CHAPTER THREE

Student's Solutions Manual

Anna Amirdjanova
University of Michigan, Ann Arbor

Neil A. Weiss
Arizona State University

FOR

A Course In Probability

Neil A. Weiss
Arizona State University

PEARSON
Addison Wesley

Boston San Francisco New York
London Toronto Sydney Tokyo Singapore Madrid
Mexico City Munich Paris Cape Town Hong Kong Montreal

Publisher: Greg Tobin
Editor-in-Chief: Deirdre Lynch
Associate Editor: Sara Oliver Gordus
Editorial Assistant: Christina Lepre
Production Coordinator: Kayla Smith-Tarbox
Senior Author Support/Technology Specialist: Joe Vetere
Compositor: Carol A. Weiss
Accuracy Checker: Delray Schultz
Proofreader: Carol A. Weiss

Preface

This *Student's Solutions Manual* (SSM) is designed to be used with the book *A Course in Probability* (CIP) by Neil A. Weiss. The SSM provides complete and detailed solutions to every fourth end-of-section exercise and to all review exercises. Thus, for end-of-section exercises, you will find solutions to problems numbered 1, 5, 9, . . . , whereas, solutions to all review problems are presented.

The solutions in this SSM employ precisely the same notation, format, and style as those given to the examples in CIP. Consequently, you need only concentrate on the exercise solutions themselves without having to struggle with new notations or conventions.

We would like to express our appreciation to all the people at Addison-Wesley who helped make this SSM possible. In particular, we thank Deirdre Lynch, Sara Oliver Gordus, Christina Lepre, Kayla Smith-Tarbox, and Joe Vetere. And, in addition, we thank Carol Weiss for her outstanding job of composition and proofreading.

Ann Arbor, Michigan *A.A.*
Prescott, Arizona *N.A.W.*

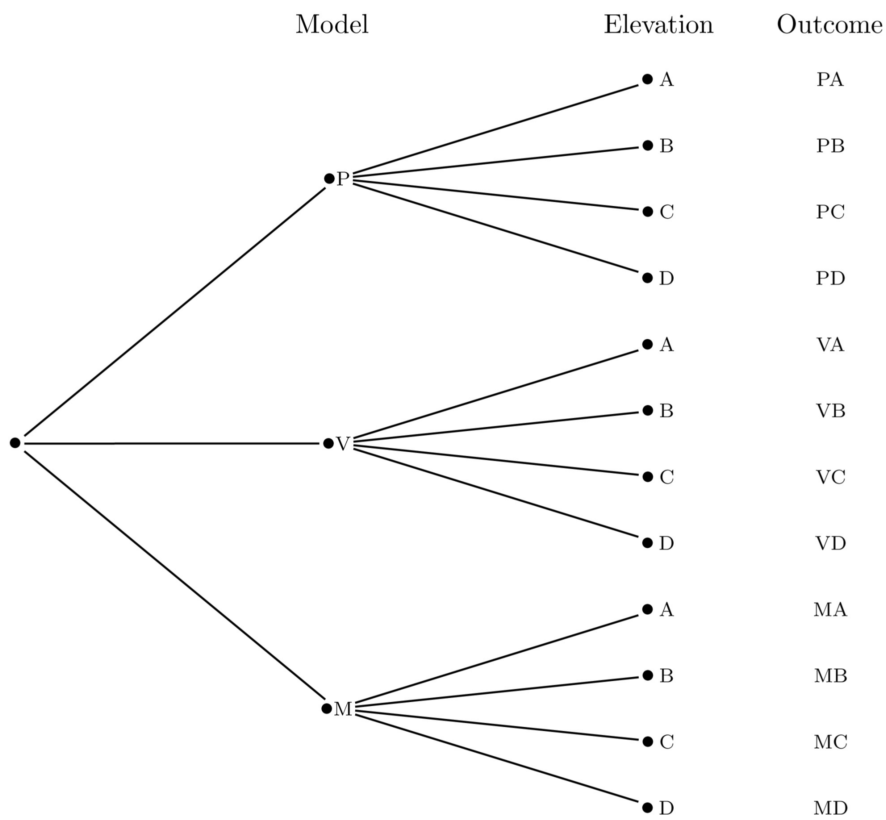
Chapter 3

Combinatorial Probability

3.1 The Basic Counting Rule

Basic Exercises

3.1
a) A tree diagram showing the possible choices for the selection of a home, including both model and elevation is as follows:



b) The total number of choices for the selection of a home, including both model and elevation, equals the number of branches at the end of the tree diagram in part (a), which is 12.
c) There are two actions ($r = 2$): selecting a model and selecting an elevation. Because there are three possibilities for the model, $m_1 = 3$; and because there are four possibilities for the elevation of each

model, $m_2 = 4$. Therefore, by the BCR, the total number of possibilities, including both model and elevation, is $m_1 \cdot m_2 = 3 \cdot 4 = 12$.

3.5 We have eight actions here—the choices of the eight decimal digits to constitute an identification number.
a) With no restrictions whatsoever, there are 10 possibilities for each digit and, hence, by the BCR, the total number of possibilities is

$$\underbrace{10 \cdot 10 \cdots 10}_{\text{eight times}} = 10^8 = 100{,}000{,}000.$$

b) If no digit can occur twice (i.e., more than once), then there are 10 possibilities for the first digit, 9 for the second, ..., 3 for the eighth. Hence, by the BCR, the total number of possibilities is

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 1{,}814{,}400.$$

c) If no digit can agree with its predecessor, then there are 10 possibilities for the first digit, 9 for the second, 9 for the third, ..., 9 for the eighth. Hence, by the BCR, the total number of possibilities is

$$10 \cdot \underbrace{9 \cdot 9 \cdots 9}_{\text{seven times}} = 10 \cdot 9^7 = 47{,}829{,}690.$$

d) If no digit can agree with either of its two immediate predecessors, then there are 10 possibilities for the first digit, 9 for the second, 8 for the third, 8 for the fourth, ..., 8 for the eighth. Hence, by the BCR, the total number of possibilities is

$$10 \cdot 9 \cdot \underbrace{8 \cdot 8 \cdots 8}_{\text{six times}} = 10 \cdot 9 \cdot 8^6 = 23{,}592{,}960.$$

3.9 Recall that a die is a cube on which a different number of dots, from one to six, is painted on each of the six faces. When we speak of a number on a die, we mean the number of dots facing up. Here, we are rolling two dice, one black and the other gray.
a) There are six possibilities each for the numbers on the two dice. Hence, by the BCR, the number of possible outcomes for the random experiment of rolling two dice is $6 \cdot 6 = 36$.
b) For the sum of the dice to be 5, we have four possibilities for the number on the black die (1, 2, 3, or 4). Once we have specified the number on the black die, the number on the gray die is determined (it must be 5 less the number on the black die). Hence, by the BCR, the number of possible outcomes in which the sum of the dice is 5 is $4 \cdot 1 = 4$.
c) For doubles to be rolled, we have six possibilities for the number on the black die. Once we have specified the number on the black die, the number on the gray die is determined (it must be the same as the number on the black die). Hence, by the BCR, the number of possible outcomes in which doubles are rolled is $6 \cdot 1 = 6$.
d) For the sum of the dice to be even, we have six possibilities for the number on the black die. Once we have specified the number on the black die, we have three possibilities for the number on the gray die (1, 3, or 5 if the number on the black die is odd; 2, 4, or 6 if the number on the black die is even). Hence, by the BCR, the number of possible outcomes in which the sum of the dice is even is $6 \cdot 3 = 18$.

3.13
a) From the BCR, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ possible arrangements altogether. By symmetry, the number of arrangements in which c is before d equals the number of arrangements in which d is before c. As either c must be before d or d must be before c, we see that the number of arrangements of the former (and the latter) is $120/2 = 60$.

b) If d is not first, then there are four possibilities for the first person in line (any person other than d), four for the second (any person other than the person chosen to be first in line), three for the third, two for the fourth, and one for the fifth. Therefore, by the BCR, there are $4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 96$ arrangements in which d is not first.

3.17 Each possible handshake is determined by an unordered pair of different numbers between 1 and n, inclusive, representing the two people who shake hands. Each such unordered pair corresponds to two ordered pairs, obtained from the two possible orders of the two numbers. From the BCR, there are $n \cdot (n-1)$ ordered pairs of different numbers between 1 and n, inclusive. Hence, the number of unordered pairs (i.e., the number of handshakes) is $n(n-1)/2$.

Advanced Exercises

3.21
a) The possible subsets of $\Omega = \{a, b, c\}$ are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$. Hence, we see that a set with three elements has eight subsets.
b) To form a subset of $\Omega = \{a, b, c\}$, we can, for each of the three elements, choose to either include the element or not include the element. Thus, there are two possibilities for element a (either include it in the subset or not), two possibilities for element b, and two possibilities for element c. Consequently, by the BCR, there are $2 \cdot 2 \cdot 2 = 8$ possibilities altogether for forming a subset of Ω. In other words, a set with three elements has eight subsets.
c) To form a subset of Ω, we can, for each of its n elements, choose to either include the element or not include the element. Thus, there are two possibilities for the first element of Ω (either include it in the subset or not), two possibilities for the second element of Ω, and so forth. Consequently, by the BCR, there are

$$\underbrace{2 \cdot 2 \cdots 2}_{n \text{ times}} = 2^n$$

possibilities altogether for forming a subset of Ω. In other words, a set with n elements has 2^n subsets.

3.2 Permutations and Combinations

Basic Exercises

3.25 Selecting the first, second, and third place winners from the 18 films is equivalent to specifying a permutation of three objects from a collection of 18 objects; the first object is the film selected to be the first-place film, the second object is the film selected to be the second-place film, and the third object is the film selected to be the third-place film. Thus, the number of possibilities for the first, second, and third place winners is $(18)_3$, the number of possible permutations of three objects from a collection of 18 objects. From the permutations rule,

$$(18)_3 = \frac{18!}{(18-3)!} = \frac{18!}{15!} = \frac{18 \cdot 17 \cdot 16 \cdot 15!}{15!} = 18 \cdot 17 \cdot 16 = 4896.$$

3.29 Selecting three of the returns for auditing from the 18 returns is equivalent to specifying a combination of three objects (the three returns to be audited) from a collection of 18 objects (the 18 returns). Thus, the number of possibilities for the selection of the three returns is $\binom{18}{3}$, the number of possible combinations of three objects from a collection of 18 objects. From the combinations rule,

$$\binom{18}{3} = \frac{18!}{3!\,(18-3)!} = \frac{18!}{3!\,15!} = \frac{18 \cdot 17 \cdot 16 \cdot 15!}{3!\,15!} = \frac{18 \cdot 17 \cdot 16}{3!} = 816.$$

3.33

a) The number of different outcomes possible is the number of ways that we can choose 10 balls from 80 balls, which is $\binom{80}{10}$. Applying the combinations rule, we find that $\binom{80}{10} = 1{,}646{,}492{,}110{,}120$.

b) We can think of the 80 balls as being divided into two groups, the 20 balls with the numbers that the player specifies and the other 60 balls. For the player to have all 10 numbers selected, the 10 balls must be chosen from the first batch, which can be done in $\binom{20}{10}$ ways. Applying the combinations rule, we find that $\binom{20}{10} = 184{,}756$.

3.37 There are several ways to proceed here. One way is to note that, to each five-card draw hand, there corresponds 5! five-card stud hands, obtained from the 5! permutations of the five cards in the five-card draw hand among themselves. As $5! = 120$, we get, by referring to the solution to Exercise 3.36, the required numbers of possibilities:

$$\begin{aligned}
\text{Straight flush: } & 120 \cdot 40 = 4{,}800 \\
\text{Four of a kind: } & 120 \cdot 624 = 74{,}880 \\
\text{Full house: } & 120 \cdot 3744 = 449{,}280 \\
\text{Flush: } & 120 \cdot 5108 = 612{,}960 \\
\text{Straight: } & 120 \cdot 10{,}200 = 1{,}224{,}000 \\
\text{Three of a kind: } & 120 \cdot 54{,}912 = 6{,}589{,}440 \\
\text{Two pair: } & 120 \cdot 123{,}552 = 14{,}826{,}240 \\
\text{One pair: } & 120 \cdot 1{,}098{,}240 = 131{,}788{,}800
\end{aligned}$$

3.41

a) The quantity $\binom{n}{k}$ represents the number of ways that k distinct objects can be chosen from n objects without regard to order. However, that number of ways equals the number of ways that we can specify the $n-k$ distinct objects (from the n objects) that will not be chosen, which is $\binom{n}{n-k}$. Consequently, we see that $\binom{n}{k} = \binom{n}{n-k}$.

b) The quantity $\binom{n}{k,\,n-k}$ represents the number of possible ordered partitions of n objects into two distinct groups of sizes k and $n-k$. But that number equals the number of ways that we can choose k distinct objects from the n objects to constitute the first group, which is $\binom{n}{k}$. Consequently, we see that $\binom{n}{k,\,n-k} = \binom{n}{k}$.

3.45

a) There are $\binom{20}{8}$ possibilities for the choice of the eight committee members. Then there are $(8)_3$ possibilities for the choice of the three officers from the eight chosen committee members. Hence, by the BCR, combinations rule, and permutations rule, the total number of possibilities is

$$\binom{20}{8} \cdot (8)_3 = 125{,}970 \cdot 336 = 42{,}325{,}920.$$

b) There are $\binom{20}{5}$ possibilities for the choice of the five nonofficer committee members. Then there are $(15)_3$ possibilities for the choice of the three officer committee members from the 15 remaining candidates. Hence, by the BCR, combinations rule, and permutations rule, the total number of possibilities is

$$\binom{20}{5} \cdot (15)_3 = 15{,}504 \cdot 2{,}730 = 42{,}325{,}920.$$

c) There are $(20)_3$ possibilities for the choice of the three officer committee members. Then there are $\binom{17}{5}$ possibilities for the choice of the five nonofficer committee members from the 17 remaining candidates. Hence, by the BCR, permutations rule, and combinations rule, the total number of possibilities is

$$(20)_3 \cdot \binom{17}{5} = 6840 \cdot 6188 = 42{,}325{,}920.$$

d) From a list of n candidates, you must choose a committee of k and, from among those k, you must choose j officers. The three quantities presented in the equations in the statement of part (d) provide expressions for the number of ways of doing that by using the steps indicated in parts (a)–(c), respectively. Hence, those three quantities must be equal.

e) From the combinations and permutations rules, we have

$$\binom{n}{k}(k)_j = \frac{n!}{k!\,(n-k)!} \cdot \frac{k!}{(k-j)!} = \frac{n!}{(n-k)!\,(k-j)!},$$

$$\binom{n}{k-j}\big(n-(k-j)\big)_j = \frac{n!}{(k-j)!\,\big(n-(k-j)\big)!} \cdot \frac{\big(n-(k-j)\big)!}{\Big(\big(n-(k-j)\big)-j\Big)!} = \frac{n!}{(n-k)!\,(k-j)!},$$

$$(n)_j\binom{n-j}{k-j} = \frac{n!}{(n-j)!} \cdot \frac{(n-j)!}{(k-j)!\,\big((n-j)-(k-j)\big)!} = \frac{n!}{(n-k)!\,(k-j)!}.$$

Consequently, we see that all three quantities presented in the equations in the statement of part (d) equal $n!\big/\big((n-k)!(k-j)!\big)$ and, hence, are equal to each other.

Theory Exercises

3.49 We have

$$(a_1 + a_2 + \cdots + a_m)^n = \underbrace{(a_1 + a_2 + \cdots + a_m)(a_1 + a_2 + \cdots + a_m)\cdots(a_1 + a_2 + \cdots + a_m)}_{n \text{ times}}.$$

Let $n_1, n_2, \ldots, n_m$ be nonnegative integers whose sum equals n and consider the term $a_1^{n_1}a_2^{n_2}\cdots a_m^{n_m}$ in the expansion of $(a_1 + a_2 + \cdots + a_m)^n$. To obtain that term, we must choose a_1 from exactly n_1 of the n factors (shown in the preceding display), choose a_2 from exactly n_2 of the remaining $n - n_1$ factors, and so forth. The number of ways we can do that is $\binom{n}{n_1, n_2, \ldots, n_m}$, the number of ordered partitions of n objects into m distinct groups of sizes $n_1, n_2, \ldots, n_m$. Hence, $a_1^{n_1}a_2^{n_2}\cdots a_m^{n_m}$ occurs in $\binom{n}{n_1, n_2, \ldots, n_m}$ summands in the expansion of $(a_1 + a_2 + \cdots + a_m)^n$. Consequently,

$$(a_1 + a_2 + \cdots + a_m)^n = \sum \binom{n}{n_1, n_2, \ldots, n_m} a_1^{n_1}a_2^{n_2}\cdots a_m^{n_m},$$

where the sum is taken over all nonnegative integers, $n_1, n_2, \ldots, n_m$, whose sum equals n.

Advanced Exercises

3.53

a) The number of nonnegative-integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 10$ is, in view of Exercise 3.52,

$$\binom{10+4-1}{4-1} = \binom{13}{3} = 286.$$

b) The number of positive-integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 10$ is, in view of Exercise 3.51,

$$\binom{10-1}{4-1} = \binom{9}{3} = 84.$$

3.3 Applications of Counting Rules to Probability

Note: Unless otherwise specified, we use the fact that a classical probability model is appropriate for each of the random experiments in the exercises of this section and, hence, that Equation (3.4) on page 110 is appropriate.

Basic Exercises

3.57 To solve these two problems, we can either consider the order of the cards dealt or not. We will do the latter. With that in mind, the number of possible outcomes for the random experiment of dealing four cards from an ordinary deck of 52 playing cards is $\binom{52}{4} = 270{,}725$; that is, $N(\Omega) = 270{,}725$.
a) Let A denote the event that the denominations of the four cards dealt are all the same. For event A to occur, there are 13 possibilities for the denomination of the four cards. Hence, we have $N(A) = 13$ and, therefore,

$$P(A) = \frac{N(A)}{N(\Omega)} = \frac{13}{270{,}725} \approx 0.0000480.$$

b) Let B denote the event that the denominations of the four cards dealt are all different. For event B to occur, there are $\binom{13}{4}$ possibilities for the four denominations and then four possibilities each for the suit of the card in each of the denominations. Hence, we have $N(B) = \binom{13}{4} \cdot 4^4 = 183{,}040$ and, therefore,

$$P(B) = \frac{N(B)}{N(\Omega)} = \frac{183{,}040}{270{,}725} \approx 0.676.$$

3.61 From the assumptions in the problem statement, a classical probability model is appropriate here.
a) There are 365 possibilities for the birthday of the first student, 365 possibilities for the birthday of the second student, ..., and 365 possibilities for the birthday of the 38th student. Therefore, we see that $N(\Omega) = 365^{38}$. Let E denote the event that at least two students in the class have the same birthday. Then E^c is the event that all the students have different birthdays. For event E^c to occur, there are 365 possibilities for the birthday of the first student, 364 possibilities for the birthday of the second student, ..., and 328 possibilities for the birthday of the 38th student. Thus, $N(E^c) = (365)_{38}$. Consequently, by the complementation rule,

$$P(E) = 1 - P(E^c) = 1 - \frac{N(E^c)}{N(\Omega)} = 1 - \frac{(365)_{38}}{365^{38}} \approx 0.864.$$

b) Arguing in precisely the same way as in part (a), we find that $N(\Omega) = 365^N$ and $N(E^c) = (365)_N$. Hence, by the complementation rule,

$$P(E) = 1 - P(E^c) = 1 - \frac{N(E^c)}{N(\Omega)} = 1 - \frac{(365)_N}{365^N}.$$

c) Applying part (b) for $N = 1, 2, \ldots, 70$, we obtain the following table for the probability, p_N, that at least two students in a class of N have the same birthday:

N	p_N	N	p_N	N	p_N	N	p_N	N	p_N
1	0.00000	15	0.25290	29	0.68097	43	0.92392	57	0.99012
2	0.00274	16	0.28360	30	0.70632	44	0.93289	58	0.99166
3	0.00820	17	0.31501	31	0.73045	45	0.94098	59	0.99299
4	0.01636	18	0.34691	32	0.75335	46	0.94825	60	0.99412
5	0.02714	19	0.37912	33	0.77497	47	0.95477	61	0.99509
6	0.04046	20	0.41144	34	0.79532	48	0.96060	62	0.99591
7	0.05624	21	0.44369	35	0.81438	49	0.96578	63	0.99660
8	0.07434	22	0.47570	36	0.83218	50	0.97037	64	0.99719
9	0.09462	23	0.50730	37	0.84873	51	0.97443	65	0.99768
10	0.11695	24	0.53834	38	0.86407	52	0.97800	66	0.99810
11	0.14114	25	0.56870	39	0.87822	53	0.98114	67	0.99844
12	0.16702	26	0.59824	40	0.89123	54	0.98388	68	0.99873
13	0.19441	27	0.62686	41	0.90315	55	0.98626	69	0.99896
14	0.22310	28	0.65446	42	0.91403	56	0.98833	70	0.99916

d) Referring to the preceding table, we see that 23 is the smallest class size for which the probability that at least two students in the class have the same birthday exceeds 0.5.

3.65 The number of possible samples of size n with replacement from a population of size N is N^n. Thus, $N(\Omega) = N^n$.
a) Let E denote the event that no member of the population is selected more than once. As event E can occur in $(N)_n$ ways, we have

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{(N)_n}{N^n}.$$

b) We have

$$\begin{aligned}
\lim_{N\to\infty} \frac{(N)_n}{N^n} &= \lim_{N\to\infty} \frac{N(N-1)(N-2)\cdots(N-n+1)}{N \cdot N \cdots N} \\
&= \lim_{N\to\infty} \frac{N}{N} \cdot \frac{N-1}{N} \cdot \frac{N-2}{N} \cdots \frac{N-(n-1)}{N} \\
&= \lim_{N\to\infty} \left(1 - \frac{1}{N}\right)\left(1 - \frac{2}{N}\right)\cdots\left(1 - \frac{n-1}{N}\right) \\
&= 1 \cdot 1 \cdots 1 = 1.
\end{aligned}$$

Consequently, as $N \to \infty$, the probability that no member of the population is selected more than once approaches 1. The interpretation is that, when the population size is large relative to the sample size, selecting a member of the population more than once is unlikely.

3.69
a) The number of possibilities for the order in which the subject writes the numbers is the number of possible permutations of 10 numbers among themselves, which is $10! = 3{,}628{,}800$.
b) If the subject doesn't have ESP, but is just guessing, then each of the 3,628,800 possible outcomes is equally likely to be the one written down by the subject. Hence, the probability that the subject writes the numbers in the correct order is $1/3{,}628{,}800 \approx 0.000000276$.

3.73

a) As a five-card draw hand consists of five cards drawn without replacement and without regard to order from an ordinary deck of 52 playing cards, the number of such hands possible is $\binom{52}{5} = 2{,}598{,}960$. And, because the cards are drawn at random, a classical probability model is appropriate here. Dividing each of the answers in Exercise 3.36 by 2,598,960, we therefore obtain the following probabilities:

Straight flush: 0.0000154

Four of a kind: 0.000240

Full house: 0.00144

Flush: 0.00197

Straight: 0.00392

Three of a kind: 0.0211

Two pair: 0.0475

One pair: 0.423

b) No, for the hands considered in part (a), the five-card stud poker probabilities (where the order in which the cards are received matters) are the same as the five-card draw poker probabilities. Indeed, as we mentioned on page 114, for random samples without replacement, you can compute probabilities based on either ordered samples or unordered samples provided that you are consistent throughout the solution. In the present case, this fact can be easily seen as follows. Each possible five-card draw hand corresponds to 5! five-card stud hands, obtained by permuting the five cards in the former. Hence, each five-card stud hand occurs in 5! times as many ways as the corresponding five-card draw hand. Using subscripts d and s for draw and stud, respectively, we see that, for each hand E,

$$P(E_s) = \frac{N(E_s)}{N(\Omega_s)} = \frac{5! \cdot N(E_d)}{5! \cdot N(\Omega_d)} = \frac{N(E_d)}{N(\Omega_d)},$$

thus showing that probabilities are the same for five-card stud hands and five-card draw hands.

Theory Exercises

3.77

a) For sampling without replacement, there are a total of $\binom{N}{n}$ possible samples of size n from a population of size N, where we are ignoring order. Thus, $N(\Omega) = \binom{N}{n}$. We note that Np members of the population have the specified attribute and $N(1-p)$ members don't have the specified attribute. Hence, event E_k can occur in as many ways as k members can be selected from the Np members with the specified attribute, which is $\binom{Np}{k}$, and $n-k$ members can be selected from the $N(1-p)$ members without the specified attribute, which is $\binom{N(1-p)}{n-k}$. Therefore, we have $N(E_k) = \binom{Np}{k}\binom{N(1-p)}{n-k}$ and, consequently,

$$P(E_k) = \frac{N(E_k)}{N(\Omega)} = \frac{\dbinom{Np}{k}\dbinom{N(1-p)}{n-k}}{\dbinom{N}{n}}.$$

b) For sampling with replacement, there are a total of N^n possible samples of size n from a population of size N. Therefore, $N(\Omega) = N^n$. We note that Np members of the population have the specified attribute and $N(1-p)$ members don't have the specified attribute. For event E_k to occur, there are $\binom{n}{k}$ possibilities for when the k members with the specified attribute are selected, $(Np)^k$ possibilities for the k members (which can be duplicated) sampled with the specified attribute, and $\big(N(1-p)\big)^{n-k}$ possibilities for the $n-k$ members without the specified attribute. Thus, we have $N(E_k) = \binom{n}{k}(Np)^k\big(N(1-p)\big)^{n-k}$ and, consequently,

$$P(E_k) = \frac{N(E_k)}{N(\Omega)} = \frac{\binom{n}{k}(Np)^k\big(N(1-p)\big)^{n-k}}{N^n}$$
$$= \binom{n}{k}\frac{(Np)^k}{N^k}\frac{\big(N(1-p)\big)^{n-k}}{N^{n-k}} = \binom{n}{k}p^k(1-p)^{n-k}.$$

c) For convenience, set $q = 1-p$. We have

$$\frac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}} = \frac{\dfrac{(Np)_k}{k!}\cdot\dfrac{(Nq)_{n-k}}{(n-k)!}}{\dfrac{(N)_n}{n!}} = \binom{n}{k}\frac{(Np)_k\cdot(Nq)_{n-k}}{(N)_n}$$
$$= \binom{n}{k}\frac{(Np)(Np-1)\cdots(Np-k+1)\cdot(Nq)(Nq-1)\cdots\big(Nq-(n-k)+1\big)}{N(N-1)\cdots(N-n+1)}$$
$$= \binom{n}{k}\frac{p\left(p-\frac{1}{N}\right)\cdots\left(p-\frac{k-1}{N}\right)\cdot q\left(q-\frac{1}{N}\right)\cdots\left(q-\frac{n-k-1}{N}\right)}{1\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{n-1}{N}\right)},$$

where, in the last equation, we divided both numerator and denominator by N^n. Hence,

$$\lim_{N\to\infty}\frac{\binom{Np}{k}\binom{N(1-p)}{n-k}}{\binom{N}{n}} = \lim_{N\to\infty}\frac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}} = \binom{n}{k}\frac{p^k\cdot q^{n-k}}{1} = \binom{n}{k}p^k(1-p)^{n-k}.$$

d) We interpret part (c) as follows: If the sample size is small relative to the population size, then, when sampling is done without replacement, the probability that exactly k members of the sample have the specified attribute can be well approximated by the corresponding probability when sampling is done with replacement. This result makes sense because, when the sample size is small relative to the population size, there is little difference between sampling without replacement and sampling with replacement.

Advanced Exercises

3.81

a) We first observe that all balls get placed in the box before noon. Indeed, for each $n \in \mathcal{N}$, the balls numbered $10n-9$ through $10n$ are placed in the box at $1/2^{n-1}$ minute to noon. As all balls placed in the box other than balls 10, 20, 30, ... are still in the box at noon, we see that infinitely many balls are in the box at noon.

b) As in part (a), all balls get placed in the box before noon. However, in this scenario, all balls are also removed from the box before noon. Indeed, ball n is removed from the box at $1/2^{n-1}$ minute to noon.

c) For this part, we need the following result:

$$\prod_{j=1}^{\infty} \frac{9j}{9j+1} = 0. \tag{$*$}$$

For convenience, we set $a_j = 9j/(9j+1)$, $j \in \mathcal{N}$. Note that, because $\left(\prod_{j=1}^{\infty} a_j\right)^{-1} = \prod_{j=1}^{\infty} a_j^{-1}$, it suffices to show that $\prod_{j=1}^{\infty} a_j^{-1} = \infty$ in order to establish Equation $(*)$. For $n \in \mathcal{N}$, we have

$$\prod_{j=1}^{n} a_j^{-1} = \prod_{j=1}^{n} \frac{9j+1}{9j} = \prod_{j=1}^{n} \left(1 + \frac{1}{9j}\right) > \sum_{j=1}^{n} \frac{1}{9j} = \frac{1}{9} \sum_{j=1}^{n} \frac{1}{j}.$$

Therefore,

$$\prod_{j=1}^{\infty} a_j^{-1} = \lim_{n\to\infty} \prod_{j=1}^{n} a_j^{-1} \geq \frac{1}{9} \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j} = \frac{1}{9} \sum_{j=1}^{\infty} \frac{1}{j} = \infty,$$

where the last equation follows from the well-known divergence of the harmonic series. Thus, we have shown that Equation $(*)$ holds.

Now, let $k \in \mathcal{N}$ and denote by A_k the event that ball k is in the box at noon. We note that ball k gets placed in the box at $2^{m_k - 1}$ minute to noon, where m_k is the smallest integer such that $k \leq 10m_k$, that is, $m_k = \lceil k/10 \rceil$. For $n \geq m_k$, let A_{nk} denote the event that ball k is in the box after the nth withdrawal. During the time of the jth withdrawal, there are $9j+1$ balls in the box. Event A_{nk} occurs if and only if ball k is not removed at withdrawals $m_k, m_k+1, \ldots, n$. The number of ways that can happen is $(9m_k)\big(9(m_k+1)\big)\cdots(9n)$. Hence,

$$P(A_{nk}) = \frac{(9m_k)\big(9(m_k+1)\big)\cdots(9n)}{(9m_k+1)\big(9(m_k+1)+1\big)\cdots(9n+1)} = \prod_{j=m_k}^{n} a_j.$$

Events A_{nk}, $n \geq m_k$, are nonincreasing and their intersection is A_k. Therefore, from Proposition 2.11(b) on page 74 and Equation $(*)$,

$$P(A_k) = \lim_{n\to\infty} P(A_{nk}) = \lim_{n\to\infty} \prod_{j=m_k}^{n} a_j = \prod_{j=1}^{m_k-1} a_j^{-1} \lim_{n\to\infty} \prod_{j=1}^{n} a_j = \prod_{j=1}^{m_k-1} a_j^{-1} \prod_{j=1}^{\infty} a_j = 0.$$

Next, let E denote the event that the box is empty at noon. Event E^c is that at least one ball is in the box at noon, that is, $E^c = \bigcup_{k=1}^{\infty} A_k$. Applying the nonnegativity axiom and Boole's inequality (Exercise 2.75), we get that

$$0 \leq P(E^c) = P\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} P(A_k) = \sum_{k=1}^{\infty} 0 = 0.$$

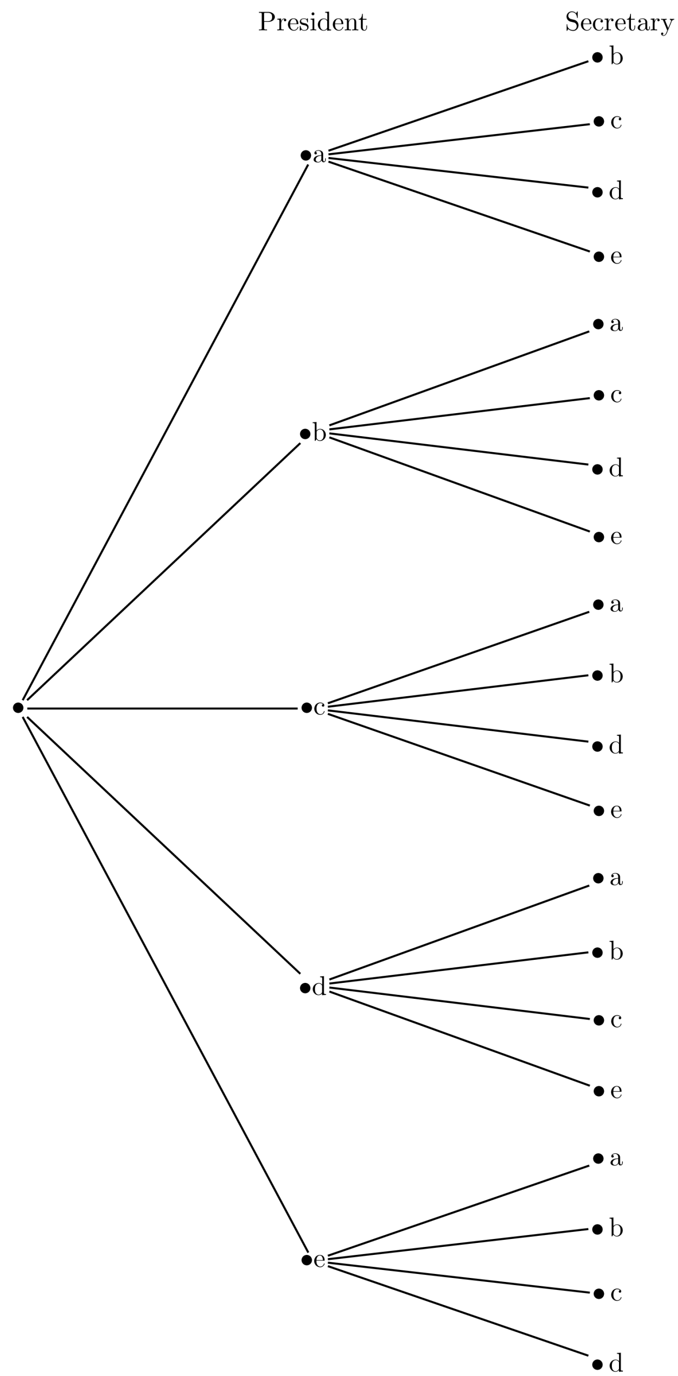
We therefore see that $P(E^c) = 0$ and, hence, from the complementation rule, that $P(E) = 1$. In other words, with probability 1, the box is empty at noon.

Review Exercises for Chapter 3

Basic Exercises

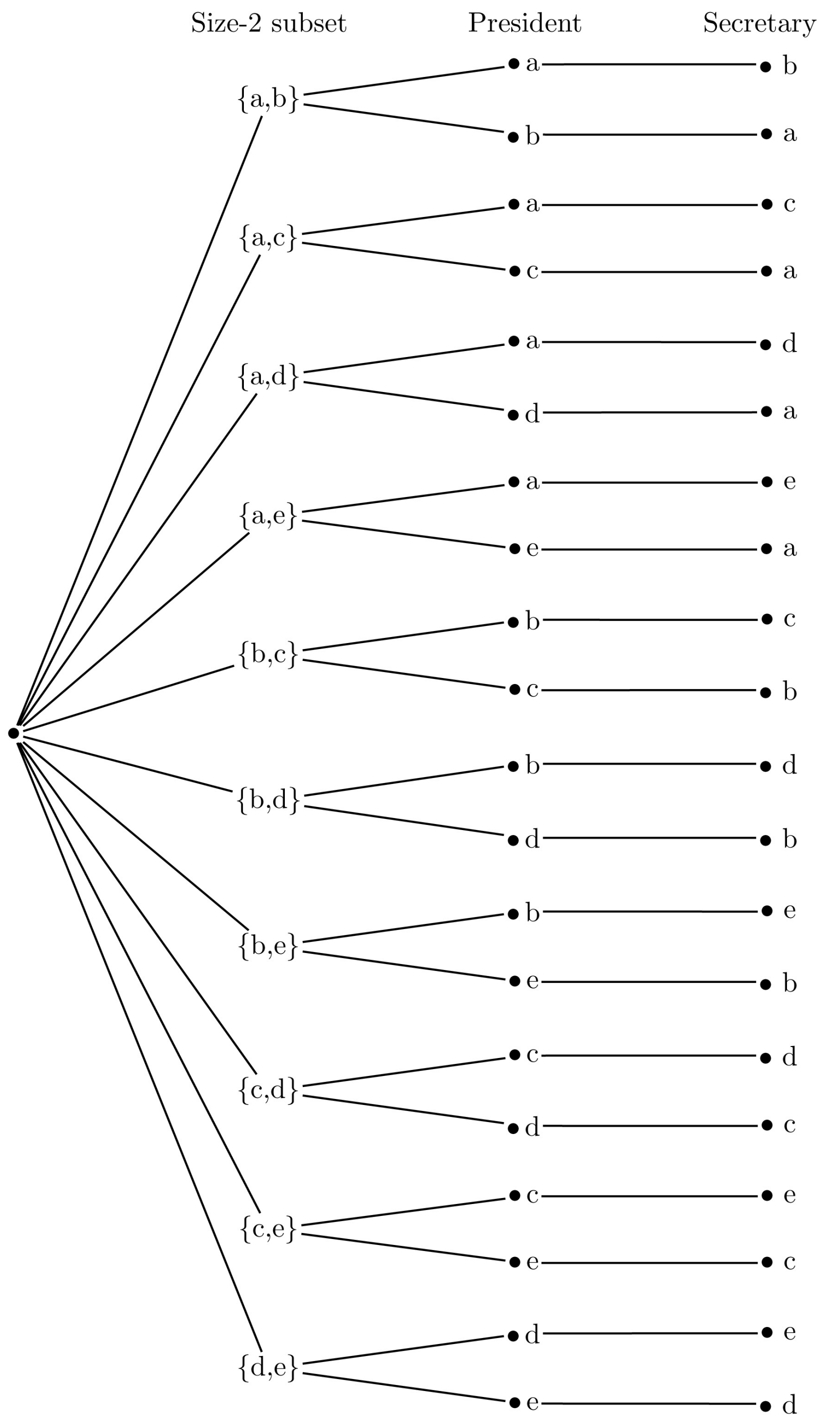
3.82

a) A tree diagram representing the process whose first step is the choice of a president and whose second step is the choice of a secretary is as follows:

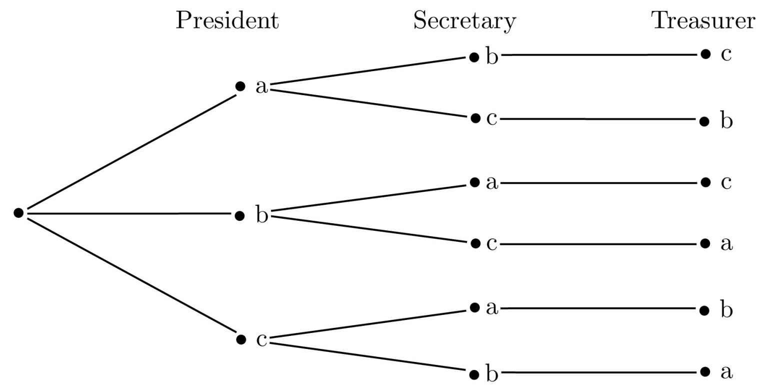


b) The required tree diagram is identical to the one in part (a) except that the headings "President" and "Secretary" should be interchanged.

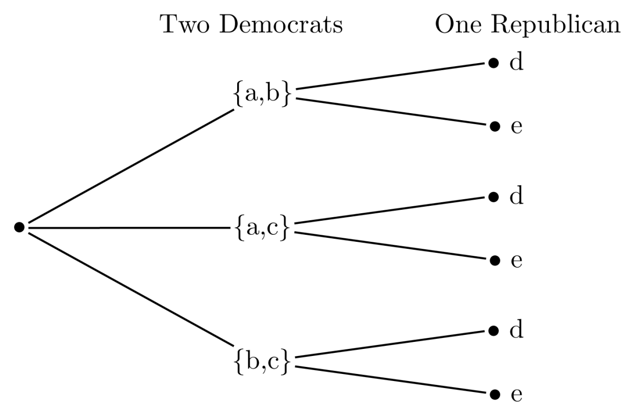
c) A tree diagram representing the process whose first step is the choice of two of the five candidates and whose second step is the choice of which of the two will be president and which will be secretary is as follows:



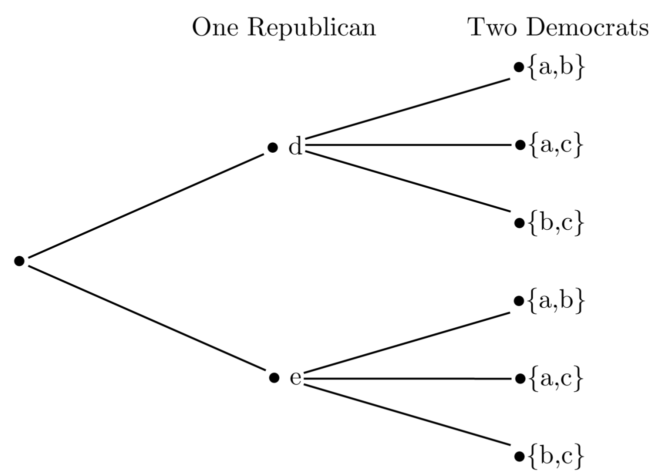
d) A tree diagram representing the process of choosing first the president, next the secretary, and then the treasurer, from among a, b, and c, is as follows:



e) A tree diagram representing the process whose first step is the choice of the two Democrats and whose second step is the choice of the one Republican is as follows:



A tree diagram representing the process whose first step is the choice of the one Republican and whose second step is the choice of the two Democrats is as follows:



3.83

a) If no letter can be used more than once, there are 10 possibilities for the first letter in the sequence, nine for the second, ..., five for the sixth. Hence, by the BCR, the total number of possibilities is

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151{,}200.$$

b) If letters can be used more than once, there are 10 possibilities each for the six letters in the sequence. Hence, by the BCR, the total number of possibilities is $10^6 = 1{,}000{,}000$.

c) If a length-6 sequence of letters is randomly chosen from among the letters a through j, with repetitions allowed, then, from part (b), we know that $N(\Omega) = 1{,}000{,}000$. Let E denote the event that there are no repetitions. From part (a), we know that $N(E) = 151{,}200$. As a classical probability model is appropriate here (because the letters are randomly chosen), we conclude that

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{151{,}200}{1{,}000{,}000} = 0.1512.$$

3.84 As there are 6 options for the first question, 8 for the second, 5 for the third, 19 for the fourth, 16 for the fifth, and 14 for the sixth, the BCR implies that there are

$$6 \cdot 8 \cdot 5 \cdot 19 \cdot 16 \cdot 14 = 1{,}021{,}440$$

possibilities for answering all six questions.

3.85 Selecting the first-string, second-string, and third-string quarterbacks from the 15 applicants is equivalent to specifying a permutation of three objects from a collection of 15 objects; the first object is the applicant selected to be the first-string quarterback, the second object is the applicant selected to be the second-string quarterback, and the third object is the applicant selected to be the third-string quarterback. Thus, the number of possibilities for the first-string, second-string, and third-string quarterbacks is $(15)_3$, the number of possible permutations of three objects from a collection of 15 objects. From the permutations rule,

$$(15)_3 = \frac{15!}{(15-3)!} = \frac{15!}{12!} = \frac{15 \cdot 14 \cdot 13 \cdot 12!}{12!} = 15 \cdot 14 \cdot 13 = 2730.$$

3.86 From page 101: "When we refer simply to 'a sample' from a finite population, we mean an unordered sample without replacement unless specifically stated otherwise." Hence, we want to determine the number of possible unordered samples of size 6 without replacement from a population of size 45. From Example 3.13 on page 100, the number of possibilities is

$$\begin{aligned}\binom{45}{6} &= \frac{45!}{6!\,(45-6)!} = \frac{45!}{6!\,39!} = \frac{45 \cdot 44 \cdot 43 \cdot 42 \cdot 41 \cdot 40 \cdot 39!}{6!\,39!} = \frac{45 \cdot 44 \cdot 43 \cdot 42 \cdot 41 \cdot 40}{6!} \\ &= \frac{45 \cdot 44 \cdot 43 \cdot 42 \cdot 41 \cdot 40}{720} = 8{,}145{,}060.\end{aligned}$$

3.87

a) If the fields for the three outfielders are specified, there are eight possibilities for the pitcher, three for the catcher, and $(10)_3$ for the outfielders. Hence, by the BCR, the total number of possibilities is

$$8 \cdot 3 \cdot (10)_3 = 8 \cdot 3 \cdot 720 = 17{,}280.$$

b) If the fields for the three outfielders aren't specified, there are eight possibilities for the pitcher, three for the catcher, and $\binom{10}{3}$ for the outfielders. Hence, by the BCR, the total number of possibilities is

$$8 \cdot 3 \cdot \binom{10}{3} = 8 \cdot 3 \cdot 120 = 2880.$$

3.88 Selecting 5 centers, 10 forwards, and 10 guards from the 50 basketball players is equivalent to specifying an ordered partition of 50 objects (the 50 players) into four distinct groups (centers, forwards, guards, and non-selected) of sizes 5, 10, 10, and 25. From the partitions rule, the number of such ordered partitions is

$$\binom{50}{5,\ 10,\ 10,\ 25} = \frac{50!}{5!\,10!\,10!\,25!} \approx 1.24 \times 10^{24}.$$

3.89 Forming four committees, consisting of 10, 7, 15, and 8 members, from the 100 members of the U.S. Senate is equivalent to specifying an ordered partition of 100 objects (the 100 senators) into five distinct groups (the four committees and the non-selected) of sizes 10, 7, 15, 8, and 60. From the partitions rule, the number of such ordered partitions is

$$\binom{100}{10,\ 7,\ 15,\ 8,\ 60} = \frac{100!}{10!\,7!\,15!\,8!\,60!} \approx 1.16 \times 10^{49}.$$

3.90

a) For a 12-horse race, the number of possible quinella wagers is $\binom{12}{2}$, the number of possible combinations of two objects (the horses picked to finish first and second) from a collection of 12 objects (the 12 horses in the race). Applying the combinations rule, we find that

$$\binom{12}{2} = \frac{12!}{2!\,(12-2)!} = \frac{12!}{2!\,10!} = \frac{12 \cdot 11}{2} = 66.$$

b) For a 12-horse race, the number of possible trifecta wagers is $(12)_3$, the number of possible permutations of three objects (the horses picked to finish first, second, and third, respectively) from a collection of 12 objects (the 12 horses in the race). Applying the permutations rule, we find that

$$(12)_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!} = 12 \cdot 11 \cdot 10 = 1320.$$

3.91 Each way that the die can come up 4 twice, 5 three times, 6 once, and 1 twice corresponds to an ordered partition of eight objects (the eight rolls of the die) into four distinct groups (the positions of the 4s, 5s, 6s, and 1s) of sizes 2, 3, 1, and 2, and vice versa. From the partitions rule, the number of such ordered partitions is

$$\binom{8}{2,\ 3,\ 1,\ 2} = \frac{8!}{2!\,3!\,1!\,2!} = 1680.$$

3.92

a) Assigning three stenographers to the executive suite, three to the marketing department, and four to a general stenographic pool is equivalent to specifying an ordered partition of 10 objects (the 10 stenographers) into three distinct groups of sizes 3, 3, and 4. From the partitions rule, the number of such ordered partitions is

$$\binom{10}{3,\ 3,\ 4} = \frac{10!}{3!\,3!\,4!} = 4200.$$

b) The number of ways of assigning the three stenographers selected for the executive suite to the president, executive vice president, and financial vice president is 3!. Hence, by the BCR, if the three stenographers in the executive suite are to be assigned to the president, executive vice president, and financial vice president, with the remaining seven stenographers assigned as in part (a), then the number of possibilities is

$$\binom{10}{3,\,3,\,4} \cdot 3! = 4200 \cdot 6 = 25{,}200.$$

c) The number of ways of assigning the three stenographers selected for the marketing department to the general manager, manager for domestic marketing, and manager for foreign marketing is 3!. Hence, by the BCR, if the three stenographers in the executive suite are to be assigned as in part (b), the three in the marketing department are to be assigned to the general manager, manager for domestic marketing, and manager for foreign marketing, with the remaining four stenographers assigned as in part (a), then the number of possibilities is

$$\binom{10}{3,\,3,\,4} \cdot 3! \cdot 3! = 4200 \cdot 6 \cdot 6 = 151{,}200.$$

3.93
a) Let us label the 11 positions for the letters, from left to right, by the numbers 1–11, respectively. Then the number of distinct arrangements of the letters in the word MISSISSIPPI is the number of possible ordered partitions of those 11 numbers into four distinct groups, one of size 1 for the "M," one of size 4 for the "I"s, one of size 4 for the "S"s, and one of size 2 for the "P"s. Therefore, by the partitions rule, Proposition 3.5 on page 104, the number of distinct arrangements of the letters in the word MISSISSIPPI is

$$\binom{11}{1,\,4,\,4,\,2} = \frac{11!}{1!\,4!\,4!\,2!} = 34{,}650.$$

b) Proceeding as in part (a), we find that the number of distinct arrangements of the letters in the word MASSACHUSETTS is

$$\binom{13}{1,\,2,\,4,\,1,\,1,\,1,\,1,\,2} = \frac{13!}{1!\,2!\,4!\,1!\,1!\,1!\,1!\,2!} = 64{,}864{,}800.$$

3.94 The number of possible five-card draw poker hands is $\binom{52}{5}$, the number of possible combinations of five objects (the five cards dealt) from a collection of 52 objects (the 52 cards in the deck). Thus, $N(\Omega) = \binom{52}{5}$. Let E denote the event that you get at least one ace on the deal. Then event E^c is that you get no aces. The number of ways this latter event can occur is the number of ways that five cards can be chosen from the 48 cards that are not aces, which is $\binom{48}{5}$. Therefore, $N(E^c) = \binom{48}{5}$ and, hence, by the complementation rule,

$$P(E) = 1 - P(E^c) = 1 - \frac{N(E^c)}{N(\Omega)} = 1 - \frac{\binom{48}{5}}{\binom{52}{5}} \approx 0.341.$$

3.95 The number of possible bridge hands is $\binom{52}{13}$, the number of possible combinations of 13 objects (the 13 cards dealt) from a collection of 52 objects (the 52 cards in the deck). Thus, $N(\Omega) = \binom{52}{13}$.
a) Let A denote the event that a bridge hand has exactly two of the four aces. For event A to occur, there are $\binom{4}{2}$ possibilities for which two aces are in the hand and $\binom{48}{11}$ possibilities for the 11 non-ace cards in

the hand. Hence, by the BCR,

$$P(A) = \frac{N(A)}{N(\Omega)} = \frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} \approx 0.213.$$

b) Let B denote the event that a bridge hand has an 8-4-1 distribution. For event B to occur, there are, by the partitions rule, $\binom{4}{1,1,1,1}$ possibilities for the suits in the hand that contain eight cards, four cards, one card, and zero cards. Then there are $\binom{13}{8}$ possibilities for the denominations of the eight cards in the hand's eight-card suit, $\binom{13}{4}$ possibilities for the denominations of the four cards in the hand's four-card suit, $\binom{13}{1}$ possibilities for the denomination of the one card in the hand's one-card suit, and $\binom{13}{0}$ possibilities for the denominations of the zero cards in the hand's zero-card suit. Hence, by the BCR,

$$P(B) = \frac{N(B)}{N(\Omega)} = \frac{\binom{4}{1,\,1,\,1,\,1}\binom{13}{8}\binom{13}{4}\binom{13}{1}\binom{13}{0}}{\binom{52}{13}} \approx 0.000452.$$

c) Let C denote the event that a bridge hand has a 5-5-2-1 distribution. For event C to occur, there are, by the partitions rule, $\binom{4}{2,1,1}$ possibilities for the two suits in the hand that contain five cards each, the one suit in the hand that contains two cards, and the one suit in the hand that contains one card. Then there are $\binom{13}{5}$ possibilities each for the denominations of the five cards in each of the hand's two five-card suits, $\binom{13}{2}$ possibilities for the denominations of the two cards in the hand's two-card suit, and $\binom{13}{1}$ possibilities for the denomination of the one card in the hand's one-card suit. Hence, by the BCR,

$$P(C) = \frac{N(C)}{N(\Omega)} = \frac{\binom{4}{2,\,1,\,1}\binom{13}{5}\binom{13}{5}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}} \approx 0.0317.$$

d) Let D denote the event that a bridge hand is void in a specified suit. For event D to occur, there are $\binom{39}{13}$ possibilities for the 13 non–specified-suit cards in the hand. Hence,

$$P(D) = \frac{N(D)}{N(\Omega)} = \frac{\binom{39}{13}}{\binom{52}{13}} \approx 0.0128.$$

e) Let E denote the event that a bridge hand is void in at least one suit. Also, let E_1, E_2, E_3, and E_4 denote the events that a bridge hand is void in spades, hearts, diamonds, and clubs, respectively. Observe that $E = \bigcup_{n=1}^{4} E_n$. We will apply the inclusion–exclusion principle to determine $P(E)$. From part (d), we know that, for each k,

$$P(E_k) = \frac{\binom{39}{13}}{\binom{52}{13}}.$$

For $i \neq j$, event $E_i \cap E_j$ is that a hand is void in two specified suits. Hence,

$$P(E_i \cap E_j) = \frac{N(E_i \cap E_j)}{N(\Omega)} = \frac{\binom{26}{13}}{\binom{52}{13}}.$$

And, for $i \neq j \neq k$, event $E_i \cap E_j \cap E_k$ is that a hand is void in three specified suits. Hence,

$$P(E_i \cap E_j \cap E_k) = \frac{N(E_i \cap E_j \cap E_k)}{N(\Omega)} = \frac{\binom{13}{13}}{\binom{52}{13}}.$$

Noting that $E_1 \cap E_2 \cap E_3 \cap E_4 = \emptyset$, we conclude, in view of the inclusion–exclusion principle and Exercise 3.38, that

$$P(E) = P\left(\bigcup_{n=1}^{4} E_n\right) = \binom{4}{1} \cdot \frac{\binom{39}{13}}{\binom{52}{13}} - \binom{4}{2} \cdot \frac{\binom{26}{13}}{\binom{52}{13}} + \binom{4}{3}\frac{\binom{13}{13}}{\binom{52}{13}} - \binom{4}{4} \cdot 0$$

$$= \frac{4\binom{39}{13} - 6\binom{26}{13} + 4\binom{13}{13}}{\binom{52}{13}} \approx 0.0511.$$

3.96 The number of possible bridge hands dealt to the four players equals the number of ways that the 52 cards can be partitioned into four distinct groups of 13 cards each, one group each for North, South, East, and West. Hence, from the partitions rule, we have that $N(\Omega) = \binom{52}{13,13,13,13}$.

a) Let A denote the event that each player gets one ace. For event A to occur, there are 4! possibilities for which players get which aces. Then the remaining 48 cards must be partitioned into four distinct groups of 12 cards each, which, by the partitions rule, can be done in $\binom{48}{12,12,12,12}$ ways. Therefore, by the BCR,

$$P(A) = \frac{N(A)}{N(\Omega)} = \frac{4! \cdot \binom{48}{12,\, 12,\, 12,\, 12}}{\binom{52}{13,\, 13,\, 13,\, 13}} \approx 0.105.$$

b) Let B denote the event that North and South together have exactly three aces. We count the number of ways that event B can occur by finding first the number of ways in which the 26 cards dealt to North and South combined can contain exactly three aces, next the number of ways those 26 cards can be partitioned equally between North and South, and then the number of ways the remaining 26 cards can be partitioned equally between East and West. These numbers are $\binom{4}{3}\binom{48}{23}$, $\binom{26}{13,13}$, and $\binom{26}{13,13}$, respectively. Hence, by the BCR,

$$P(B) = \frac{N(B)}{N(\Omega)} = \frac{\binom{4}{3}\binom{48}{23}\binom{26}{13,\, 13}\binom{26}{13,\, 13}}{\binom{52}{13,\, 13,\, 13,\, 13}} \approx 0.250.$$

Alternatively, we can obtain $P(B)$ by finding the probability that, among 26 randomly dealt cards from an ordinary deck of 52 cards, exactly three are aces. Thus,

$$P(B) = \frac{\binom{4}{3}\binom{48}{23}}{\binom{52}{26}} \approx 0.250.$$

3.97 Let E denote the event that it takes at least three draws to get the first red ball or, equivalently, that the first two balls drawn are black. The number of possible outcomes for the first two draws is $26 \cdot 25$; that is, $N(\Omega) = 26 \cdot 25$. For event E to occur, there are 16 possibilities for the first draw and 15 possibilities for the second draw. Therefore, $N(E) = 16 \cdot 15$ and, hence,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{16 \cdot 15}{26 \cdot 25} = \frac{240}{650} = \frac{24}{65} \approx 0.369.$$

3.98

a) The number of possible outcomes for the two rolls of the dice is $6 \cdot 6 = 36$; that is, $N(\Omega) = 36$. Let A, B, and C denote the events that the green die shows a larger number, equal number, and smaller number than the red die, respectively. Observe that events A, B, and C are mutually exclusive and that their union is Ω. For event B to occur, there are six possibilities for the green die and then one possibility for the red die. Hence, $N(B) = 6$. By symmetry, $N(C) = N(A)$ and, so, $P(C) = P(A)$. Thus,

$$\begin{aligned} 1 = P(\Omega) &= P(A \cup B \cup C) = P(A) + P(B) + P(C) = 2P(A) + P(B) \\ &= 2P(A) + \frac{N(B)}{N(\Omega)} = 2P(A) + \frac{6}{36} = 2P(A) + \frac{1}{6}. \end{aligned}$$

Consequently, $P(A) = (1 - 1/6)/2 = 5/12$.

b) The number of possible outcomes for the three rolls of the dice is $6 \cdot 6 \cdot 6 = 216$; that is, $N(\Omega) = 216$. Let E denote the event that at least two of the three dice show the same number. Then event E^c is that all three dice show different numbers. For event E^c to occur, there are six, five, and four possibilities for the first, second, and third dice, respectively. Therefore, $N(E^c) = 6 \cdot 5 \cdot 4 = 120$ and, hence, by the complementation rule,

$$P(E) = 1 - P(E^c) = 1 - \frac{N(E^c)}{N(\Omega)} = 1 - \frac{120}{216} = \frac{4}{9}.$$

3.99 Let E denote the event that the sum of the numbers on the two chips selected is 10.

a) If the sampling is without replacement, there are 10 possibilities for the first chip selected and nine possibilities for the second chip selected. Thus, $N(\Omega) = 10 \cdot 9 = 90$. For event E to occur, there are eight possibilities for the the first chip selected (any one of the chips numbered 1–9 except for the chip numbered 5) and then one possibility for the second chip selected. Therefore, $N(E) = 8$ and, hence,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{8}{90} = \frac{4}{45} \approx 0.0889.$$

b) If the sampling is with replacement, there are 10 possibilities for each of the two chips selected. Thus, $N(\Omega) = 10 \cdot 10 = 100$. For event E to occur, there are nine possibilities for the the first chip selected (any one of the chips numbered 1–9) and then one possibility for the second chip selected. Therefore, $N(E) = 9$ and, hence,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{9}{100} = 0.09.$$

3.100 Let E denote the event that one red, two white, and three blue balls are obtained.
a) If the sampling is without replacement (and without regard to order), there are $\binom{60}{6}$ possible outcomes; that is, $N(\Omega) = \binom{60}{6}$. Event E can occur in as many ways as we can choose one red ball from the 10 red balls, two white balls from the 20 white balls, and three blue balls from the 30 blue balls. Therefore, $N(E) = \binom{10}{1}\binom{20}{2}\binom{30}{3}$ and, hence,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{\binom{10}{1}\binom{20}{2}\binom{30}{3}}{\binom{60}{6}} \approx 0.154.$$

b) If the sampling is with replacement, then there are $(60)^6$ possible outcomes; that is, $N(\Omega) = (60)^6$. For event E to occur, there are $\binom{6}{1,2,3}$ possibilities for deciding when the one red ball, two white balls, and three blue balls are chosen; then there are $(10)^1$, $(20)^2$, and $(30)^3$ possibilities for which one red ball, two white balls, and three blue balls are chosen, respectively. Therefore, $N(E) = \binom{6}{1,2,3}(10)^1(20)^2(30)^3$ and, hence,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{\binom{6}{1,\,2,\,3}(10)^1(20)^2(30)^3}{(60)^6} \approx 0.139.$$

3.101 Let E denote the event that the kth item inspected is the last defective item in the lot. We present two methods for determining $P(E)$.

Method 1: Suppose we consider only the first k items inspected. There are $(N)_k$ possibilities for the first k items inspected; that is, $N(\Omega) = (N)_k$. For event E to occur, the first $k-1$ items inspected must include exactly $M-1$ defective items and the kth item inspected must be the remaining defective item. And, for that to happen, there are $\binom{k-1}{M-1}$ possibilities for when the first $M-1$ defective items are inspected during the first $k-1$ inspections; then there are $(M)_{M-1}$ possibilities for which $M-1$ defective items are inspected during the first $k-1$ inspections and $(N-M)_{(k-1)-(M-1)}$ possibilities for which $(k-1)-(M-1)$ nondefective items are inspected during the first $k-1$ inspections. Therefore, $N(E) = \binom{k-1}{M-1} \cdot (M)_{M-1} \cdot (N-M)_{(k-1)-(M-1)}$ and, hence,

$$\begin{aligned} P(E) = \frac{N(E)}{N(\Omega)} &= \frac{\binom{k-1}{M-1}(M)_{M-1}(N-M)_{(k-1)-(M-1)}}{(N)_k} = \frac{\binom{k-1}{M-1}M!\,(N-M)_{k-M}}{(N)_k} \\ &= \frac{\binom{k-1}{M-1}M!\,(N-M)!\,(N-k)!}{\big((N-M)-(k-M)\big)!\,N!} = \frac{\binom{k-1}{M-1}}{\binom{N}{M}}. \end{aligned}$$

Method 2: Suppose we consider only when the defective items are inspected, of which there are $\binom{N}{M}$ possibilities; that is, $N(\Omega) = \binom{N}{M}$. For event E to occur, exactly $M-1$ defective items must be among the first $k-1$ inspected, which can happen in $\binom{k-1}{M-1}$ ways. Therefore, $N(E) = \binom{k-1}{M-1}$ and, hence,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{\binom{k-1}{M-1}}{\binom{N}{M}}.$$

3.102 The number of possible six-number tickets is $\binom{42}{6}$; that is, $N(\Omega) = \binom{42}{6} = 5{,}245{,}786$. We can think of the 42 numbers as being partitioned into two groups, your six numbers (Group 1) and the other 36 numbers (Group 2).

a) Let A denote the event that you win the jackpot. Event A occurs if and only if all six winning numbers come from Group 1, which can happen in $\binom{6}{6} = 1$ way. Hence,

$$P(A) = \frac{N(A)}{N(\Omega)} = \frac{1}{5{,}245{,}786} \approx 0.000000191.$$

b) Let B denote the event that your ticket contains exactly four winning numbers. Event B occurs if and only if four of the winning numbers come from Group 1 and two come from Group 2, which can happen in $\binom{6}{4}\binom{36}{2} = 9450$ ways. Hence,

$$P(B) = \frac{N(B)}{N(\Omega)} = \frac{9450}{5{,}245{,}786} \approx 0.00180.$$

c) Let C denote the event that you don't win a prize. Event C occurs if and only if two or fewer winning numbers come from Group 1, which can happen in $\binom{6}{0}\binom{36}{6} + \binom{6}{1}\binom{36}{5} + \binom{6}{2}\binom{36}{4} = 5{,}093{,}319$ ways. Hence,

$$P(C) = \frac{N(C)}{N(\Omega)} = \frac{5{,}093{,}319}{5{,}245{,}786} \approx 0.971.$$

d) Let D denote the event that you win the jackpot if you buy N (different) tickets. For $1 \le k \le N$, let A_k denote the event that you win the jackpot with your kth ticket. Events $A_1, \dots, A_N$ are mutually exclusive and their union is D. Hence, by the additivity axiom and part (a),

$$P(D) = P\left(\bigcup_{k=1}^{N} A_k\right) = \sum_{k=1}^{N} P(A_k) = NP(A) \approx 0.000000191N.$$

3.103 Let E denote the event that no man sits across from his wife. Observe that the random experiment under consideration here is entirely analogous to the one in Example 3.21 on page 114 with $N = 4$, where, in this case, the men replace the women and the women replace the keys. Hence, from part (c) of that example, we see that $P(E^c) = \sum_{n=1}^{4} (-1)^{n+1}/n!$. Consequently, from the complementation rule,

$$P(E) = 1 - P(E^c) = 1 - \sum_{n=1}^{4} \frac{(-1)^{n+1}}{n!} = \frac{3}{8}.$$

3.104 Assuming, as we must, that each person is equally likely to get off at any one of the eight floors, a classical probability model is appropriate here. There are eight possibilities each for the floor at which each of the six people exit the elevator. Thus, $N(\Omega) = 8^6$. Let E denote the event that no two people get off at the same floor. For event E to occur, there are eight possibilities for the first person, seven for the second person, and so forth. Therefore, $N(E) = (8)_6$ and, hence,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{(8)_6}{8^6} \approx 0.0769.$$

3.105 Without regard to order, there are $\binom{N}{n}$ possible samples of size n without replacement from a population of size N; thus, $N(\Omega) = \binom{N}{n}$. Let E denote the event that k specified members of the population are included in the sample. Event E can occur in as many ways as $n - k$ members of the population can be chosen from the $N-k$ nonspecified members, which is $\binom{N-k}{n-k}$. Hence,

$$P(E)=\frac{N(E)}{N(\Omega)}=\frac{\dbinom{N-k}{n-k}}{\dbinom{N}{n}}.$$

3.106 Let E denote the event that the two samples have exactly k members in common.
a) Using the first sample to mark the population, there are two groups: Group 1, consisting of the n members in the first sample, and Group 2, consisting of the $N-n$ members not in the first sample. For event E to occur, the second sample must consist of k members from Group 1 and $m-k$ members from Group 2, which can happen in $\binom{n}{k}\cdot\binom{N-n}{m-k}$ ways. Hence,

$$P(E)=\frac{\dbinom{n}{k}\dbinom{N-n}{m-k}}{\dbinom{N}{m}}.$$

b) Using the second sample to mark the population, there are two groups: Group 1, consisting of the m members in the second sample, and Group 2, consisting of the $N-m$ members not in the second sample. For event E to occur, the first sample must consist of k members from Group 1 and $n-k$ members from Group 2, which can happen in $\binom{m}{k}\cdot\binom{N-m}{n-k}$ ways. Hence,

$$P(E)=\frac{\dbinom{m}{k}\dbinom{N-m}{n-k}}{\dbinom{N}{n}}.$$

c) As $P(E)$ equals both expressions given on the right of the displays in parts (a) and (b), those two expressions must be equal.
d) We have

$$\begin{aligned}\frac{\dbinom{n}{k}\dbinom{N-n}{m-k}}{\dbinom{N}{m}}&=\frac{\dfrac{n!}{k!\,(n-k)!}\,\dfrac{(N-n)!}{(m-k)!\,\big((N-n)-(m-k)\big)!}}{\dfrac{N!}{m!\,(N-m)!}}\\&=\frac{\dfrac{m!}{k!\,(m-k)!}\,\dfrac{(N-m)!}{(n-k)!\,(N-n-m+k)!}}{\dfrac{N!}{n!\,(N-n)!}}\\&=\frac{\dfrac{m!}{k!\,(m-k)!}\,\dfrac{(N-m)!}{(n-k)!\,\big((N-m)-(n-k)\big)!}}{\dfrac{N!}{n!\,(N-n)!}}=\frac{\dbinom{m}{k}\dbinom{N-m}{n-k}}{\dbinom{N}{n}}.\end{aligned}$$

3.107
a) There are six possibilities each for the n tosses. Consequently, by the BCR, the number of possible outcomes is 6^n; that is, $N(\Omega)=6^n$.

b) Let E denote the event that none of the first $n-1$ tosses are 6 and the nth toss is a 6. For event E to occur, there are five possibilities each for the first $n-1$ tosses and one possibility for the nth toss. Hence, by the BCR, we have that $N(E) = 5^{n-1} \cdot 1 = 5^{n-1}$.
c) Assuming the die is balanced, a classical probability model is appropriate here. Thus, from parts (a) and (b),

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{5^{n-1}}{6^n}.$$

Advanced Exercises

3.108 For each $n \in \mathcal{N}$, let A_n denote the event that the first 6 occurs on the nth toss. We note that $A_1, A_2, \ldots$ are mutually exclusive and, from Exercise 3.107(c), we know that $P(A_n) = 5^{n-1}/6^n$.
a) Let A denote the event that a 6 is eventually tossed. Then $A = \bigcup_{n=1}^{\infty} A_n$. Applying the additivity axiom and the formula for a geometric series, we conclude that

$$P(A) = P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \frac{5^{n-1}}{6^n} = \frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^{n-1} = \frac{1}{6} \cdot \frac{1}{1-5/6} = 1.$$

b) Let B denote the event that the first 6 occurs on an odd-numbered toss. Then $B = \bigcup_{n=1}^{\infty} A_{2n-1}$ and, hence, by the additivity axiom and the formula for a geometric series,

$$P(B) = P\left(\bigcup_{n=1}^{\infty} A_{2n-1}\right) = \sum_{n=1}^{\infty} P(A_{2n-1}) = \sum_{n=1}^{\infty} \frac{5^{(2n-1)-1}}{6^{2n-1}}$$
$$= \frac{1}{6}\sum_{n=1}^{\infty}\left(\left(\frac{5}{6}\right)^2\right)^{n-1} = \frac{1}{6} \cdot \frac{1}{1-(5/6)^2} = \frac{6}{11}.$$

Let C denote the event that the first 6 occurs on an even-numbered toss. Then events B and C are mutually exclusive and $A = B \cup C$. Hence,

$$1 = P(A) = P(B \cup C) = P(B) + P(C) = \frac{6}{11} + P(C).$$

Therefore, $P(C) = 5/11$.

3.109
a) For the second sample, there are $\binom{N}{n}$ possible outcomes; that is, $N(\Omega) = \binom{N}{n}$. Let E denote the event that you get exactly k tagged deer in the second sample. Event E can occur in as many ways as k deer can be chosen from the M tagged deer and $n-k$ deer can be chosen from the $N-M$ untagged deer, which is $\binom{M}{k}\binom{N-M}{n-k}$. Hence,

$$P(E) = \frac{N(E)}{N(\Omega)} = \frac{\dbinom{M}{k}\dbinom{N-M}{n-k}}{\dbinom{N}{n}}.$$

b) We are assuming that $M = 10$, $n = 8$, and that there are exactly three tagged deer in the second sample. The proportion of tagged deer in the second sample is $3/8$, which provides an estimate for the proportion of tagged deer in the population, $10/N$. Thus, $3/8 \approx 10/N$ or

$$N \approx \frac{8}{3} \cdot 10 \approx 26.7.$$

Hence, we would estimate that there are about 27 deer in the population.

3.110

a) Suppose that $0!$ is defined to be some number other than 1. Then, with $n = 0$, we have

$$(n+1)! = (0+1)! = 1! = 1 \neq 0! = 1 \cdot 0! = (0+1) \cdot 0! = (n+1) \cdot n!.$$

Hence, the equation $(n+1)! = (n+1) \cdot n!$ would not hold for $n = 0$.

b) Let d denote the number that appears in the display when you press the CLEAR key. When you press "14, ENTER," the display will show the product of d and 14, that is, $14d$; next, when you subsequently press "3, ENTER," the display will show the product of $14d$ and 3, that is, $14d \cdot 3 = 42d$; then, when you subsequently press "6, ENTER," the display will show the product of $42d$ and 6, that is, $42d \cdot 6 = 252d$. As we know from the problem statement, the display actually shows 252. Hence, we must have $252d = 252$, implying that $d = 1$. Thus, the product of no numbers gives 1. Now, for each $n \in \mathcal{N}$, the quantity $n!$ is the product of n numbers (specifically, of $1, 2, \ldots, n$). So, for consistency, $0!$ should be the product of no numbers, which, as we have seen, should be 1. Also, for each $n \in \mathcal{N}$, the quantity 9^n is the product of n numbers (specifically, of 9 with itself n times). So, again, for consistency, 9^0 should be the product of no numbers, which, as we have seen, should be 1.

c) The number of ways to choose three men and no women from a group of five men and two women is $\binom{5}{3}\binom{2}{0}$, and the number of ways to choose three men from a group of five men and two women is $\binom{5}{3}$. Consequently, we must have $\binom{5}{3}\binom{2}{0} = \binom{5}{3}$, or $\binom{2}{0} = 1$. Hence, by the combinations rule,

$$1 = \binom{2}{0} = \frac{2!}{0!\,(2-0)!} = \frac{2!}{0!\,2!} = \frac{1}{0!},$$

which means that $0! = 1$.

3.111 Let $\mathcal{P}$ denote the collection of all possible polygonal lines as described in the problem statement.

a) The number of members of $\mathcal{P}$ is $\binom{2N}{N}$, the number of ways that we can choose N locations (say, for where the line segments go up) from $2N$ locations. Thus, $N(\Omega) = N(\mathcal{P}) = \binom{2N}{N}$.

b) Each member of $\mathcal{P}$ starts at $(0, 0)$ and consists of $2N$ line segments. As the x-coordinates of successive line segments differ by 1 unit, the x-coordinate of the terminal point must be $2N$. And, because half of the line segments go up 1 unit and half go down 1 unit, the y-coordinate of the terminal point must be 0. Hence, the terminal point of each member of $\mathcal{P}$ must be $(2N, 0)$.

c) Event E occurs if and only if there is never a time when the number of people who have reached the box office with only \$20 bills exceeds the number of people who have reached the box office with only \$10 bills, which happens if and only if it is always the case that the number of line segments that have gone up never exceeds the number of line segments that have gone down, and that happens if and only if the polygonal line doesn't go above the x-axis.

d) Let $\mathcal{L}$ denote the collection of all members of $\mathcal{P}$ that go above the x-axis and let $\mathcal{L}^*$ denote the collection of all possible polygonal lines (consisting of line segments as described in the problem statement) that start at $(0, 0)$ and terminate at $(2N, 2)$. Per the hint, for each $L \in \mathcal{L}$, consider the polygonal line L^* that coincides with L up to the first point at which L goes above the x-axis and then is the mirror image of L relative to the line $y = 1$ from then on. From part (b), we know that the terminal point of L is $(2N, 0)$. Therefore, the terminal point of L^* must be $(2N, 2)$ and, hence, $L^* \in \mathcal{L}^*$. Clearly, the correspondence between $\mathcal{L}$ and $\mathcal{L}^*$ just described is one-to-one and onto. In particular, then, we have $N(\mathcal{L}) = N(\mathcal{L}^*)$. However, for each member of $\mathcal{L}^*$, the number of the $2N$ line segments that go up must exceed by 2 the number that go down, which means that $N+1$ go up and $N-1$ go down. Hence, the number of members of $\mathcal{L}^*$ is $\binom{2N}{N+1}$, the number of ways that we can choose $N+1$ locations (for where the line segments go up) from $2N$ locations. Consequently, $N(\mathcal{L}) = N(\mathcal{L}^*) = \binom{2N}{N+1}$.

e) From parts (c) and (d), we see that $N(E^c) = N(\mathcal{L}) = \binom{2N}{N+1}$ and, therefore, in view of part (a),

$$P(E^c) = \frac{N(E^c)}{N(\Omega)} = \frac{\dbinom{2N}{N+1}}{\dbinom{2N}{N}} = \frac{N}{N+1}.$$

Applying now the complementation rule, we conclude that

$$P(E) = 1 - P(E^c) = 1 - \frac{N}{N+1} = \frac{1}{N+1}.$$