Chapter 1: Speaking Mathematically

Many college students appear to have difficulty using and interpreting language involving if-then statements and quantification. Section 1.1 is a gentle introduction to the relation between informal and formal ways of expressing important kinds of mathematical statements. Experience with the exercises in the section is meant as a warm-up to prepare students to master the linguistic aspects of learning mathematics to help them come to understand the meaning of mathematical statements and evaluate their truth or falsity in later chapters. Sections 1.2 and 1.3 form a brief introduction to the language of sets, relations, and functions. Covering them at the beginning of the course can help students relate discrete mathematics to the pre-calculus or calculus they have studied previously while broadening their perspective to include a larger proportion of discrete examples.

Proofs of set properties, such as the distributive laws, and proofs of properties of relations and functions, such as transitivity and surjectivity, are considerably more complex than the examples used in this book to introduce students to the idea of mathematical proof. Thus set theory as a theory is left to Chapter 6, properties of functions to Chapter 7, and properties of relations to Chapter 8. Instructors who wish to do so could cover Section 1.2 just before starting Chapter 6 and Section 1.3 just before starting Chapter 7.

An aspect of students' backgrounds that may surprise college and university mathematics instructors concerns their understanding of the meaning of "real number." When asked to evaluate the truth or falsity of a statement about real numbers, it is not unusual for students to think only of integers. Thus an informal description of the relationship between real numbers and points on a number line is given in Section 1.2 on page 8 to illustrate that there are many real numbers between any pair of consecutive integers, Examples 3.3.5 and 3.3.6 on page 121 show that while there is a smallest positive integer there is no smallest positive real number, and the discussion on pages 433 and 434 (preceding the proof of the uncountability of the real numbers between 0 and 1) describes a procedure for approximating the (possibly infinite) decimal expansion for an arbitrarily chosen point on a number line.

Section 1.1

2. *a.* a remainder of 2 when it is divided by 5 and a remainder of 3 when it is divided by 6

   b. an integer n; n is divided by 6 the remainder is 3

4. *a.* a real number; greater than r *b.* real number r; there is a real number s

6. *a.* s is negative *b.* negative; the cube root of s is negative (*Or*: $\sqrt[3]{s}$ is negative)

   c. is negative; $\sqrt[3]{s}$ is negative (*Or*: the cube root of s is negative)

7. *b.* There is a real number whose square is less than itself.

   True. For example, $(1/2)^2 = 1/4 < 1/2$.

   d. The absolute value of the sum of any two numbers is less than or equal to the sum of their absolute values.

   True. This is known as the triangle inequality. It is discussed in Section 4.4.

9. *a.* have at most two real solutions *b.* has at most two real solutions *c.* has at most two real solutions *d.* is a quadratic equation; has at most two real solutions *e.* E has at most two real solutions

11. *a.* have positive square roots *b.* a positive square root *c.* r is a square root for e

13. *a.* real number; product with every real number equals zero

    b. with every real number equals zero *c.* $ab = 0$

Section 1.2

2. *b.* The set of all real numbers x such that x is less than or equal to zero or x is greater than or equal to 1

   d. The set of all positive integers n such that n is a factor of 6

4. *a.* Yes: $\{2\}$ is the set whose only element is 2. *b.* One: 2 is the only element in this set *c.* Two: The two elements are 0 and $\{0\}$ *d.* Yes: $\{0\}$ is one of the elements listed in the set. *e.* No: The only elements listed in the set are $\{0\}$ and $\{1\}$, and 0 is not equal to either of these.

5. The only sets that are equal to each other are A and D.

   A contains the integers 0, 1, and 2 and nothing else.

   B contains all the real numbers that are greater than or equal to -1 and less than 3.

   C contains all the real numbers that are greater than -1 and less than 3. Thus -1 is in B but not in C.

   D contains all the integers greater than -1 and less than 3. Thus D contains the integers 0, 1, and 2 and nothing else, and so $D = \{0, 1, 2\} = A$.

   E contains all the positive integers greater than -1 and less than 3. Hence E contains the integers 1 and 2 and nothing else, that is, $E = \{1, 2\}$.

6. T_2 and T_{-3} each have two elements, and T_0 and T_1 each have one element.

   Justification: $T_2 = \{2, 2^2\} = \{2, 4\}$, $T_{-3} = \{-3, (-3)^2\} = \{-3, 9\}$, $T_1 = \{1, 1^2\} = \{1, 1\} = \{1\}$, and $T_0 = \{0, 0^2\} = \{0, 0\} = \{0\}$.

7. *b.* $T = \{m \in \mathbf{Z} \mid m = 1+(-1)^k \text{ for some integer } k\} = \{0, 2\}$. Exercises in Chapter 4 explore the fact that $(-1)^k = -1$ when k is odd and $(-1)^k = 1$ when k is even. So $1+(-1)^k = 1+(-1) = 0$ when k is odd, and $1 + (-1)^k = 1 + 1 = 2$ when k is even.

   e. There are no elements in W because there are no integers that are both greater than 1 and less than -3.

   f. $X = \mathbf{Z}$ because every integer u satisfies at least one of the conditions $u \leq 4$ or $u \geq 1$.

8. *b.* Yes, because every element in C is in A. *c.* Yes, because every element in C is in C.

   d.. Yes, because although every element in C is in A, A contains elements that are not in C.

9. *c.* No: The only elements in $\{1, 2\}$ are 1 and 2, and $\{2\}$ is not equal to either of these.

   d. Yes: $\{3\}$ is one of the elements listed in $\{1, \{2\}, \{3\}\}$.

   e. Yes: $\{1\}$ is the set whose only element is 1.

   g. Yes: The only element in $\{1\}$ is 1, and 1 is an element in $\{1, 2\}$.

   h. No: The only elements in $\{\{1\}, 2\}$ are $\{1\}$ and 2, and 1 is not equal to either of these.

   j. Yes: The only element in $\{1\}$ is 1, which is is an element in $\{1\}$. So every element in $\{1\}$ is in $\{1\}$.

10. *b.* No: For two ordered pairs to be equal, the elements in each pair must occur in the same order. In this case the first element of the first pair is 5, whereas the first element of the second pair is -5, and the second element of the first pair is -5 whereas the second element of the second pair is 5.

    d. Yes The first elements of both pairs equal $\frac{1}{2}$, and the second elements of both pairs equal -8.

12. All four sets have nine elements.

    a. $S \times T = \{(2,1), (2,3), (2,5), (4,1), (4,3), (4,5), (6,1), (6,3), (6,5)\}$

    b. $T \times S = \{(1,2), (3,2), (5,2), (1,4), (3,4), (5,4), (1,6), (3,6), (5,6)\}$

    c. $S \times S = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\}$

    d. $T \times T = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5)\}$

Section 1.3

2. *a.* $2\ S\ 2$ because $\frac{1}{2} - \frac{1}{2} = 0$, which is an integer.

    $-1\ S\ -1$ because $\frac{1}{-1} - \frac{1}{-1} = 0$, which is an integer.

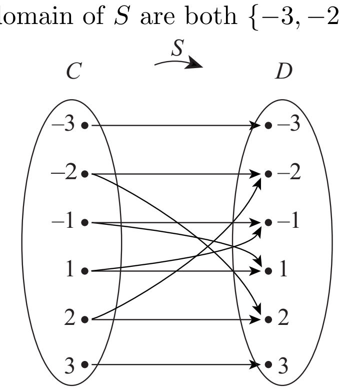
    $3\ S\ 3$ because $\frac{1}{3} - \frac{1}{3} = 0$, which is an integer.

    $3 \not S -3$ because $\frac{1}{3} - \frac{1}{-3} = \frac{2}{3}$, which is not an integer.

    b. $S = \{(-3,-3), (-2,-2), (-1,-1), (1,1), (2,2), (3,3), (1,-1), (-1,1), (2,-2), (-2,2)\}$

    c. The domain and co-domain of S are both $\{-3,-2,-1,1,2,3\}$.

    d.



4. *a.* $2\ V\ 6$ because $\frac{2-6}{4} = \frac{-4}{4} = -1$, which is an integer.

    $-2\ V\ -6$ because $\frac{-2-(-6)}{4} = \frac{4}{4} = 1$, which is an integer.

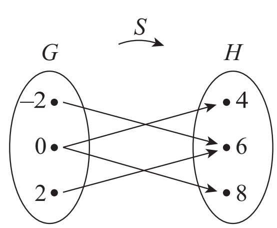
    $0 \not V 6$ because $\frac{0-6}{4} = \frac{-6}{4}$, which is not an integer.

    $2 \not V 4$ because $\frac{2-4}{4} = \frac{-2}{4}$, which is not an integer.

    b. $V = \{(-2,6), (0,4), (0,8), (2,6)\}$

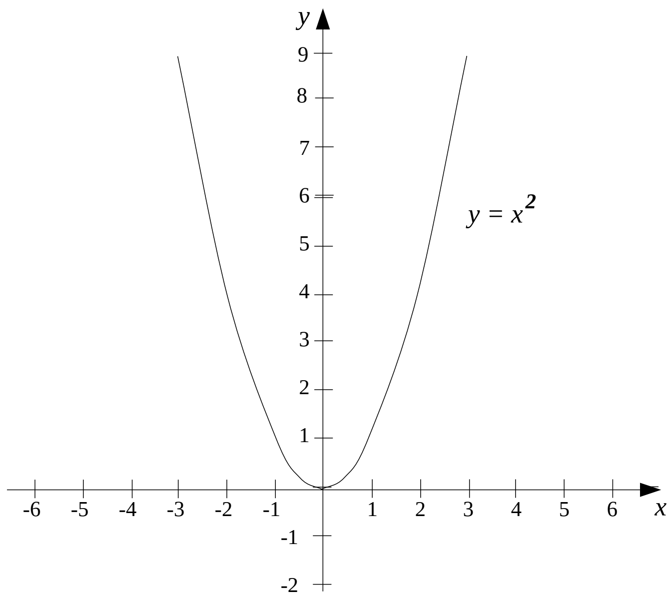
    c. Domain of $V = \{-2,0,2\}$, co-domain of $V = \{4,6,8\}$

    d.

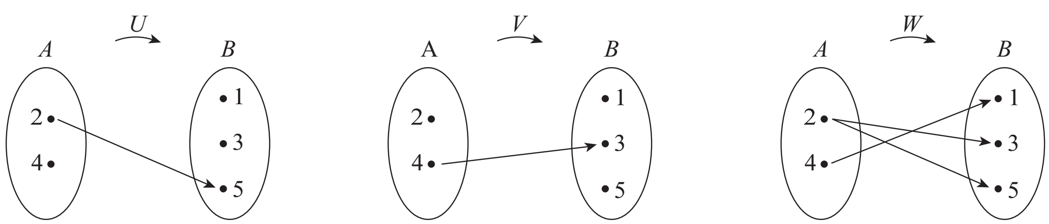


6. *a*. $(2, 4) \in R$ because $4 = 2^2$.
   $(4, 2) \notin R$ because $2 \neq 4^2$.
   $(-3, 9) \in R$ because $9 = (-3)^2$.
   $(9, -3) \notin R$ because $-3 \neq 9^2$.

   b.



8. *a*.



   b. None of U, V, or W are functions.

   U is not a function because $(4, y)$ is not in U for any y in B, and so U does not satisfy property (1) of the definition of function.

   V is not a function because $(2, y)$ is not in V for any y in B, and so V does not satisfy property (1) of the definition of function.

   W is not a function because both $(2, 3)$ and $(2, 5)$ are in W and $3 \neq 5$, and so W does not satisfy property (2) of the definition of function.

10. The following sets are relations from $\{a, b\}$ to $\{x, y\}$ that are not functions:
    $\{(a, x)\}$, $\{(a, y)\}$, $\{(b, x)\}$, $\{(b, y)\}$, $\{(a, x), (a, y)\}$, $\{(b, x), (b, y)\}$, $\{(a, x), (a, y), (b, x)\}$, $\{(a, x), (a, y), (b, y)\}$, $\{(b, x), (b, y), (a, x)\}$, $\{(b, x), (b, y), (a, y)\}$, $\{(a, x), (a, y), (b, x), (b, y)\}$.

12. T is not a function because, for example, both $(0, 1)$ and $(0, -1)$ are in T but $1 \neq -1$. Many other examples could be given showing that T does not satisfy property 2 of the definition of function.

14. *a*. Domain of $G = \{1, 2, 3, 4\}$, co-domain of $G = \{a, b, c, d\}$

    b. $G(1) = G(2) = G(3) = G(4) = c$

15. *c*. This diagram does not determine a function because 4 is related to both 1 and 2, which violates property (2) of the definition of function.

    d. This diagram defines a function; both properties (1) and (2) are satisfied.

    e. This diagram does not determine a function because 2 is in the domain but it is not related to any element in the co-domain.

17. $g(-1000) = -999$, $g(0) = 1$, $g(999) = 1000$

18. $h(-\frac{12}{5}) = h(\frac{0}{1}) = h(\frac{9}{17}) = 2$

20. For all $x \in \mathbf{R}$, $K(x) = (x-1)(x-3)+1 = (x^2-4x+3)+1 = x^2+4x+4 = (x-2)^2 = H(x)$. Therefore, by definition of equality of functions, $H = K$.

Chapter 2: The Logic of Compound Statements

The ability to reason using the principles of logic is essential for solving problems in abstract mathematics and computer science and for understanding the reasoning used in mathematical proof and disproof. Because a significant number of students who come to college have had limited opportunity to develop this ability, a primary aim of Chapters 2 and 3 is to help students develop an inner voice that speaks with logical precision. Consequently, the various rules used in logical reasoning are developed both symbolically and in the context of their somewhat limited but very important use in everyday language. Exercise sets for Sections 2.1–2.3 and 3.1–3.4 contain sentences for students to negate, write the contrapositive for, and so forth. Virtually all students benefit from doing these exercises. Another aim of Chapters 2 and 3 is to teach students the rudiments of symbolic logic as a foundation for a variety of upper-division courses. Symbolic logic is used in, among others, the study of digital logic circuits, relational databases, artificial intelligence, and program verification.

Suggestions

1. In Section 2.1 a surprising number of students apply De Morgan's law to write the negation of, for example, "$1 < x \leq 3$" as "$1 \geq x > 3$." You may find that it takes some effort to teach them to avoid making this mistake.

2. In Sections 2.1 and 2.4, students have more difficulty than you might expect simplifying statement forms and circuits. Only through trial and error can you learn the extent to which this is the case at your institution. If it is, you might either assign only the easier exercises or build in extra time to teach students how to do the more complicated ones. Discussion of simplification techniques occurs again in Chapter 6 in the context of set theory. At this later point in the course most students are able to deal with it successfully.

3. In ordinary English, the phrase "only if" is often used as a synonym for "if and only if." But it is possible to find informal sentences in which the intuitive interpretation is the same as the logical definition, and it is helpful to give examples of such statements when you introduce the logical definition. For instance, it is not hard to get students to agree that "The team will win the championship only if it wins the semifinal game" means the same as "If the team does not win the semifinal game then it will not win the championship." Once students see this, you can suggest that they remember this translation when they encounter more abstract statements of the form "A only if B."

Through guided discussion, students will also come to agree that the statement "Winning the semifinal game is a necessary condition for winning the championship" translates to "If the team does not win the semifinal game then it will not win the championship." They can be encouraged to use this (or a similar statement) as a reference to help develop intuition for general statements of the form "A is a necessary condition for B."

With students who have weaker backgrounds, you may find yourself tying up excessive amounts of class time discussing "only if" and "necessary and sufficient conditions." You might just assign the easier exercises, or you might assign exercises on these topics to be done for extra credit (putting corresponding extra credit problems on exams) and use the results to help distinguish A from B students. It is probably best not to omit these topics altogether, though, because the language of "only if" and "necessary and sufficient conditions" is a standard part of the technical vocabulary of textbooks used in upper-division courses, as well as occurring regularly in non-mathematical writing.

4. In Section 2.3, many students mistakenly conclude that an argument is valid if, when they compute the truth table, they find a single row in which both the premises and the conclusion are true. The source of students' difficulty appears to be their tendency to ignore quantification and to misinterpret if-then statements as "and" statements. Since the definition of validity includes both a universal quantifier and if-then, it is helpful to go back over the definition and the procedures for testing for validity and invalidity after discussing the general topic of universal conditional statements in Section 3.1. As a practical measure to help students assess validity and invalidity correctly, the first example in Section 2.3 is of an invalid argument whose truth table has eight rows, several of which have true premises and a true conclusion. In addtition, to further focus students' attention on the situations where all the premises are true, the truth values for the conclusions of arguments are simply omitted when at least one premise is false.

5. In Section 2.3, you might suggest that students just familiarize themselves with, but not memorize, the various forms of valid arguments covered in Section 2.3. It is wise, however, to have them learn the terms modus ponens and modus tollens (because these are used in some upper-division computer science courses) and converse and inverse errors (because these errors are so common).

Section 2.1

2. common form: If p then q.
   $\sim q$
   Therefore, $\sim p$.

   b. all prime numbers are odd; 2 is odd

4. common form: If p then q.
   If q then r.
   Therefore, if p then r.

   b. a polynomial; differentiable; is continuous

5. *b.* The truth or falsity of this sentence depends on the reference for the pronoun "she." Considered on its own, the sentence cannot be said to be either true or false, and so it is not a statement.

   c. This sentence is false; hence it is a statement.

   d. This is not a statement because its truth or falsity depends on the value of x.

7. $m \wedge \sim c$

8. *b.* $\sim w \wedge (h \wedge s)$

   c. $\sim w \wedge \sim h \wedge \sim s$

   e. $w \wedge \sim (h \wedge s)$ ($w \wedge (\sim h \vee \sim s)$ is also acceptable)

9. $(n \vee k) \wedge \sim (n \wedge k)$

10. *b.* $p \wedge \sim q$ *d.* $(\sim p \wedge q) \wedge \sim r$ *e.* $\sim p \vee (q \wedge r)$

13.

p	q	$p \wedge q$	$p \vee q$	$\sim (p \wedge q)$	$\sim (p \wedge q) \vee (p \vee q)$
T	T	T	T	F	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	F	T	T

15.

p	q	r	$\sim q$	$\sim q \vee r$	$p \wedge (\sim q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	T	T	F
F	F	F	T	T	F

17.

p	q	$p \wedge q$	$\sim p$	$\sim q$	$\sim (p \wedge q)$	$\sim p \wedge \sim q$	
T	T	T	F	F	F	F	
T	F	F	F	T	T	F	$\leftarrow$
F	T	F	T	F	T	F	$\leftarrow$
F	F	F	T	T	T	T	

different truth values in rows 2 and 3

The truth table shows that $\sim (p \wedge q)$ and $\sim p \wedge \sim q$ do not always have the same truth values. Therefore they are not logically equivalent.

19.

p	$\mathbf{t}$	$p \wedge \mathbf{t}$	p
T	T	T	T
F	T	F	F

same truth values

The truth table shows that $p \wedge \mathbf{t}$ and p always have the same truth values. Thus they are logically equivalent. This proves the identity law for $\wedge$.

20.

p	$\mathbf{c}$	$p \wedge \mathbf{c}$	$p \vee \mathbf{c}$	
T	F	F	T	$\leftarrow$
F	F	F	F	

different truth values in row 1

The truth table shows that $p \wedge \mathbf{c}$ and $p \vee \mathbf{c}$ do not always have the same truth values. Thus they are not logically equivalent.

22.

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	T

same truth values

The truth table shows that $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ always have the same truth values. Therefore they are logically equivalent. This proves the distributive law for $\wedge$ over $\vee$.

24.

p	q	r	$p \vee q$	$p \wedge r$	$(p \vee q) \vee (p \wedge r)$	$(p \vee q) \wedge r$	
T	T	T	T	T	T	T	
T	T	F	T	F	T	F	$\leftarrow$
T	F	T	T	T	T	T	$\leftarrow$
T	F	F	T	F	T	F	
F	T	T	T	F	T	T	
F	T	F	T	F	T	F	$\leftarrow$
F	F	T	F	F	F	F	
F	F	F	F	F	F	F	

different truth values

The truth table shows that $(p \vee q) \vee (p \wedge r)$ and $(p \vee q) \wedge r$ have different truth values in rows 2, 3, and 6. Hence they are not logically equivalent.

26. Sam is not an orange belt or Kate is not a red belt.

28. The units digit of 4^{67} is not 4 and it is not 6.

29. This computer program does not have a logical error in the first ten lines and it is not being run with an incomplete data set.

30. The dollar is not at an all-time high or the stock market is not at a record low.

31. The train is not late and my watch is not fast.

33. $-10 \geq x$ or $x \geq 2$

35. $x > -1$ and $x \leq 1$

37. $0 \leq x$ or $x < -7$

39. The statement's logical form is $(p \wedge q) \vee ((r \wedge s) \wedge t)$, so its negation has the form

$$\begin{aligned} \sim((p \wedge q) \vee ((r \wedge s) \wedge t)) &\equiv \; \sim(p \wedge q) \wedge \sim((r \wedge s) \wedge t)) \\ &\equiv (\sim p \vee \sim q) \wedge (\sim(r \wedge s) \vee \sim t)) \\ &\equiv (\sim p \vee \sim q) \wedge ((\sim r \vee \sim s) \vee \sim t)). \end{aligned}$$

Thus a negation is ($\mathit{num\_orders} \geq 50$ or $\mathit{num\_instock} \leq 300$) and (($50 > \mathit{num\_orders}$ or $\mathit{num\_orders} \geq 75$) or $\mathit{num\_instock} \leq 500$).

42.

p	q	r	$\sim p$	$\sim q$	$\sim p \wedge q$	$q \wedge r$	$((\sim p \wedge q) \wedge (q \wedge r))$	$((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$
T	T	T	F	F	F	T	F	F
T	T	F	F	F	F	F	F	F
T	F	T	F	T	F	F	F	F
T	F	F	F	T	F	F	F	F
F	T	T	T	F	T	T	T	F
F	T	F	T	F	T	F	F	F
F	F	T	T	T	F	F	F	F
F	F	F	T	T	F	F	F	F

all $F's$

Since all the truth values of $((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$ are F, $((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$ is a contradiction.

43.

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$p \wedge \sim q$	$(\sim p \vee q) \vee (p \wedge \sim q)$
T	T	F	F	T	F	T
T	F	F	T	F	T	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

all $T's$

Since all the truth values of $(\sim p \vee q) \vee (p \wedge \sim q)$ are T, $(\sim p \vee q) \vee (p \wedge \sim q)$ is a tautology.

45. Let b be "Bob is a double math and computer science major," m be "Ann is a math major," and a be "Ann is a double math and computer science major." Then the two statements can be symbolized as follows: a. $(b \wedge m) \wedge \sim a$ and b. $\sim (b \wedge a) \wedge (m \wedge b)$. *Note*: The entries in the truth table assume that a person who is a double math and computer science major is also a math major and a computer science major.

b	m	a	$\sim a$	$b \wedge m$	$m \wedge b$	$b \wedge a$	$\sim (b \wedge a)$	$(b \wedge m) \wedge \sim a$	$\sim (b \wedge a) \wedge (m \wedge b)$
T	T	T	F	T	T	T	F	F	F
T	T	F	T	F	T	F	T	T	T
T	F	T	F	T	F	T	F	F	F
T	F	F	T	F	F	F	T	F	F
F	T	T	F	F	F	F	T	F	F
F	T	F	T	F	F	F	T	F	F
F	F	T	F	F	F	F	T	F	F
F	F	F	T	F	F	F	T	F	F

same truth values

The truth table shows that $(b \wedge m) \wedge \sim a$ and $\sim (b \wedge a) \wedge (m \wedge b)$ always have the same truth values. Hence they are logically equivalent.

46. *b*. Yes.

p	q	r	$p \oplus q$	$q \oplus r$	$(p \oplus q) \oplus r$	$p \oplus (q \oplus r)$
T	T	T	F	F	T	T
T	T	F	F	T	F	F
T	F	T	T	T	F	F
T	F	F	T	F	T	T
F	T	T	T	F	F	F
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

same truth values

The truth table shows that $(p \oplus q) \oplus r$ and $p \oplus (q \oplus r)$ always have the same truth values. So they are logically equivalent.

c. Yes.

p	q	r	$p \oplus q$	$p \wedge r$	$q \wedge r$	$(p \oplus q) \wedge r$	$(p \wedge r) \oplus (q \wedge r)$
T	T	T	F	T	T	F	F
T	T	F	F	F	F	F	F
T	F	T	T	T	F	T	T
T	F	F	T	F	F	F	F
F	T	T	T	F	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

same truth values

The truth table shows that $(p \oplus q) \wedge r$ and $(p \wedge r) \oplus (q \wedge r)$ always have the same truth values. So they are logically equivalent.

49. *a.* the commutative law for $\vee$ *b.* the distributive law

c. the negation law for $\wedge$ *d.* the identity law for $\vee$

51. *Solution 1*:
$$\begin{aligned} p \wedge (\sim q \vee p) &\equiv p \wedge (p \vee \sim q) && \text{commutative law for } \vee \\ &\equiv p && \text{absorption law} \end{aligned}$$

Solution 2:
$$\begin{aligned} p \wedge (\sim q \vee p) &\equiv (p \wedge \sim q) \vee (p \wedge p) && \text{distributive law} \\ &\equiv (p \wedge \sim q) \vee p && \text{identity law for } \wedge \\ &\equiv p && \text{by exercise 50.} \end{aligned}$$

52.
$$\begin{aligned} \sim (p \vee \sim q) \vee (\sim p \wedge \sim q) &\equiv (\sim p \wedge \sim (\sim q)) \vee (\sim p \wedge \sim q) && \text{De Morgan's law} \\ &\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) && \text{double negative law} \\ &\equiv \sim p \wedge (q \vee \sim q) && \text{distributive law} \\ &\equiv \sim p \wedge \mathbf{t} && \text{negation law for } \vee \\ &\equiv \sim p && \text{identity law for } \wedge \end{aligned}$$

54.
$$\begin{aligned} (p \wedge (\sim (\sim p \vee q))) \vee (p \wedge q) &\equiv (p \wedge (\sim (\sim p) \wedge \sim q)) \vee (p \wedge q) && \text{De Morgan's law} \\ &\equiv (p \wedge (p \wedge \sim q)) \vee (p \wedge q) && \text{double negative law} \\ &\equiv ((p \wedge p) \wedge \sim q)) \vee (p \wedge q) && \text{associative law for } \wedge \\ &\equiv (p \wedge \sim q)) \vee (p \wedge q) && \text{idempotent law for } \wedge \\ &\equiv p \wedge (\sim q \vee q) && \text{distributive law} \\ &\equiv p \wedge (q \vee \sim q) && \text{commutative law for } \vee \\ &\equiv p \wedge \mathbf{t} && \text{negation law for } \vee \\ &\equiv p && \text{identity law for } \wedge \end{aligned}$$

Section 2.2

2. If I catch the 8:05 bus, then I am on time for work.

4. If you don't fix my ceiling, then I won't pay my rent.

6.

p	q	$\sim p$	$\sim p \wedge q$	$p \vee q$	$(p \vee q) \vee (\sim p \wedge q)$	$(p \vee q) \vee (\sim p \wedge q) \to q$
T	T	F	F	T	T	T
T	F	F	F	T	T	F
F	T	T	T	T	T	T
F	F	T	F	F	F	T

8.

p	q	r	$\sim p$	$\sim p \vee q$	$\sim p \vee q \to r$
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	T	T	T
F	F	F	T	T	F

10.

p	q	r	$p \to r$	$q \to r$	$(p \to r) \leftrightarrow (q \to r)$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

11.

p	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$p \wedge q \to r$	$(p \to (q \to r)) \leftrightarrow (p \wedge q \to r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

13. *b.*

p	q	$\sim q$	$p \to q$	$\sim (p \to q)$	$p \wedge \sim q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

same truth values

The truth table shows that $\sim (p \to q)$ and $p \wedge \sim q$ always have the same truth values. Hence they are logically equivalent.

14. *a.*

p	q	r	$\sim q$	$\sim r$	$q \vee r$	$p \wedge \sim q$	$p \wedge \sim r$	$p \to q \vee r$	$p \wedge \sim q \to r$	$p \wedge \sim r \to q$
T	T	T	F	F	T	F	F	T	T	T
T	T	F	F	T	T	F	T	T	T	T
T	F	T	T	F	T	T	F	T	T	T
T	F	F	T	T	F	T	T	F	F	F
F	T	T	F	F	T	F	F	T	T	T
F	T	F	F	T	T	F	F	T	T	T
F	F	T	T	F	T	F	F	T	T	T
F	F	F	T	T	F	F	F	T	T	T

same truth values

The truth table shows that the three statement forms $p \to q \vee r$, $p \wedge \sim q \to r$, and $p \wedge \sim r \to q$ always have the same truth values. Thus they are all logically equivalent.

b. If n is prime and n is not odd, then n is 2.

And: If n is prime and n is not 2, then n is odd.

15.

p	q	r	$q \to r$	$p \to q$	$p \to (q \to r)$	$(p \to q) \to r$	
T	T	T	T	T	T	T	
T	T	F	F	T	F	F	
T	F	T	T	F	T	T	
T	F	F	T	F	T	T	
F	T	T	T	T	T	T	
F	T	F	F	T	T	F	$\leftarrow$
F	F	T	T	T	T	F	$\leftarrow$
F	F	F	T	T	T	F	$\leftarrow$

different truth values

The truth table shows that $p \to (q \to r)$ and $(p \to q) \to r$ do not always have the same truth values. (They differ for the combinations of truth values for p, q, and r shown in rows 6, 7, and 8.) Therefore they are not logically equivalent.

17. Let p represent "2 is a factor of n," q represent "3 is a factor of n," and r represent "6 is a factor of n." The statement "If 2 is a factor of n and 3 is a factor of n, then 6 is a factor of n" has the form $p \wedge q \to r$. And the statement "2 is not a factor of n or 3 is a not a factor of n or 6 is a factor of n" has the form $\sim p \vee \sim q \vee r$.

p	q	r	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge q \to r$	$\sim p \vee \sim q \vee r$
T	T	T	F	T	T	T	T
T	T	F	F	T	T	F	F
T	F	T	F	F	F	T	T
T	F	F	F	F	F	T	T
F	T	T	T	T	F	T	T
F	T	F	T	T	F	T	T
F	F	T	T	F	F	T	T
F	F	F	T	F	F	T	T

same truth values

The truth table shows that $p \wedge q \to r$ and $\sim p \vee \sim q \vee r$ always have the same truth values. Therefore they are logically equivalent.

18. *Part 1*: Let p represent "It walks like a duck," q represent "It talks like a duck," and r represent "It is a duck." The statement "If it walks like a duck and it talks like a duck, then it is a duck" has the form $p \wedge q \to r$. And the statement "Either it does not walk like a duck or it does not talk like a duck or it is a duck" has the form $\sim p \vee \sim q \vee r$.

p	q	r	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \vee \sim q$	$p \wedge q \to r$	$(\sim p \vee \sim q) \vee r$
T	T	T	F	F	T	F	T	T
T	T	F	F	F	T	F	F	F
T	F	T	F	T	F	T	T	T
T	F	F	F	T	F	T	T	T
F	T	T	T	F	F	T	T	T
F	T	F	T	F	F	T	T	T
F	F	T	T	T	F	T	T	T
F	F	F	T	T	F	T	T	T

same truth values

The truth table shows that $p \wedge q \to r$ and $(\sim p \vee \sim q) \vee r$ always have the same truth values. Thus the following statements are logically equivalent: "If it walks like a duck and it talks like a duck, then it is a duck" and "Either it does not walk like a duck or it does not talk like a duck or it is a duck."

Part 2: The statement "If it does not walk like a duck and it does not talk like a duck then it is not a duck" has the form $\sim p \wedge \sim q \to \sim r$.

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \wedge q$	$\sim p \wedge \sim q$	$p \wedge q \to r$	$(\sim p \wedge \sim q) \to \sim r$	
T	T	T	F	F	F	T	F	T	T	
T	T	F	F	F	T	T	F	F	T	←
T	F	T	F	T	F	F	F	T	T	
T	F	F	F	T	T	F	F	T	T	
F	T	T	T	F	F	F	F	T	T	
F	T	F	T	F	T	F	F	T	T	
F	F	T	T	T	F	F	T	T	F	←
F	F	F	T	T	T	F	T	T	T	

different truth values

The truth table shows that $p \wedge q \to r$ and $(\sim p \wedge \sim q) \to \sim r$ do not always have the same truth values. (They differ for the combinations of truth values of p, q, and r shown in rows 2 and 7.) Thus they are not logically equivalent, and so the statement "If it walks like a duck and it talks like a duck, then it is a duck" is not logically equivalent to the statement "If it does not walk like a duck and it does not talk like a duck then it is not a duck." In addition, because of the logical equivalence shown in Part 1, we can also conclude that the following two statements are not logically equivalent: "Either it does not walk like a duck or it does not talk like a duck or it is a duck" and "If it does not walk like a duck and it does not talk like a duck then it is not a duck."

20. *b*. Today is New Year's Eve and tomorrow is not January.

    c. The decimal expansion of r is terminating and r is not rational.

    e. x is nonnegative and x is not positive and x is not 0.

    Or: x is nonnegative but x is not positive and x is not 0.

    Or: x is nonnegative and x is neither positive nor 0.

    g. n is divisible by 6 and either n is not divisible by 2 or n is not divisible by 3.

21. By the truth table for $\to$, $p \to q$ is false if, and only if, p is true and q is false. Under these circumstances, (*b*) $p \vee q$ is true and (*c*) $q \to p$ is also true.

22. *b*. If tomorrow is not January, then today is not New Year's Eve.

    c. If r is not rational, then the decimal expansion of r is not terminating.

e. If x is not positive and x is not 0, then x is not nonnegative.

Or: If x is neither positive nor 0, then x is negative.

g. If n is not divisible by 2 or n is not divisible by 3, then n is not divisible by 6.

23. *b*. *Converse*: If tomorrow is January, then today is New Year's Eve.

    Inverse: If today is not New Year's Eve, then tomorrow is not January.

    c. *Converse*: If r is rational then the decimal expansion of r is terminating.

    Inverse: If the decimal expansion of r is not terminating, then r is not rational.

    e. *Converse*: If x is positive or x is 0, then x is nonnegative.

    Inverse: If x is not nonnegative, then both x is not positive and x is not 0.

    Or: If x is negative, then x is neither positive nor 0.

25.

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim p \to \sim q$	
T	T	F	F	T	T	
T	F	F	T	F	T	$\leftarrow$
F	T	T	F	T	F	$\leftarrow$
F	F	T	T	T	T	

different truth values

The truth table shows that $p \to q$ *and* $\sim p \to \sim q$ have different truth values in rows 2 and 3, so they are not logically equivalent. Thus a conditional statement is not logically equivalent to its inverse.

27.

p	q	$\sim p$	$\sim q$	$q \to p$	$\sim p \to \sim q$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	T	T

same truth values

The truth table shows that $q \to p$ *and* $\sim p \to \sim q$ always have the same truth values, so they are logically equivalent. Thus the converse and inverse of a conditional statement are logically equivalent to each other.

28. The if-then form of "I say what I mean" is "If I mean something, then I say it."

    The if-then form of "I mean what I say" is "If I say something, then I mean it."

    Thus "I mean what I say" is the converse of "I say what I mean." The two statements are not logically equivalent.

30. The corresponding tautology is $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \wedge (q \vee r) \leftrightarrow$ $(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T	T
T	F	T	T	F	T	T	T	T
T	F	F	F	F	F	F	F	T
F	T	T	T	F	F	F	F	T
F	T	F	T	F	F	F	F	T
F	F	T	T	F	F	F	F	T
F	F	F	F	F	F	F	F	T

all T's

The truth table shows that $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$ is always true. Hence it is a tautology.

31. The corresponding tautology is $(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \wedge q) \rightarrow r)$.

p	q	r	$q \rightarrow r$	$p \wedge q$	$p \rightarrow (q \rightarrow r)$	$(p \wedge q) \rightarrow r)$	$p \rightarrow (q \rightarrow r) \leftrightarrow (p \wedge q) \rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	T
T	F	T	T	F	T	T	T
T	F	F	T	F	T	T	T
F	T	T	T	F	T	T	T
F	T	F	F	F	T	T	T
F	F	T	T	F	T	T	T
F	F	F	T	F	T	T	T

all T's

The truth table shows that $(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \wedge q) \rightarrow r)$ is always true. Hence it is a tautology.

33. If this integer is even, then it equals twice some integer, and if this integer equals twice some integer, then it is even.

35. If Sam is not an expert sailor, then he will not be allowed on Signe's racing boat.

    If Sam is allowed on Signe's racing boat, then he is an expert sailor.

36. The Personnel Director did not lie. By using the phrase "only if," the Personnel Director set forth conditions that were necessary but not sufficient for being hired: if you did not satisfy those conditions then you would not be hired. The Personnel Director's statement said nothing about what would happen if you did satisfy those conditions.

38. If it doesn't rain, then Ann will go.

39. *b.* If a security code is not entered, then the door will not open.

41. If this triangle has two 45° angles, then it is a right triangle.

43. If Jim does not do his homework regularly, then Jim will not pass the course.

    If Jim passes the course, then he will have done his homework regularly.

45. If this computer program produces error messages during translation, then it is not correct.

    If this computer program is correct, then it does not produce error messages during translation.

46. *c.* must be true *d.* not necessarily true *e.* must be true *f.* not necessarily true

    Note: To solve this problem, it may be helpful to imagine a compound whose boiling point is greater than 150° C. For concreteness, suppose it is 200° C. Then the given statement would be true for this compound, but statements *a*, *d*, and *f* would be false.

48. *a.*
$$\begin{aligned} p\vee \sim q \to r\vee q &\equiv\ \sim(p\vee\sim q)\vee(r\vee q) && \textit{[an acceptable answer]}\\ &\equiv (\sim p\wedge\sim(\sim q))\vee(r\vee q) && \text{by De Morgan's law}\\ & && \textit{[another acceptable answer]}\\ &\equiv (\sim p\wedge q)\vee(r\vee q) && \text{by the double negative law}\\ & && \textit{[another acceptable answer]} \end{aligned}$$

b.
$$\begin{aligned} p\vee \sim q \to r\vee q &\equiv (\sim p\wedge q)\vee(r\vee q) && \text{by part (a)}\\ &\equiv\ \sim(\sim(\sim p\wedge q)\wedge\sim(r\vee q)) && \text{by De Morgan's law}\\ &\equiv\ \sim(\sim(\sim p\wedge q)\wedge(\sim r\wedge\sim q)) && \text{by De Morgan's law} \end{aligned}$$

The steps in the answer to part (b) would also be acceptable answers for part (a).

50. *a.*
$$\begin{aligned} (p\to(q\to r))\leftrightarrow((p\wedge q)\to r) &\equiv [\sim p\vee(q\to r)]\leftrightarrow[\sim(p\wedge q)\vee r]\\ &\equiv [\sim p\vee(\sim q\vee r)]\leftrightarrow[\sim(p\wedge q)\vee r]\\ &\equiv\ \sim[\sim p\vee(\sim q\vee r)]\vee[\sim(p\wedge q)\vee r]\\ &\qquad \wedge\sim[\sim(p\wedge q)\vee r]\vee[\sim p\vee(\sim q\vee r)] \end{aligned}$$

b. By part (a), De Morgan's law, and the double negative law,
$$\begin{aligned} (p\to(q\to r))\leftrightarrow((p\wedge q)\to r) &\equiv\ \sim[\sim p\vee(\sim q\vee r)]\vee[\sim(p\wedge q)\vee r]\\ &\qquad \wedge\sim[\sim(p\wedge q)\vee r]\vee[\sim p\vee(\sim q\vee r)]\\ &\equiv\ \sim[\sim p\vee(\sim q\vee r)]\wedge\sim[\sim(p\wedge q)\vee r]\\ &\qquad \wedge\sim\sim[(p\wedge q)\wedge\sim r]\wedge\sim[\sim p\vee(\sim q\vee r)]\\ &\equiv\ \sim\sim[p\wedge\sim(\sim q\vee r)]\wedge[(p\wedge q)\wedge\sim r]\\ &\qquad \wedge\sim\sim[(p\wedge q)\wedge\sim r]\wedge[p\wedge\sim(\sim q\vee r)]\\ &\equiv\ \sim\sim[p\wedge(q\wedge\sim r)]\wedge[(p\wedge q)\wedge\sim r]\\ &\qquad \wedge\sim\sim[(p\wedge q)\wedge\sim r]\wedge[p\wedge(q\wedge\sim r)]. \end{aligned}$$

The steps in the answer to part (b) would also be acceptable answers for part (a).

51. Yes. As in exercises 47-50, the following logical equivalences can be used to rewrite any statement form in a logically equivalent way using only $\sim$ and $\wedge$:

$$\begin{aligned} &p\to q\equiv\ \sim p\vee q && p\leftrightarrow q\equiv(\sim p\vee q)\wedge(\sim q\vee p)\\ &p\vee q\equiv\ \sim(\sim p\wedge\sim q) && \sim(\sim p)\equiv p \end{aligned}$$

The logical equivalence $p\wedge q\equiv\ \sim(\sim p\vee\sim q)$ can then be used to rewrite any statement form in a logically equivalent way using only $\sim$ and $\vee$.

Section 2.3

2. $1 - 0.99999...$ is less than every positive real number.

4. This figure is not a quadrilateral.

5. They did not telephone.

9.

						premises			*conclusion*	
p	q	r	$\sim q$	$\sim r$	$p\wedge q$	$p\wedge q\to\sim r$	$p\vee\sim q$	$\sim q\to p$	$\sim r$	
T	T	T	F	F	T	F	T	T		
T	T	F	F	T	T	T	T	T	T	*critical row*
T	F	T	T	F	F	T	T	T	F	*critical row*
T	F	F	T	T	F	T	T	T	T	*critical row*
F	T	T	F	F	F	T	F	T		
F	T	F	F	T	F	T	F	T		
F	F	T	T	F	F	T	T	F		
F	F	F	T	T	F	T	T	F		

Rows 2, 3, and 4 of the truth table are the critical rows in which all the premises are true, but row 3 shows that it is possible for an argument of this form to have true premises and a false conclusion. Hence the argument form is invalid.

10.

				premises		*conclusion*	
p	q	r	$p \vee \sim q$	$p \to r$	$q \to r$	$p \vee q \to r$	
T	T	T	T	T	T	T	← *critical row*
T	T	F	T	F	F		
T	F	T	T	T	T	T	← *critical row*
T	F	F	T	F	T		
F	T	T	T	T	T	T	← *critical row*
F	T	F	T	T	F		
F	F	T	F	T	T	T	← *critical row*
F	F	F	F	T	T	T	← *critical row*

Rows 1, 3, 5, 7, and 8 of the truth table represent the situations in which all the premises are true, and in all of these rows the conclusion is also true. Therefore, the argument form is valid.

11.

							premises		*conclusion*	
p	q	r	$\sim p$	$\sim q$	$\sim r$	$q \vee r$	$p \to q \vee r$	$\sim q \vee \sim r$	$\sim p \vee \sim r$	
T	T	T	F	F	F	T	T	F		
T	T	F	F	F	T	T	T	T	T	← *critical row*
T	F	T	F	T	F	T	T	T	F	← *critical row*
T	F	F	F	T	T	F	F	T		
F	T	T	T	F	F	T	T	F		
F	T	F	T	F	T	T	T	T	T	← *critical row*
F	F	T	T	T	F	T	T	T	T	← *critical row*
F	F	F	T	T	T	F	T	T	T	← *critical row*

Rows 2, 3, 6, 7, and 8 of the truth table represent the situations in which all the premises are true, but row 3 shows that it is possible for an argument of this form to have true premises and a false conclusion. Hence the argument form is invalid.

12. *b.*

		premises		*conclusion*	
p	q	$p \to q$	$\sim p$	$\sim q$	
T	T	T	F		
T	F	F	F		
F	T	T	T	F	← *critical row*
F	F	T	T	T	← *critical row*

Rows 3, and 4 of the truth table represent the situations in which all the premises are true, but row 3 shows that it is possible for an argument of this form to have true premises and a false conclusion. Hence the argument form is invalid.

13.

		premises		*conclusion*	
p	q	$p \to q$	$\sim q$	$\sim p$	
T	T	T	F		
T	F	F	T		
F	T	T	F		
F	F	T	T	T	← *critical row*

Row 4 of the truth table represents the only situation in which all the premises are true, and in this row the conclusion is also true. Therefore, the argument form (modus tollens) is valid.

15.

		premise	*conclusion*	
p	q	q	$p \vee q$	
T	T	T	T	*critical row*
T	F	F		
F	T	T	T	*critical row*
F	F	F		

The truth table shows that in the two situations (represented by rows 1 and 3) in which the premise is true, the conclusion is also true. Therefore, the the second version of generalization is valid.

16.

		premise	*conclusion*	
p	q	$p \wedge q$	p	
T	T	T	T	*critical row*
T	F	F		
F	T	F		
F	F	F		

The truth table shows that in the only situation (represented by row 1) in which both premises are true, the conclusion is also true. Therefore, the the first version of specialization is valid.

17.

		premise	*conclusion*	
p	q	$p \wedge q$	q	
T	T	T	T	*critical row*
T	F	F		
F	T	F		
F	F	F		

The truth table shows that in the only situation (represented by row 1) in which both premises are true, the conclusion is also true. Therefore, the second version of specialization is valid.

19.

		premises		*conclusion*	
p	q	$p \vee q$	$\sim p$	q	
T	T	T	F		
T	F	T	F		
F	T	T	T	T	*critical row*
F	F	F	T		

The truth table shows that in the only situation (represented by row 3) in which both premises are true, the conclusion is also true. Therefore, the the second version of elimination is valid.

20.

			premises		*conclusion*	
p	q	r	$p \to q$	$q \to r$	$p \to r$	
T	T	T	T	T	T	*critical row*
T	T	F	T	F		
T	F	T	F	T		
T	F	F	F	T		
F	T	T	T	T	T	*critical row*
F	T	F	T	F		
F	F	T	T	T	T	*critical row*
F	F	F	T	T	T	*critical row*

The truth table shows that in the four situations (represented by rows 1, 5, 7, and 8) in which both premises are true, the conclusion is also true. Therefore, the argument form (transitivity) is valid.

21.

			premises			*conclusion*	
p	q	r	$p \vee q$	$p \to r$	$q \to r$	r	
T	T	T	T	T	T	T	← *critical row*
T	T	F	T	F	F		
T	F	T	T	T	T	T	← *critical row*
T	F	F	T	F	T		
F	T	T	T	T	T	T	← *critical row*
F	T	F	T	T	F		
F	F	T	F	T	T		
F	F	F	F	T	T		

The truth table shows that in the three situations (represented by rows 1, 3, 5) in which all three premises are true, the conclusion is also true. Therefore, proof by division into cases is valid.

23. form: $p \vee q$
$p \to r$
$\therefore \quad q \vee \sim r$

				premises		*conclusion*	
p	q	r	$\sim r$	$p \vee q$	$p \to r$	$q \vee \sim r$	
T	T	T	F	T	T	T	← *critical row*
T	T	F	T	T	F		
T	F	T	F	T	T	F	← *critical row*
T	F	F	T	T	F		
F	T	T	F	T	T	T	← *critical row*
F	T	F	T	T	T	T	← *critical row*
F	F	T	F	F	T		
F	F	F	T	F	T		

Rows 1, 3, 5, and 6 represent the situations in which both premises are true, but in row 3 the conclusion is false. Hence, it is possible for an argument of this form to have true premises and a false conclusion, and so the given argument is invalid.

28. form: $p \to q$ invalid, converse error
q
$\therefore \quad p$

29. form: $p \to q$ invalid, inverse error
$\sim p$
$\therefore \quad \sim q$

30. form: $p \to q$ invalid, converse error
q
$\therefore \quad p$

31. form: $p \wedge q$ valid, generalization

$\therefore$ q

32. form: $p \to r$ valid, proof by division into cases

$q \to r$

$\therefore$ $p \vee q \to r$

33. A valid argument with a false conclusion must have at least one false premise. In the following example, the second premise is false. (The first premise is true because its hypothesis is false.)

If the square of every real number is positive, then no real number is negative.

The square of every real number is positive.

Therefore, no real number is negative.

34. An invalid argument with a true conclusion can have premises that are either true or false. In the following example the first premise is true for either one of following two reasons: its hypothesis is false and its conclusion is true.

If the square of every real number is positive, then some real numbers are positive.

Some real numbers are positive.

Therefore, the square of every real number is positive.

35. A correct answer should indicate that for a valid argument, any argument of the same form that has true premises has a true conclusion, whereas for an invalid argument, it is possible to find an argument of the same form that has true premises and a false conclusion. The validity of an argument does not depend on whether the conclusion is true or not. The validity of an argument only depends on the formal relationship between its premises and its conclusion.

38. *b.* 1. Suppose C is a knight.
2. $\therefore$ C is a knave (because what C said was true).
3. $\therefore$ C is both a knight and a knave (by (1) and (2)), which is a contradiction.
4. $\therefore$ C is not a knight (because by the contradiction rule the supposition is false).
5. $\therefore$ What C says is false (because since C is not a knight he is a knave and knaves always speak falsely).
6. $\therefore$ At least one of C or D is a knight (by De Morgan's law).
7. $\therefore$ D is a knight (by (4) and (6) and elimination).
8. $\therefore$ C is a knave and D is a knight (by (4) and (7)).

To check that the problem situation is not inherently contradictory, note that if C is a knave and D is a knight, then each could have spoken as reported.

c. There is one knave. E and F cannot both be knights because then both would also be knaves (since each would have spoken the truth), which is a contradiction. Nor can E and F both be knaves because then both would be telling the truth which is impossible for knaves. Hence, the only possible answer is that one is a knight and the other is a knave. But in this case both E and F could have spoken as reported, without contradiction.

d. The following is one of many solutions.
1. The statement made by U must be false because if it were true then U would not be a knight (since none would be a knight), but since he spoke the truth he would be a knight and this would be a contradiction.
2. $\therefore$ there is at least one knight, and U is a knave (since his statement that there are no knights is false).
3. Suppose Z spoke the truth. Then so did W (since if there is exactly one knight then it is also true that there are at most three knights). But this implies that there are at least two knights, which contradicts $Z's$ statement. Hence Z cannot have spoken the truth.

4. $\therefore$ there are at least two knights, and Z is a knave (since his statement that there is exactly one knight is false). Also $X's$ statement is false because since both U and Z are knaves it is impossible for there to be exactly five knights. Hence X also is a knave.
5. $\therefore$ there are at least three knaves (U, Z, and X), and so there are at most three knights.
6. $\therefore$ $W's$ statement is true, and so W is a knight.
7. Suppose V spoke the truth. Then V, W, and Y are all knights (otherwise there would not be at least three knights because U, Z, and X are known to be knaves). It follows that Y spoke the truth. But Y said that exactly two were knights. This contradicts the result that V, W, and Y are all knights.
8. $\therefore$ V cannot have spoken the truth, and so V is a knave.
9. $\therefore$ U, Z, X, and V are all knaves, and so there are at most two knights.
10. Suppose that Y is a knave. Then the only knight is W, which means that Z spoke the truth. But we have already seen that this is impossible. Hence Y is a knight.
11. By 6, 9, and 10, the only possible solution is that U, Z, X, and V are knaves and W and Y are knights. Examination of the statements shows that this solution is consistent: in this case, the statements of U, Z, X, and V are false and those of W and Y are true.

40. Suppose Socko is telling the truth. Then Fats is also telling the truth because if Lefty killed Sharky then Muscles didn't kill Sharky. Consequently, two of the men were telling the truth, which contradicts the fact that all were lying except one. Therefore, Socko is not telling the truth: Lefty did not kill Sharky. Hence Muscles is telling the truth and all the others are lying. It follows that Fats is lying, and so Muscles killed Sharky.

42. (1)

	$q \to r$	premise b
	$\sim r$	premise d
$\therefore$	$\sim q$	by modus tollens

(2)

	$p \vee q$	premise a
	$\sim q$	by (1)
$\therefore$	p	by elimination

(3)

	$\sim q \to u \wedge s$	premise e
	$\sim q$	by (1)
$\therefore$	$u \wedge s$	by modus ponens

(4)

	$u \wedge s$	by (3)
$\therefore$	s	by specialization

(5)

	p	by (2)
	s	by (4)
$\therefore$	$p \wedge s$	by conjunction

(6)

	$p \wedge s \to t$	premise c
	$p \wedge s$	by (5)
$\therefore$	t	by modus ponens

44. (1)
$\sim q \vee s$ premise d
$\sim s$ premise e
$\therefore \sim q$ by elimination

(2)
$p \rightarrow q$ premise a
$\sim q$ by (1)
$\therefore \sim p$ by modus tollens

(3)
$r \vee s$ premise b
$\sim s$ premise e
$\therefore r$ by elimination

(4)
$\sim p$ by (2)
r by (3)
$\therefore \sim p \wedge r$ by conjunction

(5)
$\sim p \wedge r \rightarrow u$ premise f
$\sim p \wedge r$ by (4)
$\therefore u$ by modus ponens

(6)
$\sim s \rightarrow \sim t$ premise c
$\sim s$ premise e
$\therefore \sim t$ by modus ponens

(7)
$w \vee t$ premise g
$\sim t$ by (6)
$\therefore w$ by elimination

(8)
u by (5)
w by (7)
$\therefore u \wedge w$ by conjunction

Section 2.4

2. $R = 1$

4. $S = 1$

6. The input/output table is as follows:

Input		**Output**
P	Q	R
1	1	0
1	0	1
0	1	0
0	0	0

8. The input/output table is as follows:

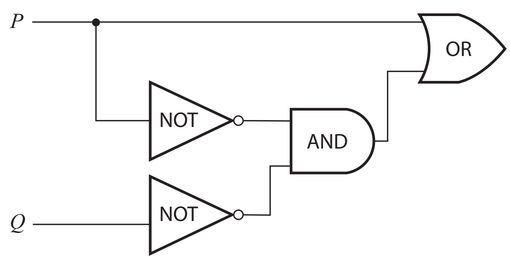
Input			Output
P	Q	R	S
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

10. $(P \vee Q) \wedge \sim Q$

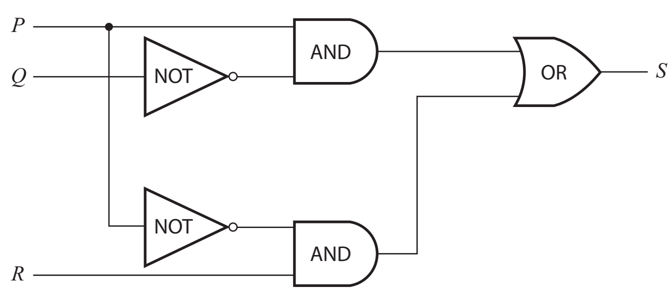
12. $(P \vee Q) \vee \sim (Q \wedge R)$

14.

15.

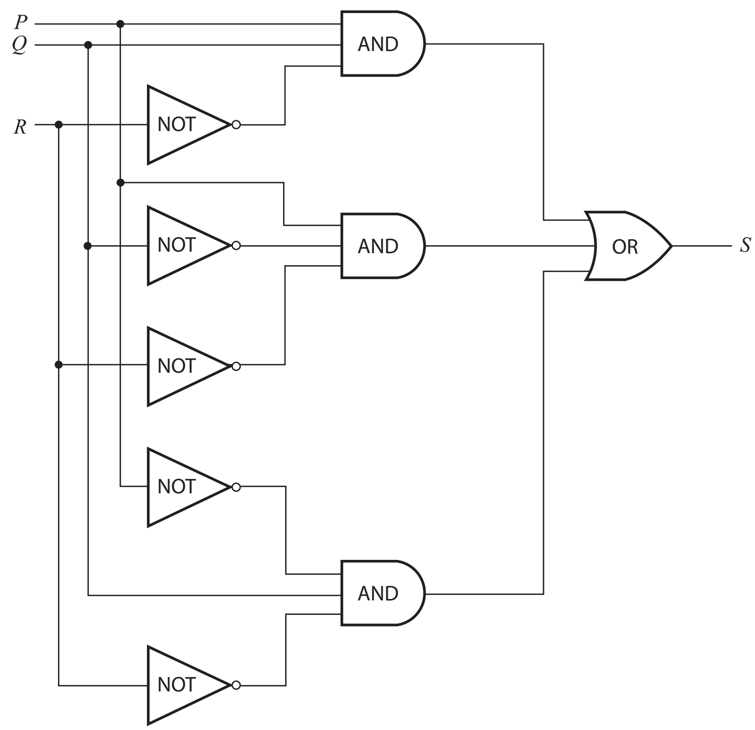


17.



19. *a*. $(P \wedge Q \wedge \sim R) \vee (P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge Q \wedge \sim R)$

b. One circuit (among many) having the given input/output table is the following:



21. *a*. $(P \wedge Q \wedge \sim R) \vee (\sim P \wedge Q \wedge R) \vee (\sim P \wedge Q \wedge \sim R)$

b. One circuit (among many) having the given input/output table is the following: