Discrete Mathematics with Applications, 4th Edition
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Test Bank Questions

Chapter 1

1. Fill in the blanks to rewrite the following statement with variables: Is there an integer with a remainder of 1 when it is divided by 4 and a remainder of 3 when it is divided by 7?

   (a) Is there an integer n such that n has ______?

   (b) Does there exist ______ such that if n is divided by 4 the remainder is 1 and if ______?

2. Fill in the blanks to rewrite the following statement with variables:

   Given any positive real number, there is a positive real number that is smaller.

   (a) Given any positive real number r, there is ______ s such that s is ______.

   (b) For any ______ , ______ such that $s < r$.

3. Rewrite the following statement less formally, without using variables:

   There is an integer n such that $1/n$ is also an integer.

4. Fill in the blanks to rewrite the following statement:

   For all objects T, if T is a triangle then T has three sides.

   (a) All triangles ______.

   (b) Every triangle ______ .

   (c) If an object is a triangle, then it ______.

   (d) If T ______ , then T ______.

   (e) For all triangles T, ______.

5. Fill in the blanks to rewrite the following statement:

   Every real number has an additive inverse.

   (a) All real numbers ______.

   (b) For any real number x, there is ______ for x.

   (c) For all real numbers x, there is real number y such that ______.

6. Fill in the blanks to rewrite the following statement:

   There is a positive integer that is less than or equal to every positive integer.

   (a) There is a positive integer m such that m is ______.

   (b) There is a ______ such that ______ every positive integer.

   (c) There is a positive integer m which satisfies the property that given any positive integer n, m is ______.

7. (a) Write in words how to read the following out loud $\{n \in \mathbf{Z} \mid n \text{ is a factor of } 9\}$.

   (b) Use the set-roster notation to indicate the elements in the set.

8. (a) Is $\{5\} \in \{1, 3, 5\}$?
   (b) Is $\{5\} \subseteq \{1, 3, 5\}$?
   (c) Is $\{5\} \in \{\{1\}, \{3\}, \{5\}\}$?
   (d) Is $\{5\} \subseteq \{\{1\}, \{3\}, \{5\}\}$?

9. Let $A = \{a, b, c\}$ and $B = \{u, v\}$. Write a. $A \times B$ and b. $B \times A$.

10. Let $A = \{3, 5, 7\}$ and $B = \{15, 16, 17, 18\}$, and define a relation R from A to B as follows: For all $(x, y) \in A \times B$,

$$(x, y) \in R \quad \Leftrightarrow \quad \frac{y}{x} \text{ is an integer.}$$

   (a) Is $3\ R\ 15$? Is $3\ R\ 16$? Is $(7, 17) \in R$? Is $(3, 18) \in R$?
   (b) Write R as a set of ordered pairs.
   (c) Write the domain and co-domain of R.
   (d) Draw an arrow diagram for R.
   (e) Is R a function from A to B? Explain.

11. Define a relation R from $\mathbf{R}$ to $\mathbf{R}$ as follows: For all $(x, y) \in \mathbf{R} \times \mathbf{R}$,$(x, y) \in R$ if, and only if, $x = y^2 + 1$.

   (a) Is $(2, 5) \in R$? Is $(5, 2) \in R$? Is $(-3)\ R\ 10$? Is $10\ R\ (-3)$?]
   (b) Draw the graph of R in the Cartesian plane.
   (c) Is R a function from A to B? Explain.

12. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Define a function $G\colon A \to B$ as follows:

$$G = \{(1, b), (2, c), (3, b), (4, c)\}.$$

   (a) Find $G(2)$.
   (b) Draw an arrow diagram for G.

13. Define functions F and G from $\mathbf{R}$ to $\mathbf{R}$ by the following formulas:

$$F(x) = (x + 1)(x - 3) \quad \text{and} \quad G(x) = (x - 2)^2 - 7.$$

   Does $F = G$? Explain.

Chapter 2

1. Which of the following is a negation for "Jim is inside and Jan is at the pool."

   (a) Jim is inside or Jan is not at the pool.
   (b) Jim is inside or Jan is at the pool.
   (c) Jim is not inside or Jan is at the pool.
   (d) Jim is not inside and Jan is not at the pool.
   (e) Jim is not inside or Jan is not at the pool.

2. Which of the following is a negation for "Jim has grown or Joan has shrunk."

   (a) Jim has grown or Joan has shrunk.
   (b) Jim has grown or Joan has not shrunk.
   (c) Jim has not grown or Joan has not shrunk.
   (d) Jim has grown and Joan has shrunk.
   (e) Jim has not grown and Joan has not shrunk.
   (f) Jim has grown and Joan has not shrunk.

3. Write a negation for each of the following statements:

   (a) The variable S is undeclared and the data are out of order.
   (b) The variable S is undeclared or the data are out of order.
   (c) If Al was with Bob on the first, then Al is innocent.
   (d) $-5 \leq x < 2$ (where x is a particular real number)

4. Are the following statement forms logically equivalent: $p \vee q \rightarrow p$ and $p \vee (\sim p \wedge q)$? Include a truth table and a few words explaining how the truth table supports your answer.

5. State precisely (but concisely) what it means for two statement forms to be logically equivalent.

6. Write the following two statements in symbolic form and determine whether they are logically equivalent. Include a truth table and a few words explaining how the truth table supports your answer.

   If Sam bought it at Crown Books, then Sam didn't pay full price.

   Sam bought it at Crown Books or Sam paid full price.

7. Write the following two statements in symbolic form and determine whether they are logically equivalent. Include a truth table and a few words explaining how the truth table supports your answer.

   If Sam is out of Schlitz, then Sam is out of beer.

   Sam is not out of beer or Sam is not out of Schlitz.

8. Write the converse, inverse, and contrapositive of "If Ann is Jan's mother, then Jose is Jan's cousin."

9. Write the converse, inverse, and contrapositive of "If Ed is Sue's father, then Liu is Sue's cousin."

10. Write the converse, inverse, and contrapositive of "If Al is Tom's cousin, then Jim is Tom's grandfather."

11. Rewrite the following statement in if-then form without using the word "necessary": Getting an answer of 10 for problem 16 is a necessary condition for solving problem 16 correctly.

12. State precisely (but concisely) what it means for a form of argument to be valid.

13. Consider the argument form:

$$\begin{array}{rl} & p \to \sim q \\ & q \to \sim p \\ \therefore & p \vee q \end{array}$$

Use the truth table below to determine whether this form of argument is valid or invalid. Include a truth table and a few words explaining how the truth table supports your answer.

p	q	$\sim p$	$\sim q$	$p \to \sim q$	$q \to \sim p$	$p \vee q$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	T	F

14. Consider the argument form:

$$\begin{array}{rl} & p \wedge \sim q \to r \\ & p \vee q \\ & q \to p \\ \text{Therefore} & r. \end{array}$$

Use the truth table below to determine whether this argument form is valid or invalid. Annotate the table (as appropriate) and include a few words explaining how the truth table supports your answer.

p	q	r	$\sim q$	$p \wedge \sim q$	$p \wedge \sim q \to r$	$p \vee q$	$q \to p$	r
T	T	T	F	F	T	T	T	T
T	T	F	F	F	T	T	T	F
T	F	T	T	T	T	T	T	T
T	F	F	T	T	F	T	T	F
F	T	T	F	F	T	T	F	T
F	T	F	F	F	T	T	F	F
F	F	T	T	F	T	F	T	T
F	F	F	T	F	T	F	T	F

15. Determine whether the following argument is valid or invalid. Include a truth table and a few words explaining why the truth table shows validity or invalidity.

If Hugo is a physics major or if Hugo is a math major, then he needs to take calculus.

Hugo needs to take calculus or Hugo is a math major.

Therefore, Hugo is a physics major or Hugo is a math major.

16. Determine whether the following argument is valid or invalid. Include a truth table and a few words explaining why the truth table shows validity or invalidity.

If 12 divides 709,438 then 3 divides 709,438.
If the sum of the digits of 709,438 is divisible by 9 then 3 divides 709,438.
The sum of the digits of 709,438 is not divisible by 9.
Therefore, 12 does not divide 709,438.

17. Write the form of the following argument. Is the argument valid or invalid? Justify your answer.

If 54,587 is a prime number, then 17 is not a divisor of 54,587.
17 is a divisor of 54,587.
Therefore, 54,587 is not a prime number.

18. Write the form of the following argument. Is the argument valid or invalid? Justify your answer.

    If Ann has the flu, then Ann has a fever.
    Ann has a fever.
    Therefore, Ann has the flu.

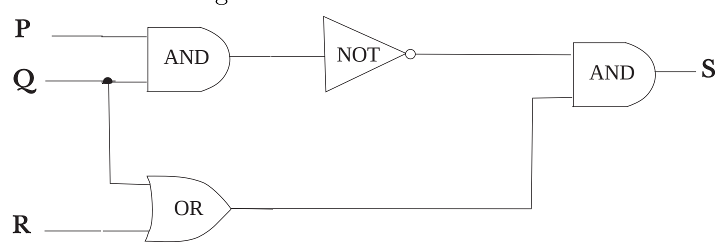
19. On the island of knights and knaves, you meet three natives, A, B, and C, who address you as follows:

    A: At least one of us is a knave.

    B: At most two of us are knaves.

    What are A, B, and C?

20. Consider the following circuit.



    (a) Find the output of the circuit corresponding to the input $P = 1,\ Q = 0$, and $R = 1$.

    (b) Write the Boolean expression corresponding to the circuit.

21. Write 110101_2 in decimal form.

22. Write 75 in binary notation.

23. Draw the circuit that corresponds to the following Boolean expression: $(P \wedge Q) \vee (\sim P \wedge \sim Q)$.(*Note for students who have studied some circuit design*: Do not simplify the circuit; just draw the one that exactly corresponds to the expression.)

24. Find a circuit with the following input/output table.

P	Q	R	S
1	1	1	0
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

25. Find $10111_2 + 1011_2$.

26. Write 100110_2 in decimal form.

27. Write the 8-bit two's complement for 49.

Chapter 3

1. Rewrite the following statement in the form ∀ ______ x, if ______ then ______ (where each of the second two blanks are sentences involving the variable x)

   Every valid argument with true premises has a true conclusion.

2. Consider the statement "The square of any odd integer is odd."

   (a) Rewrite the statement in the form ∀______ n, ______. (Do not use the words "if" or "then.")

   (b) Rewrite the statement in the form ∀______ n, if ______ then ______. (Make sure you use the variable n when you fill in each of the second two blanks.)

   (c) Write a negation for the statement.

3. Rewrite the following statement formally. Use variables and include both quantifiers ∀ and ∃ in your answer.

   Every rational number can be written as a ratio of some two integers.

4. Rewrite the following statement formally. Use variables and include both quantifiers ∀ and ∃ in your answer.

   Every even integer greater than 2 can be written as a sum of two prime numbers.

5. Which of the following is a negation for "Given any real numbers a and b, if a and b are rational then a/b is rational."

   (a) There exist real numbers a and b such that a and b are not rational and a/b is not rational.

   (b) Given any real numbers a and b, if a and b are not rational then a/b is not rational.

   (c) There exist real numbers a and b such that a and b are not rational and a/b is rational.

   (d) Given any real numbers a and b, if a and b are rational then a/b is not rational.

   (e) There exist real numbers a and b such that a and b are rational and a/b is not rational.

   (f) Given any real numbers a and b, if a and b are not rational then a/b is rational.

6. Which of the following is a negation for "For all real numbers r, there exists a number s such that $rs > 10$."

   (a) There exists a real number r such that for all real numbers s, $rs \not> 10$.

   (b) For all real numbers r, there does not exist a number s such that $rs > 10$.

   (c) There exists real numbers r and s such that $rs \not> 10$.

   (d) For all real numbers r and s, $rs \not> 10$.

   (e) There exists a real number r and there does not exist a real number s such that $rs \not> 10$.

   (f) For all real numbers r, there exists a number s such that $rs \not> 10$.

   (g) There exists a real number r such that there does not exist a real number s with $rs \neq 1$.

7. Which of the following is a negation for "There exists a real number x such that for all real numbers y, $xy > y$."
   (a) There exists a real number x such that for all real numbers y, $xy \leq y$.
   (b) There exists a real number y such that for all real numbers x, $xy \leq y$.
   (c) There exist real numbers x and y such that $xy \leq y$.
   (d) For all real numbers x there exists a real number y such that $xy \leq y$.
   (e) For all real numbers y there exists a real number x such that $xy \leq y$.
   (f) For all real numbers x and y, $xy \leq y$.

8. Which of the following is a negation for "For any integer n, if n is composite, then n is even or $n > 2$."
   (a) For any integer n, if n is composite, then n is not even or $n \leq 2$.
   (b) For any integer n, if n is not composite, then n is not even or $n \leq 2$.
   (c) For any integer n, if n is not composite, then n is not even and $n \leq 2$.
   (d) For any integer n, if n is not composite, then n is even and $n \leq 2$.
   (e) For any integer n, if n is not composite, then n is not even and $n \leq 2$.
   (f) There exists an integer n such that if n is composite, then n is not even and $n \leq 2$.
   (g) There exists an integer n such that n is composite and n is not even and $n \leq 2$.
   (h) There exists an integer n such that if n is not composite, then n is not even and $n \leq 2$.
   (i) There exists an integer n such that n is composite and n is even and $n \leq 2$.
   (j) There exists an integer n such that if n is not composite, then n is not even or $n \leq 2$.

9. Write negations for each of the following statements:
   (a) For all integers n, if n is prime then n is odd.
   (b) $\forall$ real numbers x, if $x < 1$ then $\dfrac{1}{x} > 1$.
   (c) For all integers a and b, if a^2 divides b^2 then a divides b.
   (d) $\forall$ real numbers x, if $x(x-2) > 0$ then $x > 2$ or $x < 0$.
   (e) $\forall$ real numbers x, if $x(x-2) \leq 0$ then $0 \leq x \leq 2$.
   (f) For all real numbers x and y with $x < y$, there exists an integer n such that $x \leq n \leq y$.

10. Let T be the statement

$$\forall \text{ real numbers } x, \text{ if } -1 < x \leq 0 \text{ then } x+1 > 0.$$

   (a) Write the converse of T.
   (b) Write the contrapositive of T.

11. Rewrite the following statement in if-then form without using the word "only": A graph with n vertices is a tree only if it has $n-1$ edges.

12. Are the following two statements logically equivalent? Justify your answer.
   (a) A real number is less than 1 only if its reciprocal is greater than 1.
   (b) Having a reciprocal greater than 1 is a sufficient condition for a real number to be less than 1.

13. For each of the following statements, (1) write the statement informally without using variables or the symbols $\forall$ or $\exists$, and (2) indicate whether the statement is true or false and briefly justify your answer.

    (a) $\forall$ integers a, $\exists$ an integer b such that $a + b = 0$.

    (b) $\exists$ an integer a such that $\forall$ integers b, $a + b = 0$.

14. For each of the following statements, (1) write the statement informally without using variables or the symbols $\forall$ or $\exists$, and (2) indicate whether the statement is true or false and briefly justify your answer.

    (a) $\forall$ real numbers x, $\exists$ a real number y such that $x < y$.

    (b) $\exists$ a real number y such that $\forall$ real numbers x, $x < y$.

15. Is the following argument valid or invalid? Justify your answer.

    All real numbers have nonnegative squares.
    The number i has a negative square.
    Therefore, the number i is not a real number.

16. Is the following argument valid or invalid? Justify your answer.

    All prime numbers greater than 2 are odd.
    The number a is not prime.
    Therefore, the number a is not odd.

Chapter 4

1. State precisely (but concisely) what it means for an integer n to be odd.

2. Find a counterexample to show that the following statement is false:

   For all nonzero real numbers a, b, c and d, $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{a+c}{b+d}$.

3. Consider the following statement:

   Statement A: $\forall$ integers m and n, if $2m + n$ is odd then m and n are both odd.

   (a) Write a negation for Statement A.

   (b) Disprove Statement A. That is, show that Statement A is false.

4. If m and n are integers, is $6m^2 + 34n - 18$ an even integer? Justify your answer.

5. Show that the following statement is false: The product of any two irrational numbers is irrational.

6. State precisely (but concisely) what it means for a number r to be rational.

7. Is 605.83 a rational number? Justify your answer.

8. Is 56.745 a rational number? Justify your answer.

9. State precisely (but concisely) what it means for an integer n to be divisible by an integer d.

10. Is 0 divisible by 3? Justify your answer.

11. Does 12 divide 72? Justify your answer.

12. Outline a proof of the following statement by writing the "starting point" and the "conclusion to be shown" in a proof of the statement.

$$\forall \text{ real numbers } r \text{ and } s, \text{ if } r \text{ and } s \text{ are rational then } r - s \text{ is rational.}$$

That is, complete the sentences below.

Proof: Suppose ______.

We must show that ______.

13. Prove the following statement directly from the definitions of the terms. Do not use any other facts previously proved in class or in the text or in the exercises.

$$\text{For all rational numbers } r, \text{ and } s, \text{if } s \neq 0, \text{ then } \frac{2r}{5s} \text{ is a rational number.}$$

14. Prove the following statement directly from the definitions of the terms. Do not use any other facts previously proved in class or in the text or in the exercises.

$$\text{For all integers } a, \ b, \text{ and } c, \text{ if } a \mid b \text{ and } a \mid c, \text{ then } a \mid (5b + 3c).$$

15. Prove the following statement directly from the definition of divisibility. Do not use any other facts previously proved in class or in the text or in the exercises.For all integers a, and b, if a divides b then a^3 divides b^3.

16. Prove the statement below directly from the definitions of the terms. Do not use any other facts previously proved in class or in the text or in the exercises.

$$\text{For all integers } n, \ n^2 + n + 1 \text{ is odd.}$$

17. Prove the following statement: The sum of any two consecutive integers can be written in the form $4n + 1$ for some integer n.

18. Prove the following statement: For all real numbers x, $\lfloor x - 2 \rfloor = \lfloor x \rfloor - 2$.

19. Prove the following statement: There is no smallest positive rational number.

20. Prove the following statement by contradiction: For all real numbers r and s, if r is rational and s is irrational, then $r + 2s$ is irrational.

21. Consider the following statement: For all integers n, if n^3 is even then n is even.

    (a) Prove the statement either by contradiction or by contraposition. Clearly indicate which method you are using.

    (b) If you used proof by contradiction in part (a), write what you would "suppose" and what you would "show" to prove the statement by contraposition. If you used proof by contraposition. in part (a), write what you would "suppose" and what you would "show" to prove the statement by contradiction.

22. Consider the following statement: For all real numbers r, if r^3 is irrational then r is irrational.
    (a) Prove the statement either by contradiction or by contraposition. Clearly indicate which method you are using.
    (b) If you used proof by contradiction in part (a), write what you would "suppose" and what you would "show" to prove the statement by contraposition. If you used proof by contraposition in part (a), write what you would "suppose" and what you would "show" to prove the statement by contradiction.

23. Consider the following statement: For all integers n, if n^3 is odd then n is odd.
    (a) Prove the statement either by contradiction or by contraposition. Clearly indicate which method you are using.
    (b) If you used proof by contradiction in part (a), write what you would "suppose" and what you would "show" to prove the statement by contraposition. If you used proof by contraposition in part (a), write what you would "suppose" and what you would "show" to prove the statement by contradiction.

24. True or false? For any irrational number r, r^2 is irrational. Justify your answer.

25. Fill in the blanks of the following proof by contradiction that $7+4\sqrt{2}$ is an irrational number. (You may use the fact that $\sqrt{2}$ is irrational.)

    Proof: Suppose not. That is, suppose that $7+4\sqrt{2}$ is ______. By definition of rational, $7+4\sqrt{2}=\dfrac{a}{b}$, where______. Multiplying both sides by b gives

    $$7b+4b\sqrt{2}=a,$$

    so if we subtract $7b$ from both sides we have

    $$4b\sqrt{2}= \_\_\_\_\_\_.$$

    Dividing both sides by $4b$ gives

    $$\sqrt{2}= \_\_\_\_\_\_.$$

    But then $\sqrt{2}$ would be a rational number because ______. This contradicts our knowledge that $\sqrt{2}$ is irrational. Hence ______.

26. Prove by contradiction that $4+3\sqrt{2}$ is an irrational number. (You may use the fact that $\sqrt{2}$ is irrational.)

27. Use the Euclidean algorithm to find the greatest common divisor of 284 and 168. Show your work.

28. Use the Euclidean algorithm to calculate the greatest common divisor of 10,673 and 11,284. Show your work.

Chapter 5

1. Compute $\sum_{k=0}^{3}\dfrac{1}{2^k}$.

2. Compute $\sum_{k=1}^{4}k^2$.

3. Use summation notation to rewrite the following: $1^3-2^3+3^3-4^3+5^3$.

4. Use a summation symbol to rewrite the following: $1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\dfrac{1}{5}-\dfrac{1}{6}$

5. Transform the following summation by making the change of variable $j = k + 1$:

$$\sum_{k=1}^{n} \frac{k^2}{n}.$$

6. Transform the following summation by making the change of variable $i = k + 1$:

$$\sum_{k=0}^{n} \frac{k^2}{k+n}.$$

7. Use repeated division by 2 to find the binary representation of the number 103_2. Show your work.

8. Use the formula

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

(for all real numbers $r \neq 1$ and for all integers $n \geq 0$) to find

$$2 + 2^2 + 2^3 + \cdots + 2^m$$

where m is an integer that is at least 1.

9. For each integer $n \geq 3$, let $P(n)$ be the equation

$$\boxed{3 + 4 + 5 + \cdots + n = \frac{(n-2)(n+3)}{2}.} \leftarrow P(n)$$

(Recall that by definition $3 + 4 + 5 + \cdots + n = \sum_{i=3}^{n} i.$)

(a) Is $P(3)$ true? Justify your answer.

(b) In the inductive step of a proof that $P(n)$ is true for all integers $n \geq 3$, we suppose $P(k)$ is true (this is the inductive hypothesis), and then we show that $P(k+1)$ is true. Fill in the blanks below to write what we suppose and what we must show for this particular equation.

Proof that for all integers $k \geq 3$, if the equation is true for $n = k$ then it is true for $n = k + 1$:

Let k be any integer that is greater than or equal to 3, and suppose that ______.
We must show that ______.

(c) Finish the proof started in (b) above.

10. For each integer $n \geq 3$, let $P(n)$ be the equation

$$\boxed{2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \frac{(n-2)(n^2 + 2n + 3)}{3}.} \leftarrow P(n)$$

(Recall that by definition $2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \sum_{i=3}^{n} (i-1) \cdot i.$)

(a) Is $P(3)$ true? Justify your answer.

(b) In the inductive step of a proof that $P(n)$ is true for all integers $n \geq 3$, we suppose $P(k)$ is true (this is the inductive hypothesis), and then we show that $P(k+1)$ is true. Fill in the blanks below to write what we suppose and what we must show for this particular equation.

Proof that for all integers $k \geq 3$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer that is greater than or equal to 3, and suppose that ______.
We must show that ______.

(c) Finish the proof started in (b) above.

11. For each integer $n \geq 0$, let $P(n)$ be the equation

$$\boxed{1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}.} \leftarrow P(n)$$

(Recall that by definition $1 + 3 + 3^2 + \cdots + 3^n = \sum_{i=0}^{n} 3^i$.)

(a) Is $P(0)$ true? Justify your answer.

(b) In the inductive step of a proof that $P(n)$ is true for all integers $n \geq 3$, we suppose $P(k)$ is true (this is the inductive hypothesis), and then we show that $P(k+1)$ is true. Fill in the blanks below to write what we suppose and what we must show for this particular equation.

Proof that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer that is greater than or equal to 0, and suppose that ______.
We must show that ______.

(c) Finish the proof started in (b) above.

12. Use mathematical induction to prove that for all integers $n \geq 1$,

$$4 + 8 + 12 + \cdots + 4n = 2n^2 + 2n.$$

13. Use mathematical induction to prove that for all integers $n \geq 3$,

$$3 + 4 + 5 + \cdots + n = \frac{(n-2)(n+3)}{2}.$$

14. Use mathematical induction to prove that for all integers $n \geq 3$,

$$2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \frac{(n-2)(n^2 + 2n + 3)}{3}.$$

15. Use mathematical induction to prove that for all integers $n \geq 0$,

$$1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}.$$

16. Use mathematical induction to prove that for all integers $n \geq 0$, $8^n - 1$ is divisible by 7.

17. Use mathematical induction to prove that for all integers $n \geq 5$, $1 + 4n < 2^n$.

18. Use strong mathematical induction to prove that for all integers $n \geq 2$, either n is prime or n is a product of prime numbers.

19. A sequence $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

$$a_0 = 2, \quad a_1 = 9$$
$$a_k = 5a_{k-1} - 6a_{k-2} \text{ for all integers } k \geq 2.$$

Use strong mathematical induction to prove that for all integers $n \geq 0$, $a_n = 5 \cdot 3^n - 3 \cdot 2^n$.

20. A sequence $s_1, s_2, s_3, \ldots$ is defined recursively as follows:

$$\begin{aligned} s_k &= 5s_{k-1} + (s_{k-2})^2 \quad \text{for all integers } k \geq 3 \\ s_1 &= 4 \\ s_2 &= 8 \end{aligned}$$

Use (strong) mathematical induction to prove that s_n is divisible by 4 for all integers $n \geq 1$.

21. The following **while** loop is annotated with a pre- and post-condition and also a loop invariant. Use the loop invariant theorem to prove the correctness of the loop with respect to the pre- and post-conditions.

[*Pre-condition: product* $=$ *A*[*1*] *and* $i = 1$]

while $(i \neq m)$

1. $i := i + 1$
2. *product* $:=$ *product*$\cdot$ $A[i]$

end while

[*Post-condition: product* $= A[1] \cdot A[2] \cdots A[m]$]

loop invariant: $I(n)$ is "$i = n + 1$ and *product* $:= A[1] \cdot A[2] \cdots A[n+1]$"

22. In a *Double Tower of Hanoi with Adjacency Requirement* there are three poles in a row and $2n$ disks, two of each of n different sizes, where n is any positive integer. Initially pole A (at one end of the row) contains all the disks, placed on top of each other in *pairs* of decreasing size. Disks may only be transferred one-by-one from one pole to an *adjacent* pole and at no time may a larger disk be placed on top of a smaller one. However a disk may be placed on top of another one of the same size. Let C be the pole at the other end of the row and let

$$s_n = \begin{bmatrix} \text{the minimum number of moves} \\ \text{needed to transfer a tower of } 2n \\ \text{disks from pole } A \text{ to pole } C \end{bmatrix}.$$

(a) Find s_1 and s_2.

(b) Find a recurrence relation expressing s_k in terms of s_{k-1} for all integers $k \geq 2$. Justify your answer carefully.

23. In a *Triple Tower of Hanoi*, there are three poles in a row and $3n$ disks, *three* of each of n different sizes, where n is any positive integer. Initially, one of the poles contains all the disks placed on top of each other in triples of decreasing size. Disks are transferred one by one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, a disk may be placed on top of one of the same size. Let t_n be the minimum number of moves needed to transfer a tower of $3n$ disks from one pole to another. Find a recurrence relation for $t_1, t_2, t_3, \ldots$. Justify your answer carefully.

24. A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions: (a) Rabbit pairs are not fertile during their first *two* months of life, but thereafter they give birth to *four* new male/female pairs at the end of every month; (b) No deaths occur. Let $s_n =$ the number of pairs of rabbits alive at the end of month n, for each integer $n \geq 1$, and let $s_0 = 1$. Find a recurrence relation for $s_0, s_1, s_{2,\ldots}$.Justify your answer carefully.

25. Suppose a certain amount of money is deposited into an account paying 4% annual interest, compounded quarterly. For each positive integer n, let $S_n =$ the amount on deposit at the end of the nth quarter, and let S_0 be the initial amount deposited.

    (a) Find a recurrence relation for $S_0, S_1, S_2, \ldots$, assuming no additional deposits or withdrawals for a 4-year period.

    (b) If $S_0 = \$5000$, find the amount of money on deposit at the end of 4 years.

    (c) Find the APR for the account.

26. A sequence $a_1, a_2, a_3, \ldots$ is defined as follows:

$$a_1 = 3, \qquad \text{and} \qquad a_k = 4a_{k-1} + 2 \quad \text{for all integers } k \geq 2.$$

    (a) Find a_1, a_2, and a_3.

    (b) Supposing that $a_5 = 4^4 \cdot 3 + 4^3 \cdot 2 + 4^2 \cdot 2 + 4 \cdot 2 + 2$, find a similar numerical expression for a_6 by substituting the right-hand side of this equation in place of a_5 in the equation

$$a_6 = 4 \cdot a_5 + 2.$$

    (c) Guess an explicit formula for a_n.

27. A sequence $c_0, c_1, c_2, \ldots$ is defined as follows:

$$c_0 = 1 \qquad \text{and} \qquad c_k = 7c_{k-1} + 2 \quad \text{for each integer } k \geq 1.$$

    (a) Find c_1 and c_2.

    (b) Use one of the reference formulas given at the end of this exam to simplify the expression

$$7^n + 2 \cdot 7^{n-1} + \cdots + 2 \cdot 7^2 + 2 \cdot 7 + 2.$$

    (c) Use iteration to guess an explicit formula for the sequence $c_0, c_1, c_2, \ldots$.

28. Use iteration to find an explicit formula for the sequence $b_0, b_1, b_2, \ldots$ defined recursively as follows:

$$\begin{aligned} b_k &= 2b_{k-1} + 3 \quad \text{for all integers } k \geq 1 \\ b_0 &= 1. \end{aligned}$$

    If appropriate, simplify your answer using one of the following reference formulas:

    (a) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ for all integers $n \geq 1$.

    (b) $1 + r + r^2 + \cdots + r^m = \dfrac{r^{m+1} - 1}{r - 1}$ for all integers $m \geq 0$ and all real numbers $r \neq 1$.

29. A sequence is defined recursively as follows:

$$a_0 = 2 \qquad \text{and} \qquad a_k = 4a_{k-1} + 1 \quad \text{for all } k \geq 1.$$

    It is proposed that an explicit formula for this sequence is

$$a_n = \frac{7 \cdot 4^n - 1}{3}.$$

    Use mathematical induction to check whether this proposed formula is correct.

30. A sequence is defined recursively as follows:

$$\begin{aligned} s_k &= 5s_{k-1}+1 \quad \text{for all integers } k \geq 1 \\ s_0 &= 1. \end{aligned}$$

Use mathematical induction to verify that this sequence satisfies the explicit formula

$$s_n = \frac{5^{n+1}-1}{4} \qquad \text{for all integers } n \geq 0.$$

31. Use the recursive definition of summation together with mathematical induction to prove that for all positive integers n, if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$\sum_{k=1}^{n}(2a_k - 3b_k) = 2\sum_{k=1}^{n} a_k - 3\sum_{k=1}^{n} b_k.$$

32. A sequence $a_0, a_1, a_2, \ldots$ satisfies the recurrence relation $a_k = 4a_{k-1} - 3a_{k-2}$ with initial conditions $a_0 = 1$ and $a_1 = 2$. Find an explicit formula for the sequence.

33. A sequence $b_1, b_2, b_3, \ldots$ satisfies the recurrence relation $b_k = 2b_{k-1} + 8b_{k-2}$ with initial conditions $b_1 = 1$ and $b_2 = 0$. Find an explicit formula for the sequence.

34. A sequence $c_0, c_1, c_2, \ldots$ satisfies the recurrence relation $c_k = 6c_{k-1} - 9c_{k-2}$ with initial conditions $c_0 = 1$ and $c_1 = 6$. Find an explicit formula for the sequence.

35. A sequence $d_1, d_2, d_3, \ldots$ satisfies the recurrence relation $d_k = 8d_{k-1} - 16d_{k-2}$ with initial conditions $d_1 = 0$ and $d_2 = 1$. Find an explicit formula for the sequence.

36. Define a set S recursively as follows:

    I. BASIS: $11 \in S$

    II. RECURSION:

    a. If $s \in S$, then $0s \in S$ and $s0 \in S$

    b. If x is any string (including the null string) such that $1x1 \in S$, then $10x1 \in S$ and $1x01 \in S$

    III. RESTRICTION: No strings other than those derived from I and II are in S.

    a. Is $00100010 \in S$? b. Is $011011 \in S$?

37. Define a set S recursively as follows:

    I. BASIS: $\epsilon \in S$

    II. RECURSION: If s and t are in S, then

    a. $0s \in S$ b. $s0 \in S$ c. $1s1t \in S$ d. $s1t1 \in S$

    III. RESTRICTION: No strings other than those derived from I and II are in S.

    Use structural induction to prove that every string in S contains an even number of 1's.

Chapter 6

1. Let A and B be sets. Define precisely (but concisely) what it means for A to be a subset of B.

2. Write a negation for the following statement:

   For all x, if $x \in A \cap B$ then $x \in B$.

3. Fill in the blanks in the following sentence: If A, B and C are any sets, then by definition of set difference $x \in A - (B \cap C)$ if, and only if, x ______ and x ______.

4. (a) Is $2 \subseteq \{2, 4, 6\}$?
   (b) Is $\{3\} \in \{1, 3, 5\}$?

5. If $X = \{u, v\}$, what is the power set of X?

6. Fill in the blanks:

   (a) Given sets A and B, to prove that $(A - B) \cup (A \cap B) \subseteq A$, we suppose that $x \in$ ______ and we must show that $x \in$ ______.

   (b) By definition of union, to say that $x \in (A - B) \cup (A \cap B)$ means that ______.

7. Define sets A and B as follows: $A = \{n \in \mathbf{Z} \mid n = 8r - 3 \text{ for some integer } r\}$ and $B = \{m \in \mathbf{Z} \mid m = 4s + 1 \text{ for some integer } s\}$.

   (a) Is $A \subseteq B$?
   (b) Is $B \subseteq A$?

   Justify your answers carefully. (In other words, provide a proof if the statement is true or a disproof if the statement is false.)

8. Let $X = \{l \in \mathrm{Z} \mid l = 5a + 2 \text{ for some integer } a\}$, $Y = \{m \in \mathrm{Z} \mid m = 4b + 3 \text{ for some integer } b\}$, and $Z = \{n \in \mathrm{Z} \mid n = 4c - 1 \text{ for some integer } c\}$.

   (a) Is $X \subseteq Y$?
   (b) Is $Y \subseteq Z$?

   Justify your answers carefully. (In other words, provide a proof if the statement is true or a disproof if the statement is false.)

9. The following is an outline of a proof that $(A \cup B)^c \subseteq A^c \cap B^c$. Fill in the blanks.

   Proof: Given sets A and B, to prove that $(A \cup B)^c \subseteq A^c \cap B^c$, we suppose $x \in \underline{\quad (a) \quad}$ and then we show that $x \in \underline{\quad (b) \quad}$. So suppose that $\underline{\quad (c) \quad}$. Then by definition of complement, $\underline{\quad (d) \quad}$. So by definition of union, it is not the case that (x is in A or x is in B). Consequently, x is not in A $\underline{\quad (e) \quad}$ x is not in B because of De Morgan's law of logic. In symbols, this says that $x \notin A$ and $x \notin B$. So by definition of complement, $x \in \underline{\quad (f) \quad}$ and $x \in \underline{\quad (g) \quad}$. Thus, by definition of intersection, $x \in \underline{\quad (h) \quad}$. *[as was to be shown]*.

10. Prove the following statement using an element argument and reasoning directly from the definitions of union, intersection, set difference.

    For all sets A, B, and C, $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

11. Disprove the following statement by finding a counterexample.

    For all sets A, B, and C, $A \cup (B \cap C) \subseteq (A \cup B) \cap C$.

12. Consider the statement

    For all sets A and B, $(A - B) \cap B = \emptyset$.

    The proof below is the beginning of a proof using the element method for prove that a set equals the empty set. Complete the proof without using any of the set properties from Theorem 6.2.2.

    Proof: Suppose the given statement is false. Then there exist sets A and B such that $(A - B) \cap B \neq \emptyset$. Thus there is an element x in $(A - B) \cap B$. By definition of intersection,. . . .

13. Consider the statement

$$\text{For all sets } A \text{ and } B,\ (A - B) \cap B = \emptyset.$$

Complete the proof begun below in which the given statement is derived algebraically from the properties listed in Theorem 6.2.2. Be sure to give a reason for every step that exactly justifies what was done in the step:

<u>Proof</u>:

Let A and B be any sets. Then the left-hand side of the equation to be shown is

$$\begin{aligned}
(A - B) \cap B &= (A \cap B^c) \cap B && \text{by the ______ law} \\
&= ______ && \text{by the ______ law} \\
&= ______ && \text{by the ______ law} \\
&= ______ && \text{by the ______ law} \\
&= ______ && \text{by the ______ law}
\end{aligned}$$

which is the right-hand side of the equation to be shown. *[Hence the given statement is true.]*

(The number of lines in the outline shown above are just meant to be suggestive. To complete the proof you may need more lines or you may be able to do it with fewer lines. Use however many lines as you need.)

14. (a) Prove the following statement using the element method for prove that a set equals the empty set: For all sets A and B, $A \cap (B - A) = \emptyset$.

    (b) Use the properties in Theorem 6.2.2 to prove the statement in part (a). Be sure to give a reason for every step.

15. Derive the following result "algebraically" using the properties listed in Theorem 6.2.2. Give a reason for every step.

$$\text{For all sets } A,\ B, \text{ and } C,\ (A \cup C) - B = (A - B) \cup (C - B).$$

16. Derive the following result. You may do so either "algebraically" using the properties listed in Theorem 6.2.2, being sure to give a reason for every step, or you may use the element method for proving a set equals the empty set.

$$\text{For all sets } B \text{ and } C,\ (B - C) - B = \emptyset.$$

17. Use the element method for proving a set equals the empty set to prove that

$$\text{For all sets } A \text{ and } C,\ (A - C) \cap (C - A) = \emptyset.$$

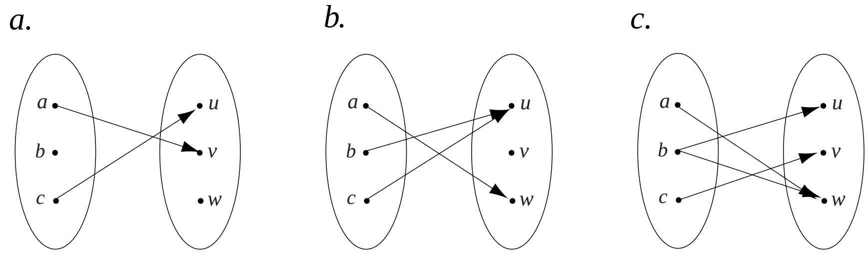
18. Prove that for all sets A and $B_1, B_2, \ldots B_n$,

$$A - \bigcap_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (A - B_i).$$

19. Is the following sentence a statement: This sentence is false or $-2^2 = 4$. Justify your answer.

Chapter 7

1. Let $X = \{a, b, c\}$ and $Y = \{u, v\}$. Which of the following arrow diagrams define functions from X to Y?



2. Fill in the blanks: $\log_3(\frac{1}{9}) =$ ______ because ______.

3. Is $\log_2 5 = \log_{16} 625$? Why or why not?

4. Let $J_5 = \{0, 1, 2, 3, 4\}$ and define a function $g\colon J_5 \times J_5 \to J_5 \times J_5$ as follows: For all $(a, b) \in J_5 \times J_5$,
$$g(a, b) = ((5a - 3)\ mod\ 5, (4b + 2)\ mod\ 5).$$
Find $g(3, 4)$.

5. Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{u, v, w, x, y\}$, and define $h\colon X \to Y$ as follows:
$$h(1) = v,\ h(2) = x,\ h(3) = v,\ h(4) = v,\ h(5) = y.$$
   (a) Draw an arrow diagram for h.
   (b) Let $A = \{1, 2\}$, $C = \{x, v\}$, $D = \{w\}$, and $E = \{w, y\}$. Find
$$h(A),\ h(X),\ h^{-1}(C),\ h^{-1}(D),\ h^{-1}(E), \text{ and } h^{-1}(Y).$$

6. Let f be a function from a set X to a set Y. Define precisely (but concisely) what it means for f to be one-to-one.

7. Let f be a function from a set X to a set Y. Define precisely (but concisely) what it means for f to be onto.

8. Let $A = B = \{1, 2, 3\}$, and consider the function $f\colon A \to B$ defined as follows: $f(1) = 3$, $f(2) = 1$, $f(3) = 3$. Is f onto? Why or why not?

9. (a) Draw an arrow diagram for a function that is onto but not one-to-one.
   (b) Draw an arrow diagram for a function that is one-to-one but not onto.

10. Define a function $f\colon \mathbf{R} - \{0\} \longrightarrow \mathbf{R}$ by the formula $f(x) = \dfrac{x+3}{x}$ for all nonzero real numbers x. Prove that f is one-to-one.

11. Let S be the set of all strings in 0's and 1's, and define a function $g\colon S \longrightarrow \mathbf{Z}$ as follows: for each string s in S,
$$g(s) = \text{the number of 1's in } s \text{ minus the number of 0's in } s.$$
   (a) What is $g(101011)$? $g(00100)$?
   (b) Is g one-to-one? Prove or give a counterexample.
   (c) Is g onto? Prove or give a counterexample.

12. Let S be the set of all strings in 0's and 1's, and define a function g: $S \longrightarrow \mathbf{Z}$ as follows: for each string s in S,

$$g(s) = \text{the number of 0's in } s.$$

   (a) What is $g(101011)$? $g(00100)$?
   (b) Is g one-to-one? Prove or give a counterexample.
   (c) Is g onto? Prove or give a counterexample.

13. Let S be the set of all strings in 0's and 1's, and define a function $F : S \rightarrow \mathbf{Z}^{nonneg}$ as follows: for all strings s in S,

$$F(s) = \text{the number of 1's in } s.$$

   (a) What is $F(001000)$? $F(111001)$? $F(10101)$? $F(0100)$?
   (b) Is F one-to-one? Prove or give a counterexample.
   (c) Is F onto? Prove or give a counterexample.
   (d) Is F a one-to-one correspondence? If so, find F^{-1}.

14. Define F: $\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \times \mathbf{R}$ as follows: $F(x, y) = (3y - 1, 1 - x)$ for all (x, y) in $\mathbf{R} \times \mathbf{R}$.

   (a) $F(0, 0) = ?$ $F(1, 4) = ?$
   (b) Is F one-to-one? Prove or give a counterexample.
   (c) Is F onto? Prove or give a counterexample.
   (d) Is F a one-to-one correspondence? If not, explain why not. If yes, find F^{-1}.

15. Let S be the set of all nonzero real numbers. Define a function g from S to S by the formula $g(x) = \dfrac{1}{x}$, for all nonzero real numbers x.

   (a) Show that g is a one-to-one correspondence from S to S.
   (b) Find g^{-1}.

16. Let S be the set of all even integers, and define a function f: $\mathbf{Z} \longrightarrow \mathbf{S}$ as follows:

$$f(n) = 2n \qquad \text{for all integers } n.$$

   (a) Prove that f is one-to-one and onto
   (b) Find a formula for the inverse function f^{-1}.
   (c) Does the set of all even integers have the same cardinality as the set of all integers? Why or why not?

17. Define a function f: $\mathbf{R} \longrightarrow \mathbf{R}$ as follows: for all real numbers x,

$$f(x) = 16x - 5.$$

   Then f is both one-to-one and onto. Find the inverse function f^{-1}.

18. Prove that if f: $X \rightarrow Y$ and g: $Y \rightarrow Z$ are one-to-one, then $g \circ f$: $X \rightarrow Z$ is also one-to-one.

19. Prove that if f: $X \rightarrow Y$ and g: $Y \rightarrow Z$ are onto, then $g \circ f$: $X \rightarrow Z$ is also onto.

20. Is the set of all squares of positive integers countable? That is, is the set $S = \{m \in \mathbf{Z} \mid m = k^2$ for some positive integer $k\}$ a countable set. Justify your answer.

21. Prove that the set of all integers and the set of all odd integers have the same cardinality.

Chapter 8

1. Define a relation R from $\{a, b, c\}$ to $\{u, v\}$ as follows: $R = \{(a, v), (b, u), (b, v), (c, u)\}$.

   (a) Draw an arrow diagram for R.

   (b) Is R a function? Why or why not?

2. Define a relation R from $\{a, b, c\}$ to $\{u, v\}$ as follows: $R = \{(a, u), (b, u), (c, v)\}$.

   (a) Draw an arrow diagram for R.

   (b) Is R a function? Why or why not?

3. Define a relation R from $\{a, b, c\}$ to $\{u, v\}$ as follows: $R = \{(a, v), (b, u)\}$.

   (a) Draw an arrow diagram for R.

   (b) Is R a function? Why or why not?

4. Define a relation T from $\mathbf{R}$ to $\mathbf{R}$ as follows: for all $(x, y) \in \mathbf{R} \times \mathbf{R}$, $x\ T\ y \Leftrightarrow y > x + 1$.

   (a) Is $(1, 0) \in T$? Is $(0, 1) \in T$? Is $(-2, 5) \in T$? Is $(-3, -4) \in T$?

   (b) Sketch the graph of T in the Cartesian plane.

5. Let $A = \{0, 1, 2, 3\}$ and define a relation R on A as follows:
   $R = \{(0, 2), (0, 3), (2, 0), (2, 1)\}$.

   (a) Draw the directed graph of R.

   (b) Is R reflexive? Explain.

   (c) Is R symmetric? Explain.

   (d) Is R transitive? Explain.

6. Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows: for all $x, y \in A$,

   $$x\ R\ y \Leftrightarrow 3 \mid (x - y).$$

   (a) Is $7\ R\ 2$? Is $7\ R\ 4$? Is $2\ R\ 5$? Is $8\ R\ 8$?

   (b) Draw the directed graph of R.

7. Let $A = \{3, 4, 5, 6, 7\}$ and define a relation R on A as follows: for all $x, y \in A$,

   $$x\ R\ y \Leftrightarrow 2 \mid (x - y).$$

   (a) Is $6\ R\ 3$? Is $4\ R\ 6$?

   (b) Draw the directed graph of R.

8. Let $B = \{0, 1, 2, 3\}$ and define a relation U on B by

   $$U = \{(0, 2), (0, 3), (2, 0), (2, 1)\}.$$

   Is U transitive? Justify your answer.

9. Define a relation R on the set $\{1, 2, 3, 4\}$ as follows:

$$T = \{(1, 4), (2, 3), (2, 4), (4, 1), (2, 1), (1, 2), (3, 2)\}$$

   (a) Is R symmetric? Justify your answer.
   (b) Is R transitive? Justify your answer.

10. Define a relation S on the set of positive integers as follows: for all positive integers m and n,

$$m\ S\ n \Leftrightarrow m \mid n.$$

   (a) Is S reflexive? Justify your answer.
   (b) Is S symmetric? Justify your answer.

11. Let $B = \{0, 1, 2, 3\}$ and define a relation U on B by

$$U = \{(0, 2), (0, 3), (1, 2)\}.$$

   Is U transitive? Justify your answer.

12. Let R be the relation defined on the set of all integers $\mathbf{Z}$ as follows: for all integers m and n,

$$m\ R\ n \Longleftrightarrow m - n \text{ is divisible by } 5.$$

   (a) Is R reflexive? Prove or give a counterexample.
   (b) Is R symmetric? Prove or give a counterexample.
   (c) Is R transitive? Prove or give a counterexample.

13. Let $A = \{1, 2, 3, 4\}$. The following relation R is an equivalence relation on A:

$$R = \{(1, 1), (1, 3), (1, 4), (2, 2), (3, 1), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4)\}.$$

   (a) Draw the directed graph of R.
   (b) Find the distinct equivalence classes of R.

14. Let S be the set of all strings of 0's and 1's of length 3. Define a relation R on S as follows: for all strings s and t in S,

$$s\ R\ t \quad \Longleftrightarrow \quad \begin{array}{l}\text{the two left-most characters}\\ \text{of } s \text{ are the same as the two}\\ \text{left-most characters of } t.\end{array}$$

   (a) Prove that R is an equivalence relation on S.
   (b) Find the distinct equivalence classes of R.

15. Define a relation T on $\mathbf{R}$ as follows: for all x and y in $\mathbf{R}$, $x\ T\ y$ if and only if $x^2 = y^2$. Then T is an equivalence relation on $\mathbf{R}$.

   (a) Prove that T is an equivalence relation on $\mathbf{R}$.
   (b) Find the distinct equivalence classes of T.

16. Prove directly from the definition of congruence modulo n that if a, c, and n are integers, $n > 1$, and $a \equiv c \pmod{n}$, then $a^3 \equiv c^3 \pmod{n}$.

17. Use the fact that $29 = 16 + 8 + 4 + 1$ to compute 18^{29} *mod* 65.

18. Find a positive inverse for 7 modulo 48. (That is, find a positive integer n such that $7n \equiv 1 \pmod{48}$.)

19. An RSA cipher has public key $pq = 65$ and $e = 7$.

    (a) Translate the message YES into its numeric equivalent, and use the formula $C = M^e \pmod{pq}$ to encrypt the message.

    (b) Decrypt the ciphertext 50 41 and translate the result into letters of the alphabet to discover the message.

Chapter 9

1. On each of three consecutive days the National Weather Service announces that there is a 50-50 chance of rain. Assuming that the National Weather Service is correct, what is the probability that it rains on at most one of the three days? Justify your answer. (*Hint*: Represent the outcome that it rains on day 1 and doesn't rain on days 2 and 3 as RNN.)

2. How many elements are in the one-dimensional array shown below?

$$A[7],\ A[8],\ \ldots\ ,\ A\left[\left\lfloor \frac{145}{2} \right\rfloor\right]$$

3. In a certain state, license plates each consist of 2 letters followed by 3 digits.

    (a) How many different license plates are there?

    (b) How many different license plates are there that have no repeated letters or digits?

4. In a certain state, license plates each consist of 2 letters followed by either 3 or 4 digits. How many different license plates are there that have no repeated letters or digits?

5. Suppose there are three routes from Byrne Hall to McGaw Hall and five routes from McGaw Hall to Monroe Hall. How many ways is it possible to travel from Byrne Hall to Monroe Hall by way of McGaw Hall?

6. In a certain discrete math class, three quizzes were given. Out of the 30 students in the class:

    15 scored 12 or above on quiz #1,
    12 scored 12 or above on quiz #2,
    18 scored 12 or above on quiz #3,
    7 scored 12 or above on quizzes #1 and #2,
    11 scored 12 or above on quizzes #1 and #3,
    8 scored 12 or above on quizzes #2 and #3,
    4 scored 12 or above on quizzes #1, #2, and #3.

    (a) How many scored 12 or above on at least one quiz?

    (b) How many scored 12 or above on quizzes 1 and 2 but not 3?

7. Consider the set S of all strings of a's and b's. For each integer $n \geq 0$, let

$$\boxed{a_n = \text{the number of strings of length n that do not contain the pattern } bb.}$$

    Find a recurrence relation for $a_1, a_2, a_3, \ldots$. Explain your answer carefully.

8. If five integers are chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, must there be at least two integers with the property that the larger minus the smaller is 2? Write an answer that would convince a good but skeptical fellow student who has learned the statement of the pigeonhole principle but not seen an application like this one. Either describe the pigeons, the pigeonholes, and how the pigeons get to the pigeonholes, or describe a function by giving its domain, co-domain, and how elements of the domain are related to elements of the co-domain.

9. Given any set of 30 integers, must there be two that have the same remainder when they are divided by 25? Write an answer that would convince a good but skeptical fellow student who has learned the statement of the pigeonhole principle but not seen an application like this one. Either describe the pigeons, the pigeonholes, and how the pigeons get to the pigeonholes, or describe a function by giving its domain, co-domain, and how elements of the domain are related to elements of the co-domain.

10. Given any set of 15 integers, must there be two that have the same remainder when divided by 12? Write an answer that would convince a good but skeptical fellow student who has learned the statement of the pigeonhole principle but not seen an application like this one. Either describe the pigeons, the pigeonholes, and how the pigeons get to the pigeonholes, or describe a function by giving its domain, co-domain, and how elements of the domain are related to elements of the co-domain.

11. Let T be the set $\{3, 4, 5, 6, 7, 8, 9, 10\}$ and suppose five integers are chosen from T. Must two of these integers have the property that the difference of the larger minus the smaller equals 2? Why or why not? Write an answer that would convince a good but skeptical fellow student who has learned the statement of the pigeonhole principle but not seen an application like this one. Either describe the pigeons, the pigeonholes, and how the pigeons get to the pigeonholes, or describe a function by giving its domain, co-domain, and how elements of the domain are related to elements of the co-domain.

12. If six integers are chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, must there be at least two integers with the property that the sum of the smaller plus the larger is 11? Why or why not? Write an answer that would convince a good but skeptical fellow student who has learned the statement of the pigeonhole principle but not seen an application like this one. Either describe the pigeons, the pigeonholes, and how the pigeons get to the pigeonholes, or describe a function by giving its domain, co-domain, and how elements of the domain are related to elements of the co-domain.

13. A club has seven members. Three are to be chosen to go as a group to a national meeting.

    (a) How many distinct groups of three can be chosen?

    (b) If the club contains four men and three women, how many distinct groups of three contain two men and one woman?

    (c) If the club contains four men and three women, how many distinct groups of three contain at most two men?

    (d) If the club contains four men and three women, how many distinct groups of three contain at least one woman?

    (e) If the club contains four men and three women, what is the probability that a distinct group of three will contain at least one woman?

    (f) If two members of the club refuse to travel together as part of the group (but each is willing to go if the other does not), how many distinct groups of three can be chosen?

    (g) If two members of the club insists on either traveling together or not going at all, How many distinct groups of three can be chosen?

14. Suppose that a fair coin is tossed ten times.

    (a) How many ways can at least eight heads be obtained?

    (b) What is the probability of obtaining at least eight heads?

15. A large pile of coins consists of pennies, nickels, dimes, and quarters (at least 20 of each).

    (a) How many different collections of 20 coins can be chosen?

    (b) How many different collections of 20 coins chosen at random will contain at least 3 coins of each type?

    (c) What is the probability that a collection of 20 coins chosen at random will contain at least 3 coins of each type?

16. Prove for all integers n, k, and r with $n \geq k \geq r$ that $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$.

17. The binomial theorem states that for any real numbers a and b,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \qquad \text{for any integer } n \geq 0.$$

    Use this theorem to compute $(2x-1)^5$.

18. The binomial theorem states that for any real numbers a and b,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \qquad \text{for any integer } n \geq 0.$$

    Use this theorem to show that for any integer $n \geq 0$, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k} 2^k = 1$.

19. Express the following sum in closed form (without using a summation symbol and without using an ellipsis . . .): $\sum_{k=0}^{n} \binom{n}{k} 7^k$.

20. Let A, B, and C be events in a sample space S such that $S = A \cup B \cup C$. Suppose that $P(A) = 0.3$, $P(B) = 0.6$, and $P(A \cap B) = 0.2$. Find each of the following.

    (a) $P(A \cup B)$

    (b) $P(C)$

    (c) $P(A^c \cup B^c)$

21. An urn contains four balls numbered 1, 3, 4, and 6. If a person selects a set of two balls at random, what is the expected value of the product of the numbers on the balls?

22. Suppose A and B are events in a sample space S, and $P(A|B) = 1/2$ and $P(B) = 1/3$. What is $P(A \cap B)$?

23. A teacher offers ten possible assignments for extra credit in a course but requires students to choose them, without looking, from a hat. Six assignments involve library research and four are computer programming exercises. Suppose that a student chooses two assignments, one after the other, at random without replacement.

    (a) What is the probability that both assignments are computer programming exercises?

    (b) What is the probability that at least one of the assignments is a computer programming exercise?

24. A screening test for a certain disease is used in a large population of people of whom 1 in 1000 actually have the disease. Suppose that the false positive rate is 1% and the false negative rate is 0.5%. Thus a person who has the disease tests positive for it 99.5% of the time, and a person who does not have the disease tests negative for it 99% of the time.

    (a) What is the probability that a randomly chosen person who tests positive for the disease actually has the disease?

    (b) What is the probability that a randomly chosen person who tests negative for the disease actually has the disease?

25. A coin is loaded so that the probability of heads is 0.55 and the probability of tails is 0.45. Suppose the coin is tossed twice and the results of the tosses are independent.

    (a) What is the probability of obtaining exactly two heads?

    (b) What is the probability of obtaining exactly one head?

    (c) What is the probability of obtaining no heads?

    (d) What is the probability of obtaining at least one head?

Chapter 10

1. If a graph has vertices of degrees 1, 1, 2, 3, and 3, how many edges does it have? Why?

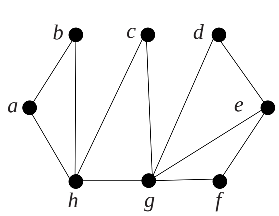
2. For each of (a)–(c) below, either draw a graph with the specified properties or else explain why no such graph exists.

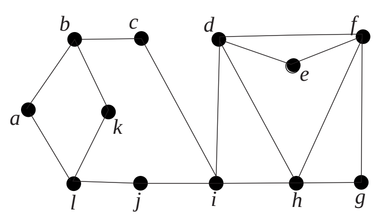
    (a) Graph with six vertices of degrees 1, 1, 2, 2, 2, and 3.

    (b) Graph with four vertices of degrees 1, 2, 2, and 5.

    (c) Simple graph with five vertices of degrees 1, 1, 1, 1, and 5.

3. Determine whether each of the following graphs has an Euler circuit. If it does have an Euler circuit, find such a circuit. If it does not have an Euler circuit, explain why you can be 100% sure that it does not.





4. Determine whether each of the following graphs has a Hamiltonian circuit. If it does have an Hamiltonian circuit, find such a circuit. If it does not have an Hamiltonian circuit, explain why you can be 100% sure that it does not.