1 PRECALCULUS REVIEW

1.1 Real Numbers, Functions, and Graphs

Preliminary Questions

1. Give an example of numbers a and b such that $a < b$ and $|a| > |b|$.

SOLUTION Take $a = -3$ and $b = 1$. Then $a < b$ but $|a| = 3 > 1 = |b|$.

2. Which numbers satisfy $|a| = a$? Which satisfy $|a| = -a$? What about $|-a| = a$?

SOLUTION The numbers $a \geq 0$ satisfy $|a| = a$ and $|-a| = a$. The numbers $a \leq 0$ satisfy $|a| = -a$.

3. Give an example of numbers a and b such that $|a + b| < |a| + |b|$.

SOLUTION Take $a = -3$ and $b = 1$. Then

$$|a + b| = |-3 + 1| = |-2| = 2, \qquad \text{but} \qquad |a| + |b| = |-3| + |1| = 3 + 1 = 4$$

Thus, $|a + b| < |a| + |b|$.

4. Are there numbers a and b such that $|a + b| > |a| + |b|$?

SOLUTION No. By the triangle inequality, $|a + b| \leq |a| + |b|$ for all real numbers a and b.

5. What are the coordinates of the point lying at the intersection of the lines $x = 9$ and $y = -4$?

SOLUTION The point $(9, -4)$ lies at the intersection of the lines $x = 9$ and $y = -4$.

6. In which quadrant do the following points lie?

(a) $(1, 4)$ **(b)** $(-3, 2)$ **(c)** $(4, -3)$ **(d)** $(-4, -1)$

SOLUTION

(a) Because both the x- and y-coordinates of the point $(1, 4)$ are positive, the point $(1, 4)$ lies in the first quadrant.

(b) Because the x-coordinate of the point $(-3, 2)$ is negative but the y-coordinate is positive, the point $(-3, 2)$ lies in the second quadrant.

(c) Because the x-coordinate of the point $(4, -3)$ is positive but the y-coordinate is negative, the point $(4, -3)$ lies in the fourth quadrant.

(d) Because both the x- and y-coordinates of the point $(-4, -1)$ are negative, the point $(-4, -1)$ lies in the third quadrant.

7. What is the radius of the circle with equation $(x - 7)^2 + (y - 8)^2 = 9$?

SOLUTION The circle with equation $(x - 7)^2 + (y - 8)^2 = 9$ has radius 3.

8. The equation $f(x) = 5$ has a solution if (choose one):

(a) 5 belongs to the domain of f.

(b) 5 belongs to the range of f.

SOLUTION The correct response is **(b)**: the equation $f(x) = 5$ has a solution if 5 belongs to the range of f.

9. What kind of symmetry does the graph have if $f(-x) = -f(x)$?

SOLUTION If $f(-x) = -f(x)$, then the graph of f is symmetric with respect to the origin.

10. Is there a function that is both even and odd?

SOLUTION Yes. The constant function $f(x) = 0$ for all real numbers x is both even and odd because

$$f(-x) = 0 = f(x)$$

and

$$f(-x) = 0 = -0 = -f(x)$$

for all real numbers x.

Exercises

1. Which of the following equations is incorrect?

(a) $3^2 \cdot 3^5 = 3^7$

(b) $(\sqrt{5})^{4/3} = 5^{2/3}$

(c) $3^2 \cdot 2^3 = 1$

(d) $(2^{-2})^{-2} = 16$

SOLUTION

(a) This equation is correct: $3^2 \cdot 3^5 = 3^{2+5} = 3^7$.

(b) This equation is correct: $(\sqrt{5})^{4/3} = (5^{1/2})^{4/3} = 5^{(1/2)\cdot(4/3)} = 5^{2/3}$.

(c) This equation is incorrect: $3^2 \cdot 2^3 = 9 \cdot 8 = 72 \neq 1$.

(d) This equation is correct: $(2^{-2})^{-2} = 2^{(-2)\cdot(-2)} = 2^4 = 16$.

2. Rewrite as a whole number (without using a calculator):

(a) 7^0

(b) $10^2(2^{-2} + 5^{-2})$

(c) $\frac{(4^3)^5}{(4^5)^3}$

(d) $27^{4/3}$

(e) $8^{-1/3} \cdot 8^{5/3}$

(f) $3 \cdot 4^{1/4} - 12 \cdot 2^{-3/2}$

SOLUTION

(a) $7^0 = 1$

(b) $10^2(2^{-2} + 5^{-2}) = 100(1/4 + 1/25) = 25 + 4 = 29$

(c) $(4^3)^5/(4^5)^3 = 4^{15}/4^{15} = 1$

(d) $(27)^{4/3} = (27^{1/3})^4 = 3^4 = 81$

(e) $8^{-1/3} \cdot 8^{5/3} = (8^{1/3})^5/8^{1/3} = 2^5/2 = 2^4 = 16$

(f) $3 \cdot 4^{1/4} - 12 \cdot 2^{-3/2} = 3 \cdot 2^{1/2} - 3 \cdot 2^2 \cdot 2^{-3/2} = 0$

3. Use the binomial expansion formula to expand $(2 - x)^7$.

SOLUTION Using the binomial expansion formula,

$$
\begin{aligned}
(2 - x)^7 &= \frac{7!}{7!0!}2^7(-x)^0 + \frac{7!}{6!1!}2^6(-x) + \frac{7!}{5!2!}2^5(-x)^2 + \frac{7!}{4!3!}2^4(-x)^3 + \frac{7!}{3!4!}2^3(-x)^4 \\
&\quad + \frac{7!}{2!5!}2^2(-x)^5 + \frac{7!}{1!6!}2(-x)^6 + \frac{7!}{0!7!}2^0(-x)^7 \\
&= 128 - 448x + 672x^2 - 560x^3 + 280x^4 - 84x^5 + 14x^6 - x^7
\end{aligned}
$$

4. Use the binomial expansion formula to expand $(x + 1)^9$.

SOLUTION Using the binomial expansion formula,

$$
\begin{aligned}
(x + 1)^9 &= \frac{9!}{9!0!}x^9 + \frac{9!}{8!1!}x^8 + \frac{9!}{7!2!}x^7 + \frac{9!}{6!3!}x^6 + \frac{9!}{5!4!}x^5 + \frac{9!}{4!5!}x^4 + \frac{9!}{3!6!}x^3 + \frac{9!}{2!7!}x^2 + \frac{9!}{1!8!}x + \frac{9!}{0!9!} \\
&= x^9 + 9x^8 + 36x^7 + 84x^6 + 126x^5 + 126x^4 + 84x^3 + 36x^2 + 9x + 1
\end{aligned}
$$

5. Which of (a)–(d) are true for $a = 4$ and $b = -5$?

(a) $-2a < -2b$

(b) $|a| < -|b|$

(c) $ab < 0$

(d) $\frac{1}{a} < \frac{1}{b}$

SOLUTION

(a) True

(b) False; $|a| = 4 > -5 = -|b|$

(c) True

(d) False; $\frac{1}{a} = \frac{1}{4} > -\frac{1}{5} = \frac{1}{b}$

6. Which of (a)–(d) are true for $a = -3$ and $b = 2$?

(a) $a < b$

(b) $|a| < |b|$

(c) $ab > 0$

(d) $3a < 3b$

SOLUTION

(a) True

(b) False; $|a| = 3 > 2 = |b|$

(c) False; $(-3)(2) = -6 < 0$

(d) True

In Exercises 7–12, express the interval in terms of an inequality involving absolute value.

7. $[-2, 2]$

SOLUTION $|x| \le 2$

8. $(-4, 4)$

SOLUTION $|x| < 4$

9. $(0, 4)$

SOLUTION The midpoint of the interval is $c = (0 + 4)/2 = 2$, and the radius is $r = (4 - 0)/2 = 2$; therefore, $(0, 4)$ can be expressed as $|x - 2| < 2$.

10. $[-4, 0]$

SOLUTION The midpoint of the interval is $c = (-4 + 0)/2 = -2$, and the radius is $r = (0 - (-4))/2 = 2$; therefore, the interval $[-4, 0]$ can be expressed as $|x + 2| \le 2$.

11. $[-1, 8]$

SOLUTION The midpoint of the interval is $c = (-1 + 8)/2 = \frac{7}{2}$, and the radius is $r = (8 - (-1))/2 = \frac{9}{2}$; therefore, the interval $[-1, 8]$ can be expressed as $\left|x - \frac{7}{2}\right| \le \frac{9}{2}$.

12. $(-2.4, 1.9)$

SOLUTION The midpoint of the interval is $c = (-2.4 + 1.9)/2 = -0.25$, and the radius is $r = (1.9 - (-2.4))/2 = 2.15$; therefore, the interval $(-2.4, 1.9)$ can be expressed as $|x + 0.25| < 2.15$.

In Exercises 13–16, write the inequality in the form $a < x < b$.

13. $|x| < 8$

SOLUTION $-8 < x < 8$

14. $|x - 12| < 8$

SOLUTION $-8 < x - 12 < 8$ so $4 < x < 20$

15. $|2x + 1| < 5$

SOLUTION $-5 < 2x + 1 < 5$ so $-6 < 2x < 4$ and $-3 < x < 2$

16. $|3x - 4| < 2$

SOLUTION $-2 < 3x - 4 < 2$ so $2 < 3x < 6$ and $\frac{2}{3} < x < 2$

In Exercises 17–22, express the set of numbers x satisfying the given condition as an interval.

17. $|x| < 4$

SOLUTION $(-4, 4)$

18. $|x| \le 9$

SOLUTION $[-9, 9]$

19. $|x - 4| < 2$

SOLUTION The expression $|x - 4| < 2$ is equivalent to $-2 < x - 4 < 2$. Therefore, $2 < x < 6$, which represents the interval $(2, 6)$.

20. $|x + 7| < 2$

SOLUTION The expression $|x + 7| < 2$ is equivalent to $-2 < x + 7 < 2$. Therefore, $-9 < x < -5$, which represents the interval $(-9, -5)$.

21. $|4x - 1| \le 8$

SOLUTION The expression $|4x - 1| \le 8$ is equivalent to $-8 \le 4x - 1 \le 8$ or $-7 \le 4x \le 9$. Therefore, $-\frac{7}{4} \le x \le \frac{9}{4}$, which represents the interval $[-\frac{7}{4}, \frac{9}{4}]$.

22. $|3x + 5| < 1$

SOLUTION The expression $|3x + 5| < 1$ is equivalent to $-1 < 3x + 5 < 1$ or $-6 < 3x < -4$. Therefore, $-2 < x < -\frac{4}{3}$, which represents the interval $(-2, -\frac{4}{3})$.

In Exercises 23–26, describe the set as a union of finite or infinite intervals.

23. $\{x : |x - 4| > 2\}$

SOLUTION $x - 4 > 2$ or $x - 4 < -2 \Rightarrow x > 6$ or $x < 2 \Rightarrow (-\infty, 2) \cup (6, \infty)$

24. $\{x : |2x + 4| > 3\}$

SOLUTION $2x + 4 > 3$ or $2x + 4 < -3 \Rightarrow 2x > -1$ or $2x < -7 \Rightarrow (-\infty, -\frac{7}{2}) \cup (-\frac{1}{2}, \infty)$

25. $\{x : |x^2 - 1| > 2\}$

SOLUTION $x^2 - 1 > 2$ or $x^2 - 1 < -2 \Rightarrow x^2 > 3$ or $x^2 < -1$ (this will never happen) $\Rightarrow x > \sqrt{3}$ or $x < -\sqrt{3} \Rightarrow$ $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

26. $\{x : |x^2 + 2x| > 2\}$

SOLUTION $x^2 + 2x > 2$ or $x^2 + 2x < -2 \Rightarrow x^2 + 2x - 2 > 0$ or $x^2 + 2x + 2 < 0$. For the first case, the zeroes are

$$x = -1 \pm \sqrt{3} \Rightarrow (-\infty, -1 - \sqrt{3}) \cup (-1 + \sqrt{3}, \infty).$$

For the second case, note there are no real zeros. Because the parabola opens upward and its vertex is located above the x-axis, there are no values of x for which $x^2 + 2x + 2 < 0$. Hence, the solution set is $(-\infty, -1 - \sqrt{3}) \cup (-1 + \sqrt{3}, \infty)$.

27. Match (a)–(f) with (i)–(vi).

(a) $a > 3$

(b) $|a - 5| < \frac{1}{3}$

(c) $\left|a - \frac{1}{3}\right| < 5$

(d) $|a| > 5$

(e) $|a - 4| < 3$

(f) $1 \le a \le 5$

(i) a lies to the right of 3.
(ii) a lies between 1 and 7.
(iii) The distance from a to 5 is less than $\frac{1}{3}$.
(iv) The distance from a to 3 is at most 2.
(v) a is less than 5 units from $\frac{1}{3}$.
(vi) a lies either to the left of -5 or to the right of 5.

SOLUTION

(a) On the number line, numbers greater than 3 appear to the right; hence, $a > 3$ is equivalent to the numbers to the right of 3: **(i)**.

(b) $|a - 5|$ measures the distance from a to 5; hence, $|a - 5| < \frac{1}{3}$ is satisfied by those numbers less than $\frac{1}{3}$ of a unit from 5: **(iii)**.

(c) $|a - \frac{1}{3}|$ measures the distance from a to $\frac{1}{3}$; hence, $|a - \frac{1}{3}| < 5$ is satisfied by those numbers less than 5 units from $\frac{1}{3}$: **(v)**.

(d) The inequality $|a| > 5$ is equivalent to $a > 5$ or $a < -5$; that is, either a lies to the right of 5 or to the left of -5: **(vi)**.

(e) The interval described by the inequality $|a - 4| < 3$ has a center at 4 and a radius of 3; that is, the interval consists of those numbers between 1 and 7: **(ii)**.

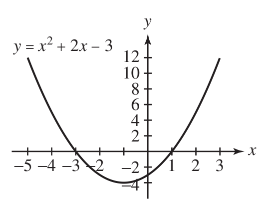
(f) The interval described by the inequality $1 < x < 5$ has a center at 3 and a radius of 2; that is, the interval consists of those numbers less than 2 units from 3: **(iv)**.

28. Describe $\left\{x : \frac{x}{x+1} < 0\right\}$ as an interval. *Hint:* Consider the sign of x and $x + 1$ individually.

SOLUTION Case 1: $x < 0$ and $x + 1 > 0$. This implies that $x < 0$ and $x > -1 \Rightarrow -1 < x < 0$.
Case 2: $x > 0$ and $x < -1$ for which there is no such x. Thus, solution set is therefore $(-1, 0)$.

29. Describe $\{x : x^2 + 2x < 3\}$ as an interval. *Hint:* Plot $y = x^2 + 2x - 3$.

SOLUTION The inequality $x^2 + 2x < 3$ is equivalent to $x^2 + 2x - 3 < 0$. The graph of $y = x^2 + 2x - 3$ is shown here. From this graph, it follows that $x^2 + 2x - 3 < 0$ for $-3 < x < 1$. Thus, the set $\{x : x^2 + 2x < 3\}$ is equivalent to the interval $(-3, 1)$.



30. Describe the set of real numbers satisfying $|x - 3| = |x - 2| + 1$ as a half-infinite interval.

SOLUTION Case 1: If $x \ge 3$, then $|x - 3| = x - 3$, $|x - 2| = x - 2$, and the equation $|x - 3| = |x - 2| + 1$ reduces to $x - 3 = x - 2 + 1$ or $-3 = -1$. As this is never true, the given equation has no solution for $x \ge 3$.

Case 2: If $2 \le x < 3$, then $|x - 3| = -(x - 3) = 3 - x$, $|x - 2| = x - 2$, and the equation $|x - 3| = |x - 2| + 1$ reduces to $3 - x = x - 2 + 1$ or $x = 2$.

Case 3: If $x < 2$, then $|x - 3| = -(x - 3) = 3 - x$, $|x - 2| = -(x - 2) = 2 - x$, and the equation $|x - 3| = |x - 2| + 1$ reduces to $3 - x = 2 - x + 1$ or $1 = 1$. As this is always true, the given equation holds for all $x < 2$.

Combining the results from all three cases, it follows that the set of real numbers satisfying $|x - 3| = |x - 2| + 1$ is equivalent to the half-infinite interval $(-\infty, 2]$.

31. Show that if $a > b$, and $a, b \neq 0$, then $b^{-1} > a^{-1}$, provided that a and b have the same sign. What happens if $a > 0$ and $b < 0$?

SOLUTION Case 1a: If a and b are both positive, then $a > b \Rightarrow 1 > \frac{b}{a} \Rightarrow \frac{1}{b} > \frac{1}{a}$.
Case 1b: If a and b are both negative, then $a > b \Rightarrow 1 < \frac{b}{a}$ (since a is negative) $\Rightarrow \frac{1}{b} > \frac{1}{a}$ (again, since b is negative).
Case 2: If $a > 0$ and $b < 0$, then $\frac{1}{a} > 0$ and $\frac{1}{b} < 0$ so $\frac{1}{b} < \frac{1}{a}$. (See Exercise 6f for an example of this.)

32. Which x satisfies both $|x - 3| < 2$ and $|x - 5| < 1$?

SOLUTION $|x - 3| < 2 \Rightarrow -2 < x - 3 < 2 \Rightarrow 1 < x < 5$. Also $|x - 5| < 1 \Rightarrow 4 < x < 6$. Since we want an x that satisfies both of these, we need the intersection of the two solution sets, that is, $4 < x < 5$.

33. Show that if $|a - 5| < \frac{1}{2}$ and $|b - 8| < \frac{1}{2}$, then
$|(a + b) - 13| < 1$. *Hint:* Use the triangle inequality ($|a + b| \leq |a| + |b|$).

SOLUTION

$$\begin{aligned} |a + b - 13| &= |(a - 5) + (b - 8)| \\ &\leq |a - 5| + |b - 8| \quad \text{(by the triangle inequality)} \\ &< \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

34. Suppose that $|x - 4| \leq 1$.
(a) What is the maximum possible value of $|x + 4|$?
(b) Show that $|x^2 - 16| \leq 9$.

SOLUTION
(a) $|x - 4| \leq 1$ guarantees $3 \leq x \leq 5$. Thus, $7 \leq x + 4 \leq 9$, so $|x + 4| \leq 9$.
(b) $|x^2 - 16| = |x - 4| \cdot |x + 4| \leq 1 \cdot 9 = 9$

35. Suppose that $|a - 6| \leq 2$ and $|b| \leq 3$.
(a) What is the largest possible value of $|a + b|$?
(b) What is the smallest possible value of $|a + b|$?

SOLUTION $|a - 6| \leq 2$ guarantees $4 \leq a \leq 8$, and $|b| \leq 3$ guarantees $-3 \leq b \leq 3$, so $1 \leq a + b \leq 11$. Based on this information,
(a) the largest possible value of $|a + b|$ is 11; and
(b) the smallest possible value of $|a + b|$ is 1.

36. Prove that $|x| - |y| \leq |x - y|$. *Hint:* Apply the triangle inequality to y and $x - y$.

SOLUTION First note

$$|x| = |x - y + y| \leq |x - y| + |y|$$

by the triangle inequality. Subtracting $|y|$ from both sides of this inequality yields

$$|x| - |y| \leq |x - y|$$

37. Express $r_1 = 0.\overline{27}$ as a fraction. *Hint:* $100r_1 - r_1$ is an integer. Then express $r_2 = 0.2666\ldots$ as a fraction.

SOLUTION Let $r_1 = 0.\overline{27}$. We observe that $100r_1 = 27.\overline{27}$. Therefore, $100r_1 - r_1 = 27.\overline{27} - 0.\overline{27} = 27$ and

$$r_1 = \frac{27}{99} = \frac{3}{11}$$

Now, let $r_2 = 0.2\overline{666}$. Then $10r_2 = 2.\overline{666}$ and $100r_2 = 26.\overline{666}$. Therefore, $100r_2 - 10r_2 = 26.\overline{666} - 2.\overline{666} = 24$ and

$$r_2 = \frac{24}{90} = \frac{4}{15}$$

38. Represent $1/7$ and $4/27$ as repeating decimals.

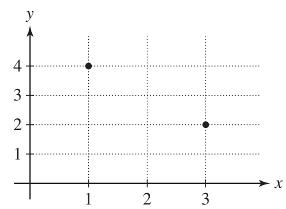
SOLUTION $\frac{1}{7} = 0.\overline{142857}$; $\frac{4}{27} = 0.\overline{148}$

39. Plot each pair of points and compute the distance between them:
(a) $(1, 4)$ and $(3, 2)$
(b) $(2, 1)$ and $(2, 4)$

SOLUTION

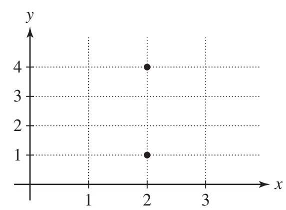
(a) The points $(1, 4)$ and $(3, 2)$ are plotted in the figure. The distance between the points is

$$d = \sqrt{(3-1)^2 + (2-4)^2} = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$



(b) The points $(2, 1)$ and $(2, 4)$ are plotted in the figure. The distance between the points is

$$d = \sqrt{(2-2)^2 + (4-1)^2} = \sqrt{9} = 3$$



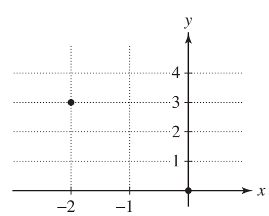
40. Plot each pair of points and compute the distance between them:

(a) $(0, 0)$ and $(-2, 3)$ **(b)** $(-3, -3)$ and $(-2, 3)$

SOLUTION

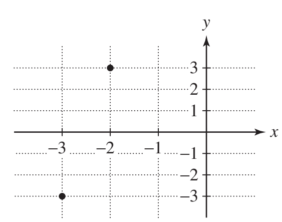
(a) The points $(0, 0)$ and $(-2, 3)$ are plotted in the figure. The distance between the points is

$$d = \sqrt{(-2-0)^2 + (3-0)^2} = \sqrt{4+9} = \sqrt{13}$$



(b) The points $(-3, -3)$ and $(-2, 3)$ are plotted in the figure. The distance between the points is

$$d = \sqrt{(-3-(-2))^2 + (-3-3)^2} = \sqrt{1+36} = \sqrt{37}$$



41. Find the equation of the circle with center $(2, 4)$:

(a) With radius $r = 3$

(b) That passes through $(1, -1)$

SOLUTION (a) The equation of the indicated circle is $(x-2)^2 + (y-4)^2 = 3^2 = 9$.

(b) First, determine the radius as the distance from the center to the indicated point on the circle:

$$r = \sqrt{(2-1)^2 + (4-(-1))^2} = \sqrt{26}$$

Thus, the equation of the circle is $(x-2)^2 + (y-4)^2 = 26$.

42. Find all points in the xy-plane with integer coordinates located at a distance 5 from the origin. Then find all points with integer coordinates located at a distance 5 from $(2, 3)$.

SOLUTION

- To be located a distance 5 from the origin, the points must lie on the circle $x^2 + y^2 = 25$. This leads to 12 points with integer coordinates:

$$\begin{array}{cccc} (5,0) & (-5,0) & (0,5) & (0,-5) \\ (3,4) & (-3,4) & (3,-4) & (-3,-4) \\ (4,3) & (-4,3) & (4,-3) & (-4,-3) \end{array}$$

- To be located a distance 5 from the point $(2,3)$, the points must lie on the circle $(x-2)^2 + (y-3)^2 = 25$, which implies that we must shift the points listed 2 units to the right and 3 units up. This gives the 12 points

$$\begin{array}{cccc} (7,3) & (-3,3) & (2,8) & (2,-2) \\ (5,7) & (-1,7) & (5,-1) & (-1,-1) \\ (6,6) & (-2,6) & (6,0) & (-2,0) \end{array}$$

43. Determine the domain and range of the function

$$f : \{r,s,t,u\} \to \{A,B,C,D,E\}$$

defined by $f(r) = A$, $f(s) = B$, $f(t) = B$, $f(u) = E$.

SOLUTION The domain is the set $D = \{r,s,t,u\}$; the range is the set $R = \{A,B,E\}$.

44. Give an example of a function whose domain D has three elements and whose range R has two elements. Does a function exist whose domain D has two elements and whose range R has three elements?

SOLUTION Define f by $f : \{a,b,c\} \to \{1,2\}$, where $f(a) = 1$, $f(b) = 1$, $f(c) = 2$.

There is no function whose domain has two elements and range has three elements. If that happened, one of the domain elements would get assigned to more than one element of the range, which would contradict the definition of a function.

In Exercises 45–52, find the domain and range of the function.

45. $f(x) = -x$

SOLUTION D: all reals; R: all reals

46. $g(t) = t^4$

SOLUTION D: all reals; R: $\{y : y \ge 0\}$

47. $f(x) = x^3$

SOLUTION D: all reals; R: all reals

48. $g(t) = \sqrt{2-t}$

SOLUTION D: $\{t : t \le 2\}$; R: $\{y : y \ge 0\}$

49. $f(x) = |x|$

SOLUTION D: all reals; R: $\{y : y \ge 0\}$

50. $h(s) = \frac{1}{s}$

SOLUTION D: $\{s : s \ne 0\}$; R: $\{y : y \ne 0\}$

51. $f(x) = \frac{1}{x^2}$

SOLUTION D: $\{x : x \ne 0\}$; R: $\{y : y > 0\}$

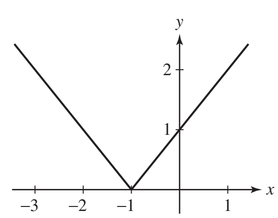
52. $g(t) = \frac{1}{\sqrt{1-t}}$

SOLUTION D: $\{t : t < 1\}$; R: $\{y : y > 0\}$

In Exercises 53–56, determine where f is increasing.

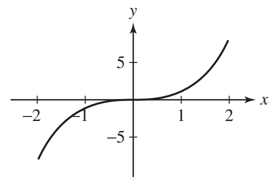
53. $f(x) = |x + 1|$

SOLUTION A graph of the function $y = |x + 1|$ is shown. From the graph, we see that the function is increasing on the interval $(-1, \infty)$.



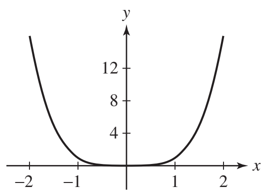
54. $f(x) = x^3$

SOLUTION A graph of the function $y = x^3$ is shown. From the graph, we see that the function is increasing for all real numbers.



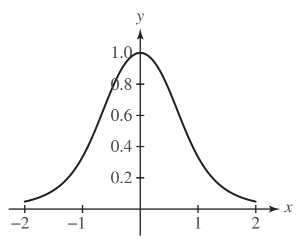
55. $f(x) = x^4$

SOLUTION A graph of the function $y = x^4$ is shown. From the graph, we see that the function is increasing on the interval $(0, \infty)$.



56. $f(x) = \frac{1}{x^4 + x^2 + 1}$

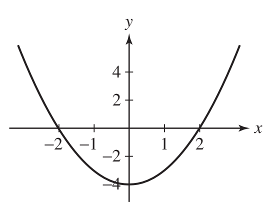
SOLUTION A graph of the function $y = \frac{1}{x^4 + x^2 + 1}$ is shown. From the graph, we see that the function is increasing on the interval $(-\infty, 0)$.



In Exercises 57–62, find the zeros of f and sketch its graph by plotting points. Use symmetry and increase/decrease information where appropriate.

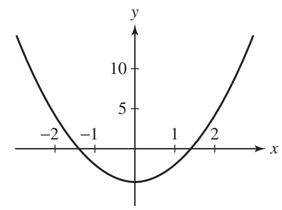
57. $f(x) = x^2 - 4$

SOLUTION Zeros: ± 2
Increasing: $x > 0$
Decreasing: $x < 0$
Symmetry: $f(-x) = f(x)$ (even function); so, y-axis symmetry



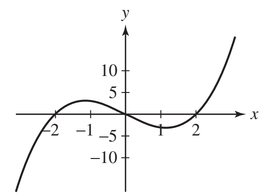
58. $f(x) = 2x^2 - 4$

SOLUTION Zeros: $\pm\sqrt{2}$
Increasing: $x > 0$
Decreasing: $x < 0$
Symmetry: $f(-x) = f(x)$ (even function); so, y-axis symmetry



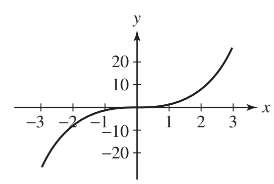
59. $f(x) = x^3 - 4x$

SOLUTION Zeros: 0, ±2; symmetry: $f(-x) = -f(x)$ (odd function); so, origin symmetry



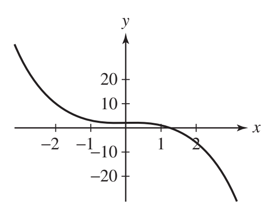
60. $f(x) = x^3$

SOLUTION Zeros: 0; increasing for all x; symmetry: $f(-x) = -f(x)$ (odd function); so, origin symmetry



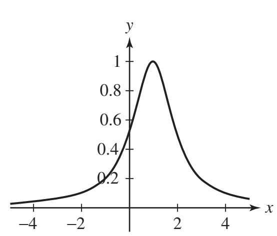
61. $f(x) = 2 - x^3$

SOLUTION This is an x-axis reflection of x^3 translated up 2 units. There is one zero at $x = \sqrt[3]{2}$.



62. $f(x) = \frac{1}{(x-1)^2+1}$

SOLUTION This is the graph of $\frac{1}{x^2+1}$ translated to the right 1 unit. The function has no zeros.



63. Which of the curves in Figure 27 is the graph of a function of x?

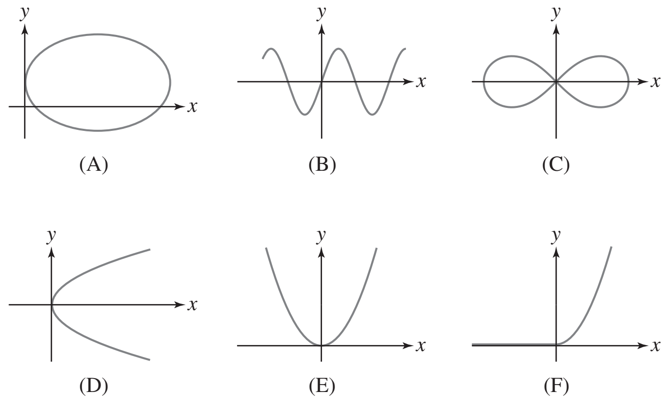


FIGURE 27

SOLUTION (B), (E), and (F) are graphs of functions. (A), (C), and (D) all fail the vertical line test.

64. Of the curves in Figure 27 that are graphs of functions, which is the graph of an odd function? Of an even function?

SOLUTION (B) is the graph of an odd function because the graph is symmetric about the origin; (E) is the graph of an even function because the graph is symmetric about the y-axis.

65. Determine whether the function is even, odd, or neither.

(a) $f(x) = x^5$ **(b)** $g(t) = t^3 - t^2$ **(c)** $F(t) = \frac{1}{t^4 + t^2}$

SOLUTION

(a) Because $f(-x) = (-x)^5 = -x^5 = -f(x)$, $f(x) = x^5$ is an odd function.

(b) Because $g(-t) = (-t)^3 - (-t)^2 = -t^3 - t^2$ equals neither $g(t)$ nor $-g(t)$, $g(t) = t^3 - t^2$ is neither an even function nor an odd function.

(c) Because $F(-t) = \frac{1}{(-t)^4 + (-t)^2} = \frac{1}{t^4 + t^2} = F(t)$, $F(t) = \frac{1}{t^4 + t^2}$ is an even function.

66. Determine whether the function is even, odd, or neither.

(a) $f(x) = 2x - x^2$ **(b)** $k(w) = (1 - w)^3 + (1 + w)^3$

(c) $f(t) = \frac{1}{t^4 + t + 1} - \frac{1}{t^4 - t + 1}$ **(d)** $g(t) = 2^t - 2^{-t}$

SOLUTION

(a) Because $f(-x) = 2(-x) - (-x)^2 = -2x - x^2$ equals neither $f(x)$ nor $-f(x)$, $f(x) = 2x - x^2$ is neither an even nor an odd function.

(b) Because $k(-w) = (1 - (-w))^3 + (1 + (-w))^3 = (1 + w)^3 + (1 - w)^3 = k(w)$, $k(w) = (1 - w)^3 + (1 + w)^3$ is an even function.

(c) Because

$$f(-t) = \frac{1}{(-t)^4 + (-t) + 1} - \frac{1}{(-t)^4 - (-t) + 1} = \frac{1}{t^4 - t + 1} - \frac{1}{t^4 + t + 1}$$
$$= -\left(\frac{1}{t^4 + t + 1} - \frac{1}{t^4 - t + 1}\right) = -f(t)$$

$f(t) = \frac{1}{t^4 + t + 1} - \frac{1}{t^4 - t + 1}$ is an odd function.

(d) Because $g(-t) = 2^{-t} - 2^{-(-t)} = 2^{-t} - 2^t = -(2^t - 2^{-t}) = -g(t)$, $g(t) = 2^t - 2^{-t}$ is an odd function.

67. Write $f(x) = 2x^4 - 5x^3 + 12x^2 - 3x + 4$ as the sum of an even and an odd function.

SOLUTION Let $g(x) = 2x^4 + 12x^2 + 4$ and $h(x) = -5x^3 - 3x$. Then

$$g(-x) = 2(-x)^4 + 12(-x)^2 + 4 = 2x^4 + 12x^2 + 4 = g(x)$$

so that g is an even function,

$$h(-x) = -5(-x)^3 - 3(-x) = 5x^3 + 3x = -h(x)$$

so that h is an odd function, and $f(x) = g(x) + h(x)$.

68. Assume that p is a function that is defined for all x.

(a) Prove that if f is defined by $f(x) = p(x) + p(-x)$ then f is even.

(b) Prove that if g is defined by $g(x) = p(x) - p(-x)$ then g is odd.

SOLUTION

(a) Let $f(x) = p(x) + p(-x)$. Then

$$f(-x) = p(-x) + p(-(-x)) = p(-x) + p(x) = f(x)$$

Because $f(-x) = f(x)$, it follows that f is an even function.

(b) Let $g(x) = p(x) - p(-x)$. Then

$$g(-x) = p(-x) - p(-(-x)) = p(-x) - p(x) = -(p(x) - p(-x)) = -g(x)$$

Because $g(-x) = -g(x)$, it follows that g is an odd function.

69. Assume that p is a function that is defined for $x > 0$ and satisfies $p(a/b) = p(b) - p(a)$. Prove that $f(x) = p\left(\frac{2-x}{2+x}\right)$ is an odd function.

SOLUTION Let $f(x) = p\left(\frac{2-x}{2+x}\right)$. Then

$$f(-x) = p\left(\frac{2-(-x)}{2-x}\right) = p\left(\frac{2+x}{2-x}\right) = p(2+x) - p(2-x) = -(p(2-x) - p(2+x)) = -p\left(\frac{2-x}{2+x}\right) = -f(x)$$

Because $f(-x) = -f(x)$, it follows that f is an odd function.

70. State whether the function is increasing, decreasing, or neither.

(a) Surface area of a sphere as a function of its radius

(b) Temperature at a point on the equator as a function of time

(c) Price of an airline ticket as a function of the price of oil

(d) Pressure of the gas in a piston as a function of volume

SOLUTION

(a) Increasing **(b)** Neither **(c)** Increasing **(d)** Decreasing

In Exercises 71–76, let f be the function shown in Figure 28.

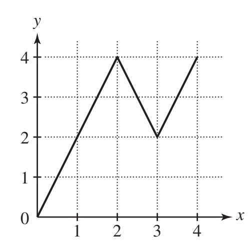


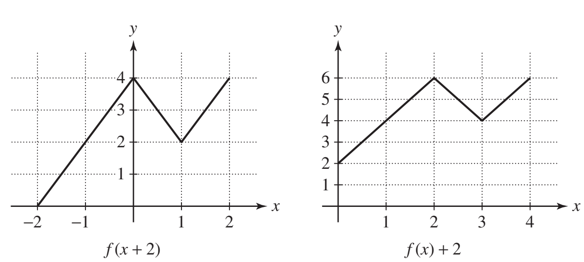
FIGURE 28

71. Find the domain and range of f.

SOLUTION D: $[0, 4]$; R: $[0, 4]$

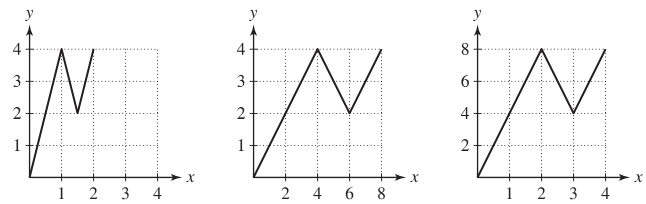
72. Sketch the graphs of $y = f(x + 2)$ and $y = f(x) + 2$.

SOLUTION The graph of $y = f(x + 2)$ is obtained by shifting the graph of $y = f(x)$ 2 units to the left (see the graph below on the left). The graph of $y = f(x) + 2$ is obtained by shifting the graph of $y = f(x)$ 2 units up (see the graph below on the right).



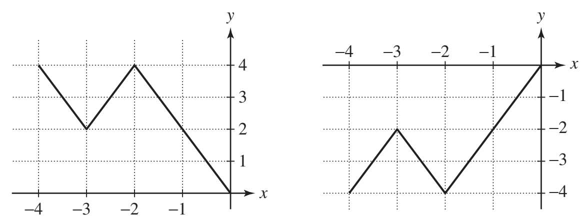
73. Sketch the graphs of $y = f(2x)$, $y = f(\frac{1}{2}x)$, and $y = 2f(x)$.

SOLUTION The graph of $y = f(2x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below in the middle). The graph of $y = 2f(x)$ is obtained by stretching the graph of $y = f(x)$ vertically by a factor of 2 (see the graph below on the right).


$f(2x)$ $f(x/2)$ $2f(x)$

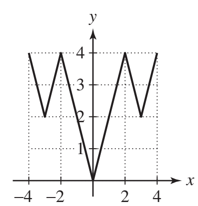
74. Sketch the graphs of $y = f(-x)$ and $y = -f(-x)$.

SOLUTION The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ across the y-axis (see the graph below on the left). The graph of $y = -f(-x)$ is obtained by reflecting the graph of $y = f(x)$ across both the x- and y-axes, or equivalently, about the origin (see the graph below on the right).


$f(-x)$ $-f(-x)$

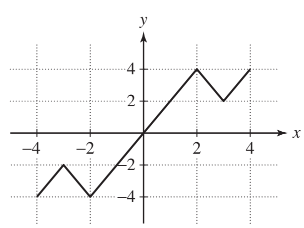
75. Extend the graph of f to $[-4, 4]$ so that it is an even function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an even function, reflect the graph of $f(x)$ across the y-axis (see the graph).



76. Extend the graph of f to $[-4, 4]$ so that it is an odd function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an odd function, reflect the graph of $f(x)$ through the origin (see the graph).



77. Suppose that f has domain $[4, 8]$ and range $[2, 6]$. Find the domain and range of:

(a) $y = f(x) + 3$ **(b)** $y = f(x + 3)$

(c) $y = f(3x)$ **(d)** $y = 3f(x)$

SOLUTION

(a) $f(x) + 3$ is obtained by shifting $f(x)$ upward 3 units. Therefore, the domain remains $[4, 8]$, while the range becomes $[5, 9]$.

(b) $f(x + 3)$ is obtained by shifting $f(x)$ left 3 units. Therefore, the domain becomes $[1, 5]$, while the range remains $[2, 6]$.

(c) $f(3x)$ is obtained by compressing $f(x)$ horizontally by a factor of 3. Therefore, the domain becomes $[\frac{4}{3}, \frac{8}{3}]$, while the range remains $[2, 6]$.

(d) $3f(x)$ is obtained by stretching $f(x)$ vertically by a factor of 3. Therefore, the domain remains $[4, 8]$, while the range becomes $[6, 18]$.

78. Let $f(x) = x^2$. Sketch the graph over $[-2, 2]$ of:

(a) $y = f(x + 1)$

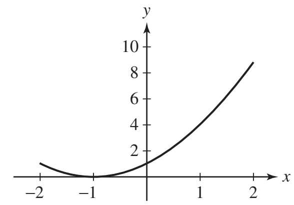
(b) $y = f(x) + 1$

(c) $y = f(5x)$

(d) $y = 5f(x)$

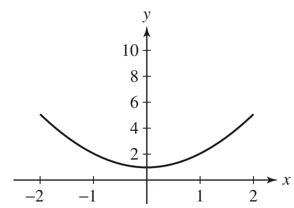
SOLUTION

(a) The graph of $y = f(x + 1)$ is obtained by shifting the graph of $y = f(x)$ 1 unit to the left.



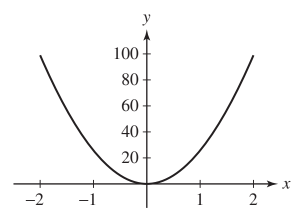
$f(x + 1)$

(b) The graph of $y = f(x) + 1$ is obtained by shifting the graph of $y = f(x)$ 1 unit up.



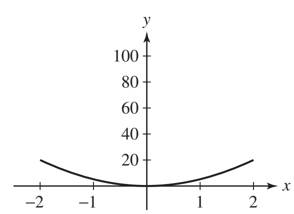
$f(x) + 1$

(c) The graph of $y = f(5x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 5.



$f(5x)$

(d) The graph of $y = 5f(x)$ is obtained by stretching the graph of $y = f(x)$ vertically by a factor of 5.



$5f(x)$

79. Suppose that the graph of $f(x) = x^4 - x^2$ is compressed horizontally by a factor of 2 and then shifted 5 units to the right.

(a) What is the equation for the new graph?

(b) What is the equation if you first shift by 5 and then compress by 2?

(c) GU Verify your answers by plotting your equations.

SOLUTION

(a) Let $f(x) = x^4 - x^2$. After compressing the graph of f horizontally by a factor of 2, we obtain the function $g(x) = f(2x) = (2x)^4 - (2x)^2$. Shifting the graph 5 units to the right then yields

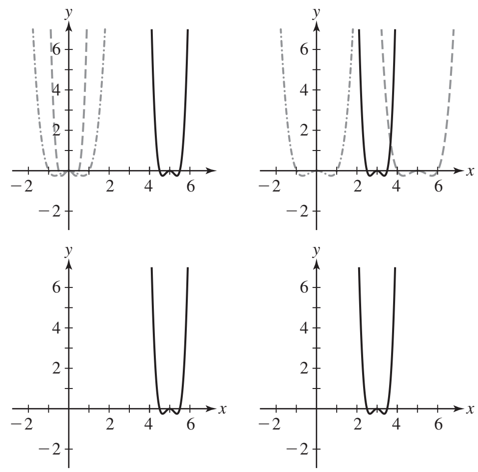
$$h(x) = g(x - 5) = (2(x - 5))^4 - (2(x - 5))^2 = (2x - 10)^4 - (2x - 10)^2$$

(b) Let $f(x) = x^4 - x^2$. After shifting the graph 5 units to the right, we obtain the function $g(x) = f(x - 5) = (x - 5)^4 - (x - 5)^2$. Compressing the graph horizontally by a factor of 2 then yields

$$h(x) = g(2x) = (2x - 5)^4 - (2x - 5)^2$$

(c) The figure below at the top left shows the graphs of $y = x^4 - x^2$ (the dash-dot curve), the graph compressed horizontally by a factor of 2 (the dashed curve), and then shifted right 5 units (the solid curve). Compare this last graph with the graph of $y = (2x - 10)^4 - (2x - 10)^2$ shown at the bottom left.

The figure below at the top right shows the graphs of $y = x^4 - x^2$ (the dash-dot curve), the graph shifted right 5 units (the dashed curve), and then compressed horizontally by a factor of 2 (the solid curve). Compare this last graph with the graph of $y = (2x - 5)^4 - (2x - 5)^2$ shown at the bottom right.



80. Figure 29 shows the graph of $f(x) = |x| + 1$. Match the functions (a)–(e) with their graphs (i)–(v).

(a) $y = f(x - 1)$ **(b)** $y = -f(x)$ **(c)** $y = -f(x) + 2$

(d) $y = f(x - 1) - 2$ **(e)** $y = f(x + 1)$

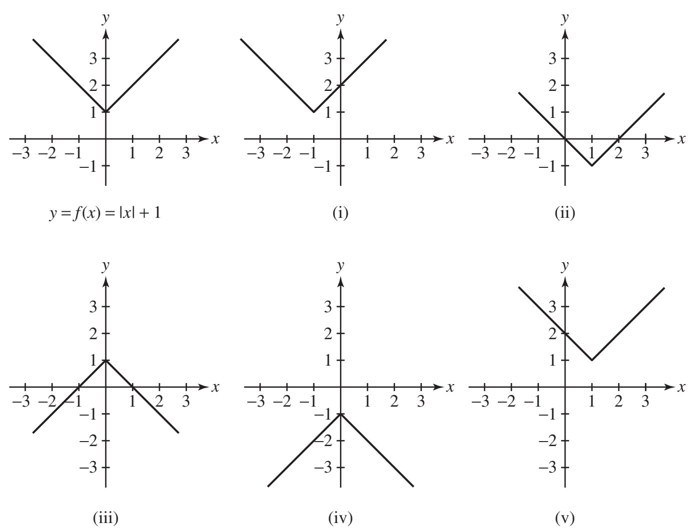


FIGURE 29

SOLUTION

(a) Shift graph to the right 1 unit: (v)

(b) Reflect graph across x-axis: (iv)

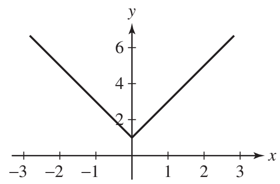
(c) Reflect graph across x-axis and then shift up 2 units: (iii)

(d) Shift graph to the right one unit and down 2 units: (ii)

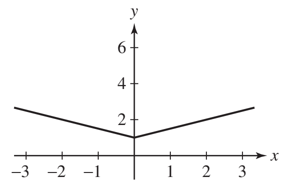
(e) Shift graph to the left 1 unit: (i)

81. Sketch the graph of $y = f(2x)$ and $y = f(\frac{1}{2}x)$, where $f(x) = |x| + 1$ (Figure 29).

SOLUTION The graph of $y = f(2x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the right).



$f(2x)$



$f(x/2)$

82. Find the function f whose graph is obtained by shifting the parabola $y = x^2$ by 3 units to the right and 4 units down, as in Figure 30.

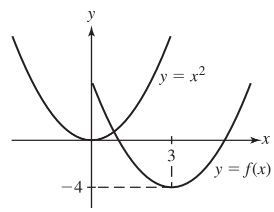
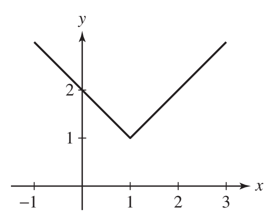


FIGURE 30

SOLUTION The new function is $f(x) = (x - 3)^2 - 4$.

83. Define $f(x)$ to be the larger of x and $2 - x$. Sketch the graph of f. What are its domain and range? Express $f(x)$ in terms of the absolute value function.

SOLUTION



The graph of $y = f(x)$ is shown. Clearly, the domain of f is the set of all real numbers while the range is $\{y \mid y \geq 1\}$. Notice the graph has the standard V shape associated with the absolute value function, but the base of the V has been translated to the point $(1, 1)$. Thus, $f(x) = |x - 1| + 1$.

84. For each curve in Figure 31, state whether it is symmetric with respect to the y-axis, the origin, both, or neither.

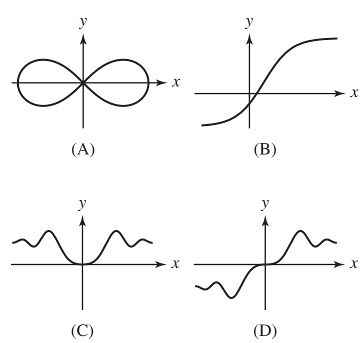


FIGURE 31

SOLUTION

(A) Both

(B) Neither

(C) y-axis

(D) Origin

85. Show that the sum of two even functions is even and the sum of two odd functions is odd.

SOLUTION Even: $(f + g)(-x) = f(-x) + g(-x) \stackrel{\text{even}}{=} f(x) + g(x) = (f + g)(x)$

Odd: $(f + g)(-x) = f(-x) + g(-x) \stackrel{\text{odd}}{=} -f(x) + -g(x) = -(f + g)(x)$

86. Suppose that f and g are both odd. Which of the following functions are even? Which are odd?

(a) $y = f(x)g(x)$
(b) $y = f(x)^3$
(c) $y = f(x) - g(x)$
(d) $y = \frac{f(x)}{g(x)}$

SOLUTION

(a) $f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) \Rightarrow$ even
(b) $f(-x)^3 = [-f(x)]^3 = -f(x)^3 \Rightarrow$ odd
(c) $f(-x) - g(-x) = -f(x) + g(x) = -(f(x) - g(x)) \Rightarrow$ odd
(d) $\frac{f(-x)}{g(-x)} = \frac{-f(x)}{-g(x)} = \frac{f(x)}{g(x)} \Rightarrow$ even

87. Prove that the only function whose graph is symmetric with respect to both the y-axis and the origin is the function $f(x) = 0$.

SOLUTION A circle of radius 1 with its center at the origin is symmetrical with respect to both the y-axis and the origin.

The only function having both symmetries is $f(x) = 0$. For if f is symmetric with respect to the y-axis, then $f(-x) = f(x)$. If f is also symmetric with respect to the origin, then $f(-x) = -f(x)$. Thus, $f(x) = -f(x)$ or $2f(x) = 0$. Finally, $f(x) = 0$.

Further Insights and Challenges

88. Prove the triangle inequality ($|a + b| \le |a| + |b|$) by adding the two inequalities:

$$-|a| \le a \le |a|, \qquad -|b| \le b \le |b|$$

SOLUTION Adding the indicated inequalities gives

$$-(|a| + |b|) \le a + b \le |a| + |b|$$

and this is equivalent to $|a + b| \le |a| + |b|$.

89. Show that a fraction $r = a/b$ in lowest terms has a *finite* decimal expansion if and only if

$$b = 2^n 5^m \quad \text{for some } n, m \ge 0$$

Hint: Observe that r has a finite decimal expansion when $10^N r$ is an integer for some $N \ge 0$ (and hence b divides 10^N).

SOLUTION Suppose r has a finite decimal expansion. Then there exists an integer $N \ge 0$ such that $10^N r$ is an integer, call it k. Thus, $r = k/10^N$. Because the only prime factors of 10 are 2 and 5, it follows that when r is written in lowest terms, its denominator must be of the form $2^n 5^m$ for some integers $n, m \ge 0$.

Conversely, suppose $r = a/b$ is written in lowest terms with $b = 2^n 5^m$ for some integers $n, m \ge 0$. Then $r = \frac{a}{b} = \frac{a}{2^n 5^m}$ or $2^n 5^m r = a$. If $m \ge n$, then $2^m 5^m r = a2^{m-n}$ or $r = \frac{a2^{m-n}}{10^m}$ and thus r has a finite decimal expansion (less than or equal to m terms, to be precise). On the other hand, if $n > m$, then $2^n 5^n r = a5^{n-m}$ or $r = \frac{a5^{n-m}}{10^n}$ and once again r has a finite decimal expansion.

90. Let $p = p_1 \dots p_s$ be an integer with digits $p_1, \dots, p_s$. Show that

$$\frac{p}{10^s - 1} = 0.\overline{p_1 \dots p_s}$$

Use this to find the decimal expansion of $r = \frac{2}{11}$. Note that

$$r = \frac{2}{11} = \frac{18}{10^2 - 1}$$

SOLUTION Let $p = p_1 \dots p_s$ be an integer with digits $p_1, \dots, p_s$, and let $\overline{p} = 0.\overline{p_1 \dots p_s}$. Then

$$10^s \overline{p} - \overline{p} = p_1 \dots p_s.\overline{p_1 \dots p_s} - 0.\overline{p_1 \dots p_s} = p_1 \dots p_s = p$$

Thus,

$$\frac{p}{10^s - 1} = \overline{p} = 0.\overline{p_1 \dots p_s}$$

Consider the rational number $r = 2/11$. Because

$$r = \frac{2}{11} = \frac{18}{99} = \frac{18}{10^2 - 1}$$

it follows that the decimal expansion of r is $0.\overline{18}$.

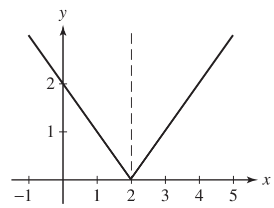
91. A function f is symmetric with respect to the vertical line $x = a$ if $f(a - x) = f(a + x)$.

(a) Draw the graph of a function that is symmetric with respect to $x = 2$.

(b) Show that if f is symmetric with respect to $x = a$, then $g(x) = f(x + a)$ is even.

SOLUTION

(a) There are many possibilities, one of which is



$y = |x - 2|$

(b) Let $g(x) = f(x + a)$. Then

$$\begin{aligned} g(-x) = f(-x + a) &= f(a - x) \\ &= f(a + x) \qquad \text{(symmetry with respect to } x = a\text{)} \\ &= g(x) \end{aligned}$$

Thus, $g(x)$ is even.

92. Formulate a condition for f to be symmetric with respect to the point $(a, 0)$ on the x-axis.

SOLUTION In order for $f(x)$ to be symmetric with respect to the point $(a, 0)$, the value of f at a distance x units to the right of a must be opposite the value of f at a distance x units to the left of a. In other words, $f(x)$ is symmetrical with respect to $(a, 0)$ if $f(a + x) = -f(a - x)$.

1.2 Linear and Quadratic Functions

Preliminary Questions

1. What is the slope of the line $y = -4x - 9$?

SOLUTION The slope of the line $y = -4x - 9$ is -4, given by the coefficient of x.

2. Are the lines $y = 2x + 1$ and $y = -2x - 4$ perpendicular?

SOLUTION The slopes of perpendicular lines are negative reciprocals of one another. Because the slope of $y = 2x + 1$ is 2 and the slope of $y = -2x - 4$ is -2, these two lines are *not* perpendicular.

3. When is the line $ax + by = c$ parallel to the y-axis? To the x-axis?

SOLUTION The line $ax + by = c$ will be parallel to the y-axis when $b = 0$ and parallel to the x-axis when $a = 0$.

4. Suppose $y = 3x + 2$. What is Δy if x increases by 3?

SOLUTION Because $y = 3x + 2$ is a linear function with slope 3, increasing x by 3 will lead to $\Delta y = 3(3) = 9$.

5. What is the minimum of $f(x) = (x + 3)^2 - 4$?

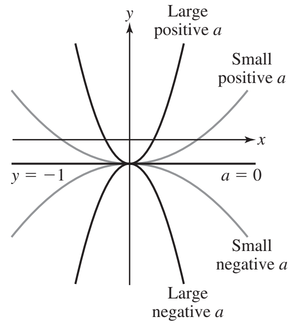
SOLUTION Because $(x + 3)^2 \geq 0$, it follows that $(x + 3)^2 - 4 \geq -4$. Thus, the minimum value of $(x + 3)^2 - 4$ is -4.

6. What is the result of completing the square for $f(x) = x^2 + 1$?

SOLUTION Because there is no x term in $x^2 + 1$, completing the square on this expression leads to $(x - 0)^2 + 1$.

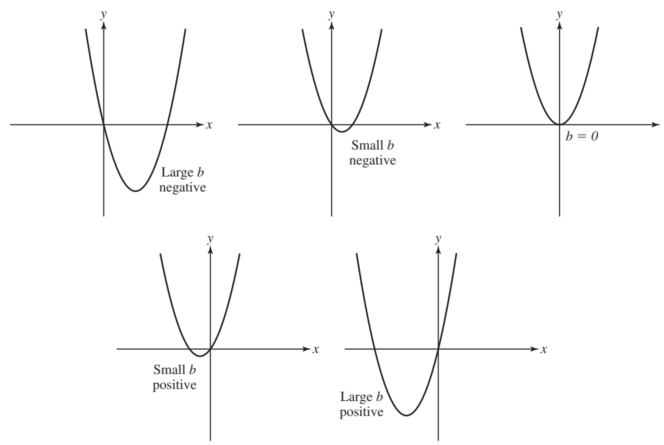
7. Describe how the parabolas $y = ax^2 - 1$ change as a changes from $-\infty$ to ∞.

SOLUTION First, note that the graph of $y = ax^2 - 1$ passes through the point $(0, -1)$ for all values of a. When a is large negative, the parabola $y = ax^2 - 1$ opens downward and is narrow. As a increases toward zero, the parabola becomes wider but remains opening downward. When $a = 0$, the parabola has "opened out" to become the horizontal line $y = -1$. For positive a, the parabola opens upward, wide for small values of a but then becoming more and more narrow as a increases. See the figure.



8. Describe how the parabolas $y = x^2 + bx$ change as b changes from $-\infty$ to ∞.

SOLUTION The parabolas open upward and have x-intercepts at $(0, 0)$ and $(-b, 0)$. Moreover, the vertex is located at $(-b/2, -b^2/4)$. For b negative and increasing toward zero, the nonzero x-intercept is positive and moves toward the origin, while the vertex moves up and to the left toward the origin. When $b = 0$, there is a single x-intercept, which is also the vertex, at the origin. For b positive and increasing, the nonzero x-intercept is negative and moves away from the origin, while the vertex moves down and to the left. See the figures.



Exercises

In Exercises 1–4, find the slope, the y-intercept, and the x-intercept of the line with the given equation.

1. $y = 3x + 12$

SOLUTION Because the equation of the line is given in slope-intercept form, the slope is the coefficient of x and the y-intercept is the constant term; that is, $m = 3$ and the y-intercept is 12. To determine the x-intercept, substitute $y = 0$ and then solve for x: $0 = 3x + 12$ or $x = -4$.

2. $y = 4 - x$

SOLUTION Because the equation of the line is given in slope-intercept form, the slope is the coefficient of x and the y-intercept is the constant term; that is, $m = -1$ and the y-intercept is 4. To determine the x-intercept, substitute $y = 0$ and then solve for x: $0 = 4 - x$ or $x = 4$.

3. $4x + 9y = 3$

SOLUTION To determine the slope and y-intercept, we first solve the equation for y to obtain the slope-intercept form. This yields $y = -\frac{4}{9}x + \frac{1}{3}$. From here, we see that the slope is $m = -\frac{4}{9}$ and the y-intercept is $\frac{1}{3}$. To determine the x-intercept, substitute $y = 0$ and solve for x: $4x = 3$ or $x = \frac{3}{4}$.

4. $y - 3 = \frac{1}{2}(x - 6)$

SOLUTION The equation is in point-slope form, so we see that $m = \frac{1}{2}$. Substituting $x = 0$ yields $y - 3 = -3$ or $y = 0$. Thus, the x- and y-intercepts are both 0.

In Exercises 5–8, find the slope of the line.

5. $y = 3x + 2$

SOLUTION $m = 3$

6. $y = 3(x - 9) + 2$

SOLUTION $m = 3$

7. $3x + 4y = 12$

SOLUTION First, solve the equation for y to obtain the slope-intercept form. This yields $y = -\frac{3}{4}x + 3$. The slope of the line is therefore $m = -\frac{3}{4}$.

8. $3x + 4y = -8$

SOLUTION First, solve the equation for y to obtain the slope-intercept form. This yields $y = -\frac{3}{4}x - 2$. The slope of the line is therefore $m = -\frac{3}{4}$.

In Exercises 9–20, find the equation of the line with the given description.

9. Slope 3, y-intercept 8

SOLUTION Using the slope-intercept form for the equation of a line, we have $y = 3x + 8$.

10. Slope −2, y-intercept 3

SOLUTION Using the slope-intercept form for the equation of a line, we have $y = -2x + 3$.

11. Slope 3, passes through $(7, 9)$

SOLUTION Using the point-slope form for the equation of a line, we have $y - 9 = 3(x - 7)$ or $y = 3x - 12$.

12. Slope −5, passes through $(0, 0)$

SOLUTION Using the point-slope form for the equation of a line, we have $y - 0 = -5(x - 0)$ or $y = -5x$.

13. Horizontal, passes through $(0, -2)$

SOLUTION A horizontal line has a slope of 0. Using the point-slope form for the equation of a line, we have $y - (-2) = 0(x - 0)$ or $y = -2$.

14. Passes through $(-1, 4)$ and $(2, 7)$

SOLUTION The slope of the line that passes through $(-1, 4)$ and $(2, 7)$ is

$$m = \frac{7 - 4}{2 - (-1)} = 1$$

Using the point-slope form for the equation of a line, we have $y - 7 = 1(x - 2)$ or $y = x + 5$.

15. Parallel to $y = 3x - 4$, passes through $(1, 1)$

SOLUTION Because the equation $y = 3x - 4$ is in slope-intercept form, we can readily identify that it has a slope of 3. Parallel lines have the same slope, so the slope of the requested line is also 3. Using the point-slope form for the equation of a line, we have $y - 1 = 3(x - 1)$ or $y = 3x - 2$.

16. Passes through $(1, 4)$ and $(12, -3)$

SOLUTION The slope of the line that passes through $(1, 4)$ and $(12, -3)$ is

$$m = \frac{-3 - 4}{12 - 1} = \frac{-7}{11}$$

Using the point-slope form for the equation of a line, we have $y - 4 = -\frac{7}{11}(x - 1)$ or $y = -\frac{7}{11}x + \frac{51}{11}$.

17. Perpendicular to $3x + 5y = 9$, passes through $(2, 3)$

SOLUTION We start by solving the equation $3x + 5y = 9$ for y to obtain the slope-intercept form for the equation of a line. This yields

$$y = -\frac{3}{5}x + \frac{9}{5}$$

from which we identify the slope as $-\frac{3}{5}$. Perpendicular lines have slopes that are negative reciprocals of one another, so the slope of the desired line is $m_{\perp} = \frac{5}{3}$. Using the point-slope form for the equation of a line, we have $y - 3 = \frac{5}{3}(x - 2)$ or $y = \frac{5}{3}x - \frac{1}{3}$.

18. Vertical, passes through $(-4, 9)$

SOLUTION A vertical line has the equation $x = c$ for some constant c. Because the line needs to pass through the point $(-4, 9)$, we must have $c = -4$. The equation of the desired line is then $x = -4$.

19. Horizontal, passes through $(8, 4)$

SOLUTION A horizontal line has slope 0. Using the point-slope form for the equation of a line, we have $y - 4 = 0(x - 8)$ or $y = 4$.

20. Slope 3, x-intercept 6

SOLUTION If the x-intercept is 6, then the line passes through the point $(6, 0)$. Using the point-slope form for the equation of a line, we have $y - 0 = 3(x - 6)$ or $y = 3x - 18$.

21. Find the equation of the perpendicular bisector of the segment joining $(1, 2)$ and $(5, 4)$ (Figure 12). *Hint:* The midpoint Q of the segment joining (a, b) and (c, d) is $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$.

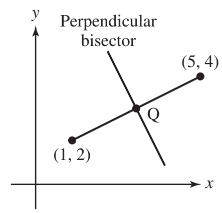


FIGURE 12

SOLUTION The slope of the segment joining $(1, 2)$ and $(5, 4)$ is

$$m = \frac{4-2}{5-1} = \frac{1}{2}$$

and the midpoint of the segment (Figure 12) is

$$\text{midpoint} = \left(\frac{1+5}{2}, \frac{2+4}{2}\right) = (3, 3)$$

The perpendicular bisector has slope $-1/m = -2$ and passes through $(3, 3)$, so its equation is $y - 3 = -2(x - 3)$ or $y = -2x + 9$.

22. Intercept–Intercept Form Show that if $a, b \neq 0$, then the line with x-intercept $x = a$ and y-intercept $y = b$ has equation (Figure 13)

$$\frac{x}{a} + \frac{y}{b} = 1$$

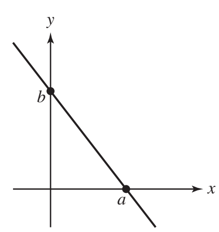


FIGURE 13

SOLUTION The line passes through the points $(a, 0)$ and $(0, b)$. Thus, $m = -\frac{b}{a}$. Using the point-slope form for the equation of a line yields $y - 0 = -\frac{b}{a}(x - a) \Rightarrow y = -\frac{b}{a}x + b \Rightarrow \frac{b}{a}x + y = b \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$.

23. Find an equation of the line with x-intercept $x = 4$ and y-intercept $y = 3$.

SOLUTION From Exercise 22, $\frac{x}{4} + \frac{y}{3} = 1$ or $3x + 4y = 12$.

24. Find y such that $(3, y)$ lies on the line of slope $m = 2$ through $(1, 4)$.

SOLUTION In order for the point $(3, y)$ to lie on the line through $(1, 4)$ of slope 2, the slope of the segment connecting $(1, 4)$ and $(3, y)$ must have slope 2. Therefore,

$$m = \frac{y-4}{3-1} = \frac{y-4}{2} = 2 \Rightarrow y - 4 = 4 \Rightarrow y = 8$$

25. Determine whether there exists a constant c such that the line $x + cy = 1$:

(a) Has slope 4

(b) Passes through $(3, 1)$

(c) Is horizontal

(d) Is vertical

SOLUTION

(a) Rewriting the equation of the line in slope-intercept form gives $y = -\frac{x}{c} + \frac{1}{c}$. To have slope 4 requires $-\frac{1}{c} = 4$ or $c = -\frac{1}{4}$.

(b) Substituting $x = 3$ and $y = 1$ into the equation of the line gives $3 + c = 1$ or $c = -2$.

(c) From (a), we know the slope of the line is $-\frac{1}{c}$. There is no value for c that will make this slope equal to 0.

(d) With $c = 0$, the equation becomes $x = 1$. This is the equation of a vertical line.

26. Determine whether there exists a constant c such that the line $cx - 2y = 4$:

(a) Has slope 4
(b) Passes through $(1, -4)$
(c) Is horizontal
(d) Is vertical

SOLUTION

(a) Rewriting the equation of the line in slope-intercept form gives $y = \frac{c}{2}x - 2$. To have slope 4 requires $\frac{c}{2} = 4$ or $c = 8$.
(b) Substituting $x = 1$ and $y = -4$ into the equation of the line gives $c + 8 = 4$ or $c = -4$.
(c) A horizontal line has slope zero. From (a), we know the slope of the line is $\frac{c}{2}$, so to have slope zero requires $\frac{c}{2} = 0$ or $c = 0$.
(d) A vertical line has equation $x = a$ for some constant a. There is no value for the constant c that will make the equation $cx - 2y = 4$ have the correct form.

27. Suppose that the number of Bob's Bits computers that can be sold when its price is P (in dollars) is given by a linear function $N(P)$, where $N(1000) = 10{,}000$ and $N(1500) = 7500$.

(a) Determine $N(P)$.
(b) What is the slope of the graph of $N(P)$, including units? Describe what the slope represents.
(c) What is the change ΔN in the number of computers sold if the price is increased by $\Delta P = \$100$?

SOLUTION

(a) We first determine the slope of the line:

$$m = \frac{10000 - 7500}{1000 - 1500} = \frac{2500}{-500} = -5$$

Knowing that $N(1000) = 10000$, it follows that

$$N - 10000 = -5(P - 1000) \quad \text{or} \quad N(P) = -5P + 15000$$

(b) The slope of the graph of $N(P)$ is -5 computers/dollar. The slope represents the rate of change in the number of computers sold with respect to the price of the computer. In particular, five fewer computers are sold for every \$1 increase in price of the computer.
(c) $\Delta N = -5\Delta P = -5(100) = -500$. If the price of the computer is increased by \$100, 500 fewer computers will be sold.

28. Suppose that the demand for Colin's kidney pies is linear in the price P. Further, assume that he can sell 100 pies when the price is \$5.00 and 40 pies when the price is \$10.00.

(a) Determine the demand N (number of pies sold) as a function of the price P (in dollars).
(b) What is the slope of the graph of $N(P)$, including units? Describe what the slope represents.
(c) Determine the revenue $R = N \times P$ for prices $P = 5, 6, 7, 8, 9, 10$ and then choose a price to maximize the revenue.

SOLUTION

(a) We first determine the slope of the line:

$$m = \frac{100 - 40}{5 - 10} = \frac{60}{-5} = -12$$

Knowing that $N(5) = 100$, it follows that

$$N - 100 = -12(P - 5) \quad \text{or} \quad N(P) = -12P + 160$$

(b) The slope of the graph of $N(P)$ is -12 pies/dollar. The slope represents the rate of change in the number of pies sold with respect to the price of the pie. In particular, 12 fewer pies are sold for every \$1 increase in price of the pie.
(c) The table displays the revenue for prices $P = 5, 6, 7, 8, 9, 10$.

Price (P) (in dollars)	Demand (N) (number of pies sold)	Revenue ($N \times P$) (in dollars)
5	100	500
6	88	528
7	76	532
8	64	512
9	52	468
10	40	400

To determine the price that will maximize revenue, note that the revenue function, $R(P)$, is given by

$$R(P) = N(P) \times P = (160 - 12P)P = -12P^2 + 160P$$

Completing the square on the revenue function yields

$$R(P) = -12\left(P^2 - \frac{40}{3}P + \frac{400}{9}\right) + \frac{1600}{9} = -12\left(P - \frac{20}{3}\right)^2 + \frac{1600}{6}$$

From here, it follows that revenue is maximized when $P = \frac{20}{3}$, or roughly \$6.67.

29. In each case, identify the slope and give its meaning with the appropriate units.

(a) The function $N = -70t + 5000$ models the enrollment at Maple Grove College during the fall of 2018, where N represents the number of students and t represents the time in weeks since the start of the semester.

(b) The function $C = 3.5n + 700$ represents the cost (in dollars) to rent the Shakedown Street Dance Hall for an evening if n people attend the dance.

SOLUTION

(a) The slope is -70 students/week. During the fall of 2018, enrollment dropped by 70 students per week.

(b) The slope is 3.5 dollars/person. The rental cost for the Shakedown Street Dance Hall increases \$3.50 for each additional person in attendance.

30. In each case, identify the slope and give its meaning with the appropriate units.

(a) The function $N = 3.9T - 178.8$ models the the number of times, N, that a cricket chirps in a minute when the temperature is $T°$ Celsius.

(b) The function $V = 47,500d$ gives the volume (V, in gallons) of molasses in the storage tank in relation to the depth (d, in feet) of the molasses.

SOLUTION

(a) The slope is 3.9 chirps per minute/°C. The cricket chirp rate increases by 3.9 chirps per minute for every 1°C increase in temperature.

(b) The slope is 47,500 gal/ft. The volume of molasses in the storage tank increases by 47,500 gal for every 1-ft increase in depth of the molasses in the tank.

31. Materials expand when heated. Consider a metal rod of length L_0 at temperature T_0. If the temperature is changed by an amount ΔT, then the rod's length approximately changes by $\Delta L = \alpha L_0 \Delta T$, where α is the thermal expansion coefficient and ΔT is not an extreme temperature change. For steel, $\alpha = 1.24 \times 10^{-5}\,°\text{C}^{-1}$.

(a) A steel rod has length $L_0 = 40$ cm at $T_0 = 40°$C. Find its length at $T = 90°$C.

(b) Find its length at $T = 50°$C if its length at $T_0 = 100°$C is 65 cm.

(c) Express length L as a function of T if $L_0 = 65$ cm at $T_0 = 100°$C.

SOLUTION

(a) With $T = 90°$C and $T_0 = 40°$C, $\Delta T = 50°$C. Therefore,

$$\Delta L = \alpha L_0 \Delta T = (1.24 \times 10^{-5})(40)(50) = 0.0248 \qquad \text{and} \qquad L = L_0 + \Delta L = 40.0248 \text{ cm}$$

(b) With $T = 50°$C and $T_0 = 100°$C, $\Delta T = -50°$C. Therefore,

$$\Delta L = \alpha L_0 \Delta T = (1.24 \times 10^{-5})(65)(-50) = -0.0403 \qquad \text{and} \qquad L = L_0 + \Delta L = 64.9597 \text{ cm}$$

(c) $L = L_0 + \Delta L = L_0 + \alpha L_0 \Delta T = L_0(1 + \alpha\Delta T) = 65(1 + \alpha(T - 100))$

32. Do the points $(0.5, 1)$, $(1, 1.2)$, $(2, 2)$ lie on a line?

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{1.2 - 1}{1 - 0.5} = \frac{0.2}{0.5} = 0.4$$

while the second pair of data points yields a slope of

$$\frac{2 - 1.2}{2 - 1} = \frac{0.8}{1} = 0.8$$

Because the slopes are not equal, the three points do not lie on a line.

33. Find b such that $(2, -1)$, $(3, 2)$, and $(b, 5)$ lie on a line.

SOLUTION The slope of the line determined by the points $(2, -1)$ and $(3, 2)$ is

$$\frac{2 - (-1)}{3 - 2} = 3$$

To lie on the same line, the slope between $(3, 2)$ and $(b, 5)$ must also be 3. Thus, we require

$$\frac{5 - 2}{b - 3} = \frac{3}{b - 3} = 3$$

or $b = 4$.

34. Find an expression for the velocity v as a linear function of t that matches the following data:

t (s)	0	2	4	6
v (m/s)	39.2	58.6	78	97.4

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{58.6 - 39.2}{2 - 0} = 9.7$$

while the second pair of data points yields a slope of

$$\frac{78 - 58.6}{4 - 2} = 9.7$$

and the last pair of data points yields a slope of

$$\frac{97.4 - 78}{6 - 4} = 9.7$$

Thus, the data suggest a linear function with slope 9.7. Finally,

$$v - 39.2 = 9.7(t - 0) \Rightarrow v = 9.7t + 39.2$$

35. The period T of a pendulum is measured for pendulums of several different lengths L. Based on the following data, does T appear to be a linear function of L?

L (cm)	20	30	40	50
T (s)	0.9	1.1	1.27	1.42

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{1.1 - 0.9}{30 - 20} = 0.02$$

while the second pair of data points yields a slope of

$$\frac{1.27 - 1.1}{40 - 30} = 0.017$$

and the last pair of data points yields a slope of

$$\frac{1.42 - 1.27}{50 - 40} = 0.015$$

Because the three slopes are not equal, T does not appear to be a linear function of L.

36. Show that f is linear of slope m if and only if

$$f(x + h) - f(x) = mh \quad \text{(for all } x \text{ and } h\text{)}$$

That is to say, prove the following two statements:

(a) f is linear of slope m implies that $f(x + h) - f(x) = mh$ (for all x and h).

(b) $f(x + h) - f(x) = mh$ (for all x and h) implies that f is linear of slope m.

SOLUTION

(a) First, suppose $f(x)$ is linear. Then the slope between $(x, f(x))$ and $(x + h, f(x + h))$ is

$$m = \frac{f(x + h) - f(x)}{h} \Rightarrow mh = f(x + h) - f(x)$$

(b) Conversely, suppose $f(x + h) - f(x) = mh$ for all x and for all h. Then

$$m = \frac{f(x + h) - f(x)}{h} = \frac{f(x + h) - f(x)}{x + h - x}$$

which is the slope between $(x, f(x))$ and $(x + h, f(x + h))$. Since this is true for all x and h, f must be linear (it has constant slope).

37. Find the roots of the quadratic polynomials:

(a) $f(x) = 4x^2 - 3x - 1$

(b) $f(x) = x^2 - 2x - 1$

SOLUTION

(a) $x = \frac{3 \pm \sqrt{9-4(4)(-1)}}{2(4)} = \frac{3 \pm \sqrt{25}}{8} = 1$ or $-\frac{1}{4}$

(b) $x = \frac{2 \pm \sqrt{4-(4)(1)(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$

In Exercises 38–45, complete the square and find the minimum or maximum value of the quadratic function.

38. $y = x^2 + 2x + 5$

SOLUTION $y = x^2 + 2x + 1 - 1 + 5 = (x+1)^2 + 4$; therefore, the minimum value of the quadratic polynomial is 4, and this occurs at $x = -1$.

39. $y = x^2 - 6x + 9$

SOLUTION $y = (x-3)^2$; therefore, the minimum value of the quadratic polynomial is 0, and this occurs at $x = 3$.

40. $y = -9x^2 + x$

SOLUTION $y = -9(x^2 - x/9) = -9(x^2 - \frac{x}{9} + \frac{1}{324}) + \frac{9}{324} = -9(x - \frac{1}{18})^2 + \frac{1}{36}$; therefore, the maximum value of the quadratic polynomial is $\frac{1}{36}$, and this occurs at $x = \frac{1}{18}$.

41. $y = x^2 + 6x + 2$

SOLUTION $y = x^2 + 6x + 9 - 9 + 2 = (x+3)^2 - 7$; therefore, the minimum value of the quadratic polynomial is -7, and this occurs at $x = -3$.

42. $y = 2x^2 - 4x - 7$

SOLUTION $y = 2(x^2 - 2x + 1 - 1) - 7 = 2(x^2 - 2x + 1) - 7 - 2 = 2(x-1)^2 - 9$; therefore, the minimum value of the quadratic polynomial is -9, and this occurs at $x = 1$.

43. $y = -4x^2 + 3x + 8$

SOLUTION $y = -4x^2 + 3x + 8 = -4(x^2 - \frac{3}{4}x + \frac{9}{64}) + 8 + \frac{9}{16} = -4(x - \frac{3}{8})^2 + \frac{137}{16}$; therefore, the maximum value of the quadratic polynomial is $\frac{137}{16}$, and this occurs at $x = \frac{3}{8}$.

44. $y = 3x^2 + 12x - 5$

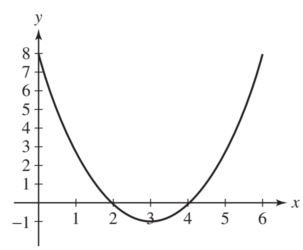
SOLUTION $y = 3(x^2 + 4x + 4) - 5 - 12 = 3(x+2)^2 - 17$; therefore, the minimum value of the quadratic polynomial is -17, and this occurs at $x = -2$.

45. $y = 4x - 12x^2$

SOLUTION $y = -12(x^2 - \frac{x}{3}) = -12(x^2 - \frac{x}{3} + \frac{1}{36}) + \frac{1}{3} = -12(x - \frac{1}{6})^2 + \frac{1}{3}$; therefore, the maximum value of the quadratic polynomial is $\frac{1}{3}$, and this occurs at $x = \frac{1}{6}$.

46. Sketch the graph of $y = x^2 - 6x + 8$ by plotting the roots and the minimum point.

SOLUTION $y = x^2 - 6x + 9 - 9 + 8 = (x-3)^2 - 1$ so the vertex is located at $(3, -1)$ and the roots are $x = 2$ and $x = 4$. This is the graph of x^2 moved right 3 units and down 1 unit.



47. Sketch the graph of $y = x^2 + 4x + 6$ by plotting the minimum point, the y-intercept, and one other point.

SOLUTION $y = x^2 + 4x + 4 - 4 + 6 = (x+2)^2 + 2$ so the minimum occurs at $(-2, 2)$. If $x = 0$, then $y = 6$ and if $x = -4$, $y = 6$. This is the graph of x^2 moved left 2 units and up 2 units.