

Chapter 2

2.1 Classify each of the following signals as finite or infinite. For the finite signals, find the smallest integer N such that $x(k) = 0$ for $|k| > N$.

- (a) $x(k) = \mu(k + 5) - \mu(k - 5)$
- (b) $x(k) = \sin(.2\pi k)\mu(k)$
- (c) $x(k) = \min(k^2 - 9, 0)\mu(k)$
- (d) $x(k) = \mu(k)\mu(-k)/(1 + k^2)$
- (e) $x(k) = \tan(\sqrt{2}\pi k)[\mu(k) - \mu(k - 100)]$
- (f) $x(k) = \delta(k) + \cos(\pi k) - (-1)^k$
- (g) $x(k) = k^{-k} \sin(.5\pi k)$

Solution

- (a) finite, $N = 5$
- (b) infinite
- (c) finite, $N = 2$
- (d) finite, $N = 1$
- (e) finite, $N = 99$
- (f) finite, $N = 0$
- (g) infinite

2.2 Classify each of the following signals as causal or noncausal.

- (a) $x(k) = \max\{k, 0\}$
- (b) $x(k) = \sin(.2\pi k)\mu(-k)$
- (c) $x(k) = 1 - \exp(-k)$
- (d) $x(k) = \text{mod}(k, 10)$
- (e) $x(k) = \tan(\sqrt{2}\pi k)[\mu(k) + \mu(k - 100)]$
- (f) $x(k) = \cos(\pi k) + (-1)^k$
- (g) $x(k) = \sin(.5\pi k)/(1 + k^2)$

Solution

- (a) causal

- (b) noncausal
- (c) noncausal
- (d) noncausal
- (e) causal
- (e) causal
- (f) noncausal

2.3 Classify each of the following signals as periodic or aperiodic. For the periodic signals, find the period, M .

- (a) $x(k) = \cos(.02\pi k)$
- (b) $x(k) = \sin(.1k) \cos(.2k)$
- (c) $x(k) = \cos(\sqrt{3}k)$
- (d) $x(k) = \exp(j\pi/8)$
- (e) $x(k) = \text{mod}(k, 10)$
- (f) $x(k) = \sin^2(.1\pi k)\mu(k)$
- (g) $x(k) = j^{2k}$

Solution

- (a) periodic, $M = 100$
- (b) nonperiodic, ($\tau = 20\pi$)
- (c) nonperiodic, ($\tau = 2\pi/\sqrt{3}$)
- (d) periodic, $M = 16$
- (e) periodic, $M = 10$
- (f) nonperiodic, (causal)
- (g) periodic, $M = 2$

2.4 Classify each of the following signals as bounded or unbounded.

- (a) $x(k) = k \cos(.1\pi k)/(1 + k^2)$
- (b) $x(k) = \sin(.1k) \cos(.2k)\delta(k - 3)$
- (c) $x(k) = \cos(\pi k^2)$
- (d) $x(k) = \tan(.1\pi k)[\mu(k) - \mu(k - 10)]$
- (e) $x(k) = k^2/(1 + k^2)$
- (f) $x(k) = k \exp(-k)\mu(k)$

Solution

- (a) bounded
- (b) bounded
- (c) bounded
- (d) unbounded
- (e) bounded
- (f) bounded

2.5 For each of the following signals, determine whether or not it is bounded. For the bounded signals, find a bound, B_x .

- (a) $x(k) = [1 + \sin(5\pi k)]\mu(k)$
- (b) $x(k) = k(.5)^k\mu(k)$
- (c) $x(k) = \left[\frac{(1+k)\sin(10k)}{1 + (.5)^k} \right] \mu(k)$
- (d) $x(k) = [1 + (-1)^k] \cos(10k)\mu(k)$

Solution

- (a) bounded, $B_x = 1$
- (b) The following are the first few values of $x(k)$.

k	$x(k)$
0	0
1	1/2
2	1/2
3	3/8
4	4/16
5	5/25

Thus $x(k)$ is bounded with $B_x = .5$.

- (c) unbounded
- (d) bounded, $B_x = 2$.

2.6 Consider the following sum of causal exponentials.

$$x(k) = [c_1 p_1^k + c_2 p_2^k] \mu(k)$$

(a) Using the inequalities in Appendix 2, show that

$$|x(k)| \leq |c_1| \cdot |p_1|^k + |c_2| \cdot |p_2|^k$$

(b) Show that $x(k)$ is absolutely summable if $|p_1| < 1$ and $|p_2| < 1$. Find an upper bound on $\|x\|_1$

(c) Suppose $|p_1| < 1$ and $|p_2| < 1$. Find an upper bound on the energy E_x .

Solution

(a) Using Appendix 2

$$\begin{aligned} |x(k)| &= |[c_1(p_1)^k + c_2(p_2)^k]\mu(k)| \\ &= |c_1(p_1)^k + c_2(p_2)^k| \cdot |\mu(k)| \\ &= |c_1(p_1)^k + c_2(p_2)^k| \\ &\leq |c_1(p_1)^k| + |c_2(p_2)^k| \\ &= |c_1| \cdot |p_1|^k + |c_2| \cdot |p_2|^k \\ &= |c_1| \cdot |p_1|^k + |c_2| \cdot |p_2|^k \end{aligned}$$

(b) Suppose $|p_1| < 1$ and $|p_2| < 1$. Then using (a) and the geometric series in (2.2.14)

$$\begin{aligned} \|x\|_1 &= \sum_{k=-\infty}^{\infty} |x(k)| \\ &\leq \sum_{k=0}^{\infty} |c_1| \cdot |p_1|^k + |c_2| \cdot |p_2|^k \\ &= |c_1| \sum_{k=0}^{\infty} |p_1|^k + |c_2| \sum_{k=0}^{\infty} |p_2|^k \\ &= \frac{|c_1|}{1 - |p_1|} + \frac{|c_2|}{1 - |p_2|} \end{aligned}$$

(c) Using (b) and (2.2.7) through (2.2.9)

$$\begin{aligned}
E_x &= \|x\|_2^2 \\
&\leq \|x\|_1^2 \\
&\leq \frac{|c_1|}{1 - |p_1|} + \frac{|c_2|}{1 - |p_2|}
\end{aligned}$$

2.7 Find the average power of the following signals.

- (a) $x(k) = 10$
- (b) $x(k) = 20\mu(k)$
- (c) $x(k) = \text{mod}(k, 5)$
- (d) $x(k) = a \cos(\pi k/8) + b \sin(\pi k/8)$
- (e) $x(k) = 100[\mu(k + 10) - \mu(k - 10)]$
- (f) $x(k) = j^k$

Solution

Using (2.2.10)-(2.2.12) and Appendix 2

- (a) $P_x = 100$
- (b) $P_x = 400$
- (c) $P_x = (1 + 4 + 9 + 16)/5 = 6$
- (d)

$$\begin{aligned}
[a \cos(\pi k/8) + b \sin(\pi k/8)]^2 &= a^2 \cos^2(\pi k/8) + 2ab \cos(\pi k/8) \sin(\pi k/8) + b^2 \sin^2(\pi k/8) \\
&= a^2 \left[\frac{1 + \cos(\pi k/4)}{2} \right] + ab \sin(\pi k/4) + b^2 \left[\frac{1 - \cos(\pi k/4)}{2} \right]
\end{aligned}$$

Thus

$$P_x = \frac{a^2 + b^2}{2}$$

- (e) $P_x = 10^4$

(f)

$$\begin{aligned} P_x &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N |j^k|^2 \\ &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N (|j|^k)^2 \\ &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^N 1 \\ &= 1 \end{aligned}$$

2.8 Classify each of the following systems as linear or nonlinear.

- (a) $y(k) = 4[y(k-1) + 1]x(k)$
- (b) $y(k) = 6kx(k)$
- (c) $y(k) = -y(k-2) + 10x(k+3)$
- (d) $y(k) = .5y(k) - 2y(k-1)$
- (e) $y(k) = .2y(k-1) + x^2(k)$
- (f) $y(k) = -y(k-1)x(k-1)/10$

Solution

- (a) nonlinear (product term)
- (b) linear
- (c) linear
- (d) linear
- (e) nonlinear (input term)
- (f) nonlinear (product term)

2.9 Classify each of the following systems as time-invariant or time-varying.

- (a) $y(k) = [x(k) - 2y(k-1)]^2$
- (b) $y(k) = \sin[\pi y(k-1)] + 3x(k-2)$
- (c) $y(k) = (k+1)y(k-1) + \cos[.1\pi x(k)]$
- (d) $y(k) = .5y(k-1) + \exp(-k/5)\mu(k)$
- (e) $y(k) = \log[1 + x^2(k-2)]$

(f) $y(k) = kx(k - 1)$

Solution

- (a) time-invariant
- (b) time-invariant
- (c) time-varying
- (d) time-varying
- (e) time-invariant
- (f) time-varying

2.10 Classify each of the following systems as causal or noncausal.

- (a) $y(k) = [3x(k) - y(k - 1)]^3$
- (b) $y(k) = \sin[\pi y(k - 1)] + 3x(k + 1)$
- (c) $y(k) = (k + 1)y(k - 1) + \cos[.1\pi x(k^2)]$
- (d) $y(k) = .5y(k - 1) + \exp(-k/5)\mu(k)$
- (e) $y(k) = \log[1 + y^2(k - 1)x^2(k + 2)]$
- (f) $h(k) = \mu(k + 3) - \mu(k - 3)$

Solution

- (a) causal
- (b) noncausal
- (c) causal
- (d) causal
- (e) noncausal
- (f) noncausal

2.11 Consider the following system that consists of a gain of A and a delay of d samples.

$$y(k) = Ax(k - d)$$

- (a) Find the impulse response $h(k)$ of this system.
- (b) Classify this system as FIR or IIR.

- (c) Is this system BIBO stable? If so, find $\|h\|_1$.
- (d) For what values of A and d is this a passive system?
- (e) For what values of A and d is this an active system?
- (f) For what values of A and d is this a lossless system?

Solution

- (a) $h(k) = A\delta(k - d)$
- (b) FIR
- (c) Yes, it is BIBO stable with $\|h\|_1 = |A|$.
- (d)

$$\begin{aligned}
 E_y &= \sum_{k=-\infty}^{\infty} y^2(k) \\
 &= \sum_{k=-\infty}^{\infty} [Ax(k - d)]^2 \\
 &= A^2 \sum_{k=-\infty}^{\infty} x^2(k - d) \\
 &= A^2 \sum_{i=-\infty}^{\infty} x^2(i) \quad , \quad i = k - d \\
 &= A^2 E_x
 \end{aligned}$$

This is a passive system for $|A| < 1$.

- (e) This is an active system for $|A| > 1$
- (f) This is a lossless system for $|A| = 1$

2.12 Consider the following linear time-invariant discrete-time system S .

$$y(k) - y(k - 2) = 2x(k)$$

- (a) Find the characteristic polynomial of S and express it in factored form.
- (b) Write down the general form of the zero-input response, $y_{zi}(k)$.
- (c) Find the zero-input response when $y(-1) = 4$ and $y(-2) = -1$.

Solution

(a)

$$\begin{aligned}a(z) &= z^2 - 1 \\ &= (z - 1)(z + 1)\end{aligned}$$

(b)

$$\begin{aligned}y_{zi}(k) &= c_1(p_1)^k + c_2(p_2)^k \\ &= c_1 + c_2(-1)^k\end{aligned}$$

(c) Evaluating part (b) at the two initial conditions yields

$$\begin{aligned}c_1 - c_2 &= 4 \\ c_1 + c_2 &= -1\end{aligned}$$

Adding the equations yields $2c_1 = 3$ or $c_1 = 1.5$. Subtracting the first equation from the second yields $2c_2 = -5$ or $c_2 = -2.5$. Thus the zero-input response is

$$y_{zi}(k) = 1.5 - 2.5(-1)^k$$

✓ **2.13** Consider the following linear time-invariant discrete-time system S .

$$y(k) = 1.8y(k-1) - .81y(k-2) - 3x(k-1)$$

- Find the characteristic polynomial $a(z)$ and express it in factored form.
- Write down the general form of the zero-input response, $y_{zi}(k)$.
- Find the zero-input response when $y(-1) = 2$ and $y(-2) = 2$.

Solution

(a)

$$\begin{aligned}a(z) &= z^2 - 1.8z + .81 \\ &= (z - .9)^2\end{aligned}$$

(b)

$$\begin{aligned}y_{zi}(k) &= (c_1 + c_2k)p^k \\ &= (c_1 + c_2k).9^k\end{aligned}$$

(c) Evaluating part (b) at the two initial conditions yields

$$\begin{aligned}(c_1 - c_2).9^{-1} &= 2 \\ (c_1 - 2c_2).9^{-2} &= 2\end{aligned}$$

or

$$\begin{aligned}c_1 - c_2 &= 1.8 \\ c_1 - 2c_2 &= 1.62\end{aligned}$$

Subtracting the second equation from the first yields $c_2 = .18$. Subtracting the second equation from two times the first yields $c_1 = 1.98$. Thus the zero-input response is

$$y_{zi}(k) = (1.98 + .18k).9^k$$

2.14 Consider the following linear time-invariant discrete-time system S .

$$y(k) = -.64y(k-2) + x(k) - x(k-2)$$

- Find the characteristic polynomial $a(z)$ and express it in factored form.
- Write down the general form of the zero-input response, $y_{zi}(k)$, expressing it as a real signal.

(c) Find the zero-input response when $y(-1) = 3$ and $y(-2) = 1$.

Solution

(a)

$$\begin{aligned} a(z) &= z^2 + .64 \\ &= (z - .8j)(z + .8j) \end{aligned}$$

(b) In polar form the roots are $z = .8 \exp(\pm j\pi/2)$. Thus

$$\begin{aligned} y_{zi}(k) &= r^k [c_1 \cos(k\theta) + c_2 \sin(k\theta)] \\ &= .8^k [c_1 \cos(k\pi/2) + c_2 \sin(\pi k/2)] \end{aligned}$$

(c) Evaluating part (b) at the two initial conditions yields

$$\begin{aligned} .8^{-1}c_2(-1) &= 3 \\ .8^{-2}c_1(-1) &= 1 \end{aligned}$$

Thus $c_2 = -3(.8)$ and $c_1 = -1(.64)$. Hence the zero-input response is

$$y_{zi}(k) = -(.8)^k [.64 \cos(\pi k/2) + 2.4 \sin(\pi k/2)]$$

2.15 Consider the following linear time-invariant discrete-time system S .

$$y(k) - 2y(k-1) + 1.48y(k-2) - .416y(k-3) = 5x(k)$$

- Find the characteristic polynomial $a(z)$. Using the MATLAB function *roots*, express it in factored form.
- Write down the general form of the zero-input response, $y_{zi}(k)$.

- (c) Write the equations for the unknown coefficient vector $c \in R^3$ as $Ac = y_0$, where $y_0 = [y(-1), y(-2), y(-3)]^T$ is the initial condition vector.

Solution

(a)

$$a(z) = z^3 - 2z^2 + 1.48z - .416$$

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a = [1 -2 1.48 -.416]
r = roots(a)
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$$a(z) = (z - .8)(z - .6 - .4j)(z - .6 + .4j)$$

- (b) The complex roots in polar form are $p_{2,3} = r \exp(\pm j\theta)$ where

$$\begin{aligned} r &= \sqrt{.6^2 + .4^2} \\ &= .7211 \\ \theta &= \arctan(\pm .4/.6) \\ &= \pm .588 \end{aligned}$$

Thus the form of the zero-input response is

$$\begin{aligned} y_{zi}(k) &= c_1(p_1)^k + r^k[c_2 \cos(k\theta) + c_3 \sin(k\theta)] \\ &= c_1(.8)^k + .7211^k[c_2 \cos(.588k) + c_3 \sin(.588k)] \end{aligned}$$

- (c) Let $c \in R^3$ be the unknown coefficient vector, and $y_0 = [y(-1), y(-2), y(-3)]^T$. Then $Ac = y_0$ or

$$\begin{bmatrix} .8^{-1} & .7211^{-1} \cos(-.588) & .7211^{-1} \sin(-.588) \\ .8^{-2} & .7211^{-2} \cos[-2(.588)] & .7211^{-2} \sin[-2(.588)] \\ .8^{-3} & .7211^{-3} \cos[-3(.588)] & .7211^{-3} \sin[-3(.588)] \end{bmatrix} c = y_0$$

2.16 Consider the following linear time-invariant discrete-time system S .

$$y(k) - .9y(k-1) = 2x(k) + x(k-1)$$

- (a) Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$.
- (b) Write down the general form of the zero-state response, $y_{zs}(k)$, when the input is $x(k) = 3(.4)^k \mu(k)$.
- (c) Find the zero-state response.

Solution

(a)

$$\begin{aligned} a(z) &= z - .9 \\ b(z) &= 2z + 1 \end{aligned}$$

(b)

$$\begin{aligned} y_{zs}(k) &= [d_0(p_0)^k + d_1(p_1)^k] \mu(k) \\ &= [d_0(.4)^k + d_1(.9)^k] \mu(k) \end{aligned}$$

(c)

$$\begin{aligned} d_0 &= \left. \frac{Ab(z)}{a(z)} \right|_{z=p_0} \\ &= \frac{3[2(.4) + 1]}{.4 - .9} \\ &= \frac{5.4}{-.5} \\ &= -10.8 \\ d_1 &= \left. \frac{A(z - p_1)b(z)}{(z - p_0)a(z)} \right|_{z=p_1} \\ &= \frac{3[2(.9) + 1]}{.5} \\ &= \frac{8.4}{.5} \\ &= 16.8 \end{aligned}$$

Thus the zero-state response is

$$y_{zs}(k) = [-10.8(.4)^k + 16.8(.9)^k]\mu(k)$$

2.17 Consider the following linear time-invariant discrete-time system S .

$$y(k) = y(k-1) - .24y(k-2) + 3x(k) - 2x(k-1)$$

- (a) Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$.
- (b) Suppose the input is the unit step, $x(k) = \mu(k)$. Write down the general form of the zero-state response, $y_{zs}(k)$.
- (c) Find the zero-state response to the unit step input.

Solution

(a)

$$\begin{aligned}a(z) &= z^2 - z + .24 \\b(z) &= 3z - 2\end{aligned}$$

(b) The factored form of $a(z)$ is

$$a(z) = (z - .6)(z - .4)$$

Thus the form of the zero-state response to a unit step input is

$$y_{zs}(k) = [d_0 + d_1(.6)^k + d_2(.4)^k]\mu(k)$$

(c)

$$\begin{aligned}d_0 &= \left. \frac{Ab(z)}{a(z)} \right|_{z=p_0} \\&= \frac{3-2}{(1-.6)(1-.4)} \\&= \frac{1}{.24} \\&= 4.167 \\d_1 &= \left. \frac{A(z-p_1)b(z)}{(z-p_0)a(z)} \right|_{z=p_1} \\&= \frac{3(.6)-2}{(.6-1)(.6-.4)} \\&= \frac{-.2}{-.08} \\&= 2.5 \\d_2 &= \left. \frac{A(z-p_2)b(z)}{(z-p_0)a(z)} \right|_{z=p_2} \\&= \frac{3(.4)-2}{(.4-1)(.4-.6)} \\&= \frac{-.8}{-.12} \\&= 6.667\end{aligned}$$

Thus the zero-state response is

$$y_{zs}(k) = [4.167 + 2.5(.6)^k + 6.667(.4)^k]\mu(k)$$

2.18 Consider the following linear time-invariant discrete-time system S .

$$y(k) = y(k-1) - .21y(k-2) + 3x(k) + 2x(k-2)$$

- Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$. Express $a(z)$ in factored form.
- Write down the general form of the zero-input response, $y_{zi}(k)$.
- Find the zero-input response when the initial condition is $y(-1) = 1$ and $y(-2) = -1$.

- (d) Write down the general form of the zero-state response when the input is $x(k) = 2(.5)^{k-1}\mu(k)$.
- (e) Find the zero-state response using the input in (d).
- (f) Find the complete response using the initial condition in (c) and the input in (d).

Solution

(a)

$$\begin{aligned} a(z) &= z^2 - z + .21 \\ &= (z - .3)(z - .7) \\ b(z) &= 3z^2 + 2 \end{aligned}$$

(b) The general form of the zero-input response is

$$\begin{aligned} y_{zi}(k) &= c_1(p_1)^k + c_2(p_2)^k \\ &= c_1(.3)^k + c_2(.7)^k \end{aligned}$$

(c) Using (b) and applying the initial conditions yields

$$\begin{aligned} c_1(.3)^{-1} + c_2(.7)^{-1} &= 1 \\ c_1(.3)^{-2} + c_2(.7)^{-2} &= -1 \end{aligned}$$

Clearing the denominators,

$$\begin{aligned} .7c_1 + .3c_2 &= .21 \\ .49c_1 + .09c_2 &= -.0441 \end{aligned}$$

Subtracting the second equation from seven times the first equation yields $2.01c_2 = 1.51$. Subtracting $.3$ times the first equation from the second yields $.28c_1 = -.127$. Thus the zero-input response is

$$y_{zi}(k) = -.454(.3)^k + .751(.7)^k$$

(d) First note that

$$\begin{aligned}x(k) &= 2(.5)^{k-1}\mu(k) \\ &= 4(.5)^k\mu(k)\end{aligned}$$

The general form of the zero-state response is

$$y_{zs}(k) = [d_0(.5)^k + d_1(.3)^k + d_2(.7)^k]\mu(k)$$

(e)

$$\begin{aligned}d_0 &= \left. \frac{Ab(z)}{a(z)} \right|_{z=p_0} \\ &= \frac{4[3(.5)^2 + 2]}{(.5 - .3)(.5 - .7)} \\ &= \frac{4(2.75)}{-.04} \\ &= -275 \\ d_1 &= \left. \frac{A(z - p_1)b(z)}{(z - p_0)a(z)} \right|_{z=p_1} \\ &= \frac{4[3(.3)^2 + 2]}{(.3 - .5)(.3 - .7)} \\ &= \frac{4(2.27)}{.08} \\ &= 113.5 \\ d_2 &= \left. \frac{A(z - p_2)b(z)}{(z - p_0)a(z)} \right|_{z=p_2} \\ &= \frac{4[3(.7)^2 + 2]}{(.7 - .5)(.7 - .3)} \\ &= \frac{4(2.63)}{.08} \\ &= 131.5\end{aligned}$$

Thus the zero-state response is

$$y_{zs}(k) = [-275(.5)^k + 113.5(.3)^k + 131.5(.7)^k]\mu(k)$$

(f) By superposition, the complete response is

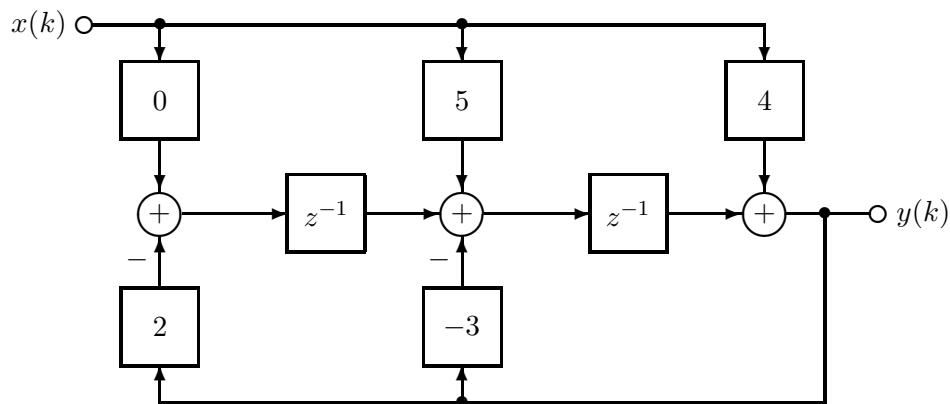
$$\begin{aligned} y(k) &= y_{zi}(k) + y_{zs}(k) \\ &= -.454(.3)^k + .751(.7)^k + [-275(.5)^k + 113.5(.3)^k + 131.5(.7)^k]\mu(k) \end{aligned}$$

2.19 Consider the following linear time-invariant discrete-time system S . Sketch a block diagram of this IIR system.

$$y(k) = 3y(k-1) - 2y(k-2) + 4x(k) + 5x(k-1)$$

Solution

$$\begin{aligned} a &= [1, -3, 2] \\ b &= [4, 5, 0] \end{aligned}$$



Problem 2.19

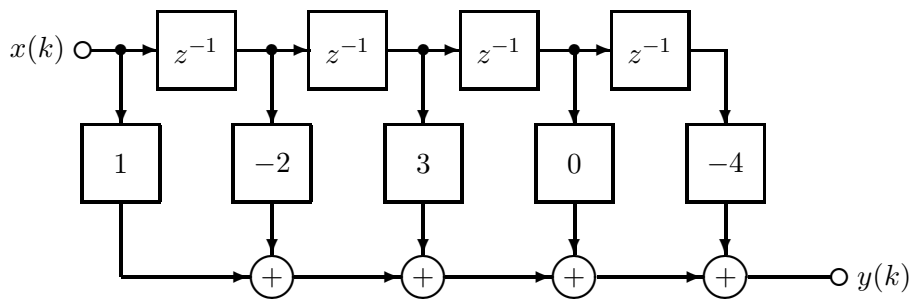
2.20 Consider the following linear time-invariant discrete-time system S . Sketch a block diagram of this FIR system.

$$y(k) = x(k) - 2x(k-1) + 3x(k-2) - 4x(k-4)$$

Solution

$$a = [1, 0, 0]$$

$$b = [1, -2, 3, 0, -4]$$



Problem 2.20

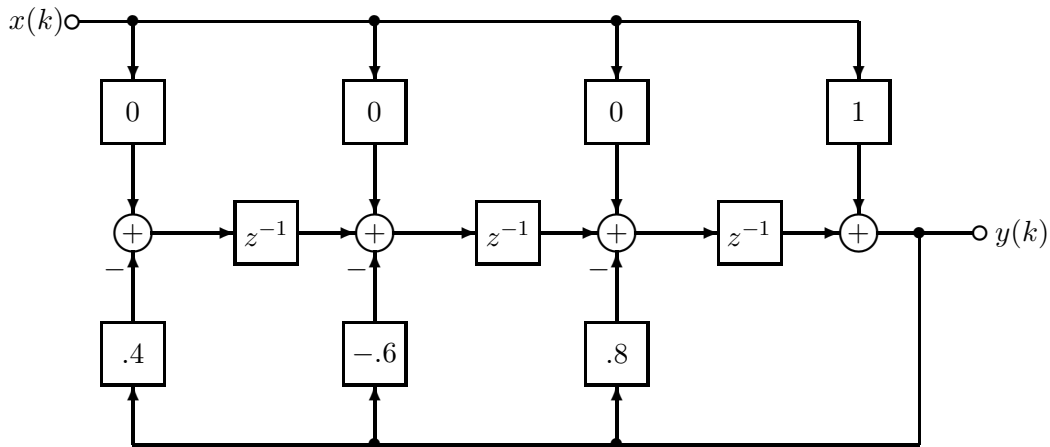
2.21 Consider the following linear time-invariant discrete-time system S called an *auto-regressive* system. Sketch a block diagram of this system.

$$y(k) = x(k) - .8y(k-1) + .6y(k-2) - .4y(k-3)$$

Solution

$$a = [1, .8, -.6, .4]$$

$$b = [1, 0, 0, 0]$$



Problem 2.21

2.22 Consider the block diagram shown in Figure 2.32.

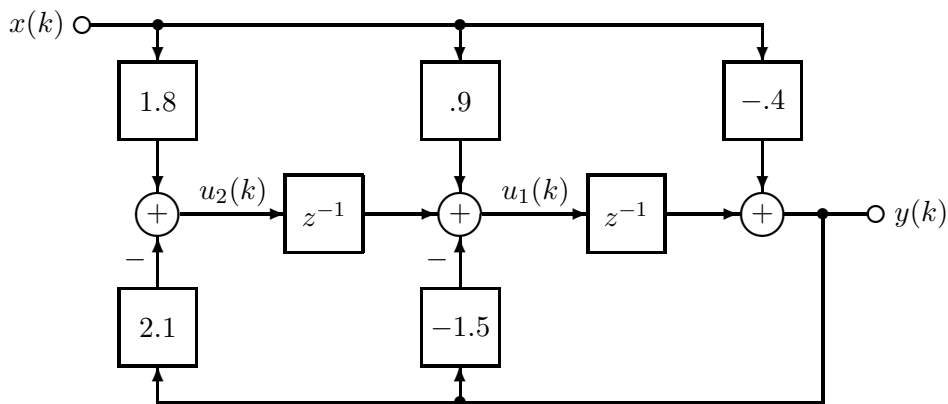


Figure 2.32 A Block Diagram of the System in Problem 2.22

- Write a single difference equation description of this system.
- Write a system of difference equations for this system for $u_i(k)$ for $1 \leq i \leq 2$ and $y(k)$.

Solution

- By inspection of Figure 2.32

$$y(k) = -0.4x(k) + 0.9x(k-1) + 1.8x(k-2) + 1.5y(k-1) - 2.1y(k-2)$$

(b) The equivalent system of equations is

$$\begin{aligned}u_2(k) &= 1.8x(k) - 2.1y(k) \\u_1(k) &= .9x(k) + 1.5y(k) + u_2(k - 1) \\y(k) &= -.4x(k) + u_1(k - 1)\end{aligned}$$

2.23 Consider the following linear time-invariant discrete-time system S .

$$y(k) = .6y(k - 1) + x(k) - .7x(k - 1)$$

- (a) Find the characteristic polynomial and the input polynomial.
- (b) Write down the form of the impulse response, $h(k)$.
- (c) Find the impulse response.

Solution

(a)

$$\begin{aligned}a(z) &= z - .6 \\b(z) &= z - .7\end{aligned}$$

(b)

$$h(k) = d_0\delta(k) + d_1(.6)^k\mu(k)$$

(c)

$$\begin{aligned}d_0 &= \left. \frac{b(z)}{a(z)} \right|_{z=0} \\&= \frac{-.7}{-.6} \\&= 1.167 \\d_1 &= \left. \frac{(z - p_1)b(z)}{za(z)} \right|_{z=p_1} \\&= \frac{.6 - .7}{.6} \\&= -.167\end{aligned}$$

Thus the impulse response is

$$h(k) = 1.167\delta(k) - .167(.6)^k\mu(k)$$

2.24 Consider the following linear time-invariant discrete-time system S .

$$y(k) = -.25y(k-2) + x(k-1)$$

- (a) Find the characteristic polynomial and the input polynomial.
- (b) Write down the form of the impulse response, $h(k)$.
- (c) Find the impulse response. Use the identities in Appendix 2 to express $h(k)$ in real form.

Solution

(a)

$$\begin{aligned}a(z) &= z^2 + .25 \\b(z) &= z\end{aligned}$$

(b) First note that

$$a(z) = (z - .5j)(z + .5j)$$

Thus the form of the impulse response is

$$h(k) = d_0\delta(k) + [d_1(.5j)^k + d_2(-.5j)^k]\mu(k)$$

(c)

$$\begin{aligned}d_0 &= \left. \frac{b(z)}{a(z)} \right|_{z=0} \\ &= 0 \\ d_1 &= \left. \frac{(z - p_1)b(z)}{za(z)} \right|_{z=p_1} \\ &= \frac{.5j}{.5j(j)} \\ &= -j \\ d_2 &= \left. \frac{(z - p_2)b(z)}{za(z)} \right|_{z=p_2} \\ &= \frac{-.5j}{-.5j(-j)} \\ &= j\end{aligned}$$

Thus from Appendix 2 the impulse response is

$$\begin{aligned}h(k) &= [-j(.5j)^k + j(-.5j)^k]\mu(k) \\ &= 2\text{Re}[-j(.5j)^k]\mu(k) \\ &= -2\text{Re}[(.5)^k(j)^{k+1}]\mu(k) \\ &= -2(.5)^k\text{Re}\{\exp(j\pi/2)\}^{k+1}\mu(k) \\ &= -2(.5)^k\text{Re}[\exp[j(k+1)\pi/2]]\mu(k) \\ &= -2(.5)^k\cos[(k+1)\pi/2]\mu(k)\end{aligned}$$

2.25 Consider the following linear time-invariant discrete-time system S . Suppose $0 < m \leq n$ and the characteristic polynomial $a(z)$ has simple nonzero roots.

$$y(k) = \sum_{i=0}^m b_i x(k-i) - \sum_{i=1}^n a_i y(k-i)$$

- Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$.
- Find a constraint on $b(z)$ that ensures that the impulse response $h(k)$ does not contain an impulse term.

Solution

(a)

$$\begin{aligned}a(z) &= z^n + a_1 z^{n-1} + \cdots + a_n \\b(z) &= b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}\end{aligned}$$

(b) The coefficient of the impulse term is

$$\begin{aligned}d_0 &= \left. \frac{b(z)}{a(z)} \right|_{z=0} \\ &= \frac{b(0)}{a(0)}\end{aligned}$$

Thus

$$\begin{aligned}d_0 \neq 0 &\Leftrightarrow b(0) \neq 0 \\ &\Leftrightarrow m = n\end{aligned}$$

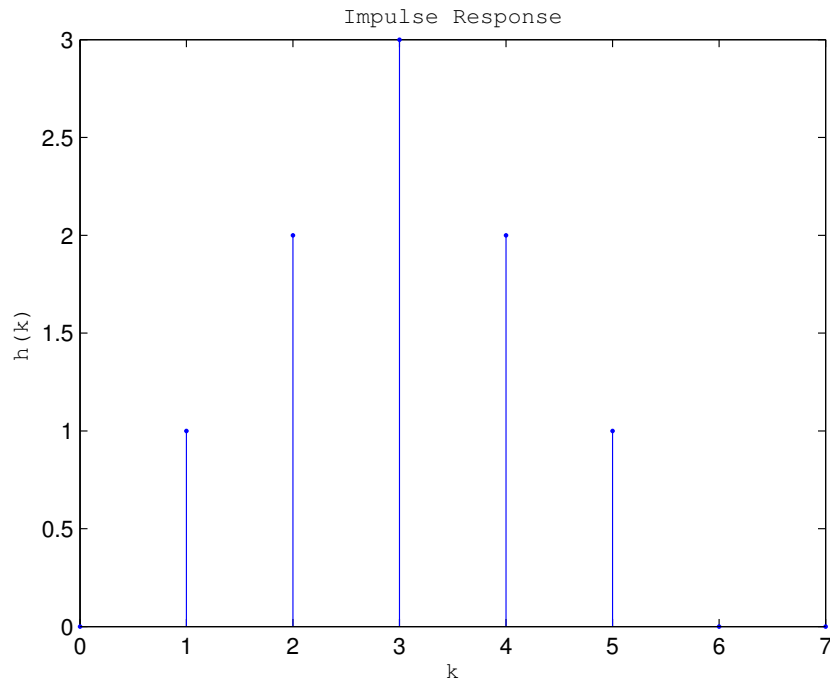
2.26 Consider the following linear time-invariant discrete-time system S . Compute and sketch the impulse response of this FIR system.

$$y(k) = u(k-1) + 2u(k-2) + 3u(k-3) + 2u(k-4) + u(k-5)$$

Solution

By inspection, the impulse response is

$$h(k) = [0, 1, 2, 3, 2, 1, 0, 0, \dots]$$



Problem 2.26

2.27 Using the definition of linear convolution, show that for any signal $h(k)$

$$h(k) \star \delta(k) = h(k)$$

Solution

From Definition 2.3 we have

$$\begin{aligned} h(k) \star \delta(k) &= \sum_{i=-\infty}^{\infty} h(i)x(k-i) \\ &= \sum_{i=-\infty}^{\infty} h(i)\delta(k-i) \\ &= h(k) \end{aligned}$$

2.28 Use Definition 2.3 and the commutative property to show that the linear convolution operator is associative.

$$f(k) \star [g(k) \star h(k)] = [f(k) \star g(k)] \star h(k)$$

Solution

From Definition 2.3 we have

$$\begin{aligned} d_1(k) &= f(k) \star [g(k) \star h(k)] \\ &= \sum_{m=-\infty}^{\infty} f(m) \left[\sum_{i=-\infty}^{\infty} g(i)h(k-m-i) \right] \\ &= \sum_{m=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} f(m)g(i)h(k-m-i) \end{aligned}$$

Next, using the commutative property

$$\begin{aligned} d_2(k) &= [f(k) \star g(k)] \star h(k) \\ &= h(k) \star [f(k) \star g(k)] \\ &= \sum_{i=-\infty}^{\infty} h(i) \left[\sum_{m=-\infty}^{\infty} f(m)g(k-i-m) \right] \\ &= \sum_{i=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(i)f(m)g(k-i-m) \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(k-n-m)f(m)g(n) \quad , \quad n = k - i - m \\ &= \sum_{m=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} f(m)g(i)h(k-m-i) \quad , \quad i = n \end{aligned}$$

Thus $d_2(k) = d_1(k)$.

2.29 Use Definition 2.3 to show that the linear convolution operator is distributive.

$$f(k) \star [g(k) + h(k)] = f(k) \star g(k) + f(k) \star h(k)$$

Solution

$$\begin{aligned}d(k) &= f(k) \star [g(k) + h(k)] \\&= \sum_{i=-\infty}^{\infty} f(i)[g(k-i) + h(k-i)] \\&= \sum_{i=-\infty}^{\infty} f(i)g(k-i) + f(i)h(k-i) \\&= \sum_{i=-\infty}^{\infty} f(i)g(k-i) + \sum_{i=-\infty}^{\infty} f(i)h(k-i) \\&= f(k) \star g(k) + f(k) \star h(k)\end{aligned}$$

2.30 Suppose $h(k)$ and $x(k)$ are defined as follows.

$$\begin{aligned}h &= [2, -1, 0, 4]^T \\x &= [5, 3, -7, 6]^T\end{aligned}$$

- (a) Let $y_c(k) = h(k) \circ x(k)$. Find the circular convolution matrix $C(x)$ such that $y_c = C(x)h$.
(b) Use $C(x)$ to find $y_c(k)$.

Solution

- (a) Using (2.7.9) and Example 2.14 as a guide, the 4×4 circular convolution matrix is

$$\begin{aligned}C(x) &= \begin{bmatrix} x(0) & x(3) & x(2) & x(1) \\ x(1) & x(0) & x(3) & x(2) \\ x(2) & x(1) & x(0) & x(3) \\ x(3) & x(2) & x(1) & x(0) \end{bmatrix} \\&= \begin{bmatrix} 5 & 6 & -7 & 3 \\ 3 & 5 & 6 & -7 \\ -7 & 3 & 5 & 6 \\ 6 & -7 & 3 & 5 \end{bmatrix}\end{aligned}$$

(b) Using (2.7.10) and the results from part (a)

$$\begin{aligned}
 y_c &= C(x)h \\
 &= \begin{bmatrix} 5 & 6 & -7 & 3 \\ 3 & 5 & 6 & -7 \\ -7 & 3 & 5 & 6 \\ 6 & -7 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 4 \end{bmatrix} \\
 &= \begin{bmatrix} 16 \\ -27 \\ 7 \\ 39 \end{bmatrix}
 \end{aligned}$$

This can be verified using the DSP Companion function *f_conv*.

2.31 Suppose $h(k)$ and $x(k)$ are the following signals of length L and M , respectively.

$$\begin{aligned}
 h &= [3, 6, -1]^T \\
 x &= [2, 0, -4, 5]^T
 \end{aligned}$$

- Let h_z and x_z be zero-padded versions of $h(k)$ and $x(k)$ of length $N = L + M - 1$. Construct h_z and x_z .
- Let $y_c(k) = h_z(k) \circ x_z(k)$. Find the circular convolution matrix $C(x_z)$ such that $y_c = C(x_z)h_z$.
- Use $C(x_z)$ to find $y_c(k)$.
- Use $y_c(k)$ to find the linear convolution $y(k) = h(k) \star x(k)$ for $0 \leq k < N$.

Solution

(a) Here

$$\begin{aligned}
 N &= L + M - 1 \\
 &= 3 + 4 - 1 \\
 &= 6
 \end{aligned}$$

Thus the zero-padded versions of $h(k)$ and $x(k)$ are

$$\begin{aligned}
 h_z &= [3, 6, -1, 0, 0, 0]^T \\
 x_z &= [2, 0, -4, 5, 0, 0]^T
 \end{aligned}$$

(b) Using (2.7.9) and the results from part (a), the $N \times N$ circular convolution matrix is

$$\begin{aligned}
 C(x_z) &= \begin{bmatrix} x_z(0) & x_z(5) & x_z(4) & x_z(3) & x_z(2) & x_z(1) \\ x_z(1) & x_z(0) & x_z(5) & x_z(4) & x_z(3) & x_z(2) \\ x_z(2) & x_z(1) & x_z(0) & x_z(5) & x_z(4) & x_z(3) \\ x_z(3) & x_z(2) & x_z(1) & x_z(0) & x_z(5) & x_z(4) \\ x_z(4) & x_z(3) & x_z(2) & x_z(1) & x_z(0) & x_z(5) \\ x_z(5) & x_z(4) & x_z(3) & x_z(2) & x_z(1) & x_z(0) \end{bmatrix} \\
 &= \begin{bmatrix} 2 & 0 & 0 & 5 & -4 & 0 \\ 0 & 2 & 0 & 0 & 5 & -4 \\ -4 & 0 & 2 & 0 & 0 & 5 \\ 5 & -4 & 0 & 2 & 0 & 0 \\ 0 & 5 & -4 & 0 & 2 & 0 \\ 0 & 0 & 5 & -4 & 0 & 2 \end{bmatrix}
 \end{aligned}$$

(c) Using (2.7.9), the circular convolution of $h_z(k)$ with $x_z(k)$ is

$$\begin{aligned}
 y_z(k) &= C(x_z)h_z \\
 &= \begin{bmatrix} 2 & 0 & 0 & 5 & -4 & 0 \\ 0 & 2 & 0 & 0 & 5 & -4 \\ -4 & 0 & 2 & 0 & 0 & 5 \\ 5 & -4 & 0 & 2 & 0 & 0 \\ 0 & 5 & -4 & 0 & 2 & 0 \\ 0 & 0 & 5 & -4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} 6 \\ 12 \\ -14 \\ -9 \\ 34 \\ -5 \end{bmatrix}
 \end{aligned}$$

(d) Using (2.7.14) and the results of part (c), the linear convolution $y(k) = h(k) \star x(k)$ is

$$\begin{aligned}
 y(k) &= h_z(k) \circ x_z(k) \\
 &= C(x_z)h_z \\
 &= [6, 12, -14, -9, 34, -5]^T
 \end{aligned}$$

This can be verified using the DSP Companion function *f_conv*.

- 2.32 Consider a linear discrete-time system S with input x and output y . Suppose S is driven by an input $x(k)$ for $0 \leq k < L$ to produce a zero-state output $y(k)$. Use deconvolution to find the impulse response $h(k)$ for $0 \leq k < L$ if $x(k)$ and $y(k)$ are as follows.

$$\begin{aligned}x &= [2, 0, -1, 4]^T \\y &= [6, 1, -4, 3]^T\end{aligned}$$

Solution

Using (2.7.15) and Example 2.16 as a guide

$$\begin{aligned}h(0) &= \frac{y(0)}{x(0)} \\&= \frac{6}{2} \\&= 3\end{aligned}$$

Applying (2.7.18) with $k = 1$ yields

$$\begin{aligned}h(1) &= \frac{y(1) - h(0)x(1)}{x(0)} \\&= \frac{1 - 3(0)}{2} \\&= .5\end{aligned}$$

Applying (2.7.18) with $k = 2$ yields

$$\begin{aligned}h(2) &= \frac{y(2) - h(0)x(2) - h(1)x(1)}{x(0)} \\&= \frac{-4 - 3(-1) - .5(0)}{2} \\&= -.5\end{aligned}$$

Finally, applying (2.7.18) with $k = 3$ yields

$$\begin{aligned} h(3) &= \frac{y(3) - h(0)x(3) - h(1)x(2) - h(2)x(1)}{x(0)} \\ &= \frac{3 - 3(4) - .5(-1) + .5(0)}{2} \\ &= -4.25 \end{aligned}$$

Thus the impulse response of the discrete-time system is

$$h(k) = [3, .5, -.5, -4.25]^T, \quad 0 \leq k < 4$$

This can be verified using the DSP Companion function *f_conv*.

2.33 Suppose $x(k)$ and $y(k)$ are the following finite signals.

$$\begin{aligned} x &= [5, 0, -4]^T \\ y &= [10, -5, 7, 4, -12]^T \end{aligned}$$

- Write the polynomials $x(z)$ and $y(z)$ whose coefficient vectors are x and y , respectively. The leading coefficient corresponds to the highest power of z .
- Using long division, compute the quotient polynomial $q(z) = y(z)/x(z)$.
- Deconvolve $y(k) = h(k) \star x(k)$ to find $h(k)$, using (2.7.15) and (2.7.18). Compare the result with $q(z)$ from part (b).

Solution

(a)

$$\begin{aligned} x(z) &= 5z^2 - 4 \\ y(z) &= 10z^4 - 5z^3 + 7z^2 + 4z - 12 \end{aligned}$$

(b)

$$\begin{array}{r} 5z^2 - 4 \quad | \quad \begin{array}{l} 2z^2 - z + 3 \\ \hline 10z^4 - 5z^3 + 7z^2 + 4z - 12 \\ \hline 10z^4 - 0z^3 - 8z^2 \\ \hline -5z^3 + 15z^2 + 4z \\ \hline -5z^3 - 0z^2 + 4z \\ \hline 15z^2 + 0z - 12 \\ \hline 15z^2 + 0z - 12 \\ \hline 0 \end{array} \end{array}$$

Thus the quotient polynomial is

$$q(z) = 2z^2 - z + 3$$

(c) Using (2.7.15) and Example 2.16 as a guide

$$\begin{aligned} q(0) &= \frac{y(0)}{x(0)} \\ &= \frac{-12}{-4} \\ &= 3 \end{aligned}$$

Applying (2.7.18) with $k = 1$ yields

$$\begin{aligned} q(1) &= \frac{y(1) - q(0)x(1)}{x(0)} \\ &= \frac{4 - 3(0)}{-4} \\ &= -1 \end{aligned}$$

Applying (2.7.18) with $k = 2$ yields

$$\begin{aligned} q(2) &= \frac{y(2) - q(0)x(2) - q(1)x(1)}{x(0)} \\ &= \frac{7 - 3(5) - (-1)0}{-4} \\ &= 2 \end{aligned}$$

Thus $q = [2, -1, 3]$ and the quotient polynomial is

$$q(z) = 2z^2 - z + 3$$

This can be verified using the MATLAB function *deconv*.

2.34 Some books use the following alternative way to define the linear cross-correlation of an L point signal $y(k)$ with an M -point signal $x(k)$. Using a change of variable, show that this is equivalent to Definition 2.5

$$r_{yx}(k) = \frac{1}{L} \sum_{n=0}^{L-1-k} y(n+k)x(n)$$

Solution

Consider the change of variable $i = n + k$. Then $n = i - k$ and

$$\begin{aligned} r_{yx}(k) &= \frac{1}{L} \sum_{n=0}^{L-1-k} y(n+k)x(n) \Big|_{i=n+k} \\ &= \frac{1}{L} \sum_{i=k}^{L-1} y(i)x(i-k) \end{aligned}$$

Since $x(n) = 0$ for $n < 0$, the lower limit of the sum can be changed to zero without affecting the result. Thus,

$$r_{yx}(k) = \frac{1}{L} \sum_{i=0}^{L-1} y(i)x(i-k) \quad , \quad 0 \leq k < L$$

This is identical to Definition 2.5.

2.35 Suppose $x(k)$ and $y(k)$ are defined as follows.

$$\begin{aligned}x &= [5, 0, -10]^T \\y &= [1, 0, -2, 4, 3]^T\end{aligned}$$

- (a) Find the linear cross-correlation matrix $D(x)$ such that $r_{yx} = D(x)y$.
- (b) Use $D(x)$ to find the linear cross-correlation $r_{yx}(k)$.
- (c) Find the normalized linear cross-correlation $\rho_{yx}(k)$.

Solution

- (a) Using (2.8.2) and Example 2.18 as a guide, the linear cross-correlation matrix is

$$\begin{aligned}D(x) &= \frac{1}{5} \begin{bmatrix} x(0) & x(1) & x(2) & 0 & 0 \\ 0 & x(0) & x(1) & x(2) & 0 \\ 0 & 0 & x(0) & x(1) & x(2) \\ 0 & 0 & 0 & x(0) & x(1) \\ 0 & 0 & 0 & 0 & x(0) \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 5 & 0 & -10 & 0 & 0 \\ 0 & 5 & 0 & -10 & 0 \\ 0 & 0 & 5 & 0 & -10 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

- (b) Using (2.8.3) and the results from part (a)

$$\begin{aligned}
r_{yx} &= D(x)y \\
&= \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 4 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ -8 \\ -8 \\ 4 \\ 3 \end{bmatrix}
\end{aligned}$$

This can be verified using the DSP Companion function *f_corr*.

(c) Using (2.8.5) we have $L = 5$ and $M = 3$. Also from Definition 2.5

$$\begin{aligned}
r_{yy}(0) &= \frac{1}{L} \sum_{i=0}^{L-1} y^2(i) \\
&= \frac{1 + 0 + 4 + 16 + 9}{5} \\
&= 6 \\
r_{xx}(0) &= \frac{1}{M} \sum_{i=0}^{M-1} x^2(i) \\
&= \frac{25 + 0 + 100}{3} \\
&= 41.67
\end{aligned}$$

Finally, from (4.49) the normalized cross-correlation of $x(k)$ with $y(k)$ is

$$\begin{aligned}
\rho_{yx}(k) &= \frac{r_{yx}(k)}{\sqrt{(M/L)r_{xx}(0)r_{yy}(0)}} \\
&= \frac{r_{yx}(k)}{\sqrt{.6(6)41.67}} \\
&= [.408, -.653, -.653, .327, .245]^T
\end{aligned}$$

This can be verified using the DSP Companion function *f_corr*.

✓ 2.36 Suppose $y(k)$ is as follows.

$$y = [5, 7, -2, 4, 8, 6, 1]^T$$

- (a) Construct a 3-point signal $x(k)$ such that $r_{yx}(k)$ reaches its peak positive value at $k = 3$ and $|x(0)| = 1$.
- (b) Construct a 4-point signal $x(k)$ such that $r_{yx}(k)$ reaches its peak negative value at $k = 2$ and $|x(0)| = 1$.

Solution

- (a) Recall that the cross-correlation $r_{yx}(k)$ measures the degree which $x(k)$ is similar to a subsignal of $y(k)$. In order for $r_{yx}(k)$ to reach its maximum positive value at $k = 3$, one must have maximum positive correlation starting at $k = 3$. Thus for some positive constant α it is necessary that

$$\begin{aligned}x &= \alpha[y(3), y(4), y(5)]^T \\ &= \alpha[4, 8, 6]^T\end{aligned}$$

The constraint, $|x(0)| = 1$, implies that the positive scale factor must be $\alpha = 1/4$. Thus

$$x = [1, 2, 1.5]^T$$

- (b) In order for $r_{yx}(k)$ to reach its maximum negative value at $k = 2$, one must have maximum negative correlation starting at $k = 2$. Thus for some positive constant α we need

$$\begin{aligned}x &= -\alpha[y(2), y(3), y(4), y(5)]^T \\ &= \alpha[2, -4, -8, -6]^T\end{aligned}$$

The constraint, $|x(0)| = 1$, implies that the positive scale factor must be $\alpha = 1/2$. Thus

$$x = [1, -2, -4, -3]^T$$

The answers to (a) and (b) can be verified using the DSP Companion function *f_corr*.

2.37 Suppose $x(k)$ and $y(k)$ are defined as follows.

$$\begin{aligned}x &= [4, 0, -12, 8]^T \\y &= [2, 3, 1, -1]^T\end{aligned}$$

- (a) Find the circular cross-correlation matrix $E(x)$ such that $c_{yx} = E(x)y$.
- (b) Use $E(x)$ to find the circular cross-correlation $c_{yx}(k)$.
- (c) Find the normalized circular cross-correlation $\sigma_{yx}(k)$.

Solution

- (a) Using Definition 2.6, $c_{yx}(k)$ is just $1/N$ times the dot product of y with x rotated right by k samples. Thus the k th row of $E(x)$ is the vector x rotated right by k samples.

$$\begin{aligned}E(x) &= \frac{1}{4} \begin{bmatrix} x(0) & x(1) & x(2) & x(3) \\ x(3) & x(0) & x(1) & x(2) \\ x(2) & x(3) & x(0) & x(1) \\ x(1) & x(2) & x(3) & x(0) \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 4 & 0 & -12 & 8 \\ 8 & 4 & 0 & -12 \\ -12 & 8 & 4 & 0 \\ 0 & -12 & 8 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -3 & 2 \\ 2 & 1 & 0 & -3 \\ -3 & 2 & 1 & 0 \\ 0 & -3 & 2 & 1 \end{bmatrix}\end{aligned}$$

- (b) Using Definition 2.6 and the results from part (a)

$$\begin{aligned}c_{yx} &= E(x)y \\ &= \begin{bmatrix} 1 & 0 & -3 & 2 \\ 2 & 1 & 0 & -3 \\ -3 & 2 & 1 & 0 \\ 0 & -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 10 \\ 1 \\ -8 \end{bmatrix}\end{aligned}$$

This can be verified using the DSP Companion function *f_corr*.

(c) Using (2.8.7), $N = 4$. Also from Definition 2.6

$$\begin{aligned}
 c_{yy}(0) &= \frac{1}{N} \sum_{i=0}^{N-1} y^2(i) \\
 &= \frac{4 + 9 + 1 + 1}{4} \\
 &= 3.75 \\
 c_{xx}(0) &= \frac{1}{N} \sum_{i=0}^{N-1} x^2(i) \\
 &= \frac{16 + 0 + 144 + 64}{4} \\
 &= 56
 \end{aligned}$$

Finally, from (2.8.7) the normalized circular cross-correlation of $y(k)$ with $x(k)$ is

$$\begin{aligned}
 \sigma_{yx}(k) &= \frac{c_{yx}(k)}{\sqrt{c_{xx}(0)c_{yy}(0)}} \\
 &= \frac{c_{yx}(k)}{\sqrt{3.75(56)}} \\
 &= [-.207, .690, .069, -.552]^T
 \end{aligned}$$

This can be verified using the DSP Companion function *f_corr*.

2.38 Suppose $y(k)$ is as follows.

$$y = [8, 2, -3, 4, 5, 7]^T$$

- (a) Construct a 6-point signal $x(k)$ such that $\sigma_{yx}(2) = 1$ and $|x(0)| = 6$.
- (b) Construct a 6-point signal $x(k)$ such that $\sigma_{yx}(3) = -1$ and $|x(0)| = 12$.

Solution

- (a) Recall that normalized circular cross-correlation, $-1 \leq \sigma_{yx}(k) \leq 1$, measures the degree which a rotated version of a signal $x(k)$ is similar to the signal $y(k)$. In order for $\sigma_{yx}(k)$ to reach its maximum positive value at $k = 2$, one must have maximum positive correlation starting at $k = 2$. Thus for some positive constant α it is necessary that

$$\begin{aligned} x &= \alpha[y(2), y(3), y(4), y(5), y(0), y(1)]^T \\ &= \alpha[-3, 4, 5, 7, 8, 2]^T \end{aligned}$$

The constraint, $|x(0)| = 6$, implies that the positive scale factor must be $\alpha = 2$. Thus

$$x = [-6, 8, 10, 14, 16, 4]^T$$

Because y and x are of the same length, this will result is $\sigma_{yx}(2) = 1$ which can be verified by using the DSP Companion function *f_corr*.

- (b) In order for $\sigma_{yx}(k)$ to reach its maximum negative value at $k = 3$, one must have maximum negative correlation starting at $k = 3$. Thus for some positive constant α

$$\begin{aligned} x &= -\alpha[y(3), y(4), y(5), y(0), y(1), y(2)]^T \\ &= \alpha[4, 5, 7, 8, 2, -3]^T \end{aligned}$$

The constraint, $|x(0)| = 12$, implies that the positive scale factor must be $\alpha = 3$. Thus

$$x = [12, 15, 21, 24, 6, -9]^T$$

Because y and x are of the same length, this will result is $\sigma_{yx}(3) = -1$ which can be verified by using the DSP Companion function *f_corr*.

2.39 Let $x(k)$ be an N -point signal with average power P_x .

- (a) Show that $r_{xx}(0) = c_{xx}(0) = P_x$
 (b) Show that $\rho_{xx}(0) = \sigma_{xx}(0) = 1$

Solution

(a) The average power of $x(k)$ is

$$P_x = \frac{1}{N} \sum_{k=0}^{N-1} x^2(k)$$

From Definition 2.5, the auto-correlation of an N -point signal is

$$\begin{aligned} r_{xx}(0) &= \frac{1}{N} \sum_{i=0}^{N-1} x(i)x(i-0) \\ &= \frac{1}{N} \sum_{i=0}^{N-1} x^2(i) \\ &= P_x \end{aligned}$$

From Definition 2.6, the circular auto-correlation of an N -point signal with periodic extension $x_p(k)$ is

$$\begin{aligned} c_{xx}(0) &= \frac{1}{N} \sum_{i=0}^{N-1} x(i)x_p(i-0) \\ &= \frac{1}{N} \sum_{i=0}^{N-1} x(i)x_p(i) \\ &= \frac{1}{N} \sum_{i=0}^{N-1} x^2(i) \\ &= P_x \end{aligned}$$

(b) From (2.8.5), the normalized auto-correlation of an N -point signal is

$$\begin{aligned} \rho_{xx}(0) &= \frac{r_{xx}(0)}{\sqrt{(N/N)r_{xx}(0)r_{xx}(0)}} \\ &= 1 \end{aligned}$$

From (2.8.7), the normalized circular auto-correlation of an N -point signal is

$$\begin{aligned}\sigma_{xx}(0) &= \frac{c_{xx}(0)}{\sqrt{c_{xx}(0)c_{xx}(0)}} \\ &= 1\end{aligned}$$

2.40 This problem establishes the normalized circular cross-correlation inequality, $|\sigma_{yx}(k)| \leq 1$. Let $x(k)$ and $y(k)$ be sequences of length N where $x_p(k)$ is the periodic extension of $x(k)$.

(a) Consider the signal $u(i, k) = ay(i) + x_p(i - k)$ where a is arbitrary. Show that

$$\frac{1}{N} \sum_{i=0}^{N-1} [ay(i) + x_p(i - k)]^2 = a^2 c_{yy}(0) + 2ac_{yx}(k) + c_{xx}(0) \geq 0$$

(b) Show that the inequality in part (a) can be written in matrix form as

$$[a, 1] \begin{bmatrix} c_{yy}(0) & c_{yx}(k) \\ c_{yx}(k) & c_{xx}(0) \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} \geq 0$$

(c) Since the inequality in part (b) holds for any a , the 2×2 coefficient matrix $C(k)$ is positive semi-definite, which means that $\det[C(k)] \geq 0$. Use this fact to show that

$$c_{yx}^2(k) \leq c_{xx}(0)c_{yy}(0) \quad , \quad 0 \leq k < N$$

(d) Use the results from part (c) and the definition of normalized cross-correlation to show that

$$-1 \leq \sigma_{yx}(k) \leq 1 \quad , \quad 0 \leq k < N$$

Solution

(a)

$$\begin{aligned}\frac{1}{N} \sum_{i=0}^{N-1} u^2(i, k) &= \frac{1}{N} \sum_{i=0}^{N-1} [ay(i) + x_p(i-k)]^2 \\ &= \frac{1}{N} \sum_{i=0}^{N-1} a^2 y^2(i) + 2ay(i)x_p(i-k) + x_p^2(i-k) \\ &= \frac{a^2}{N} \sum_{i=0}^{N-1} y^2(i) + \frac{2a}{N} \sum_{i=0}^{N-1} y(i)x_p(i-k) + \frac{1}{N} \sum_{i=0}^{N-1} x_p^2(i-k) \\ &= a^2 c_{yy}(0) + 2ac_{yx}(k) + \frac{1}{N} \sum_{i=0}^{N-1} x^2(i) \\ &= a^2 c_{yy}(0) + 2ac_{yx}(k) + c_{xx}(0) \\ &\geq 0\end{aligned}$$

(b)

$$\begin{aligned}[a, 1] \begin{bmatrix} c_{yy}(0) & c_{yx}(k) \\ c_{yx}(k) & c_{xx}(0) \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} &= [a, 1] \begin{bmatrix} ac_{yy}(0) + c_{yx}(k) \\ ac_{yx}(k) + c_{xx}(0) \end{bmatrix} \\ &= a^2 c_{yy}(0) + ac_{yx}(k) + ac_{yx}(k) + c_{xx}(0) \\ &= a^2 c_{yy}(0) + 2ac_{yx}(k) + c_{xx}(0)\end{aligned}$$

(c) The coefficient matrix $C(k)$ from part (b) is positive semi-definite and therefore $\det[C(k)] \geq 0$. But

$$\begin{aligned}\det[C(k)] &= \det \left\{ \begin{bmatrix} c_{yy}(0) & c_{yx}(k) \\ c_{yx}(k) & c_{xx}(0) \end{bmatrix} \right\} \\ &= c_{yy}(0)c_{xx}(0) - c_{yx}^2(k) \\ &\geq 0\end{aligned}$$

Thus

$$c_{yx}^2(k) \leq c_{xx}(0)c_{yy}(0) \quad , \quad 0 \leq k < N$$

(d) Using (2.8.7) and the results from part (c)

$$\begin{aligned}
 |\sigma_{yx}(k)| &= \left| \frac{c_{yx}(k)}{\sqrt{c_{xx}(0)c_{yy}(0)}} \right| \\
 &= \left| \sqrt{\frac{c_{yx}^2(k)}{c_{xx}(0)c_{yy}(0)}} \right| \\
 &\leq 1
 \end{aligned}$$

Thus

$$-1 \leq \sigma_{yx}(k) \leq 1 \quad , \quad 0 \leq k < N$$

2.41 Consider the following FIR system.

$$y(k) = \sum_{i=0}^5 (1+i)^2 x(k-i)$$

Let $x(k)$ be a bounded input with bound B_x . Show that $y(k)$ is bounded with bound $B_y = cB_x$. Find the minimum scale factor, c .

Solution

$$\begin{aligned}
 |y(k)| &= \left| \sum_{i=0}^5 (1+i)^2 x(k-i) \right| \\
 &\leq \sum_{i=0}^5 |(1+i)^2 x(k-i)| \\
 &= \sum_{i=0}^5 |(1+i)^2| \cdot |x(k-i)| \\
 &\leq B_x \sum_{i=0}^5 |(1+i)^2| \\
 &= \|h\|_1 B_x
 \end{aligned}$$

Here

$$\begin{aligned}\|h\|_1 &= \sum_{i=0}^5 (1+i)^2 \\ &= 1 + 4 + 9 + 16 + 25 + 36 \\ &= 93\end{aligned}$$

Thus

$$B_y = 93B_x$$

2.42 Consider a linear time-invariant discrete-time system S with the following impulse response. Find conditions on A and p that guarantee that S is BIBO stable.

$$h(k) = Ap^k \mu(k)$$

Solution

The system S is BIBO stable if and only if $\|h\|_1 < \infty$. Here

$$\begin{aligned}\|h\|_1 &= \sum_{k=-\infty}^{\infty} |h(k)| \\ &= \sum_{k=0}^{\infty} Ap^k \\ &= A \sum_{k=0}^{\infty} p^k \\ &= \frac{A}{1-p} \quad , \quad |p| < 1\end{aligned}$$

Thus S is BIBO stable if and only if $|p| < 1$. There is no constraint on A .

2.43 From Proposition 2.1, a linear time-invariant discrete-time system S is BIBO stable if and only if the impulse response $h(k)$ is absolutely summable, that is, $\|h\|_1 < \infty$. Show that $\|h\|_1 < \infty$ is necessary for stability. That is, suppose that S is stable but $h(k)$ is not absolutely summable. Consider the following input, where $h^*(k)$ denotes the complex conjugate of $h(k)$ (Proakis and Manolakis, 1992).

$$x(k) = \begin{cases} \frac{h^*(k)}{|h(k)|} & , \quad h(k) \neq 0 \\ 0 & , \quad h(k) = 0 \end{cases}$$

- (a) Show that $x(k)$ is bounded by finding a bound B_x .
- (b) Show that S is not BIBO stable by showing that $y(k)$ is unbounded at $k = 0$.

Solution

- (a) Since $x(k) = 0$ when $h(k) = 0$, consider the case when $h(k) \neq 0$.

$$\begin{aligned} |x(k)| &= \left| \frac{h^*(k)}{|h(k)|} \right| \\ &= \frac{|h^*(k)|}{|h(k)|} \\ &= \frac{|h(k)|}{|h(k)|} \\ &= 1 \end{aligned}$$

Thus $x(k)$ is bounded with $B_x = 1$.

(b)

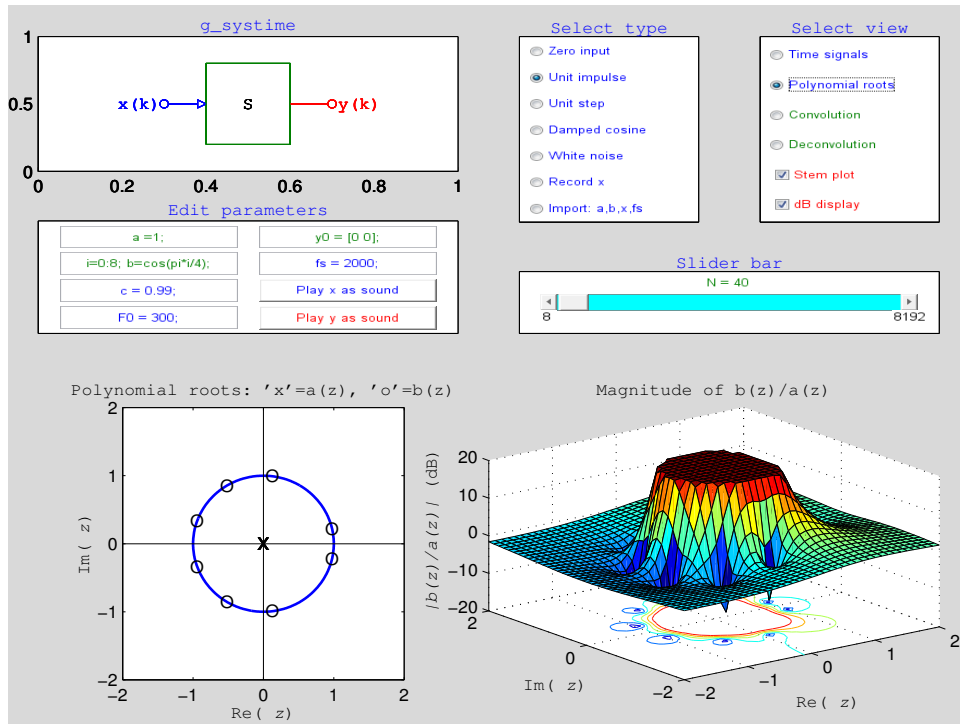
$$\begin{aligned} |y(0)| &= |h(k) \star x(k)|_{k=0} \\ &= \left| \sum_{i=-\infty}^{\infty} h(i)x(-i) \right| \\ &= \left| \sum_{i=-\infty}^{\infty} \frac{h(i)h^*(-i)}{|h(-i)|} \right| \\ &= \sum_{i=-\infty}^{\infty} \frac{|h(i)| \cdot |h^*(-i)|}{|h(-i)|} \\ &= \sum_{i=-\infty}^{\infty} |h(i)| \\ &= \|h\|_1 \\ &= \infty \end{aligned}$$

2.44 Consider the following discrete-time system. Use GUI module *g_systime* to simulate this system. Hint: You can enter the *b* vector in the edit box by using two statements on one line: `i=0:8; b=cos(pi*i/4)`

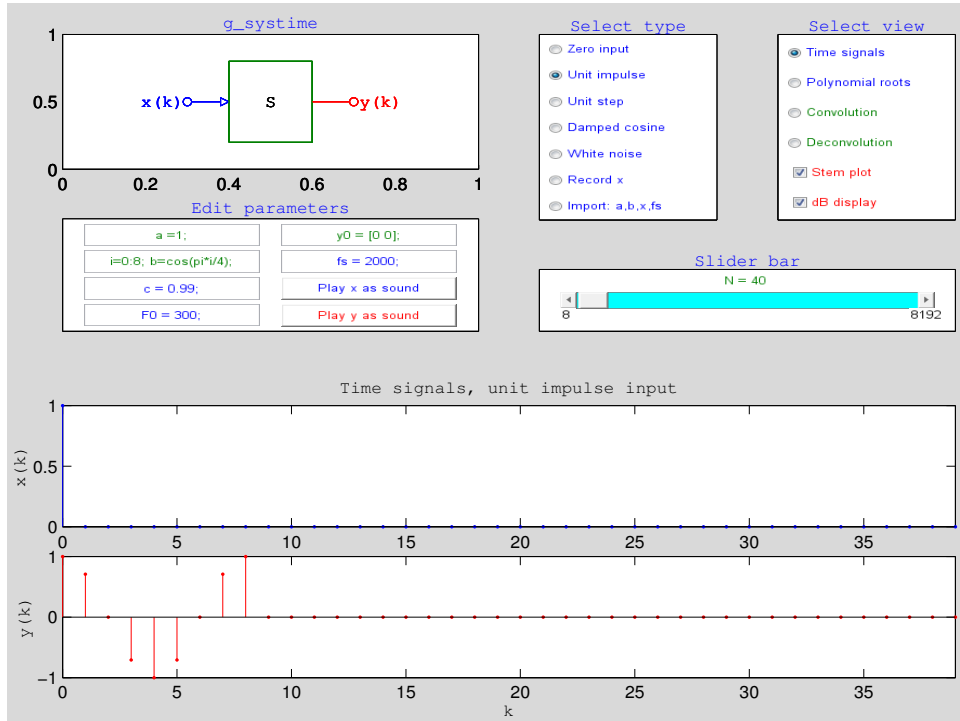
$$y(k) = \sum_{i=0}^8 \cos(\pi i/4)x(k-i)$$

- (a) Plot the polynomial roots
- (b) Plot and the impulse response using $N = 40$.

Solution



Problem 2.44 (a) Polynomial Roots



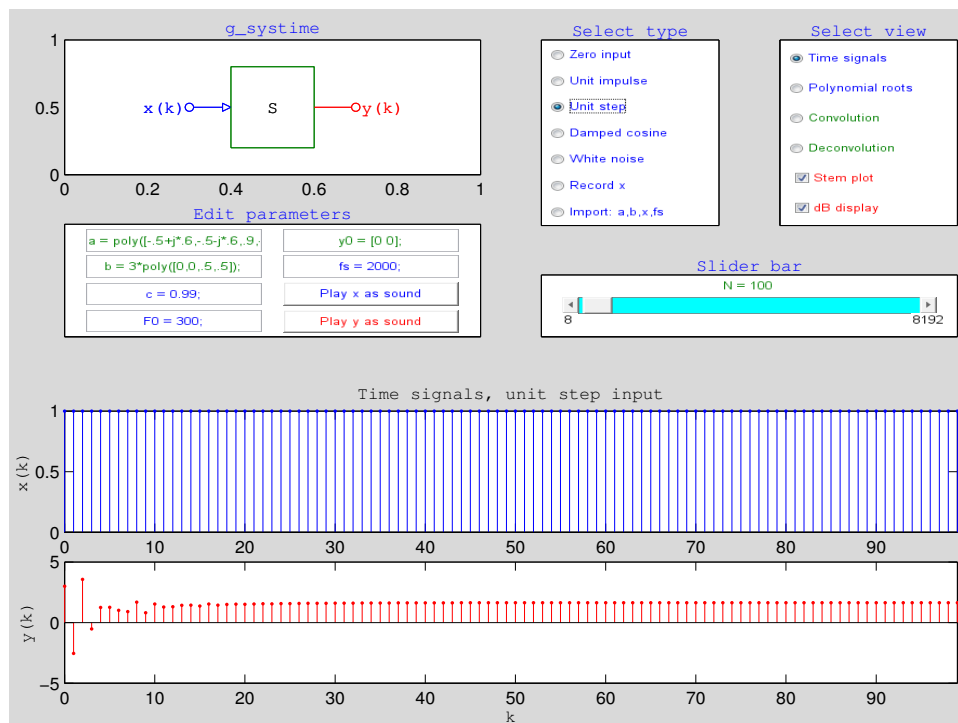
Problem 2.44 (b) Impulse Response

2.45 Consider a discrete-time system with the following characteristic and input polynomials. Use GUI module *g_systime* to plot the step response using $N = 100$ points. The MATLAB *poly* function can be used to specify coefficient vectors a and b in terms of their roots, as discussed in Section 2.9.

$$a(z) = (z + .5 \pm j.6)(z - .9)(z + .75)$$

$$b(z) = 3z^2(z - .5)^2$$

Solution



Problem 2.45 Step Response

✓ 2.46 Consider the following linear discrete-time system.

$$y(k) = 1.7y(k-2) - .72y(k-4) + 5x(k-2) + 4.5x(k-4)$$

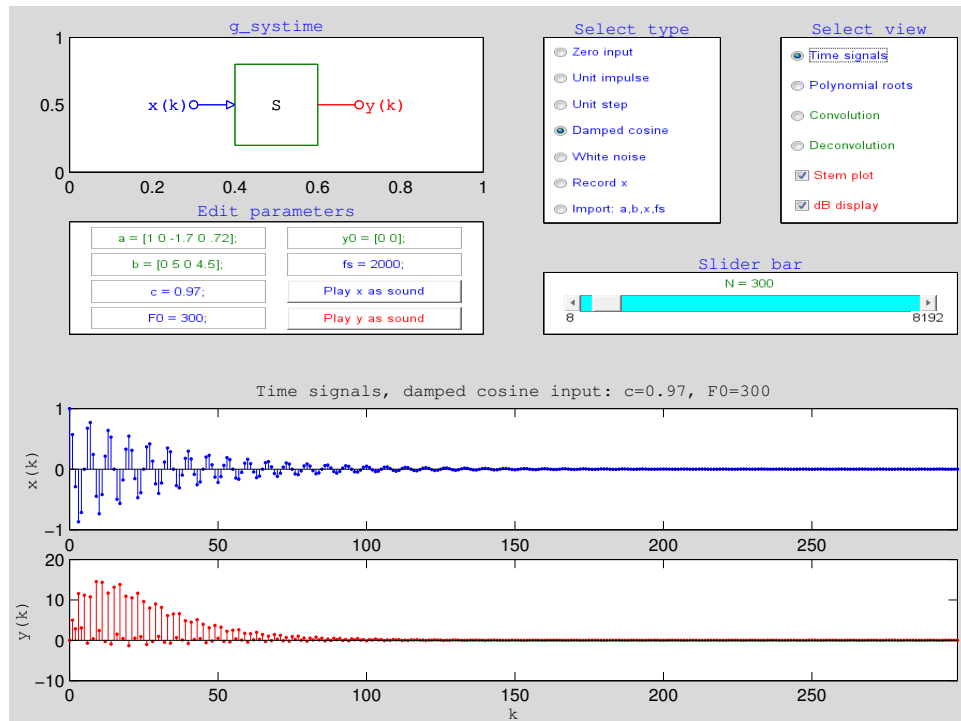
Use GUI module *g_systime* to plot the following damped cosine input and the zero-state response to it using $N = 30$. To determine F_0 , set $2\pi F_0 kT = .3\pi k$ and solve for F_0/f_s where $T = 1/f_s$.

$$x(k) = .97^k \cos(.3\pi k)$$

Solution

$$2\pi F_0 kT = .3\pi k$$

Thus $2F_0T = .3$ or $F_0 = .15f_s$. If $f_s = 2000$, then $F_0 = 300$.



Problem 2.46 Input and Output

2.47 Consider the following linear discrete-time system.

$$y(k) = -.4y(k-1) + .19y(k-2) - .104y(k-3) + 6x(k) - 7.7x(k-1) + 2.5x(k-2)$$

Create a MAT-file called *prob2_47* that contains $fs = 100$, the appropriate coefficient vectors a and b , and the following input samples, where $v(k)$ is white noise uniformly distributed over $[-.2, .2]$. Uniform white noise can be generated with the MATLAB function *rand*.

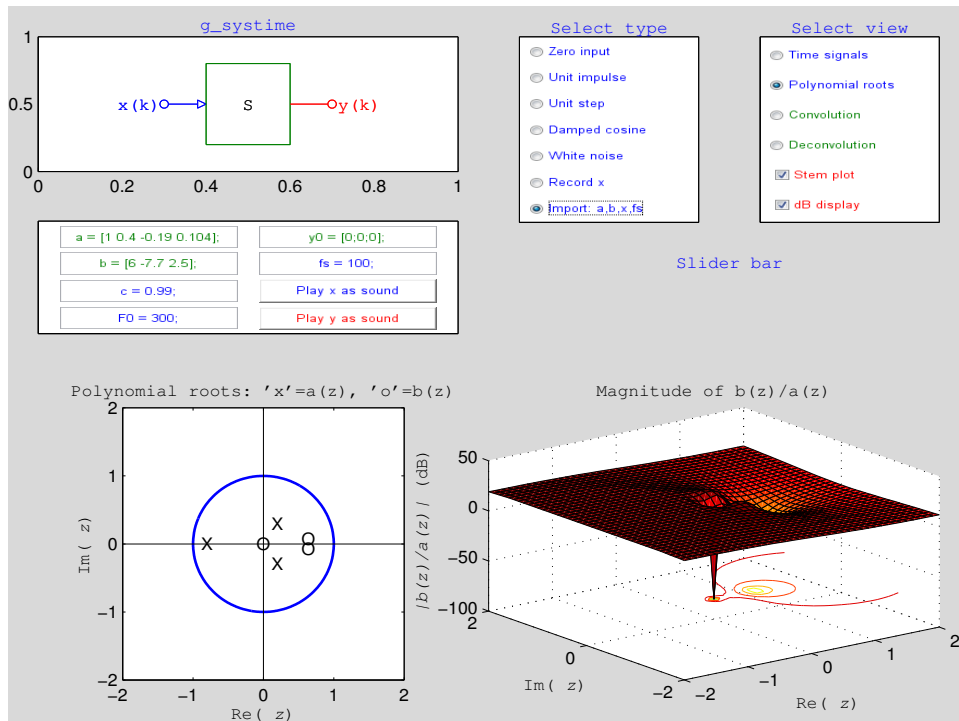
$$x(k) = k \exp(-k/50) + v(k) \quad , \quad 0 \leq k < 500$$

- Print the MATLAB program used to create *prob2_47.mat*.
- Use GUI module *g_systime* and the Import option to plot the roots of the characteristic polynomial and the input polynomial.
- Plot the zero-state response on the input $x(k)$.

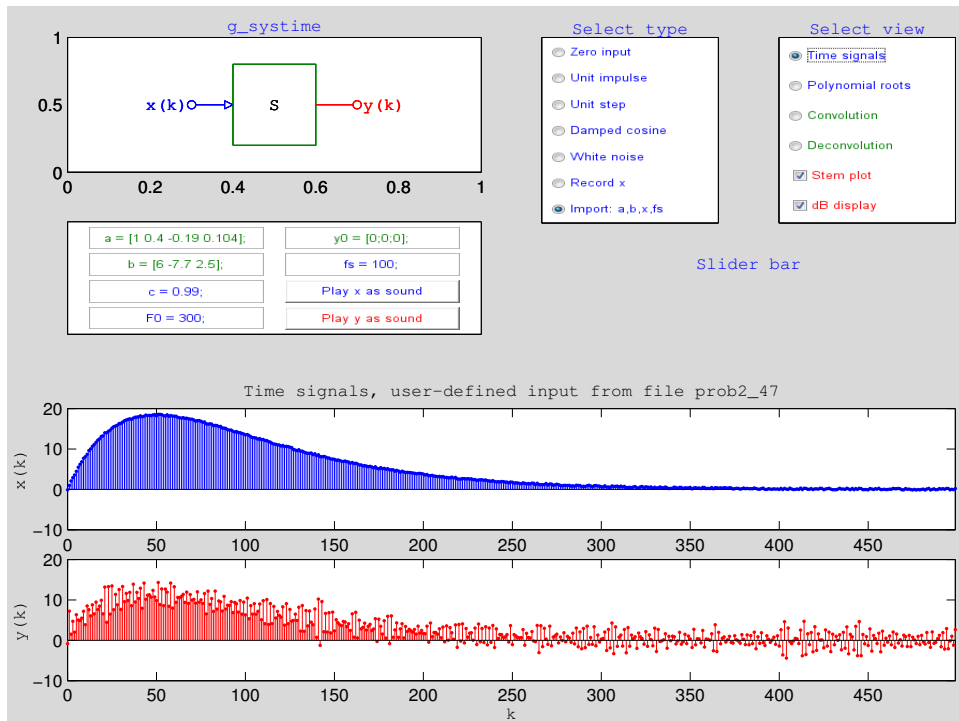
Solution

(a) % Problem 2.47

```
f_header('Problem 2.47: Create MAT file')
fs = 100;
a = [1 .4 -.19 .104]
b = [6 -7.7 2.5];
N = 500;
v = -.2 + .4*rand(1,N);
k = 0:N-1;
x = k .* exp(-k/50) + v;
save prob2_47 fs a b x
what
```



Problem 2.47 (b) Polynomial Roots



Problem 2.47 (c) Input and Output

2.48 Consider the following discrete-time system, which is a narrow band *resonator* filter with sampling frequency of $f_s = 800$ Hz.

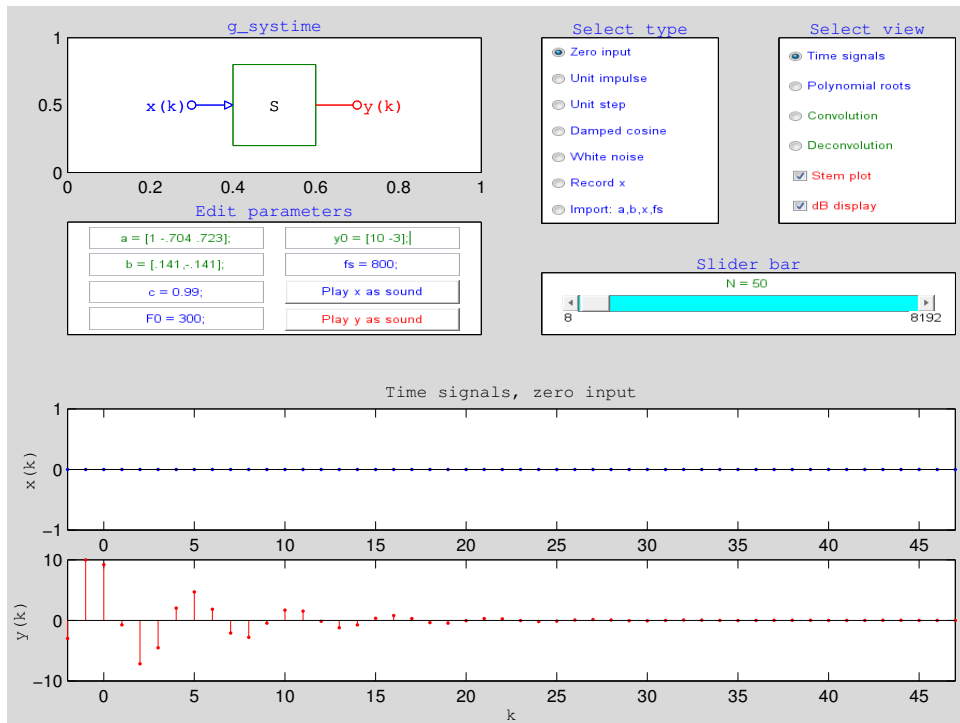
$$y(k) = .704y(k-1) - .723y(k-2) + .141x(k) - .141x(k-2)$$

Use GUI module *g_systime* to find the zero-input response for the following initial conditions. In each case plot $N = 50$ points.

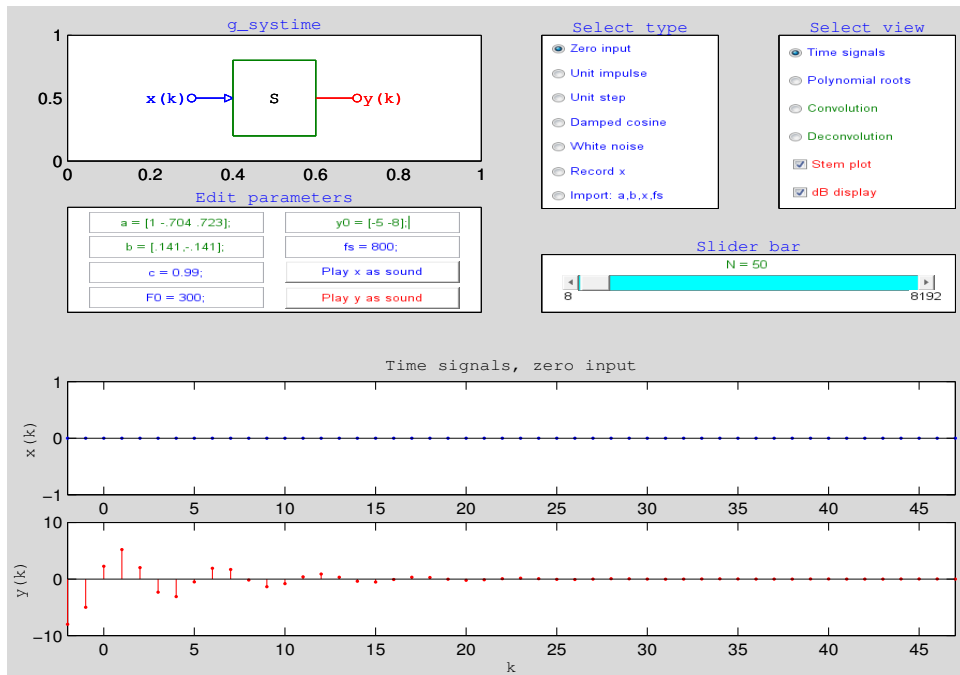
(a) $y_0 = [10, -3]^T$

(b) $y_0 = [-5, -8]^T$

Solution



Problem 2.48 (a) Zero-input Response



Problem 2.48 (b) Zero-input Response

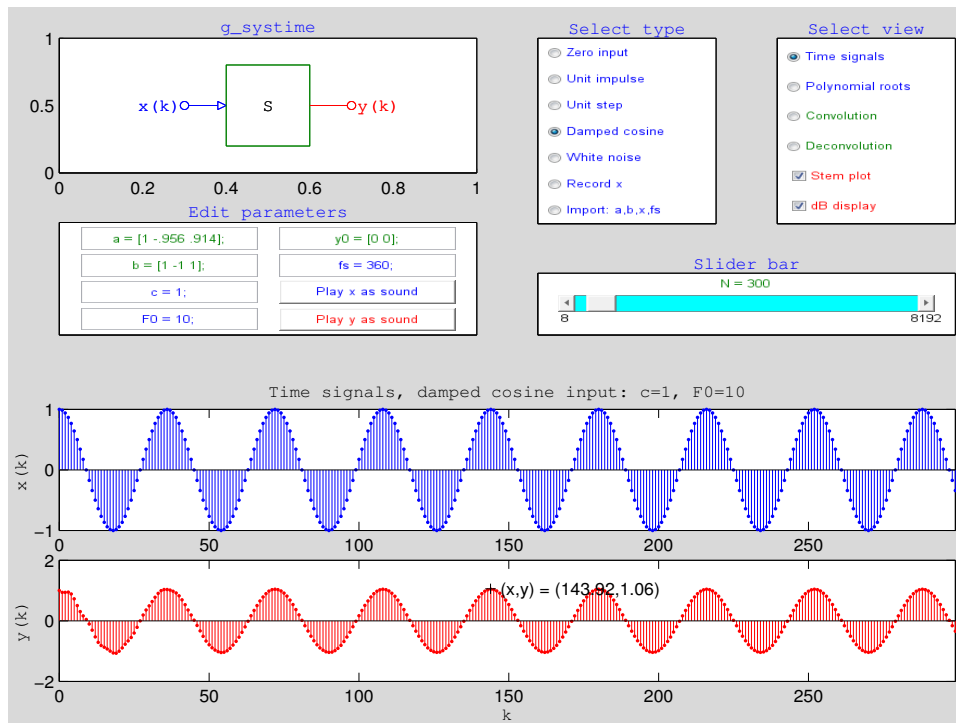
2.49 Consider the following discrete-time system, which is a *notch* filter with sampling interval $T = 1/360$ sec.

$$y(k) = .956y(k-1) - .914y(k-2) + x(k) - x(k-1) + x(k-2)$$

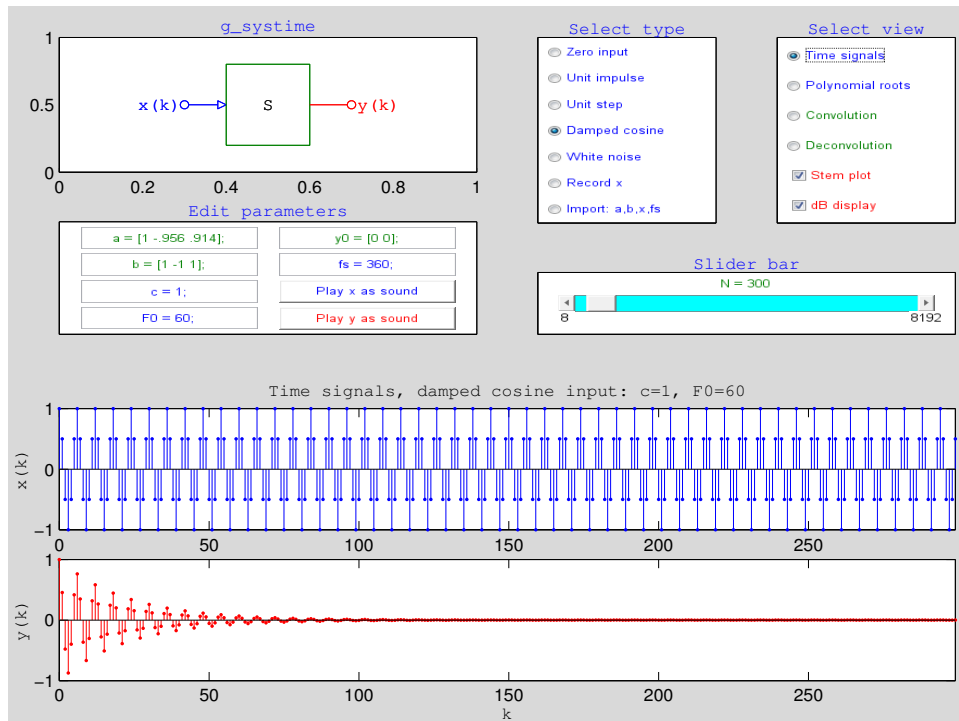
Use GUI module *g_systime* to find the output corresponding to the sinusoidal input $x(k) = \cos(2\pi F_0 k T)\mu(k)$. Do the following cases. Use the caliper option to estimate the steady state amplitude in each case.

- (a) Plot the output when $F_0 = 10$ Hz.
- (b) Plot the output when $F_0 = 60$ Hz.

Solution



Problem 2.49 (a) $F_0 = 10$ Hz



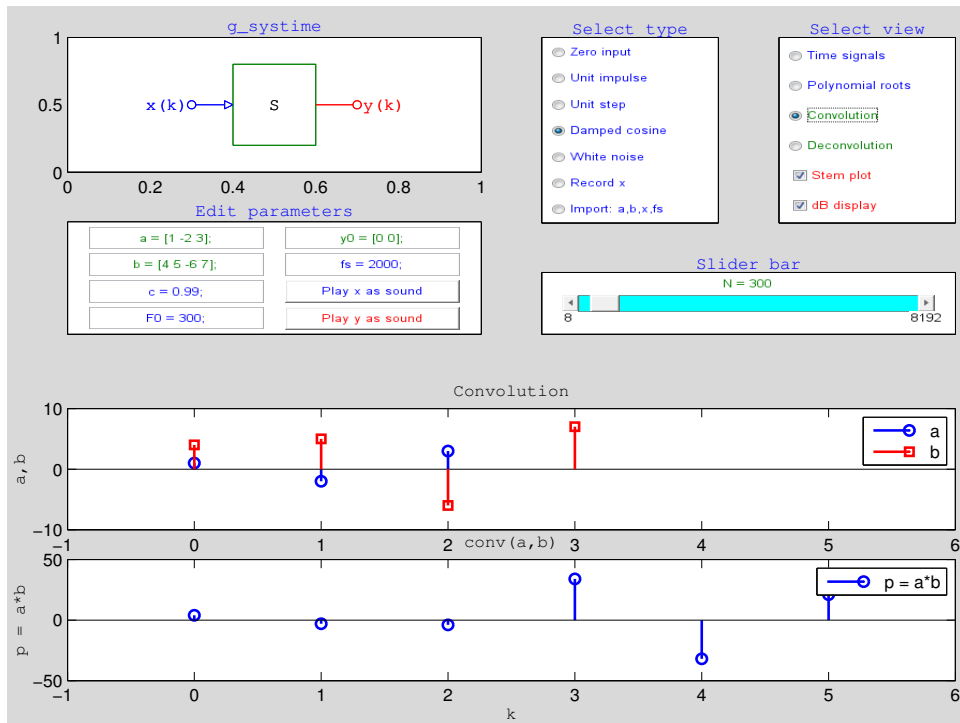
Problem 2.49 (b) $F_0 = 60$ Hz

2.50 Consider the following two polynomials. Use *g_systime* to compute, plot, and Export to a data file the coefficients of the product polynomial $c(z) = a(z)b(z)$. Then Import the saved file and display the coefficients of the product polynomial.

$$a(z) = z^2 - 2z + 3$$

$$b(z) = 4z^3 + 5z^2 - 6z + 7$$

Solution



Problem 2.50 Polynomial Multiplication

product =
 4 -3 -4 34 -32 21

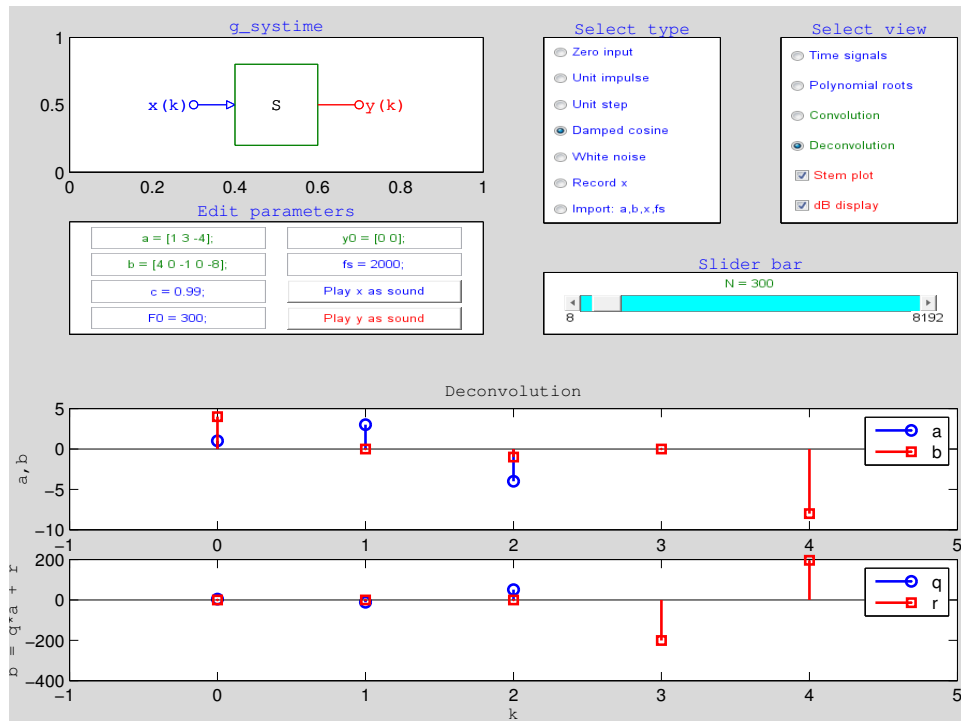
2.51 Consider the following two polynomials. Use `g_systime` to compute, plot, and Export to a data file the coefficients of the quotient polynomial $q(z)$ and the remainder polynomial $r(z)$ where $b(z) = q(z)a(z) + r(z)$. Then Import the saved file and display the coefficients of the quotient and remainder polynomials.

$$a(z) = z^2 + 3z - 4$$

$$b(z) = 4z^4 - z^2 - 8$$

Solution

quotient =
 4 -12 51
 remainder =
 0 0 0 -201 196

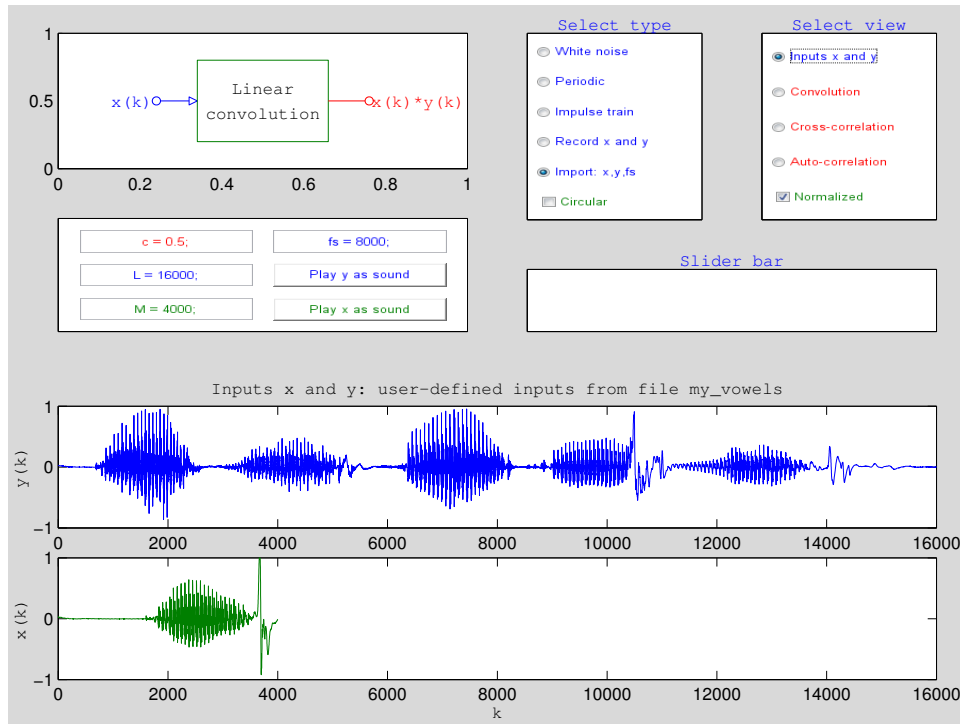


Problem 2.51 Polynomial Division

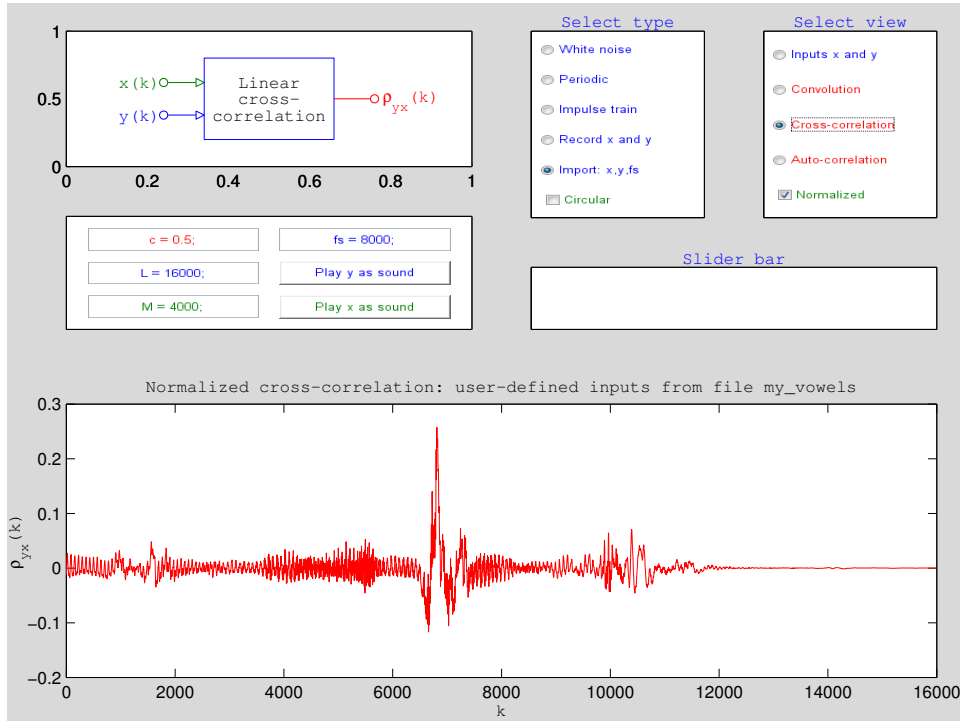
√ 2.52 Use the GUI module *g_correlate* to record the sequence of vowels “A”, “E”, “I”, “O”, “U” in y . Play y to make sure you have a good recording of all five vowels. Then record the vowel “O” in x . Play x back to make sure you have a good recording of “O” that sounds similar to the “O” in y . Export the results to a MAT-file named *my_vowels*.

- (a) Plot the inputs x and y showing the vowels.
- (b) Plot the normalized cross-correlation of y with x using the *Caliper* option to mark the peak which should show the location of x in y .
- (c) Based on the plots in (a), estimate the lag d_1 that would be required to get the “O” in x to align with the “O” in y . Compare this with the peak location d_2 in (b). Find the percent error relative to the estimated lag d_1 . There will be some error due to the overlap of x with adjacent vowels and co-articulation effects in creating y .

Solution



Problem 2.52 (a) The Vowels A, E, I, O, U



Problem 2.52 (b) Normalized Cross-correlation of x with y

- (c) From part (a), the start of O in x is approximately $o_x = 9000$, and the start of O in y is approximately $o_y = 1800$. Thus the translation of y required to get a match with x is

$$\begin{aligned}d_1 &= o_x - o_y \\ &\approx 9000 - 1800 \\ &= 7200\end{aligned}$$

The peak in part (b) is at $d_2 = 6807$. Thus the percent error in finding the location of O in x is

$$\begin{aligned}E &= \frac{100(d_2 - d_1)}{d_1} \\ &= \frac{100(6807 - 7200)}{7200} \\ &= -5.46\%\end{aligned}$$

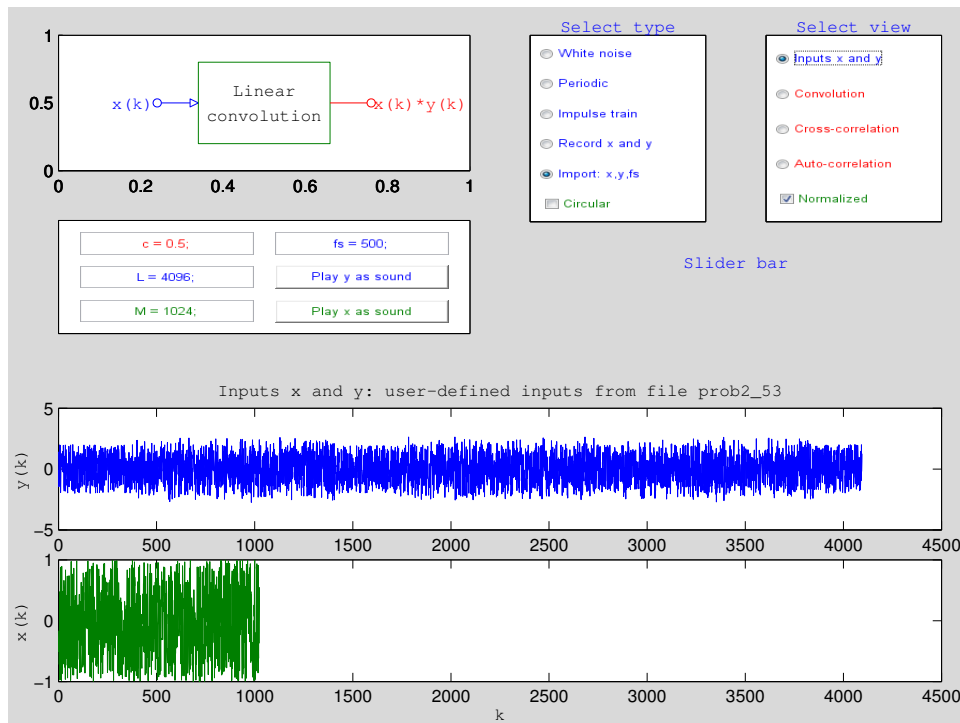
2.53 The file `prob2_53.mat` contains two signals, x and y , and their sampling frequency, fs . Use the GUI module `g_correlate` to Import x , y , and fs .

- (a) Plot $x(k)$ and $y(k)$.
- (b) Plot the normalized linear cross-correlation $\rho_{yx}(k)$. Does $y(k)$ contain any scaled and shifted versions of $x(k)$? Determine how many, and use the Caliper option to estimate the locations of $x(k)$ within $y(k)$.

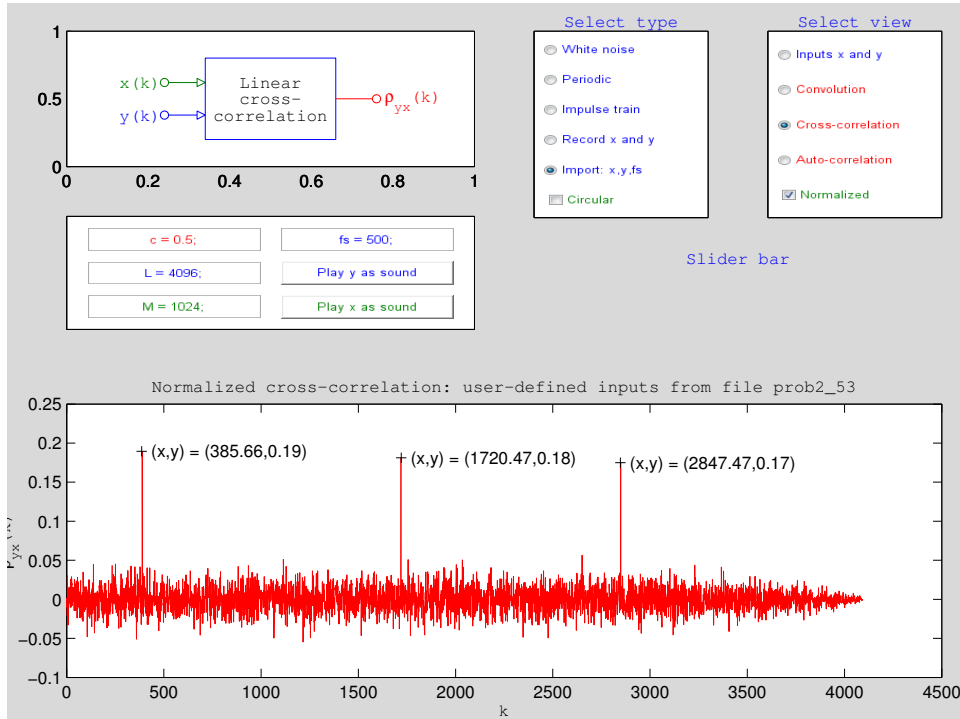
Solution

From the plot of $\rho_{xy}(k)$, there are three scaled and shifted versions of $y(k)$ within $x(k)$. They are located at

$$k = [388, 1718, 2851]$$



Problem 2.53 (a)



Problem 2.53 (b)

2.54 Consider the following discrete-time system.

$$y(k) = .95y(k-1) + .035y(k-2) - .462y(k-3) + .351y(k-4) + .5x(k) - .75x(k-1) - 1.2x(k-2) + .4x(k-3) - 1.2x(k-4)$$

Write a MATLAB program that uses *filter* and *plot* to compute and plot the zero-state response of this system to the following input. Plot both the input and the output on the same graph.

$$x(k) = (k+1)^2(.8)^k\mu(k) \quad , \quad 0 \leq k \leq 100$$

Solution

```
% Problem 2.54

% Initialize

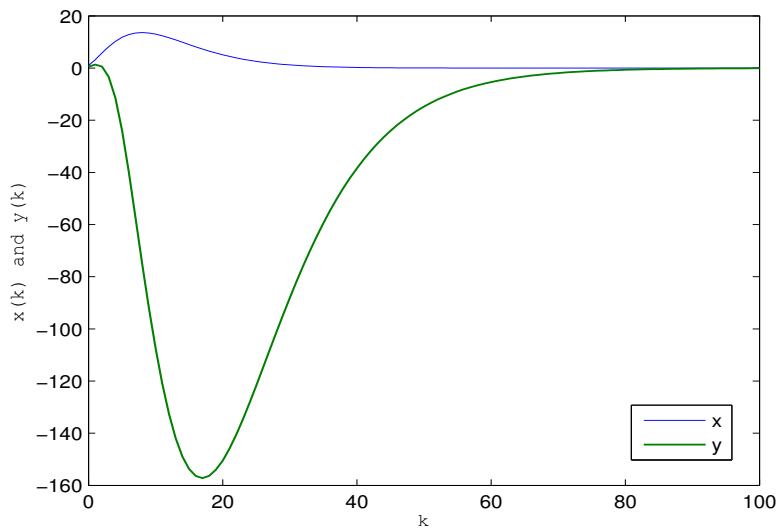
f_header('Problem 2.54')
a = [1 -.95 -.035 .462 -.351]
b = [.5 -.75 -1.2 .4 -1.2]
N =101;
k = 0 : N-1;
x = (k+1).^2 .* (.8).^k;

% Find zero-state response

y = filter (b,a,x);

% Plot input and output

figure
h = plot (k,x,k,y);
set (h(2),'LineWidth',1.0)
f_labels ('','k','x(k) and y(k)')
legend ('x','y')
f_wait
```



Problem 2.54 Input and Zero-State Response

2.55 Consider the following discrete-time system.

$$\begin{aligned}
 a(z) &= z^4 - .3z^3 - .57z^2 + .115z + .0168 \\
 b(z) &= 10(z + .5)^3
 \end{aligned}$$

This system has four simple nonzero roots. Therefore the zero-input response consists of a sum of the following four natural mode terms.

$$y_{zi}(k) = c_1 p_1^k + c_2 p_2^k + c_3 p_3^k + c_4 p_4^k$$

The coefficients can be determined from the initial condition

$$y_0 = [y(-1), y(-2), y(-3), y(-4)]^T$$

Setting $y_{zi}(-k) = y(-k)$ for $1 \leq k \leq 4$ yields the following linear algebraic system in the coefficient vector $c = [c_1, c_2, c_3, c_4]^T$.

$$\begin{bmatrix} p_1^{-1} & p_2^{-1} & p_3^{-1} & p_4^{-1} \\ p_1^{-2} & p_2^{-2} & p_3^{-2} & p_4^{-2} \\ p_1^{-3} & p_2^{-3} & p_3^{-3} & p_4^{-3} \\ p_1^{-4} & p_2^{-4} & p_3^{-4} & p_4^{-4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = y_0$$

Write a MATLAB program that uses *roots* to find the roots of the characteristic polynomial and then solves this linear algebraic system for the coefficient vector *c* using the MATLAB left division or `\` operator when the initial condition is *y*₀. Print the roots and the coefficient vector *c*. Use *stem* to plot the zero-input response *y*_{zi}(*k*) for $0 \leq k \leq 40$.

Solution

```
% Problem 2.55

% Initialize

f_header('Problem 2.55')
a = [1 -.3 -.57 .115 .0168]
y = [2 -1 0 3]'
n = 4;

% Construct coefficient matrix

p = roots(a)
A = zeros(n,n);
for i = 1 : n
    for k = 1 : n
        A(i,k) = p(k)^(-i);
    end
end

% Find coefficient vector c

c = A \ y

% Compute zero-input response

N =41;
k = 0 : N-1;
y_0 = zeros(1,N);
for i = 1 : n
```

```

    y_0 = y_0 + c(i) .* k;
end

% Plot it

figure
stem (k,y_0,'filled','.')
f_labels ('','k','y_0(k)')
f_wait

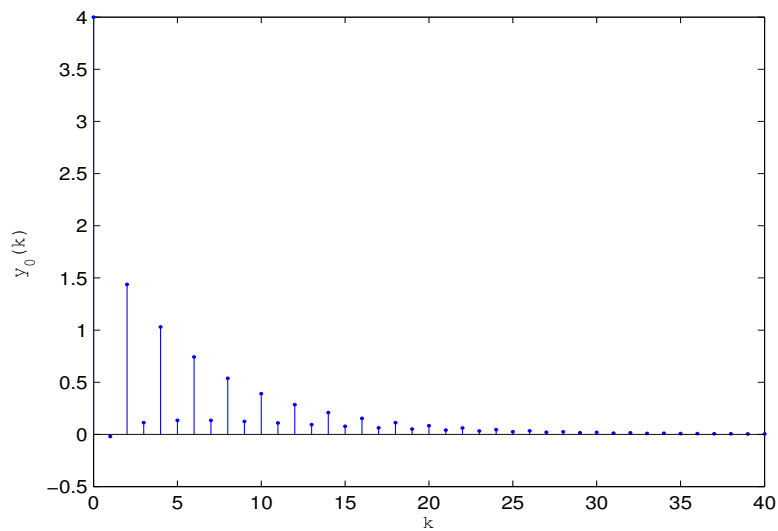
```

Program Output:

```

p =
    -0.7000
     0.8000
     0.3000
    -0.1000
c =
    -0.8195
     0.8720
    -0.0742
     0.0013

```



Problem 2.55 Zero-Input Response to Initial Condition

- √ **2.56** Consider the discrete-time system in Problem 2.55. Write a MATLAB program that uses the DSP Companion function `f_filter0` to compute the zero-input response to the following initial condition. Use `stem` to plot the zero-input response $y_{zi}(k)$ for $-4 \leq k \leq 40$.

$$y_0 = [y(-1), y(-2), y(-3), y(-4)]^T$$

Solution

```
% Problem 2.56

% Initialize

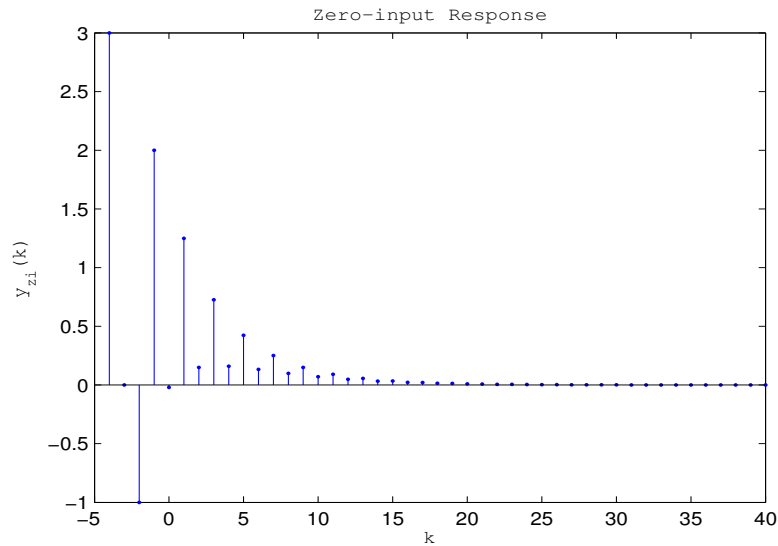
f_header('Problem 2.56')
a = [1 -.3 -.57 .115 .0168]
b = 10*poly([-0.5,-0.5,-0.5])
y0 = [2 -1 0 3]'
n = 4;

% Solve system

N = 41;
x = zeros(1,N);
y_zi = f_filter0(b,a,x,y0);

% Plot it

figure
k = [-n : N-1];
stem(k,y_zi,'filled','k')
f_labels('Zero-input Response','k','y_{zi}(k)')
f_wait
```



Problem 2.56 Zero-input Response

2.57 Consider the following running average filter.

$$y(k) = \frac{1}{10} \sum_{i=0}^9 x(k-i) \quad , \quad 0 \leq k \leq 100$$

Write a MATLAB program that performs the following tasks.

- (a) Use *filter* and *plot* to compute and plot the zero-state response to the following input, where $v(k)$ is a random white noise uniformly distributed over $[-.1, .1]$. Plot $x(k)$ and $y(k)$ below one another. Uniform white noise can be generated using the MATLAB function *rand*.

$$x(k) = \exp(-k/20) \cos(\pi k/10) \mu(k) + v(k)$$

- (b) Add a third curve to the graph in part (a) by computing and plotting the zero-state response using *conv* to perform convolution.

Solution

The transfer function of this FIR filter is

$$H(z) = .1 \sum_{i=0}^9 z^{-i}$$

```

% Problem 2.57

% Initialize

f_header('Problem 2.57')
m = 9;
b = .1*ones(1,m+1);
a = 1;
N =101;
k = 0 : N-1;
c = .1;
x = exp(-k/20) .* cos(pi*k/10) + f_randu(1,N,-c,c);

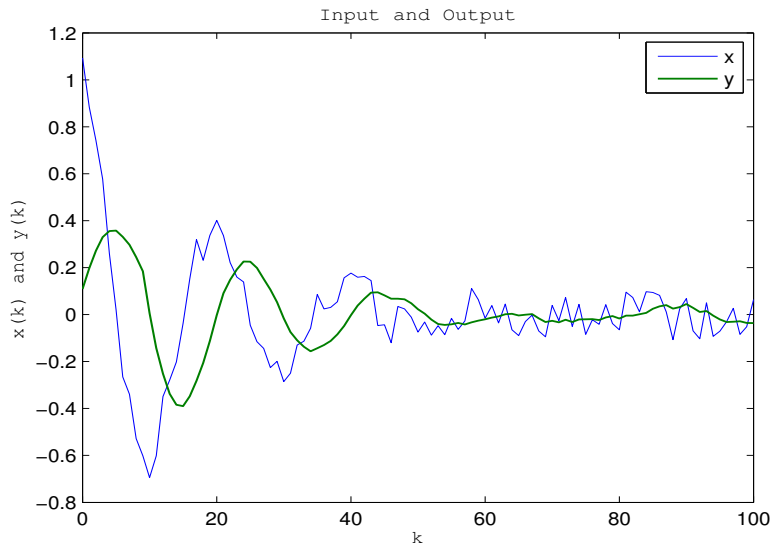
% Find zero-state response

y = filter (b,a,x);

% Plot input and output

figure
h = plot (k,x,k,y);
set (h(2),'LineWidth',1.0)
f_labels ('Input and Output','k','x(k) and y(k)')
legend ('x','y')
f_wait

```



Problem 2.57 Running Average Filter of Order $m = 9$

2.58 Consider the following FIR filter. Write a MATLAB program that performs the following tasks.

$$y(k) = \sum_{i=0}^{20} \frac{(-1)^i x(k-i)}{10+i^2}$$

- (a) Use the function *filter* to compute and plot the impulse response $h(k)$ for $0 \leq k < N$ where $N = 50$.
- (b) Compute and plot the following periodic input.

$$x(k) = \sin(.1\pi k) - 2 \cos(.2\pi k) + 3 \sin(.3\pi k) \quad , \quad 0 \leq k < N$$

- (c) Use *conv* to compute the zero-state response to the input $x(k)$ using convolution. Also compute the zero-state response to $x(k)$ using *filter*. Plot both responses on the same graph using a legend.

Solution

```

% Problem 2.58

% Construct filter

f_header('Problem 2.58')
i = 0 : 20;
b = (-1).^2 ./ (10 + i.^2);
a = 1;

% Construct input

N = 50;
k = 0 : N-1;
x = sin(.1*pi*k) - 2*cos(.2*pi*k) + 3*sin(.3*pi*k);

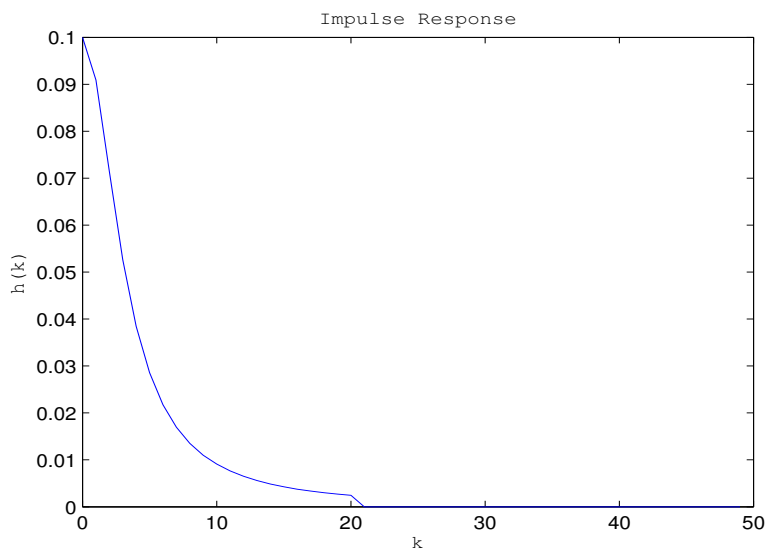
% Compute and plot impulse response

delta = [1,zeros(1,N-1)];
h = filter (b,a,delta);
figure
plot (k,h)
f_labels ('Impulse Response','k','h(k)')
f_wait

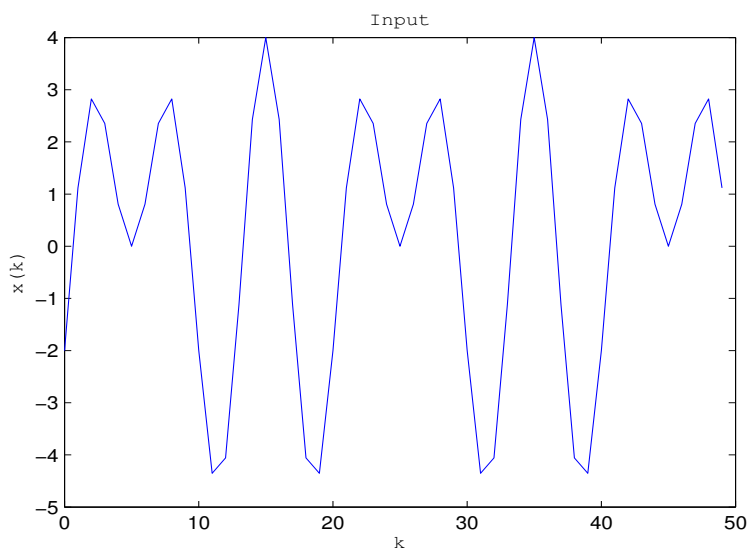
% Compute and plot zero-state response using convolution

figure
plot (k,x)
f_labels ('Input','k','x(k)')
f_wait
circ = 0;
y1 = f_conv (h,x,circ);
k1 = 0 : length(y1)-1;
y2 = filter (b,a,x);
k2 = 0 : N-1;
hp = plot (k1,y1,k2,y2);
set (hp(2),'LineWidth',1.5)
f_labels ('Zero State Response','k','y(k)')
legend ('Using f\_conv','Using filter')
f_wait

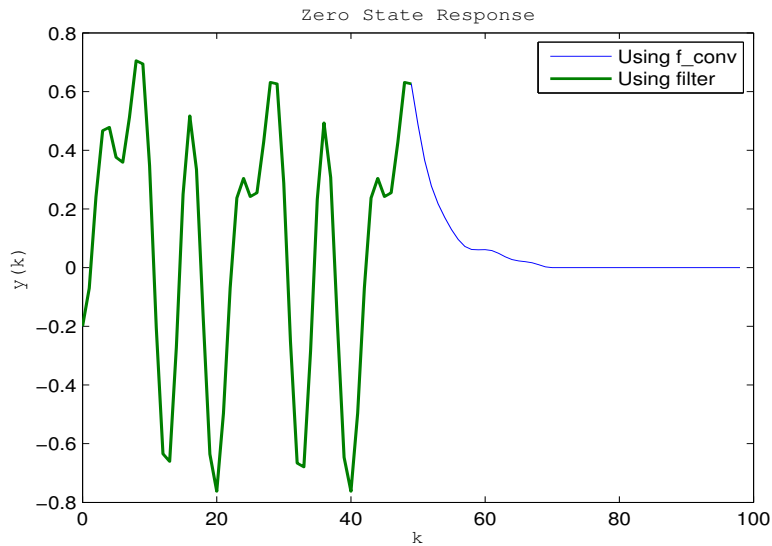
```



Problem 2.58 (a) Impulse Response



Problem 2.58 (b) Periodic Input



Problem 2.58 (c) Zero-State Response

2.59 Consider the following pair of signals.

$$h = [1, 2, 3, 4, 5, 4, 3, 2, 1]^T$$

$$x = [2, -1, 3, 4, -5, 0, 7, 9, -6]^T$$

Verify that linear convolution and circular convolution produce different results by writing a MATLAB program that uses the DSP Companion function *f_conv* to compute the linear convolution $y(k) = h(k) \star x(k)$ and the circular convolution $y_c(k) = h(k) \circ x(k)$. Plot $y(k)$ and $y_c(k)$ below one another on the same screen.

Solution

```
% Problem 2.59

% Initialize

f_header('Problem 2.59')
h = [1 2 3 4 5 4 3 2 1]
x = [2 -1 3 4 -5 0 7 9 -6]

% Compute convolutions
```

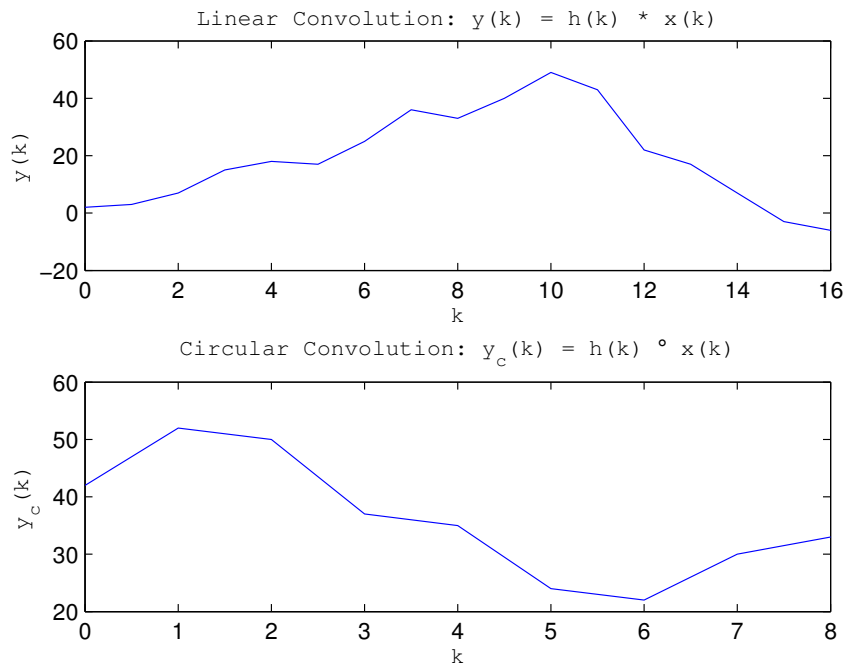
```

y = f_conv (h,x,0);
y_c = f_conv (h,x,1);

% Plot them

figure
subplot (2,1,1)
k = 0 : length(y)-1;
plot (k,y)
f_labels ('Linear Convolution: y(k) = h(k) * x(k)', 'k', 'y(k)')
subplot (2,1,2)
k = 0 : length(y_c)-1;
plot (k,y_c)
f_labels ('Circular Convolution: y_c(k) = h(k) \circ x(k)', 'k', 'y_c(k)')
f_wait

```



Problem 2.59 Linear and Circular Convolution

2.60 Consider the following pair of signals.

$$\begin{aligned}h &= [1, 2, 4, 8, 16, 8, 4, 2, 1]^T \\x &= [2, -1, -4, -4, -1, 2]^T\end{aligned}$$

Verify that linear convolution can be achieved by zero padding and circular convolution by writing a MATLAB program that pads these signals with an appropriate number of zeros and uses the DSP Companion function *f_conv* to compare the linear convolution $y(k) = h(k) \star x(k)$ with the circular convolution $y_{zc}(k) = h_z(k) \circ x_z(k)$. Plot the following.

- The zero-padded signals $h_z(k)$ and $x_z(k)$ on the same graph using a legend.
- The linear convolution $y(k) = h(k) \star x(k)$.
- The zero-padded circular convolution $y_{zc}(k) = h_z(k) \circ x_z(k)$.

Solution

```
% Problem 2.60

% Initialize

f_header('Problem 2.60')
h = [1 2 4 8 16 8 4 2 1];
x = [2 -1 -4 -4 -1 2];

% Construct and plot zero-padded signals

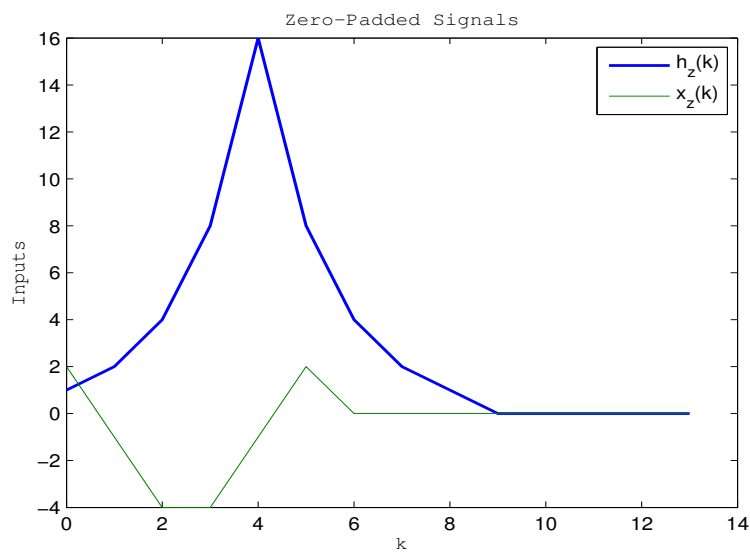
L = length(h);
M = length(x);
h_z = [h, zeros(1,M-1)]
x_z = [x, zeros(1,L-1)]
figure
k = 0 : length(h_z)-1;
hp = plot (k,h_z,k,x_z);
set (hp(1), 'LineWidth', 1.5)
f_labels ('Zero-Padded Signals', 'k', 'Inputs')
legend ('h_z(k)', 'x_z(k)')
f_wait

% Compute and plot convolutions
```

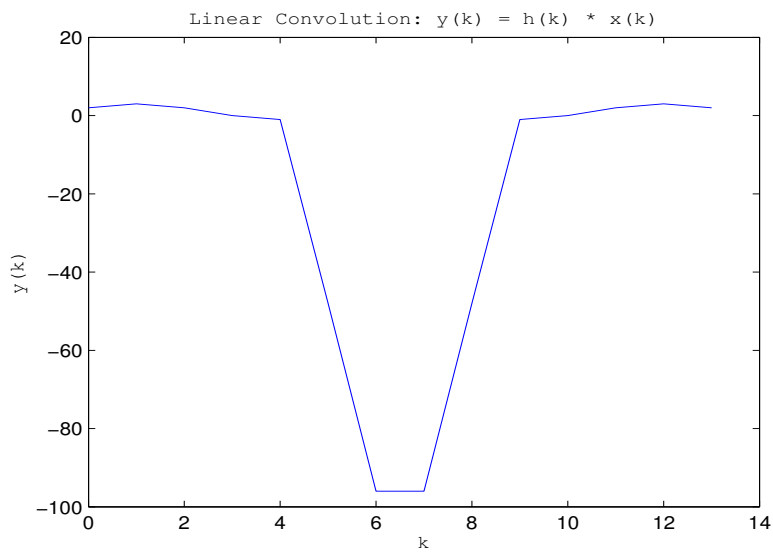
```

y = f_conv (h,x,0);
y_zc = f_conv (h_z,x_z,1);
figure
plot (k,y)
f_labels ('Linear Convolution:  $y(k) = h(k) * x(k)$ ','k','y(k)')
f_wait
figure
plot (k,y_zc)
f_labels ('Circular Convolution:  $y_{\{zc\}}(k) = h_z(k) \circ x_z(k)$ ','k','y_{\{zc\}}(k)')
f_wait

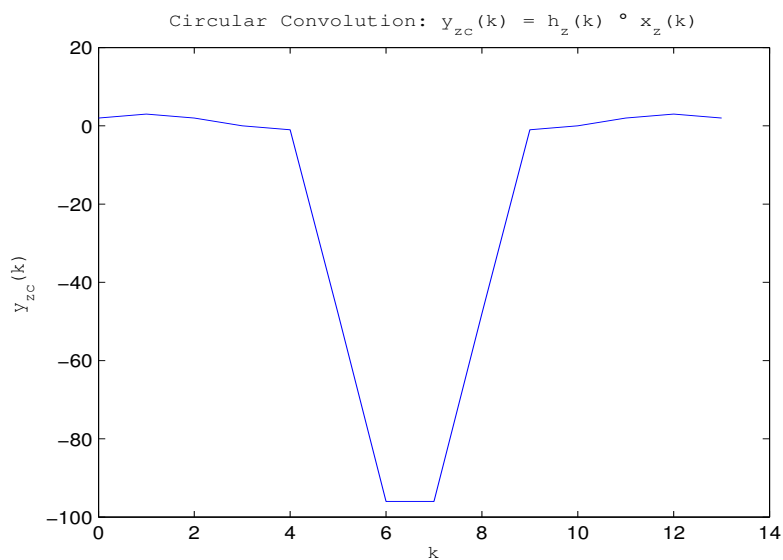
```



Problem 2.60 (a) Zero-padded Signals



Problem 2.60 (b) Linear Convolution



Problem 2.60 (c) Zero-padded Circular Convolution

2.61 Consider the following polynomials

$$\begin{aligned}a(z) &= z^4 + 4z^3 + 2z^2 - z + 3 \\b(z) &= z^3 - 3z^2 + 4z - 1 \\c(z) &= a(z)b(z)\end{aligned}$$

Let $a \in R^5$, $b \in R^4$ and $c \in R^8$ be the coefficient vectors of $a(z)$, $b(z)$ and $c(z)$, respectively.

- Find the coefficient vector of $c(z)$ by direct multiplication by hand.
- Write a MATLAB program that uses *conv* to find the coefficient vector of $c(z)$ by computing c as the linear convolution of a with b .
- In the program, show that a can be recovered from b and c by using the MATLAB function *deconv* to perform deconvolution.

Solution

```
% Problem 2.61

% Initialize

f_header('Problem 2.61')
a = [1 4 2 -1 3]
b = [1 -3 4 -1]

% Construct coefficient vector of product polynomial

c = conv (a,b)

% Recover coefficients of a from b and c

[a,r] = deconv (c,a)
```

- Using direct multiplication, $C(z) = A(z)B(z)$, we have

$$\begin{aligned}
A(z)B(z) &= \frac{z^4 + 4z^3 + 2z^2 - z + 3}{z^3 - 3z^2 + 4z - 1} \\
&= \frac{z^7 + 4z^6 + 2z^5 - z^4 + 3z^3}{-3z^6 - 12z^5 - 6z^4 + 3z^3 - 9z^2} \\
&\quad \frac{4z^5 + 16z^4 + 8z^3 - 4z^2 + 12z}{-z^4 - 4z^3 - 2z^2 + z - 3} \\
&= \frac{z^7 + z^6 - 6z^5 + 8z^4 + 10z^3 - 15z^2 + 13z - 3}{-3z^6 - 12z^5 - 6z^4 + 3z^3 - 9z^2}
\end{aligned}$$

Thus the coefficient vector of the product polynomial is

$$c = [1, 1, -6, 8, 10, -15, 13, -3]^T$$

(b) The program output for c using *conv* is

$$\begin{array}{r}
\mathbf{c} = \\
1 \quad 1 \quad -6 \quad 8 \quad 10 \quad -15 \quad 13 \quad -3
\end{array}$$

(c) The program output for a using *deconv* is

$$\begin{array}{r}
\mathbf{a} = \\
1 \quad -3 \quad 4 \quad -1
\end{array}$$

2.62 Consider the following pair of signals.

$$\begin{aligned}
x &= [2, -4, 3, 7, 6, 1, 9, 4, -3, 2, 7, 8]^T \\
y &= [3, 2, 1, 0, -1, -2, -3, -2, -1, 0, 1, 2]^T
\end{aligned}$$

Verify that linear cross-correlation and circular cross-correlation produce different results by writing a MATLAB program that uses the DSP Companion function *f_corr* to compute the linear cross-correlation, $r_{yx}(k)$, and the circular cross-correlation, $c_{yx}(k)$. Plot $r_{yx}(k)$ and $c_{yx}(k)$ below one another on the same screen.

Solution

```

% Problem 2.62

% Initialize

f_header('Problem 2.62')
x = [3 2 1 0 -1 -2 -3 -2 -1 0 1 2]
y = [2 -4 3 7 6 1 9 4 -3 2 7 8]

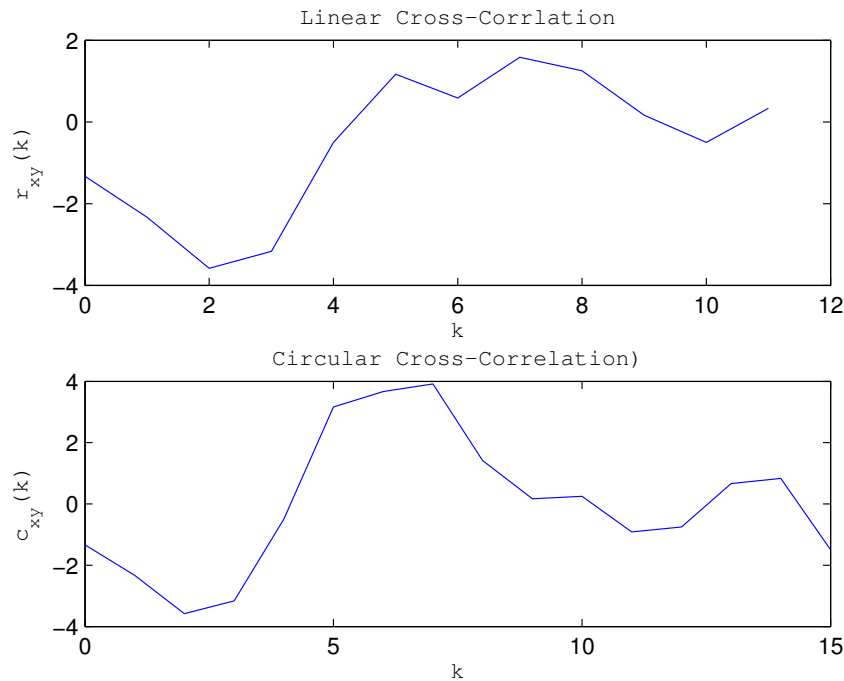
% Compute cross-correlations

r_xy = f_corr (x,y,0,0);
c_xy = f_corr (x,y,1,0);

% Plot them

figure
subplot (2,1,1)
k = 0 : length(r_xy)-1;
plot (k,r_xy)
f_labels ('Linear Cross-Correlation','k','r_{xy}(k)')
subplot (2,1,2)
k = 0 : length(c_xy)-1;
plot (k,c_xy)
f_labels ('Circular Cross-Correlation','k','c_{xy}(k)')
f_wait

```



Problem 2.62 Linear and Circular Cross-Correlation

✓ 2.63 Consider the following pair of signals.

$$y = [1, 8, -3, 2, 7, -5, -1, 4]^T$$

$$x = [2, -3, 4, 0, 5]^T$$

Verify that linear cross-correlation can be achieved by zero-padding and circular cross-correlation by writing a MATLAB program that pads these signals with an appropriate number of zeros and uses the DSP Companion function f_corr to compute the linear cross-correlation $r_{yx}(k)$ and the circular cross-correlation $c_{yzx_z}(k)$. Plot the following.

- The zero-padded signals $x_z(k)$ and $y_z(k)$ on the same graph using a legend.
- The linear cross-correlation $r_{yx}(k)$ and the scaled zero-padded circular cross-correlation $(N/L)c_{yzx_z}(k)$ on the same graph using a legend.

Solution

```
% Problem 2.63
```

```

% Initialize

f_header('Problem 2.63')
y = [1 8 -3 2 7 -5 -1 4]
x = [2 -3 4 0 5]

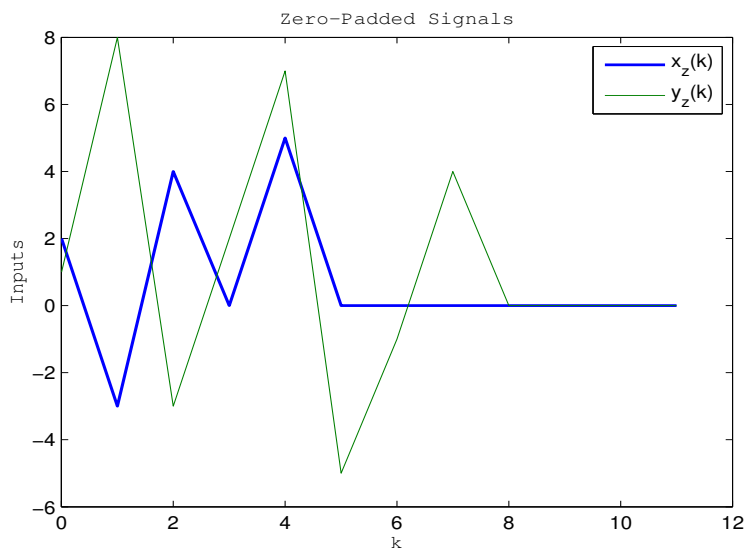
% Construct and plot zero-padded signals

L = length(y);
M = length(x);
x_z = [x, zeros(1,L-1)];
y_z = [y, zeros(1,M-1)];
figure
N = length(y_z);
k = 0 : N-1;
hp = plot (k,x_z,k,y_z);
set (hp(1),'LineWidth',1.5)
f_labels ('Zero-Padded Signals','k','Inputs')
legend ('x_z(k)','y_z(k)')
f_wait

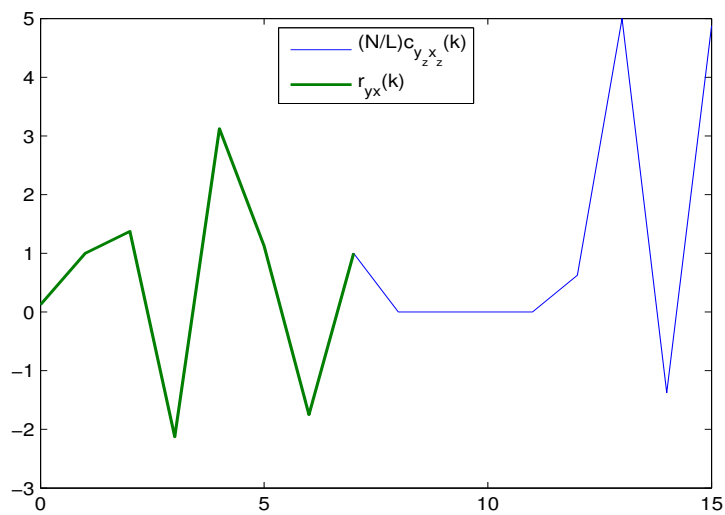
% Compute and plot cross-correlations

r_yx = f_corr (y,x,0,0);
R_yx = (N/L)*f_corr (y_z,x_z,1,0);
kr = 0 : length(r_yx)-1;
kR = 0 : length(R_yx)-1;
figure
h = plot (kR,R_yx,kr,r_yx);
set (h(2),'LineWidth',1.5)
legend ('(N/L)c_{y_zx_z}(k)', 'r_{yx}(k)', 'Location','North')
f_wait

```

Problem 2.63 (a) Zero-Padded Signals



Problem 2.63 (b) Cross-Correlations

2.64 Consider the following pair of signals of length $N = 8$.

$$\begin{aligned}x &= [2, -4, 7, 3, 8, -6, 5, 1]^T \\y &= [3, 1, -5, 2, 4, 9, 7, 0]^T\end{aligned}$$

Write a MATLAB program that performs the following tasks.

- Use the DSP Companion function `f_corr` to compute the circular cross-correlation, $c_{yx}(k)$.
- Compute and print $u(k) = x(-k)$ using the periodic extension, $x_p(k)$.
- Verify that $c_{yx}(k) = [y(k) \circ x(-k)]/N$ by using the DSP Companion function `f_conv` to compute and plot the scaled circular convolution, $w(k) = [u(k) \circ x(k)]/N$. Plot $c_{yx}(k)$ and $w(k)$ below one another on the same screen.

Solution

```
% Problem 2.64

% Initialize

f_header('Problem 2.64')
y = [3 1 -5 2 4 9 7 0]
x = [2 -4 7 3 8 -6 5 1]

% Compute and plot circular cross-correlation

c_yx = f_corr (y,x,1,0);

% Construct u(k) = x(-k) using periodic extension x_p(k)

N = length(x);
u = [x(1), x(N:-1:2)]

% Compute and plot scaled circular convolution

w = f_conv (y,u,1)/N;
figure
subplot(2,1,1)
kc = 0 : length(c_yx)-1;
plot (kc,c_yx)
f_labels ('Circular Cross-correlation of y(k) with x(k)', 'k', 'c_{yx}(k)')
```

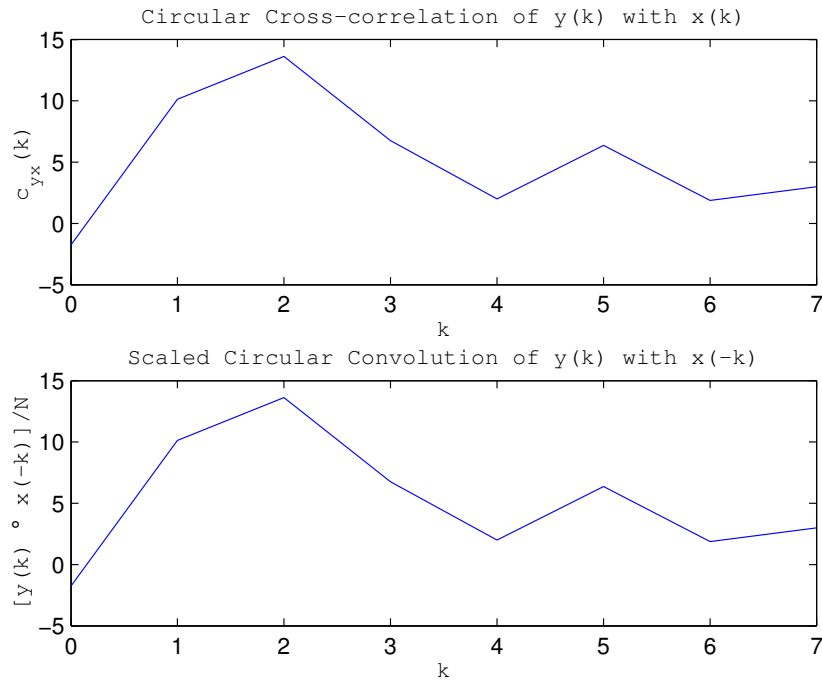
```

subplot(2,1,2)
kw = 0 : length(w)-1;
plot (kw,w)
f_labels ('Scaled Circular Convolution of y(k) with x(-k)', 'k', '[y(k) \circ x(-k)]/N')
f_wait

```

(b) The signal $u(k) = x(-k)$ using the periodic extension $x_p(k)$ is

u =
2 1 5 -6 8 3 7 -4



Problem 2.64 (c) Scaled Circular Convolution

