CHAPTER 3

POWER SERIES METHODS

SECTION 3.1

INTRODUCTION AND REVIEW OF POWER SERIES

The power series method consists of substituting a series $y = \Sigma c_n x^n$ into a given differential equation in order to determine what the coefficients $\{c_n\}$ must be in order that the power series will satisfy the equation. It might be pointed out that, if we find a recurrence relation in the form $c_{n+1} = \phi(n)c_n$, then we can determine the radius of convergence ρ of the series solution directly from the recurrence relation

$$\rho = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{\phi(n)}\right|.$$

In Problems 1–10 we give first a recurrence relation that can be used to find the radius of convergence and to calculate the succeeding coefficients $c_1, c_2, c_3, \cdots$ in terms of the arbitrary constant c_0. Then we give the series itself.

1. $c_{n+1} = \dfrac{c_n}{n+1}$; it follows that $c_n = \dfrac{c_0}{n!}$ and $\rho = \lim\limits_{n\to\infty}(n+1) = \infty$.

$$y(x) = c_0\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) = c_0\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = c_0 e^x$$

2. $c_{n+1} = \dfrac{4c_n}{n+1}$; it follows that $c_n = \dfrac{4^n c_0}{n!}$ and $\rho = \lim\limits_{n\to\infty}\dfrac{n+1}{4} = \infty$.

$$y(x) = c_0\left(1 + 4x + 8x^2 + \frac{32x^3}{3} + \frac{32x^4}{4} + \cdots\right)$$

$$= c_0\left(1 + \frac{4x}{1!} + \frac{4^2x^2}{2!} + \frac{4^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots\right) = c_0 e^{4x}$$

3. $c_{n+1} = -\dfrac{3c_n}{2(n+1)}$; it follows that $c_n = \dfrac{(-1)^n 3^n c_0}{2^n n!}$ and $\rho = \lim\limits_{n\to\infty}\dfrac{2(n+1)}{3} = \infty$.

$$y(x) = c_0\left(1 - \frac{3x}{2} + \frac{9x^2}{8} - \frac{9x^3}{16} + \frac{27x^4}{128} - \cdots\right)$$

$$= c_0\left(1-\frac{3x}{1!2}+\frac{3^2x^2}{2!2^2}-\frac{3^3x^3}{3!2^3}+\frac{3^4x^4}{4!2^4}-\cdots\right) = c_0e^{-3x/2}$$

4. When we substitute $y = \Sigma c_n x^n$ into the equation $y'+2xy = 0$, we find that

$$c_1+\sum_{n=0}^{\infty}\left[(n+2)c_{n+2}+2c_n\right]x^{n+1} = 0.$$

Hence $c_1 = 0$ — which we see by equating constant terms on the two sides of this equation — and $c_{n+2} = -\dfrac{2c_n}{n+2}$. It follows that

$$c_1 = c_3 = c_5 = \cdots = c_{\text{odd}} = 0 \quad \text{and} \quad c_{2k} = \frac{(-1)^k c_0}{k!}.$$

Hence

$$y(x) = c_0\left(1-x^2+\frac{x^4}{2}-\frac{x^6}{3}+\cdots\right) = c_0\left(1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots\right) = c_0e^{-x^2}$$

and $\rho = \infty$.

5. When we substitute $y = \Sigma c_n x^n$ into the equation $y' = x^2 y$, we find that

$$c_1+2c_2x+\sum_{n=0}^{\infty}\left[(n+3)c_{n+3}-c_n\right]x^{n+1} = 0.$$

Hence $c_1 = c_2 = 0$ — which we see by equating constant terms and x-terms on the two sides of this equation — and $c_3 = \dfrac{c_n}{n+3}$. It follows that

$$c_{3k+1} = c_{3k+2} = 0 \quad \text{and} \quad c_{3k} = \frac{c_0}{3\cdot 6\cdot\cdots\cdot(3k)} = \frac{c_0}{k!3^k}.$$

Hence

$$y(x) = c_0\left(1+\frac{x^3}{3}+\frac{x^6}{18}+\frac{x^9}{162}+\cdots\right) = c_0\left(1+\frac{x^3}{1!3}+\frac{x^6}{2!3^2}+\frac{x^9}{3!3^3}+\cdots\right) = c_0e^{(x^3/3)}$$

and $\rho = \infty$.

6. $c_{n+1} = \dfrac{c_n}{2}$; it follows that $c_n = \dfrac{c_0}{2^n}$ and $\rho = \lim\limits_{n\to\infty} 2 = 2$.

$$y(x) = c_0\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\frac{x^4}{16}+\cdots\right)$$

$$= c_0\left[1+\left(\frac{x}{2}\right)+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^4+\cdots\right] = \frac{c_0}{1-\frac{x}{2}} = \frac{2c_0}{2-x}$$

7. $c_{n+1} = 2c_n$; it follows that $c_n = 2^n c_0$ and $\rho = \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2}$.

$$y(x) = c_0\left(1+2x+4x^2+8x^3+16x^4+\cdots\right)$$
$$= c_0\left[1+(2x)+(2x)^2+(2x)^3+(2x)^4+\cdots\right] = \frac{c_0}{1-2x}$$

8. $c_{n+1} = -\frac{(2n-1)c_n}{2n+2}$; it follows that $\rho = \lim_{n\to\infty}\frac{2n+2}{2n-1} = 1$.

$$y(x) = c_0\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\cdots\right)$$

Separation of variables gives $y(x) = c_0\sqrt{1+x}$.

9. $c_{n+1} = \frac{(n+2)c_n}{n+1}$; it follows that $c_n = (n+1)c_0$ and $\rho = \lim_{n\to\infty}\frac{n+1}{n+2} = 1$.

$$y(x) = c_0\left(1+2x+3x^2+4x^3+5x^4+\cdots\right)$$

Separation of variables gives $y(x) = \frac{c_0}{(1-x)^2}$.

10. $c_{n+1} = \frac{(2n-3)c_n}{2n+2}$; it follows that $\rho = \lim_{n\to\infty}\frac{2n+2}{2n-3} = 1$.

$$y(x) = c_0\left(1-\frac{3x}{2}+\frac{3x^2}{8}+\frac{x^3}{16}+\frac{3x^4}{128}+\cdots\right)$$

Separation of variables gives $y(x) = c_0(1-x)^{3/2}$.

In Problems 11–14 the differential equations are second-order, and we find that the two initial coefficients c_0 and c_1 are both arbitrary. In each case we find the even-degree coefficients in terms of c_0 and the odd-degree coefficients in terms of c_1. The solution series in these problems are all recognizable power series that have infinite radii of convergence.

11. $c_{n+1} = \frac{c_n}{(n+1)(n+2)}$; it follows that $c_{2k} = \frac{c_0}{(2k)!}$ and $c_{2k+1} = \frac{c_1}{(2k+1)!}$.

$$y(x) = c_0\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)+c_1\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots\right) = c_0\cosh x + c_1\sinh x$$

12. $c_{n+1} = \dfrac{4c_n}{(n+1)(n+2)}$; it follows that $c_{2k} = \dfrac{2^{2k}c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{2^{2k}c_1}{(2k+1)!}$.

$$\begin{aligned} y(x) &= c_0\left(1+2x^2+\frac{2x^4}{3}+\frac{4x^6}{45}+\cdots\right)+c_1\left(x+\frac{2x^3}{3}+\frac{2x^5}{15}+\frac{4x^7}{315}+\cdots\right) \\ &= c_0\left(1+\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}+\frac{(2x)^6}{6!}+\cdots\right)+\frac{c_1}{2}\left((2x)+\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}+\frac{(2x)^7}{7!}+\cdots\right) \\ &= c_0\cosh 2x+\frac{c_1}{2}\sinh 2x \end{aligned}$$

13. $c_{n+1} = -\dfrac{9c_n}{(n+1)(n+2)}$; it follows that $c_{2k} = \dfrac{(-1)^k 3^{2k}c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{(-1)^k 3^{2k}c_1}{(2k+1)!}$.

$$\begin{aligned} y(x) &= c_0\left(1-\frac{9x^2}{2}+\frac{27x^4}{8}-\frac{81x^6}{80}+\cdots\right)+c_1\left(x-\frac{3x^3}{2}+\frac{27x^5}{40}-\frac{81x^7}{560}+\cdots\right) \\ &= c_0\left(1-\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}-\frac{(3x)^6}{6!}+\cdots\right)+\frac{c_1}{3}\left((3x)-\frac{(3x)^3}{3!}+\frac{(3x)^5}{5!}-\frac{(3x)^7}{7!}+\cdots\right) \\ &= c_0\cos 3x+\frac{c_1}{3}\sin x \end{aligned}$$

14. When we substitute $y = \Sigma c_n x^n$ into $y''+y-x = 0$ and split off the terms of degrees 0 and 1, we get

$$(2c_2+c_0)+(6c_3+c_1-1)\,x+\sum_{n=2}^{\infty}[(n+1)(n+2)c_{n+2}+c_n]x^n = 0.$$

Hence $c_2 = -\dfrac{c_0}{2}$, $c_3 = -\dfrac{c_1-1}{6}$, and $c_{n+2} = -\dfrac{c_n}{(n+1)(n+2)}$ for $n \geq 2$. It follows that

$$\begin{aligned} y(x) &= c_0+c_0\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+c_1x+(c_1-1)\left(-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right) \\ &= x+c_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+(c_1-1)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right) \\ &= x+c_0\cos x+(c_1-1)\sin x. \end{aligned}$$

15. Assuming a power series solution of the form $y = \Sigma c_n x^n$, we substitute it into the differential equation $xy'+y=0$ and find that $(n+1)c_n = 0$ for all $n \geq 0$. This implies that $c_n = 0$ for all $n \geq 0$, which means that the only power series solution of our differential equation is the trivial solution $y(x) \equiv 0$. Therefore the equation has no *non-trivial* power series solution.

16. Assuming a power series solution of the form $y = \Sigma c_n x^n$, we substitute it into the differential equation $2xy' = y$ and find that $2nc_n = c_n$ for all $n \geq 0$. This implies that $0c_0 = c_0,\ 2c_1 = c_1,\ 4c_2 = c_2, \cdots$, and hence that $c_n = 0$ for all $n \geq 0$, which means that the only power series solution of our differential equation is the trivial solution $y(x) \equiv 0$. Therefore the equation has no *non-trivial* power series solution.

17. Assuming a power series solution of the form $y = \Sigma c_n x^n$, we substitute it into the differential equation $x^2 y' + y = 0$. We find that $c_0 = c_1 = 0$ and that $c_{n+1} = -nc_n$ for $n \geq 1$, so it follows that $c_n = 0$ for all $n \geq 0$. Just as in Problems 15 and 16, this means that the equation has no *non-trivial* power series solution.

18. When we substitute and assumed power series solution $y = \Sigma c_n x^n$ into $x^3 y' = 2y$, we find that $c_0 = c_1 = c_2 = 0$ and that $c_{n+2} = nc_n/2$ for $n \geq 1$. Hence $c_n = 0$ for all $n \geq 0$, just as in Problems 15–17.

In Problems 19–22 we first give the recurrence relation that results upon substitution of an assumed power series solution $y = \Sigma c_n x^n$ into the given second-order differential equation. Then we give the resulting general solution, and finally apply the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ to determine the desired particular solution.

19. $c_{n+2} = -\dfrac{2^2 c_n}{(n+1)(n+2)}$ for $n \geq 0$, so $c_{2k} = \dfrac{(-1)^k 2^{2k} c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{(-1)^k 2^{2k} c_1}{(2k+1)!}$.

$$y(x) = c_0\left(1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \cdots\right) + c_1\left(x - \frac{2^2x^3}{3!} + \frac{2^4x^5}{5!} - \frac{2^6x^7}{7!} + \cdots\right)$$

$c_0 = y(0) = 0$ and $c_1 = y'(0) = 3$, so

$$\begin{aligned} y(x) &= 3\left(x - \frac{2^2x^3}{3!} + \frac{2^4x^5}{5!} - \frac{2^6x^7}{7!} + \cdots\right) \\ &= \frac{3}{2}\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots\right] = \frac{3}{2}\sin 2x. \end{aligned}$$

20. $c_{n+2} = \dfrac{2^2 c_n}{(n+1)(n+2)}$ for $n \geq 0$, so $c_{2k} = \dfrac{2^{2k} c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{2^{2k} c_1}{(2k+1)!}$.

$$y(x) = c_0\left(1 + \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} + \frac{2^6x^6}{6!} + \cdots\right) + c_1\left(x + \frac{2^2x^3}{3!} + \frac{2^4x^5}{5!} + \frac{2^6x^7}{7!} + \cdots\right)$$

$c_0 = y(0) = 2$ and $c_1 = y'(0) = 0$, so

$$y(x) = 2\left(1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} + \cdots\right) = 2\cosh 2x.$$

21. $c_{n+1} = \dfrac{2nc_n - c_{n-1}}{n(n+1)}$ for $n \geq 1$; with $c_0 = y(0) = 0$ and $c_1 = y'(0) = 1$, we obtain

$c_2 = 1,\ c_3 = \dfrac{1}{2},\ c_4 = \dfrac{1}{6} = \dfrac{1}{3!},\ c_5 = \dfrac{1}{24} = \dfrac{1}{4!},\ c_6 = \dfrac{1}{120} = \dfrac{1}{5!}$. Evidently $c_n = \dfrac{1}{(n-1)!}$, so

$$y(x) = x + x^2 + \frac{x^3}{2!} + \frac{x^3}{3!} + \frac{x^5}{4!} + \cdots = x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = xe^x.$$

22. $c_{n+1} = -\dfrac{nc_n - 2c_{n-1}}{n(n+1)}$ for $n \geq 1$; with $c_0 = y(0) = 1$ and $c_1 = y'(0) = -2$, we obtain

$c_2 = 2,\ c_3 = -\dfrac{4}{3} = -\dfrac{2^3}{3!},\ c_4 = \dfrac{2}{3} = \dfrac{2^4}{4!},\ c_5 = -\dfrac{4}{15} = -\dfrac{2^5}{5!}$. Apparently $c_n = \pm\dfrac{2^n}{n!}$, so

$$y(x) = 1 - (2x) + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} - \frac{(2x)^5}{5!} + \cdots = e^{-2x}.$$

23. $c_0 = c_1 = 0$ and the recursion relation

$$(n^2 - n + 1)c_n + (n-1)c_{n-1} = 0$$

for $n \geq 2$ imply that $c_n = 0$ for $n \geq 0$. Thus any assumed power series solution $y = \Sigma c_n x^n$ must reduce to the trivial solution $y(x) \equiv 0$.

24. **(a)** The fact that $y(x) = (1+x)^\alpha$ satisfies the differential equation $(1+x)y' = \alpha y$ follows immediately from the fact that $y'(x) = \alpha(1+x)^{\alpha-1}$.

(b) When we substitute $y = \Sigma c_n x^n$ into the differential equation $(1+x)y' = \alpha y$ we get the recurrence formula

$$c_{n+1} = \frac{(\alpha - n)c_n}{n+1}.\ c_{n+1} = (\alpha - n)c_n/(n+1).$$

Since $c_0 = 1$ because of the initial condition $y(0) = 1$, the binomial series (Equation (12) in the text) follows.

(c) The function $(1+x)^\alpha$ and the binomial series must agree on $(-1, 1)$ because of the uniqueness of solutions of linear initial value problems.

25. Substitution of $\sum_{n=0}^{\infty} c_n x^n$ into the differential equation $y'' = y' + y$ leads routinely — via shifts of summation to exhibit x^n-terms throughout — to the recurrence formula

$$(n+2)(n+1)c_{n+2} = (n+1)c_{n+1} + c_n,$$

and the given initial conditions yield $c_0 = 0 = F_0$ and $c_1 = 1 = F_1$. But instead of proceeding immediately to calculate explicit values of further coefficients, let us first multiply the recurrence relation by $n!$. This trick provides the relation

$$(n+2)!c_{n+2} = (n+1)!c_{n+1} + n!c_n,$$

that is, the Fibonacci-defining relation $F_{n+2} = F_{n+1} + F_n$ where $F_n = n!c_n$, so we see that $c_n = F_n / n!$ as desired.

26. This problem is pretty fully outlined in the textbook. The only hard part is squaring the power series:

$$\begin{aligned}\left(1 + c_3x^3 + c_5x^5 + c_7x^7 + c_9x^9 + c_{11}x^{11} + \cdots\right)^2 \\ = x^2 + 2c_3x^4 + \left(c_3^2 + 2c_5\right)x^6 + \left(2c_3c_5 + 2c_7\right)x^8 + \\ \left(c_5^2 + 2c_3c_7 + 2c_9\right)x^{10} + \left(2c_5c_7 + 2c_3c_9 + 2c_{11}\right)x^{12} + \cdots\end{aligned}$$

27. **(b)** The roots of the characteristic equation $r^3 = 1$ are $r_1 = 1$, $r_2 = \alpha = (-1 + i\sqrt{3})/2$, and $r_3 = \beta = (-1 - i\sqrt{3})/2$. Then the general solution is

$$y(x) = Ae^x + Be^{\alpha x} + Ce^{\beta x}. \qquad (*)$$

Imposing the initial conditions, we get the equations

$$\begin{aligned} A + B + C &= 1 \\ A + \alpha B + \beta C &= 1 \\ A + \alpha^2 B + \beta^2 C &= -1. \end{aligned}$$

The solution of this system is $A = 1/3$, $B = (1 - i\sqrt{3})/3$, $C = (1 + i\sqrt{3})/3$. Substitution of these coefficients in (*) and use of Euler's relation $e^{i\theta} = \cos\theta + i\sin\theta$ finally yields the desired result.

SECTION 3.2

SERIES SOLUTIONS NEAR ORDINARY POINTS

Instead of deriving in detail the recurrence relations and solution series for Problems 1 through 15, we indicate where some of these problems and answers originally came from. Each of the differential equations in Problems 1–10 is of the form

$$(Ax^2 + B)y'' + Cxy' + Dy = 0$$

with selected values of the constants A, B, C, D. When we substitute $y = \Sigma c_n x^n$, shift indices where appropriate, and collect coefficients, we get

$$\sum_{n=0}^{\infty}\left[An(n-1)c_n + B(n+1)(n+2)c_{n+2} + Cnc_n + Dc_n\right]x^n = 0.$$

Thus the recurrence relation is

$$c_{n+2} = -\frac{An^2 + (C-A)n + D}{B(n+1)(n+2)}c_n \quad \text{for } n \geq 0.$$

It yields a solution of the form

$$y = c_0\, y_{\text{even}} + c_1\, y_{\text{odd}}$$

where y_{even} and y_{odd} denote series with terms of even and odd degrees, respectively. The even-degree series $c_0 + c_2x^2 + c_4x^4 + \cdots$ converges (by the ratio test) provided that

$$\lim_{n\to\infty}\left|\frac{c_{n+2}x^{n+2}}{c_nx^n}\right| = \left|\frac{Ax^2}{B}\right| < 1.$$

Hence its radius of convergence is at least $\rho = \sqrt{|B/A|}$, as is that of the odd-degree series $c_1x + c_3x^3 + c_5x^4 + \cdots$. (See Problem 6 for an example in which the radius of convergence is, surprisingly, greater than $\sqrt{|B/A|}$.)

In Problems 1–15 we give first the recurrence relation and the radius of convergence, then the resulting power series solution.

1. $c_{n+2} = c_n;\qquad \rho = 1;\qquad c_0 = c_2 = c_4 = \cdots;\qquad c_1 = c_3 = c_4 = \cdots$

$$y(x) = c_0\sum_{n=0}^{\infty}x^{2n} + c_1\sum_{n=0}^{\infty}x^{2n+1} = \frac{c_0 + c_1x}{1-x^2}$$

2. $c_{n+2} = -\frac{1}{2}c_n;\qquad \rho = 2;\qquad c_{2n} = \frac{(-1)^n c_0}{2^n};\qquad c_{2n+1} = \frac{(-1)^n c_1}{2^n}$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n} + c_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^n}$$

3. $c_{n+2} = -\frac{c_n}{(n+2)};\qquad \rho = \infty;$

$$c_{2n} = \frac{(-1)^n c_0}{(2n)(2n-2)\cdots\cdots 4\cdot 2} = \frac{(-1)^n c_0}{n!2^n};$$

$$c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)(2n-1)\cdots\cdots 5\cdot 3} = \frac{(-1)^n c_1}{(2n+1)!!}$$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!2^n} + c_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!!}$$

4. $$c_{n+2} = -\frac{n+4}{n+2}c_n; \qquad \rho = 1$$

$$c_{2n} = \left(-\frac{2n+2}{2n}\right)\left(-\frac{2n}{2n-2}\right)\cdots\cdots\left(-\frac{6}{4}\right)\left(-\frac{4}{2}\right)c_0 = (-1)^n\frac{2n+2}{2}c_0 = (-1)^n(n+1)c_0$$

$$c_{2n} = \left(-\frac{2n+3}{2n+1}\right)\left(-\frac{2n+1}{2n-1}\right)\cdots\cdots\left(-\frac{7}{5}\right)\left(-\frac{5}{3}\right)c_0 = (-1)^n\frac{2n+3}{3}c_1$$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n} + \frac{1}{3}c_1\sum_{n=0}^{\infty}(-1)^n(2n+3)x^{2n+1}$$

5. $$c_{n+2} = \frac{nc_n}{3(n+2)}; \qquad \rho = 3; \qquad c_2 = c_4 = c_6 = \cdots = 0$$

$$c_{2n+1} = \frac{2n-1}{3(2n+1)}\cdot\frac{2n-3}{3(2n-1)}\cdots\cdots\frac{3}{3(5)}\cdot\frac{1}{3(3)}c_1 = \frac{c_1}{(2n+1)3^n}$$

$$y(x) = c_0 + c_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)3^n}$$

6. $$c_{n+2} = \frac{(n-3)(n-4)}{(n+1)(n+2)}c_n$$

The factor $(n-3)$ in the numerator yields $c_5 = c_7 = c_9 = \cdots = 0$, and the factor $(n-4)$ yields $c_6 = c_8 = c_{10} = \cdots = 0$. Hence y_{even} and y_{odd} are both polynomials with radius of convergence $\rho = \infty$.

$$y(x) = c_0(1+6x^2+x^4) + c_1(x+x^3)$$

7. $$c_{n+2} = -\frac{(n-4)^2}{3(n+1)(n+2)}c_n; \qquad \rho \geq \sqrt{3}$$

The factor $(n-4)$ yields $c_6 = c_8 = c_{10} = \cdots = 0$, so y_{even} is a 4th-degree polynomial.

We find first that $c_3 = -c_1/2$ and $c_5 = c_1/120$, and then for $n \geq 3$ that

$$c_{2n+1} = \left(-\frac{(2n-5)^2}{3(2n)(2n+1)}\right)\left(-\frac{(2n-7)^2}{3(2n-2)(2n-1)}\right)\cdots\cdots\left(-\frac{1^2}{3(6)(7)}\right)c_5 =$$

$$= (-1)^{n-2}\frac{\left[(2n-5)!!\right]^2}{3^{n-2}(2n+1)(2n-1)\cdots\cdots 7\cdot 6}\cdot\frac{c_1}{120} = 9\cdot(-1)^n\frac{\left[(2n-5)!!\right]^2}{3^n(2n+1)!}c_1$$

$$y(x) = c_0\left(1-\frac{8}{3}x^2+\frac{8}{27}x^4\right)+c_1\left[x-\frac{1}{2}x^3+\frac{1}{120}x^5+9\sum_{n=3}^{\infty}\frac{[(2n-5)!!]^2(-1)^n}{(2n+1)!\,3^n}x^{2n+1}\right]$$

8. $$c_{n+2} = \frac{(n-4)(n+4)}{2(n+1)(n+2)}c_n; \qquad \rho \geq \sqrt{2}$$

We find first that $c_3 = -5c_1/4$ and $c_5 = 7c_1/32$, and then for $n \geq 3$ that

$$c_{2n+1} = \left(\frac{(2n-5)(2n+3)}{2(2n)(2n+1)}\right)\left(\frac{(2n-7)(2n+1)}{2(2n-2)(2n-1)}\right)\cdots\cdots\left(\frac{1\cdot 9}{2(6)(7)}\right)c_5 =$$

$$= \frac{(2n-5)!!(2n+3)(2n+1)\cdots\cdots 9}{2^{n-2}(2n+1)(2n)\cdots\cdots 7\cdot 6}\cdot\frac{7c_1}{32} = 4\cdot\frac{5!}{7\cdot 5\cdot 3}\cdot\frac{7}{32}\cdot\frac{(2n-5)!!(2n+3)!!}{2^n(2n+1)!}c_1$$

$$c_{2n+1} = \frac{(2n-5)!!(2n+3)!!}{2^n(2n+1)!}c_1$$

$$y(x) = c_0\left(1-4x^2+2x^4\right)+c_1\left[x-\frac{5}{4}x^3+\frac{7}{32}x^5+\sum_{n=3}^{\infty}\frac{(2n-5)!!(2n+3)!!}{(2n+1)!\,2^n}x^{2n+1}\right]$$

9. $$c_{n+2} = \frac{(n+3)(n+4)}{(n+1)(n+2)}c_n; \qquad \rho = 1$$

$$c_{2n} = \frac{(2n+1)(2n+2)}{(2n-1)(2n)}\cdot\frac{(2n-1)(2n)}{(2n-3)(2n-2)}\cdots\cdots\frac{3\cdot 4}{1\cdot 2}c_0 = \frac{1}{2}(n+1)(2n+1)c_0$$

$$c_{2n+1} = \frac{(2n+2)(2n+3)}{(2n)(2n+1)}\cdot\frac{(2n)(2n+1)}{(2n-2)(2n-1)}\cdots\cdots\frac{4\cdot 5}{2\cdot 3}c_1 = \frac{1}{3}(n+1)(2n+3)c_1$$

$$y(x) = c_0\sum_{n=0}^{\infty}(n+1)(2n+1)x^{2n}+\frac{1}{3}c_1\sum_{n=0}^{\infty}(n+1)(2n+3)x^{2n+1}$$

10. $$c_{n+2} = -\frac{(n-4)}{3(n+1)(n+2)}c_n; \qquad \rho = \infty$$

The factor $(n-4)$ yields $c_6 = c_8 = c_{10} = \cdots = 0$, so y_{even} is a 4th-degree polynomial. We find first that $c_3 = c_1/6$ and $c_5 = c_1/360$, and then for $n \geq 3$ that

$$c_{2n+1} = \frac{-(2n-5)}{3(2n+1)(2n)} \cdot \frac{-(2n-3)}{3(2n-1)(2n-2)} \cdot \cdots \cdot \frac{-1}{3(7)(6)} c_5$$

$$= \frac{(2n-5)!!(-1)^{n-2}}{3^{n-2}(2n+1)(2n)\cdot\cdots\cdot(7)(6)} \cdot \frac{c_1}{360} =$$

$$= \frac{3^2 \cdot 5!}{360} \cdot \frac{(2n-5)!!(-1)^n}{3^n(2n+1)(2n)\cdot\cdots\cdot(7)(6)\cdot 5!} \cdot c_1 = 3 \cdot \frac{(2n-5)!!(-1)^n}{3^n(2n+1)!} c_1$$

$$y(x) = c_0\left(1+\frac{2}{3}x^2+\frac{1}{27}x^4\right)+c_1\left[x+\frac{1}{6}x^3+\frac{1}{360}x^5+3\sum_{n=3}^{\infty}\frac{(2n-5)!!(-1)^n}{(2n+1)!\,3^n}x^{2n+1}\right]$$

11. $c_{n+2} = \dfrac{2(n-5)}{5(n+1)(n+2)}c_n; \qquad \rho = \infty$

The factor $(n-5)$ yields $c_7 = c_9 = c_{11} = \cdots = 0$, so y_{odd} is a 5th-degree polynomial. We find first that $c_2 = -c_1$, $c_4 = c_0/10$ and $c_6 = c_0/750$, and then for $n \geq 4$ that

$$c_{2n} = \frac{2(2n-7)}{5(2n)(2n-1)} \cdot \frac{2(2n-5)}{5(2n-2)(2n-3)} \cdot \cdots \cdot \frac{2(1)}{5(8)(7)} c_6$$

$$= \frac{2^{n-3}(2n-7)!!}{5^{n-3}(2n)(2n-1)\cdot\cdots\cdot(8)(7)} \cdot \frac{c_0}{750} =$$

$$= \frac{5^3 \cdot 6!}{2^3 \cdot 750} \cdot \frac{2^n(2n-7)!!}{5^n(2n)(2n)\cdot\cdots\cdot(8)(7)\cdot 6!} \cdot c_1 = 15 \cdot \frac{2^n(2n-7)!!}{5^n(2n)!} c_0$$

$$y(x) = c_1\left(x-\frac{4x^3}{15}+\frac{4x^5}{375}\right)+c_0\left[1-x^2+\frac{x^4}{10}+\frac{x^6}{750}+15\sum_{n=4}^{\infty}\frac{(2n-7)!!\,2^n}{(2n)!\,5^n}x^{2n}\right]$$

12. $c_{n+3} = \dfrac{c_n}{n+2}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0$, so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also.

$$y(x) = c_0\left[1+\sum_{n=1}^{\infty}\frac{x^{3n}}{2\cdot 5\cdot\cdots\cdot(3n-1)}\right]+c_1\sum_{n=0}^{\infty}\frac{x^{3n+1}}{n!\,3^n}$$

13. $c_{n+3} = -\dfrac{c_n}{n+3}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0$, so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also.

$$y(x) = c_0\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{n!\,3^n}+c_1\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+1}}{1\cdot 4\cdot\cdots\cdot(3n+1)}$$

14. $c_{n+3} = -\dfrac{c_n}{(n+2)(n+3)}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0$, so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also. Then

$$c_{3n} = \frac{-1}{(3n)(3n-1)} \cdot \frac{-1}{(3n-3)(3n-4)} \cdots \frac{-1}{3 \cdot 2} c_0 = \frac{(-1)^n c_0}{3^n n! \cdot (3n-1)(3n-4) \cdots 5 \cdot 2},$$

$$c_{3n+1} = \frac{-1}{(3n+1)(3n)} \cdot \frac{-1}{(3n-2)(3n-3)} \cdots \frac{-1}{4 \cdot 3} c_1 = \frac{(-1)^n c_1}{3^n n! \cdot (3n+1)(3n-2) \cdots 4 \cdot 1}.$$

$$y(x) = c_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{3n}}{3^n n! \cdot 2 \cdot 5 \cdots (3n-1)} \right] + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3^n n! \cdot 1 \cdot 4 \cdots (3n+1)}$$

15. $c_{n+4} = -\dfrac{c_n}{(n+3)(n+4)}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = c_3 = 0$, so the recurrence relation yields $c_6 = c_{10} = \cdots = 0$ and $c_7 = c_{11} = \cdots = 0$ also. Then

$$c_{4n} = \frac{-1}{(4n)(4n-1)} \cdot \frac{-1}{(4n-4)(4n-5)} \cdots \frac{-1}{4 \cdot 3} c_0 = \frac{(-1)^n c_0}{4^n n! \cdot (4n-1)(4n-5) \cdots 5 \cdot 3},$$

$$c_{3n+1} = \frac{-1}{(4n+1)(4n)} \cdot \frac{-1}{(4n-3)(4n-4)} \cdots \frac{-1}{5 \cdot 4} c_1 = \frac{(-1)^n c_1}{4^n n! \cdot (4n+1)(4n-3) \cdots 9 \cdot 5}.$$

$$y(x) = c_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{4^n n! \cdot 3 \cdot 7 \cdots (4n-1)} \right] + c_1 \left[x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+1}}{4^n n! \cdot 5 \cdot 9 \cdots (4n+1)} \right]$$

16. The recurrence relation is $c_{n+2} = -\dfrac{n-1}{n+1} c_n$ for $n \geq 1$. The factor $(n-1)$ in the numerator yields $c_3 = c_5 = c_7 = \cdots = 0$. When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = c_0$, and then the recurrence relation gives

$$c_{2n} = -\frac{2n-3}{2n-1} \cdot -\frac{2n-5}{2n-3} \cdots -\frac{3}{5} \cdot -\frac{1}{3} c_2 = \frac{(-1)^{n-1}}{2n-1} c_0.$$

Hence

$$y(x) = c_1 x + c_0 \left(1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \cdots \right)$$

$$= c_1 x + c_0 + c_0 x \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) = c_1 x + c_0 \left(1 + x \tan^{-1} x \right).$$

With $c_0 = y(0) = 0$ and $c_1 = y'(0) = 1$ we obtain the particular solution $y(x) = x$.

17. The recurrence relation

$$c_{n+2} = -\frac{(n-2)c_n}{(n+1)(n+2)}$$

yields $c_2 = c_0 = y(0) = 1$ and $c_4 = c_6 = \cdots = 0$. Because $c_1 = y'(0) = 0$, it follows also that $c_1 = c_3 = c_5 = \cdots = 0$. Thus the desired particular solution is $y(x) = 1 + x^2$.

18. The substitution $t = x - 1$ yields $y'' + ty' + y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{c_n}{n+2}.$$

for $n \geq 0$, so the solution series has radius of convergence $\rho = \infty$. The initial conditions give $c_0 = 2$ and $c_1 = 0$, so $c_{\text{odd}} = 0$ and it follows that

$$y = 2\left(1 - \frac{t^2}{2} + \frac{t^4}{2\cdot 4} - \frac{t^6}{2\cdot 4\cdot 6} + \cdots\right),$$

$$y(x) = 2\left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^4}{2\cdot 4} - \frac{(x-1)^6}{2\cdot 4\cdot 6} + \cdots\right) = 2\sum_{n=0}^{\infty}\frac{(-1)^n (x-1)^{2n}}{n!2^n}.$$

19. The substitution $t = x - 1$ yields $(1 - t^2)y'' - 6ty' - 4y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = \frac{n+4}{n+2}c_n.$$

for $n \geq 0$, so the solution series has radius of convergence $\rho = 1$, and therefore converges if $-1 < t < 1$. The initial conditions give $c_0 = 0$ and $c_1 = 1$, so $c_{\text{even}} = 0$ and

$$c_{2n+1} = \frac{2n+3}{2n+1}\cdot\frac{2n+1}{2n-1}\cdot\cdots\cdot\frac{7}{5}\cdot\frac{5}{3}c_1 = \frac{2n+3}{3}.$$

Thus

$$y = \frac{1}{3}\sum_{n=0}^{\infty}(2n+3)t^{2n+1} = \frac{1}{3}\sum_{n=0}^{\infty}(2n+3)(x-1)^{2n+1},$$

and the x-series converges if $0 < x < 2$.

20. The substitution $t = x - 3$ yields $(t^2 + 1)y'' - 4ty' + 6y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}c_n$$

for $n \geq 0$. The initial conditions give $c_0 = 2$ and $c_1 = 0$. It follows that $c_{\text{odd}} = 0$, $c_2 = -6$ and $c_4 = c_6 = \cdots = 0$, so the solution reduces to

$$y = 2 - 6t^2 = 2 - 6(x-3)^2.$$

21. The substitution $t = x+2$ yields $(4t^2+1)y'' = 8y$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{4(n-2)}{(n+2)}c_n$$

for $n \geq 0$. The initial conditions give $c_0 = 1$ and $c_1 = 0$. It follows that $c_{\text{odd}} = 0$, $c_2 = 4$ and $c_4 = c_6 = \cdots = 0$, so the solution reduces to

$$y = 2 + 4t^2 = 1 + 4(x+2)^2.$$

22. The substitution $t = x+3$ yields $(t^2-9)y'' + 3ty' - 3y = 0$, with primes now denoting differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = \frac{(n+3)(n-1)}{9(n+1)(n+2)}c_n$$

for $n \geq 0$. The initial conditions give $c_0 = 0$ and $c_1 = 2$. It follows that $c_{\text{even}} = 0$ and $c_3 = c_5 = \cdots = 0$, so

$$y = 2t = 2x + 6.$$

In Problems 23–26 we first derive the recurrence relation, and then calculate the solution series $y_1(x)$ with $c_0 = 1$ and $c_1 = 0$ as well as the solution series $y_2(x)$ with $c_0 = 0$ and $c_1 = 1$.

23. Substitution of $y = \Sigma c_n x^n$ yields

$$c_0 + 2c_2 + \sum_{n=1}^{\infty}\left[c_{n-1} + c_n + (n+1)(n+2)c_{n+2}\right]x^n = 0,$$

so

$$c_2 = -\frac{1}{2}c_0, \qquad c_{n+2} = -\frac{c_{n-1}+c_n}{(n+1)(n+2)} \quad \text{for } n \geq 1.$$

$$y_1(x) = 1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots; \qquad y_2(x) = x - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{120} + \cdots$$

24. Substitution of $y = \Sigma c_n x^n$ yields

$$-2c_2 + \sum_{n=1}^{\infty}\left[2c_{n-1} + n(n+1)c_n - (n+1)(n+2)c_{n+2}\right]x^n = 0,$$

so

$$c_2 = 0, \qquad c_{n+2} = \frac{2c_{n-1} + n(n+1)c_n}{(n+1)(n+2)} \quad \text{for} \quad n \geq 1.$$

$$y_1(x) = 1 + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^6}{45} + \cdots; \qquad y_2(x) = x + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{5} + \cdots$$

25. Substitution of $y = \Sigma c_n x^n$ yields

$$2c_2 + 6c_3 x + \sum_{n=2}^{\infty} \left[c_{n-2} + (n-1)c_{n-1} + (n+1)(n+2)c_{n+2} \right] x^n = 0,$$

so

$$c_2 = c_3 = 0, \qquad c_{n+2} = -\frac{c_{n-2} + (n-1)c_{n-1}}{(n+1)(n+2)} \quad \text{for} \quad n \geq 2.$$

$$y_1(x) = 1 - \frac{x^4}{12} + \frac{x^7}{126} + \frac{x^8}{672} + \cdots; \qquad y_2(x) = x - \frac{x^4}{12} - \frac{x^5}{20} + \frac{x^7}{126} + \cdots$$

26. Substitution of $y = \Sigma c_n x^n$ yields

$$2c_2 + 6c_3 x + 12c_4 x^2 + (2c_2 + 20c_5)x^3 + \sum_{n=4}^{\infty} \left[c_{n-4} + (n-1)(n-2)c_{n-1} + (n+1)(n+2)c_{n+2} \right] x^n = 0,$$

so

$$c_2 = c_3 = c_4 = c_5 = 0, \qquad c_{n+2} = -\frac{c_{n-4} + (n-1)(n-2)c_{n-1}}{(n+1)(n+2)} \quad \text{for} \quad n \geq 4.$$

$$y_1(x) = 1 - \frac{x^6}{30} + \frac{x^9}{72} - \frac{29x^{12}}{3960} + \cdots; \qquad y_2(x) = x - \frac{x^7}{42} + \frac{x^{10}}{90} - \frac{41x^{13}}{6552} + \cdots$$

27. Substitution of $y = \Sigma c_n x^n$ yields

$$c_0 + 2c_2 + (2c_1 + 6c_3)x + \sum_{n=2}^{\infty} \left[2c_{n-2} + (n+1)c_n + (n+1)(n+2)c_{n+2} \right] x^n = 0,$$

so

$$c_2 = -\frac{c_0}{2}, \quad c_3 = -\frac{c_1}{3}, \qquad c_{n+2} = -\frac{2c_{n-2} + (n+1)c_n}{(n+1)(n+2)} \quad \text{for} \quad n \geq 2.$$

With $c_0 = y(0) = 1$ and $c_1 = y'(0) = -1$, we obtain

$$y(x) = 1 - x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{24} + \frac{x^5}{30} + \frac{29x^6}{720} - \frac{13x^7}{630} - \frac{143x^8}{40320} + \frac{31x^9}{22680} + \cdots.$$

Finally, $x = 0.5$ gives

$$\begin{aligned} y(0.5) &= 1 - 0.5 - 0.125 + 0.041667 - 0.002604 + 0.001042 \\ &\qquad + 0.000629 - 0.000161 - 0.000014 + 0.000003 + \cdots \\ y(0.5) &\approx 0.415562 \approx 0.4156. \end{aligned}$$

28. When we substitute $y = \Sigma c_n x^n$ and $e^{-x} = \sum (-1)^n x^n / n!$ and then collect coefficients of the terms involving 1, x, x^2, and x^3, we find that

$$c_2 = -\frac{c_0}{2}, \quad c_3 = \frac{c_0 - c_1}{6}, \quad c_4 = \frac{c_1}{12}, \quad c_5 = -\frac{3c_0 + 2c_1}{120}.$$

With the choices $c_0 = 1$, $c_1 = 0$ and $c_0 = 0$, $c_1 = 1$ we obtain the two series solutions

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{40} + \cdots \quad \text{and} \quad y_2(x) = x - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{60} + \cdots.$$

29. When we substitute $y = \Sigma c_n x^n$ and $\cos x = \sum (-1)^n x^{2n} / (2n)!$ and then collect coefficients of the terms involving $1, x, x^2, \cdots, x^6$, we obtain the equations

$$c_0 + 2c_2 = 0, \quad c_1 + 6c_3 = 0, \quad 12c_4 = 0, \quad -2c_3 + 20c_5 = 0,$$

$$\frac{1}{12}c_2 - 5c_4 + 30c_6 = 0, \quad \frac{1}{4}c_3 - 9c_5 + 42c_6 = 0,$$

$$-\frac{1}{360}c_2 + \frac{1}{2}c_4 - 14c_6 + 56c_8 = 0.$$

Given c_0 and c_1, we can solve easily for $c_2, c_3, \cdots, c_8$ in turn. With the choices $c_0 = 1$, $c_1 = 0$ and $c_0 = 0$, $c_1 = 1$ we obtain the two series solutions

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^6}{720} + \frac{13x^8}{40320} + \cdots \quad \text{and} \quad y_2(x) = x - \frac{x^3}{6} - \frac{x^5}{60} - \frac{13x^7}{5040} + \cdots.$$

30. When we substitute $y = \Sigma c_n x^n$ and $\sin x = \Sigma (-1)^n x^{2n+1}/(2n+1)!$, and then collect coefficients of the terms involving $1, x, x^2, \cdots, x^5$, we obtain the equations

$$c_0 + c_1 + 2c_2 = 0, \quad c_1 + 2c_2 + 6c_3 = 0, \quad -\frac{c_1}{6} + c_2 + 3c_3 + 12c_4 = 0,$$

$$-\frac{c_2}{3} + c_3 + 4c_4 + 20c_5 = 0, \quad \frac{c_1}{120} - \frac{c_3}{2} + c_4 + 5c_5 + 30c_6 = 0.$$

Given c_0 and c_1, we can solve easily for $c_2, c_3, \cdots, c_6$ in turn. With the choices $c_0 = 1$, $c_1 = 0$ and $c_0 = 0$, $c_1 = 1$ we obtain the two series solutions

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{60} + \frac{x^6}{180} + \cdots \text{ and } y_2(x) = x - \frac{x^2}{2} + \frac{x^4}{18} - \frac{7x^5}{360} + \frac{x^6}{900} + \cdots.$$

33. Substitution of $y = \Sigma c_n x^n$ in Hermite's equation leads in the usual way to the recurrence formula

$$c_{n+2} = -\frac{2(\alpha - n)c_n}{(n+1)(n+2)}.$$

Starting with $c_0 = 1$, this formula yields

$$c_2 = -\frac{2\alpha}{2!}, \quad c_4 = +\frac{2^2\alpha(\alpha-2)}{4!}, \quad c_6 = -\frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}, \ \ldots.$$

Starting with $c_1 = 1$, it yields

$$c_3 = -\frac{2(\alpha-1)}{3!}, \quad c_5 = +\frac{2^2(\alpha-1)(\alpha-3)}{5!}, \quad c_7 = -\frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}, \ \ldots.$$

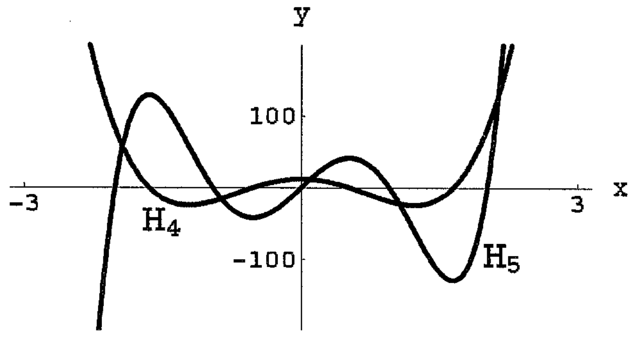
This gives the desired even-term and odd-term series y_1 and y_2. If α is an integer, then obviously one series or the other has only finitely many non-zero terms. For instance, with $\alpha = 4$ we get

$$y_1(x) = 1 - \frac{2\cdot 4}{2}x^2 + \frac{2^2\cdot 4\cdot 2}{24}x^4 = 1 - 4x^2 + \frac{4}{3}x^4 = \frac{1}{12}\left(16x^4 - 48x^2 + 12\right),$$

and with $\alpha = 5$ we get

$$y_2(x) = x - \frac{2\cdot 4}{6}x^3 + \frac{2^2\cdot 4\cdot 2}{120}x^5 = x - \frac{4}{3}x^3 + \frac{4}{15}x^5 = \frac{1}{120}\left(32x^5 - 160x^3 + 120\right).$$

The figure below shows the interlaced zeros of the 4th and 5th Hermite polynomials.



34. Substitution of $y = \Sigma c_n x^n$ in the Airy equation leads upon shift of index and collection of terms to

$$2c_2 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)c_{n+2} - c_{n-1}\right]x^n = 0.$$

The identity principle then gives $c_2 = 0$ and the recurrence formula

$$c_{n+3} = \frac{c_n}{(n+2)(n+3)}.$$

Because of the "3-step" in indices, it follows that $c_2 = c_5 = c_8 = c_{11} = \cdots = 0$. Starting with $c_0 = 1$, we calculate

$$c_3 = \frac{1}{2\cdot 3} = \frac{1}{3!} =, \quad c_6 = \frac{1}{3!\cdot 5\cdot 6} = \frac{1\cdot 4}{6!}, \quad c_9 = \frac{1\cdot 4}{6!\cdot 8\cdot 9} = \frac{1\cdot 4\cdot 7}{9!}, \ \ldots.$$

Starting with $c_1 = 1$, we calculate

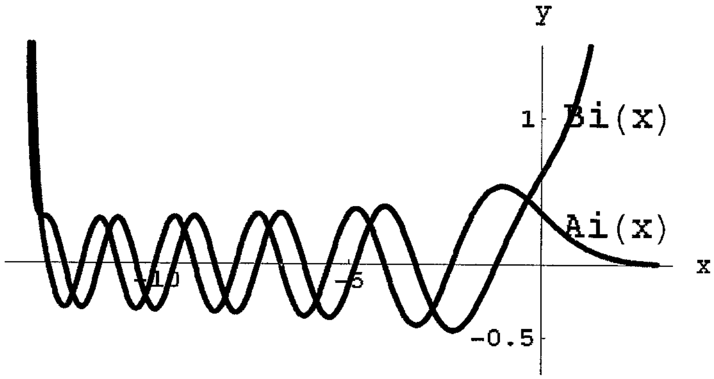
$$c_4 = \frac{1}{3\cdot 4} = \frac{2}{4!}, \quad c_7 = \frac{2}{4!\cdot 6\cdot 7} = \frac{2\cdot 5}{7!}, \quad c_{10} = \frac{2\cdot 5}{7!\cdot 9\cdot 10} = \frac{2\cdot 5\cdot 8}{10!}, \ \ldots.$$

Evidently we are building up the coefficients

$$c_{3k} = \frac{1\cdot 4\cdots(3k-2)}{(3k)!} \quad \text{and} \quad c_{3k+1} = \frac{2\cdot 5\cdots(3k-1)}{(3k+1)!}$$

that appear in the desired series for $y_1(x)$ and $y_2(x)$. Finally, the Mathematica commands

```
A[1] = 1/6; A[k_] := A[k - 1]/(3k (3k - 1))
B[1] = 1/12; B[k_] := B[k - 1]/(3k (3k + 1))
n = 40;
y1 = 1 + Sum[A[k] x^(3k), {k, 1, n}];
y2 = x + Sum[B[k] x^(3k+1), {k, 1, n}];
yA = y1/(3^(2/3) Gamma[2/3]) - y2/(3^(1/3) Gamma[1/3]);
yB = y1/(3^(1/6) Gamma[2/3]) + y2/(3^(-1/6) Gamma[1/3]);
Plot[ {yA, yB}, {x, -13.5, 3}, PlotRange → {-0.75, 1.5}];
```



produce the figure above. But with $n = 50$ (instead of $n = 40$) terms we get a figure that is visually indistinguishable from Figure 3.2.3 in the textbook.

35. **(a)** If

$$y_0 = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^{3n} n!} x^{2n} = 1 + \sum_{n=1}^{\infty} a_n z^n$$

where $a_n = \dfrac{(2n-1)!!}{2^{3n} n!}$, then the radius of convergence of the series in $z = x^2$ is

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(2n-1)!!/2^{3n} n!}{(2n+1)!!/2^{3n+3}(n+1)!} = \lim_{n\to\infty} \frac{2^3(n+1)}{2n+1} = 4.$$

Thus the series in z converges if $-4 < z = x^2 < 4$, so the series $y_0(x)$ converges if $-2 < x < 2$, and thus has radius of convergence equal to 2.

(b) If

$$y_1 = x\left(1 + \sum_{n=1}^{\infty} \frac{n!}{2^n (2n+1)!!} x^{2n}\right) = x\left(1 + \sum_{n=1}^{\infty} b_n z^n\right)$$

where $b_n = \dfrac{n!}{2^n (2n+1)!!}$, then the radius of convergence of the series in z is

$$\rho = \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right| = \lim_{n\to\infty} \frac{n!/2^n (2n+1)!!}{(n+1)!/2^{n+1}(2n+3)!!} = \lim_{n\to\infty} \frac{2(2n+3)}{n+1} = 4.$$

Hence it follows as in part (a) that the series $y_1(x)$ has radius of convergence equal to 2.

SECTION 3.3

REGULAR SINGULAR POINTS

1. Upon division of the given differential equation by x we see that $P(x) = 1 - x^2$ and $Q(x) = (\sin x)/x$. Because both are analytic at $x = 0$ — in particular, $(\sin x)/x \to 1$ as $x \to 0$ because

$$\frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!} = 1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots$$

— it follows that $x = 0$ is an ordinary point.

2. Division of the differential equation by x yields

$$y''+xy'+\frac{e^x-1}{x}y = 0.$$

Because the function

$$\frac{e^x-1}{x} = \frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}-1\right) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!} = 1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots$$

is analytic at the origin, we see that $x = 0$ is an ordinary point.

3. When we rewrite the given equation in the standard form of Equation (3) in this section, we see that $p(x) = (\cos x)/x$ and $q(x) = x$. Because $(\cos x)/x \to \infty$ as $x \to 0$ it follows that $p(x)$ is not analytic at $x = 0$, so $x = 0$ is an irregular singular point.

4. When we rewrite the given equation in the standard form of Equation (3), we have $p(x) = 2/3$ and $q(x) = (1-x^2)/3x$. Since $q(x)$ is not analytic at the origin, $x = 0$ is an irregular singular point.

5. In the standard form of Equation (3) we have $p(x) = 2/(1+x)$ and $q(x) = 3x^2/(1+x)$. Both are analytic $x = 0$, so $x = 0$ is a regular singular point. The indicial equation is

$$r(r-1)+2r = r^2+r = r(r+1) = 0,$$

so the exponents are $r_1 = 0$ and $r_2 = -1$.

6. In the standard form of Equation (3) we have $p(x) = 2/(1-x^2)$ and $q(x) = -2/(1-x^2)$, so $x = 0$ is a regular singular point with $p_0 = 2$ and $q_0 = -2$. The indicial equation is $r^2+r-2 = 0$, so the exponents are $r = -2, 1$.

7. In the standard form of Equation (3) we have $p(x) = (6 \sin x)/x$ and $q(x) = 6$, so $x = 0$ is a regular singular point with $p_0 = q_0 = 6$. The indicial equation is $r^2 + 5r + 6 = 0$, so the exponents are $r_1 = -2$ and $r_2 = -3$.

8. In the standard form of Equation (3) we have $p(x) = 21/(6+2x)$ and $q(x) = 9(x^2-1)/(6+2x)$, so $x = 0$ is a regular singular point with $p_0 = 7/2$ and $q_0 = -3/2$. The indicial equation simplifies to $2r^2 + 5r - 3 = 0$, so the exponents are $r = -3, 1/2$.

9. The only singular point of the differential equation $y'' + \frac{x}{1-x}y' + \frac{x^2}{1-x}y = 0$ is $x = 1$. Upon substituting $t = x - 1$, $x = t + 1$ we get the transformed equation

$y'' - \frac{t+1}{t}y' - \frac{(t+1)^2}{t}y = 0$, where primes now denote differentiation with respect to t.

In the standard form of Equation (3) we have $p(t) = -(1+t)$ and $q(t) = -t(1+t)^2$. Both these functions are analytic, so it follows that $x = 1$ is a regular singular point of the original equation.

10. The only singular point of the differential equation $y'' + \frac{2}{x-1}y' + \frac{1}{(x-1)^2}y = 0$ is $x = 1$. Upon substituting $t = x - 1$, $x = t + 1$ we get the transformed equation

$y'' + \frac{2}{t}y' + \frac{1}{t^2}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) \equiv 2$ and $q(t) \equiv 1$. Both these functions are analytic, so it follows that $x = 1$ is a regular singular point of the original equation.

11. The only singular points of the differential equation $y'' - \frac{2x}{1-x^2}y' + \frac{12}{1-x^2}y = 0$ are $x = +1$ and $x = -1$.

$\boldsymbol{x = +1}$: Upon substituting $t = x - 1$, $x = t + 1$ we get the transformed equation

$y'' + \frac{2(t+1)}{t(t+2)}y' - \frac{12}{t(t+2)}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = \frac{2(t+1)}{t+2}$ and $q(t) = -\frac{12t}{t+2}$. Both these functions are analytic at $t = 0$, so it follows that $x = +1$ is a regular singular point of the original equation.

$\boldsymbol{x = -1}$: Upon substituting $t = x + 1$, $x = t - 1$ we get the transformed equation

$y'' + \frac{2(t-1)}{t(t-2)}y' - \frac{12}{t(t-2)}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = \frac{2(t-1)}{t-2}$ and $q(t) = -\frac{12t}{t-2}$.

Both these functions are analytic at $t = 0$, so it follows that $x = -1$ is a regular singular point of the original equation.

12. The only singular point of the differential equation $y'' + \frac{3}{x-2}y' + \frac{x^3}{(x-2)^3}y = 0$ is $x = 2$. Upon substituting $t = x - 2$, $x = t + 2$ we get the transformed equation $y'' + \frac{3}{t}y' + \frac{(t+2)^3}{t^3}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) \equiv 3$ and $q(t) = \frac{(t+2)^3}{t}$. Because q is *not* analytic at $t = 0$, it follows that $x = 2$ is an irregular singular point of the original equation.

13. The only singular points of the differential equation $y'' + \frac{1}{x-2}y' + \frac{1}{x+2}y = 0$ are $x = +2$ and $x = -2$.

$\boldsymbol{x = +2}$: Upon substituting $t = x - 2$, $x = t + 2$ we get the transformed equation $y'' + \frac{1}{t+4}y' + \frac{1}{t}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = \frac{t}{t+4}$ and $q(t) = t$. Both these functions are analytic at $t = 0$, so it follows that $x = +2$ is a regular singular point of the original equation.

$\boldsymbol{x = -2}$: Upon substituting $t = x + 2$, $x = t - 2$ we get the transformed equation $y'' + \frac{1}{t}y' + \frac{1}{t-4}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) \equiv 1$ and $q(t) = \frac{t^2}{t-4}$. Both these functions are analytic at $t = 0$, so it follows that $x = -2$ is a regular singular point of the original equation.

14. The only singular points of the differential equation $y'' + \frac{x^2+9}{(x^2-9)^2}y' + \frac{x^2+4}{(x^2-9)^2}y = 0$ are $x = +3$ and $x = -3$.

$\boldsymbol{x = +3}$: Upon substituting $t = x - 3$, $x = t + 3$ we get the transformed equation $y'' + \frac{t^2+6t+13}{t^2(t^2+6)^2}y' + \frac{t^2+6t+18}{t^2(t^2+6)^2}y = 0$, where primes now denote differentiation with respect to t. Because $p(t) = \dfrac{t^2 + 6t + 13}{t(t^2 + 6)^2}$ is *not* analytic at $t = 0$, it follows that $x = 3$ is an irregular singular point of the original equation.

$x = -3$: Upon substituting $t = x + 3$, $x = t - 3$ we get the transformed equation $y'' + \dfrac{t^2 - 6t + 13}{t^2(t^2 - 6)^2} y' + \dfrac{t^2 - 6t + 18}{t^2(t^2 - 6)^2} y = 0$, where primes now denote differentiation with respect to t. Because $p(t) = \dfrac{t^2 - 6t + 13}{t(t^2 - 6)^2}$ is *not* analytic at $t = 0$, it follows that $x = -3$ is an irregular singular point of the original equation.

15. The only singular point of the differential equation $y'' - \dfrac{x^2 - 4}{(x-2)^2} y' + \dfrac{x+2}{(x-2)^2} y = 0$ is $x = 2$. Upon substituting $t = x - 2$, $x = t + 2$ we get the transformed equation $y'' - \dfrac{t+4}{t} y' + \dfrac{t+4}{t^2} y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = -(t+4)$ and $q(t) = t + 4$. Both these functions are analytic, so it follows that $x = 2$ is a regular singular point of the original equation.

16. The only singular points of the differential equation $y'' + \dfrac{3x+2}{x^3(1-x)} y' + \dfrac{1}{x^2(1-x)} y = 0$ are $x = 0$ and $x = 1$.

$x = 0$: In the standard form of Equation (3) we have $p(x) = \dfrac{3x+2}{x^2(1-x)}$ and $q(x) = \dfrac{1}{1-x}$. Since p is not analytic at $x = 0$, it follows that $x = 0$ is an irregular singular point.

$x = 1$: Upon substituting $t = x - 1$, $x = t + 1$ we get the transformed equation $y'' - \dfrac{3t+5}{(t+1)^3} y' - \dfrac{t}{(t+1)^2} y = 0$, where primes now denote differentiation with respect to t. Both $p(t) \equiv -\dfrac{t(3t+5)}{(t+1)^3}$ and $q(t) = -\dfrac{t^3}{(t+1)^2}$ are analytic at $t = 0$, so it follows that $x = 1$ is a regular singular point of the original equation.

Each of the differential equations in Problems 17–20 is of the form

$$Axy'' + By' + Cy = 0$$

with indicial equation $Ar^2 + (B - A)r = 0$. Substitution of $y = \Sigma c_n x^{n+r}$ into the differential equation yields the recurrence relation

$$c_n = -\frac{C c_{n-1}}{A(n+r)^2 + (B-A)(n+r)}$$

for $n \geq 1$. In these problems the exponents $r_1 = 0$ and $r_2 = (A - B)/A$ do *not* differ by an integer, so this recurrence relation yields two linearly independent Frobenius series solutions when we apply it separately with $r = r_1$ and with $r = r_2$.

17. With exponent $r_1 = 0$: $c_n = -\dfrac{c_{n-1}}{4n^2 - 2n}$

$$y_1(x) = x^0\left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \cdots\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\sqrt{x}\right)^{2n}}{(2n)!} = \cos\sqrt{x}$$

With exponent $r_2 = \dfrac{1}{2}$: $c_n = -\dfrac{c_{n-1}}{4n^2 + 2n}$

$$y_2(x) = x^{1/2}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \cdots\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\sqrt{x}\right)^{2n+1}}{(2n+1)!} = \sin\sqrt{x}$$

18. With exponent $r_1 = 0$: $c_n = \dfrac{c_{n-1}}{2n^2 + n}$

$$y_1(x) = x^0\left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \cdots\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!(2n+1)!!}$$

With exponent $r_2 = -\dfrac{1}{2}$: $c_n = \dfrac{c_{n-1}}{2n^2 - n}$

$$y_2(x) = x^{-1/2}\left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \cdots\right) = \frac{1}{\sqrt{x}}\left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!(2n-1)!!}\right]$$

19. With exponent $r_1 = 0$: $c_n = \dfrac{c_{n-1}}{2n^2 - 3n}$

$$y_1(x) = x^0\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \frac{x^4}{360} - \cdots\right) = 1 - x - \sum_{n=2}^{\infty} \frac{x^n}{n!(2n-3)!!}$$

With exponent $r_2 = \dfrac{3}{2}$: $c_n = \dfrac{c_{n-1}}{2n^2 + 3n}$

$$y_2(x) = x^{3/2}\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \frac{x^4}{83160} + \cdots\right) = x^{3/2}\left[1 + 3\sum_{n=1}^{\infty} \frac{x^n}{n!(2n+3)!!}\right]$$

20. With exponent $r_1 = 0$: $c_n = -\dfrac{2c_{n-1}}{3n^2 - n}$

$$y_1(x) = x^0\left(1 - x + \frac{x^2}{5} - \frac{x^3}{60} + \frac{x^4}{1320} - \cdots\right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^n}{n! \cdot 2 \cdot 5 \cdot \cdots \cdot (3n-1)}$$

With exponent $r_2 = \dfrac{1}{3}$: $c_n = -\dfrac{2c_{n-1}}{3n^2 + n}$

$$y_2(x) = x^{1/3}\left(1 - \frac{x}{2} + \frac{x^2}{14} - \frac{x^3}{210} + \frac{x^4}{5460} - \cdots\right) = x^{1/3} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n! \cdot 1 \cdot 4 \cdot \cdots \cdot (3n+1)}$$

The differential equations in Problems 21–24 are all of the form

$$Ax^2y'' + Bxy' + (C + Dx^2)y = 0 \qquad (1)$$

with indical equation

$$\phi(r) = Ar^2 + (B - A)r + C = 0. \qquad (2)$$

Substitution of $y = \Sigma c_n x^{n+r}$ into the differential equation yields

$$\phi(r)c_0x^r + \phi(r+1)c_1x^{r+1} + \sum_{n=2}^{\infty}\left[\phi(r+n)c_n + Dc_{n-2}\right]x^{n+r} = 0. \qquad (3)$$

In each of Problems 21–24 the exponents r_1 and r_2 do *not* differ by an integer. Hence when we substitute either $r = r_1$ or $r = r_2$ into Equation (*) above, we find that c_0 is arbitrary because $\phi(r)$ is then zero, that $c_1 = 0$ — because its coefficient $\phi(r+1)$ is then nonzero — and that

$$c_n = -\frac{Dc_{n-2}}{\phi(r+n)} = -\frac{Dc_{n-2}}{A(n+r)^2 + (B-A)(n+r) + C} \qquad (4)$$

for $n \geq 2$. Thus this recurrence formula yields two linearly independent Frobenius series solutions when we apply it separately with $r = r_1$ and with $r = r_2$.

21. With exponent $r_1 = 1$: $c_1 = 0$, $c_n = \dfrac{2c_{n-2}}{n(2n+3)}$

$$y_1(x) = x^1\left(1 + \frac{x^2}{7} + \frac{x^4}{154} + \frac{x^6}{6930} + \cdots\right) = x\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n! \cdot 7 \cdot 11 \cdot \cdots \cdot (4n+3)}\right]$$

With exponent $r_2 = -\dfrac{1}{2}$: $c_1 = 0$, $c_n = \dfrac{2c_{n-2}}{n(2n-3)}$

$$y_2(x) = x^{-1/2}\left(1 + x^2 + \frac{x^4}{10} + \frac{x^6}{270} + \cdots\right) = \frac{1}{\sqrt{x}}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n! \cdot 1 \cdot 5 \cdot \cdots \cdot (4n-3)}\right]$$