2 LIMITS AND THE DERIVATIVE

EXERCISE 2-1

2. $x^2 - 64 = (x-8)(x+8)$

4. $x^2 + 5x - 36 = (x+9)(x-4)$

6. $x^3 + 15x^2 + 50x = x(x^2 + 15x + 50) = x(x+5)(x+10)$

8. $20x^2 + 11x - 3 = (4x+3)(5x-1)$

10. $f(-1.5) = 2$

12. $f(1.25) = 1.75$

14. (A) $\lim_{x\to 1^-} f(x) = 2$ (B) $\lim_{x\to 1^+} f(x) = 2$ (C) $\lim_{x\to 1} f(x) = 2$ (D) $f(1) = 2$

16. (A) $\lim_{x\to 4^-} f(x) = 4$ (B) $\lim_{x\to 4^+} f(x) = 4$ (C) $\lim_{x\to 4} f(x) = 4$
(D) $f(4)$ does not exist (E) Yes, define $f(4) = 4$

18. $g(0.1) = 1$

20. $g(2.5) = 1.5$

22. (A) $\lim_{x\to 2^-} g(x) = 2$ (B) $\lim_{x\to 2^+} g(x) = 2$ (C) $\lim_{x\to 2} g(x) = 2$ (D) $g(2) = 2$

24. (A) $\lim_{x\to 4^-} g(x) = 0$ (B) $\lim_{x\to 4^+} g(x) = 0$ (C) $\lim_{x\to 4} g(x) = 0$ (D) $g(4) = 0$

26. (A) $\lim_{x\to -2^+} f(x) = 3$ (B) $\lim_{x\to -2^-} f(x) = -3$
(C) Since $\lim_{x\to -2^+} f(x) \neq \lim_{x\to -2^-} f(x)$, $\lim_{x\to -2} f(x)$ does not exist.
(D) $f(-2) = -3$
(E) No, the limit does not exist, so it cannot be equal to any possible value of $f(-2)$.

28. (A) $\lim_{x\to 2^+} f(x) = -3$ (B) $\lim_{x\to 2^-} f(x) = 3$
(C) $\lim_{x\to 2} f(x)$ does not exist since $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$
(D) $f(2) = 3$ (E) No, $\lim_{x\to 2} f(x)$ does not exist.

30. $3x \to -6$ as $x \to -2$; thus $\lim_{x\to -2} 3x = -6$

32. $x - 3 \to 5 - 3 = 2$ as $x \to 5$; thus $\lim_{x\to 5} (x - 3) = 2$

34. $x(x + 3) \to (-1)(-1 + 3) = -2$ as $x \to -1$; thus $\lim_{x\to -1} x(x + 3) = -2$

36. $x - 2 \to 4 - 2 = 2$ as $x \to 4$; thus $\lim_{x\to 4} \frac{x-2}{x} = \frac{2}{4} = \frac{1}{2}$

38. $\sqrt{16-7x} \to \sqrt{16-7(0)} = \sqrt{16} = 4$ as $x \to 0$; thus $\lim_{x\to 0} \sqrt{16-7x} = 4$

40. $\lim_{x\to 1} 2g(x) = 2\lim_{x\to 1} g(x) = 2(4) = 8$

42. $\lim_{x\to 1}[g(x) - 3f(x)] = \lim_{x\to 1} g(x) - 3\lim_{x\to 1} f(x) = 4 - 3(-5) = 19$

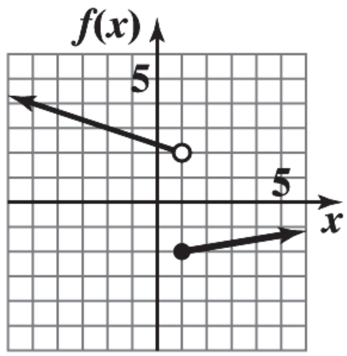
44. $\lim_{x\to 1} \frac{3-f(x)}{1-4g(x)} = \frac{\lim_{x\to 1}[3-f(x)]}{\lim_{x\to 1}[1-4g(x)]} = \frac{3-\lim_{x\to 1} f(x)}{1-4\lim_{x\to 1} g(x)} = \frac{3-(-5)}{1-4(4)} = -\frac{8}{15}$

46. $\lim_{x\to 1} \sqrt[3]{2x+2f(x)} = \sqrt[3]{\lim_{x\to 1}[2x+2f(x)]}$

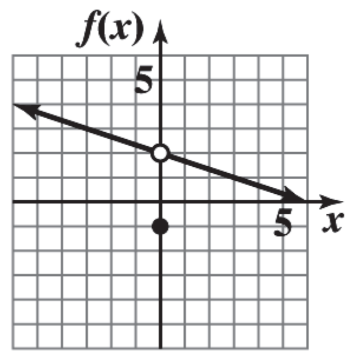
$= \sqrt[3]{2\lim_{x\to 1} x + 2\lim_{x\to 1} f(x)}$

$= \sqrt[3]{2-10} = -2$

48.



50.



52. $f(x) = \begin{cases} 2+x & \text{if } x \le 0 \\ 2-x & \text{if } x > 0 \end{cases}$

(A) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2-x) = 2$

(B) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2+x) = 2$

(C) $\lim_{x\to 0} f(x) = 2$ since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 2$

(D) $f(0) = 2 + 0 = 2$

54. $f(x) = \begin{cases} x+3 & \text{if } x < -2 \\ \sqrt{x+2} & \text{if } x > -2 \end{cases}$

(A) $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \sqrt{x+2} = 0$

(B) $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (x+3) = 1$

(C) $\lim_{x\to -2} f(x)$ does not exist since $\lim_{x\to -2^+} f(x) \ne \lim_{x\to -2^-} f(x)$

(D) $f(-2)$ does not exist; f is not defined at $x = -2$.

56. $f(x) = \begin{cases} \frac{x}{x+3} & \text{if } x < 0 \\ \frac{x}{x-3} & \text{if } x > 0 \end{cases}$

(A) $\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{x}{x+3}$ does not exist since $x = -3$ is a non-removable zero of the denominator.

(B) $\lim_{x\to 0} f(x) = \lim_{x\to 0^-} \frac{x}{x+3} = \lim_{x\to 0^+} \frac{x}{x+3} = 0$

(C) $\lim_{x\to 3} f(x)$ does not exist, since $\lim_{x\to 3^+} f(x)$ does not exist.

58. $f(x) = \dfrac{x-3}{|x-3|} = \begin{cases} \dfrac{x-3}{-(x-3)} = -1 & \text{if } x < 3 \\ \dfrac{x-3}{x-3} = 1 & \text{if } x > 3 \end{cases}$

(Note: Observe that for $x < 3$, $|x-3| = 3 - x = -(x-3)$ and for $x > 3$, $|x-3| = x-3$)

(A) $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} 1 = 1$ (B) $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (-1) = -1$

(C) $\lim\limits_{x\to 3} f(x)$ does not exist, since $\lim\limits_{x\to 3^+} f(x) \neq \lim\limits_{x\to 3^-} f(x)$

(D) $f(3)$ does not exist; f is not defined at $x = 3$.

60. $f(x) = \dfrac{x+3}{x^2+3x} = \dfrac{x+3}{x(x+3)}$

(A) $\lim\limits_{x\to -3} \dfrac{x+3}{x(x+3)} = \lim\limits_{x\to -3} \dfrac{1}{x} = -\dfrac{1}{3}$ (B) $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{1}{x}$ does not exist. (C) $\lim\limits_{x\to 3} \dfrac{1}{x} = \dfrac{1}{3}$

62. $f(x) = \dfrac{x^2+x-6}{x+3} = \dfrac{(x+3)(x-2)}{(x+3)}$

(A) $\lim\limits_{x\to -3} f(x) = \lim\limits_{x\to -3} \dfrac{(x+3)(x-2)}{(x+3)} = \lim\limits_{x\to -3} (x-2) = -5$

(B) $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{x^2+x-6}{x+3} = \dfrac{-6}{3} = -2$

(C) $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} \dfrac{x^2+x-6}{x+3} = \dfrac{0}{5} = 0$

64. $f(x) = \dfrac{x^2-1}{(x+1)^2} = \dfrac{(x-1)(x+1)}{(x+1)^2}$

(A) $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} \dfrac{(x-1)(x+1)}{(x+1)^2} = \lim\limits_{x\to -1} \dfrac{x-1}{x+1}$ does not exist since $\lim\limits_{x\to -1} (x-1) = -2$ but $\lim\limits_{x\to -1} (x+1) = 0$.

(B) $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{x^2-1}{(x+1)^2} = \dfrac{-1}{1} = -1$ (C) $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{x^2-1}{(x+1)^2} = \dfrac{0}{4} = 0$

66. $f(x) = \dfrac{3x^2+2x-1}{x^2+3x+2} = \dfrac{(3x-1)(x+1)}{(x+2)(x+1)}$

(A) $\lim\limits_{x\to -3} f(x) = \lim\limits_{x\to -3} \dfrac{3x^2+2x-1}{x^2+3x+2} = \dfrac{20}{2} = 10$

(B) $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} \dfrac{(3x-1)(x+1)}{(x+2)(x+1)} = \lim\limits_{x\to -1} \dfrac{3x-1}{x+2} = \dfrac{-4}{1} = -4$

(C) $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} \dfrac{3x^2+2x-1}{x^2+3x+2} = \dfrac{15}{12} = \dfrac{5}{4}$

68. True: $\lim_{x\to 1}\frac{f(x)}{g(x)}=\frac{\lim_{x\to 1}f(x)}{\lim_{x\to 1}g(x)}=\frac{1}{1}=1$

70. Not always true. For example, the statement is false for $f(x)=\frac{1}{x}$.

72. Not always true. For example, the statement is false for $f(x)=\begin{cases}-1 & x\le 0\\ 1 & x>0\end{cases}$

74. $\lim_{x\to 2}\frac{x-5}{x+2}$ does not have the form $\frac{0}{0}$; $\lim_{x\to 2}\frac{x-5}{x+2}=\frac{-3}{4}$.

76. $\lim_{x\to 9}\frac{x^2-5x-36}{x-9}$ has the form $\frac{0}{0}$; $\frac{x^2-5x-36}{x-9}=\frac{(x-9)(x+4)}{x-9}=x+4$, provided $x\ne 9$.

Therefore $\lim_{x\to 9}\frac{x^2-5x-36}{x-9}=\lim_{x\to 9}(x+4)=13$.

78. $\lim_{x\to 10}\frac{x^2-15x+50}{(x-10)^2}$ has the form $\frac{0}{0}$; $\frac{x^2-15x+50}{(x-10)^2}=\frac{(x-5)(x-10)}{(x-10)^2}=\frac{x-5}{x-10}$, provided $x\ne 10$.

Therefore $\lim_{x\to 10}\frac{x^2-15x+50}{(x-10)^2}$ does not exist.

80. $\lim_{x\to -3}\frac{x+3}{x-3}$ does not have the form $\frac{0}{0}$; $\lim_{x\to -3}\frac{x+3}{x-3}=\frac{0}{-6}=0$.

82. $f(x) = 5x - 1$

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{5(2+h)-1-(10-1)}{h}=\lim_{h\to 0}\frac{10+5h-1-9}{h}=\lim_{h\to 0}\frac{5h}{h}=\lim_{h\to 0}5=5$$

84. $f(x) = x^2 - 2$

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{(2+h)^2-2-(4-2)}{h}=\lim_{h\to 0}\frac{4+4h+h^2-2-2}{h}$$

$$=\lim_{h\to 0}\frac{4h+h^2}{h}=\lim_{h\to 0}(4+h)=4$$

86. $f(x) = -4x + 13$

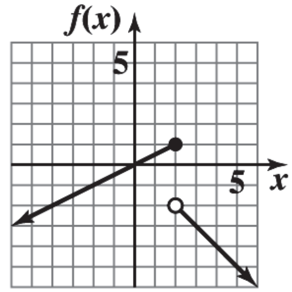
$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{-4(2+h)+13-[-4(2)+13]}{h}=\lim_{h\to 0}\frac{-4h}{h}=-4$$

88. $f(x) = -3|x|$

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{-3|2+h|-[-3(2)]}{h}=\lim_{h\to 0}\frac{-3(2+h)+6}{h}=\lim_{h\to 0}\frac{-3h}{h}=-3$$

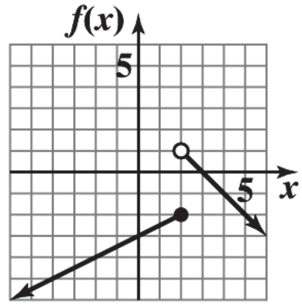
90. (A) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (0.5x) = 1$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-x) = -2$



(B) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (-3 + 0.5x) = -2$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3 - x) = 1$



(C) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (-3m + 0.5x) = -3m + 1$

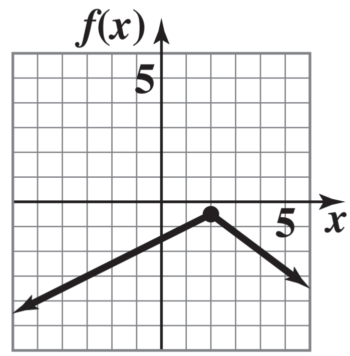
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3m - x) = 3m - 2$

$-3m + 1 = 3m - 2$

$6m = 3$

$m = \frac{1}{2} = 0.5$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = -0.5$

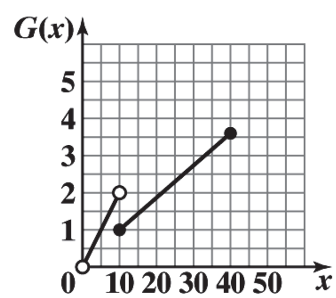


(D) The graph in (A) is broken when it jumps from (2, 1) down to (2, –2), the graph in (B) is also broken when it jumps from (2, –2) up to (2, 1), while the graph in (C) is one continuous piece with no jumps or breaks.

92. (A) If a state-to-state long distance call lasts x minutes, then for $0 < x < 10$, the charge will be $0.18x$ and for $x \geq 10$, the charge will be $0.09x$. Thus,

$$G(x) = \begin{cases} 0.18x, & 0 < x < 10 \\ 0.09x, & x \geq 10 \end{cases}$$

(B)



(C) As x approaches 10 from the left, $G(x)$ approaches 1.8, thus, the left limit of $G(x)$ at $x = 10$ exists, $\lim_{x\to 10^-} G(x) = 1.8$.

Similarly, $\lim_{x\to 10^+} G(x) = 0.90$. However, $\lim_{x\to 10} G(x)$ does not exist, since $\lim_{x\to 10^-} G(x) \neq \lim_{x\to 10^+} G(x)$.

94. For calls lasting more than 20 minutes, the charge for the service given in Problem 91 is $0.07x - 0.41$ whereas for that of Problem 92 is $0.09x$. It is clear that the latter is more expensive than the former.

96. (A) Let x be the volume of a purchase before the discount is applied. Then $P(x)$ is given by:

$$P(x) = \begin{cases} x & \text{if} \quad 0 \le x < 300 \\ 300 + 0.97(x - 300) = 0.97x + 9 & \text{if} \quad 300 \le x < 1{,}000 \\ 0.97(1{,}000) + 9 + 0.95(x - 1{,}000) = 0.95x + 29 & \text{if} \quad 1{,}000 \le x < 3{,}000 \\ 0.95(3{,}000) + 29 + 0.93(x - 3{,}000) = 0.93x + 89 & \text{if} \quad 3{,}000 \le x < 5{,}000 \\ 0.93(5{,}000) + 89 + 0.90(x - 5{,}000) = 0.90x + 239 & \text{if} \quad x \ge 5{,}000 \end{cases}$$

(B) $\lim_{x \to 1{,}000^-} P(x) = 0.97(1{,}000) + 9 = 979$

$\lim_{x \to 1{,}000^+} P(x) = 0.95(1{,}000) + 29 = 979$

Thus, $\lim_{x \to 1{,}000} P(x) = 979$

$\lim_{x \to 3{,}000^-} P(x) = 0.95(3{,}000) + 29 = 2{,}879$

$\lim_{x \to 3{,}000^+} P(x) = 0.93(3{,}000) + 89 = 2{,}879$

Thus, $\lim_{x \to 3{,}000} P(x) = 2{,}879$

(C) For $0 \le x < 300$, they produce the same price. For $x \ge 300$, the one in Problem 95 produces a lower price.

98. From Problem 97, we have:

$$F(x) = \begin{cases} 20x & \text{if} \quad 0 < x \le 4{,}000 \\ 80{,}000 & \text{if} \quad x > 4{,}000 \end{cases}$$

Thus

$$A(x) = \frac{F(x)}{x} = \begin{cases} 20 & \text{if} \quad 0 < x \le 4{,}000 \\ \dfrac{80{,}000}{x} & \text{if} \quad x > 4{,}000 \end{cases}$$

$$\lim_{x \to 4{,}000^-} A(x) = \lim_{x \to 4{,}000^+} A(x) = 20 = \lim_{x \to 4{,}000} A(x)$$

$$\lim_{x \to 8{,}000^-} A(x) = \lim_{x \to 8{,}000^+} A(x) = \frac{80{,}000}{8{,}000} = 10 = \lim_{x \to 8{,}000} A(x)$$

EXERCISE 2-2

2. $x = 5$

4. $y = 1$

6. $y + 4 = -3(x - 8)$ (point-slope form); $3x + y = 20$

8. Slope: $m = \dfrac{30 - 20}{1 - (-1)} = 5$; $y - 20 = 5[x - (-1)]$ (point-slope form); $-5x + y = 25$

10. $\lim_{x \to -\infty} f(x) = \infty$

12. $\lim_{x \to -2^-} f(x) = \infty$

14. $\lim_{x \to 2^+} f(x) = \infty$

16. $\lim_{x \to 2} f(x)$ does not exist

18. $f(x) = \dfrac{x^2}{x+3}$

(A) $\lim_{x\to -3^-} \dfrac{x^2}{x+3} = -\infty$; as x approaches –3 from the left, the denominator is negatively approaching 0 and the numerator is positively approaching $(-3)^2 = 9$.

(B) $\lim_{x\to -3^+} \dfrac{x^2}{x+3} = \infty$; numerator approaches $(-3)^2 = 9$ and denominator is positively approaching 0.

(C) Since left and right limits at –3 are not equal, $\lim_{x\to -3} f(x)$ does not exist.

20. $f(x) = \dfrac{2x+2}{(x+2)^2}$

(A) $\lim_{x\to -2^-} \dfrac{2x+2}{(x+2)^2} = -\infty$; as x approaches –2 from the left, the denominator is positively approaching 0 and the numerator is negatively approaching $2(-2) + 2 = -2$.

(B) $\lim_{x\to -2^+} \dfrac{2x+2}{(x+2)^2} = -\infty$; as x approaches –2 from the right, the denominator is positively approaching 0 and the numerator is negatively approaching $2(-2) + 2 = -2$.

(C) Since $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) = -\infty$, we can say that $\lim_{x\to -2} f(x) = -\infty$.

22. $f(x) = \dfrac{x^2+x+2}{x-1}$

(A) $\lim_{x\to 1^-} \dfrac{x^2+x+2}{x-1} = -\infty$; as x approaches 1, the numerator approaches 4 and the denominator negatively approaches 0.

(B) $\lim_{x\to 1^+} \dfrac{x^2+x+2}{x-1} = \infty$; in this case the denominator positively approaches 0.

(C) $\lim_{x\to 1} \dfrac{x^2+x+2}{x-1}$ does not exist.

24. $f(x) = \dfrac{x^2+x-2}{(x+2)}$

$f(x) = \dfrac{(x-1)(x+2)}{(x+2)}$

(A) $\lim_{x\to -2^-} \dfrac{(x-1)(x+2)}{(x+2)} = \lim_{x\to -2^-} (x-1) = -3$

(B) $\lim_{x\to -2^+} \dfrac{(x-1)(x+2)}{(x+2)} = \lim_{x\to -2^+} (x-1) = -3$

(C) $\lim_{x\to -2} \dfrac{(x-1)(x+2)}{(x+2)} = \lim_{x\to -2} (x-1) = -3$ or we can say that left and right limits at $x = -2$ exist and are equal, therefore $\lim_{x\to -2} f(x)$ exists and is equal to the common value –3.

26. $p(x) = 10 - x^6 + 7x^3 = -x^6 + 7x^3 + 10$

(A) Leading term: $-x^6$ (B) $\lim_{x\to\infty} p(x) = \lim_{x\to\infty}(-x^6) = -\infty$ (C) $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty}(-x^6) = -\infty$

28. $p(x) = -x^5 + 2x^3 + 9x$

(A) Leading term: $-x^5$ (B) $\lim_{x\to\infty} p(x) = \lim_{x\to\infty}(-x^5) = -\infty$ (C) $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty}(-x^5) = \infty$

30. $p(x) = 5x + x^3 - 8x^2 = x^3 - 8x^2 + 5x$

(A) Leading term: x^3 (B) $\lim_{x\to\infty} p(x) = \lim_{x\to\infty}(x^3) = \infty$ (C) $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty}(x^3) = -\infty$

32. $p(x) = 1 + 4x^2 + 4x^4 = 4x^4 + 4x^2 + 1$

(A) Leading term: $4x^4$ (B) $\lim_{x\to\infty} p(x) = \lim_{x\to\infty}(4x^4) = \infty$ (C) $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty}(4x^4) = \infty$

34. $g(x) = \dfrac{x}{4-x}$.

Note that $n(4) = 4$, $d(4) = 0$, so $x = 4$ is a point of discontinuity (in fact it is the only point at which $g(x)$ is discontinuous.)

As x approaches 4 from the left, $g(x)$ approaches ∞ and as x approaches 4 from the right, $g(x)$ approaches $-\infty$.

The only vertical asymptote is $x = 4$.

36. $k(x) = \dfrac{x^2-9}{x^2+9}$.

$k(x)$ does not have any discontinuity point since $d(x) > 0$ for all x and hence no vertical asymptotes.

38. $G(x) = \dfrac{x^2+9}{9-x^2}$

$G(x) = \dfrac{x^2+9}{(3-x)(3+x)}$

Since $n(-3) = (-3)^2 + 9 = 18$, $n(3) = (3)^2 + 9 = 18$;

$d(-3) = d(3) = 0$. $G(x)$ is discontinuous at $x = -3$ and at $x = 3$.

$\lim_{x\to-3^-} G(x) = -\infty$, $\lim_{x\to-3^+} G(x) = \infty$, $\lim_{x\to3^-} G(x) = \infty$, $\lim_{x\to3^+} G(x) = -\infty$

Vertical asymptotes: $x = -3$, $x = 3$.

40. $K(x) = \dfrac{x^2+2x-3}{x^2-4x+3} = \dfrac{(x+3)(x-1)}{(x-3)(x-1)}$. Discontinuous at $x = 1$ and $x = 3$.

$\lim_{x\to1} K(x) = \lim_{x\to1} \dfrac{x+3}{x-3} = -2$

$\lim_{x\to3^-} K(x) = -\infty$, $\lim_{x\to3^+} K(x) = \infty$

Vertical asymptote: $x = 3$

42. $S(x) = \dfrac{6x+9}{x^4+6x^3+9x^2}$.

$S(x) = \dfrac{3(2x+3)}{x^2(x^2+6x+9)} = \dfrac{3(2x+3)}{x^2(x+3)^2}$

$n(0) = 9$, $n(-3) = 3(-6+3) = -9$, $d(0) = 0$, $d(-3) = 0$, so $S(x)$ is discontinuous at $x = -3$ and at $x = 0$.

$\lim_{x\to 0^-} S(x) = \infty$, $\lim_{x\to 0^+} S(x) = \infty$, $\lim_{x\to 0} S(x) = \infty$

$\lim_{x\to -3^-} S(x) = -\infty$, $\lim_{x\to -3^+} S(x) = -\infty$, $\lim_{x\to -3} S(x) = -\infty$

Vertical asymptotes: $x = -3$, $x = 0$.

44. (A) $f(5) = \dfrac{2-3(5)^3}{7+4(5)^3} = -\dfrac{373}{507} \approx -0.736$

(B) $f(10) = \dfrac{2-3(10)^3}{7+4(10)^3} = -\dfrac{2{,}998}{4{,}007} \approx -0.748$

(C) $\lim_{x\to\infty} \dfrac{2-3x^3}{7+4x^3} = \lim_{x\to\infty} \dfrac{-3x^3}{4x^3} = \lim_{x\to\infty} \dfrac{\frac{2}{x^3}-3}{\frac{7}{x^3}+4}$ (Divide numerator and denominator by x^3.)

$= \dfrac{0-3}{0+4} = \dfrac{-3}{4}$.

46. (A) $f(-8) = \dfrac{5(-8)+11}{7(-8)^3-2} = \dfrac{-29}{-3{,}586} = \dfrac{29}{3{,}586} \approx 0.008$

(B) $f(-16) = \dfrac{5(-16)+11}{7(-16)^3-2} = \dfrac{-69}{-28{,}674} = \dfrac{69}{28{,}674} \approx 0.002$

(C) $\lim_{x\to\infty} \dfrac{5x+11}{7x^3-2} = \lim_{x\to\infty} \dfrac{\frac{5}{x^2}+\frac{11}{x^3}}{7-\frac{2}{x^3}}$ (Divide numerator and denominator by x^3.)

$= \dfrac{0+0}{7-0} = 0$

48. (A) $f(-3) = \dfrac{4(-3)^7-8(-3)}{6(-3)^4+9(-3)^2} = -\dfrac{8{,}724}{567} \approx -15.386$

(B) $f(-6) = \dfrac{4(-6)^7-8(-6)}{6(-6)^4+9(-6)^2} = -\dfrac{1{,}119{,}696}{8{,}100} \approx -138.234$

(C) $\lim_{x\to-\infty} \dfrac{4x^7-8x}{6x^4+9x^2} = \lim_{x\to-\infty} \dfrac{4x^3-\frac{8}{x^3}}{6+\frac{9}{x^2}}$ (Divide numerator and denominator by x^4.)

As $x \to -\infty$, $4x^3 - \dfrac{8}{x^3} \to -\infty$ and $6 + \dfrac{9}{x^2} \to 6$. Therefore, $\lim_{x\to-\infty} \dfrac{4x^7-8x}{6x^4+9x^2} = -\infty$.

50. (A) $f(-50)=\dfrac{3+(-50)}{5+4(-50)}=\dfrac{47}{195}\approx 0.241$

(B) $f(-100)=\dfrac{3+(-100)}{5+4(-100)}=\dfrac{97}{395}\approx 0.246$

(C) $\lim\limits_{x\to-\infty}\dfrac{3+x}{5+4x}=\lim\limits_{x\to\infty}\dfrac{\frac{3}{x}+1}{\frac{5}{x}+4}$ (Divide numerator and denominator by x.)

$=\dfrac{0+1}{0+4}=\dfrac{1}{4}$

52. $f(x)=\dfrac{3x+2}{x-4}$

$\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\dfrac{3x+2}{x-4}=\lim\limits_{x\to\infty}\dfrac{3+\frac{2}{x}}{1-\frac{4}{x}}=\dfrac{3+0}{1-0}=3$

So $y=3$ is the horizontal asymptote.
Vertical asymptote: $x=4$ (since $n(4)=14$, $d(4)=0$).

54. $f(x)=\dfrac{x^2-1}{x^2+2}$.

$\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\dfrac{x^2-1}{x^2+2}=\lim\limits_{x\to\infty}\dfrac{1-\frac{1}{x^2}}{1+\frac{2}{x^2}}$ (Dividing the numerator and denominator by x^2.)

$=\dfrac{1-0}{1+0}=1$

So, the horizontal asymptote is: $y=1$.
$d(x)=x^2+2>0$ so, there are no vertical asymptotes.

56. $f(x)=\dfrac{x}{x^2-4}=\dfrac{x}{(x-2)(x+2)}$

$\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\dfrac{x}{x^2-4}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}}{1-\frac{4}{x^2}}=\dfrac{0}{1-0}=0,$

so the horizontal asymptote is: $y=0$.
Since $n(-2)=-2$, $n(2)=2$, $d(-2)=d(2)=0$, we have two vertical asymptotes: $x=-2$, $x=2$.

58. $f(x)=\dfrac{x^2+9}{x}$

$\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\dfrac{x^2+9}{x}=\lim\limits_{x\to\infty}\dfrac{1+\frac{9}{x^2}}{\frac{1}{x}}=\dfrac{1+0}{0}=\infty$

So, there are no horizontal asymptotes. Since $n(0)=9$, $d(0)=0$,
$x=0$ is the only vertical asymptote.

60. $f(x) = \dfrac{x+5}{x^2}$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+5}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{5}{x^2}}{1} = \frac{0+0}{1} = 0,$$

so the horizontal asymptote is: $y = 0$.

Since $n(0) = 5$, $d(0) = 0$, $x = 0$ is the vertical asymptote.

62. $f(x) = \dfrac{2x^2+7x+12}{2x^2+5x-12}$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2+7x+12}{2x^2+5x-12} = \lim_{x\to\infty} \frac{2x^2}{2x^2} = 1,$$

so, $y = 1$ is the horizontal asymptote.

Since $n(-4) = 16$, $n\left(\frac{3}{2}\right) = 27$, $d(-4) = d\left(\frac{3}{2}\right) = 0$, $x = -4$ and $x = \frac{3}{2}$ are the vertical asymptotes.

64. $f(x) = \dfrac{x^2-x-12}{2x^2+5x-12}$

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x^2-x-12}{2x^2+5x-12} = \lim\limits_{x\to\infty} \dfrac{x^2}{2x^2} = \dfrac{1}{2}$, so $y = \dfrac{1}{2}$ is the horizontal asymptote. Since $n(-4) = 8$, $n\left(\frac{3}{2}\right) = -11.25$, $d(-4) = d\left(\frac{3}{2}\right) = 0$, $x = -4$ and $x = \frac{3}{2}$ are the vertical asymptotes.

66. $f(x) = \dfrac{3+4x+x^2}{5-x}$; $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{3+4x+x^2}{5-x} = \lim\limits_{x\to\infty} \dfrac{x^2}{-x} = \lim\limits_{x\to\infty} (-x) = -\infty$

68. $f(x) = \dfrac{4x+1}{5x-7}$; $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{4x+1}{5x-7} = \lim\limits_{x\to\infty} \dfrac{4x}{5x} = \dfrac{4}{5}$

70. $f(x) = \dfrac{2x+3}{x^2-1}$; $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{2x+3}{x^2-1} = \lim\limits_{x\to-\infty} \dfrac{2x}{x^2} = 0$

72. $f(x) = \dfrac{6-x^4}{1+2x}$; $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{6-x^4}{1+2x} = \lim\limits_{x\to-\infty} \dfrac{-x^4}{2x} = \infty$

74. $f(x) = 4-5x-x^3$;

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} (4-5x-x^3) = \lim\limits_{x\to\infty} (-x^3) = -\infty$; $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} (4-5x-x^3) = \lim\limits_{x\to-\infty} (-x^3) = \infty$

76. $f(x) = \dfrac{9x^2+6x+1}{4x^2+4x+1}$;

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{9x^2+6x+1}{4x^2+4x+1} = \lim\limits_{x\to\infty} \dfrac{9x^2}{4x^2} = \dfrac{9}{4}$; $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{9x^2+6x+1}{4x^2+4x+1} = \lim\limits_{x\to-\infty} \dfrac{9x^2}{4x^2} = \dfrac{9}{4}$

78. False: $f(x) = \dfrac{1}{(x-2)(x+2)} = \dfrac{1}{x^2-4}$ has two vertical asymptotes.

80. True: Theorem 4 gives three possible cases, two of which give exactly one horizontal asymptote and one of which gives no horizontal asymptote.

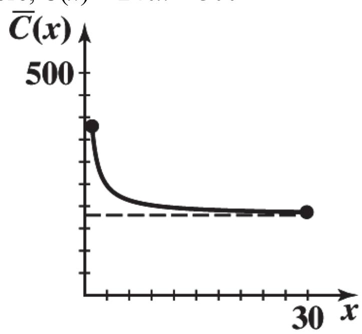
82. False: $f(x) = \dfrac{x^2 + 2x}{x^2 + x + 2}$ crosses the horizontal asymptote $y = 1$ at $x = 2$.

84. $\lim_{x\to-\infty} (a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0) = \infty$ if $a_n > 0$ and n an even positive integer, or $a_n < 0$ and n an odd positive integer.
$\lim_{x\to-\infty} (a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0) = -\infty$ if $a_n > 0$ and n is an odd positive integer or $a_n < 0$ and n is an even positive integer.

86. (A) Since $C(x)$ is a linear function of x, it can be written in the form
$C(x) = mx + b$
Since the fixed costs are \$300, $b = 300$.
Also, $C(20) = 5100$, so
$5100 = m(20) + 300$
$20m = 4800$
$m = 240$
Therefore, $C(x) = 240x + 300$

(B) $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{240x + 300}{x}$

(C)



(D) $\overline{C}(x) = \dfrac{240x + 300}{x} = \dfrac{240 + \dfrac{300}{x}}{1}$

As x increases, the numerator tends to 240 and the denominator is 1. Therefore, $\overline{C}(x)$ tends to 240 or \$240 per board. Therefore, $\overline{C}(x)$ tends to \$240 per board.

88. (A) $C_e(x) = 4{,}000 + 932x$; $\overline{C}_e(x) = \dfrac{4{,}000}{x} + 932$

(B) $C_c(x) = 2{,}700 + 1{,}332x$; $\overline{C}_c(x) = \dfrac{2{,}700}{x} + 1{,}332$

(C) $C_e(x) = C_c(x)$ implies that $4{,}000 + 932x = 2{,}700 + 1{,}332x$ or $400x = 1300$ and

$x = \dfrac{1{,}300}{400} = 3.25$ years

(D) $\overline{C}_e(x) = \overline{C}_c(x)$ implies that

$$\frac{4{,}000}{x} + 932 = \frac{2{,}700}{x} + 1{,}332$$

Multiplying both sides by x results in the same equation as in (C) and hence the answer is $x = 3.25$ years.

(E) $\lim_{x\to\infty} \overline{C}_e(x) = \lim_{x\to\infty} \left(\dfrac{4{,}000}{x} + 932\right) = 0 + 932 = 932$

$\lim_{x\to\infty} \overline{C}_c(x) = \lim_{x\to\infty} \left(\dfrac{2{,}700}{x} + 1{,}332\right) = 0 + 1{,}332 = 1{,}332$

90. $C(t) = \dfrac{5t(t+50)}{t^3+100}$

$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \dfrac{5t^2+250t}{t^3+100}$ (Divide numerator and denominator by t^3.)

$= \lim_{t\to\infty} \dfrac{\frac{5}{t}+\frac{250}{t^2}}{1+\frac{100}{t^3}} = \dfrac{0+0}{1+0} = 0$

The long term drug concentration is 0 mg/ml.

92. $N(t) = \dfrac{100t}{t+9}$, $t \geq 0$

(A) $N(6) = \dfrac{100(6)}{6+9} = \dfrac{600}{15} \approx 40$ components/day

(B) $70 = \dfrac{100t}{t+9}$ or

$70t + 630 = 100t$

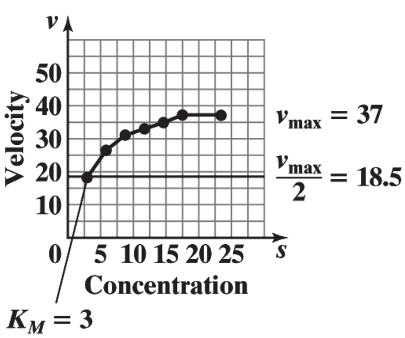
$30t = 630$

$t = \dfrac{630}{30} = 21$ days

(C) $\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \dfrac{100t}{t+9} = \lim_{t\to\infty} \dfrac{100}{1+\frac{9}{t}} = \dfrac{100}{1+0} = 100$

The maximum number of components an employee can produce in consecutive days is 100 components.

94. (A) $v_{max} = 37$, $K_M = 3$



(B) $v(s) = \dfrac{37s}{3+s}$

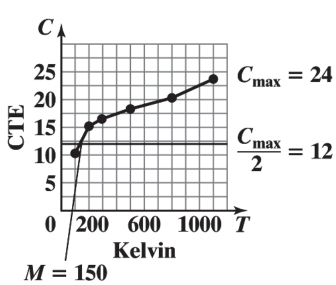
(C) For $s = 9$, $v = \dfrac{37(9)}{3+9} = 27.75$

For $v = 32$, $32 = \dfrac{37s}{3+s}$

or $96 + 32s = 37s$

and $s = \dfrac{96}{5} = 19.2$

96. (A) $C_{max} = 24$, $M = 150$



(B) $C(T) = \dfrac{24T}{150+T}$

(C) For $T = 600$,

$C = \dfrac{(24)(600)}{150+600} = 19.2$

For $C = 12$, $12 = \dfrac{24T}{150+T}$ or

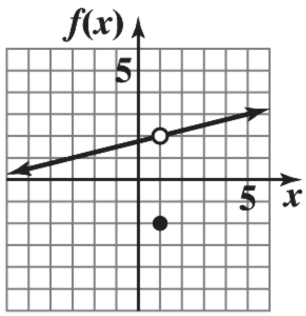
$1800 + 12T = 24T$,

$T = 150$.

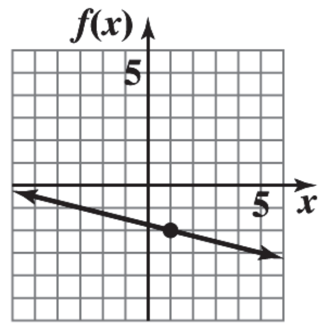
EXERCISE 2-3

2. $(-8,-4]$ **4.** $[0.1, 0.3]$ **6.** $(-\infty,-4]\cup[4,\infty)$ **8.** $(-\infty,-6)\cup[9,\infty)$

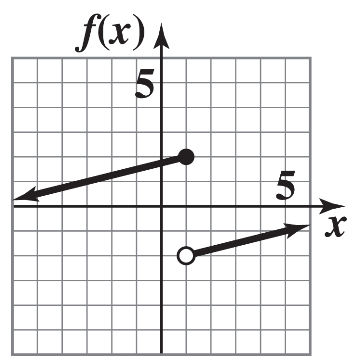
10. f is discontinuous at $x = 1$ since $\lim_{x\to 1} f(x) \neq f(1)$



12. f is discontinuous at $x = 1$ since $\lim_{x\to 1} f(x) = f(1)$



14. f is discontinuous at $x = 1$, since $\lim_{x\to 1} f(x)$ does not exist



16. $f(0.1) = 1.1$

18. $f(-0.9) = 0.1$

20. (A) $\lim_{x\to 2^-} f(x) = 2$ (B) $\lim_{x\to 2^+} f(x) = 2$ (C) $\lim_{x\to 2} f(x) = 2$

$\left(\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 2\right)$

(D) $f(2)$ does not exist; f is not defined at $x = 2$. (E) No, since f is not even defined at $x = 2$.

22. (A) $\lim_{x\to -1^-} f(x) = 0$ (B) $\lim_{x\to -1^+} f(x) = 0$

(C) $\lim_{x\to -1} f(x) = 0$ $\left(\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = 0\right)$

(D) $f(-1) = 0$ (E) Yes, since $\lim_{x\to -1} f(x) = f(0)$.

24. $g(-2.1)=1.9$

26. $g(-1.9)=3.95$

28. (A) $\lim_{x\to -2^-} g(x)=2$ (B) $\lim_{x\to -2^+} g(x)=4$

(C) $\lim_{x\to -2} g(x)$ does not exist, since $\lim_{x\to -2^-} g(x) \neq \lim_{x\to -2^+} g(x)$

(D) $g(-2)$ does not exist; g is not defined at $x = -2$.

(E) No, since g is not even defined at $x = -2$.

30. (A) $\lim_{x\to 4^-} g(x)=1$ (B) $\lim_{x\to 4^+} g(x)=1$ (C) $\lim_{x\to 4} g(x)=1$

$\left(\lim_{x\to 4^-} g(x) = \lim_{x\to 4^+} g(x) = 1\right)$

(D) $g(4) = 1$.

(E) Yes, since $\lim_{x\to 4} f(x) = f(4)$

32. $h(x) = 4 - 2x$ is a polynomial function. Therefore, f is continuous for all x [Theorem 1(C)].

34. $k(x) = \dfrac{2x}{x-4}$ is a rational function and the denominator $x - 4$ is 0 when $x = 4$. Thus, k is continuous for all x except $x = 4$ [Theorem 1(D)].

36. $n(x) = \dfrac{x-2}{(x-3)(x+1)}$ is a rational function and the denominator $(x - 3)(x + 1)$ is 0 when $x = 3$ or $x = -1$. Thus, n is continuous for all x except $x = 3$, $x = -1$ [Theorem 1(D)].

38. $G(x) = \dfrac{1-x^2}{x^2+1}$

$G(x)$ is a rational function and its denominator is never zero, hence by Theorem 1(D), $G(x)$ is continuous for all x.

40. $N(x) = \dfrac{x^2+4}{4-25x^2}$

$N(x)$ is a rational function and according to Theorem 1(D), $N(x)$ is continuous for all x except $x = \pm\dfrac{2}{5}$ which make the denominator 0.

42. $f(x)=\dfrac{2x+7}{5x-1}$; f is discontinuous at $x=\dfrac{1}{5}$; $f(x)=0$ at $x=\dfrac{-7}{2}$. Partition numbers $\dfrac{1}{5}, \dfrac{-7}{2}$.

44. $f(x)=\dfrac{x^2+4}{x^2-9}$; f is discontinuous at $x=3,-3$; $f(x)\neq 0$ for all x. Partition numbers $3, -3$.

46. $f(x)=\dfrac{x^3+x}{x^2-x-42}=\dfrac{x(x^2+1)}{(x-7)(x+6)}$; f is discontinuous at $x=7,-6$; $f(x)=0$ at $x=0$. Partition numbers $-6, 0, 7$.

48. $x^2 - 2x - 8 < 0$

Let $f(x) = x^2 - 2x - 8 = (x - 4)(x + 2)$.
Then f is continuous for all x and $f(-2) = f(4) = 0$.
Thus, $x = -2$ and $x = 4$ are partition numbers.

Test Numbers

x	$f(x)$
-3	$7(+)$
0	$-8(-)$
5	$7(+)$

Thus, $x^2 - 2x - 8 < 0$ for: $-2 < x < 4$ (inequality notation), $(-2, 4)$ (interval notation)

50. $x^2 + 7x > -10$ or $x^2 + 7x + 10 > 0$

Let $f(x) = x^2 + 7x + 10 = (x + 2)(x + 5)$.
Then f is continuous for all x and $f(-5) = f(-2) = 0$.

Thus, $x = -5$ and $x = -2$ are partition numbers.

Test Numbers

x	$f(x)$
-6	$4(+)$
-4	$-2(-)$
0	$10(+)$

Thus, $x^2 + 7x + 10 > 0$ for: $x < -5$ or $x > -2$ (inequality notation), $(-\infty, -5) \cup (-2, \infty)$ (interval notation)

52. $x^4 - 9x^2 > 0$

$x^4 - 9x^2 = x^2(x^2 - 9)$

Since $x^2 > 0$ for $x \neq 0$, then $x^4 - 9x^2 > 0$ if $x^2 - 9 > 0$ or $x^2 > 9$
or "$x < -3$ or $x > 3$" or $(-\infty, -3) \cup (3, \infty)$.

54. $\dfrac{x-4}{x^2+2x} < 0$

Let $f(x) = \dfrac{x-4}{x^2+2x} = \dfrac{x-4}{x(x+2)}$. Then f is discontinuous at $x = 0$ and
$x = -2$ and $f(4) = 0$. Thus, $x = -2$, $x = 0$, and $x = 4$ are partition numbers.

Test Numbers

x	$f(x)$
-3	$-\frac{7}{3}(-)$
-1	$5(+)$
1	$-1(-)$
5	$\frac{1}{35}(+)$

Thus, $\dfrac{x-4}{x^2+2x} < 0$ for: $x < -2$ or $0 < x < 4$ (inequality notation), $(-\infty, -2) \cup (0, 4)$ (interval notation)

56. (A) $g(x) > 0$ for $x < -4$ or $x > 4$; $(-\infty, -4) \cup (4, \infty)$.

(B) $g(x) < 0$ for $-4 < x < 1$ or $1 < x < 4$; $(-4, 1) \cup (1, 4)$.

58. $f(x) = x^4 - 4x^2 - 2x + 2$. Partition numbers: $x_1 \approx 0.5113$, $x_2 \approx 2.1209$

(A) $f(x) > 0$ on $(-\infty, 0.5113) \cup (2.1209, \infty)$

(B) $f(x) < 0$ on $(0.5113, 2.1209)$

60. $f(x) = \dfrac{x^3 - 5x + 1}{x^2 - 1}$. Partition numbers: $x_1 \approx -2.3301$, $x_2 \approx -1$, $x_3 \approx 0.2016$, $x_4 = 1$, $x_5 \approx 2.1284$

(A) $f(x) > 0$ on $(-2.3301, -1) \cup (0.2016, 1) \cup (2.1284, \infty)$.

(B) $f(x) < 0$ on $(-\infty, -2.3301) \cup (-1, 0.2016) \cup (1, 2.1284)$.

62. $\sqrt{7-x}$

Let $f(x) = 7 - x$. Then $\sqrt{7-x} = \sqrt[2]{f(x)}$ is continuous whenever $f(x)$ is continuous and nonnegative [Theorem 1(F)]. Since $f(x) = 7 - x$ is continuous for all x [Theorem 1(C)] and $f(x) \ge 0$ for $x \le 7$, $\sqrt{7-x}$ is continuous on $(-\infty, 7]$.

64. $\sqrt[3]{x-8}$

Let $f(x) = x - 8$. Then $\sqrt[3]{x-8} = \sqrt[3]{f(x)}$ is continuous whenever $f(x)$ is continuous [Theorem 1(E)]. Since $f(x) = x - 8$ is continuous for all x [Theorem 1(C)], $\sqrt[3]{x-8}$ is continuous on $(-\infty, \infty)$.

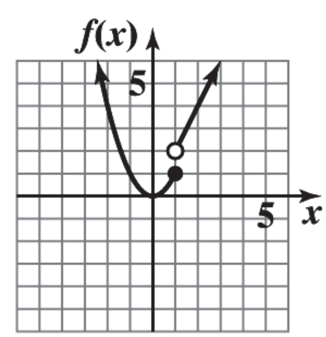
66. $\sqrt{4-x^2}$

Let $f(x) = 4 - x^2$. Then $\sqrt{4-x^2} = \sqrt[2]{f(x)}$ is continuous whenever $f(x)$ is continuous and nonnegative [Theorem 1(F)]. Since $f(x) = 4 - x^2$ is continuous for all x [Theorem 1(C)] and $f(x)$ is nonnegative on $[-2, 2]$, $\sqrt{4-x^2}$ is continuous on $[-2, 2]$.

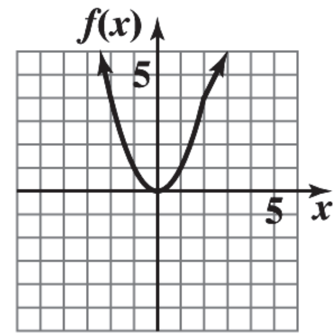
68. $\sqrt[3]{x^2+2}$

Let $f(x) = x^2 + 2$. Then $\sqrt[3]{x^2+2} = \sqrt[3]{f(x)}$ is continuous whenever $f(x)$ is continuous [Theorem 1(E)]. Since $f(x) = x^2 + 2$ is continuous for all x [Theorem 1(C)], $\sqrt[3]{x^2+2}$ is continuous on $(-\infty, \infty)$.

70. The graph of f is shown at the right. This function is discontinuous at $x = 1$.
[$\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = 2$;
Thus, $\lim_{x\to 1} f(x)$ does not exist.]

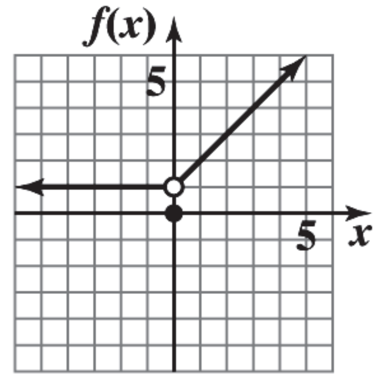


72. The graph of f is shown at the right.
This function is continuous for all x. $[\lim_{x\to 2} f(x) = f(2) = 4]$



74. The graph of f is shown at the right.

This function is discontinuous at $x = 0$,
since $\lim_{x\to 0} f(x) = 1 \neq f(0) = 0$



76. (A) Since $\lim_{x\to 2^+} f(x) = f(2) = 2$, f is continuous from the right at $x = 2$.

(B) Since $\lim_{x\to 2^-} f(x) = 1 \neq f(2) = 2$, f is not continuous from the left at $x = 2$.

(C) f is continuous on the open interval (1, 2).

(D) f is *not* continuous on the closed interval [1, 2] since $\lim_{x\to 2^-} f(x) = 1 \neq f(2) = 2$, i.e., f is not continuous from the left at $x = 2$.

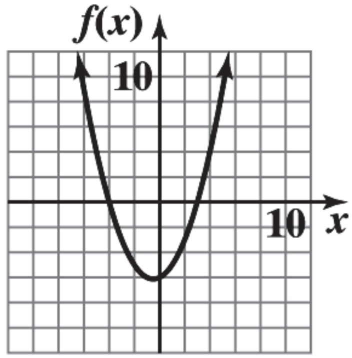
(E) f is continuous on the half-closed interval [1, 2).

78. True: If $r(x) = \dfrac{n(x)}{d(x)}$ is a rational function and $d(x)$ has degree n, then $r(x)$ has at most n points of discontinuity.

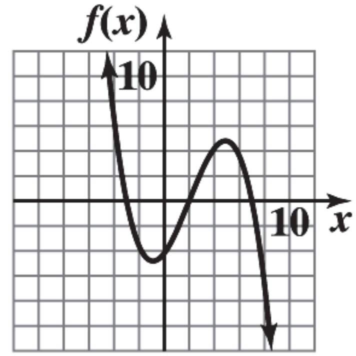
80. True: Continuous on (0, 2) means continuous at every real number x in (0, 2), including $x = 1$.

82. False. The greatest integer function has infinitely many points of discontinuity. See Prob. 75.

84. x intercepts: $x = -4, 3$



86. x intercepts: $x = -3, 2, 7$

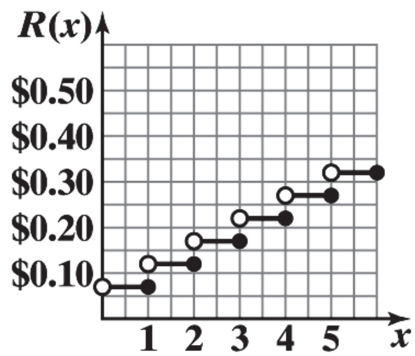


88. $f(x) = \dfrac{6}{x-4} \neq 0$ for all x. This does not contradict Theorem 2 because f is not continuous on (2, 7); f is discontinuous at $x = 4$.

90. (A)

$$R(x) = \begin{cases} 0.07 & \text{if } 0 < x \le 1 \\ 0.12 & \text{if } 1 < x \le 2 \\ 0.17 & \text{if } 2 < x \le 3 \\ 0.22 & \text{if } 3 < x \le 4 \\ 0.27 & \text{if } 4 < x \le 5 \\ 0.32 & \text{if } 5 < x \le 6 \\ \vdots & \qquad \vdots \end{cases}$$

(B)



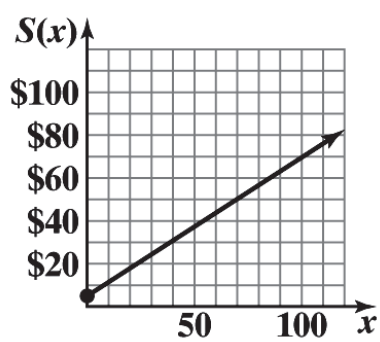
(C) $\lim_{x \to 3.5} R(x) = 0.22 = R(3.5)$; Thus, $R(x)$ is continuous at $x = 3.5$.

$\lim_{x \to 3} R(x)$ does not exist; $R(3) = 0.17$; Thus, $R(x)$ is not continuous at $x = 3$.

92. If x is a positive integer, then $S(x) = R(x) + 0.05$.
$S(x) = R(x)$ for all other values of x in the domain of R.

94. (A) $S(x) = \begin{cases} 5 + 0.69x \text{ if } 0 \le x \le 5 \\ 5.2 + 0.65x \text{ if } 5 < x \le 50 \\ 6.2 + 0.63x \text{ if } 50 < x \end{cases}$

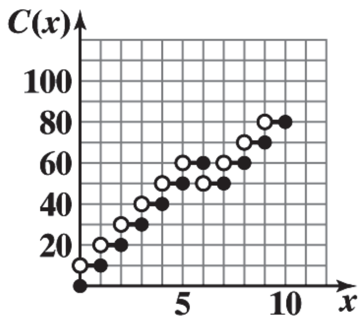
(B) The graph of S is:



(C) $\lim_{x \to 5} S(x) = 8.45 = S(5)$; thus, $S(x)$ is continuous at $x = 5$.

$\lim_{x \to 50} S(x) = 37.7 = S(50)$; thus, $S(x)$ is continuous at $x = 50$.

96. (A) The graph of $C(x)$ is:



(B) From the graph, $\lim_{x \to 4.5} C(x) = 50$ and $C(4.5) = 50$.

(C) From the graph, $\lim_{x \to 8} C(x)$ does not exist; $C(8) = 60$.

(D) Since $\lim_{x \to 4.5} C(x) = 50 = C(4.5)$, $C(x)$ is continuous at $x = 4.5$.
Since $\lim_{x \to 8} C(x)$ does not exist and $C(8) = 60$, $C(x)$ is not continuous at $x = 8$.

98. (A) From the graph, p is discontinuous at $t = t_2$, and $t = t_4$.

(B) $\lim_{t \to t_1} p(t) = 10$; $p(t_1) = 10$.

(C) $\lim_{t \to t_2} p(t) = 30$, $p(t_2) = 10$.

(D) $\lim_{t \to t_4} p(t)$ does not exist; $p(t_4) = 80$.

EXERCISE 2-4

2. Slope $m = \frac{8-11}{1-(-1)} = \frac{-3}{2}, \ -1.5$

4. Slope $m = \frac{3-(-3)}{4-(-12)} = \frac{6}{16} = \frac{3}{8}$; 0.375

6. $\frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

8. $\frac{1-\sqrt{2}}{5+\sqrt{2}} = \frac{1-\sqrt{2}}{5+\sqrt{2}} \cdot \frac{5-\sqrt{2}}{5-\sqrt{2}} = \frac{7-6\sqrt{2}}{23} = \frac{7}{23} - \frac{6}{23}\sqrt{2}$

10. (A) $\frac{f(-1)-f(-2)}{-1-(-2)} = \frac{4-1}{1} = 3$ is the slope of the secant line through $(-2, f(-2))$ and $(-1, f(-1))$.

(B) $\frac{f(-2+h)-f(-2)}{h} = \frac{5-(-2+h)^2-1}{h} = \frac{5-[4-4h+h^2]-1}{h}$

$= \frac{5-4+4h-h^2-1}{h} = \frac{4h-h^2}{h} = 4-h$;

slope of the secant line through $(-2, f(-2))$ and $(-2+h, f(-2+h))$

(C) $\lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h} = \lim_{h\to 0}(4-h) = 4$;

slope of the tangent line at $(-2, f(-2))$

12. $f(x) = 3x^2$

(A) Slope of secant line through $(2, f(2))$ and $(5, f(5))$:

$$\frac{f(5)-f(2)}{5-2} = \frac{3(5)^2-3(2)^2}{5-2} = \frac{75-12}{3} = \frac{63}{3} = 21$$

(B) Slope of secant line through $(2, f(2))$ and $(2+h, f(2+h))$:

$$\frac{3(2+h)^2-3(2)^2}{2+h-2} = \frac{3(4+4h+h^2)-12}{h} = \frac{12+12h+3h^2-12}{h} = \frac{12h+3h^2}{h} = 12 + 3h$$

(C) Slope of the graph at $(2, f(2))$: $\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}(12+3h) = 12.$

14. (A) Distance traveled for $0 \le t \le 4$: $352(1.5) = 528$; average velocity: $v = \frac{528}{4} = 132$ mph .

(B) $\frac{f(4)-f(0)}{4-0} = \frac{528}{4} = 132.$

(C) Slope at $x = 4$: $m = 150$. Equation of tangent line at $(4, f(4))$: $y - 528 = 150(x-4)$ or $y = 150x - 72.$

16. $f(x) = \frac{1}{1+x^2}$; $f(2) = \frac{1}{5} = 0.2$. Equation of tangent line: $y - 0.2 = -0.16(x-2)$ or $y = -0.16x + 3.4.$

18. $f(x) = x^4$; $f(-1) = 1$. Equation of tangent line: $y - 1 = -4(x+1)$ or $y = -4x - 3.$

20. $f(x) = 9$

Step 1. Find $f(x + h)$.

$f(x + h) = 9$

Step 2. Find $f(x + h) - f(x)$.

$f(x + h) - f(x) = 9 - 9 = 0$

Step 3. Find $\frac{f(x+h)-f(x)}{h}$.

$$\frac{f(x+h)-f(x)}{h} = \frac{0}{h} = 0$$

Step 4. Find $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(0) = 0$

Thus, if $f(x) = 9$, then $f'(x) = 0$, $f'(1) = 0$, $f'(2) = 0$, $f'(3) = 0$.

22. $f(x) = 4 - 6x$

Step 1. $f(x + h) = 4 - 6(x + h) = 4 - 6x - 6h$

Step 2. $f(x + h) - f(x) = (4 - 6x - 6h) - (4 - 6x)$
$= 4 - 6x - 6h - 4 + 6x = -6h$

Step 3. $\frac{f(x+h)-f(x)}{h} = \frac{-6h}{h} = -6$

Step 4. $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(-6) = -6$

$f'(1) = -6$, $f'(2) = -6$, $f'(3) = -6$

24. $f(x) = 2x^2 + 8$

Step 1. $f(x + h) = 2(x + h)^2 + 8 = 2(x^2 + 2xh + h^2) + 8$
$= 2x^2 + 4xh + 2h^2 + 8$

Step 2. $f(x + h) - f(x) = (2x^2 + 4xh + 2h^2 + 8) - (2x^2 + 8)$
$= 2x^2 + 4xh + 2h^2 + 8 - 2x^2 - 8$
$= 4xh + 2h^2$

Step 3. $\frac{f(x+h)-f(x)}{h} = \frac{4xh+2h^2}{h} = 4x + 2h$

Step 4. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} (4x + 2h) = 4x$
$f'(1) = 4, \quad f'(2) = 8, \quad f'(3) = 12$

26. $f(x) = x^2 + 4x + 7$

Step 1. $f(x + h) = (x + h)^2 + 4(x + h) + 7 = x^2 + 2xh + h^2 + 4x + 4h + 7$

Step 2. $f(x + h) - f(x) = (x^2 + 2xh + h^2 + 4x + 4h + 7) - (x^2 + 4x + 7)$
$= 2xh + 4h + h^2 = h(2x + 4 + h)$

Step 3. $\frac{f(x+h)-f(x)}{h} = \frac{h(2x+4+h)}{h} = 2x + 4 + h$

Step 4. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} (2x + 4 + h) = 2x + 4$
$f'(1) = 6, \quad f'(2) = 8, \quad f'(3) = 10$

28. $f(x) = 2x^2 + 5x + 1$

Step 1. $f(x + h) = 2(x + h)^2 + 5(x + h) + 1$
$= 2(x^2 + 2xh + h^2) + 5x + 5h + 1$
$= 2x^2 + 4xh + 2h^2 + 5x + 5h + 1$

Step 2. $f(x + h) - f(x) = (2x^2 + 4xh + 2h^2 + 5x + 5h + 1) - (2x^2 + 5x + 1)$
$= h(4x + 5 + 2h)$

Step 3. $\frac{f(x+h)-f(x)}{h} = \frac{h(4x+5+2h)}{h} = 4x + 5 + 2h$

Step 4. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} (4x + 5 + 2h) = 4x + 5$
$f'(1) = 9, \quad f'(2) = 13, \quad f'(3) = 17$

30. $f(x) = -x^2 + 9x - 2$

Step 1. $f(x + h) = -(x + h)^2 + 9(x + h) - 2$
$= -(x^2 + 2xh + h^2) + 9x + 9h - 2$
$= -x^2 - 2xh - h^2 + 9x + 9h - 2$

Step 2. $f(x + h) - f(x) = (-x^2 - 2xh - h^2 + 9x + 9h - 2) - (-x^2 + 9x - 2)$
$= -2xh + 9h - h^2 = h(-2x + 9 - h)$

Step 3. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{h(-2x+9-h)}{h} = -2x + 9 - h$

Step 4. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} (-2x + 9 - h) = -2x + 9$ f'

$f'(1) = 7, \quad f'(2) = 5, \quad f'(3) = 3$

32. $f(x) = -2x^3 + 5$

Step 1. $f(x + h) = -2(x + h)^3 + 5 = -2(x^3 + 3x^2h + 3xh^2 + h^3) + 5$
$= -2x^3 - 6x^2h - 6xh^2 - 2h^3 + 5$

Step 2. $f(x + h) - f(x) = -2x^3 - 6x^2h - 6xh^2 - 2h^3 + 5 - (-2x^3 + 5)$
$= -6x^2h - 6xh^2 - 2h^3$
$= -2h(3x^2 + 3xh + h^2)$

Step 3. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{-2h(3x^2+3xh+h^2)}{h} = -2(3x^2 + 3xh + h^2)$

Step 4. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \{-2(3x^2 + 3xh + h^2)\} = -6x^2$
$f'(1) = -6, \quad f'(2) = -24, \quad f'(3) = -54$

34. $f(x) = \dfrac{6}{x} - 2$

Step 1. $f(x + h) = \dfrac{6}{x+h} - 2$

Step 2. $f(x + h) - f(x) = \left(\dfrac{6}{x+h} - 2\right) - \left(\dfrac{6}{x} - 2\right)$
$= \dfrac{6}{x+h} - \dfrac{6}{x} = \dfrac{6x - 6x - 6h}{x(x+h)} = \dfrac{-6h}{x(x+h)}$

Step 3. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\frac{-6h}{x(x+h)}}{h} = -\dfrac{6}{x(x+h)}$

Step 4. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{-6}{x(x+h)} = -\dfrac{6}{x^2}$

$f'(1) = -6, \quad f'(2) = -\dfrac{6}{4} = -\dfrac{3}{2}, \quad f'(3) = -\dfrac{6}{9} = -\dfrac{2}{3}$

36. $f(x) = 3 - 7\sqrt{x}$

Step 1. $f(x+h) = 3 - 7\sqrt{x+h}$

Step 2. $f(x+h) - f(x) = (3 - 7\sqrt{x+h}) - (3 - 7\sqrt{x}) = 7(\sqrt{x} - \sqrt{x+h})$

Step 3. $\frac{f(x+h)-f(x)}{h} = \frac{7(\sqrt{x}-\sqrt{x+h})}{h} = \frac{7(\sqrt{x}-\sqrt{x+h})}{h} \cdot \frac{(\sqrt{x}+\sqrt{x+h})}{(\sqrt{x}+\sqrt{x+h})}$

$= \frac{7(x-(x+h))}{h(\sqrt{x}+\sqrt{x+h})} = \frac{7(x-x-h)}{h(\sqrt{x}+\sqrt{x+h})}$

$= \frac{-7h}{h(\sqrt{x}+\sqrt{x+h})} = \frac{-7}{\sqrt{x}+\sqrt{x+h}}$

Step 4. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\left(\frac{-7}{\sqrt{x}+\sqrt{x+h}}\right) = \frac{-7}{2\sqrt{x}}$

$f'(1) = -\frac{7}{2}$, $f'(2) = -\frac{7}{2\sqrt{2}} = -\frac{7\sqrt{2}}{4}$, $f'(3) = -\frac{7}{2\sqrt{3}} = -\frac{7\sqrt{3}}{6}$

38. $f(x) = 16\sqrt{x+9}$

Step 1. $f(x+h) = 16\sqrt{x+h+9}$

Step 2. $f(x+h) - f(x) = 16\sqrt{x+h+9} - 16\sqrt{x+9}$
$= 16(\sqrt{x+h+9} - \sqrt{x+9})$

Step 3. $\frac{f(x+h)-f(x)}{h} = \frac{16(\sqrt{x+h+9}-\sqrt{x+9})}{h}$

$= \frac{16(\sqrt{x+h+9}-\sqrt{x+9})}{h} \cdot \frac{(\sqrt{x+h+9}+\sqrt{x+9})}{(\sqrt{x+h+9}+\sqrt{x+9})}$

$= \frac{16((x+h+9)-(x+9))}{h(\sqrt{x+h+9}+\sqrt{x+9})}$

$= \frac{16h}{h(\sqrt{x+h+9}+\sqrt{x+9})} = \frac{16}{\sqrt{x+h+9}+\sqrt{x+9}}$

Step 4. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{16}{\sqrt{x+h+9}+\sqrt{x+9}} = \frac{16}{2\sqrt{x+9}} = \frac{8}{\sqrt{x+9}}$

$f'(1) = \frac{8}{\sqrt{10}} = \frac{4\sqrt{10}}{5}$, $f'(2) = \frac{8}{\sqrt{11}} = \frac{8\sqrt{11}}{11}$, $f'(3) = \frac{8}{\sqrt{12}} = \frac{4\sqrt{3}}{3}$

40. $f(x)=\dfrac{1}{x+4}$.

Step 1. $f(x+h)=\dfrac{1}{x+4+h}$

Step 2. $f(x+h)-f(x)=\dfrac{1}{x+4+h}-\dfrac{1}{x+4}=\dfrac{x+4-(x+4+h)}{(x+4+h)(x+4)}=\dfrac{-h}{(x+4+h)(x+4)}$

Step3. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{-h}{h(x+4+h)(x+4)}=\dfrac{-1}{(x+4+h)(x+4)}$

Step 4. $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\dfrac{-1}{(x+4+h)(x+4)}=\dfrac{-1}{(x+4)^2}$.

$f'(1)=\dfrac{-1}{25},\quad f'(2)=\dfrac{-1}{36},\quad f'(3)=\dfrac{-1}{49}$

42. $f(x)=\dfrac{x}{x+2}$

Step1. $f(x+h)=\dfrac{x+h}{x+2+h}$

Step 2. $f(x+h)-f(x)=\dfrac{x+h}{x+2+h}-\dfrac{x}{x+2}=\dfrac{(x+h)(x+2)-x(x+2+h)}{(x+2+h)(x+2)}=\dfrac{2h}{(x+2+h)(x+2)}$

Step 3. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{2h}{h(x+2+h)(x+2)}=\dfrac{2}{(x+2+h)(x+2)}$

Step 4. $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\dfrac{2}{(x+2+h)(x+2)}=\dfrac{2}{(x+2)^2}$.

$f'(1)=\dfrac{2}{9},\quad f'(2)=\dfrac{2}{16}=\dfrac{1}{8},\quad f'(3)=\dfrac{2}{25}$

44. $y=f(x)=x^2+x$

(A) $f(2)=2^2+2=6, f(4)=4^2+4=20$

Slope of secant line: $\dfrac{f(4)-f(2)}{4-2}=\dfrac{20-6}{2}=\dfrac{14}{2}=7$

(B) $f(2)=6, f(2+h)=(2+h)^2+(2+h)=4+4h+h^2+2+h$
$=6+5h+h^2$

Slope of secant line: $\dfrac{f(2+h)-f(2)}{h}=\dfrac{6+5h+h^2-6}{h}$

$=\dfrac{5h+h^2}{h}=5+h$

(C) Slope of tangent line at $(2, f(2))$:

$\lim_{h\to 0}\dfrac{f(2+h)-f(2)}{h}=\lim_{h\to 0}(5+h)=5$

(D) Equation of tangent line at $(2, f(2))$:
$y-f(2)=f'(2)(x-2)$ or $y-6=5(x-2)$ and $y=5x-4$.

46. $f(x) = x^2 + x$

(A) Average velocity: $\dfrac{f(4)-f(2)}{4-2} = \dfrac{(4)^2+4-((2)^2+2)}{2} = \dfrac{16+4-6}{2} = 7$ meters per second

(B) Average velocity: $\dfrac{f(2+h)-f(2)}{h} = \dfrac{(2+h)^2+(2+h)-6}{h} = \dfrac{4+4h+h^2+2+h-6}{h}$

$= \dfrac{5h+h^2}{h} = 5 + h$ meters per second

(C) Instantaneous velocity: $\lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = \lim\limits_{h\to 0}(5+h) = 5$ meters per second

48. $F'(x)$ does not exist at $x = b$.

50. $F'(x)$ does exist at $x = d$.

52. $F'(x)$ does not exist at $x = f$.

54. $F'(x)$ does not exist at $x = h$.

56. $f(x) = x^2 + 2x$

(A) Step 1. Simplify $\dfrac{f(x+h)-f(x)}{h}$.

$\dfrac{f(x+h)-f(x)}{h} = \dfrac{(x+h)^2+2(x+h)-(x^2+2x)}{h}$

$= \dfrac{x^2+2xh+h^2+2x+2h-x^2-2x}{h}$

$= \dfrac{2xh+h^2+2h}{h} = 2x + 2 + h$

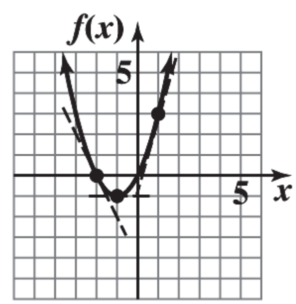
Step 2. Evaluate $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

$\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}(2x+2+h) = 2x + 2$

Therefore, $f'(x) = 2x + 2$.

(B) $f'(-2) = -2$, $f'(-1) = 0$,

(C) $f'(1) = 4$



58. To find $v = f'(x)$, use the two-step process for the given distance function, $f(x) = 8x^2 - 4x$.

Step 1. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{8(x+h)^2-4(x+h)-(8x^2-4x)}{h}$

$= \dfrac{8(x^2+2xh+h^2)-4x-4h-8x^2+4x}{h}$

$$= \frac{8x^2 + 16xh + 8h^2 - 4x - 4h - 8x^2 + 4x}{h}$$

$$= \frac{16xh - 4h + 8h^2}{h} = 16x - 4 + 8h$$

Step 2. $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} (16x - 4 + 8h) = 16x - 4$

Thus, the velocity, $v = f'(x) = 16x - 4$

$f'(1) = 12$ feet per second, $f'(3) = 44$ feet per second, $f'(5) = 76$ feet per second

60. (A) The graphs of g and h are vertical translations of the graph of f. All Three functions should have the same derivatives; they differ from each other by a constant.

(B) $m(x) = -x^2 + c$

Step 1. $m(x + h) = -(x + h)^2 + c = -x^2 - 2xh - h^2 + c$

Step 2. $m(x + h) - m(x) = (-x^2 - 2xh - h^2 + c) - (-x^2 + c)$
$= -x^2 - 2xh - h^2 + c + x^2 - c = -2xh - h^2$

Step 3. $\frac{m(x+h) - m(x)}{h} = \frac{-2xh - h^2}{h} = -2x - h$

Step 4. $m'(x) = \lim_{h\to 0} \frac{m(x+h) - m(x)}{h} = \lim_{h\to 0} (-2x - h) = -2x$

62. True: $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{m(x+h) + b - (mx + b)}{h}$

$$= \lim_{h\to 0} \frac{mx + mh + b - mx - b}{h} = \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m$$

64. Let $c \in (a,b)$. We wish to show that $\lim_{x\to c} f(x) = f(c)$. If we let $h = x - c$, then $x = h + c$, and this statement is equivalent to $\lim_{h\to 0} f(c+h) = f(c)$, which is in turn equivalent to $\lim_{h\to 0} \left(f(c+h) - f(c)\right) = 0$. Since $f'(x)$ exists at every point in the interval, we know that $f'(c)$ is defined and

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

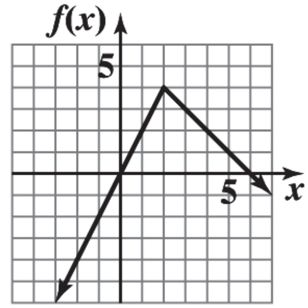
$$\left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}\right) = \left(\lim_{h\to 0} h\right) f'(c)$$

$$\lim_{h\to 0} h\left(\frac{f(c+h) - f(c)}{h}\right) = 0$$

$$\lim_{h\to 0} \left(f(c+h) - f(c)\right) = 0$$

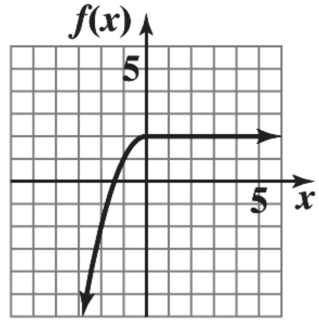
66. False. For example, $f(x) = |x|$ has a sharp corner at $x = 0$, but is continuous there.

68. The graph of $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ 6 - x & \text{if } x \geq 2 \end{cases}$ is:



f is not differentiable at $x = 2$ because the graph of f has a sharp corner at this point.

70. $f(x) = \begin{cases} 2 - x^2 & \text{if } x \leq 0 \\ 2 & \text{if } x > 0 \end{cases}$



It is clear that $f\,'(x) = \begin{cases} -2x & \text{if } x < 0 \\ 0 & \text{if } x > 2 \end{cases}$

Thus, the only question is $f\,'(0)$.
Since $\lim_{x\to 0^-} f\,'(x) = \lim_{x\to 0^-} (-2x) = 0$ and

$\lim_{x\to 0^+} f\,'(x) = \lim_{x\to 0^+} (0) = 0$, f is differentiable at 0 as well;
f is differentiable for all real numbers.

72. $f(x) = 1 - |x|$

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{1 - |0+h| - (1 - |0|)}{h} = \lim_{h\to 0} -\frac{|h|}{h}$$

The limit does not exist. Thus, f is not differentiable at $x = 0$.

74. $f(x) = x^{2/3}$

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{(0+h)^{2/3} - 0^{2/3}}{h} = \lim_{h\to 0} \frac{h^{2/3}}{h} = \lim_{h\to 0} \frac{1}{h^{1/3}}$$

The limit does not exist. Thus, f is not differentiable at $x = 0$.

76. $f(x) = \sqrt{1 + x^2}$

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{1 + (0+h)^2} - \sqrt{1 + 0^2}}{h} = \lim_{h\to 0} \frac{\sqrt{1+h^2} - 1}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1+h^2} - 1}{h} \cdot \frac{\sqrt{1+h^2} + 1}{\sqrt{1+h^2} + 1} = \lim_{h\to 0} \frac{1 + h^2 - 1}{h\left[\sqrt{1+h^2} + 1\right]} = \lim_{h\to 0} \frac{h}{\sqrt{1+h^2} + 1} = \frac{0}{2} = 0$$

f is differentiable at $x = 0$ and $f\,'(0) = 0$.

78. $y = 16x^2$

Now, if y = 1,024 ft, then

$$16x^2 = 1{,}024$$
$$x^2 = \frac{1{,}024}{16} = 64$$
$$x = 8 \text{ sec.}$$

$y' = 32x$ and at $x = 8$, $y' = 32(8) = 256$ ft/sec.

80. $P(x) = 45x - 0.025x^2 - 5{,}000,\ 0 \le x \le 2{,}400.$

(A) Average change $= \dfrac{P(850) - P(800)}{850 - 800}$

$$= \frac{[45(850) - 0.025(850)^2 - 5{,}000] - [45(800) - 0.025(800)^2 - 5{,}000]}{50}$$

$$= \frac{45(850) - 0.025(850)^2 - 45(800) + 0.025(800)^2}{50}$$

$$= \frac{54{,}250 - 54{,}062.5}{50} = \frac{187.5}{50} = \$3.75$$

(B) $P(x) = 45x - 0.025x^2 - 5{,}000$

Step 1. $P(x + h) = 45(x + h) - 0.025(x + h)^2 - 5{,}000$
$= 45x + 45h - 0.025x^2 - 0.05xh - 0.025h^2 - 5{,}000$

Step 2. $P(x + h) - P(x) = (45x + 45h - 0.025x^2 - 0.05xh - 0.025h^2 - 5{,}000) - (45x - 0.025x^2 - 5{,}000)$
$= 45h - 0.05xh - 0.025h^2$

Step 3. $\dfrac{P(x+h) - P(x)}{h} = \dfrac{45h - 0.05xh - 0.025h^2}{h} = 45 - 0.05x - 0.025h$

Step 4. $P'(x) = \lim\limits_{h\to 0} \dfrac{P(x+h) - P(x)}{h} = \lim\limits_{h\to 0} (45 - 0.05x - 0.025h) = 45 - 0.05x$

(C) $P(800) = 45(800) - 0.025(800)^2 - 5{,}000 = 15{,}000$
$P'(800) = 45 - 0.05(800) = 5;$

At a production level of 800 car seats, the profit is \$15,000 and is increasing at the rate of \$5 per seat.

82. $S(t) = 2\sqrt{t+6}$

(A) Step 1. $S(t + h) = 2\sqrt{t+h+6}$

Step 2. $S(t + h) - S(t) = 2[\sqrt{t+h+6} - \sqrt{t+6}\,]$

$$= 2[\sqrt{t+h+6} - \sqrt{t+6}\,] \cdot \frac{\sqrt{t+h+6} + \sqrt{t+6}}{\sqrt{t+h+6} + \sqrt{t+6}}$$

$$= \frac{2[(t+h+6) - (t+6)]}{\sqrt{t+h+6} + \sqrt{t+6}} = \frac{2h}{\sqrt{t+h+6} + \sqrt{t+6}}$$

Step 3. $\dfrac{S(t+h) - S(t)}{h} = \dfrac{\dfrac{2h}{\sqrt{t+h+6} + \sqrt{t+6}}}{h} = \dfrac{2}{\sqrt{t+h+6} + \sqrt{t+6}}$

Step 4. $S'(t) = \lim\limits_{h\to 0} \dfrac{S(t+h) - S(t)}{h} = \lim\limits_{h\to 0} \dfrac{2}{\sqrt{t+h+6} + \sqrt{t+6}} = \dfrac{2}{2\sqrt{t+6}} = \dfrac{1}{\sqrt{t+6}}$

(B) $S(10) = 2\sqrt{10+6} = 2\sqrt{16} = 2(4) = 8;$

$S'(10) = \dfrac{1}{\sqrt{10+6}} = \dfrac{1}{\sqrt{16}} = \dfrac{1}{4} = 0.25$

After 10 months, the total sales are \$8 million and are increasing at the rate of \$0.25 million = \$250,000 per month.

(C) The estimated total sales are \$8.25 million after 11 months and \$8.5 million after 12 months.

84. (A) $p(t) = 48t^2 - 37t + 1{,}698$

Step 1. $p(t+h) = 48(t+h)^2 - 37(t+h) + 1{,}698$

$= 48(t^2 + 2th + h^2) - 37t - 37h + 1{,}698$

$= 48t^2 + 96th + 48h^2 - 37t - 37h + 1{,}698$

Step 2. $p(t+h) - p(t) = 48t^2 + 96th + 48h^2 - 37t - 37h + 1{,}698 - (48t^2 - 37t + 1{,}698)$

$= 96th + 48h^2 - 37h$

Step 3. $\dfrac{p(t+h) - p(t)}{h} = \dfrac{96th + 48h^2 - 37h}{h} = 96t + 48h - 37$

Step 4. $p'(t) = \lim_{h\to 0} \dfrac{p(t+h) - p(t)}{h} = \lim_{h\to 0}(96t + 48h - 37) = 96t - 37$

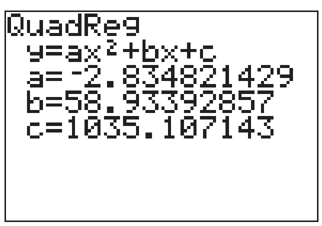
(B) 2022 corresponds to $t = 12$. Thus

$p(12) = 48(12)^2 - 37(12) + 1{,}698 = 8{,}166$

$p'(12) = 96(12) - 37 = 1{,}115$

In 2022, 8,166 thousand tons of copper will be consumed and this quantity is increasing at the rate of 1,115 thousand tons/year.

86. (A) Quadratic regression model



$C(x) \approx -2.835x^2 + 58.934x + 1035.107, \quad C'(x) \approx -5.670x + 58.934.$

(B) $C(20) = -2.835(20)^2 + 58.934(20) + 1035.107 \approx 1{,}079.9$;

$C'(20) = -5.670(20) + 58.934 = -54.5$

In 2020, 1,079.9 billion kilowatts will be sold and the amount sold is decreasing at the rate of 54.5 billion kilowatts per year.

88. (A) $F(t) = 98 + \dfrac{4}{t+1}$

Step 1. $F(t+h) = 98 + \dfrac{4}{t+h+1}$

Step 2. $F(t+h) - F(t) = \left(98 + \dfrac{4}{t+h+1}\right) - \left(98 + \dfrac{4}{t+1}\right) = \dfrac{4}{t+h+1} - \dfrac{4}{t+1}$

$= 4\left[\dfrac{(t+1) - (t+h+1)}{(t+h+1)(t+1)}\right] = \dfrac{-4h}{(t+h+1)(t+1)}$

Step 3. $\dfrac{F(t+h)-F(t)}{h} = \dfrac{\dfrac{-4h}{(t+h+1)(t+1)}}{h} = \dfrac{-4}{(t+h+1)(t+1)}$

Step 4. $F'(t) = \lim_{h\to 0} \dfrac{F(t+h)-F(t)}{h} = \lim_{h\to 0} \dfrac{-4}{(t+h+1)(t+1)} = \dfrac{-4}{(t+1)^2}$

(B) $F(3) = 99$, $F'(3) = \dfrac{-4}{16} = \dfrac{-1}{4}$. The body temperature 3 hours after taking the medicine is 99^{o} and is decreasing at the rate of 0.25^{o} per hour.

EXERCISE 2-5

2. $\sqrt[3]{x} = x^{1/3}$

4. $\dfrac{1}{x} = x^{-1}$

6. $\dfrac{1}{(x^5)^2} = \dfrac{1}{x^{10}} = x^{-10}$

8. $\dfrac{1}{\sqrt[5]{x}} = \dfrac{1}{x^{1/5}} = x^{-1/5}$

10. $\dfrac{d}{dx}3 = 0$ (Derivative of a constant rule.)

12. $y = x^6$

$y' = 6x^{6-1} = 6x^5$ (Power rule)

14. $g(x) = x^5$

$g'(x) = 5x^{5-1} = 5x^4$ (Power rule)

16. $y = x^{-8}$

$\dfrac{dy}{dx} = -8x^{-8-1} = -8x^{-9}$ (Power rule)

18. $f(x) = x^{9/2}$

$f'(x) = \dfrac{9}{2}x^{9/2-1} = \dfrac{9}{2}x^{7/2}$ (Power rule)

20. $y = \dfrac{1}{x^{12}} = x^{-12}$

$y' = -12x^{-12-1} = -12x^{-13}$ (Power rule)

22. $\dfrac{d}{dx}(-2x^3) = -2(3x^2)$ (constant times a function rule)

$= -6x^2$

24. $f(x) = .8x^4$

$f'(x) = (.8)(4x^3) = 3.2x^3$

26. $y = \dfrac{x^5}{25}$

$y' = \dfrac{1}{25}(5x^4) = \dfrac{x^4}{5}$

28. $h(x) = 5g(x)$; $h'(2) = 5g'(2) = 5(-1) = -5$

30. $h(x) = g(x) - f(x)$; $h'(2) = g'(2) - f'(2) = -1 - 3 = -4$

32. $h(x) = -4f(x) + 5g(x) - 9$; $h'(2) = -4f'(2) + 5g'(2) = -4(3) + 5(-1) = -17$

34. $\frac{d}{dx}(-4x + 9) = \frac{d}{dx}(-4x) + \frac{d}{dx}(9) = -4 + 0 = -4$

36. $y = 2 + 5t - 8t^3$

$\frac{dy}{dt} = 0 + 5 - 24t^2 = 5 - 24t^2$

38. $g(x) = 5x^{-7} - 2x^{-4}$

$g'(x) = (5) \cdot (-7)x^{-8} - (2) \cdot (-4)x^{-5}$

$= -35x^{-8} + 8x^{-5}$

40. $\frac{d}{du}(2u^{4.5} - 3.1u + 13.2) = (2) \cdot (4.5)u^{3.5} - 3.1 + 0 = 9u^{3.5} - 3.1$

42. $F(t) = 0.2t^3 - 3.1t + 13.2$

$F'(t) = (0.2) \cdot (3)t^2 - 3.1 + 0 = 0.6t^2 - 3.1$

44. $w = \frac{7}{5u^2} = \frac{7}{5}u^{-2}$

$w' = \left(\frac{7}{5}\right) \cdot (-2)u^{-3} = -\frac{14}{5}u^{-3}$

46. $\frac{d}{dx}\left(\frac{5x^3}{4} - \frac{2}{5x^3}\right) = \frac{d}{dx}\left(\left(\frac{5}{4}\right)x^3 - \left(\frac{2}{5}\right)x^{-3}\right) = \left(\frac{5}{4}\right) \cdot (3)x^2 - \left(\frac{2}{5}\right) \cdot (-3)x^{-4} = \frac{15}{4}x^2 + \frac{6}{5}x^{-4}$

48. $H(w) = \frac{5}{w^6} - 2\sqrt{w} = 5w^{-6} - 2w^{1/2}$

$H'(w) = (5) \cdot (-6)w^{-7} - (2) \cdot \left(\frac{1}{2}\right)w^{-1/2} = -30w^{-7} - w^{-1/2}$

50. $\frac{d}{du}(8u^{3/4} + 4u^{-1/4}) = (8) \cdot \left(\frac{3}{4}\right)u^{-1/4} + (4) \cdot \left(-\frac{1}{4}\right)u^{-5/4} = 6u^{-1/4} - u^{-5/4}$

52. $F(t) = \frac{5}{t^{1/5}} - \frac{8}{t^{3/2}} = 5t^{-1/5} - 8t^{-3/2}$

$F'(t) = (5) \cdot \left(-\frac{1}{5}\right)t^{-6/5} - (8) \cdot \left(-\frac{3}{2}\right)t^{-5/2} = -t^{-6/5} + 12t^{-5/2}$

54. $w = \frac{10}{\sqrt[5]{u}} = 10u^{-1/5}$

$w' = (10) \cdot \left(-\frac{1}{5}\right)u^{-6/5} = -2u^{-6/5}$

56. $\frac{d}{dx}\left(2.8x^{-3} - \frac{0.6}{\sqrt[3]{x^2}} + 7\right) = \frac{d}{dx}(2.8x^{-3} - 0.6x^{-2/3} + 7) = (2.8) \cdot (-3)x^{-4} - (0.6) \cdot \left(-\frac{2}{3}\right)x^{-5/3} + 0$

$= -8.4x^{-4} + 0.4x^{-5/3}$

58. $f(x) = 2x^2 + 8x$

(A) $f'(x) = 4x + 8$

(B) Slope of the graph of f at $x = 2$: $f'(2) = 4(2) + 8 = 16$
Slope of the graph of f at $x = 4$: $f'(4) = 4(4) + 8 = 24$

(C) Tangent line at $x = 2$: $y - y_1 = m(x - x_1)$
$x_1 = 2$
$y_1 = f(2) = 2(2)^2 + 8(2) = 24$
$m = f'(2) = 16$
Thus, $y - 24 = 16(x - 2)$ or $y = 16x - 8$
Tangent line at $x = 4$: $y - y_1 = m(x - x_1)$
$x_1 = 4$
$y_1 = f(4) = 2(4)^2 + 8(4) = 64$
$m = f'(4) = 24$
Thus, $y - 64 = 24(x - 4)$ or $y = 24x - 32$

(D) The tangent line is horizontal at the values $x = c$ such that $f'(c) = 0$. Thus, we must solve the following:
$f'(x) = 4x + 8 = 0$
$4x = -8$
$x = -2$

60. $f(x) = x^4 - 32x^2 + 10$

(A) $f'(x) = 4x^3 - 64x$

(B) Slope of the graph of f at $x = 2$: $f'(2) = 4(2)^3 - 64(2) = -96$
Slope of the graph of f at $x = 4$: $f'(4) = 4(4)^3 - 64(4) = 0$

(C) Tangent line at $x = 2$: $y - y_1 = m(x - x_1)$, where
$x_1 = 2$, $y_1 = f(2) = (2)^4 - 32(2)^2 + 10 = -102$, $m = -96$
$y + 102 = -96(x - 2)$ or $y = -96x + 90$
Tangent line at $x = 4$ is a horizontal line since the slope $m = 0$. Therefore, the equation of the tangent
line at $x = 4$ is: $y = f(4) = (4)^4 - 32(4)^2 + 10 = -246$

(D) Solve $f'(x) = 0$ for x:
$4x^3 - 64x = 0$
$4x(x^2 - 16) = 0$
$4x(x + 4)(x - 4) = 0$
$x = -4$, $x = 0$, $x = 4$

62. $f(x) = 80x - 10x^2$

(A) $v = f'(x) = 80 - 20x$

(B) $v\big|_{x=0} = f'(0) = 80$ ft/sec.

$v\big|_{x=3} = f'(3) = 80 - 20(3) = 20$ ft/sec.

(C) Solve $v = f'(x) = 0$ for x:

$80 - 20x = 0$

$20x = 80$

$x = 4$ seconds

64. $f(x) = x^3 - 9x^2 + 24x$

(A) $v = f'(x) = 3x^2 - 18x + 24$

(B) $v\big|_{x=0} = f'(0) = 24$ ft/sec.

$v\big|_{x=3} = f'(3) = 3(3)^2 - 18(3) + 24 = -3$ ft/sec.

(C) Solve $v = f'(x) = 0$ for x:

$3x^2 - 18x + 24 = 0$ or $x^2 - 6x + 8 = 0$

$(x - 2)(x - 4) = 0$

$x = 2, \; x = 4$ seconds

66. $f'(x) = 2x + 1 - \frac{5}{\sqrt{x}}$; $f'(x) = 0$ at $x \approx 1.5247$.

68. $f'(x) = 4x^{1/3} - 4x + 4$; $f'(x) = 0$ at $x \approx 2.3247$.

70. $f'(x) = 0.08x^3 - 0.18x^2 - 1.56x + 0.94$; $f'(x) = 0$ at $x \approx -3.7626,\ 0.5742,\ 5.4384$.

72. $f'(x) = x^3 - 7.8x^2 + 16.2x - 10$; $f'(x) = 0$ at $x \approx 1.2391,\ 1.6400,\ 4.9209$.

74. The tangent line to the graph of a parabola at the vertex is a horizontal line. Therefore, to find the x coordinate of the vertex,
we solve $f'(x) = 0$ for x.

76. No. The derivative is a quadratic function which can have at most two zeros.

78. $y = (2x - 5)^2$; $y' = (2)(2x - 5)(2) = 8x - 20$

80. $y = \frac{x^2 + 25}{x^2} = 1 + \frac{25}{x^2} = 1 + 25x^{-2}$; $\frac{dy}{dx} = 0 + (25) \cdot (-2)x^{-3} = -50x^{-3}$

82. $f(x) = \frac{2x^5 - 4x^3 + 2x}{x^3} = \frac{2x^5}{x^3} - \frac{4x^3}{x^3} + \frac{2x}{x^3} = 2x^2 - 4 + 2x^{-2}$; $f'(x) = 4x - 4x^{-3}$

84. False: The function $f(x) = \frac{1}{x}$ is a counter-example.

86. False: The function $f(x) = 2x$ is a counter-example.

88. $f(x) = u(x) - v(x)$

Step1. $f(x+h) = u(x+h) - v(x+h)$

Step 2. $f(x+h) - f(x) = u(x+h) - v(x+h) - [u(x) - v(x)] = u(x+h) - u(x) - [v(x+h) - v(x)]$

Step 3. $\dfrac{f(x+h) - f(x)}{h} = \dfrac{u(x+h) - u(x) - [v(x+h) - v(x)]}{h} = \dfrac{u(x+h) - u(x)}{h} - \dfrac{v(x+h) - v(x)}{h}$

Step 4. $\lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0}\left[\dfrac{u(x+h) - u(x)}{h} - \dfrac{v(x+h) - v(x)}{h}\right]$

$$= \lim_{h\to 0}\frac{u(x+h) - u(x)}{h} - \lim_{h\to 0}\frac{v(x+h) - v(x)}{h} = u'(x) - v'(x)$$

90. $S(t) = 0.015t^4 + 0.4t^3 + 3.4t^2 + 10t - 3$

(A) $S'(t) = (0.015)\cdot(4)t^3 + (0.4)\cdot(3)t^2 + (3.4)(2)t + 10 - 0 = 0.06t^3 + 1.2t^2 + 6.8t + 10$

(B) $S(4) = 0.015(4)^4 + 0.4(4)^3 + 3.4(4)^2 + 10(4) - 3 = 120.84$,
$S'(4) = 0.06(4)^3 + 1.2(4)^2 + 6.8(4) + 10 = 60.24$.

After 4 months, sales are $120.84 million and are increasing at the rate of $60.24 million per month.

(C) $S(8) = 0.015(8)^4 + 0.4(8)^3 + 3.4(8)^2 + 10(8) - 3 = 560.84$,
$S'(8) = 0.06(8)^3 + 1.2(8)^2 + 6.8(8) + 10 = 171.92$.

After 8 months, sales are $560.84 million and are increasing at the rate of $171.92 million per month.

92. $x = 10 + \dfrac{180}{p}$, $2 \le p \le 10$

For $p = 5$, $x = 10 + \dfrac{180}{5} = 10 + 36 = 46$

$x = 10 + \dfrac{180}{p} = 10 + 180p^{-1}$

$\dfrac{dx}{dp} = -180p^{-2} = -\dfrac{180}{p^2}$

For $p = 5$, $\left.\dfrac{dx}{dp}\right|_{p=5} = -\dfrac{180}{25} = -7.2$

At the $5 price level, the demand is 46 pounds and is decreasing at the rate of 7.2 pounds per dollar increase in price.

94 . (A) Cubic Regression model

```
CubicReg
 y=ax³+bx²+cx+d
 a=-4.666667E-4
 b=.0276428571
 c=.265952381
 d=25.46857143
```

(B) $F(x) \approx -0.000467x^3 + 0.027643x^2 + 0.265952x + 25.468751$

$F(50) \approx 49.5, \quad F'(50) \approx -0.5$

In 2020, 49.5% of female high-school graduates enroll in college and the percentage is decreasing at the rate of 0.5% per year.

96. $C(x) = \dfrac{0.1}{x^2} = 0.1x^{-2}$

$C'(x) = -0.2x^{-3} = -\dfrac{0.2}{x^3}$, the instantaneous rate of change of concentration at x miles.

(A) At $x = 1$, $C'(1) = -0.2$ parts per million per mile.

(B) At $x = 2$, $C'(2) = -\dfrac{0.2}{8} = -0.025$ parts per million per mile.

98. $y = 21\sqrt[3]{x^2}$, $0 \le x \le 8$.

First, find $y = 21\sqrt[3]{x^2} = 21x^{2/3}$.

Then $y' = 21\left(\dfrac{2}{3}x^{-1/3}\right) = 14x^{-1/3} = \dfrac{14}{x^{1/3}} = \dfrac{14}{\sqrt[3]{x}}$, is the rate of learning at the end of x hours.

(A) Rate of learning at the end of 1 hour:

$\dfrac{14}{\sqrt[3]{1}} = 14$ items per hour.

(B) Rate of learning at the end of 8 hours:

$\dfrac{14}{\sqrt[3]{8}} = \dfrac{14}{2} = 7$ items per hour.

EXERCISE 2-6

2. $f(x) = 0.1x + 3;\ f(7) = 0.1(7) + 3 = 3.7,\quad f(7.1) = 0.1(7.1) + 3 = 3.71$

4. $f(x) = 0.1x + 3;\ f(-10) = 0.1(-10) + 3 = 2,\quad f(-10.1) = 0.1(-10.1) + 3 = 1.99$

6. $g(x) = x^2;\ g(1) = 1^2 = 1,\quad g(1.1) = (1.1)^2 = 1.21$

8. $g(x) = x^2$; $g(5) = 5^2 = 25$, $g(4.9) = (4.9)^2 = (5-0.1)^2 = 24.01$

10. $\Delta x = x_2 - x_1 = 5 - 2 = 3$, $\Delta y = f(x_2) - f(x_1) = 3(5)^2 - 3(2)^2 = 75 - 12 = 63$

$\frac{\Delta y}{\Delta x} = \frac{63}{3} = 21$

12. $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \frac{f(2+1) - f(2)}{1} = \frac{f(3) - f(2)}{1} = \frac{3(3)^2 - 3(2)^2}{1} = 27 - 12 = 15$

14. $\Delta y = f(x_2) - f(x_1) = f(3) - f(2) = 3(3)^2 - 3(2)^2 = 27 - 12 = 15$
$\Delta x = x_2 - x_1 = 3 - 2 = 1$
$\frac{\Delta y}{\Delta x} = \frac{15}{1} = 15$

16. $y = 200x - \frac{x^2}{30}$, $dy = \left(200x - \frac{x^2}{30}\right)' dx = \left(200 - \frac{x}{15}\right) dx$

18. $y = x^3(60 - x) = 60x^3 - x^4$, $dy = (180x^2 - 4x^3)\,dx$

20. $y = 52\sqrt{x} = 52x^{1/2}$, $dy = (52x^{1/2})'dx = (26x^{-1/2})dx$

22. (A) $\frac{f(3+\Delta x) - f(3)}{\Delta x} = \frac{3(3+\Delta x)^2 - 3(3)^2}{\Delta x} = \frac{3(9 + 6\Delta x + (\Delta x)^2) - 27}{\Delta x}$

$= \frac{27 + 18\Delta x + 3(\Delta x)^2 - 27}{\Delta x} = \frac{18\Delta x + 3(\Delta x)^2}{\Delta x} = 18 + 3\Delta x$

(B) As Δx tends to zero, then, clearly, $18 + 3\Delta x$ tends to 18.
Note the values in the following table:

Δx	$18 + 3\Delta x$
1	21
0.1	18.3
0.01	18.03
0.001	18.003

24. $y = (3x + 5)^2 = 9x^2 + 30x + 25$, $dy = (18x + 30)\,dx = 6(3x+5)\,dx.$

26. $y = \frac{(x-1)^2}{x^2} = \frac{x^2 - 2x + 1}{x^2} = 1 - 2x^{-1} + x^{-2}$, $dy = (2x^{-2} - 2x^{-3})\,dx.$

28. $y = f(x) = 30 + 12x^2 - x^3$
$\Delta y = f(2 + 0.1) - f(2) = f(2.1) - f(2) = [30 + 12(2.1)^2 - (2.1)^3] - [(30 + 12(2)^2 - 2^3]$
$= 30 + 52.92 - 9.261 - 30 - 48 + 8 = 3.66$

$dy = (30 + 12x^2 - x^3)'\big|_{x=2}\, dx = (24x - 3x^2)\big|_{x=2}\,(0.1) = (24(2) - 3(2)^2)(0.1) = (48 - 12)(0.1) = 3.6$

30. $y = f(x) = 100\left(x - \dfrac{4}{x^2}\right)$

$\Delta y = f(2 - 0.1) - f(2) = f(1.9) - f(2) = 100\left(1.9 - \dfrac{4}{(1.9)^2}\right) - 100\left(2 - \dfrac{4}{2^2}\right) = 79.197 - 100 = -20.803$

$dy = \left(100\left(x - \dfrac{4}{x^2}\right)\right)'\Big|_{x=2}\ dx\ = 100\left(1 + \dfrac{8}{x^3}\right)\Big|_{x=2}\ (-0.1) = 100\left(1 + \dfrac{8}{2^3}\right)(-0.1) = -20$

32. $V = \dfrac{4}{3}\pi r^3$, $r = 5$ cm, $dr = \Delta r = 0.1$ cm.

$dV = \left(\dfrac{4}{3}\pi r^3\right)'\Big|_{r=5}\ dr = 4\pi r^2\Big|_{r=5}\ (0.1) = 31.4\ \text{cm}^3.$

34. $f(x) = x^2 + 2x + 3$; $f'(x) = 2x + 2$; $x = -2$; $\Delta x = dx$

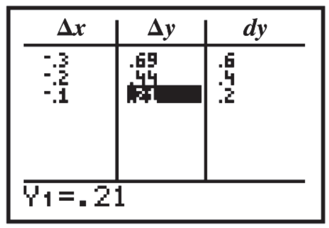
(A) $\Delta y = f(-2 + \Delta x) - f(-2)$
$= [(-2 + \Delta x)^2 + 2(-2 + \Delta x) + 3]$
$\quad - [(-2)^2 + 2(-2) + 3]$
$= 4 - 4\Delta x + (\Delta x)^2 - 4 + 2\Delta x + 3 - 4 + 4 - 3$
$= -2\Delta x + (\Delta x)^2$
$dy = f'(-2)dx = -2\ dx$

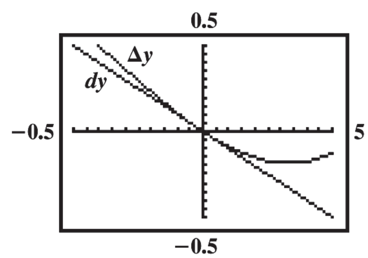
(B) $\Delta y(-0.1) = -2(-0.1) + (-0.1)^2 = 0.21$
$dy(-0.1) = -2(-0.1) = 0.2$

$\Delta y(-0.2) = -2(-0.2) + (-0.2)^2 = 0.44$
$dy(-0.2) = -2(-0.2) = 0.4$

$\Delta y(-0.3) = -2(-0.3) + (-0.3)^2 = 0.69$
$dy(-0.3) = -2(-0.3) = 0.6$

(C)

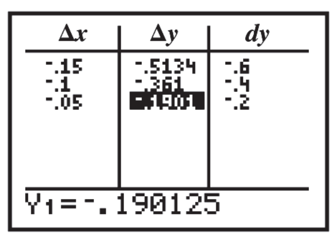




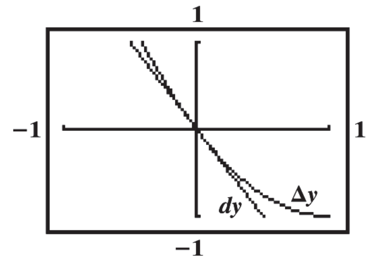
36. $f(x) = x^3 - 2x^2$; $f'(x) = 3x^2 - 4x$; $x = 2$, $\Delta x = dx$

(A) $\Delta y = f(2 + \Delta x) - f(2)$
$= [(2 + \Delta x)^3 - 2(2 + \Delta x)^2] - [2^3 - 2(2)^2]$
$= 8 + 12\Delta x + 6(\Delta x)^2 + (\Delta x)^3 - 8 - 8\Delta x - 2(\Delta x)^2 - 8 + 8$
$= 4\Delta x + 4(\Delta x)^2 + (\Delta x)^3$
$dy = f'(2)dx = 4\ dx$

(C)



(B) $\Delta y(-0.05) = 4(-0.05) + 4(-0.05)^2 + (-0.05)^3$
$= -0.1901$
$dy(-0.05) = 4(-0.05) = -0.2$
$\Delta y(-0.10) = 4(-0.10) + 4(-0.10)^2 + (-0.10)^3$
$= -0.361$
$dy(-0.10) = 4(-0.10) = -0.4$



$\Delta y(-0.15) = 4(-0.15) + 4(-0.15)^2 + (-0.15)^3 = -0.5134$
$dy(-0.15) = 4(-0.15) = -0.6$

38. False.
Example. Let $y = f(x) = x^2 + 1$. Then
$\Delta y = f(0 + \Delta x) - f(0) = f(\Delta x) - f(0) = (\Delta x)^2 + 1 - 1 = (\Delta x)^2$
$dy = f'(0)dx = 0\ dx = 0.$

40. True.
$\Delta y = f(2 + \Delta x) - f(2) = 0$ implies that
$f(2 + \Delta x) = f(2)$
Since this is true for every increment and since the right-hand side of this equation is a constant, the function $f(x)$ must be a constant function.

42. $y = (2x^2 - 4)\sqrt{x} = (2x^2 - 4)(x)^{1/2} = 2x^{5/2} - 4x^{1/2}$, $dy = (5x^{3/2} - 2x^{-1/2})\,dx.$

44. $y = f(x) = \dfrac{590}{\sqrt{x}} = 590x^{-1/2}$; $x = 64$, $\Delta x = dx = 1$.

$\Delta y = f(x + \Delta x) - f(x) = f(64 + 1) - f(64) = f(65) - f(64) = \dfrac{590}{\sqrt{65}} - \dfrac{590}{\sqrt{64}} = -0.57$

$y = f(x) = \dfrac{590}{\sqrt{x}} = 590x^{-1/2}$, $\quad f'(x) = -295x^{-3/2}$

$dy = f'(64)dx = f'(64)(1) = -295(64)^{-3/2} = -\dfrac{295}{512} = -0.576$

46. Given $D(x) = 1{,}000 - 40x^2$, $1 \le x \le 5$. Then, $D'(x) = -80x$.
The approximate change in demand dD corresponding to a change $\Delta x = dx$ in the price x is:
$dD = D'(x)dx$
Thus, letting $x = 3$ and $dx = 0.20$, we get
$dD = D'(3)(0.20) = -80(3)(0.20) = -48.$
There will be a 48-pound decrease in demand (approximately) when the price is increased from \$3.00 to \$3.20.

48. $R(x) = 200x - \dfrac{x^2}{30}$; $\quad R'(x) = 200 - \dfrac{x}{15}$

Profit $P(x) = R(x) - C(x) = 200x - \dfrac{x^2}{30} - 72{,}000 - 60x = 140x - \dfrac{x^2}{30} - 72{,}000$

$P'(x) = 140 - \dfrac{x}{15}$

Now, for $x = 1{,}500$, $\Delta x = dx = 10$, we get

$dR = R'(1{,}500)(10) = \left(200 - \frac{1{,}500}{15}\right)(10) = 1{,}000$

$dP = P'(1{,}500)(10) = \left(140 - \frac{1{,}500}{15}\right)(10) = 400$

Thus, the approximate change in revenue is \$1,000 and the approximate change in profit is \$400 if the production is increased from 1,500 to 1,510 televisions.

For $x = 4{,}500$, $\Delta x = dx = 10$, we have:

$dR = R'(4{,}500)(10) = \left(200 - \frac{4{,}500}{15}\right)(10) = -1{,}000$

$dP = P'(4{,}500)(10) = \left(140 - \frac{4{,}500}{15}\right)(10) = -1{,}600.$

Thus, the approximate change in revenue is –\$1,000 and and the approximate change in profit is –\$1,600 if the production is increased from 4,500 to 4,510 televisions.

50. $V = \frac{4}{3}\pi r^3$; $V' = 4\pi r^2$.

The approximate volume of the shell for a radius change from 5 mm to 5.3 mm is given by:

$dV = 4\pi r^2\Big|_{r=5}\ dx = 4\pi(5)^2(0.3)$ (Note: $\Delta x = dx = 0.3$ mm)

$= 94.2$ cubic millimeters

52. $T = x^2\left(1 - \frac{x}{9}\right) = x^2 - \frac{x^3}{9}$, $0 \le x \le 6$; $T' = 2x - \frac{x^2}{3}$.

(A) For $x = 2$, $\Delta x = dx = 0.1$,

$dT = \left(2x - \frac{x^2}{3}\right)\Big|_{x=2}\ dx = \left(2(2) - \frac{2^2}{3}\right)(0.1) = 0.27$ degrees

(B) For $x = 3$, $\Delta x = dx = 0.1$

$dT = \left(2x - \frac{x^2}{3}\right)\Big|_{x=3}\ dx = \left(2(3) - \frac{3^2}{3}\right)(0.1) = 0.3$ degrees

(C) For $x = 4$, $\Delta x = dx = 0.1$

$dT = \left(2x - \frac{x^2}{3}\right)\Big|_{x=4}\ dx = \left(2(4) - \frac{4^2}{3}\right)(0.1) = 0.27$ degrees

54. $y = 52\sqrt{x}$, $0 \le x \le 9$; $y = 52x^{1/2}$ and hence $y' = \frac{52}{2}x^{-1/2} = 26x^{-1/2}$.

For $x = 1$ and $\Delta x = dx = 0.1$ the approximate increase in the number of items learned is given by

$dy = y'\Big|_{x=1}\ dx = 26(1)^{-1/2}(0.1) = 2.6$ items.

Similarly, for $x = 4$, $\Delta x = dx = 0.1$, we have

$dy = y'\Big|_{x=4}\ dx = 26(4)^{-1/2}(0.1) = 1.3$ items.

EXERCISE 2-7

In Problems 2 – 8, $C(x) = 10{,}000 + 150x - 0.2x^2$.

2. $C(100) = 10{,}000 + 150(100) - 0.2(100)^2 = 25{,}000 - 2{,}000 = 23{,}000$, \$23,000

4. $C(199) = 10{,}000 + 150(199) - 0.2(199)^2 = 39{,}850 - 7{,}920.20 = 31{,}929.80$, \$31,929.80

6. Using the results in Problems 4 and 5, $C(200) - C(199) = 32{,}000 - 31{,}929.80 = 70.20$, \$70.20

8. Average cost of producing 200 bicycles: $\frac{C(200)}{200} = \frac{32{,}000}{200} = 160$, \$160

10. $C'(x) = 9.5$

12. $C'(x) = 13 - 0.4x$

14. $R'(x) = 36 - 0.06x$

16. $R'(x) = 25 - 0.10x$

18. $P'(x) = (36 - 0.06x) - 9.5 = 26.5 - 0.06x$

20. $P'(x) = (25 - 0.10x) - (13 - 0.4x) = 12 + 0.3x$

22. $\overline{R}(x) = \frac{5x - 0.02x^2}{x} = 5 - 0.02x$

24. $\overline{R}'(x) = -0.02$

26. $P'(x) = 3.9 - 0.04x$

28. $\overline{P}'(x) = -0.02 + \frac{145}{x^2}$

30. True: If $p = b - mx$ then $R(x) = xp = bx - mx^2$, and $R'(x) = b - 2mx$.

32. False: If $C(x) = 5x + 10$, then the marginal cost is $C'(x) = 5$. In this case, the average marginal cost over any interval is 5. However, the average cost is $\overline{C}(x) = 5 + \frac{10}{x}$ so the marginal average cost is $\overline{C}'(x) = -\frac{10}{x^2}$, which is not equal to 5 over the interval [1,2], for example.

34. $C(x) = 1{,}000 + 100x - 0.25x^2$

(A) The exact cost of producing the 51st guitar is:

$C(51) - C(50)$

$= 1{,}000 + 100(51) - 0.25(51)^2 - [1{,}000 + 100(50) - 0.25(50)^2]$

$= 100 - 0.25(51)^2 + 0.25(50)^2 = 74.75$ or \$74.75

(B) $C'(x) = 100 - 0.5x$

$C'(50) = 100 - 0.5(50) = 75$ or \$75.

36. $C(x) = 20{,}000 + 10x$

(A) $\overline{C}(x) = \frac{20{,}000 + 10x}{x} = \frac{20{,}000}{x} + 10 = 20{,}000x^{-1} + 10$

$\overline{C}(1{,}000) = \frac{20{,}000 + 10(1{,}000)}{1{,}000} = \frac{30{,}000}{1{,}000} = 30$ or \$30

(B) $\overline{C}'(x) = -20{,}000x^{-2} = \frac{-20{,}000}{x^2}$

$\overline{C}'(1{,}000) = \frac{-20{,}000}{(1{,}000)^2} = -0.02$ or -2¢

At a production level of 1,000 dictionaries, average cost is decreasing at the rate of 2¢ per dictionary.

(C) The average cost per dictionary if 1,001 are produced is approximately \$30.00 – \$0.02 = \$29.98.

38. $P(x) = 22x - 0.2x^2 - 400,\ 0 \le x \le 100$

(A) The exact profit from the sale of the 41st calendar is

$P(41) - P(40) = 22(41) - 0.2(41)^2 - 400 - [22(40) - 0.2(40)^2 - 400]$

$= 22 - 0.2(41)^2 + 0.2(40)^2 = 5.80$ or \$5.80

(B) $P'(x) = 22 - 0.4x$

$P'(40) = 22 - 0.4(40) = 22 - 16 = 6$ or \$6

40. $P(x) = 12x - 0.02x^2 - 1{,}000,\ 0 \le x \le 600;\quad P'(x) = 12 - 0.04x$

(A) $P'(200) = 12 - 0.04(200) = 12 - 8 = 4$ or \$4;

at a production level of 200 cameras, profit is increasing at the rate of \$4 per camera.

(B) $P'(350) = 12 - 0.04(350) = 12 - 14 = -2$ or –\$2;

at a production level of 350 cameras, profit is decreasing at the rate of \$2 per camera.

42. $P(x) = 20x - 0.02x^2 - 320,\ 0 \le x \le 1{,}000$

Average profit: $\overline{P}(x) = \frac{P(x)}{x} = 20 - 0.02x - \frac{320}{x} = 20 - 0.02x - 320x^{-1}$

(A) At $x = 40$, $\overline{P}(40) = 20 - 0.02(40) - \frac{320}{40} = 11.20$ or \$11.20.

(B) $\overline{P}'(x) = -0.02 + 320x^{-2} = -0.02 + \frac{320}{x^2}$

$\overline{P}'(40) = -0.02 + \frac{320}{(40)^2} = 0.18$ or \$0.18;

at a production level of 40 grills, the average profit per grill is increasing at the rate of \$0.18 per grill.

(C) The average profit per grill if 41 grills are produced is approximately \$11.20 + \$0.18 = \$11.38.

44. $x = 1{,}000 - 20p$

(A) $20p = 1{,}000 - x,\ p = 50 - 0.05x,\ 0 \le x \le 1{,}000$

(B) $R(x) = x(50 - 0.05x) = 50x - 0.05x^2,\ 0 \le x \le 1{,}000$

(C) $R'(x) = 50 - 0.10x$

$R'(400) = 50 - 0.10(400) = 50 - 40 = 10$;

at a production level of 400 steam irons, revenue is increasing at the rate of \$10 per steam iron.

(D) $R'(650) = 50 - 0.10(650) = 50 - 65 = -15$;
at a production level of 650 steam irons, revenue is decreasing at the rate of \$15 per steam iron.

46. $x = 9{,}000 - 30p$ and $C(x) = 150{,}000 + 30x$

(A) $30p = 9{,}000 - x,\quad p = 300 - \frac{1}{30}x,\ 0 \le x \le 9{,}000$

(B) $C'(x) = 30$

(C) $R(x) = x\left(300 - \frac{1}{30}x\right) = 300x - \frac{1}{30}x^2,\ 0 \le x \le 9{,}000$

(D) $R'(x) = 300 - \frac{1}{15}x$

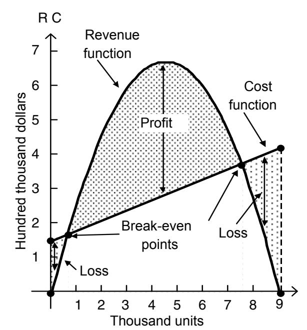
(E) $R'(3{,}000) = 300 - \frac{1}{15}(3{,}000) = 100$; at a production level of

3,000 sets, revenue is increasing at the rate of \$100 per set.

$R'(6000) = 300 - \frac{1}{15}(6{,}000) = 300 - 400 = -100$; at a production level of 6,000 sets, revenue is

decreasing at the rate of \$100 per set.

(F) The graphs of $C(x)$ and $R(x)$ are shown at the right.



To find the break-even points, set $C(x) = R(x)$:

$$150{,}000 + 30x = 300x - \frac{1}{30}x^2$$

$$x^2 - 8{,}100x + 4{,}500{,}000 = 0$$

$$(x - 600)(x - 7{,}500) = 0$$

$$x = 600 \quad \text{or} \quad x = 7{,}500$$

Now, $C(600) = 150{,}000 + 30(600) = 168{,}000$;

$C(7{,}500) = 150{,}000 + 30(7{,}500) = 375{,}000$

Thus, the break-even points are:
(600, 168,000) and (7,500, 375,000).

(G) $P(x) = R(x) - C(x) = 300x - \frac{1}{30}x^2 - (150{,}000 + 30x)$

$$= -\frac{1}{30}x^2 + 270x - 150{,}000$$

(H) $P'(x) = -\frac{1}{15}x + 270$

(I) $P'(1{,}500) = -\frac{1}{15}(1{,}500) + 270 = 170$; at a production level of 1,500 sets, profit is increasing at the

rate of \$170 per set.

$P'(4{,}500) = -\frac{1}{15}(4{,}500) + 270 = -30$; at a production level of 4,500 sets, profit is decreasing at the rate of \$30 per set.

48. (A) We are given $p = 25$ when $x = 300$ and $p = 20$ when $x = 400$. Thus, we have the pair of equations:

$25 = 300m + b$

$20 = 400m + b$

Subtracting the second equation from the first, we get $-100m = 5$. Thus, $m = -\frac{1}{20}$.

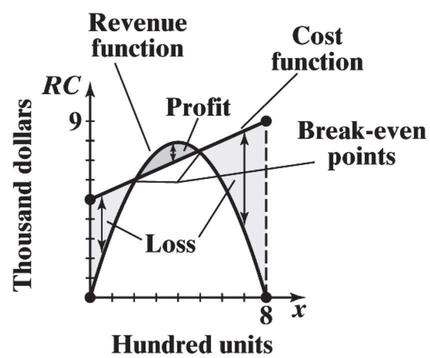
Substituting this into either equation yields $b = 40$. Therefore,

$p = -\frac{1}{20}x + 40 = 40 - \frac{x}{20}, \ 0 \le x \le 800$

(B) $R(x) = x\left(40 - \frac{x}{20}\right) = 40x - \frac{x^2}{20}, \ 0 \le x \le 800$

(C) From the financial department's estimates, $m = 5$ and $b = 5{,}000$. Thus, $C(x) = 5x + 5{,}000$.

(D) The graphs of $R(x)$ and $C(x)$ are shown at the right.



To find the break-even points, set $C(x) = R(x)$:

$$5x + 5{,}000 = 40x - \frac{x^2}{20}$$

$$x^2 - 700x + 100{,}000 = 0$$

$$(x - 200)(x - 500) = 0$$

$$x = 200 \quad \text{or} \quad x = 500$$

Now, $C(200) = 5(200) + 5{,}000 = 6{,}000$ and

$C(500) = 5(500) + 5{,}000 = 7{,}500$

Thus, the break-even points are: (200, 6,000) and (500, 7,500).

(E) $P(x) = R(x) - C(x) = 40x - \frac{x^2}{20} - (5x + 5{,}000)$

$= 35x - \frac{x^2}{20} - 5{,}000$

(F) $P'(x) = 35 - \frac{x}{10}$

$P'(325) = 35 - \frac{325}{10} = 2.5$; at a production level of 325 toasters, profit is increasing at the rate of \$2.50 per toaster.

$P'(425) = 35 - \frac{425}{10} = -7.5$; at a production level of 425 toasters, profit is decreasing at the rate of \$7.50 per toaster.

50. Total cost: $C(x) = 5x + 2{,}340$

Total revenue: $R(x) = 40x - 0.1x^2, \ 0 \le x \le 400$

(A) $R'(x) = 40 - 0.2x$

The graph of R has a horizontal tangent line at the value(s) of x where $R'(x) = 0$, i.e.

$40 - 0.2x = 0$

or $x = 200$

(B) $P(x) = R(x) - C(x) = 40x - 0.1x^2 - (5x + 2{,}340)$

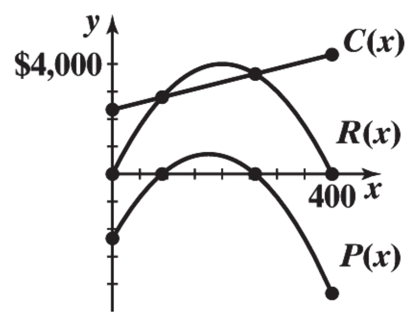
$= 35x - 0.1x^2 - 2{,}340$

(C) $P'(x) = 35 - 0.2x$. Setting $P'(x) = 0$, we have

$35 - 0.2x = 0$

or $x = 175$

(D) The graphs of $C(x)$, $R(x)$ and $P(x)$ are shown at the right.



Break-even points: $R(x) = C(x)$

$$40x - 0.1x^2 = 5x + 2{,}340$$
$$x^2 - 350x + 23{,}400 = 0$$
$$(x - 90)(x - 260) = 0$$
$$x = 90 \quad \text{or} \quad x = 260$$

Thus, the break-even points are:
(90, 2,790) and (260, 3,640).

x intercepts for P: $-0.1x^2 + 35x - 2{,}340 = 0$ or

$x^2 - 350x + 23{,}400 = 0$

which is the same as the above equation. Thus, $x = 90$ and $x = 260$ are x intercepts of P.

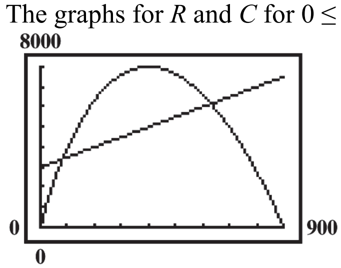
52. Demand equation: $p = 60 - 2\sqrt{x} = 60 - 2x^{1/2}$

Cost equation: $C(x) = 3{,}000 + 5x$

(A) Revenue $R(x) = xp = x(60 - 2x^{1/2})$

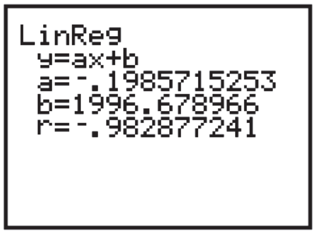
$= 60x - 2x^{3/2}$

(B) The graphs for R and C for $0 \le x \le 900$ are shown below:

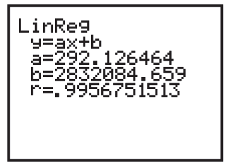


Break-even points: (81, 3,405), (631, 6,155)

54. (A)



(B) Fixed costs: \$2,832,085; variable cost: \$292



(C) Let $y = p(x)$ be the linear regression equation found in part (A) and let $y = C(x)$ be the linear regression equation found in part (B). Then revenue $R(x) = xp(x)$, and the break-even points are

$R(x) = C(x)$.

Break-even points: (2,253, 3,490,130), (6,331, 4,681,675).

(D) The company will make a profit when $2,253 \le x \le 6,331$. From part A), $p(2,253) = 740$ and $p(6,331) = 1,549$. Thus, the company will make a profit for the price range $\$740 \le p \le \$1,549$.

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Section 2-1 Introduction to Limits

Goal: To find limits of functions

Definition: Limit
We write

$$\lim_{x\to c} f(x) = L \text{ or } f(x) \to L \text{ as } x \to c$$

if the functional value $f(x)$ is close to the single real number L whenever x is close, but not equal, to c (on either side of c).

Definition: One sided limits

$\lim_{x\to c^-} f(x) = L$ is the limit of the function as x approaches the value c from the left.

$\lim_{x\to c^+} f(x) = L$ is the limit of the function as x approaches the value c from the right.

Properties of Limits:

1. $\lim_{x\to c} k = k$ for any constant k
2. $\lim_{x\to c} x = c$
3. $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
4. $\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
5. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$ for any constant k
6. $\lim_{x\to c}[f(x)\cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
7. $\lim_{x\to c}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ provided $\lim_{x\to c} g(x) \neq 0$
8. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$ (the limit value must be positive for n even.)

1 - 5 Find each limit if it exists

1. $\lim_{x\to 2}(7x+2) = (7(2)+2) = 14+2 = 16$

2. $\lim_{x\to -3} 5x = 5(-3) = -15$

3. $\lim_{x\to 5} x(4x+1) = \lim_{x\to 5} x \cdot \lim_{x\to 5}(4x+1) = 5[4(5)+1)] = 5(21) = 105$

4. $\lim_{x\to -1}\left(\frac{x+8}{x+2}\right) = \frac{\lim_{x\to -1} x+8}{\lim_{x\to -1} x+2} = \frac{-1+8}{-1+2} = \frac{7}{1} = 7$

5. $\lim_{x\to -3}\sqrt{-5x+1} = \sqrt{-5(-3)+1} = \sqrt{15+1} = \sqrt{16} = 4$

6 - 8 Find the value of the following limits given that

$\lim_{x\to 3} f(x) = 4$ and $\lim_{x\to 3} g(x) = -3.$

6. $\lim_{x\to 3} 7f(x) = 7\cdot 4 = 28$

7. $\lim_{x\to 3}[3f(x)-2g(x)] = 3(4)-2(-3) = 12+6 = 18$

8. $\lim_{x\to 3}\left(\frac{2f(x)}{3g(x)}\right) = \frac{2(4)}{3(-3)} = \frac{8}{-9} = -\frac{8}{9}$

9. Let $f(x)=\begin{cases} x^2+2 & \text{if } x<1 \\ 4x-1 & \text{if } x\geq 1 \end{cases}$. Find:

a) $\lim\limits_{x\to 1^-} f(x)$

Since we are looking for the left–hand limit, use the top function.

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x^2+2)=((1)^2+2)=3$$

b) $\lim\limits_{x\to 1^+} f(x)$

Since we are looking for the right–hand limit, use the bottom function.

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(4x-1)=(4(1)-1)=3$$

c) $\lim\limits_{x\to 1} f(x)$ Since the left– and right–hand limits are the same value, $\lim\limits_{x\to 1} f(x)=3.$

d) $f(1)$

The value of 1 is defined in the bottom function, therefore $f(1)=4(1)-1=3.$

10. Let $f(x)=\begin{cases} 5x-6 & \text{if } x\leq 2 \\ 2x+4 & \text{if } x>2 \end{cases}$. Find:

a) $\lim\limits_{x\to 2^-} f(x)$

Since we are looking for the left–hand limit, use the top function.

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(5x-6)=5(2)-6=4$$

b) $\lim\limits_{x\to 2^+} f(x)$

Since we are looking for the right–hand limit, use the bottom function.

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(2x+4)=(2(2)+4)=8$$

c) $\lim_{x\to 2} f(x)$

Since the left– and right–hand limits are not the same value, $\lim_{x\to 2} f(x)$ does not exist.

d) $f(2)$

The value of 2 is defined in the top function, therefore $f(2) = 5(2) - 6 = 4$.

11. Let $f(x) = \left(\dfrac{x^2 + 4x - 5}{x - 1}\right)$. Find

a) $\lim_{x\to 1} f(x)$

Substituting in a value of 1 will result in a zero in the denominator, therefore we must try to remove the problem by first simplifying the function:

$$f(x) = \frac{x^2 + 4x - 5}{x - 1} = \frac{(x+5)(x-1)}{x-1} = x + 5, \quad x \neq 1$$

Therefore, $\lim_{x\to 1} f(x) = \lim_{x\to 1} (x+5) = 1 + 5 = 6$

b) $\lim_{x\to -1} f(x) = \lim_{x\to -1} \dfrac{x^2 + 4x - 5}{x - 1} = \dfrac{(-1)^2 + 4(-1) - 5}{-1 - 1} = \dfrac{-8}{-2} = 4$

c) $\lim_{x\to -5} f(x) = \lim_{x\to -5} \dfrac{x^2 + 4x - 5}{x - 1} = \dfrac{(-5)^2 + 4(-5) - 5}{-5 - 1} = \dfrac{0}{-6} = 0$

12. Let $f(x)=\left(\frac{|x-2|}{x-2}\right)$. Find

a) $\lim_{x\to 2^-} f(x)$

As x approaches 2 from the left side, the value of the numerator will be positive and the value of the denominator will be negative (because the x values are smaller than 2). The limit will therefore, be a value of –1 (because they are opposite signs).

b) $\lim_{x\to 2^+} f(x)$

As x approaches 2 from the right side, the value of the numerator will be positive and the value of the denominator will be positive (because the x values are larger than 2). The limit will therefore, be a value of 1 (because they are the same sign).

c) $\lim_{x\to 2} f(x)$ does not exist because the left–hand and right–hand limits are different values.

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Section 2-2 Infinite Limits and Limits at Infinity

Goal: To find limits of functions as they approach infinity

Limits of Power Functions at Infinity:

If p is a positive real number and k is any real number except 0, then

1. $\lim_{x\to-\infty} \frac{k}{x^p} = 0$
2. $\lim_{x\to\infty} \frac{k}{x^p} = 0$
3. $\lim_{x\to-\infty} kx^p = \pm\infty$
4. $\lim_{x\to\infty} kx^p = \pm\infty$

provided that x^p is a real number for negative values of x. The limits in 3 and 4 will be either positive or negative infinity, depending on k and p.

Limits of Rational Functions at Infinity:

If $f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}, a_m \neq 0, b_n \neq 0,$

then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{a_m x^m}{b_n x^n}$ and $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{a_m x^m}{b_n x^n}$

There are three cases to consider:

1. If $m < n$, then $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$
2. If $m = n$, $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \frac{a_m}{b_n}.$
3. If $m > n$, then the limit will be ∞ or $-\infty$, depending on the values of m, n, a_m, and b_n.

1 - 3 Find each limit. Use ∞ or $-\infty$ when appropriate.

1. $f(x) = \dfrac{x}{x+4}$

a) $\lim\limits_{x \to -4^-} f(x)$

If the value is substituted into the function, we would have a zero in the denominator. Since there is no way to remove the problem, we will look at what happens around the value of –4 when approached from the left. These values are smaller than –4, therefore the denominator will approach 0 from the negative side (that is, the denominator will always be negative). Since the numerator will always be negative, $\lim\limits_{x \to -4^-} f(x) = \infty.$

b) $\lim\limits_{x \to -4^+} f(x)$

If the value is substituted into the function, we would have a zero in the denominator. Since there is no way to remove the problem, we will look at what happens around the value of –4 when approached from the right. These values are larger than –4, therefore the denominator will approach 0 from the positive side (that is, the denominator will always be positive). Since the numerator will always be negative, $\lim\limits_{x \to -4^+} f(x) = -\infty.$

c) $\lim\limits_{x \to -4} f(x)$ does not exist because the left–hand and right–hand limits are different infinite limits.

2. $f(x) = \dfrac{4x+1}{(x-5)^2}$

a) $\lim\limits_{x \to 5^-} f(x)$

If the value is substituted into the function, we would have a zero in the denominator. Since there is no way to remove the problem, we will look at what happens around the value of 5 when approached from the left. These values are smaller than 5, but the denominator value is being squared and will always be positive. Since the value of the numerator is 21 as the x value approaches 5, $\lim\limits_{x \to 5^-} f(x) = \infty.$

b) $\lim_{x \to 5^+} f(x)$

If the value is substituted into the function, we would have a zero in the denominator. Since there is no way to remove the problem, we will look at what happens around the value of 5 when approached from the right. These values are larger than 5, but the denominator value is being squared and will always be positive. Since the value of the numerator is 21 as the x value approaches 5, $\lim_{x \to 5^+} f(x) = \infty$.

c) $\lim_{x \to 5} f(x) = \infty$ because the left– and right–hand limits both approach infinity.

3. $f(x) = \dfrac{x^2 + 3x - 5}{x + 5}$

a) $\lim_{x \to -5^-} f(x)$

If the value is substituted into the function, we would have a zero in the denominator. Since there is no way to remove the problem, we will look at what happens around the value of –5 when approached from the left. These values are smaller than –5, therefore the denominator will approach 0 from the negative side (that is, the denominator will always be negative). Since the numerator is positive around –5, $\lim_{x \to -5^-} f(x) = -\infty$.

b) $\lim_{x \to -5^+} f(x)$

If the value is substituted into the function, we would have a zero in the denominator. Since there is no way to remove the problem, we will look at what happens around the value of –5 when approached from the right. These values are larger than –5, therefore the denominator will approach 0 from the positive side (that is, the denominator will always be positive). Since the numerator is positive around –5, $\lim_{x \to -5^+} f(x) = \infty$.

c) $\lim_{x \to -5} f(x)$ does not exist because the left–hand and right–hand limits are different infinite limits.

4 - 6 Find each function value and limit. Use ∞ or $-\infty$ where appropriate.

4. $f(x)=\dfrac{4x+1}{6x-3}$

a) $f(20)$

$$f(20)=\frac{4(20)+1}{6(20)-3}=\frac{81}{117}\approx 0.6923$$

b) $f(200)$

$$f(200)=\frac{4(200)+1}{6(200)-3}=\frac{801}{1197}\approx 0.6692$$

c) $\lim\limits_{x\to\infty} f(x)=\dfrac{4}{6}=\dfrac{2}{3}$ because the function is a rational expression and $m=n$, the limit is the ratio of the coefficients.

5. $f(x)=\dfrac{x-5}{3x^2+2x+2}$

a) $f(10)$

$$f(10)=\frac{10-5}{3(10)^2+2(10)+2}=\frac{5}{322}\approx 0.0155$$

b) $f(100)$

$$f(100)=\frac{100-5}{3(100)^2+2(100)+2}=\frac{95}{30,222}\approx 0.0031$$

c) $\lim\limits_{x\to\infty} f(x)=0$ because the function is a rational expression and $m<n$.

6. $f(x)=\dfrac{x^3+2x-1}{3-8x}$

a) $f(-10)$

$$f(-10)=\frac{(-10)^3+2(-10)-1}{3-8(-10)}=\frac{-1021}{83}\approx -12.3012$$

b) $f(-100)$

$$f(-100)=\frac{(-100)^3+2(-100)-1}{3-8(-100)}=\frac{-1,000,201}{803}\approx -1245.5803$$

c) $\lim\limits_{x\to-\infty} f(x)=-\infty$ because the function is a rational expression, $m>n$, and when x becomes increasingly negative, the value of the function will be increasingly negative.

7 - 9 Find the vertical and horizontal asymptotes for the following functions.

7. $f(x)=\dfrac{5x}{x+4}$

Vertical asymptotes are found by setting the denominator equal to zero. Therefore, there would be a vertical asymptote at $x+4=0 \Rightarrow x=-4$.

Horizontal asymptotes are found by finding the limit of the function as it goes to infinity. The limit of this function is 5 because $m=n$ and the limit is the ratio of the coefficients. Therefore, the horizontal asymptote is $y=5$.

8. $f(x)=\dfrac{x^2-3}{x^2+4}$

Vertical asymptotes are found by setting the denominator equal to zero. Since the denominator cannot be zero, there are no vertical asymptotes.

Horizontal asymptotes are found by finding the limit of the function as it goes to infinity. The limit of this function is 1 because $m=n$ and the limit is the ratio of the coefficients. Therefore, the horizontal asymptote is $y=1$.

9. $f(x)=\frac{x^2+4x-5}{x^2+8x+15}$

The function can be reduced as follows:

$$f(x)=\frac{x^2+4x-5}{x^2+8x+15}=\frac{(x+5)(x-1)}{(x+5)(x+3)}=\frac{x-1}{x+3},\ \ x\neq -5 \text{ and } x\neq -3$$

Vertical asymptotes are found by setting the denominator equal to zero. Therefore, there would be a vertical asymptote at $x+3=0\Rightarrow x=-3$.

Horizontal asymptotes are found by finding the limit of the function as it goes to infinity. The limit of this function is 1 because $m=n$ and the limit is the ratio of the coefficients. Therefore, the horizontal asymptote is $y=1$.

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Section 2-3 Continuity

Goal: To determine if functions are continuous at specific points and intervals

Definition: Continuity

A function *f* is continuous at the point $x = c$ if

1. $\lim_{x \to c} f(x)$ exists 2. $f(c)$ exists 3. $\lim_{x \to c} f(x) = f(c)$

Continuity Properties:

1. A constant function $f(x) = k$, where *k* is a constant, is continuous for all *x*.
2. For *n* a positive integer, $f(x) = x^n$ is continuous for all *x*.
3. A polynomial function is continuous for all *x*.
4. A rational function is continuous for all *x* except those values that make a denominator 0.
5. For *n* an odd positive integer greater than 1, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous.
6. For *n* an even positive integer, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous and nonnegative.

Constructing Sign Charts:

1. Find all partition numbers. These are all the values that make the function discontinuous or 0.
2. Plot the numbers found in step 1 on a real–number line, dividing the number line into intervals.
3. Select a test value in each open interval and evaluate $f(x)$ at each test value to determine whether $f(x)$ is positive or negative.
4. Construct a sign chart, using the real–number line in step 2.

1 - 5 Using the continuity properties, determine where each of the functions are continuous.

1. $f(x) = 3x^3 - 4x^2 + x + 7$

Since the function is a polynomial function, it is continuous for all *x*.

2. $f(x) = \dfrac{x^3 + 4x^2 + 3x - 7}{x^2 - 15x + 26}$

Since the function is a rational function, it is continuous for all *x* except when the denominator is 0. Therefore, $x^2 - 15x + 26 = 0,$ or $(x-2)(x-13) = 0,$ when $x = 2$ or $x = 13.$ So the function is continuous for all *x* except $x = 2$ or $x = 13.$

3. $f(x) = \dfrac{3x - 2}{x^2 + 3}$

Since the function is a rational function, it is continuous for all *x* except when the denominator is 0. The denominator cannot have a value of 0, therefore the function is continuous for all *x*.

4. $f(x) = \sqrt[3]{x - 10}$

Since the function is an odd positive root, the function is continuous for all values of *x* where the radicand is continuous. The radicand is a polynomial function, therefore the function is continuous for all *x*.

5. $f(x) = \sqrt{x^2 - 25}$

Since the function is an even root, the function is continuous for all values of *x* where the radicand is continuous and nonnegative. Therefore, $x^2 - 25 \geq 0$ and the points in question are $\pm 5.$ For all values between –5 and 5, the radicand would be negative, therefore, the function is continuous on the interval $(-\infty, -5] \cup [5, \infty).$

6 - 7 Use a sign chart to solve each inequality. Express answers in inequality and interval notation.

6. $4x^2 - 29x + 7 < 0$

Find the partition numbers:

$$4x^2 - 29x + 7 = 0$$
$$(4x - 1)(x - 7) = 0$$

Therefore, the partition numbers are $\frac{1}{4}$ and 7.

The number line would be broken into three parts involving the intervals $(-\infty, \frac{1}{4})$, $(\frac{1}{4}, 7)$, and $(7, \infty)$. Pick test values. We choose 0, 1, and 8 for the test values. Find the function values using these three test values.

$$f(0) = 4(0)^2 - 29(0) + 7$$
$$f(0) = 7$$

$$f(1) = 4(1)^2 - 29(1) + 7$$
$$f(1) = 4 - 29 + 7$$
$$f(1) = -18$$

$$f(8) = 4(8)^2 - 29(8) + 7$$
$$f(8) = 256 - 232 + 7$$
$$f(8) = 31$$

Only the test value 1 makes the inequality true, so the solution is $\frac{1}{4} < x < 7$, or $(\frac{1}{4}, 7)$.

7. $\dfrac{x^3+3x^2}{x+5}>0$

Find the partition numbers:

$$x^3+3x^2=0 \qquad x+5=0$$
$$x^2(x+3)=0$$

Therefore, the partition numbers are 0, –3, and –5.

The number line would be broken into four parts involving the intervals $(-\infty,-5)$, $(-5,-3)$, $(-3,0)$, and $(0,\infty)$. Pick test values. We choose –6, –4, –1, and 1 for the test values. Find the function values using these four test values.

$$f(-6)=\frac{(-6)^3+3(-6)^2}{-6+5}=\frac{-108}{-1}=108 \qquad f(-4)=\frac{(-4)^3+3(-4)^2}{-4+5}=\frac{-16}{1}=-16$$

$$f(-1)=\frac{(-1)^3+3(-1)^2}{-1+5}=\frac{2}{4}=0.5 \qquad f(1)=\frac{(1)^3+3(1)^2}{1+5}=\frac{4}{6}\approx 0.67$$

The test values of –6, –1, and 1 make the original inequality true, so the solution is $x<-5$ or $x>-3$, or $(-\infty,-5)\cup(-3,\infty)$.

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Section 2-4 The Derivative

Goal: To find the first derivative of a function using the four step process.

Definition: Average Rate of Change

For $y = f(x)$, the average rate of change from $x = a$ to $x = a + h$ is

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h},\ h \neq 0$$

where h is the distance from the initial value of x to the final value of x.

Definition: Instantaneous Rate of Change

For $y = f(x)$, the instantaneous rate of change at $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \text{ if the limit exists.}$$

This formula is also used to find the slope of a graph at the point $(a, f(a))$ and to find the first derivative of a function, $f(x)$.

Procedure: Finding the first derivative:

1. Find $f(x+h)$.
2. Find $f(x+h) - f(x)$.
3. Find $\dfrac{f(x+h) - f(x)}{h}$.
4. Find $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

1 - 3 Use the four step procedure to find $f'(x)$ and then find $f'(1)$, $f'(2)$, and $f'(3)$.

1. $f(x) = 7x - 3$

Step 1: $f(x+h) = 7(x+h) - 3 = 7x + 7h - 3$

Step 2: $f(x+h) - f(x) = 7x + 7h - 3 - (7x - 3) = 7h$

Step 3: $\dfrac{f(x+h) - f(x)}{h} = \dfrac{7h}{h} = 7$

Step 4: $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} 7 = 7$

$f'(x) = 7$, $f'(1) = 7$, $f'(2) = 7$, and $f'(3) = 7$.

2. $f(x) = -3x^2 + 5x - 6$

Step 1:
$$f(x+h) = -3(x+h)^2 + 5(x+h) - 6$$
$$= -3(x^2 + 2xh + h^2) + 5(x+h) - 6$$
$$f(x+h) = -3x^2 - 6xh - 3h^2 + 5x + 5h - 6$$

Step 2:
$$f(x+h) - f(x) = -3x^2 - 6xh - 3h^2 + 5x + 5h - 6 - (-3x^2 + 5x - 6)$$
$$f(x+h) - f(x) = -6xh - 3h^2 + 5h$$

Step 3:
$$\frac{f(x+h) - f(x)}{h} = \frac{-6xh - 3h^2 + 5h}{h}$$
$$\frac{f(x+h) - f(x)}{h} = -6x - 3h + 5$$

Step 4:
$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} (-6x - 3h + 5)$$
$$= -6x - 3(0) + 5$$
$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = -6x + 5$$

Therefore, $f'(x) = -6x + 5$

$f'(1) = -6(1)+5$ $\quad$ $f'(2) = -6(2)+5$ $\quad$ $f'(3) = -6(3)+5$

$f'(1) = -1$ $\quad$ $f'(2) = -7$ $\quad$ $f'(3) = -13$

3. $f(x) = \dfrac{2x}{x-5}$

Step 1: $f(x+h) = \dfrac{2(x+h)}{(x+h)-5} = \dfrac{2x+2h}{x+h-5}$

Step 2:
$$f(x+h) - f(x) = \frac{2x+2h}{x+h-5} - \frac{2x}{x-5}$$
$$= \frac{(2x+2h)(x-5)}{(x+h-5)(x-5)} - \frac{2x(x+h-5)}{(x+h-5)(x-5)}$$
$$f(x+h) - f(x) = \frac{-10h}{(x+h-5)(x-5)}$$

Step 3:
$$\frac{f(x+h)-f(x)}{h} = \frac{-10h}{(x+h-5)(x-5)h}$$
$$\frac{f(x+h)-f(x)}{h} = \frac{-10}{(x+h-5)(x-5)}$$

Step 4:
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{-10}{(x+h-5)(x-5)}$$
$$= \frac{-10}{(x+0-5)(x-5)}$$
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{-10}{(x-5)^2}$$

Therefore, $f'(x) = \dfrac{-10}{(x-5)^2}$

$$f'(1) = \frac{-10}{(1-5)^2} = \frac{-10}{(-4)^2} = \frac{-10}{16} = -\frac{5}{8}$$

$$f'(2) = \frac{-10}{(2-5)^2} = \frac{-10}{(-3)^2} = \frac{-10}{9} = -\frac{10}{9}$$

$$f'(3) = \frac{-10}{(3-5)^2} = \frac{-10}{(-2)^2} = \frac{-10}{4} = -\frac{5}{2}$$

4. The profit, in hundreds of dollars, from the sale of x items is given by

$$P(x) = 2x^2 - 5x + 6$$

a. Find the average rate of change of profit from $x = 2$ to $x = 4$.
b. Find the instantaneous rate of change equation using the four–step procedure.
c. Using the equation found in part b, find the instantaneous rate of change when $x = 2$ and interpret the results.

Solution:

a. First find the function values at 2 and 4. Note that $h = 4 - 2 = 2$.

$$P(2) = 2(2)^2 - 5(2) + 6 \qquad P(4) = 2(4)^2 - 5(4) + 6$$
$$P(2) = 8 - 10 + 6 \qquad P(4) = 32 - 20 + 6$$
$$P(2) = 4 \qquad P(4) = 18$$

Now use the average rate of change formula:

$$\begin{aligned}\frac{P(a+h)-P(a)}{h} &= \frac{P(2+2)-P(2)}{2}\\ &= \frac{P(4)-P(2)}{2}\\ &= \frac{18-4}{2} = 7\end{aligned}$$

b. Step 1:

$$\begin{aligned}P(x+h) &= 2(x+h)^2 - 5(x+h) + 6\\ &= 2(x^2 + 2xh + h^2) - 5(x+h) + 6\\ P(x+h) &= 2x^2 + 4xh + 2h^2 - 5x - 5h + 6\end{aligned}$$

Step 2:

$$\begin{aligned}P(x+h) - P(x) &= 2x^2 + 4xh + 2h^2 - 5x - 5h + 6 - (2x^2 - 5x + 6)\\ P(x+h) - P(x) &= 4xh + 2h^2 - 5h\end{aligned}$$

Step 3:

$$\begin{aligned}\frac{P(x+h)-P(x)}{h} &= \frac{4xh + 2h^2 - 5h}{h}\\ \frac{P(x+h)-P(x)}{h} &= 4x + 2h - 5\end{aligned}$$

Step 4: $$\lim_{h\to 0}\frac{P(x+h)-P(x)}{h}=\lim_{h\to 0}(4x+2h-5)$$
$$=4x+2(0)-5$$
$$\lim_{h\to 0}\frac{P(x+h)-P(x)}{h}=4x-5$$

Therefore, $P'(x)=4x-5$

c. $P'(2)=4(2)-5=3$ means that when the second item was sold, your profit increased by $300.

5. The distance of a particle from some fixed point is given by

$$s(t)=t^2+5t+2$$

where t is time measured in seconds.

a. Find the average velocity from $t = 4$ to $t = 6$.
b. Find the instantaneous rate of change equation using the four–step procedure.
c. Using the equation found in part b, find the instantaneous rate of change when $t = 4$ and interpret the results.

Solution:

a. First find the function values at 4 and 6:

$$s(4)=(4)^2+5(4)+2 \qquad s(6)=(6)^2+5(6)+2$$
$$s(4)=16+20+2 \qquad s(6)=36+30+2$$
$$s(4)=38 \qquad s(6)=68$$

Now use the average rate of change formula. Note that $h=6-4=2$.

$$\frac{s(t+h)-s(t)}{h}=\frac{s(4+2)-s(4)}{2}$$
$$=\frac{s(6)-s(4)}{2}$$
$$=\frac{68-38}{2}=15$$

b. Step 1:
$$s(t+h) = (t+h)^2 + 5(t+h) + 2$$
$$s(t+h) = t^2 + 2th + h^2 + 5t + 5h + 2$$

Step 2:
$$s(t+h) - s(t) = t^2 + 2th + h^2 + 5t + 5h + 2 - (t^2 + 5t + 2)$$
$$s(t+h) - s(t) = 2th + h^2 + 5h$$

Step 3:
$$\frac{s(t+h) - s(t)}{h} = \frac{2th + h^2 + 5h}{h}$$
$$\frac{s(t+h) - s(t)}{h} = 2t + h + 5$$

Step 4:
$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = \lim_{h \to 0} (2t + h + 5)$$
$$= 2t + 0 + 5$$
$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = 2t + 5$$

Therefore, $s'(t) = 2t + 5$

c. $s'(4) = 2(4) + 5 = 9$ means that after 2 seconds the particle is traveling at 9 units per second.

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Section 2-5 Basic Differentiation Properties

Goal: To find the first derivatives using the basic properties

Notation: If $y = f(x)$, then $f'(x)$, y', $\frac{dy}{dx}$ all represent the derivative of *f* at *x*.

Theorems:

1. If $y = f(x) = C$, then $f'(x) = 0$ (Constant Function Rule)
2. If $y = f(x) = x^n$, where *n* is a real number, then $f'(x) = nx^{n-1}$ (Power Rule)
3. If $y = f(x) = ku(x)$, then $f'(x) = ku'(x)$ (Constant Multiple Property)
4. If $y = f(x) = u(x) \pm v(x)$, then $f'(x) = u'(x) \pm v'(x)$ (Sum and Difference Property)

1 - 6 Find the indicated derivatives.

1. y' for $y = x^3$

   Use theorem 2 to find the derivative: $y' = 3x^2$

2. $\frac{dy}{dx}$ for $y = \frac{1}{x^7}$

   Convert the problem to the form x^n and then use theorem 2.

   $y = \frac{1}{x^7} = x^{-7}$ $\frac{dy}{dx} = -7x^{-8} = -\frac{7}{x^8}$

3. $\frac{d}{du}(2u^2 - 3u + 8)$

Use theorem 4 to break the original function into pieces, then use a combination of theorems 1–3.

$$\begin{aligned}\frac{d}{du}(2u^2 - 3u + 8) &= \frac{d}{du}2u^2 - \frac{d}{du}3u + \frac{d}{du}8 \\ &= 4u - 3 + 0 \\ &= 4u - 3\end{aligned}$$

4. $f'(x)$ if $f(x) = x^{3.2} + 7x^{2.1} - 3x + 7$

Use theorem 4 to break the original function into pieces, then use a combination of theorems 1–3.

$$\begin{aligned}f'(x) &= f'(x^{3.2}) + 7f'(x^{2.1}) - 3f'(x) + f'(7) \\ &= 3.2x^{2.2} + 7(2.1)x^{1.1} - 3 + 0 \\ &= 3.2x^{2.2} + 14.7x^{1.1} - 3\end{aligned}$$

5. $\frac{d}{dx}(\frac{1}{x^4} + 6\sqrt{x} - 5x + 8)$

Use theorem 4 to break the original function into pieces, then use a combination of theorems 1–3.

$$\begin{aligned}\frac{d}{dx}(\frac{1}{x^4} + 6\sqrt{x} - 5x + 8) &= \frac{d}{dx}\frac{1}{x^4} + \frac{d}{dx}6\sqrt{x} - \frac{d}{dx}5x + \frac{d}{dx}8 \\ &= \frac{d}{dx}x^{-4} + \frac{d}{dx}6x^{1/2} - \frac{d}{dx}5x + \frac{d}{dx}8 \\ &= -4x^{-5} + 6(\tfrac{1}{2})x^{-1/2} - 5 + 0 \\ &= -\frac{4}{x^5} + \frac{3}{\sqrt{x}} - 5\end{aligned}$$

6. $f'(x)$ if $f(x) = x^{4/5} + 4x^{3/5} - 2x^{-1/5} + 5$

Use theorem 4 to break the original function into pieces, then use a combination of theorems 1 - 3.

$$\begin{aligned}f'(x) &= f'(x^{4/5}) + 4f'(x^{3/5}) - 2f'(x^{-1/5}) + f'(5) \\ &= \tfrac{4}{5}x^{-1/5} + \tfrac{12}{5}x^{-2/5} + \tfrac{2}{5}x^{-6/5}\end{aligned}$$

7. Given the function $f(x) = 7x^2 - 9x + 2$

a. Find $f'(x)$.
b. Find the slope of the graph of f at $x = 3$.
c. Find the equation of the tangent line at $x = 3$.
d. Find the value of x where the tangent is horizontal.

Solution:

a.
$$f(x) = 7x^2 - 9x + 2$$
$$f'(x) = 7f'(x^2) - 9f'(x) + f'(2)$$
$$f'(x) = 14x - 9$$

b. Substitute the value into the equation in part a to find the slope.

$$f'(x) = 14x - 9$$
$$f'(3) = 14(3) - 9$$
$$m = 33$$

c. Substitute the given value into the original function to find the y value of the point and then use that point with the slope found in part b. The point is (3, 38).

$$y - y_1 = m(x - x_1)$$
$$y - (38) = 33(x - 3)$$
$$y - 38 = 33x - 99$$
$$y = 33x - 61$$

d. The tangent is horizontal when the first derivative has a value of 0.

$$f'(x) = 14x - 9$$
$$0 = 14x - 9$$
$$9 = 14x$$
$$0.64 \approx x$$

8 - 9 Use the following information for both problems:

If an object moves along the y axis (marked in feet) so that its position at time x (in seconds) is given by the indicated function, find:

a. The instantaneous velocity function $v = f'(x)$
b. The velocity when $x = 0$ and $x = 4$
c. The time(s) when $v = 0$

8. $f(x) = 3x^2 - 12x - 8$

Solution:

a.
$$f(x) = 3x^2 - 12x - 8$$
$$f'(x) = 3f'(x^2) - 12f'(x) - f'(8)$$
$$f'(x) = 6x - 12$$
$$v = 6x - 12$$

b.
$$f'(x) = 6x - 12 \qquad f'(x) = 6x - 12$$
$$v(x) = 6x - 12 \qquad v(x) = 6x - 12$$
$$v(0) = 6(0) - 12 \qquad v(4) = 6(4) - 12$$
$$v(0) = -12 \qquad v(4) = 12$$

c.
$$v(x) = 6x - 12$$
$$0 = 6x - 12$$
$$12 = 6x$$
$$2 = x$$

9. $f(x)=x^3-\frac{21}{2}x^2+30x$

Solution:

a. $f(x)=x^3-\frac{21}{2}x^2+30x$

$f'(x)=f'(x^3)-\frac{21}{2}f'(x^2)+30f'(x)$

$f'(x)=3x^2-21x+30$

$v=3x^2-21x+30$

b. $f'(x)=3x^2-21x+30$

$v(x)=3x^2-21x+30$

$v(0)=3(0)^2-21(0)+30$

$v(0)=0-0+30$

$v(0)=30$

$f'(x)=3x^2-21x+30$

$v(x)=3x^2-21x+30$

$v(4)=3(4)^2-21(4)+30$

$v(4)=48-84+30$

$v(4)=-6$

c. $v(x)=3x^2-21x+30$

$0=3x^2-21x+30$

$0=3(x^2-7x+10)$

$0=3(x-5)(x-2)$

$x=2,5$

10. Find $f'(x)$ if $f(x)=(5x-6)^2$

First simplify the function by using the FOIL to expand it.

$f(x)=(5x-6)^2=(5x-6)(5x-6)=25x^2-60x+36$

Now use theorem 4 to break the original function into pieces, then use a combination of theorems 1–3.

$f(x)=25x^2-60x+36$

$f'(x)=25f'(x^2)-60f'(x)+f'(36)$

$f'(x)=50x-60$

11. Find $f'(x)$ if $f(x) = \frac{9x-5}{x}$.

First simplify the function.

$$f(x) = \frac{9x-5}{x} = \frac{9x}{x} - \frac{5}{x} = 9 - 5x^{-1}$$

Now use theorem 4 to break the original function into pieces, then use a combination of theorems 1 - 3.

$$f(x) = 9 - 5x^{-1}$$
$$f'(x) = f'(9) - 5f'(x^{-1})$$
$$f'(x) = 0 + 5x^{-2}$$
$$f'(x) = \frac{5}{x^2}$$

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Section 2-6 Differentials

Goal: To use differentials to solve problems

Definition: Differentials

If $y = f(x)$ defines a differentiable function, then the differential *dy* or *df* is defined as the product of $f'(x)$ and *dx*, where $dx = \Delta x$. Symbolically,

$$dy = f'(x)dx \quad \text{or} \quad df = f'(x)dx$$

where $dx = \Delta x$.

Recall that $\Delta y = f(x + \Delta x) - f(x)$

1. Given the function $y = 3x^4$, find $\Delta x, \Delta y,$ and $\frac{\Delta y}{\Delta x}$ given $x_1 = 2$ and $x_2 = 5$.

When $x_1 = 2,\ y_1 = 3(2)^4 = 48.$ When $x_2 = 5,\ y_2 = 3(5)^4 = 1875.$

$\Delta x = 5 - 2$ $\qquad \Delta y = 1875 - 48$ $\qquad \frac{\Delta y}{\Delta x} = \frac{1827}{3} = 609$

$\Delta x = 3$ $\qquad \Delta y = 1827$

2. Given the function $y = 3x^4$, find $\Delta x, \Delta y,$ and $\frac{\Delta y}{\Delta x}$ given $x_1 = 3$ and $x_2 = 6$.

When $x_1 = 3,\ y_1 = 3(3)^4 = 243.$ When $x_2 = 6,\ y_2 = 3(6)^4 = 3888.$

$\Delta x = 6 - 3$ $\qquad \Delta y = 3888 - 243$ $\qquad \frac{\Delta y}{\Delta x} = \frac{3645}{3} = 1215$

$\Delta x = 3$ $\qquad \Delta y = 3645$

3. Given the function $y = 3x^3 - 12x^2 + 4x + 16$, find dy.

$$y = 3x^3 - 12x^2 + 4x + 16$$
$$\frac{dy}{dx} = 9x^2 - 24x + 4$$
$$dy = (9x^2 - 24x + 4)dx$$

4. Given the function $y = x^5(2 - \frac{x^3}{12})$, find dy.

$$y = x^5(2 - \tfrac{x^3}{12})$$
$$y = 2x^5 - \tfrac{1}{12}x^8$$
$$\frac{dy}{dx} = 10x^4 - \tfrac{8}{12}x^7$$
$$dy = (10x^4 - \tfrac{2}{3}x^7)dx$$

5. Given the function $y = 25 - 7x^2 - x^3$, find dy and Δy given $x = 4$ and $dx = \Delta x = 0.1$.

$$y = 25 - 7x^2 - x^3$$
$$\frac{dy}{dx} = -14x - 3x^2$$
$$dy = (-14x - 3x^2)dx$$

$$dy = (-14x - 3x^2)dx$$
$$dy = [-14(4) - 3(4)^2](0.1)$$
$$dy = (-56 - 48)(0.1)$$
$$dy = (-104)(0.1)$$
$$dy = -10.4$$

$$\Delta y = f(x + \Delta x) - f(x)$$
$$\Delta y = f(4 + 0.1) - f(4)$$
$$\Delta y = f(4.1) - f(4)$$
$$\Delta y = (25 - 7(4.1)^2 - (4.1)^3) - (25 - 7(4)^2 - (4)^3)$$
$$\Delta y = -161.591 - (-151)$$
$$\Delta y = -10.591$$

6. A company will sell N units of a product after spend x thousand dollars in advertising, as given by

$$N = 120x - x^2 \qquad 10 \le x \le 60$$

Approximately what increase in sales will result by increasing the advertising budget from $15,000 to $17,000? From $25,000 to $27,000?

$$N = 120x - x^2 \qquad dN = (120 - 2x)dx \qquad dN = (120 - 2x)dx$$
$$\frac{dN}{dx} = 120 - 2x \qquad dN = (120 - 2(15))(2) \qquad dN = (120 - 2(25))(2)$$
$$dN = (120 - 2x)dx \qquad dN = 180 \qquad dN = 140$$

Therefore, increasing the advertising budget from $15,000 to $17, 000 will result in an increase of 180 units and an increase from $25,000 to $27,000 will only result in a 140 unit increase.

7. The average pulse rate y (in beats per minute) of a healthy person x inches tall is given approximately by

$$y = \frac{590}{\sqrt{x}} \qquad 30 \le x \le 75$$

Approximately how will the pulse rate change for a change in height from 49 inches to 52 inches?

$$y = \frac{590}{\sqrt{x}} = 590x^{-1/2}$$
$$y = 590x^{-1/2} \qquad dy = (-295x^{-3/2})dx$$
$$\frac{dy}{dx} = -295x^{-3/2} \qquad dy = (-295(49)^{-3/2})(3)$$
$$dy = (-295x^{-3/2})dx \qquad dy = (-\tfrac{295}{343})(3)$$
$$dy = -\tfrac{885}{343} \approx -2.58$$

Therefore, if a person grows from 49 to 52 inches, their pulse rate would decrease by approximately 2.5 beats per minute.

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Section 2-7 Marginal Analysis in Business And Economics

Goal: To solve problems involving marginal functions in business and economics

Definition: Marginal Cost, Revenue, and Profit

If x is the number of units of a product produced in some time interval, then

total cost $= C(x)$	and	marginal cost $= C'(x)$
total revenue $= R(x)$	and	marginal revenue $= R'(x)$
total profit $= R(x) - C(x)$	and	marginal profit $= R'(x) - C'(x)$

Definition: Marginal Average Cost, Revenue, and Profit

If x is the number of units of a product produced in some time interval, then

Cost per unit: average cost $= \overline{C} = \frac{C(x)}{x}$ and marginal average cost $= \overline{C}'(x) = \frac{d}{dx}\overline{C}(x)$

Rev. per unit: average revenue $= \overline{R} = \frac{R(x)}{x}$ and marginal average revenue $= \overline{R}'(x) = \frac{d}{dx}\overline{R}(x)$

Profit per unit: average profit $= \overline{P} = \frac{P(x)}{x}$ and marginal average profit $= \overline{P}'(x) = \frac{d}{dx}\overline{P}(x)$

1 - 10 Find the indicated function if cost and revenue are given by

$C(x) = 6000 - 40x + 0.006x^2$ and $R(x) = 10{,}000x - 200x^2$

1. Marginal cost function

$$C(x) = 6000 - 40x + 0.006x^2$$
$$C'(x) = -40 + 0.012x$$

2. Average cost function

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{6000 - 40x + 0.006x^2}{x} = \frac{6000}{x} - 40 + 0.006x$$

3. Marginal average cost function

$$\overline{C}(x) = \frac{6000}{x} - 40 + 0.006x$$

$$\overline{C}(x) = 6000x^{-1} - 40 + 0.006x$$

$$\overline{C}'(x) = -6000x^{-2} + 0.006$$

$$\overline{C}'(x) = -\frac{3000}{x^2} + 0.003$$

4. Marginal revenue function

$$R(x) = 10,000x - 200x^2$$
$$R'(x) = 10,000 - 400x$$

5. Average revenue function

$$\overline{R}(x) = \frac{R(x)}{x} = \frac{10,000x - 200x^2}{x} = 10,000 - 200x$$

6. Marginal average revenue function

$$\overline{R}(x) = 10,000 - 200x$$

$$\overline{R}'(x) = -200$$

7. Profit function

$$P(x) = R(x) - C(x)$$
$$= (10,000x - 200x^2) - (6000 - 40x + 0.006x^2)$$
$$P(x) = -200.006x^2 + 10,040x - 6000$$

8. Marginal profit function

$$P(x) = -200.006x^2 + 10,040x - 6000$$
$$P'(x) = -400.012x + 10,040$$

9. Average profit function

$$\overline{P}(x) = \frac{P(x)}{x} = \frac{-200.006x^2 + 10,040x - 6000}{x} = -200.006x + 10,040 - \frac{6000}{x}$$

10. Marginal average profit function

$$\overline{P}'(x) = \overline{R}'(x) - \overline{C}'(x)$$

$$\overline{P}'(x) = (-200) - (-\tfrac{6000}{x^2} + 0.006)$$

$$\overline{P}'(x) = \tfrac{6000}{x^2} - 200.006$$

11. Consider the revenue (in dollars) of a stereo system given by

$$R(x) = \tfrac{1000}{x} + 1000x$$

a. Find the exact revenue from the sale of the 101st stereo.
b. Use marginal revenue to approximate the revenue from the sale of the 101st stereo.

<u>Solution</u>:

a. To find the exact revenue, find the revenue from the 101st and 100th and subtract their values:

$$R(x) = \tfrac{1000}{x} + 1000x$$
$$R(101) = \tfrac{1000}{101} + 1000(101)$$
$$R(101) = 101{,}009.90$$

$$R(x) = \tfrac{1000}{x} + 1000x$$
$$R(100) = \tfrac{1000}{100} + 1000(100)$$
$$R(100) = 100{,}010$$

$$R(101) - R(100) = 101009.90 - 100010 = 999.90$$

Therefore, the actual revenue from the 101st stereo was $999.90.

b. Find the marginal revenue formula and then substitute in 101.

$$R(x) = \tfrac{1000}{x} + 1000x$$
$$R(x) = 1000x^{-1} + 1000x$$
$$R'(x) = -1000x^{-2} + 1000$$
$$R'(x) = -\tfrac{1000}{x^2} + 1000$$

$$R'(x) = -\tfrac{1000}{x^2} + 1000$$
$$R'(101) = -\tfrac{1000}{(101)^2} + 1000$$
$$R'(x) = -0.098 + 1000$$
$$R'(x) = 999.90$$

Therefore, the approximate revenue from the 101st stereo was $999.90.

12. The total cost (in dollars) of manufacturing x units of a product is:

$$C(x) = 10,000 + 15x$$

a. Find the average cost per unit if 300 units are produced.
b. Find the marginal average cost at a production level of 300 units and interpret the results.
c. Use the results in parts a and b to estimate the average cost per unit if 301 units are produced.

Solution:

a. The cost of producing 300 units is $C(300) = 10,000 + 15(300) = 14,500$.

Therefore the average cost is $\overline{C}(300) = \frac{14500}{300} = \$48.33$.

b. The average cost function is $\overline{C}(x) = \frac{C(x)}{x} = \frac{10,000+15x}{x} = 10000x^{-1} + 15$.

Therefore, the marginal average cost function for producing the 300th unit would be:

$$\overline{C}(x) = 10000x^{-1} + 15$$
$$\overline{C}'(x) = -10000x^{-2}$$
$$\overline{C}(x) = -\frac{10000}{x^2}$$

$$\overline{C}(x) = -\frac{10000}{x^2}$$
$$\overline{C}(300) = -\frac{10000}{(300)^2}$$
$$\overline{C}(300) = -0.11$$

This means that the average cost is decreasing by \$0.11 per unit produced.

c. The average cost per unit for the 301st unit would be \$48.33 – \$0.11 = \$48.22.

13. The total profit (in dollars) from the sale of x units of a product is:

$$P(x) = 30x - 0.03x^2 + 200$$

a. Find the exact profit from the 201st unit sold.
b. Find the marginal profit from selling the 201st unit.

Solution:

a. To find the exact profit, find the profit from the 201st and 200th and subtract their values:

$$P(x) = 30x - 0.03x^2 + 200$$
$$P(201) = 30(201) - 0.03(201)^2 + 200$$
$$P(201) = 5017.97$$

$$P(x) = 30x - 0.03x^2 + 200$$
$$P(200) = 30(200) - 0.03(200)^2 + 200$$
$$P(200) = 5000$$

$$P(201) - P(200) = 5017.97 - 5000 = 17.97$$

Therefore, the actual profit from the 201st unit was $17.97.

b. Find the marginal profit formula and then substitute in 201.

$$P(x) = 30x - 0.03x^2 + 200$$
$$P'(x) = 30 - 0.06x$$

$$P'(x) = 30 - 0.06x$$
$$P'(201) = 30 - 0.06(201)$$
$$P'(201) = 17.94$$

Therefore, the approximate profit from the 201st stereo was $17.94.

14. The total cost and revenue (in dollars) for the production and sale of x units are given, respectively, by:

$$C(x) = 32x + 36{,}000 \quad \text{and} \quad R(x) = 300x - 0.03x^2$$

a. Find the profit function *P(x)*.
b. Determine the actual cost, revenue, and profit from making and selling 101 units.
c. Determine the marginal cost, revenue, and profit from making and selling the 101^{st} unit.

Solution:

a.
$$P(x) = R(x) - C(x)$$
$$P(x) = (300x - 0.03x^2) - (32x + 36{,}000)$$
$$P(x) = -0.03x^2 + 268x - 36{,}000$$

b.
$$C(x) = 32x + 36{,}000 \qquad C(x) = 32x + 36{,}000$$
$$C(101) = 32(101) + 36{,}000 \qquad C(100) = 32(100) + 36{,}000$$
$$C(101) = 39{,}232 \qquad C(100) = 39{,}200$$

$$R(x) = 300x - 0.03x^2 \qquad R(x) = 300x - 0.03x^2$$
$$R(101) = 300(101) - 0.03(101)^2 \qquad R(100) = 300(100) - 0.03(100)^2$$
$$R(101) = 29{,}993.97 \qquad R(100) = 29{,}700$$

$$P(x) = R(x) - C(x) \qquad P(x) = R(x) - C(x)$$
$$P(101) = R(101) - C(101) \qquad P(100) = R(100) - C(100)$$
$$P(101) = 29{,}993.97 - 39{,}232 \qquad P(100) = 29{,}700 - 39{,}200$$
$$P(101) = -9238.03 \qquad P(100) = -9500$$

Actual Cost	Actual Revenue	Actual Profit
$C(101) - C(100)$	$R(101) - R(100)$	$P(101) - P(100)$
$39{,}232 - 39{,}200$	$29{,}993.97 - 29{,}700$	$-9238.03 - (-9500)$
32	293.97	261.97

c. Marginal functions would be:

$$C(x) = 32x + 36,000 \qquad R(x) = 300x - 0.03x^2$$
$$C'(x) = 32 \qquad R'(x) = 300 - 0.06x$$

$$P'(x) = R'(x) - C'(x)$$
$$P'(x) = (300 - 0.06x) - (32)$$
$$P'(x) = -0.06x + 268$$

Marginal values for the 101st unit are:

$$C'(x) = 32 \qquad R'(x) = 300 - 0.06x$$
$$C'(101) = 32 \qquad R'(101) = 300 - 0.06(101)$$
$$R'(101) = 293.94$$

$$P'(x) = -0.06x + 268$$
$$P'(101) = -0.06(101) + 268$$
$$P'(101) = 261.94$$