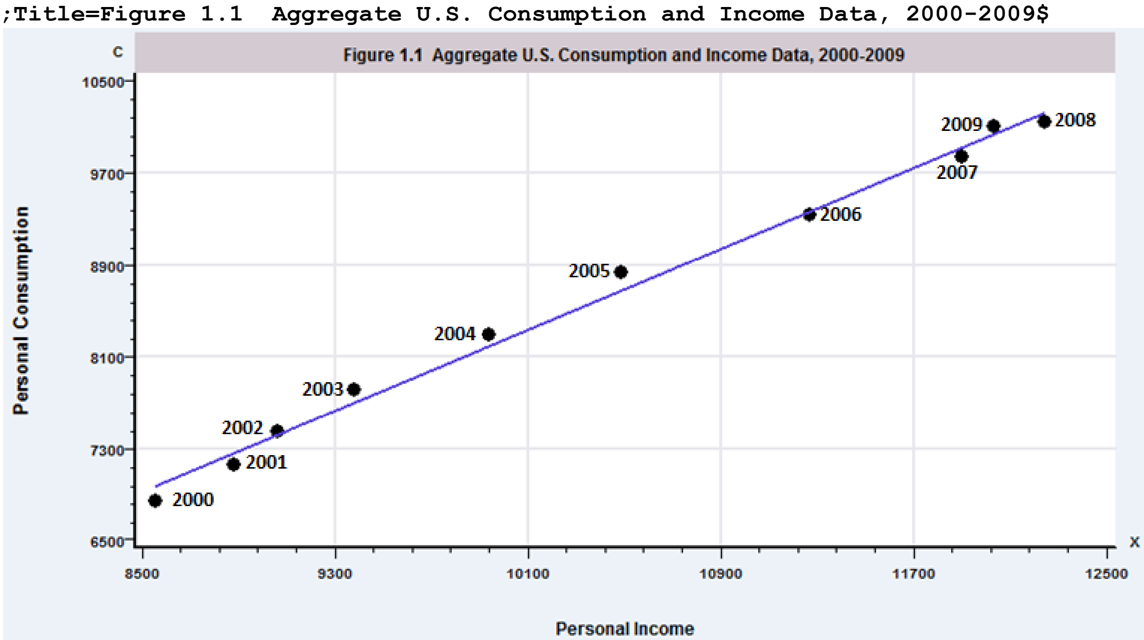
Chapter 1

Econometrics

There are no exercises or applications in Chapter 1.

Example 1.2

```
import$
Year, X,C
2000, 8559.4, 6830.4
2001, 8883.3, 7148.8
2002, 9060.1, 7439.2
2003, 9378.1, 7804.0
2004, 9937.2, 8285.1
2005, 10485.9, 8819.0
2006, 11268.1, 9322.7
2007, 11894.1, 9826.4
2008, 12238.8, 10129.9
2009, 12030.3, 10088.5
plot
;lhs=x
;rhs=c
;limits=6500,10500
;endpoints=8500,12500
;grid
;regression
;vaxis=Personal Consumption;Footer=Personal Income
;Title=Figure 1.1  Aggregate U.S. Consumption and Income Data, 2000-2009$
```



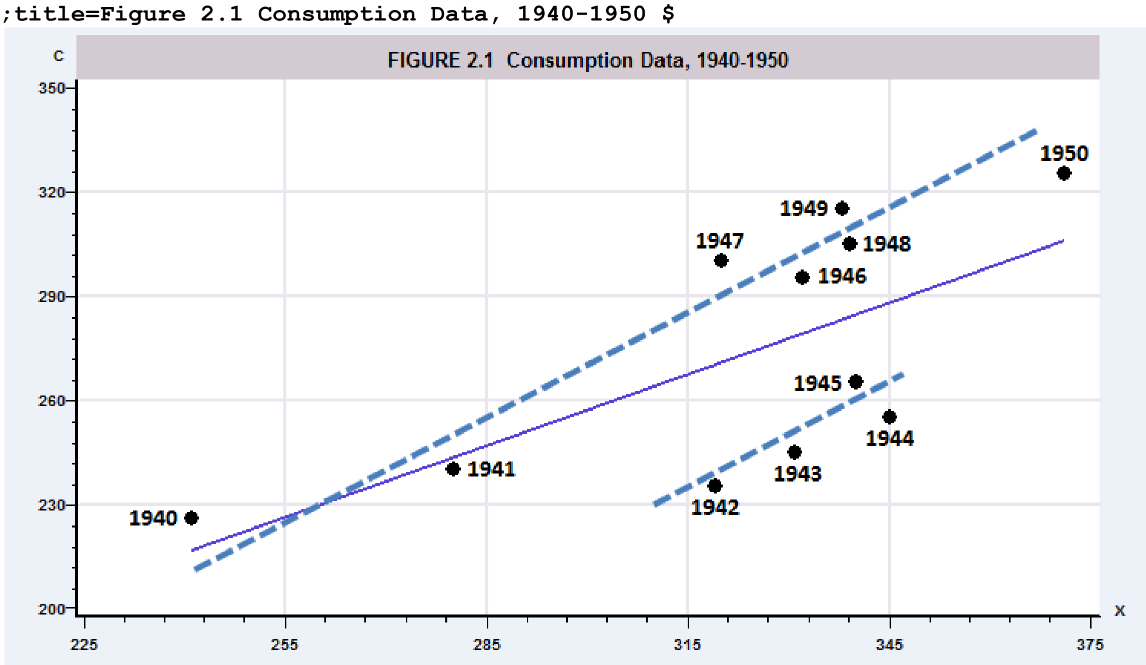
(Dates were added to the figure by editing.)

Chapter 2

The Linear Regression Model

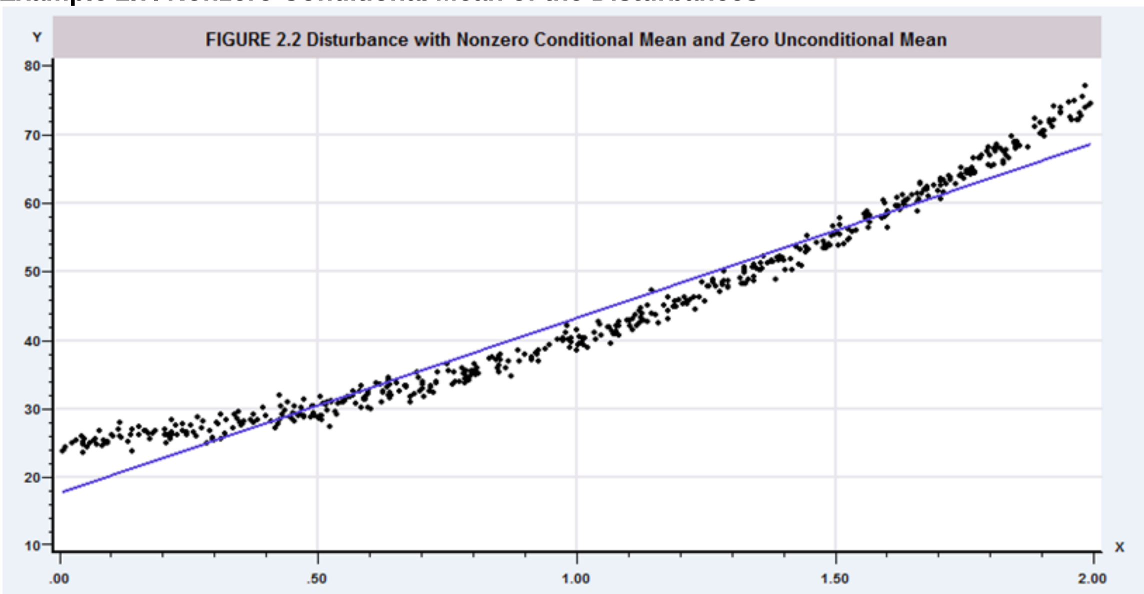
There are no exercises or applications in Chapter 2.

```
Example 2.1. Keynes's Consumption
import$
Year     X    C    W
1940   241  226   0
1941   280  240   0
1942   319  235   1
1943   331  245   1
1944   345  255   1
1945   340  265   1
1946   332  295   0
1947   320  300   0
1948   339  305   0
1949   338  315   0
1950   371  325   0
plot;lhs=x;rhs=c;limits=200,350; endpoints=225,375;regression
;title=Figure 2.1 Consumption Data, 1940-1950 $
```



(Dates and dashed lines were added by editing.)

Example 2.7. Nonzero Conditional Mean of the Disturbances



Chapter 3

Least Squares Regression

EXAMPLES – Section 3.2.2 and Table 3.2

```
Import$
YEAR RealGNP  Invest GNPDefl Interest  Infl  Trend RealInv
2000   87.1   2.034   81.9    9.23      3.4    1    2.484
2001   88.0   1.929   83.8    6.91      1.6    2    2.311
2002   89.5   1.925   85.0    4.67      2.4    3    2.265
2003   92.0   2.028   86.7    4.12      1.9    4    2.339
2004   95.5   2.277   89.1    4.34      3.3    5    2.556
2005   98.7   2.527   91.9    6.19      3.4    6    2.750
2006  101.4   2.681   94.8    7.96      2.5    7    2.828
2007  103.2   2.644   97.3    8.05      4.1    8    2.717
2008  102.9   2.425   99.2    5.09      0.1    9    2.445
2009  100.0   1.878  100.0    3.25      2.7   10    1.878
2010  102.5   2.101  101.2    3.25      1.5   11    2.076
2011  104.2   2.240  103.3    3.25      3.0   12    2.168
2012  105.6   2.479  105.2    3.25      1.7   13    2.356
2013  109.0   2.648  106.7    3.25      1.5   14    2.482
2014  111.6   2.856  108.3    3.25      0.8   15    2.637
EndData
Create ; Y  = RealInv $
Create ; T  = trend $
Create ; G  = realgnp $
Create ; R  = interest $
Create ; P  = infl $
Namelist;z=y,t,g,r,p$
Dstat  ; rhs=z$
--------+--------------------------------------------------------------------
        |                  Standard                                   Missing
Variable|         Mean    Deviation       Minimum       Maximum    Cases  Values
--------+--------------------------------------------------------------------
       Y|     2.420067      .262666         1.878         2.828       15       0
       T|          8.0     4.472136           1.0          15.0       15       0
       G|     99.41333     7.525468          87.1         111.6       15       0
       R|     5.070667     2.081351          3.25          9.23       15       0
       P|         2.26     1.092703            .1           4.1       15       0
--------+--------------------------------------------------------------------
Descriptive Statistics for   5 variables
Dstat results are matrix LASTDSTA in current project.
Regress;Lhs=y;rhs=one,t,g,r,p$
-----------------------------------------------------------------------------
Ordinary     least squares regression ............
LHS=Y        Mean                 =        2.42007
             Standard deviation   =         .26267
----------   No. of observations  =             15  DegFreedom   Mean square
Regression   Sum of Squares       =        .760908           4        .19023
Residual     Sum of Squares       =        .205002          10        .02050
Total        Sum of Squares       =        .965911          14        .06899
----------   Standard error of e  =         .14318  Root MSE          .11691
Fit          R-squared            =         .78776  R-bar squared     .70287
Model test   F[  4,    10]        =        9.27926  Prob F > F*       .00213
--------+--------------------------------------------------------------------
        |                  Standard            Prob.      95% Confidence
       Y|  Coefficient       Error       t    |t|>T*         Interval
--------+--------------------------------------------------------------------
Constant|   -6.26176***     1.93671    -3.23  .0090   -10.57700  -1.94651
```

```
       T|    -.16187***       .04739    -3.42  .0066     -.26746   -.05628
       G|     .09960***       .02421     4.11  .0021      .04566    .15355
       R|     .01972          .03380      .58  .5725     -.05559    .09503
       P|    -.01109          .03990     -.28  .7867     -.09998    .07781
--------+--------------------------------------------------------------------
***, **, * ==>  Significance at 1%, 5%, 10% level.
Model was estimated on Aug 01, 2017 at 08:37:09 AM
-----------------------------------------------------------------------------
Namelist; x=one,t,g,r,p$
Matrix  ; list;x'x$
--------+----------------------------------------------------------------------
  RESULT|            1             2             3             4             5
--------+----------------------------------------------------------------------
       1|      15.0000       120.000       1491.20       76.0600       33.9000
       2|      120.000       1240.00       12381.5       522.060       244.100
       3|      1491.20       12381.5       149038.       7453.03       3332.83
       4|      76.0600       522.060       7453.03       446.323       186.656
       5|      33.9000       244.100       3332.83       186.656       93.3300
Matrix ; list;x'y$
--------+--------------
  RESULT|            1
--------+--------------
       1|      36.3010
       2|      288.691
       3|      3612.90
       4|      188.300
       5|      82.8193
Matrix ; list;<x'x>*x'y$
--------+--------------
  RESULT|            1
--------+--------------
       1|     -6.26176
       2|     -.161870
       3|      .0996027
       4|      .0197220
       5|     -.0110883

Matrix ; list;xcor(z)$
--------+------------------------------------------
Cor.Mat.|       Y        T        G        R        P
--------+------------------------------------------
       Y| 1.00000  -.10441   .14809   .55261   .19388
       T| -.10441  1.00000   .95910  -.66317  -.39612
       G|  .14809   .95910  1.00000  -.49410  -.32384
       R|  .55261  -.66317  -.49410  1.00000   .46358
       P|  .19388  -.39612  -.32384   .46358  1.00000
Create ; dy = dev(y) $
Create ; dt = dev(t) $
Create ; dg = dev(g) $
Calc ; list ; xbr(y)
     ; xbr(t)
     ; xbr(g) $
[CALC]         =       2.4200667
[CALC]         =       8.0000000
[CALC]         =      99.4133333
Calculator: Computed   3 scalar results
Calc ; list ; sty = dt'dy
     ; sgg = dg'dg
     ; sgy = dg'dy
     ; stg = dt'dg
     ; stt = dt'dt$
[CALC] STY     =      -1.7170000
[CALC] SGG     =     792.8573333
```

```
[CALC] SGY     =       4.0982867
[CALC] STG     =     451.9000000
[CALC] STT     =     280.0000000
Calculator: Computed   5 scalar results
Calc ; list ; b2 = (sty*sgg - sgy*stg)/(stt*sgg-stg*stg)$
[CALC] B2      =       -.1806630
Calc ; list ; b3 = (sgy*stt - sty*stg)/(stt*sgg-stg*stg)$
[CALC] B3      =        .1081404
Calc ; list ; b1 = xbr(y) - b2*xbr(t)-b3*xbr(g)$
[CALC] B1      =      -6.8852242
Calc ; list ; byg = sgy / sgg $
[CALC] BYG     =        .0051690
Calc ; list ; byt = sty / stt $
[CALC] BYT     =       -.0061321
Calc ; list ; btg = stg / sgg$
[CALC] BTG     =        .5699638
Calc ; list ; r2gt=stg^2/(sgg*stt)$
[CALC] R2GT    =        .9198809
Calc ; list ; byg_t=byg-((byt*btg-r2gt*byg)/(1-r2gt))$
[CALC] BYG_T   =        .1081404
Namelist ; yvar=y $
Matrix;list;xcor(x,yvar)$
--------+--------
Cor.Mat.|       Y
--------+--------
     ONE|  .00000
       T| -.10441
       G|  .14809
       R|  .55261
       P|  .19388

Regress;quietly ; Lhs=y;rhs=one,t,g,r,p$
Matrix ; vars = diag(varb) ; sdevs=sqrt(vars)$
Matrix ; tstats = <sdevs>*b$
Matrix ; pcor = dirp(tstats,tstats) + degfrdm$
Matrix ; pci  = diri(pcor)$
Matrix ; pcor = dirp(tstats,tstats,pci)$
Matrix ; list ; pcor = esqr(pcor)$
--------+--------------
    PCOR|             1
--------+--------------
       1|       .000000
       2|       .733814
       3|       .792847
       4|       .181449
       5|      .0875491
```

Exercises

1. Let $\mathbf{X} = \begin{bmatrix} 1 & x_1 \\ ... & ... \\ 1 & x_n \end{bmatrix}$.

(a) The normal equations are given by (3-12), $\mathbf{X}'\mathbf{e} = \mathbf{0}$ (we drop the minus sign), hence for each of the columns of $\mathbf{X}$, $\mathbf{x}_k$, we know that $\mathbf{x}_k'\mathbf{e} = 0$. This implies that $\Sigma_{i=1}^n e_i = 0$ and $\Sigma_{i=1}^n x_i e_i = 0$.

(b) Use $\Sigma_{i=1}^n e_i$ to conclude from the first normal equation that $a = \bar{y} - b\bar{x}$.

(c) We know that $\Sigma_{i=1}^n e_i = 0$ and $\Sigma_{i=1}^n x_i e_i = 0$. It follows then that $\Sigma_{i=1}^n (x_i - \bar{x})e_i = 0$ because $\Sigma_{i=1}^n \bar{x}e_i = \bar{x}\Sigma_{i=1}^n e_i = 0$. Substitute e_i to obtain $\Sigma_{i=1}^n (x_i - \bar{x})(y_i - a - bx_i) = 0$

or $\Sigma_{i=1}^n (x_i - \bar{x})(y_i - \bar{y} - b(x_i - \bar{x})) = 0$

Then, $\Sigma_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) = b\Sigma_{i=1}^n (x_i - \bar{x})(x_i - \bar{x}))$ so $b = \dfrac{\Sigma_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\Sigma_{i=1}^n (x_i - \bar{x})^2}$.

(d) The first derivative vector of $\mathbf{e}'\mathbf{e}$ is $-2\mathbf{X}'\mathbf{e}$. (The normal equations.) The second derivative matrix is $\partial^2(\mathbf{e}'\mathbf{e})/\partial\mathbf{b}\partial\mathbf{b}' = 2\mathbf{X}'\mathbf{X}$. We need to show that this matrix is positive definite. The diagonal elements are $2n$ and $2\Sigma_{i=1}^n x_i^2$ which are clearly both positive. The determinant is

$[(2n)(2\Sigma_{i=1}^n x_i^2)] - (2\Sigma_{i=1}^n x_i)^2 = 4n\Sigma_{i=1}^n x_i^2 - 4(n\bar{x})^2 = 4n[(\Sigma_{i=1}^n x_i^2) - n\bar{x}^2] = 4n[(\Sigma_{i=1}^n (x_i - \bar{x})^2]$.

Note that a much simpler proof appears after (3-6).

2. Write $\mathbf{c}$ as $\mathbf{b} + (\mathbf{c} - \mathbf{b})$. Then, the sum of squared residuals based on $\mathbf{c}$ is

$$(\mathbf{y} - \mathbf{Xc})'(\mathbf{y} - \mathbf{Xc}) = [\mathbf{y} - \mathbf{X}(\mathbf{b} + (\mathbf{c} - \mathbf{b}))]'[\mathbf{y} - \mathbf{X}(\mathbf{b} + (\mathbf{c} - \mathbf{b}))] = [(\mathbf{y} - \mathbf{Xb}) + \mathbf{X}(\mathbf{c} - \mathbf{b})]'[(\mathbf{y} - \mathbf{Xb}) + \mathbf{X}(\mathbf{c} - \mathbf{b})]$$
$$= (\mathbf{y} - \mathbf{Xb})'(\mathbf{y} - \mathbf{Xb}) + (\mathbf{c} - \mathbf{b})'\mathbf{X}'\mathbf{X}(\mathbf{c} - \mathbf{b}) + 2(\mathbf{c} - \mathbf{b})'\mathbf{X}'(\mathbf{y} - \mathbf{Xb}).$$

But, the third term is zero, as $2(\mathbf{c} - \mathbf{b})'\mathbf{X}'(\mathbf{y} - \mathbf{Xb}) = 2(\mathbf{c} - \mathbf{b})\mathbf{X}'\mathbf{e} = \mathbf{0}$. Therefore,

$$(\mathbf{y} - \mathbf{Xc})'(\mathbf{y} - \mathbf{Xc}) = \mathbf{e}'\mathbf{e} + (\mathbf{c} - \mathbf{b})'\mathbf{X}'\mathbf{X}(\mathbf{c} - \mathbf{b})$$

or

$$(\mathbf{y} - \mathbf{Xc})'(\mathbf{y} - \mathbf{Xc}) - \mathbf{e}'\mathbf{e} = (\mathbf{c} - \mathbf{b})'\mathbf{X}'\mathbf{X}(\mathbf{c} - \mathbf{b}).$$

The right hand side can be written as $\mathbf{d}'\mathbf{d}$ where $\mathbf{d} = \mathbf{X}(\mathbf{c} - \mathbf{b})$, so it is necessarily positive. This confirms what we knew at the outset, least squares is least squares.

3. In the regression of $\mathbf{y}$ on $\mathbf{i}$ and $\mathbf{X}$, the coefficients on $\mathbf{X}$ are $\mathbf{b} = (\mathbf{X}'\mathbf{M}^0\mathbf{X})^{-1}\mathbf{X}'\mathbf{M}^0\mathbf{y}$. $\mathbf{M}^0 = \mathbf{I} - \mathbf{i}(\mathbf{i}'\mathbf{i})^{-1}\mathbf{i}'$ is the matrix which transforms observations into deviations from their column means. Since $\mathbf{M}^0$ is idempotent and symmetric we may also write the preceding as $[(\mathbf{X}'\mathbf{M}^{0\prime})(\mathbf{M}^0\mathbf{X})]^{-1}(\mathbf{X}'\mathbf{M}^{0\prime})(\mathbf{M}^0\mathbf{y})$ which implies that the regression of $\mathbf{M}^0\mathbf{y}$ on $\mathbf{M}^0\mathbf{X}$ produces the least squares slopes. If only $\mathbf{X}$ is transformed to deviations, we would compute $[(\mathbf{X}'\mathbf{M}^{0\prime})(\mathbf{M}^0\mathbf{X})]^{-1}(\mathbf{X}'\mathbf{M}^{0\prime})\mathbf{y}$ but, of course, this is identical. However, if only $\mathbf{y}$ is transformed, the result is $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{M}^0\mathbf{y}$ which is likely to be quite different.

4. What is the result of the matrix product $\mathbf{M}_1\mathbf{M}$ where $\mathbf{M}_1$ is defined in (3-19) and $\mathbf{M}$ is defined in (3-14)?

$$\mathbf{M}_1\mathbf{M} = (\mathbf{I} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1')(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}') = \mathbf{M} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{M}$$

There is no need to multiply out the second term. Each column of $\mathbf{MX}_1$ is the vector of residuals in the regression of the corresponding column of $\mathbf{X}_1$ on all of the columns in $\mathbf{X}$. Since that $\mathbf{x}$ is one of the columns in $\mathbf{X}$, this regression provides a perfect fit, so the residuals are zero. Thus, $\mathbf{MX}_1$ is a matrix of zeroes which implies that $\mathbf{M}_1\mathbf{M} = \mathbf{M}$.

5. The original $\mathbf{X}$ matrix has n rows. We add an additional row, $\mathbf{x}_s'$. The new $\mathbf{y}$ vector likewise has an additional element. Thus, $\mathbf{X}_{n,s} = \begin{bmatrix} \mathbf{X}_n \\ \mathbf{x}_s' \end{bmatrix}$ and $\mathbf{y}_{n,s} = \begin{bmatrix} \mathbf{y}_n \\ y_s \end{bmatrix}$. The new coefficient vector is

$\mathbf{b}_{n,s} = (\mathbf{X}_{n,s}'\mathbf{X}_{n,s})^{-1}(\mathbf{X}_{n,s}'\mathbf{y}_{n,s})$. The matrix is $\mathbf{X}_{n,s}'\mathbf{X}_{n,s} = \mathbf{X}_n'\mathbf{X}_n + \mathbf{x}_s\mathbf{x}_s'$. To invert this, use (A -66);

$(\mathbf{X}_{n,s}'\mathbf{X}_{n,s})^{-1} = (\mathbf{X}_n'\mathbf{X}_n)^{-1} - \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}$. The vector is $(\mathbf{X}_{n,s}'\mathbf{y}_{n,s}) = (\mathbf{X}_n'\mathbf{y}_n) + \mathbf{x}_s y_s$. Multiply out the four terms to get

$(\mathbf{X}_{n,s}'\mathbf{X}_{n,s})^{-1}(\mathbf{X}_{n,s}'\mathbf{y}_{n,s}) =$

$$\mathbf{b}_n - \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s\mathbf{x}_s'\mathbf{b}_n + (\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s y_s - \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s y_s$$

=

$$\mathbf{b}_n + (\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s y_s - \frac{\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s y_s - \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s\mathbf{x}_s'\mathbf{b}_n$$

$$\mathbf{b}_n + \left[1 - \frac{\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}\right](\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s y_s - \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s\mathbf{x}_s'\mathbf{b}_n$$

$$\mathbf{b}_n + \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s y_s - \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s\mathbf{x}_s'\mathbf{b}_n$$

$$\mathbf{b}_n + \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s(y_s - \mathbf{x}_s'\mathbf{b}_n)$$

6. Define the data matrix as follows: $\mathbf{X} = \begin{bmatrix} \mathbf{i} & \mathbf{x} & \mathbf{0} \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1, & \mathbf{0} \\ & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} \mathbf{y}_o \\ y_m \end{bmatrix}$. (The subscripts on the parts of **y** refer to the “observed” and “missing” rows of **X**. We will use Frish-Waugh to obtain the first two columns of the least squares coefficient vector. $\mathbf{b}_1 = (\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-1}(\mathbf{X}_1'\mathbf{M}_2\mathbf{y})$. Multiplying it out, we find that $\mathbf{M}_2$ = an identity matrix save for the last diagonal element that is equal to 0.

$\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1 = \mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_1'\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0}' & 1 \end{bmatrix}\mathbf{X}_1$. This just drops the last observation. $\mathbf{X}_1'\mathbf{M}_2\mathbf{y}$ is computed likewise. Thus, the coeffients on the first two columns are the same as if y_0 had been linearly regressed on $\mathbf{X}_1$. The denomonator of R^2 is different for the two cases (drop the observation or keep it with zero fill and the dummy variable). For the first strategy, the mean of the n-1 observations should be different from the mean of the full n unless the last observation happens to equal the mean of the first n-1.

For the second strategy, replacing the missing value with the mean of the other n-1 observations, we can deduce the new slope vector logically. Using Frisch-Waugh, we can replace the column of x’s with deviations from the means, which then turns the last observation to zero. Thus, once again, the coefficient on the x equals what it is using the earlier strategy. The constant term will be the same as well.

7. For convenience, reorder the variables so that $\mathbf{X} = [\mathbf{i}, \mathbf{P}_d, \mathbf{P}_n, \mathbf{P}_s, \mathbf{Y}]$. The three dependent variables are $\mathbf{E}_d$, $\mathbf{E}_n$, and $\mathbf{E}_s$, and $\mathbf{Y} = \mathbf{E}_d + \mathbf{E}_n + \mathbf{E}_s$. The coefficient vectors are

$\mathbf{b}_d = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{E}_d$,
$\mathbf{b}_n = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{E}_n$, and
$\mathbf{b}_s = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{E}_s$.

The sum of the three vectors is

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{E}_d + \mathbf{E}_n + \mathbf{E}_s] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}.$$

Now, **Y** is the last column of **X**, so the preceding sum is the vector of least squares coefficients in the regression of the last column of **X** on all of the columns of **X**, including the last. Of course, we get a perfect fit. In addition, $\mathbf{X}'[\mathbf{E}_d + \mathbf{E}_n + \mathbf{E}_s]$ is the last column of $\mathbf{X}'\mathbf{X}$, so the matrix product is equal to the last column of an identity matrix. Thus, the sum of the coefficients on all variables except income is 0, while that on income is 1.

8. Let $\overline{R}_K^2$ denote the adjusted R^2 in the full regression on K variables including $\mathbf{x}_k$, and let $\overline{R}_1^2$ denote the adjusted R^2 in the short regression on K-1 variables when $\mathbf{x}_k$ is omitted. Let R_K^2 and R_1^2 denote their unadjusted counterparts. Then,

$$R_K^2 = 1 - \mathbf{e}'\mathbf{e}/\mathbf{y}'\mathbf{M}^0\mathbf{y}$$

$$R_1^2 = 1 - \mathbf{e}_1'\mathbf{e}_1/\mathbf{y}'\mathbf{M}^0\mathbf{y}$$

where $\mathbf{e}'\mathbf{e}$ is the sum of squared residuals in the full regression, $\mathbf{e}_1'\mathbf{e}_1$ is the (larger) sum of squared residuals in the regression which omits $\mathbf{x}_k$, and $\mathbf{y}'\mathbf{M}^0\mathbf{y} = \Sigma_i (y_i - \overline{y})^2$.

Then, $$\overline{R}_K^2 = 1 - [(n\text{-}1)/(n\text{-}K)](1 - R_K^2)$$

and $$\overline{R}_1^2 = 1 - [(n\text{-}1)/(n\text{-}(K\text{-}1))](1 - R_1^2).$$

The difference is the change in the adjusted R^2 when $\mathbf{x}_k$ is added to the regression,

$$\overline{R}_K^2 - \overline{R}_1^2 = [(n\text{-}1)/(n\text{-}K\text{+}1)][\mathbf{e}_1'\mathbf{e}_1/\mathbf{y}'\mathbf{M}^0\mathbf{y}] - [(n\text{-}1)/(n\text{-}K)][\mathbf{e}'\mathbf{e}/\mathbf{y}'\mathbf{M}^0\mathbf{y}].$$

The difference is positive if and only if the ratio is greater than 1. After cancelling terms, we require for the adjusted R^2 to increase that $\mathbf{e}_1'\mathbf{e}_1/(n\text{-}K\text{+}1)]/[(n\text{-}K)/\mathbf{e}'\mathbf{e}] > 1$. From the previous problem, we have that $\mathbf{e}_1'\mathbf{e}_1 = \mathbf{e}'\mathbf{e} + b_K^2(\mathbf{x}_k'\mathbf{M}_1\mathbf{x}_k)$, where $\mathbf{M}_1$ is defined above and b_k is the least squares coefficient in the full regression of $\mathbf{y}$ on $\mathbf{X}_1$ and $\mathbf{x}_k$. Making the substitution, we require $[(\mathbf{e}'\mathbf{e} + b_K^2(\mathbf{x}_k'\mathbf{M}_1\mathbf{x}_k))(n\text{-}K)]/[(n\text{-}K)\mathbf{e}'\mathbf{e} + \mathbf{e}'\mathbf{e}] > 1$. Since $\mathbf{e}'\mathbf{e} = (n\text{-}K)s^2$, this simplifies to $[\mathbf{e}'\mathbf{e} + b_K^2(\mathbf{x}_k'\mathbf{M}_1\mathbf{x}_k)]/[\mathbf{e}'\mathbf{e} + s^2] > 1$. Since all terms are positive, the fraction is greater than one if and only $b_K^2(\mathbf{x}_k'\mathbf{M}_1\mathbf{x}_k) > s^2$ or $b_K^2(\mathbf{x}_k'\mathbf{M}_1\mathbf{x}_k/s^2) > 1$. The denominator is the estimated variance of b_k, so the result is proved.

9. This R^2 must be lower. The sum of squares associated with the coefficient vector which omits the constant term must be higher than the one which includes it. We can write the coefficient vector in the regression without a constant as $\mathbf{c} = (0,\mathbf{b}^*)$ where $\mathbf{b}^* = (\mathbf{W}'\mathbf{W})^{-1}\mathbf{W}'\mathbf{y}$, with $\mathbf{W}$ being the other K-1 columns of $\mathbf{X}$. Then, the result of the previous exercise applies directly.

10. We use the notation 'Var[.]' and 'Cov[.]' to indicate the sample variances and covariances. Our information is Var[N] = 1, Var[D] = 1, Var[Y] = 1.

Since $C = N + D$, Var[C] = Var[N] + Var[D] + 2Cov[N,D] = 2(1 + Cov[N,D]).

From the regressions, we have

Cov[C,Y]/Var[Y] = Cov[C,Y] = .8.

But, Cov[C,Y] = Cov[N,Y] + Cov[D,Y].

Also, Cov[C,N]/Var[N] = Cov[C,N] = .5,

but, Cov[C,N] = Var[N] + Cov[N,D] = 1 + Cov[N,D], so Cov[N,D] = -.5,

so that Var[C] = 2(1 + -.5) = 1.

And, Cov[D,Y]/Var[Y] = Cov[D,Y] = .4.

Since Cov[C,Y] = .8 = Cov[N,Y] + Cov[D,Y], Cov[N,Y] = .4.

Finally, Cov[C,D] = Cov[N,D] + Var[D] = -.5 + 1 = .5.

Now, in the regression of C on D, the sum of squared residuals is $(n\text{-}1)\{$Var[C] - (Cov[C,D]/Var[D])^{2}Var[D]$\}$ based on the general regression result $\Sigma e^2 = \Sigma(y_i - \overline{y})^2 - b^2\Sigma(x_i - \overline{x})^2$. All of the necessary figures were obtained above. Inserting these and n-1 = 20 produces a sum of squared residuals of 15.

11. Computed results are

```
Regress;lhs=realinv;rhs=one,realgnp,interest$
----------------------------------------------------------------------------
Ordinary     least squares regression ............
LHS=REALINV  Mean                  =        2.42007
             Standard deviation    =         .26267
----------   No. of observations   =             15  DegFreedom   Mean square
Regression   Sum of Squares        =        .521605            2       .26080
Residual     Sum of Squares        =        .444305           12       .03703
Total        Sum of Squares        =        .965911           14       .06899
----------   Standard error of e   =         .19242  Root MSE          .17211
```

```
Fit          R-squared            =         .54001  R-bar squared     .46335
Model test   F[  2,    12]        =        7.04388  Prob F > F*       .00947
--------+--------------------------------------------------------------------
        |                  Standard            Prob.      95% Confidence
 REALINV|  Coefficient       Error       t    |t|>T*         Interval
--------+--------------------------------------------------------------------
Constant|    -.04298         .86319     -.05  .9611    -1.92371   1.83775
 REALGNP|     .01945**       .00786     2.47  .0293      .00232    .03657
INTEREST|     .10448***      .02842     3.68  .0032      .04256    .16640
--------+--------------------------------------------------------------------
Namelist; X=one,realgnp,interest$
Matrix ; list ; x'x ; x'realinv$
  RESULT|           1            2            3
--------+------------------------------------------
       1|       15.0000      1491.20      76.0600
       2|       1491.20      149038.      7453.03
       3|       76.0600      7453.03      446.323
  RESULT|           1
--------+-------------
       1|       36.3010
       2|       3612.90
       3|       188.300
Matrix ; list ; <x'x>*x'realinv$
  RESULT|           1
--------+-------------
       1|     -.0429785
       2|      .0194467
       3|       .104480
Matrix ; list ; ba=<x'x>*x'realinv$
      BA|           1
--------+-------------
       1|     -.0429785
       2|      .0194467
       3|       .104480
Matrix ; e = realinv - x*ba$
Calc   ; list ; r2 = 1 - e'e / ((n-1)*var(realinv)) $
[CALC] R2      =         .5400140
```

12. The results cannot be correct. Since log *S/N* = log *S/Y* + log *Y/N* by simple, exact algebra, the same result must apply to the least squares regression results. That means that the second equation estimated must equal the first one plus log *Y/N*. Looking at the equations, that means that all of the coefficients would have to be identical save for the second, which would have to equal its counterpart in the first equation, plus 1. Therefore, the results cannot be correct. In an exchange between Leff and Arthur Goldberger that appeared later in the same journal, Leff argued that the difference was simple rounding error. You can see that the results in the second equation resemble those in the first, but not enough so that the explanation is credible. Further discussion about the data themselves appeared in subsequent discussion. [See Goldberger (1973) and Leff (1973).]

13. a. Consider a regresion of y on x_1, x_2 and x3. The incremental contribution of x_3 will be different depending on whether the order entered is (x_1,x_3,x_2) or (x_1,x_2,x_3), (x_2,x_1,x_3), or (x_2,x_3,x_1).
b. Use the equation above (3-31) and consider x_2 after x_1. If x_1 and x2 are orthogonal, then $\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2 = \mathbf{X}_2'\mathbf{X}_2$ and the result reduces to $R_{1.2}^2 = R_1^2 + R_2^2$. This is the if part. For only if, note that (3-31) implies that if the variables are not orthogonal, then, as observed earlier the previous result cannot hold.
c. Entering T first raises R^2 from 0.00000 to 0.01090. Entering T last raises R^2 from .54013 to .78776.

```
--------------------------------------------------
Ordinary     least squares regression ...........
T entered first R-squared            =        .01090
T not entered   R-squared            =        .54013
T entered last  R-squared            =        .78776
---------------------------------------------------
```

Application

```
?=======================================================================
? Chapter 3 Application 1
?=======================================================================
Read $
(Data appear in the text.)
Namelist ; X1 = one,educ,exp,ability$
Namelist ; X2 = mothered,fathered,sibs$
?=======================================================================
? a.
?=======================================================================
Regress  ; Lhs = wage ; Rhs = x1$
+----------------------------------------------------+
| Ordinary    least squares regression                |
| LHS=WAGE     Mean                    =   2.059333     |
|              Standard deviation      =   .2583869     |
| WTS=none     Number of observs.      =         15     |
| Model size   Parameters              =          4     |
|              Degrees of freedom      =         11     |
| Residuals    Sum of squares          =   .7633163     |
|              Standard error of e     =   .2634244     |
| Fit          R-squared               =   .1833511     |
|              Adjusted R-squared      =  -.3937136E-01 |
| Model test   F[  3,    11] (prob) =    .82 (.5080) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    1.66364000       .61855318     2.690   .0210
 EDUC    |     .01453897       .04902149      .297   .7723   12.8666667
 EXP     |     .07103002       .04803415     1.479   .1673   2.80000000
 ABILITY |     .02661537       .09911731      .269   .7933    .36600000
?=======================================================================
? b.
?=======================================================================
Regress  ; Lhs = wage ; Rhs = x1,x2$
+----------------------------------------------------+
| Ordinary    least squares regression                |
| LHS=WAGE     Mean                    =   2.059333     |
|              Standard deviation      =   .2583869     |
| WTS=none     Number of observs.      =         15     |
| Model size   Parameters              =          7     |
|              Degrees of freedom      =          8     |
| Residuals    Sum of squares          =   .4522662     |
|              Standard error of e     =   .2377673     |
| Fit          R-squared               =   .5161341     |
|              Adjusted R-squared      =   .1532347     |
| Model test   F[  6,     8] (prob) =   1.42 (.3140) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|     .04899633       .94880761      .052   .9601
 EDUC    |     .02582213       .04468592      .578   .5793   12.8666667
 EXP     |     .10339125       .04734541     2.184   .0605   2.80000000
 ABILITY |     .03074355       .12120133      .254   .8062    .36600000
 MOTHERED|     .10163069       .07017502     1.448   .1856   12.0666667
 FATHERED|     .00164437       .04464910      .037   .9715   12.6666667
 SIBS    |     .05916922       .06901801      .857   .4162   2.20000000
?=======================================================================
? c.
```

```
?=======================================================================
Regress  ; Lhs = mothered ; Rhs = x1 ; Res = meds $
Regress  ; Lhs = fathered ; Rhs = x1 ; Res = feds $
Regress  ; Lhs = sibs     ; Rhs = x1 ; Res = sibss $
Namelist ; X2S = meds,feds,sibss $
Matrix   ; list ; Mean(X2S) $
Matrix Result   has  3 rows and  1 columns.
               1
        +--------------
       1| -.1184238D-14
       2|  .1657933D-14
       3| -.5921189D-16
The means are (essentially) zero.  The sums must be zero, as these new
variables are orthogonal to the columns of X1. The first column in X1 is a
column of ones, so this means that these residuals must sum to zero.
?=======================================================================
? d.
?=======================================================================
Namelist ; X = X1,X2 $
Matrix   ; i = init(n,1,1) $
Matrix   ; M0 = iden(n) - 1/n*i*i' $
Matrix   ; b12 = <X'X>*X'wage$
Calc     ; list ; ym0y =(N-1)*var(wage) $
Matrix   ; list ; cod = 1/ym0y * b12'*X'*M0*X*b12 $
Matrix COD      has  1 rows and  1 columns.
               1
        +--------------
       1|     .51613
Matrix   ; e = wage - X*b12 $
Calc     ; list ; cod = 1 - 1/ym0y * e'e $
COD     =       .516134
The R squared is the same using either method of computation.
Calc     ; list ; RsqAd = 1 - (n-1)/(n-col(x))*(1-cod)$
RSQAD   =       .153235
? Now drop the constant
Namelist ; X0 = educ,exp,ability,X2 $
Matrix   ; i = init(n,1,1) $
Matrix   ; M0 = iden(n) - 1/n*i*i' $
Matrix   ; b120 = <X0'X0>*X0'wage$
Matrix   ; list ; cod = 1/ym0y * b120'*X0'*M0*X0*b120 $
Matrix COD      has  1 rows and  1 columns.
               1
        +--------------
       1|     .52953
Matrix   ; e0 = wage - X0*b120 $
Calc     ; list ; cod = 1 - 1/ym0y * e0'e0 $
COD     =       .515973
The R squared now changes depending on how it is computed.  It also goes up,
completely artificially.
?=======================================================================
? e.
?=======================================================================
The R squared for the full regression appears immediately below.
? f.
Regress ; Lhs = wage ; Rhs = X1,X2 $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| WTS=none     Number of observs.   =         15     |
| Model size   Parameters           =          7     |
|              Degrees of freedom   =          8     |
| Fit          R-squared            =   .5161341     |
+----------------------------------------------------+
+--------+-------------+----------------+--------+--------+----------+
```

```
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|     .04899633       .94880761      .052   .9601
 EDUC    |     .02582213       .04468592      .578   .5793   12.8666667
 EXP     |     .10339125       .04734541     2.184   .0605   2.80000000
 ABILITY |     .03074355       .12120133      .254   .8062    .36600000
 MOTHERED|     .10163069       .07017502     1.448   .1856   12.0666667
 FATHERED|     .00164437       .04464910      .037   .9715   12.6666667
 SIBS    |     .05916922       .06901801      .857   .4162   2.20000000
Regress ; Lhs = wage ; Rhs = X1,X2S $
| Ordinary    least squares regression                 |
| WTS=none     Number of observs.   =         15       |
| Model size   Parameters           =          7       |
|              Degrees of freedom   =          8       |
| Fit          R-squared            =   .5161341       |
|              Adjusted R-squared   =   .1532347       |
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    1.66364000       .55830716     2.980   .0176
 EDUC    |     .01453897       .04424689      .329   .7509   12.8666667
 EXP     |     .07103002       .04335571     1.638   .1400   2.80000000
 ABILITY |     .02661537       .08946345      .297   .7737    .36600000
 MEDS    |     .10163069       .07017502     1.448   .1856 -.118424D-14
 FEDS    |     .00164437       .04464910      .037   .9715  .165793D-14
 SIBSS   |     .05916922       .06901801      .857   .4162 -.592119D-16
```

In the first set of results, the first coefficient vector is
$\mathbf{b}_1 = (\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{M}_2\mathbf{y}$ and $\mathbf{b}_2 = (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1}\mathbf{X}_2'M_1\mathbf{y}$
In the second regression, the second set of regressors is M_1X_2, so
$\mathbf{b}_1 = (\mathbf{X}_1'\mathbf{M}_{12}\,\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{M}_{12}\mathbf{y}$ where $\mathbf{M}_{12} = \mathbf{I} - (\mathbf{M}_1\mathbf{X}_2)[(M_1\mathbf{X}_2)'(\mathbf{M}_1\mathbf{X}_2)]^{-1}(\mathbf{M}_1\mathbf{X}_2)'$
Thus, because the "M" matrix is different, the coefficient vector is different. The second set of coefficients in the second regression is
$\mathbf{b}_2 = [(\mathbf{M}_1\mathbf{X}_2)'\mathbf{M}_1(\mathbf{M}_1\mathbf{X}_2)]^{-1}\,(\mathbf{M}_1\mathbf{X}_2)\mathbf{M}_1\mathbf{y} = (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{M}_1\mathbf{y}$ because $\mathbf{M}_1$ is idempotent.