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$. $\boldsymbol{\nabla}\times\mathbf{F} = \langle -2, -2, -2\rangle$, so the surface integral is

$$\iint\limits_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = -2\int_0^{2\pi}\int_0^3 r\left(\frac{r\cos\theta}{\sqrt{16-r^2}} + \frac{r\sin\theta}{\sqrt{16-r^2}} + 1\right) dr\, d\theta = -18\pi.$$

14.7.10 The boundary of the region can be parameterized by $\langle 4\cos t, 4\sin t, 6 - 4\sin t\rangle$, so the line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle -4\sin\text{ t}, 4\sin t - 4\cos t - 6, 4\sin t - 4\cos t\rangle \cdot \langle -4\sin t, 4\cos t, -4\,\cos t\rangle\, dt$$

$$= \int_0^{2\pi}\left(16\sin^2 t + 16\sin t\,\cos t - 16\cos^2 t - 24\cos t - 16\sin t\,\cos t + 16\cos^2 t\right)\, dt$$

$$= \int_0^{2\pi}\left(16\sin^2 t - 24\cos t\right)\, dt = 16\pi.$$

$\boldsymbol{\nabla}\times\mathbf{F} = \langle 2, 1, 0\rangle$, and an outward pointing normal to the plane is $\langle 0, 1, 1\rangle$, so the surface integral is $\iint\limits_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint\limits_R 1\, dA = \int_0^{2\pi}\int_0^4 r\, dr\, d\theta = 16\pi$.

14.7.11 $\boldsymbol{\nabla}\times\mathbf{F} = \langle 1, -1, -2\rangle$; for S take the disk $x^2 + y^2 \le 12$ with upward-oriented normal vector $\langle 0, 0, 1\rangle$. Then $\iint\limits_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint\limits_R (-2)\, dA = -24\pi$.

14.7.12 $\boldsymbol{\nabla}\times\mathbf{F} = \langle -1 - x, 0, z - 1\rangle$; for S take the region $x^2 + \frac{y^2}{4} \le 1$ in the plane $z = 1$, with normal vector $\langle 0, 0, 1\rangle$; then $\iint\limits_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint\limits_R (z - 1)\, dA = 0$ because $z = 1$ on the region of integration.

14.7.13 $\boldsymbol{\nabla}\times\mathbf{F} = \langle 0, -4z, 0\rangle$. For S take the plane in the first octant, which sits over $0 \le x \le 4$, $0 \le y \le 4 - x$. The upward-pointing normal to this plane is $\langle 1, 1, 1\rangle$. Then $\iint\limits_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint\limits_R (-4z)\, dA = \int_0^4\int_0^{4-x}(-4)\,(4 - x - y)\, dy\, dx = -\frac{128}{3}$.

14.7.14 $\nabla \times \mathbf{F} = \langle 2y-2z, 0, 2y-2x\rangle$. Take S to be the square bounded by C, with normal $\langle 0,0,1\rangle$. Then $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint\limits_R (2y-2x)\ dA = \int_{-1}^{1}\int_{-1}^{1}(2y-2x)\ dy\,dx = 0$.

14.7.15 $\nabla \times \mathbf{F} = \langle 2z, -1, -2y\rangle$. Take S to be the disk $\langle 3r\cos t, 4r\cos t, 5r\sin t\rangle$ for $0 \le r \le 1$, $0 \le t \le 2\pi$. $\mathbf{t}_r \times \mathbf{t}_t = \langle 20r, -15r, 0\rangle$ is a normal vector. Then $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint\limits_R \langle 2z, -1, -2y\rangle \cdot \langle 20r, -15r, 0\rangle\, dA = \int_0^{2\pi}\int_0^1 ((10r\ \sin t)\cdot 20r + 15r)\ dr\,dt = 15\pi$.

14.7.16 $\nabla \times \mathbf{F} = \mathbf{0}$, so the surface integral is zero.

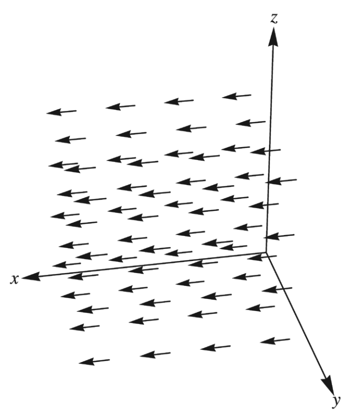
14.7.17 The boundary of the surface is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, found by setting $z = 0$. Parameterize the path by $\mathbf{r}(t) = \langle 2\cos t, 3\sin t, 0\rangle$; then $\oint \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\cdot \mathbf{r}'(t)\ dt = \int_0^{2\pi}(-4\cos t\sin t + 9\cos t\ \sin t)\ dt = \int_0^{2\pi} 5\cos t\sin t\, dt = 0$.

14.7.18 The boundary of the surface is on the plane $x = 0$, and it is the circle $y^2 + z^2 = 9$. Parameterize the circle by $\mathbf{r}(t) = \langle 0, 3\cos t, 3\sin t\rangle$; then $\mathbf{r}'(t) = \langle 0, -3\sin t, 3\cos t\rangle$. $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{\sqrt{x^2+y^2+z^2}}\langle x, y, z\rangle$, and $\oint \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}\cdot\mathbf{r}'(t)\ dt = \frac{1}{3}\int_0^{2\pi}(-9\sin t\cos t + 9\ \sin t\cos t)\ dt = 0$.

14.7.19 The boundary of the surface is the intersection of the plane $x = 3$ with the sphere $x^2+y^2+z^2 = 25$, so is the circle $y^2 + z^2 = 16$ at $x = 3$. Parametrize the circle with $x = 3$, $y = 4\cos t$ and $z = 4\sin t$. We have $\mathbf{r}'(t) = \langle 0, -4\sin t, 4\cos t\rangle$, so $\oint \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle 8\cos t, -4\sin t, 3 - 4\cos t - 4\sin t\rangle \cdot \langle 0, -4\sin t, 4\cos t\rangle\, dt = \int_0^{2\pi}\left(16\sin^2 t + 12\cos t - 16\cos^2 t - 16\sin t\cos t\right)\ dt = 0$.

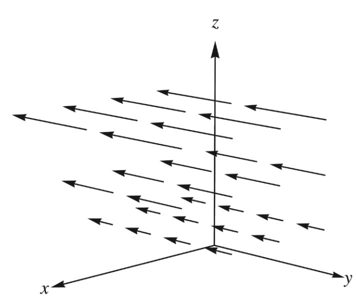
14.7.20 The boundary of the surface is given in the problem: $\mathbf{r}(t) = \langle \cos t, 2\sin t, \sqrt{3}\cos t\rangle$; so $\mathbf{r}'(t) = \langle -\sin t, 2\cos\ t, -\sqrt{3}\sin t\rangle$ and the integral is $\oint \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle \cos t + 2\sin t, 2\ \sin t + \sqrt{3}\cos t, \left(1+\sqrt{3}\right)\cos t\rangle \cdot \langle -\sin t, 2\cos t, -\sqrt{3}\ \sin t\rangle\, dt = \int_0^{2\pi}\left(-\sin t\cos t - 2\ \sin^2 t + 4\sin t\cos t + 2\sqrt{3}\cos^2 t - \left(3+\sqrt{3}\right)\cos t\sin t\right)\ dt = \int_0^{2\pi}\left(-\sqrt{3}\sin t\cos t - 2\ \sin^2 t + 2\sqrt{3}\cos^2 t\right)\ dt = 2\pi\left(\sqrt{3}-1\right)$.

14.7.21 $\nabla \times \mathbf{v} = \langle 1, 0, 0\rangle$. The curl looks like:



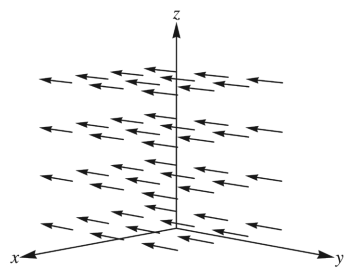
This means that the maximum rotation of the field is in the direction $\langle 1, 0, 0\rangle$. The rotation is counterclockwise looking in the negative x direction.

14.7.22 $\nabla \times \mathbf{v} = \langle 0, -2z, 0\rangle$. The curl looks like:



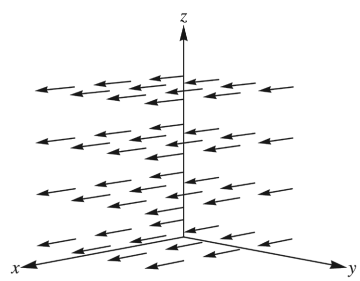
This means that the maximum rotation of the field is in the direction of the$y$-axis, and the amount of rotation, clockwise or counterclockwise, increases the further from the z-axis one gets. It rotates counterclockwise (viewed from the positive z-axis) for $z < 0$, and clockwise for $z > 0$.

14.7.23 $\nabla \times \mathbf{v} = \langle 0, -2, 0 \rangle$. The curl looks like:



This means that the maximum rotation of the field is in the direction of the y-axis. It is constant at all points, and is clockwise viewed from the positive y-axis.

14.7.24 $\nabla \times \mathbf{v} = \langle 2, 0, 0 \rangle$. The curl looks like:



The maximum rotation of the vector field is in the direction of the x-axis; it is constant at all points, and is counterclockwise viewed from the positive x-axis.

14.7.25

a. False. This is a rotation field with axis of rotation $\langle 1, 1, 2 \rangle$, but $\langle 0, 1, -1 \rangle \cdot \langle 1, 1, 2 \rangle \neq 0$, so the paddle wheel axis is not perpendicular to the axis of rotation.

b. False. It relates the curl of F, not its flux.

c. True. This is because it is conservative: it is the gradient of $ax + F(x) + by + G(y) + cz + H(z)$, where F, G, H are the antiderivatives of f, g, h, respectively.

d. True. See Theorem 14.14.

14.7.26 This is a conservative vector field, with $\varphi = x^2 - y^2 + z^2$, so the integral $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C.

14.7.27 This is a conservative vector field, so the integral around any closed curve is zero.

14.7.28 This is a conservative vector field with $\varphi = x^3 y + y^2 z^2$, so the integral around any closed curve is zero.

14.7.29 This is a conservative vector field with $\varphi = xy^2 z^3$, so the integral around any closed curve is zero.

14.7.30 The surface S is $\langle r \cos\varphi \cos t, r \sin t, r \sin\varphi \cos t \rangle$; computing the normal vector gives $\mathbf{t}_r \times \mathbf{t}_t = \langle -r \sin\varphi, 0, r \cos\varphi \rangle$. Thus the surface area is $\iint\limits_S 1\, dS = \int_0^1 \int_0^{2\pi} |\mathbf{t}_r \times \mathbf{t}_t| \, dt\, dr = \int_0^1 \int_0^{2\pi} r\, dt\, dr = \pi$. This makes sense because the surface is simply the unit circle inclined at the angle φ to the xy-plane.

14.7.31 $\mathbf{r}'(t) = \langle -\cos\varphi \sin t, \cos t, -\sin\varphi \sin t\rangle$, so that $|\mathbf{r}'(t)| = 1$. Then the length of C is $\int_C 1\, ds = \int_0^{2\pi} 1\, dt = 2\pi$, again as expected because C is just an inclined unit circle.

14.7.32 By Stokes' theorem, $\oint \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS$, $\nabla\times\mathbf{F} = \langle 0,0,2\rangle$; using $\mathbf{n}$ from Problem 30, we have $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \int_0^1\int_0^{2\pi} 2r\cos\varphi\, dt\, dr = 2\pi\cos\varphi$. This is maximum for $\varphi = 0$, when it is 2π.

14.7.33 We have

$$\oint \mathbf{F}\cdot d\mathbf{r} = \int_C \langle -y,-z,x\rangle\cdot\langle -\cos\varphi\sin t, \cos t, -\sin\varphi\,\sin t\rangle\, dt$$
$$= \int_0^{2\pi}\left(\cos\varphi\sin^2 t - \sin\varphi\cos^2 t + \cos\varphi\sin\varphi\,\cos t\sin t\right)\, dt = \pi\left(\cos\varphi - \sin\varphi\right).$$

This is maximum for $\varphi = 0$, when it is π.

14.7.34 $\nabla\times(\mathrm{a}\times\mathbf{r}) = \langle 2a_1, 2a_2, 2a_3\rangle$. Thus

$$\oint \mathbf{F}\cdot d\mathbf{r} = \iint_S \langle 2a_1,2a_2,2a_3\rangle\cdot\langle -r\sin\varphi, 0, r\cos\varphi\rangle\, dS$$
$$= 2\int_0^{2\pi}\int_0^1 \left(-a_1 r\sin\varphi + a_3 r\cos\varphi\right)\, dr\, dt = 2\pi\left(a_3\cos\varphi - a_1\sin\varphi\right).$$

This is a maximum when its derivative vanishes, i.e. when $a_3\cos\varphi - a_1\sin\varphi = 0$. Now, $\langle a_1,a_2,a_3\rangle$ points in the direction of the normal if their cross-product is zero, i.e. if $\langle a_1,a_2,a_3\rangle\times\langle -r\sin\varphi, 0, r\cos\varphi\rangle = \langle a_2\, r\cos\varphi, -a_3\, r\sin\varphi - a_1\, r\cos\varphi, a_2, r\sin\varphi\rangle = 0$. This happens when $a_2 = 0$ and $a_3\sin\varphi + a_1\cos\varphi = 0$.

14.7.35 $\nabla\times\mathbf{F} = \langle 3,0,0\rangle$. To evaluate the circulation around C, we instead (using Stokes' theorem) evaluate $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS$ for the surface of the disk of which C is a boundary. Note that $\mathbf{n} = \langle 1,1,1\rangle$, so that $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint_R 3\, dA = 3\cdot$area of $A = 48\pi$. From this calculation, it is clear that the result depended on the radius of the circle, because that affects the area of A, but not on the center of the circle.

14.7.36

a. $\nabla\times\mathbf{F} = \langle 1,1,0\rangle$, so the integrand of the surface integral is just $\frac{x+y}{z}$; by symmetry, the integral vanishes on each level curve, so it vanishes altogether.

b. On the boundary of S, we have $z = 0$, so that $\mathbf{F} = \langle 0,0,2y+x\rangle$, and thus $\mathbf{r}'(t) = \langle -\sin t, \cos t, 0\rangle$ so that the dot product and thus the line integral is zero.

14.7.37

a. The normal vectors point toward the z-axis on the curved surface of S and in the direction of $\langle 0,1,0\rangle$ on the flat surface of S.

b. To evaluate the integral, we must add up the integrals on each of the surfaces. $\nabla\times\mathbf{F} = \langle 1,1,1\rangle$. Let S_1 be the surface in the xz-plane, parameterized by $\langle x,0,z\rangle$ for $-2\le x\le 2$,$x^2\le z\le 4$; then the normal to S_1 is $\mathbf{t}_x\times\mathbf{t}_z = \langle 0,1,0\rangle$, so that the integral over S_1 is

$$\iint_{S_1}\langle 1,1,1\rangle\cdot\langle 0,-1,0\rangle\, dS = \int_{-2}^{2}\int_{x^2}^{4}(-1)\, dy\, dx = -\int_{-2}^{2}\left(4-x^2\right)\, dx = -\frac{32}{3}.$$

S_2 is the half of the paraboloid for $y\ge 0$, parameterized as $\langle r\cos u, r\sin u, r^2\rangle$, $0\le r\le 2, -\pi\le u\le 0$. The normal to S_2 is $\mathbf{t}_r\times\mathbf{t}_u = \langle -2r^2\cos u, -2r^2\,\sin u, r\rangle$. The integral over S_2 is

$$\iint_{S_1}\langle 1,1,1\rangle\cdot\langle -2r^2\cos u, -2r^2\,\sin u, r\rangle\, dS = \int_0^2\int_{-\pi}^{0}\left(-2r^2\left(\cos u + \sin u\right) + r\right)\, du\, dr = \frac{32}{3} + 2\pi.$$

Thus the total is 2π.

c. The line integral is the sum of two line integrals:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_{C_1} \mathbf{F}\cdot d\mathbf{r}_1 + \oint_{C_2} \mathbf{F}\cdot d\mathbf{r}_2$$

where $C_1 = \langle t, 0, 4\rangle$ for $-2 \le t \le 2$ and$C_2 = \langle 2\cos t, 2\sin t, 4\rangle$ for $-\pi \le t \le 0$. Then

$$\oint_{C_1} \mathbf{F}\cdot d\mathbf{r}_1 = \int_{-2}^{2} \langle 2z+y, 2x+z, 2y+x\rangle \cdot \langle 1,0,0\rangle\, dt = \int_{2}^{-2} 8\, dt = -32$$

$$\oint_{C_2} \mathbf{F}\cdot d\mathbf{r}_2 = \int_{-\pi}^{0} \langle 2z+y, 2x+z, 2y+x\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\, dt$$

$$= \int_{-\pi}^{0} \left(-16\sin t - 4\sin^2 t + 8\cos t + 8\cos^2 t\right)\, dt = 32 + 2\pi,$$

so the total line integral is 2π.

14.7.38 We have, from Stokes' theorem and Ampere's Law, $\iint_S (\nabla\times\mathbf{B})\cdot\mathbf{n}\, dS = \oint_C \mathbf{B}\cdot d\mathbf{r} = \mu I = \mu \iint_S \mathbf{J}\cdot\mathbf{n}\, dS$ Thus we have

$$\iint_S ((\nabla\times\mathbf{B}) - \mu\mathbf{J})\cdot\mathbf{n}\, dS = 0$$

for all surfaces S bounded by any given closed curve C. It is clear that given the freedom to choose C and S, that it follows that the integrand is identically zero, i.e. that for any surface S,$((\nabla\times\mathbf{B}) - \mu\mathbf{J})\cdot$n$= 0$. From this it is easy to see that we must have $\nabla\times\mathbf{B} = \mu\mathbf{J}$, since we are free to make the normal vector point in any direction at any given point by choosing S appropriately.

14.7.39 The boundary of the region is the circle C: $x^2+y^2 = 1$ for$z = 0$. With the usual parameterization, we have

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \langle \cos t - \sin t, \sin t, -\cos t\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt$$

$$= \int_0^{2\pi} \left(-\sin t\cos t + \sin^2 t + \sin t\cos t\right)\, dt = \int_0^{2\pi} \sin^2 t\, dt = \pi.$$

So the integral is independent of a.

14.7.40

a. $\nabla\times\mathbf{F} = \left\langle \frac{\partial}{\partial y}(ay) - \frac{\partial}{\partial z}(cx), \frac{\partial}{\partial z}(bz) - \frac{\partial}{\partial x}(ay), \frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(bz)\right\rangle = \langle a, b, c\rangle$.

b. The area of R is $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint_S \mathbf{n}\cdot\mathbf{n}\, dS = \iint_R |\mathbf{n}|^2\, dA =$ area of R because $|\mathbf{n}| = 1$, so that $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS =$ area of R.

c. $\mathbf{r}'(t) = \langle 5\cos t, -13\sin t, 12\cos t\rangle$, so that $\mathbf{r}(t)\times\mathbf{r}'(t) = \langle 156, 0, -65\rangle$ and thus, because $\mathbf{r}(t)\times\mathbf{r}'(t)$ is constant, it points in a constant direction, so that $\mathbf{r}$ must lie in a plane.

d. By parts (b) and (c), we have a normal vector $\langle 156, 0, -65\rangle$; its magnitude is 169, so we take $\mathbf{F} = \left\langle 0, -\frac{5x}{13}, \frac{12y}{13}\right\rangle$; then the area of R is $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \left\langle 0, -\frac{25}{13}\sin t, 12\cos t\right\rangle \cdot \langle 5\cos t, -13\sin t, 12\cos t\rangle\, dt = \int_0^{2\pi} \left(25\sin^2 t + 144\cos^2 t\right)\, dt = 169\pi$.

14.7.41

a. The boundary of this surface is the circle $x^2+y^2 = 1$ at $z = 0$, so we choose instead the surface of the disk bounded by that circle. $\nabla\times\mathbf{F} = \langle 2x, 0, -2z\rangle$, which is $\langle 2x, 0, 0\rangle$ at $z = 0$, and the normal to the disk is $\langle 0, 0, 1\rangle$. Thus, the integral is equal to $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint_S 0\, dS = 0$.

b. With the usual parameterization of the boundary circle (and remembering that $z = 0$), we have $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 0, 0, 1\rangle \cdot \langle -\sin t, \cos t, 0\rangle \, dt = 0$.

14.7.42

a. Let C be the circle $x^2 + y^2 = a^2$. Parameterize the circle in the usual way; then $\mathbf{F} = \frac{1}{a^p}\langle x, y, 0\rangle$ and $\mathbf{r}(t) = \langle a\cos t, a\sin t, 0\rangle$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{a^p}\int_0^{2\pi} \left(-a^2 \sin t \cos t + a^2 \sin t \cos t\right) \, dt = 0$.

b. Stokes' Theorem will apply when the vector field is defined throughout the disk of radius a, which happens only when $p \le 0$. In that case, $\boldsymbol{\nabla} \times \mathbf{F} = a^{-p}\langle 0, 0, 0\rangle$, so that the surface integral is zero.

14.7.43

a. $\boldsymbol{\nabla} \times \mathbf{F} = \left\langle 0 - 0, 0 - 0, \frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2}\right\rangle = \mathbf{0}$.

b. Let C be the unit circle with the usual parameterization; then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle -\sin t, \cos t, 0\rangle \cdot \langle -\sin t, \cos t, 0\rangle \, dt = 2\pi$.

c. The theorem does not apply because the vector field is not defined at the origin, which is inside the curve C. For example, the limit of the y-coordinate is different depending on the direction.

14.7.44

a. The circumference of the disk is $2\pi R$, so the average circulation is $\frac{1}{2\pi R}\iint\limits_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} \, dS$

b. As R becomes small, because $\mathbf{F}$ and thus $\boldsymbol{\nabla} \times \mathbf{F}$ are continuous, $\boldsymbol{\nabla} \times \mathbf{F}$ can be made arbitrarily close to$(\boldsymbol{\nabla} \times \mathbf{F})_P$ everywhere on S by taking R small enough. Approximately, then, $(\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} \approx (\boldsymbol{\nabla} \times \mathbf{F})_P \cdot$n, so that $\frac{1}{2\pi R}\iint\limits_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} \, dS \approx \frac{1}{2\pi R}\iint\limits_S (\boldsymbol{\nabla} \times \mathbf{F})_P \cdot \mathbf{n} \, dS = (\boldsymbol{\nabla} \times \mathbf{F})_P \cdot \mathbf{n} \frac{1}{2\pi R}\iint\limits_S 1 \, dS = (\boldsymbol{\nabla} \times \mathbf{F})_P \cdot \mathbf{n}$. As R becomes smaller, the goodness of the approximation of $\boldsymbol{\nabla} \times \mathbf{F}$ becomes better, so the value of the integral does as well.

14.7.45 By the chain rule, $\frac{df}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}$ and similarly for g, h, so

$$M_y = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + hz_{xy} + z_x\left(\frac{\partial h}{\partial y} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial y}\right) = f_y + f_z z_y + hz_{xy} + z_x\left(h_y + h_z z_y\right)$$
$$N_x = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + hz_{yx} + z_y\left(\frac{\partial h}{\partial x} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial x}\right) = g_x + g_z z_x + hz_{yx} + z_y\left(h_x + h_z z_x\right).$$

14.7.46 One argument is quite simple: a closed surface has a closed (empty!) boundary, so the integral of F over that boundary is zero. Alternatively, choose any closed curve C dividing the surface into two pieces. On one half, the outward-pointing normals give a counterclockwise orientation to the boundary (viewed from above); on the other half, they give a clockwise orientation. Thus the integral $\iint\limits_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} + \oint_{-C} \mathbf{F} \cdot d\mathbf{r}$, which is zero.

14.7.47 Let F= $\langle 0, g(y, z), h(y, z)\rangle$ be a vector field in the yz-plane; for a region R in that plane, with boundary C, the normal is $\langle 1, 0, 0\rangle$. Now, $\boldsymbol{\nabla} \times \mathbf{F} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, 0, 0\right\rangle$, so by Stokes' theorem, $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint\limits_R \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dA$.

14.8 The Divergence Theorem

14.8.1 The surface integral measures the flow across the boundary.

14.8.2 The volume integral measures the net expansion or contraction of the vector field in the region.

14.8.3 The Divergence Theorem says that the flow across the boundary equals the net expansion or contraction of the field within the solid, so that either can be computed from the other.

14.8.4 Since $\nabla \cdot \langle 2z + y, -x, -2x\rangle = 0$, the net flux is zero.

14.8.5 Since $\nabla \cdot \langle x, y, z\rangle = 3$, the Divergence theorem says that the net outward flux is equal to $\iiint\limits_D 3\, dV = 3 \cdot \text{volume of } S = 32\pi$.

14.8.6 From Example 4 (or Exercise 71 in section 14.5), the divergence is zero.

14.8.7 The outward fluxes must be equal, since by the Divergence theorem the net flux, which is the difference of the two, is zero.

14.8.8 Outward, since it is equal to the integral of $\operatorname{div} \mathbf{F}$ over the cube.

14.8.9 For the volume integral, $\boldsymbol{\nabla} \cdot \mathbf{F} = 9$, so that $\iiint\limits_D \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \iiint\limits_D 9\, dV = 9 \cdot \frac{4}{3} 2^3 \pi = 96\pi$.

For the surface integral, with the usual parameterization of the sphere,

$$\begin{aligned} \mathbf{F} \cdot \mathrm{n} &= \langle 2a \sin u \cos v, 3a \sin u \sin v, 4a \cos u\rangle \cdot \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u\rangle \\ &= 2a^3 \sin^3 u \cos^2 v + 3a^3 \sin^3 u \sin^2 v + 4a^3 \cos^2 u \sin u \end{aligned}$$

and here $a = 2$, so that

$$\begin{aligned} \iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS &= 8 \iint\limits_R \left(2 \sin^3 u \cos^2 v + 3 \sin^3 u \sin^2 v + 4 \cos^2 u \sin u\right) dA \\ &= 8 \int_0^{2\pi} \int_0^{\pi} \left(2 \sin^3 u \cos^2 v + 3 \sin^3 u \sin^2 v + 4 \cos^2 u \sin u\right) du\, dv = 96\pi. \end{aligned}$$

14.8.10 For the volume integral, $\boldsymbol{\nabla} \cdot \mathbf{F} = -3$, so that $\iiint\limits_D (-3)\, dV = -3 \cdot \text{volume of} D = -24$

For the surface integral, we have six surfaces:

$$\begin{array}{ll} S_1 : x = -1 & \mathbf{n} = \langle -1, 0, 0\rangle \\ S_2 : x = 1 & \mathbf{n} = \langle 1, 0, 0\rangle \\ S_3 : y = -1 & \mathbf{n} = \langle 0, -1, 0\rangle \\ S_4 : y = 1 & \mathbf{n} = \langle 0, 1, 0\rangle \\ S_5 : z = -1 & \mathbf{n} = \langle 0, 0, -1\rangle \\ S_6 : z = 1 & \mathbf{n} = \langle 0, 0, 1\rangle \end{array}$$

and on each of those surfaces a simple computation shows that we have $\mathbf{F} \cdot \mathbf{n} = -1$. Thus

$$\sum_{i=1}^{6} \iiint\limits_{S_i} \mathbf{F} \cdot \mathbf{n}\, dS_i = \sum_{i=1}^{6} (-1 \cdot \text{area of } S_i) = -24.$$

14.8.11 For the volume integral, $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$, so the volume integral is zero. For the surface integral, the boundary ellipsoid can be parameterized by $\langle 2 \sin u \cos v, 2\sqrt{2} \sin u \sin v, 2\sqrt{3} \cos u\rangle$, and $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v = \langle 4\sqrt{6} \sin^2 u \cos v, 4\sqrt{3} \sin^2 u \sin v, 4\sqrt{2} \sin u \cos u\rangle$ so that

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} = &\langle 2\sqrt{3} \cos u - 2\sqrt{2} \sin u \sin v, 2 \sin u \cos v, -2 \sin u \cos v\rangle \cdot \\ &\langle 4\sqrt{6} \sin^2 u \cos v, 4\sqrt{3} \sin^2 u \sin v, 4\sqrt{2} \sin u \cos u\rangle = -8 \sin^2 u \cos v \left(-2\sqrt{2} \cos u + \sqrt{3} \sin u \sin v\right) \end{aligned}$$

and then $\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^{\pi} \left(-8 \sin^2 u \cos v \left(-2\sqrt{2} \cos u + \sqrt{3} \sin u \sin v\right)\right) du\, dv = 0$.

14.8.12 For the volume integral, $\nabla \cdot \mathbf{F} = 2(x+y+z)$, so that

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = 2\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3}(x+y+z)\ dz\, dy\, dx = 0.$$

For the surface integral, we have six surfaces:

$S_1 : x = -1$	$\mathbf{n} = \langle -1, 0, 0\rangle$
$S_2 : x = 1$	$\mathbf{n} = \langle 1, 0, 0\rangle$
$S_3 : y = -2$	$\mathbf{n} = \langle 0, -1, 0\rangle$
$S_4 : y = 2$	$\mathbf{n} = \langle 0, 1, 0\rangle$
$S_5 : z = -3$	$\mathbf{n} = \langle 0, 0, -1\rangle$
$S_6 : z = 3$	$\mathbf{n} = \langle 0, 0, 1\rangle$

A short computation shows that for S_1: $\mathbf{F}\cdot\mathbf{n} = -1$, for S_2: $\mathbf{F}\cdot\mathbf{n} = 1$, for $S_{3:}$ $\mathbf{F}\cdot\mathbf{n} = -4$, for S_4: $\mathbf{F}\cdot\mathbf{n} = 4$, for S_5: $\mathbf{F}\cdot\mathbf{n} = -9$, and for S_6: $\mathbf{F}\cdot\mathbf{n} = 9$. Thus, the surface integral is zero.

14.8.13 $\nabla \cdot \mathbf{F} = 0$, so by the Divergence Theorem, the net outward flux is zero since the volume integral of ∇·F is zero.

14.8.14 $\nabla \cdot \mathbf{F} = 0$, so by the Divergence Theorem, the net outward flux is zero since the volume integral of ∇·F is zero.

14.8.15 $\nabla \cdot \mathbf{F} = 0$, so by the Divergence Theorem, the net outward flux is zero since the volume integral of ∇·F is zero.

14.8.16 If $\mathbf{a} = \langle a, b, c\rangle$, then $\mathbf{a}\times\mathbf{r}$ is the field $\mathbf{F}$ in Exercise 15, so the net outward flux is zero.

14.8.17 By the Divergence theorem, we can compute the integral of ∇·F over the ball of radius $\sqrt{6}$ $\nabla\cdot\mathbf{F} = 2$, so the volume integral is

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = 2 \cdot \text{volume of sphere of radius } \sqrt{6} = 2\cdot\frac{4}{3}\pi\cdot 6\sqrt{6} = 16\pi\sqrt{6}.$$

14.8.18 $\nabla\cdot\mathbf{F} = 2x$, so by the Divergence theorem, the outward flux is $\iiint_D 2x\, dV = \int_0^1\int_0^1\int_0^1 2x\, dx\, dy\, dz = 1$.

14.8.19 $\nabla\cdot\mathbf{F} = 4$, so by the Divergence theorem, the outward flux is 4 times the volume of the tetrahedron, which is (by the formula for the volume of a pyramid), $\frac{1}{3}$ times the area of the base times the height, or $\frac{1}{6}$. So the outward flux is $\frac{2}{3}$.

14.8.20 $\nabla \cdot \mathbf{F} = 2(x+y+z)$, so by the Divergence theorem, the outward flux is

$$\iiint_D 2(x+y+z)\ dV = 2\int_0^5\int_0^{2\pi}\int_0^{\pi} r\left(5\sin u\cos v + 5\sin u\ \sin v + 5\cos u\right)\ du\, dv\, dr = 0.$$

14.8.21 $\nabla \cdot \mathbf{F} = -4$, so by the Divergence theorem, the outward flux is -4 times the volume of the sphere, so is $-\frac{128}{3}\pi$.

14.8.22 $\nabla \cdot \mathbf{F} = 0$, so the outward flux is zero by the Divergence theorem.

14.8.23 $\nabla\cdot\mathbf{F} = 3$, so the outward flux is 3 times the volume of the paraboloid, which is $\int_0^2\int_0^{2\pi} r\left(4-r^2\right)\, d\theta\, dr$ $= 8\pi$, so the outward flux is 24π.

14.8.24 $\nabla \cdot \mathbf{F} = 3$, so the outward flux is 3 times the volume of the cone. The area of the base of the cone is 16π, so the outward flux is $3 \cdot \frac{1}{3} \cdot 16\pi \cdot 4 = 64\pi$.

14.8.25 $\nabla \cdot \mathbf{F} = -3$, so the outward flux across the boundary of D is the outward flux across the sphere of radius 4 minus that across the sphere of radius 3, which is $-3 \cdot \frac{4}{3}\pi\left(4^3 - 2^3\right) = -224\pi$.

14.8.26 $\nabla \cdot \mathbf{F} = 4\,|\mathbf{r}|$, so the outward flux across a sphere of radius r is

$$\iiint_D 4\sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} 4\rho\,\rho^2 \sin u\,d\rho\,du\,dv = 4\pi r^4.$$

Thus the net outward flux across the boundary of the given region is 60π.

14.8.27 $\nabla \cdot \mathbf{F} = \frac{2}{|\mathbf{r}|}$, so the outward flux across a sphere of radius r is

$$\iiint_D \frac{2}{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \frac{2}{\rho}\rho^2 \sin u\,d\rho\,du\,dv = 4\pi r^2.$$

Thus the net outward flux across the boundary of the given region is 12π.

14.8.28 $\nabla \cdot \mathbf{F} = 0$, so the net outward flux is zero.

14.8.29 $\nabla \cdot \mathbf{F} = 2\,(x-y+z)$. The net outward flux is thus the difference in the outward flux across the two planes, so is

$$\begin{aligned}&\iiint_D 2\,(x-y+z)\;dV\\ &= 2\left(\int_0^4\int_0^{4-x}\int_0^{4-x-y}(x-y+z)\;dz\,dy\,dx - \int_0^2\int_0^{2-x}\int_0^{2-x-y}(x-y+z)\;dz\,dy\,dx\right)\\ &= 20.\end{aligned}$$

14.8.30 $\nabla \cdot \mathbf{F} = 6$, so the net outward flux is 6 times the difference in the volumes of the cylinders, so is $6 \cdot (4\pi - \pi) \cdot 8 = 144\pi$.

14.8.31

a. False. For example, $\mathbf{F} = \langle y, 0, 0\rangle$ has $\nabla \cdot \mathbf{F} = 0$ at all points of the unit sphere, but the normal to the unit sphere, $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$ is not perpendicular to $\mathbf{F}$ at all points.

b. False. For example, any rotation field has $\nabla \cdot \mathbf{F} = 0$, so that $\iint_S \mathbf{F} \cdot \mathbf{n}\,dS = 0$ by the Divergence theorem, but $\mathbf{F}$ is not in general constant.

c. True. This is because it is bounded by $\iiint_D 1\,dV$.

14.8.32 $\nabla \cdot \mathbf{F} = 3$; we compute the outward flux from the Divergence theorem as (where S_1 is the upper hemisphere of radius a.)

$$\iiint_{S_1} \nabla \cdot \mathbf{F}\,dV = 3\iiint_{S_1} 1\,dV = 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} r^2 \sin u\,dr\,du\,dv = 2\pi a^3.$$

The the outward flux over the whole sphere is thus $4\pi a^3$.

14.8.33 Because $\nabla \cdot \mathbf{F} = 0$, the outward flux is zero from the Divergence theorem.

14.8.34 Because $\nabla \cdot \mathbf{F} = 0$, the outward flux is zero from the Divergence theorem.

14.8.35 $\nabla \cdot \mathbf{F} = 3\sin y$, so the outward flux is

$$\iiint_S 3\sin y\, dV = \int_0^{\pi/2}\int_0^1\int_0^x 3\sin y\, dz\, dx\, dy = \frac{3}{2}.$$

14.8.36

a. We have $\mathbf{F}\cdot\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|^p}\cdot\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{|\mathbf{r}|^2}{|\mathbf{r}|^{p+1}} = |\mathbf{r}|^{1-p}$. Thus the surface integral is $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = a^{1-p}\cdot$ area of sphere $= a^{1-p}\cdot 4\pi a^2 = 4\pi a^{3-p}$.

b. The conditions of the Divergence Theorem require that $\mathbf{F}$ be defined and have continuous partials everywhere inside the sphere; in particular, this must hold at the origin. Thus we must have $p \le -2$. Then the volume integral is

$$\iiint_S \frac{3-p}{|\mathbf{r}|^p}\, dV = (3-p)\int_0^a\int_0^{2\pi}\int_0^{\pi} r^{-p}\cdot r^2\sin u\, du\, dv\, dr = 4\pi a^{3-p}.$$

14.8.37

a. Either use Exercise 36(a), or compute $\mathbf{F}\cdot\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}\cdot\frac{\mathbf{r}}{|\mathbf{r}|} = 1$, so the surface integral is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \text{area of sphere} = 4\pi a^2.$$

b. $\nabla\cdot\mathbf{F} = 2\,|\mathbf{r}|^{-1}$, so if D is the shell between the spheres of radius ϵ and a, the volume integral is

$$\iiint_S 2\,|\mathbf{r}|^{-1}\, dV = 2\int_\epsilon^a\int_0^{2\pi}\int_0^{\pi} r\sin u\, du\, dv\, dr = 4\pi\left(a^2-\epsilon^2\right)$$

and $\lim_{\epsilon\to 0} 4\pi\left(a^2-\epsilon^2\right) = 4\pi a^2$.

14.8.38

a. $\frac{\partial}{\partial x}\varphi = \frac{x}{x^2+y^2+z^2}$, so that $\nabla\varphi = \frac{1}{|\mathbf{r}|^2}\langle x,y,z\rangle = \frac{\mathbf{r}}{|\mathbf{r}|^2}$.

b. $\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|^2}$, so that $\mathbf{F}\cdot\mathbf{n} = \frac{1}{|\mathbf{r}|^2}$. Then $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \frac{1}{a^2}\int_0^{2\pi}\int_0^{\pi} a^3\sin u\, du\, dv = 4\pi a$.

c. By Exercise 36, $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{1}{|\mathbf{r}|^2}$.

d. If D is the shell between the spheres of radius ϵ and$a$, the volume integral is

$$\iiint_D |\mathbf{r}|^{-2}\, dV = 2\int_\epsilon^a\int_0^{2\pi}\int_0^{\pi}\frac{1}{r^2}\, r\sin u\, du\, dv\, dr = 4\pi\left(a-\epsilon\right)$$

and $\lim_{\epsilon\to 0} 4\pi\left(a-\epsilon\right) = 4\pi a$.

14.8.39

a. By Exercise 36, the flux of $\mathbf{E}$ across a sphere of radius a is $\frac{Q}{4\pi\epsilon_0}4\pi a^{3-3} = \frac{Q}{\epsilon_0}$.

b. The net outward flux across S is the difference of the fluxes across the inner and outer spheres; but by part (a), these are equal, so the net flux across S is zero.

c. The left-hand side is the flux across the boundary of D, while the right-hand side is the sum of the charge densities at each point of D. The statement says that the flux across the boundary, up to multiplication by a constant, is the sum of the charge densities in the region.

d. By the Divergence theorem, and using part (c),

$$\frac{1}{\epsilon_0}\iiint\limits_D q\left(x,y,z\right)\,dV = \iint\limits_S \mathrm{E}\cdot\mathbf{n}\,dS = \iiint\limits_D \boldsymbol{\nabla}\cdot\mathrm{E}dV$$

and because this holds for all regions D, we conclude that $\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{q(x,y,z)}{\epsilon_0}$.

e. $\nabla^2\varphi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi = \boldsymbol{\nabla}\cdot\mathbf{E} = \frac{q(x,y,z)}{\epsilon_0}$.

14.8.40

a. By Exercise 36, the flux of $\mathbf{F}$ across a sphere of radius a is $4\pi GMa^{3-p} = 4\pi GM$.

b. Since the outward flux across a sphere, from part (a), is independent of the radius of the sphere, the outward flux across the spheres of radii a and b are equal, so their difference, which is the net flux across the spherical shell bounded by them, is zero.

c. The left hand side is the flux across the boundary of D, while the right-hand side is the sum of the mass density inside D. The statement says that the flux across the boundary is determined by (is a constant multiple of) the sum of the mass density inside D.

d. By the Divergence theorem, and using part (c),

$$4\pi G\iiint\limits_D \rho\left(x,y,z\right)\,dV = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint\limits_D \boldsymbol{\nabla}\cdot\mathbf{F}dV$$

and because this holds over all regions D, we have $\boldsymbol{\nabla}\cdot\mathbf{F} = 4\pi G\rho\left(x,y,z\right)$.

e. $\boldsymbol{\nabla}^2\varphi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi = \boldsymbol{\nabla}\cdot\mathbf{F} = 4\pi G\rho\left(x,y,z\right)$.

14.8.41 $\mathbf{F} = -\nabla\mathbf{T} = \langle -1,-2,-1\rangle$, so that $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$ and the heat flux is zero.

14.8.42 $\mathbf{F} = -\nabla\mathbf{T} = \langle -2x,-2y,-2z\rangle$, so that $\boldsymbol{\nabla}\cdot\mathbf{F} = -6$, and the heat flux is -6 times the volume of the region, or -6.

14.8.43 $\mathbf{F} = -\nabla\mathbf{T} = \langle 0,0,e^{-z}\rangle$; then $\boldsymbol{\nabla}\cdot\mathbf{F} = -e^{-z}$. The heat flux is then $\int_0^1\int_0^1\int_0^1 -e^{-z}\,dx\,dy\,dz = e^{-1}-1$.

14.8.44 From Exercise 42, $\boldsymbol{\nabla}\cdot\mathbf{F} = -6$, so the heat flux is -6 times the volume of the sphere, or -8π.

14.8.45 F$= -\nabla\mathbf{T} = \langle 200xe^{-x^2-y^2-z^2}, 200ye^{-x^2-y^2-z^2}, 200ze^{-x^2-y^2-z^2}\rangle$. Then

$$\boldsymbol{\nabla}\cdot\mathbf{F} = 200e^{-x^2-y^2-z^2}\left(3-2x^2-2y^2-2z^2\right)$$

so that

$$\iiint\limits_D \boldsymbol{\nabla}\cdot\mathbf{F}\,dV = 200\int_0^{2\pi}\int_0^{\pi}\int_0^{a} e^{-r^2}\left(3-2r^2\right)r^2\sin u\,dr\,du\,dv = 800\pi a^3 e^{-a^2}.$$

14.8.46

a. By Exercise 36, the net flux across a sphere of radius a centered at the origin is $4\pi a^{3-p}$, which is independent of a only if $p = 3$.

b. In the general case, we have $\nabla \cdot \mathbf{F} = \frac{3-p}{|\mathbf{r}|^p}$, so

$$\begin{aligned}\iiint_D \nabla \cdot \mathbf{F}\, dV &= \int_a^b \int_{\varphi_1}^{\varphi_2} \int_{\theta_1}^{\theta_2} \frac{3-p}{r^p} r^2 \sin u \, du\, dv\, dr \\ &= \left(a^{3-p} - b^{3-p}\right)\left(\varphi_1 \cos\theta_1 - \varphi_2 \cos\theta_1 - \varphi_1 \cos\theta_2 + \varphi_2 \cos\theta_2\right) \\ &= \left(a^{3-p} - b^{3-p}\right)\left(\varphi_1 - \varphi_2\right)\left(\cos\theta_1 - \cos\theta_2\right),\end{aligned}$$

and these are in general zero only if $3 - p = 0$.

14.8.47

a. $\varphi_x(x,y,z) = G'(\rho)\,\rho_x = G'(\rho) \cdot \frac{x}{\sqrt{x^2+y^2+z^2}} = G'(\rho)\,\frac{x}{\rho}$, so that $\nabla\varphi = \mathbf{F} = G'(\rho)\,\frac{\mathbf{r}}{\rho}$.

b. The normal to the sphere of radius a is $\left\langle \frac{x}{z}, \frac{y}{z}, 1\right\rangle$, so on that sphere (where $\rho = a$)

$$\mathbf{F} \cdot \mathbf{n} = G'(a)\,\frac{\frac{x^2}{z} + \frac{y^2}{z} + z}{a} = G'(a)\,\frac{\frac{a^2-z^2}{z} + z}{a} = G'(a)\,\frac{a}{z},$$

and then the surface integral over the upper hemisphere is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = aG'(a) \int_0^a \int_0^{2\pi} \frac{r}{\sqrt{a^2 - r^2}}\, d\theta\, dr = 2\pi a^2 G'(a),$$

so the total surface integral is twice that, or $4\pi a^2 G'(a)$.

c. By the Chain Rule,

$$\frac{\partial}{\partial x} G'(\rho)\,\frac{x}{\rho} = G''(\rho)\,\rho_x \frac{x}{\rho} + G'(\rho)\,\frac{y^2+z^2}{\rho^3}$$

so that (noting that $\rho_x = \frac{x}{\rho}$)

$$\nabla \cdot \mathbf{F} = G''(\rho)\left(\frac{x^2+y^2+z^2}{\rho^2}\right) + G'(\rho)\,\frac{2\left(x^2+y^2+z^2\right)}{\rho^3} = G''(\rho) + \frac{2\,G'(\rho)}{\rho}.$$

d. By the Divergence theorem, the flux is also given by

$$\begin{aligned}\iiint_D \nabla \cdot \mathbf{F}\, dV &= \int_0^a \int_0^\pi \int_0^{2\pi} \rho^2 \sin u \left(G''(\rho) + \frac{2\,G'(\rho)}{\rho}\right) dv\, du\, d\rho \\ &= 2\pi \int_0^a \int_0^\pi \sin u \left(\rho^2\, G''(\rho) + 2\rho\, G'(\rho)\right)\, du\, d\rho \\ &= 2\pi \int_0^a (-\cos u)\left(\rho^2\, G''(\rho) + 2\rho\, G'(\rho)\right)\Big|_{u=0}^{u=\pi} d\rho \\ &= 4\pi \int_0^a \left(\rho^2\, G''(\rho) + 2\rho\, G'(\rho)\right)\, d\rho.\end{aligned}$$

It remains to evaluate this integral. Using integration by parts, we have

$$\begin{aligned}4\pi\left[\int_0^a \left(\rho^2\, G''(\rho) + 2\rho\, G'(\rho)\right)\, d\rho\right] &= 4\pi\left[\rho^2\, G'(\rho)\Big|_{\rho=0}^{\rho=a} - \int_0^a 2\rho G'(\rho)\, d\rho\right] \\ &= 4\pi\left[a^2 G'(a) - \int_0^a 2\rho\, G'(\rho)\, d\rho\right]\end{aligned}$$

and the remaining integrals cancel, giving $4\pi a^2 G'(a)$ as the final result.

14.8.48

a. Rearrange the given equation and integrate over D to get

$$\iiint_D u\,\boldsymbol{\nabla}\cdot\mathbf{F}\,dS = \iiint_D \boldsymbol{\nabla}\cdot(u\,\mathbf{F})\;dS - \iiint_D \mathbf{F}\cdot\boldsymbol{\nabla}u\,dS.$$

By the Divergence theorem, the first term on the right side is equal to $\iint_S u\,\mathbf{F}\cdot\mathbf{n}\,dS$ where S is the boundary of D. The result follows.

b. If you set $\mathbf{F}=\langle f(x),0,0\rangle$ and$u(x,y,z)=g(x)$, then

$$\begin{array}{ll} \boldsymbol{\nabla}\cdot\mathbf{F}=f'(x) & u\boldsymbol{\nabla}\cdot\mathbf{F}=f'(x)\,g(x) \\ u\mathbf{F}=\langle f(x)\,g(x)\,,0,0\rangle & \nabla\cdot(u\mathbf{F})=(fg)'(x) \\ \nabla u=\langle g'(x)\,,0,0\rangle & \mathbf{F}\cdot\boldsymbol{\nabla}u=f(x)\,g'(x) \end{array}$$

so that

$$\iiint_D f'(x)\,g(x)\;dV = \iiint_D (fg)'(x)\;dV - \iiint_D f(x)\,g'(x)\;dV = f(x)\,g(x) - \iiint_D f(x)\,g'(x)\;dV,$$

which is the usual rule for integration by parts.

c. One approach is to set $u=1$ and $\mathbf{F}=\frac{1}{2}\langle x^2z^2,x^2y^2,y^2z^2\rangle$. Gauss' formula then gives

$$\iiint_D \left(x^2y+y^2z+z^2x\right) = \frac{1}{2}\iint_S \langle x^2z^2,x^2y^2,y^2z^2\rangle\,dS.$$

The integral on the right is computed by integrating over each face of the cube; on faces where x is constant, the normal is either $\langle 1,0,0\rangle$ or $\langle -1,0,0\rangle$ depending on whether x is 1 or 0; similarly for y and z. Considering x first, when $x=0$, this surface integral becomes 0 on that face (the integrand is x^2z^2 at $x=0$), while for $x=1$ it becomes z^2. In that case, the integral is $\frac{1}{3}$. This holds for each dimension, so the total integral on the right side is $3\cdot\frac{1}{3}=1$, and thus the integral on the left is $\frac{1}{2}$.

14.8.49 Suppose F$=\langle f,g\rangle$ where$f=f(x,y)$, $g=g(x,y)$, and suppose$u=u(x,y)$. Then $\boldsymbol{\nabla}\cdot\mathbf{F}=f_x+g_y$, and $\nabla u=\langle u_x,u_y\rangle$. Then we have for this case

$$\begin{aligned} \iiint_D u\,\boldsymbol{\nabla}\cdot\mathbf{F}\,dS &= \iint_R u\,(f_x+g_y)\;dA \\ \iint_S u\mathbf{F}\cdot\mathbf{n}\,dS &= \oint_C u\mathbf{F}\cdot\mathbf{n}\,ds \\ \iiint_D \mathbf{F}\cdot\boldsymbol{\nabla}u\,dS &= \iint_R (f\,u_x+g\,u_y)\;dA, \end{aligned}$$

and the result follows. Setting $u=1$ then gives $\iint_R (f_x+g_y)\;dA=\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$, which is the flux form of Green's Theorem.

14.8.50 Apply the Divergence Theorem to the vector field $\mathbf{F}=u\boldsymbol{\nabla}u$. By the product rule (Thm. 14.11), we have $\boldsymbol{\nabla}\cdot(v\boldsymbol{\nabla}v)=\boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}v+u\boldsymbol{\nabla}^2v$, so the Divergence theorem says that

$$\iiint_D \left(u\boldsymbol{\nabla}^2v+\boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}v\right)\;dV = \iiint_D \left(\boldsymbol{\nabla}\cdot(v\boldsymbol{\nabla}v)\right)\;dV = \iint_S u\boldsymbol{\nabla}v\cdot\mathbf{n}\,dS.$$

14.8.51 From Exercise 50, we have both

$$\iiint_D \left(u\nabla^2 v + \nabla u \cdot \nabla v\right)\, dV = \iint_S u\nabla v \cdot \mathbf{n}\, dS$$
$$\iiint_D \left(v\nabla^2 u + \nabla v \cdot \nabla u\right)\, dV = \iint_S v\nabla u \cdot \mathbf{n}\, dS,$$

where the second formula is obtained by switching u and v in the first formula. Subtracting the second from the first, and using the fact that the dot product is commutative and integrals are linear, we have the desired result:

$$\iiint_D \left(u\nabla^2 v - v\nabla^2 u\right)\, dV = \iint_S (u\nabla v - v\nabla u) \cdot \mathbf{n}\, dS.$$

14.8.52 A computation shows that $\nabla\varphi = \frac{-p\,\mathrm{r}}{|r|^{p+2}}$. Thus the potential field $\nabla\varphi = 0$ for $p = 0$, so certainly $\nabla^2\varphi = 0$ as well. Otherwise, for $p = 1$, we have $\nabla\varphi = \frac{-\mathbf{r}}{|r|^3}$; this is an inverse square field, and we have seen many times (e.g. Exercise 36(b)) that the divergence of such a field is zero. Thus $\nabla^2\varphi = 0$ for $p = 1$ as well.

14.8.53 The Divergence theorem applied to the field$\nabla\varphi$ says that

$$\iiint_D \left(\nabla^2\varphi\right)\, dV = \iint_S \nabla\varphi \cdot \mathbf{n}\, dS$$

and if φ is harmonic, the left side is zero.

14.8.54 Apply Green's First Identity (Exercise 49) to u and u to give

$$\iiint_D \left(u\nabla^2 u + \nabla u \cdot \nabla u\right)\, dV = \iint_S u\nabla u \cdot \mathbf{n}\, dS.$$

Because $\nabla^2 u = 0$ and $\nabla u \cdot \nabla u = |\nabla u|^2$, the result follows.

14.8.55 If $\mathbf{T}$ is a vector field $\langle t, u, v\rangle$, then by $\iint \mathbf{T}$, we mean $\langle \iint t, \iint u, \iint v\rangle$.

a. Let $\mathbf{F} = \langle f, g, h\rangle$ and suppose $\mathbf{n} = \langle n_1, n_2, n_3\rangle$. Then

$$\mathbf{n} \times \mathbf{F} = \langle n_2 h - n_3 g, n_3 f - n_1 h, n_1 g - n_2 f\rangle$$
$$\nabla \times \mathbf{F} = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right\rangle$$

Considering first the $\mathbf{i}$ component of these vectors, note that for the vector field $\mathbf{F}_1 = \langle 0, h, -g\rangle$, the Divergence theorem says that

$$\iint_S (n_2 h - n_3 g)\, dS = \iint_S \langle 0, h, -g\rangle \cdot \langle n_1, n_2, n_3\rangle\, dS = \iint_S \mathbf{F}_1 \cdot \mathbf{n}\, dS = \iiint_D (\nabla \cdot \mathbf{F}_1)\, dR$$
$$= \iint_D \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dA$$

and similarly for the second and third components.

b. Similarly to part (a), note that

$$\mathbf{n} \times \nabla\varphi = \mathbf{n} \times \langle \varphi_x, \varphi_y, \varphi_z\rangle = \langle n_2\varphi_z - n_3\varphi_y, n_3\varphi_x - n_1\varphi_z, n_1\varphi_y - n_2\varphi_x\rangle.$$

Looking first at the x component of this vector, we have

$$n_2\varphi_z - n_3\varphi_y = \langle 0, \varphi_z, -\varphi_y\rangle \cdot \langle n_1, n_2, n_3\rangle = (\nabla \times \langle \varphi, 0, 0\rangle) \cdot \langle n_1, n_2, n_3\rangle$$

so that Stokes' theorem says that, writing $\mathbf{F} = \langle \varphi, 0, 0\rangle$,

$$\iint_S (\nabla \times \langle \varphi, 0, 0\rangle) \cdot \langle n_1, n_2, n_3\rangle \, dS = \iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \langle \varphi, 0, 0\rangle \cdot d\mathbf{r} = \oint_C \varphi \, d\mathbf{r}.$$

and similarly for the second and third components.

Chapter Fourteen Review

1

a. False. The curl is $\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 2$.

b. True. The curl of a conservative vector field is zero.

c. False. For example, $\langle -y, x\rangle$ and $\langle 0, 2x\rangle$ both have curl 2.

d. False. For example, $\langle x, 0, 0\rangle$ and $\langle 0, y, 0\rangle$ both have divergence 1.

e. True. By the Divergence theorem, the integral is equal to $\iiint_D \boldsymbol{\nabla} \cdot \mathbf{F} \, dV = \iiint_D 3 \, dV$.

2

a. Choice F, because this is a radial vector field whose magnitude increases with distance from the origin.

b. Choice E, because this is a rotational field.

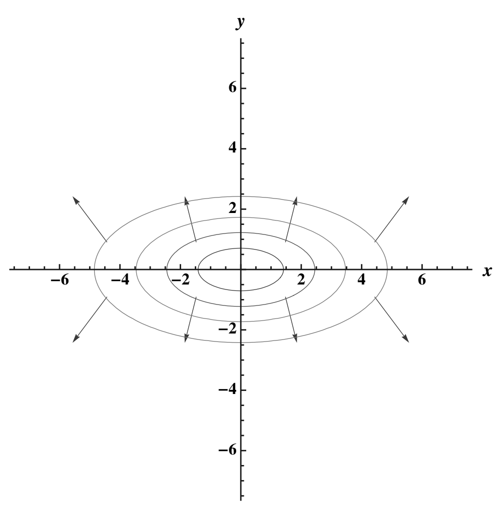
c. Choice D, because this is also a radial field, but the magnitude is constant.

d. Choice C, because this field is vertical along the line $y = x$.

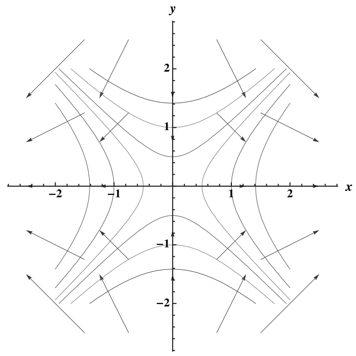
e. Choice B, because the magnitude decreases rapidly away from the origin.

f. Choice A, because this field is periodic.

3 $\nabla\varphi = \langle 2x, 8y\rangle$.



4 $\nabla\varphi = \langle x, -y\rangle$.



5 $\nabla\varphi = -\frac{\mathbf{r}}{|\mathbf{r}|^3}$

6 $\nabla\varphi = -e^{-x^2-y^2-z^2}\langle x, y, z\rangle$.

7

a. $\langle x, y\rangle$ is an outward normal; for (x, y) on the circle, $|\langle x, y\rangle| = \sqrt{x^2+y^2} = 2$, so the unit outward normal is $\frac{1}{2}\langle x, y\rangle$.

b. The normal component is $2\langle y, -x\rangle \cdot \frac{1}{2}\langle x, y\rangle = 0$.

c. The normal component is $\frac{1}{x^2+y^2}\langle x, y\rangle \cdot \frac{1}{2}\langle x, y\rangle = \frac{1}{x^2+y^2} \cdot \frac{1}{2} \cdot \left(x^2+y^2\right) = \frac{1}{2}$.

8 We have $|\mathbf{r}'(t)| = 5$, so the line integral is

$$\int_C \left(x^2 - 2xy + y^2\right)\, ds = 5\int_0^\pi \left(25\cos^2 t - 10\cos t\sin t + 25\sin^2 t\right)\, dt = 25\int_0^\pi (5 - 2\sin t\cos t)\, dt = 125\pi.$$

9 Here $|\mathbf{r}'(t)| = \sqrt{1+9+36} = \sqrt{46}$, so

$$\int_C y\, e^{-xz}\, ds = \sqrt{46}\int_0^{\ln 8} 3te^{6t^2}\, dt = \frac{\sqrt{46}}{4}\left(e^{54\,(\ln 2)^2} - 1\right).$$

10 Parameterize the line by $\langle -3t, 1+6t, 2-3t\rangle$ for $0 \le t \le 1$. Then $|\mathbf{r}'(t)| = \sqrt{9+36+9} = \sqrt{54} = 3\sqrt{6}$, so

$$\int_C \left(xz - y^2\right)\, ds = 3\sqrt{6}\int_0^1 \left(3t\,(3t-2) - (6t+1)^2\right)\, dt = -57\sqrt{6}.$$

11 For the first parameterization we have $|\mathbf{r}'(t)| = 2$, so

$$\oint_C (x - 2y - 3z)\, ds = 2\int_0^{2\pi} (2\cos t - 4\sin t)\, dt = 0.$$

For the second parameterization we have $|\mathbf{r}'(t)| = \sqrt{16t^2\sin^2(t^2) + 16t^2\cos^2(t^2)} = 4t$, so

$$\oint_C (x - 2y - 3z)\, ds = \int_0^{\sqrt{2\pi}} \left(8t\cos t^2 - 16t\sin t^2\right)\, dt = 0.$$

12

a. Parameterize the path from P to Q by $\langle 1-t, t, 0\rangle$ for $0 \le t \le 1$; then the work done is $\int_C \langle 1, 2y, -4z\rangle \cdot d\mathbf{r} = \int_0^1 \langle 1, 2t, 0\rangle \cdot \langle -1, 1, 0\rangle\, dt = \int_0^1 (2t-1)\, dt = 0$.

b. Parameterize the path from P to O by $\langle 1-t,0,0\rangle$, and the path from O to Q by $\langle 0,t,0\rangle$, both for $0\le t\le 1$. Then the work done is $\int_C \langle 1,2y,-4z\rangle\cdot d\mathbf{r} = \int_0^1 \left(\langle 1,0,0\rangle\cdot\langle -1,0,0\rangle + \langle 1,2t,0\rangle\cdot\langle 0,1,0\rangle\right)dt = \int_0^1 (2t-1)\,dt = 0$.

c. Parameterize the quarter circle by $\langle \cos t,\sin t,0\rangle$ for $0\le t\le \frac{\pi}{2}$; then the work done is $\int_C\langle 1,2y,-4z\rangle\cdot d\mathbf{r} = \int_0^{\pi/2}\langle 1,2\sin t,0\rangle\cdot\langle -\sin t,\cos t,0\rangle\,dt = \int_0^{\pi/2}(2\cos t\,\sin t-\sin t)\,dt = 0$.

d. Yes, the work is independent of the path; this is a conservative vector field with potential function $x+y^2-2z^2$.

13 Parameterize the first path by $\mathbf{r}_1(t) = \langle 0,t,0\rangle$, and the second by $\mathbf{r}_1(t)=\langle 0,1,4t\rangle$, both for $0\le t\le 1$. Then $\mathbf{r}_1'(t)=\langle 0,1,0\rangle$ and $\mathbf{r}_2'(t)=\langle 0,0,4\rangle$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \big(\langle -t,0,0\rangle\cdot\langle 0,1,0\rangle + \langle -1,4t,0\rangle\cdot\langle 0,0,4\rangle\big)\,dt = 0.$$

14 $\mathbf{r}'(t) = \langle 2t,6t,-2t\rangle$, so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^2 \frac{1}{(11t^4)^{3/2}}\langle t^2,3t^2,-t^2\rangle\cdot\langle 2t,6t,-2t\rangle\,dt = 11^{-3/2}\int_1^2 t^{-6}\cdot 22t^3\,dt = 11^{-3/2}\int_1^2 22t^{-3}\,dt = \frac{3}{44}\sqrt{11}.$$

15 The circulation is

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_0^{2\pi}\langle 2\sin t-2\cos t,2\sin t\rangle\cdot\big\langle -2\sin t,2\cos t\big\rangle\,dt = \int_0^{2\pi}\left(-4\sin^2 t+8\sin t\cos t\right)dt = -4\pi.$$

The outward flux is

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{n}\,ds &= \int_0^{2\pi}\left((2\sin t-2\cos t)(2\cos t)-(2\sin t)(-2\sin t)\right)dt\\ &= 4\int_0^{2\pi}\left(\sin t\,\cos t-\cos^2 t+\sin^2 t\right)dt = 0.\end{aligned}$$

16 The circulation is

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_0^{2\pi}\langle 2\cos t,2\sin t\rangle\cdot\langle -2\sin t,2\cos t\rangle\,dt = 0.$$

The outward flux is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi}\left((2\cos t)(2\cos t)-(2\sin t)(-2\sin t)\right)dt = 8\pi.$$

17 The circulation is

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \frac{1}{4}\int_0^{2\pi}\langle 2\cos t,2\sin t\rangle\cdot\langle -2\sin t,2\cos t\rangle\,dt = 0.$$

The outward flux is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \frac{1}{4}\int_0^{2\pi}\left((2\cos t)(2\cos t)-(2\sin t)(-2\sin t)\right)dt = 2\pi.$$

18 The circulation is

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_0^{2\pi}\langle 2\cos t-2\sin t,2\cos t\rangle\cdot\big\langle -2\sin t,2\cos t\big\rangle\,dt = 4\int_0^{2\pi}(-\sin t\cos t+1)\,dt = 4\pi.$$

The outward flux is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi}\left((2\cos t-2\sin t)(2\cos t)-(2\cos t)(-2\sin t)\right)dt = 4\int_0^{2\pi}\left(\cos^2 t\right)dt = 4\pi.$$

19 The normal to the plane $x = 0$ is $\langle 1, 0, 0\rangle$, so the flux is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{-1/2}^{1/2} \int_{-L}^{L} v_0 \left(L^2 - y^2\right) dy\, dz = \frac{4}{3} v_0 L^3..$$

20 A potential function is xy^2.

21 A potential function is $xy + yz^2$.

22 A potential function is $e^x \cos y$.

23 A potential function is xye^z.

24

a. $\mathbf{F} = \langle 2xy, x^2\rangle$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \langle 2\left(9 - t^2\right) t, \left(9 - t^2\right)^2\rangle \cdot \langle -2t, 1\rangle\, dt = \int_0^3 \left(-4t^2\left(9 - t^2\right) + \left(9 - t^2\right)^2\right) dt$ $= \int_0^3 \left(9 - 5t^2\right)\left(9 - t^2\right) dt = 0$

b. Because $\mathbf{F} = \nabla\varphi$, where $\varphi(x, y) = x^2 y$, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(3) - \varphi(0) = 0 - 0 = 0$.

25

a. $\mathbf{F} = \langle yz, xz, xy\rangle$, so

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi} \left\langle \frac{t}{\pi}\sin t, \frac{t}{\pi}\cos t, \sin t\cos t\right\rangle \cdot \left\langle -\sin t, \cos t, \frac{1}{\pi}\right\rangle dt \\ &= \int_0^{\pi} \left(\frac{t}{\pi}\left(\cos^2 t - \sin^2 t\right) + \frac{1}{\pi}\sin t\cos t\right) dt = 0.\end{aligned}$$

b. $\mathbf{F} = \nabla(xyz) = \nabla\varphi$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\cos\pi\sin\pi) - \varphi\left(\cos 0 \sin 0 \cdot \frac{0}{\pi}\right) = 0 - 0 = 0.$$

26

a. Parameterize C by the four paths $\mathbf{r}_1(t) = \langle -1 + 2t, -1\rangle$, $\mathbf{r}_2(t) = \langle 1, -1 + 2t\rangle$, $\mathbf{r}_3(t) = \langle 1 - 2t, 1\rangle$, $\mathbf{r}_4(t) = \langle -1, 1 - 2t\rangle$, for $0 \le t \le 1$. Then $\mathbf{r}_1'(t) = \langle 2, 0\rangle$, $\mathbf{r}_2'(t) = \langle 0, 2\rangle$, $\mathbf{r}_3'(t) = \langle -2, 0\rangle$, $\mathbf{r}_4'(t) = \langle 0, -2\rangle$, and

$$\begin{aligned}&\int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^1 \left(\langle -1 + 2t, 1\rangle \cdot \langle 2, 0\rangle + \langle 1, 1 - 2t\rangle \cdot \langle 0, 2\rangle + \langle 1 - 2t, -1\rangle \cdot \langle -2, 0\rangle + \langle -1, 2t - 1\rangle \cdot \langle 0, -2\rangle\right) dt \\ &= \int_0^1 (4t - 2 + 2 - 4t + 4t - 2 + 2 - 4t)\, dt = \int_0^1 0\, dt = 0.\end{aligned}$$

b. $\mathbf{F} = \nabla\varphi$, where $\varphi(x, y) = \frac{1}{2}\left(x^2 - y^2\right)$, so the integral around any closed curve is zero.

27

a. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left\langle \sin t, 4, -\cos t\right\rangle \cdot \left\langle -\sin t, \cos t, 0\right\rangle dt = \int_0^{2\pi} \left(-\sin^2 t + 4\cos t\right) dt = -\pi$.

b. The vector field is not conservative, since for example $\frac{\partial}{\partial y}(y) \neq \frac{\partial}{\partial z}(x)$.

28 For $p \neq 2$, $\mathbf{F} = \nabla\varphi$ where $\varphi = \frac{-1}{(p-2)|\mathbf{r}|^{p-2}}$, while for $p = 2$, $\varphi = \frac{1}{2}\ln\left(|\mathbf{r}|^2\right)$, as can be seen by taking the gradient. Thus $\mathbf{F}$ is conservative on all of $\mathbb{R}^2$ for $p < 0$.

29 By the circulation form of Green's Theorem,

$$\oint_C xy^2\,dx + x^2y\,dy = \iint\limits_R \left(\frac{\partial}{\partial x}\left(x^2y\right) - \frac{\partial}{\partial y}\left(xy^2\right)\right) dA = \iint\limits_R (2xy - 2xy)\ dA = 0.$$

30 By the circulation form of Green's Theorem,

$$\oint_C \left(-3y + x^{3/2}\right) dx + \left(x - y^{2/3}\right) dy = \iint\limits_R \left(\frac{\partial}{\partial x}\left(x - y^{2/3}\right) - \frac{\partial}{\partial y}\left(-3y + x^{3/2}\right)\right) dA = \iint\limits_R 4\,dA = 4\pi.$$

31 By the circulation form of Green's Theorem,

$$\begin{aligned}\oint_C \left(x^3 + xy\right) dy + \left(2y^2 - 2x^2y\right) dx &= \iint\limits_R \left(\frac{\partial}{\partial x}\left(x^3 + xy\right) - \frac{\partial}{\partial y}\left(2y^2 - 2x^2y\right)\right) dA \\ &= \iint\limits_R \left(3x^2 + y - 4y + 2x^2\right) dA = \int_{-1}^{1}\int_{-1}^{1} \left(5x^2 - 3y\right) dy\,dx = \frac{20}{3}.\end{aligned}$$

32 By the flux form of Green's Theorem, $\oint_C 3x^3\,dy - 3y^3\,dx = \iint\limits_R \left(9x^2 + 9y^2\right) dA = 9\int_0^4\int_0^{2\pi} r^3\,d\theta\,dr = 1152\pi$. Because the orientation is clockwise, the answer is -1152π.

33 The ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$; parameterize it by $\mathbf{r}(t) = \langle x, y\rangle = \langle 4\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$. Then the area of the region is

$$\frac{1}{2}\oint_C \left((4\cos t)(2\cos t) - (2\sin t)(-4\sin t)\right) dt = \int_0^{2\pi} 4\left(\cos^2 t + \sin^2 t\right) dt = 8\pi.$$

34 $dx = -3\cos^2 t\sin t\,dt$, and$dy = 3\sin^2 t\cos t\,dt$, so the area of the hypocycloid is

$$\begin{aligned}\frac{1}{2}\oint_C x\,dy - y\,dx &= \frac{1}{2}\int_0^{2\pi} \left(\left(\cos^3\ t\right)\left(3\sin^2 t\cos t\right) - \left(\sin^3\ t\right)\left(-3\cos^2 t\sin t\right)\right) dt \\ &= \frac{3}{2}\int_0^{2\pi} \left(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right) dt = \frac{3\pi}{8}.\end{aligned}$$

35

a. $\mathbf{F} = \left(x^2 + y^2\right)^{-1/2}\langle x, y\rangle$, so the circulation is

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint\limits_R \left(\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right) dA \\ &= \iint\limits_R \left(\frac{-xy}{\sqrt{x^2+y^2}} - \frac{-xy}{\sqrt{x^2+y^2}}\right) dA = 0.\end{aligned}$$

b. The flux is

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{n}\,ds &= \iint\limits_R \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right) dA \\ &= \iint\limits_R \left(\frac{y^2}{\left(x^2+y^2\right)^{3/2}} + \frac{x^2}{\left(x^2+y^2\right)^{3/2}}\right) dA = \iint\limits_R \left(\frac{1}{\sqrt{x^2+y^2}}\right) dA = \int_0^{\pi}\int_1^3 1\,dr\,d\theta = 2\pi.\end{aligned}$$

36

a. The circulation is $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_R \left(\frac{\partial}{\partial x}(x\cos y) - \frac{\partial}{\partial y}(-\sin y)\right) dA = \int_0^{\pi/2}\int_0^{\pi/2} 2\,\cos y\,dy\,dx = \pi.$

b. The flux is $\oint_C \mathbf{F}\cdot \mathbf{n}\,ds = \iint\limits_R \left(\frac{\partial}{\partial x}(-\sin y) + \frac{\partial}{\partial y}(x\cos y)\right) dA = \int_0^{\pi/2}\int_0^{\pi/2} (-x\sin y)\,dy\,dx = -\frac{1}{8}\pi^2.$

37

a. For $\mathbf{F}$ to be conservative, we must have $\frac{\partial}{\partial y}(ax+by) = \frac{\partial}{\partial x}(cx+dy)$, or $b = c$.

b. For $\mathbf{F}$ to be source-free, we must have $\frac{\partial}{\partial x}(ax+by) = -\frac{\partial}{\partial y}(cx+dy)$, or $a = -d$.

c. $\mathbf{F}$ is both conservative and source-free if $b = c$ and $a = -d$, i.e. if $\mathbf{F} = \langle ax+by, bx-ay\rangle$.

38 The divergence is $\frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$. The curl is

$$\langle \frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz), \frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(xy), \frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\rangle = \mathbf{0}.$$

The field is both source-free and irrotational.

39 The divergence is $4\,|\mathbf{r}|$. The curl is

$$\langle \frac{\partial}{\partial y}(z\,|\mathbf{r}|) - \frac{\partial}{\partial z}(y\,|\mathbf{r}|), \frac{\partial}{\partial z}(x\,|\mathbf{r}|) - \frac{\partial}{\partial x}(z\,|\mathbf{r}|), \frac{\partial}{\partial x}(y\,|\mathbf{r}|) - \frac{\partial}{\partial y}(x\,|\mathbf{r}|)\rangle = \mathbf{0}.$$

The field is irrotational but not source-free.

40 The divergence is $\frac{\partial}{\partial x}(\sin xy) + \frac{\partial}{\partial y}(\cos yz) + \frac{\partial}{\partial z}(\sin xz) = y\cos xy - z\sin yz + x\,\cos xz$. The curl is $\langle y\sin yz, -z\cos xz, -x\cos xy\rangle$. The field is neither irrotational nor source-free.

41 The divergence is $\frac{\partial}{\partial x}\left(2xy+z^4\right) + \frac{\partial}{\partial y}\left(x^2\right) + \frac{\partial}{\partial z}\left(4xz^3\right) = 2y + 12xz^2$. The curl is

$$\langle \frac{\partial}{\partial y}\left(4xz^3\right) - \frac{\partial}{\partial z}\left(x^2\right), \frac{\partial}{\partial z}\left(2xy+z^4\right) - \frac{\partial}{\partial x}\left(4xz^3\right), \frac{\partial}{\partial x}\left(x^2\right) - \frac{\partial}{\partial y}\left(2xy+z^4\right)\rangle = \mathbf{0},$$

so the field is irrotational but not source-free.

42 $|\mathbf{r}|^4 = \left(x^2+y^2+z^2\right)^2$, so

$$\nabla\left(x^2+y^2+z^2\right)^{-2} = \left\langle -4x\left(x^2+y^2+z^2\right)^{-3}, -4y\left(x^2+y^2+z^2\right)^{-3}, -4z\left(x^2+y^2+z^2\right)^{-3}\right\rangle = -\frac{4\,\mathbf{r}}{|\mathbf{r}|^6}.$$

Now

$$\frac{\partial}{\partial x}\left(-4x\left(x^2+y^2+z^2\right)^{-3}\right) = -4\left(y^2+z^2-5x^2\right)\left(x^2+y^2+z^2\right)^{-4},$$

so that

$$\nabla\cdot\nabla\,|\mathbf{r}|^{-4} = -4\left(x^2+y^2+z^2\right)^{-4}\left(y^2+z^2-5x^2+x^2+z^2-5y^2+x^2+y^2-5z^2\right) = \frac{12}{|\mathbf{r}|^6}.$$

43

a. The curl is

$$\langle \frac{\partial}{\partial y}(-y) - \frac{\partial}{\partial z}(x), \frac{\partial}{\partial z}(z) - \frac{\partial}{\partial x}(-y), \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(z)\rangle = \langle -1, 1, 1\rangle.$$

So the scalar component in the direction of $\langle 1,0,0\rangle$ is$\langle -1,1,1\rangle\cdot\langle 1,0,0\rangle = -1$, and the scalar component in the direction of $\left\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$ is$\left\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle\cdot\langle 1,0,0\rangle = 0$.

b. The scalar component of the curl is a maximum in the direction of the curl, i.e. in the direction $\langle -1,1,1\rangle$, whose unit vector is $\frac{1}{\sqrt{3}}\langle -1,1,1\rangle$.

44 The curl of the vector field is $\nabla \times \mathbf{F} = \langle 0,0,2\rangle$, and the component of the curl along a unit vector $\mathbf{n}$ is thus $\langle 0,0,2\rangle \cdot \mathbf{n}$.

a. It does not spin, because $\langle 0,0,2\rangle \cdot \langle 1,0,0\rangle = 0$.

b. The scalar component of the curl in the direction $\langle 0,0,1\rangle$ is 2.

c. It spins the fastest when the paddle wheel is aligned with $\langle 0,0,1\rangle$.

45 Parameterize the sphere by $\langle 3\sin u\cos v, 3\sin u\sin v, 3\cos u\rangle$, $0 \le u \le \frac{\pi}{2}$; $0 \le v \le 2\pi$. Then $|\mathbf{n}| = 9\sin u$, so

$$\iint_S 1\,dS = \iint_R 9\sin u\,dA = \int_0^{2\pi}\int_0^{\pi/2} 9\sin u\,du\,dv = 18\pi.$$

46 Parameterize the surface by $\langle v\cos u, v\sin u, v\rangle$ for $2 \le v \le 4, 0 \le u \le 2\pi$. Then

$$\iint_S 1\,dS = \iint_R \sqrt{2}\,v\,dA = \sqrt{2}\int_2^4\int_0^{2\pi} v\,du\,dv = 12\pi\sqrt{2}.$$

47 The volume element is $\sqrt{1^2+1^2+1^2} = \sqrt{3}$, so the area is

$$\iint_S 1\,dS = \sqrt{3}\iint_R 1\,dA = \sqrt{3}\int_{-1}^1\int_{-1}^1 1\,dx\,dy = 4\sqrt{3}.$$

48 The volume element is $\sqrt{z_x^2+z_y^2+1} = \sqrt{2\left(x^2+y^2\right)+1}$, so the integral is

$$\iint_S 1\,dS = \iint_R \sqrt{2\left(x^2+y^2\right)+1}\,dA = \int_0^2\int_0^{2\pi} r\sqrt{2r^2+1}\,d\theta\,dr = \frac{26}{3}\pi.$$

49 The volume element for $z = 2-x-y$ is $\sqrt{3}$, so the integral is

$$\iint_S (1+yz)\,dS = \sqrt{3}\iint_R (1+yz)\,dA = \sqrt{3}\int_0^2\int_0^{2-x} (1+y\,(2-x-y))\,dy\,dx = \frac{8\sqrt{3}}{3}.$$

50 The normal to the curved surface of the cylinder at (x,y,z) is $\langle 0,y,z\rangle$, so

$$\iint_S \langle 0,y,z\rangle \cdot \mathbf{n}\,dS = a\iint_R \left(y^2+z^2\right)\,dA = a^2 \cdot \text{area of } R = 32\pi a^3.$$

51 Parameterize the curved surface using spherical coordinates, so that $|\mathbf{n}| = 4\sin u$; then for the curved surface we have

$$\iint_S (x-y+z)\,dS = 8\int_0^{2\pi}\int_0^{\pi/2} (\sin u\cos v - \sin u\sin v + \cos u)\sin u\,du\,dv = 8\pi.$$

For the planar surface, $\mathbf{n} = \langle 0,-1,0\rangle$ so that $|\mathbf{n}| = 1$ and

$$\iint_S (x-y+z)\,dS = \iint_R (x-y+z)\,dA = \int_0^2\int_0^{2\pi} r\,(\cos\theta - \sin\theta)\,d\theta\,dr = 0$$

and the total integral is thus 8π.

52 The normal to the cylinder is $\langle x, y, 0\rangle$ with magnitude 1, so

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \langle x, y, z\rangle\cdot\langle x, y, 0\rangle\,dA = \iint_R \left(x^2+y^2\right)\,dA = \iint_R 1\,dA = 32\pi.$$

53 $\mathbf{F} = \left(x^2+y^2+z^2\right)^{-1/2}\langle x, y, z\rangle$; using spherical coordinates to parameterize the sphere gives

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \frac{1}{a}\langle a\sin u\cos v, a\sin u\sin v, a\,\cos u\rangle\cdot\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\sin u\cos u\rangle\,dA$$
$$= \iint_R a^2\sin u\,dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin u\,du\,dv = 4\pi a^2.$$

54

a. Using the explicit description, we have $\sqrt{z_z^2+z_y^2+1} = \sqrt{4\left(x^2+y^2\right)+1}$, so

$$\iint_S 1\,dS = \iint_R \sqrt{4\left(x^2+y^2\right)+1}\,dA = \int_0^2\int_0^{2\pi} r\sqrt{4r^2+1}\,d\theta\,dr = \frac{\pi}{6}\left(17\sqrt{17}-1\right).$$

b. Using the given parametric description, we have $|\mathbf{n}| = v\sqrt{1+4v^2}$, so

$$\iint_S 1\,dS = \iint_R v\,\sqrt{1+4v^2}\,dA = \int_0^{2\pi}\int_0^2 v\,\sqrt{1+4v^2}\,dv\,d\theta = \frac{\pi}{6}\left(17\sqrt{17}-1\right).$$

c. Using the given parametric description, we have

$$|\mathbf{t}_u\times\mathbf{t}_v| = \left|\langle-\sqrt{v}\,\sin u, \sqrt{v}\cos u, 0\rangle\times\left\langle-\frac{1}{2}v^{-1/2}\cos u, \frac{1}{2}v^{-1/2}\sin u, 1\right\rangle\right| = \frac{1}{2}\sqrt{4v+1}$$

so that

$$\iint_S 1\,dS = \frac{1}{2}\iint_R \sqrt{4v+1}\,dA = \frac{1}{2}\int_0^{2\pi}\int_0^4 \sqrt{4v+1}\,dv\,d\theta = \frac{\pi}{6}\left(17\sqrt{17}-1\right).$$

55

a. The base of S is the surface where $z = 0$, or the circle $x^2+y^2 = a^2$. Similarly, the base of the paraboloid is found by setting $z = 0$; simplifying gives again $x^2+y^2 = a^2$. The high point of the hemisphere (maximum z-coordinate) occurs when $x = y = 0$; then $z = a$. Similarly, the high point on the paraboloid also occurs when $x = y = 0$ and again this gives $z = a$.

b. The graph of the paraboloid is inside that of the hemisphere everywhere, so we would expect it to have smaller surface area. We know that the surface area of the hemisphere is $4\pi a^2\cdot\frac{1}{2} = 2\pi a^2$. For the paraboloid, we have $z_x = -\frac{2x}{a}$, $z_y = -\frac{2y}{a}$, so that $|\mathbf{n}| = \sqrt{\frac{4(x^2+y^2)}{a^2}+1}$, so

$$\iint_S 1\,dS = \frac{1}{a}\iint_R \sqrt{4\left(x^2+y^2\right)+a^2}\,dA = \frac{1}{a}\int_0^a\int_0^{2\pi} r\sqrt{4r^2+a^2}\;d\theta\,dr = \frac{\left(5\sqrt{5}-1\right)}{6}\pi a^2,$$

which is in fact smaller than the area of the hemisphere.

c. $\mathbf{n} = \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u \rangle$, so

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \langle a \sin u \cos v, a \sin u \sin v, a\ \cos u \rangle \cdot \langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u \rangle\, dA$$
$$= a^3 \int_0^{2\pi} \int_0^{\pi/2} \left(\sin^3 u\ \cos^2 v + \sin^3 u \sin^2 v + \cos^2 u \sin u\right)\, du\, dv$$
$$= a^3 \int_0^{2\pi} \int_0^{\pi/2} \left(\sin^3 u + \cos^2 u \sin u\right)\, du\, dv = a^3 \int_0^{2\pi} \int_0^{\pi/2} \sin u\, du\, dv = 2\pi a^3.$$

d. For the paraboloid, the parameterization is $\left\langle v \cos u, v \sin u, a - \frac{v^2}{a} \right\rangle$, $0 \le v \le a$, $0 \le$u$\le 2\pi$, and

$$\mathbf{n} = \langle -v \sin u, v \cos u, 0 \rangle \times \left\langle \cos u, \sin u, -\frac{2v}{a} \right\rangle = \left\langle -\frac{1}{a} 2v^2 \cos u, -\frac{1}{a} 2v^2 \sin u, -v \right\rangle,$$

so that the outward-pointing normal is $\left\langle \frac{2v^2}{a} \cos u, \frac{2v^2}{a} \sin u, v \right\rangle$ and

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \left\langle v \cos u, v \sin u, a - \frac{v^2}{a} \right\rangle \cdot \left\langle \frac{2v^2}{a} \cos u, \frac{2v^2}{a} \sin u, v \right\rangle dA$$
$$= \int_0^a \int_0^{2\pi} \left(\frac{2}{a} v^3 + av - \frac{v^3}{a} \right) du\, dv = \frac{3}{2} \pi a^3.$$

56

a. For the given r, we have $\frac{(a \cos u \sin v)^2}{a^2} + \frac{(b\ \sin u \sin v)^2}{b^2} = \cos^2 u \sin^2 v + \sin^2 u \sin^2 v + \cos^2 v = \sin^2 v + \cos^2 v = 1$

b. The normal vector is determined by $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v = \langle -bc \cos u \sin^2 v, -ac \sin u\ \sin 2v, -ab \sin v \cos v \rangle$, so that the outward pointing normal is the negative of this vector. Then

$$\iint_S 1\, dS = \int_0^{2\pi} \int_0^{\pi} \left| \langle bc \cos u \sin^2 v, ac \sin u \sin^2 v, ab\ \sin v \cos v \rangle \right|\ dv\, du$$
$$= \int_0^{2\pi} \int_0^{\pi} \sqrt{b^2 c^2 \cos^2 u\ \sin^4 v + a^2 c^2 \sin^2 u \sin^4 v + a^2 b^2 \sin^2 v\ \cos^2 v}\, dv\, du.$$

57 Parameterize $x^2 + y^2 = 4$ for $z = 0$ using $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 0 \rangle$ for$\mathbf{r} 0 \le t \le 2\pi$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\ dt = \int_0^{2\pi} \langle 0, 0, 4 \cos t \sin t \rangle \cdot \langle -2\ \sin t, 2 \cos t, 0 \rangle\, dt\ = 0.$$

58 $\mathbf{F} = \langle u^2 - v^2, u, 2v\,(6 - 2u - v) \rangle$, $\mathbf{r}\,(u, v) = \langle u, v, 6 - 2u - v \rangle$, $\mathbf{t}_u \times\ \mathbf{t}_v = \langle 2, 1, 1 \rangle$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \int_0^{6-2u} \left(2u^2 - 4v^2 + 12v - 4uv\right)\, dv\, du = \frac{39\sqrt{6}}{2}.$$

59 The boundary of this region is in the xy-plane, found by setting $z = 0$, so it is $x^2 + y^2 = 99$, the circle of radius $\sqrt{99}$ about the origin. Parameterize the circle in the usual way; then

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \langle 0, \sqrt{99} \cos t, \sqrt{99} \sin t \rangle \cdot \left\langle -\sqrt{99} \sin t, \sqrt{99} \cos t, 0 \right\rangle dt$$
$$= \int_0^{2\pi} 99 \cos^2 t\, dt = 99\pi.$$

60 The boundary of this region is the circle $x^2 + z^2 = 4$ for $y = 0$; parameterizing it in the usual way as $\langle 2\cos t, 0, 2\sin t\rangle$ gives

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 4\cos^2 t - 4\sin^2 t, 0, 4\sin t\cos t\rangle \cdot \langle -2\sin t, 0, 2\cos t\rangle\, dt$$
$$= 8\int_0^{2\pi} \left(\left(\sin^2 t - \cos^2 t\right)\sin t + \sin t\cos^2 t\right) dt = 8\int_0^{2\pi} \sin^3 t\, dt = 0.$$

61 By Stokes' theorem, the circulation around a closed curve C can be found by choosing a surface S of which C is the boundary; then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

But for $\mathbf{F} = \nabla\left(10 - x^2 + y^2 + z^2\right)$, $\nabla \times \mathbf{F} = \mathbf{0}$, so the right-hand side is zero.

62 We have $\nabla \cdot \mathbf{F} = -3$ so that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV = -3 \cdot \text{volume of cube} = -3.$$

63 $\nabla \cdot \mathbf{F} = x^2 + y^2 + z^2$, so

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \left(x^2 + y^2 + z^2\right) dV = \int_0^3 \int_0^{2\pi} \int_0^{\pi} r^2 \cdot r^2 \sin u\, du\, dv\, dr = \frac{972}{5}\pi.$$

64 $\nabla \cdot \mathbf{F} = 2(x + y + z)$, so

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D 2(x + y + z)\, dV = \int_0^8 \int_0^2 \int_0^{2\pi} 2r\,(r\cos\theta + r\sin\theta + t)\, d\theta\, dr\, dt = 256\pi.$$

65 $\nabla \cdot \mathbf{F} = 3\left(x^2 + y^2\right)$, so the outward flux across the boundary S of a hemisphere D of radius a is $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D 3\left(x^2 + y^2\right) dV = \int_0^a \int_0^{2\pi} \int_0^{\pi/2} 3r^2 \sin^2 u \cdot r^2 \sin u\, du\, dv\, dr = \frac{4}{5}\pi a^5$, so that the net flux across the region bounded by the hemispheres of radii 1 and 2 is $\frac{4}{5}\pi(32 - 1) = \frac{124}{5}\pi$.

66 $\nabla \cdot \mathbf{F} = 0$, so the flux is zero across any surface that bounds a region where F is defined and differentiable; the given region does not include zero, so is one of these. Thus the net outward flux is zero.

67 Using the Divergence theorem, $\nabla \cdot \mathbf{F} = 2x + \sin y + 2y - 2\sin y + 2z + \sin y = 2(x + y + z)$, so that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D 2(x + y + z)\, dV = \int_0^4 \int_0^1 \int_0^{1-x} 2(x + y + z)\, dy\, dx\, dz = \frac{32}{3}.$$

68

a. The normal vectors point outwards everywhere on S; that is, on the curved surface, they point upwards, and on the flat surface they point in the direction of negative x.

b. Parameterize C by two paths: $\mathbf{r}_1(t) = \langle a\cos t, a\sin t, 0\rangle$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and $\mathbf{r}_2(t) = \langle 0, a - 2t, 0\rangle$ for $0 \le t \le a$. Then $\mathbf{r}_1'(t) = \langle -a\sin t, a\cos t, 0\rangle$ and $\mathbf{r}_2'(t) = \langle 0, -2, 0\rangle$. So

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{-\pi/2}^{\pi/2} \langle -a\sin t, a\cos t, a\sin t - 2a\cos t\rangle \cdot \langle -a\sin t, a\cos t, 0\rangle\, dt$$
$$+ \int_0^a \langle 2t - a, 0, a - 2t\rangle \cdot \langle 0, -2, 0\rangle dt = \int_{-\pi/2}^{\pi/2} a^2\, dt = \pi a^2.$$

c. $\nabla \times \mathbf{F} = \langle 2, 4, 2 \rangle$. For the curved portion of S, using spherical coordinates, the normal vector is $\langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u \rangle$, and for the flat portion, the normal vector is $\langle -1, 0, 0 \rangle$. Then $\iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint\limits_{S_1} \left(2a^2 \sin^2 u \cos v + 4a^2 \sin^2 u \sin v + 2a^2 \cos u \sin u\right) dS + \iint\limits_{S_2} (-2)\, dS = 2a^2 \int_{-\pi/2}^{\pi/2} \int_0^{\pi/2} \left(\sin^2 u \cos v + 2 \sin^2 u \sin v + \sin u \cos u\right) du\, dv - \pi a^2 = 2\pi a^2 - \pi a^2 = \pi a^2$.