C H A P T E R 2
Matrices

CHAPTER 2
Matrices

Section 2.1 Operations with Matrices

2. $x = 13, y = 12$

4. $x + 2 = 2x + 6 \qquad 2y = 18$
$-4 = x \qquad y = 9$

$2x = -8 \qquad y + 2 = 11$
$x = -4 \qquad y = 9$

6. (a) $A + B = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 6+1 & -1+4 \\ 2+(-1) & 4+5 \\ -3+1 & 5+10 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 1 & 9 \\ -2 & 15 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 6-1 & -1-4 \\ 2-(-1) & 4-5 \\ -3-1 & 5-10 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 3 & -1 \\ -4 & -5 \end{bmatrix}$

(c) $2A = 2\begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 2(6) & 2(-1) \\ 2(2) & 2(4) \\ 2(-3) & 2(5) \end{bmatrix} = \begin{bmatrix} 12 & -2 \\ 4 & 8 \\ -6 & 10 \end{bmatrix}$

(d) $2A - B = \begin{bmatrix} 12 & -2 \\ 4 & 8 \\ -6 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 12-1 & -2-4 \\ 4-(-1) & 8-5 \\ -6-1 & 10-10 \end{bmatrix} = \begin{bmatrix} 11 & -6 \\ 5 & 3 \\ -7 & 0 \end{bmatrix}$

(e) $B + \frac{1}{2}A = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix} + \begin{bmatrix} 3 & -\frac{1}{2} \\ 1 & 2 \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 4 & \frac{7}{2} \\ 0 & 7 \\ -\frac{1}{2} & \frac{25}{2} \end{bmatrix}$

8. (a) $A + B = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3+0 & 2+2 & -1+1 \\ 2+5 & 4+4 & 5+2 \\ 0+2 & 1+1 & 2+0 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 0 \\ 7 & 8 & 7 \\ 2 & 2 & 2 \end{bmatrix}$

(b) $A - B = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3-0 & 2-2 & -1-1 \\ 2-5 & 4-4 & 5-2 \\ 0-2 & 1-1 & 2-0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -2 \\ -3 & 0 & 3 \\ -2 & 0 & 2 \end{bmatrix}$

(c) $2A = 2\begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2(3) & 2(2) & 2(-1) \\ 2(2) & 2(4) & 2(5) \\ 2(0) & 2(1) & 2(2) \end{bmatrix} = \begin{bmatrix} 6 & 4 & -2 \\ 4 & 8 & 10 \\ 0 & 2 & 4 \end{bmatrix}$

(d) $2A - B = 2\begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -2 \\ 4 & 8 & 10 \\ 0 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & -3 \\ -1 & 4 & 8 \\ -2 & 1 & 4 \end{bmatrix}$

(e) $B + \frac{1}{2}A = \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & 1 & -\frac{1}{2} \\ 1 & 2 & \frac{5}{2} \\ 0 & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 3 & \frac{1}{2} \\ 6 & 6 & \frac{9}{2} \\ 2 & \frac{3}{2} & 1 \end{bmatrix}$

10. (a) $A + B$ is not possible. A and B have different sizes.

(b) $A - B$ is not possible. A and B have different sizes.

(c) $2A = 2\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}$

(d) $2A - B$ is not possible. A and B have different sizes.

(e) $B + \frac{1}{2}A$ is not possible. A and B have different sizes.

12. (a) $c_{23} = 5a_{23} + 2b_{23} = 5(2) + 2(11) = 32$

(b) $c_{32} = 5a_{32} + 2b_{32} = 5(1) + 2(4) = 13$

14. Simplifying the right side of the equation produces

$$\begin{bmatrix} w & x \\ y & x \end{bmatrix} = \begin{bmatrix} -4 + 2y & 3 + 2w \\ 2 + 2z & -1 + 2x \end{bmatrix}.$$

By setting corresponding entries equal to each other, you obtain four equations.

$$\begin{aligned} w &= -4 + 2y \\ x &= 3 + 2w \\ y &= 2 + 2z \\ x &= -1 + 2x \end{aligned} \Rightarrow \begin{cases} -2y + w = -4 \\ x - 2w = 3 \\ y - 2z = 2 \\ x = 1 \end{cases}$$

The solution to this linear system is: $x = 1$, $y = \frac{3}{2}$, $z = -\frac{1}{4}$, and $w = -1$.

16. (a) $AB = \begin{bmatrix} 2 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 2(4) + (-2)(2) & 2(1) + (-2)(-2) \\ -1(4) + 4(2) & -1(1) + 4(-2) \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & -9 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 4(2) + 1(-1) & 4(-2) + 1(4) \\ 2(2) + (-2)(-1) & 2(-2) + (-2)(4) \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 6 & -12 \end{bmatrix}$

18. (a) $AB = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1(1) + (-1)(2) + 7(1) & 1(1) + (-1)(1) + 7(-3) & 1(2) + (-1)(1) + 7(2) \\ 2(1) + (-1)(2) + 8(1) & 2(1) + (-1)(1) + 8(-3) & 2(2) + (-1)(1) + 8(2) \\ 3(1) + 1(2) + (-1)(1) & 3(1) + 1(1) + (-1)(-3) & 3(2) + 1(1) + (-1)(2) \end{bmatrix} = \begin{bmatrix} 6 & -21 & 15 \\ 8 & -23 & 19 \\ 4 & 7 & 5 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1(1) + 1(2) + 2(3) & 1(-1) + 1(-1)1 + 2(1) & 1(7) + 1(8) + 2(-1) \\ 2(1) + 1(2) + 1(3) & 2(-1) + 1(-1) + 1(1) & 2(7) + 1(8) + 1(-1) \\ 1(1) + (-3)(2) + 2(3) & 1(-1) + (-3)(-1) + 2(1) & 1(7) + (-3)(8) + 2(-1) \end{bmatrix} = \begin{bmatrix} 9 & 0 & 13 \\ 7 & -2 & 21 \\ 1 & 4 & -19 \end{bmatrix}$

20. (a) $AB = \begin{bmatrix} 3 & 2 & 1 \\ -3 & 0 & 4 \\ 4 & -2 & -4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3(1) + 2(2) + 1(1) & 3(2) + 2(-1) + 1(-2) \\ -3(1) + 0(2) + 4(1) & -3(2) + 0(-1) + 4(-2) \\ 4(1) + (-2)(2) + (-4)(1) & 4(2) + (-2)(-1) + (-4)(-2) \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 1 & -14 \\ -4 & 18 \end{bmatrix}$

(b) BA is not defined because B is 3×2 and A is 3×3.

22. (a) $AB = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -1(2) & -1(1) & -1(3) & -1(2) \\ 2(2) & 2(1) & 2(3) & 2(2) \\ -2(2) & -2(1) & -2(3) & -2(2) \\ 1(2) & 1(1) & 1(3) & 1(2) \end{bmatrix} = \begin{bmatrix} -2 & -1 & -3 & -2 \\ 4 & 2 & 6 & 4 \\ -4 & -2 & -6 & -4 \\ 2 & 1 & 3 & 2 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 2 & 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(-1) + 1(2) + 3(-2) + 2(1) \end{bmatrix} = \begin{bmatrix} -4 \end{bmatrix}$

24. (a) AB is not defined because A is 2×2 and B is 3×2.

(b) $BA = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2(2) + 1(5) & 2(-3) + 1(2) \\ 1(2) + 3(5) & 1(-3) + 3(2) \\ 2(2) + (-1)(5) & 2(-3) + (-1)(2) \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ 17 & 3 \\ -1 & -8 \end{bmatrix}$

26. (a) $AB = \begin{bmatrix} 2 & 1 & 2 \\ 3 & -1 & -2 \\ -2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 4 & 0 & 1 & 3 \\ -1 & 2 & -3 & -1 \\ -2 & 1 & 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2(4)+1(-1)+2(-2) & 2(0)+1(2)+2(1) & 2(1)+1(-3)+2(4) & 2(3)+1(-1)+2(3) \\ 3(4)+(-1)(-1)+(-2)(-2) & 3(0)+(-1)(2)+(-2)(1) & 3(1)+(-1)(-3)+(-2)(4) & 3(3)+(-1)(-1)+(-2)(3) \\ -2(4)+1(-1)+(-2)(-2) & -2(0)+1(2)+(-2)(1) & -2(1)+1(-3)+(-2)(4) & -2(3)+1(-1)+(-2)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 7 & 11 \\ 17 & -4 & -2 & 4 \\ -5 & 0 & -13 & -13 \end{bmatrix}$$

(b) BA is not defined because B is 3×4 and A is 3×3.

28. (a) AB is not defined because A is 2×5 and B is 2×2.

(b) $BA = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 & -2 & 4 \\ 6 & 13 & 8 & -17 & 20 \end{bmatrix}$

$$= \begin{bmatrix} 1(1)+6(6) & 1(0)+6(13) & 1(3)+6(8) & 1(-2)+6(-17) & 1(4)+6(20) \\ 4(1)+2(6) & 4(0)+2(13) & 4(3)+2(8) & 4(-2)+2(-17) & 4(4)+2(20) \end{bmatrix}$$

$$= \begin{bmatrix} 37 & 78 & 51 & -104 & 124 \\ 16 & 26 & 28 & -42 & 56 \end{bmatrix}$$

30. $C + E$ is not defined because C and E have different sizes.

32. $-4A$ is defined and has size 3×4 because A has size 3×4.

34. BE is defined. Because B has size 3×4 and E has size 4×3, the size of BE is 3×3.

36. $2D + C$ is defined and has size 4×2 because $2D$ and C have size 4×2.

38. As a system of linear equations, $A\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned} x_1 + 2x_2 + x_3 + 3x_4 &= 0 \\ x_1 - x_2 \qquad\quad + x_4 &= 0. \\ x_2 - x_3 + 2x_4 &= 0 \end{aligned}$$

Use Gauss-Jordan elimination on the augmented matrix for this system.

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Choosing $x_4 = t$, the solution is $x_1 = -2t$, $x_2 = -t$, $x_3 = t$, and $x_4 = t$, where t is any real number.

40. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

42. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} -4 & 9 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -13 \\ 12 \end{bmatrix}.$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} -4 & 9 & -13 \\ 1 & -3 & 12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -23 \\ 0 & 1 & -\frac{35}{3} \end{bmatrix}$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -23 \\ -\frac{35}{3} \end{bmatrix}$.

44. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 1 & 1 & -3 & -1 \\ -1 & 2 & 0 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$.

46. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} 1 & -1 & 4 \\ 1 & 3 & 0 \\ 0 & -6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17 \\ -11 \\ 40 \end{bmatrix}.$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 4 & 17 \\ 1 & 3 & 0 & -11 \\ 0 & -6 & 5 & 40 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$.

48. In matrix form $A\mathbf{x} = \mathbf{b}$, the system is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}.$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & -1 & 1 & -1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

50. The augmented matrix row reduces as follows.

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ -1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are an infinite number of solutions. For example, $x_3 = 0,\ x_2 = 2,\ x_1 = -3$.

So, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$.

52. The augmented matrix row reduces as follows.

$$\begin{bmatrix} -3 & 5 & -22 \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 10 \\ 0 & 9 & -18 \\ 0 & -4 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -3 & 10 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

So,

$$\mathbf{b} = \begin{bmatrix} -22 \\ 4 \\ 32 \end{bmatrix} = 4\begin{bmatrix} -3 \\ 3 \\ 4 \end{bmatrix} + (-2)\begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}.$$

54. Expanding the left side of the equation produces

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 2a_{11} - a_{21} & 2a_{12} - a_{22} \\ 3a_{11} - 2a_{21} & 3a_{12} - 2a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and you obtain the system

$$\begin{aligned} 2a_{11} \quad - a_{21} \quad\quad &= 1 \\ 2a_{12} \quad\quad - a_{22} &= 0 \\ 3a_{11} \quad - 2a_{21} \quad\quad &= 0 \\ 3a_{12} \quad\quad - 2a_{22} &= 1. \end{aligned}$$

Solving by Gauss-Jordan elimination yields

$a_{11} = 2,\ a_{12} = -1,\ a_{21} = 3,$ and $a_{22} = -2.$

So, you have $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$.

56. Expanding the left side of the matrix equation produces

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2a + 3b & a + b \\ 2c + 3d & c + d \end{bmatrix} = \begin{bmatrix} 3 & 17 \\ 4 & -1 \end{bmatrix}.$$

You obtain two systems of linear equations (one involving a and b and the other involving c and d).

$$\begin{aligned} 2a + 3b &= 3 \\ a + b &= 17, \end{aligned}$$

and

$$\begin{aligned} 2c + 3d &= 4 \\ c + d &= -1. \end{aligned}$$

Solving by Gauss-Jordan elimination yields $a = 48$, $b = -31$, $c = -7$, and $d = 6$.

58. $AA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

60. $AB = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix}$

$$= \begin{bmatrix} 3(-7) + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + (-5)4 + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} -21 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly,

$$BA = \begin{bmatrix} -21 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

62. (a) $AB = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} \\ a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \\ a_{33}b_{31} & a_{33}b_{32} & a_{33}b_{33} \end{bmatrix}$

The ith row of B has been multiplied by a_{ii}, the ith diagonal entry of A.

(b) $BA = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{22}b_{12} & a_{33}b_{13} \\ a_{11}b_{21} & a_{22}b_{22} & a_{33}b_{23} \\ a_{11}b_{31} & a_{22}b_{32} & a_{33}b_{33} \end{bmatrix}$

The ith column of B has been multiplied by a_{ii}, the ith diagonal entry of A.

(c) If $a_{11} = a_{22} = a_{33}$, then $AB = a_{11}B = BA$.

64. The trace is the sum of the elements on the main diagonal.

$1 + 1 + 1 = 3$

66. The trace is the sum of the elements on the main diagonal.

$1 + 0 + 2 + (-3) = 0$

68. Let $AB = [c_{ij}]$, where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Then, $Tr(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{ki}\right)$.

Similarly, if $BA = [d_{ij}]$, $d_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj}$. Then $Tr(BA) = \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} b_{ik}a_{ki}\right) = Tr(AB)$.

70. $AB = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha(-\sin\beta) - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \sin\alpha(-\sin\beta) + \cos\alpha\cos\beta \end{bmatrix}$

$BA = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & \cos\beta(-\sin\alpha) - \sin\beta\cos\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & \sin\beta(-\sin\alpha) + \cos\beta\cos\alpha \end{bmatrix}$

So, you see that $AB = BA = \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}$.

72. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Then the matrix equation $AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is equivalent to

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This equation implies that

$$a_{11}b_{11} + a_{12}b_{21} - b_{11}a_{11} - b_{12}a_{21} = a_{12}b_{21} - b_{12}a_{21} = 1$$
$$a_{21}b_{12} + a_{22}b_{22} - b_{21}a_{12} - b_{22}a_{22} = a_{21}b_{12} - b_{21}a_{12} = 1$$

which is impossible. So, the original equation has no solution.

74. Assume that A is an $m \times n$ matrix and B is a $p \times q$ matrix. Because the product AB is defined, you know that $n = p$. Moreover, because AB is square, you know that $m = q$. Therefore, B must be of order $n \times m$, which implies that the product BA is defined.

76. Let rows s and t be identical in the matrix A. So, $a_{sj} = a_{tj}$ for $j = 1, \ldots, n$. Let $AB = [c_{ij}]$, where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Then, $c_{sj} = \sum_{k=1}^{n} a_{sk}b_{kj}$, and $c_{tj} = \sum_{k=1}^{n} a_{tk}b_{kj}$. Because $a_{sk} = a_{tk}$ for $k = 1, \ldots, n$, rows s and t of AB are the same.

78. (a) No, the matrices have different sizes.

(b) No, the matrices have different sizes.

(c) Yes; No, BA is undefined.

80. $1.2\begin{bmatrix} 70 & 50 & 25 \\ 35 & 100 & 70 \end{bmatrix} = \begin{bmatrix} 84 & 60 & 30 \\ 42 & 120 & 84 \end{bmatrix}$

82. (a) Multiply the matrix for 2010 by $\frac{1}{3090}$. This produces a matrix giving the information as percents of the total population.

$$A = \frac{1}{3090}\begin{bmatrix} 12{,}306 & 35{,}240 & 7830 \\ 16{,}095 & 41{,}830 & 9051 \\ 27{,}799 & 72{,}075 & 14{,}985 \\ 5698 & 13{,}717 & 2710 \\ 12{,}222 & 31{,}867 & 5901 \end{bmatrix} \approx \begin{bmatrix} 3.98 & 11.40 & 2.53 \\ 5.21 & 13.54 & 2.93 \\ 9.00 & 23.33 & 4.85 \\ 1.84 & 4.44 & 0.88 \\ 3.96 & 10.31 & 1.91 \end{bmatrix}$$

Multiply the matrix for 2013 by $\frac{1}{3160}$. This produces a matrix giving the information as percents of the total population.

$$B = \frac{1}{3160}\begin{bmatrix} 12{,}026 & 35{,}471 & 8446 \\ 15{,}772 & 41{,}985 & 9791 \\ 27{,}954 & 73{,}703 & 16{,}727 \\ 5710 & 14{,}067 & 3104 \\ 12{,}124 & 32{,}614 & 6636 \end{bmatrix} \approx \begin{bmatrix} 3.81 & 11.23 & 2.67 \\ 4.99 & 13.29 & 3.10 \\ 8.85 & 23.32 & 5.29 \\ 1.81 & 4.45 & 0.98 \\ 3.84 & 10.32 & 2.10 \end{bmatrix}$$

(b)

$$B - A = \begin{bmatrix} 3.81 & 11.23 & 2.67 \\ 4.99 & 13.29 & 3.10 \\ 8.85 & 23.32 & 5.29 \\ 1.81 & 4.45 & 0.98 \\ 3.84 & 10.32 & 2.10 \end{bmatrix} - \begin{bmatrix} 3.98 & 11.40 & 2.53 \\ 5.21 & 13.54 & 2.93 \\ 9.00 & 23.33 & 4.85 \\ 1.84 & 4.44 & 0.88 \\ 3.96 & 10.31 & 1.91 \end{bmatrix} = \begin{bmatrix} -0.18 & -0.18 & 0.14 \\ -0.22 & -0.25 & 0.17 \\ -0.15 & -0.001 & 0.44 \\ -0.04 & 0.01 & 0.11 \\ -0.12 & 0.01 & 0.19 \end{bmatrix}$$

(c) The 65+ age group is projected to show relative growth from 2010 to 2013 over all regions because its column in $B - A$ contains all positive percents.

84. $AB = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline -1 & -2 & -3 & -4 \\ -5 & -6 & -7 & -8 \end{array}\right]$

86. (a) True. The number of elements in a row of the first matrix must be equal to the number of elements in a column of the second matrix. See page 43 of the text.

(b) True. See page 45 of the text.

Section 2.2 Properties of Matrix Operations

2. $\begin{bmatrix} 6 & 8 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} -11 & -7 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 + 0 + (-11) & 8 + 5 + (-7) \\ -1 + (-3) + 2 & 0 + (-1) + (-1) \end{bmatrix} = \begin{bmatrix} -5 & 6 \\ -2 & -2 \end{bmatrix}$

4. $\frac{1}{2}\left(\begin{bmatrix} 5 & -2 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 14 & 6 & -18 & 9 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 5 + 14 & -2 + 6 & 4 + (-18) & 0 + 9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 19 & 4 & -14 & 9 \end{bmatrix} = \begin{bmatrix} \frac{19}{2} & 2 & -7 & \frac{9}{2} \end{bmatrix}$

6.
$$-1\begin{bmatrix} 4 & 11 \\ -2 & -1 \\ 9 & 3 \end{bmatrix} + \frac{1}{6}\left(\begin{bmatrix} -5 & -1 \\ 3 & 4 \\ 0 & 13 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ -9 & -1 \\ 6 & -1 \end{bmatrix}\right) = \begin{bmatrix} -4 & -11 \\ 2 & 1 \\ -9 & -3 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} -5 + 7 & -1 + 5 \\ 3 + (-9) & 4 + (-1) \\ 0 + 6 & 13 + (-1) \end{bmatrix}$$
$$= \begin{bmatrix} -4 & -11 \\ 2 & 1 \\ -9 & -3 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 2 & 4 \\ -6 & 3 \\ 6 & 12 \end{bmatrix} = \begin{bmatrix} -4 & -11 \\ 2 & 1 \\ -9 & -3 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -1 & \frac{1}{2} \\ 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -4 + \frac{1}{3} & -11 + \frac{2}{3} \\ 2 + (-1) & 1 + \frac{1}{2} \\ -9 + 1 & -3 + 2 \end{bmatrix} = \begin{bmatrix} -\frac{11}{3} & -\frac{31}{3} \\ 1 & \frac{3}{2} \\ -8 & -1 \end{bmatrix}$$

8. $A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

10. $(a + b)B = (3 + (-4))\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = (-1)\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$

12. $(ab)O = (3)(-4)\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (-12)\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

14. (a)
$$X = 3A - 2B$$
$$= \begin{bmatrix} -6 & -3 \\ 3 & 0 \\ 9 & -12 \end{bmatrix} - \begin{bmatrix} 0 & 6 \\ 4 & 0 \\ -8 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & -9 \\ -1 & 0 \\ 17 & -10 \end{bmatrix}$$

(b)
$$2X = 2A - B$$
$$2X = \begin{bmatrix} -4 & -2 \\ 2 & 0 \\ 6 & -8 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$$
$$2X = \begin{bmatrix} -4 & -5 \\ 0 & 0 \\ 10 & -7 \end{bmatrix}$$
$$X = \begin{bmatrix} -2 & -\frac{5}{2} \\ 0 & 0 \\ 5 & -\frac{7}{2} \end{bmatrix}$$

(c)
$$2X + 3A = B$$
$$2X + \begin{bmatrix} -6 & -3 \\ 3 & 0 \\ 9 & -12 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$$
$$2X = \begin{bmatrix} 6 & 6 \\ -1 & 0 \\ -13 & 11 \end{bmatrix}$$
$$X = \begin{bmatrix} 3 & 3 \\ -\frac{1}{2} & 0 \\ -\frac{13}{2} & \frac{11}{2} \end{bmatrix}$$

(d)
$$2A + 4B = -2X$$
$$\begin{bmatrix} -4 & -2 \\ 2 & 0 \\ 6 & -8 \end{bmatrix} + \begin{bmatrix} 0 & 12 \\ 8 & 0 \\ -16 & -4 \end{bmatrix} = -2X$$
$$\begin{bmatrix} -4 & 10 \\ 10 & 0 \\ -10 & -12 \end{bmatrix} = -2X$$
$$\begin{bmatrix} 2 & -5 \\ -5 & 0 \\ 5 & 6 \end{bmatrix} = X$$

16. $$c(CB) = (-2)\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\right) = (-2)\begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 2 & 6 \end{bmatrix}$$

18. $$C(BC) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & -1 \end{bmatrix}$$

20. $$B(C + O) = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & -1 \end{bmatrix}$$

22. $$B(cA) = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\left((-2)\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -2 & -4 & -6 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -10 & 0 \\ 2 & 0 & 10 \end{bmatrix}$$

24. (a) $$(AB)C = \left(\begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & -5 & 0 \\ -2 & 3 & 3 \end{bmatrix}\right)\begin{bmatrix} -3 & 4 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 26 & 6 \\ 7 & -14 & -9 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 0 \\ -12 & 5 \end{bmatrix}$$

(b) $$A(BC) = \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix}\left(\begin{bmatrix} 1 & -5 & 0 \\ -2 & 3 & 3 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}\right) = \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} -3 & -1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 18 & 0 \\ -12 & 5 \end{bmatrix}$$

26. $$AB = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{8} \end{bmatrix}$$
$$BA = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix} \neq AB$$

28. $$AC = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -6 & 9 \\ 16 & -8 & 12 \\ 4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 3 \\ 5 & 4 & 4 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix} = BC$$

But $A \neq B$.

30. $$AB = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

But $A \neq O$ and $B \neq O$.

32. $AT = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

34. $A + IA = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$

36. $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

So, $A^4 = \left(A^2\right)^2 = I_2^2 = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

38. In general, $AB \neq BA$ for matrices.

40. $D^T = \begin{bmatrix} 6 & -7 & 19 \\ -7 & 0 & 23 \\ 19 & 23 & -32 \end{bmatrix}^T = \begin{bmatrix} 6 & -7 & 19 \\ -7 & 0 & 23 \\ 19 & 23 & -32 \end{bmatrix}$

42. $(AB)^T = \left(\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} 1 & 1 \\ -4 & -2 \end{bmatrix}^T = \begin{bmatrix} 1 & -4 \\ 1 & -2 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}^T = \begin{bmatrix} -3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & -2 \end{bmatrix}$

44. $(AB)^T = \left(\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 4 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix}\right)^T = \begin{bmatrix} 4 & 0 & -7 \\ 2 & 4 & 7 \\ 4 & 2 & 2 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 4 & 2 \\ -7 & 7 & 2 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix}^T \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 4 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 4 & 2 \\ -7 & 7 & 2 \end{bmatrix}$

46. (a) $A^T A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 11 \\ 11 & 21 \end{bmatrix}$

(b) $AA^T = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ -1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 25 & -8 \\ 2 & -8 & 4 \end{bmatrix}$

48. (a) $A^T A = \begin{bmatrix} 4 & 2 & 14 & 6 \\ -3 & 0 & -2 & 8 \\ 2 & 11 & 12 & -5 \\ 0 & -1 & -9 & 4 \end{bmatrix}\begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 252 & 8 & 168 & -104 \\ 8 & 77 & -70 & 50 \\ 168 & -70 & 294 & -139 \\ -104 & 50 & -139 & 98 \end{bmatrix}$

(b) $AA^T = \begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix}\begin{bmatrix} 4 & 2 & 14 & 6 \\ -3 & 0 & -2 & 8 \\ 2 & 11 & 12 & -5 \\ 0 & -1 & -9 & 4 \end{bmatrix} = \begin{bmatrix} 29 & 30 & 86 & -10 \\ 30 & 126 & 169 & -47 \\ 86 & 169 & 425 & -28 \\ -10 & -47 & -28 & 141 \end{bmatrix}$

50.

$$A^{17} = \begin{bmatrix} (1)^{17} & 0 & 0 & 0 & 0 \\ 0 & (-1)^{17} & 0 & 0 & 0 \\ 0 & 0 & (1)^{17} & 0 & 0 \\ 0 & 0 & 0 & (-1)^{17} & 0 \\ 0 & 0 & 0 & 0 & (1)^{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

52. $$A^{20} = \begin{bmatrix} (1)^{20} & 0 & 0 & 0 & 0 \\ 0 & (-1)^{20} & 0 & 0 & 0 \\ 0 & 0 & (1)^{20} & 0 & 0 \\ 0 & 0 & 0 & (-1)^{20} & 0 \\ 0 & 0 & 0 & 0 & (1)^{20} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

54. Because $A^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 27 \end{bmatrix} = \begin{bmatrix} 2^3 & 0 & 0 \\ 0 & (-1)^3 & 0 \\ 0 & 0 & (3)^3 \end{bmatrix}$, you have $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

56. (a) False. In general, for $n \times n$ matrices A and B it is *not* true that $AB = BA$. For example, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

Then $AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = BA$.

(b) False. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = AC$, but $B \neq C$.

(c) True. See Theorem 2.6, part 2 on page 57.

58.

$aX + A(bB) = b(AB + IB)$	Original equation
$aX + (Ab)B = b(AB + B)$	Associative property; property of the identity matrix
$aX + bAB = bAB + bB$	Property of scalar multiplication; distributive property
$aX + bAB + (-bAB) = bAB + bB + (-bAB)$	Add $-bAB$ to both sides.
$aX = bAB + bB + (-bAB)$	Additive inverse
$aX = bAB + (-bAB) + bB$	Commutative property
$aX = bB$	Additive inverse
$X = \frac{b}{a}B$	Divide by a.

60. $$f(A) = -10\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 5\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix} - 2\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}^2 + \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}^3$$

$$= -\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} + \begin{bmatrix} 10 & 5 & -5 \\ 5 & 0 & 10 \\ -5 & 5 & 15 \end{bmatrix} - 2\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}^2$$

$$= \begin{bmatrix} 0 & 5 & -5 \\ 5 & -10 & 10 \\ -5 & 5 & 5 \end{bmatrix} - 2\begin{bmatrix} 6 & 1 & -3 \\ 0 & 3 & 5 \\ -4 & 2 & 12 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 6 & 1 & -3 \\ 0 & 3 & 5 \\ -4 & 2 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 & -5 \\ 5 & -10 & 10 \\ -5 & 5 & 5 \end{bmatrix} - \begin{bmatrix} 12 & 2 & -6 \\ 0 & 6 & 10 \\ -8 & 4 & 24 \end{bmatrix} + \begin{bmatrix} 16 & 3 & -13 \\ -2 & 5 & 21 \\ -18 & 8 & 44 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & -12 \\ 3 & -11 & 21 \\ -15 & 9 & 25 \end{bmatrix}$$

62. $(cd)A = (cd)[a_{ij}] = [(cd)a_{ij}] = [c(da_{ij})] = c[da_{ij}] = c(dA)$

64. $(c + d)A = (c + d)[a_{ij}] = [(c + d)a_{ij}] = [ca_{ij} + da_{ij}] = [ca_{ij}] + [da_{ij}] = c[a_{ij}] + d[a_{ij}] = cA + dA$

66. (a) To show that $A(BC) = (AB)C$, compare the ijth entries in the matrices on both sides of this equality. Assume that A has size $n \times p$, B has size $p \times r$, and C has size $r \times m$. Then the entry in the kth row and the jth column of BC is $\sum_{l=1}^{r} b_{kl}c_{lj}$. Therefore, the entry in ith row and jth column of $A(BC)$ is

$$\sum_{k=1}^{p} a_{ik} \sum_{l=1}^{r} b_{kl}c_{lj} = \sum_{k,\, l} a_{ik}b_{kl}c_{lj}.$$

The entry in the ith row and jth column of $(AB)C$ is $\sum_{l=1}^{r} d_{il}c_{lj}$, where d_{il} is the entry of AB in the ith row and the lth column.

So, $d_{il} = \sum_{k=1}^{p} a_{ik}b_{kl}$ for each $l = 1, \ldots, r$. So, the ijth entry of $(AB)C$ is

$$\sum_{i=1}^{r} \sum_{k=1}^{p} a_{ik}b_{kl}c_{lj} = \sum_{k,\, l} a_{ik}b_{kl}c_{lj}.$$

Because all corresponding entries of $A(BC)$ and $(AB)C$ are equal and both matrices are of the same size $(n \times m)$, you conclude that $A(BC) = (AB)C$.

(b) The entry in the ith row and jth column of $(A + B)C$ is $(a_{il} + b_{il})c_{1j} + (a_{i2} + b_{i2})c_{2j} + \cdots + (a_{in} + b_{in})c_{nj}$, whereas the entry in the ith row and jth column of $AC + BC$ is $(a_{i1}c_{1j} + \cdots + a_{in}c_{nj}) + (b_{i1}c_{1j} + \cdots + b_{in}c_{nj})$, which are equal by the distributive law for real numbers.

(c) The entry in the ith row and jth column of $c(AB)$ is $c[a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}]$. The corresponding entry for $(cA)B$ is $(ca_{i1})b_{1j} + (ca_{i2})b_{2j} + \cdots + (ca_{in})b_{nj}$ and the corresponding entry for $A(cB)$ is $a_{i1}(cb_{1j}) + a_{i2}(cb_{2j}) + \cdots + a_{in}(cb_{nj})$. Because these three expressions are equal, you have shown that $c(AB) = (cA)B = A(cB)$.

68. (2) $(A + B)^T = ([a_{ij}] + [b_{ij}])^T = [a_{ij} + b_{ij}]^T = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}] = A^T + B^T$

(3) $(cA)^T = (c[a_{ij}])^T = [ca_{ij}]^T = [ca_{ji}] = c[a_{ji}] = c(A^T)$

(4) The entry in the ith row and jth column of $(AB)^T$ is $a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots a_{jn}b_{ni}$. On the other hand, the entry in the ith row and jth column of $B^T A^T$ is $b_{1i}a_{j1} + b_{2i}a_{j2} + \cdots + b_{ni}a_{jn}$, which is the same.

70. (a) Answers will vary. *Sample answer*: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

(b) Let A and B be symmetric.

If $AB = BA$, then $(AB)^T = B^T A^T = BA = AB$ and AB is symmetric.

If $(AB)^T = AB$, then $AB = (AB)^T = B^T A^T = BA$ and $AB = BA$.

72. Because $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = A^T$, the matrix is symmetric.

74. Because $-A = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} = A^T$, the matrix is skew-symmetric.

76. If $A^T = -A$ and $B^T = -B$, then $(A + B)^T = A^T + B^T = -A - B = -(A + B)$, which implies that $A + B$ is skew-symmetric.

78. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$

$$A - A^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_{12} - a_{21} & a_{13} - a_{31} & \cdots & a_{1n} - a_{n1} \\ a_{21} - a_{12} & 0 & a_{23} - a_{32} & \cdots & a_{2n} - a_{n2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} - a_{1n} & a_{n2} - a_{2n} & a_{n3} - a_{3n} & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_{12} - a_{21} & a_{13} - a_{31} & \cdots & a_{1n} - a_{n1} \\ -(a_{12} - a_{21}) & 0 & a_{23} - a_{32} & \cdots & a_{2n} - a_{n2} \\ \vdots & \vdots & \vdots & & \vdots \\ -(a_{1n} - a_{n1}) & -(a_{2n} - a_{n2}) & -(a_{3n} - a_{n3}) & \cdots & 0 \end{bmatrix}$$

So, $A - A^T$ is skew-symmetric.

Section 2.3 The Inverse of a Matrix

2. $AB = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 - 1 & 1 - 1 \\ -2 + 2 & -1 + 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 - 1 & -2 + 2 \\ 1 - 1 & -1 + 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $AB = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

6. $AB = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 3 & 6 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. Use the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}.$$

So, the inverse is

$$A^{-1} = \frac{1}{2(2) - (-2)(2)}\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

10. Use the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}.$$

So, the inverse is

$$A^{-1} = \frac{1}{(1)(-3) - (-2)(2)}\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}.$$

12. Using the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$$

you see that $ad - bc = (-1)(-3) - (1)(3) = 0.$ So, the matrix has no inverse.

14. Adjoin the identity matrix to form

$$[A \;\; I] = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 1 & 0 \\ -1 & -4 & -7 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$[I \;\; A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & -13 & 6 & 4 \\ 0 & 1 & 0 & 12 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{bmatrix}$$

16. Adjoin the identity matrix to form

$$[A \;\; I] = \begin{bmatrix} 10 & 5 & -7 & 1 & 0 & 0 \\ -5 & 1 & 4 & 0 & 1 & 0 \\ 3 & 2 & -2 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$[I \;\; A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & -10 & -4 & 27 \\ 0 & 1 & 0 & 2 & 1 & -5 \\ 0 & 0 & 1 & -13 & -5 & 35 \end{bmatrix}$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} -10 & -4 & 27 \\ 2 & 1 & -5 \\ -13 & -5 & 35 \end{bmatrix}.$$

18. Adjoin the identity matrix to form

$$[A \;\; I] = \begin{bmatrix} 3 & 2 & 5 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ -4 & 4 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, you cannot form the identity matrix on the left side. Therefore, the matrix has no inverse.

20. Adjoin the identity matrix to form

$$[A \;\; I] = \begin{bmatrix} -\frac{5}{6} & \frac{1}{3} & \frac{11}{6} & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 2 & 0 & 1 & 0 \\ 1 & -\frac{1}{2} & -\frac{5}{2} & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, you cannot form the identity matrix on the left side. Therefore, the matrix has no inverse.

22. Adjoin the identity matrix to form

$$[A \;\; I] = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 1 & 0 & 0 \\ -0.3 & 0.2 & 0.2 & 0 & 1 & 0 \\ 0.5 & 0.5 & 0.5 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$[I \;\; A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 0.8 \\ 0 & 1 & 0 & -10 & 4 & 4.4 \\ 0 & 0 & 1 & 10 & -2 & -3.2 \end{bmatrix}$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} 0 & -2 & 0.8 \\ -10 & 4 & 4.4 \\ 10 & -2 & -3.2 \end{bmatrix}.$$

24. Adjoin the identity matrix to form

$$[A \;\; I] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{bmatrix}$$

Using elementary row operations, you cannot form the identity matrix on the left side. Therefore, the matrix has no inverse.

26. Adjoin the identity matrix to form

$$[A \;\; I] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$[I \;\; A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Therefore, the inverse is

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

28. Adjoin the identity matrix to form

$$[A\ \ I] = \left[\begin{array}{rrrrrrrr} 4 & 8 & -7 & 14 & 1 & 0 & 0 & 0 \\ 2 & 5 & -4 & 6 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -7 & 0 & 0 & 1 & 0 \\ 3 & 6 & -5 & 10 & 0 & 0 & 0 & 1 \end{array}\right].$$

Using elementary row operations, reduce the matrix as follows.

$$[A\ \ I] = \left[\begin{array}{rrrrrrrr} 1 & 0 & 0 & 0 & 27 & -10 & 4 & -29 \\ 0 & 1 & 0 & 0 & -16 & 5 & -2 & 18 \\ 0 & 0 & 1 & 0 & -17 & 4 & -2 & 20 \\ 0 & 0 & 0 & 1 & -7 & 2 & -1 & 8 \end{array}\right]$$

Therefore the inverse is

$$A^{-1} = \left[\begin{array}{rrrr} 27 & -10 & 4 & -29 \\ -16 & 5 & -2 & 18 \\ -17 & 4 & -2 & 20 \\ -7 & 2 & -1 & 8 \end{array}\right].$$

30. Adjoin the identity matrix to form

$$[A\ \ I] = \left[\begin{array}{rrrrrrrr} 1 & 3 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 1 \end{array}\right].$$

Using elementary row operations, reduce the matrix as follows.

$$\left[I\ \ A^{-1}\right] = \left[\begin{array}{rrrrrrrr} 1 & 0 & 0 & 0 & 1 & -1.5 & -4 & 2.6 \\ 0 & 1 & 0 & 0 & 0 & 0.5 & 1 & -0.8 \\ 0 & 0 & 1 & 0 & 0 & 0 & -0.5 & 0.1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0.2 \end{array}\right].$$

Therefore, the inverse is

$$A^{-1} = \left[\begin{array}{rrrr} 1 & -1.5 & -4 & 2.6 \\ 0 & 0.5 & 1 & -0.8 \\ 0 & 0 & -0.5 & 0.1 \\ 0 & 0 & 0 & 0.2 \end{array}\right].$$

32. $A = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

$ad - bc = (1)(2) - (-2)(-3) = -4$

$$A^{-1} = -\frac{1}{4}\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

34. $A = \begin{bmatrix} -12 & 3 \\ 5 & -2 \end{bmatrix}$

$ad - bc = (-12)(-2) - 3(5) = 24 - 15 = 9$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} -2 & -3 \\ -5 & -12 \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} & -\frac{1}{3} \\ -\frac{5}{9} & -\frac{4}{3} \end{bmatrix}$$

36. $A = \begin{bmatrix} -\frac{1}{4} & \frac{9}{4} \\ \frac{5}{3} & \frac{8}{9} \end{bmatrix}$

$ad - bc = \left(-\frac{1}{4}\right)\left(\frac{8}{9}\right) - \left(\frac{9}{4}\right)\left(\frac{5}{3}\right) = -\frac{143}{36}$

$$A^{-1} = -\frac{36}{143}\begin{bmatrix} \frac{8}{9} & -\frac{9}{4} \\ -\frac{5}{3} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{32}{143} & \frac{81}{143} \\ \frac{60}{143} & \frac{9}{143} \end{bmatrix}$$

38. $A^{-2} = \left(A^{-1}\right)^2 = \left(\frac{1}{47}\begin{bmatrix} 6 & -7 \\ 5 & 2 \end{bmatrix}\right)^2 = \frac{1}{2209}\begin{bmatrix} 1 & -56 \\ 40 & -31 \end{bmatrix}$

$$A^{-2} = \left(A^2\right)^{-1} = \begin{bmatrix} -31 & 56 \\ -40 & 1 \end{bmatrix}^{-1} = \frac{1}{2209}\begin{bmatrix} 1 & -56 \\ 40 & -31 \end{bmatrix}$$

The results are equal.

40. $A^{-2} = \left(A^{-1}\right)^2 = \left(\frac{1}{2}\left[\begin{array}{rrr} -15 & -4 & 28 \\ -1 & 0 & 2 \\ 23 & 6 & -42 \end{array}\right]\right)^2 = \frac{1}{4}\left[\begin{array}{rrr} 873 & 228 & -1604 \\ 61 & 16 & -112 \\ -1317 & -344 & 2420 \end{array}\right]$

$$A^{-2} = \left(A^2\right)^{-1} = \left[\begin{array}{rrr} 48 & 4 & 32 \\ -29 & 48 & -17 \\ 22 & 9 & 15 \end{array}\right]^{-1} = \frac{1}{4}\left[\begin{array}{rrr} 873 & 228 & -1604 \\ 61 & 16 & -112 \\ -1317 & -344 & 2420 \end{array}\right]$$

The results are equal.

42. (a) $(AB)^{-1} = B^{-1}A^{-1}$

$$= \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}\begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix}$$

$$= \frac{1}{77}\begin{bmatrix} -4 & 9 \\ -9 & 1 \end{bmatrix}$$

(b) $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix}^T = \begin{bmatrix} -\frac{2}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$

(c) $(2A)^{-1} = \frac{1}{2}A^{-1} = \frac{1}{2}\begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & \frac{1}{14} \\ \frac{3}{14} & \frac{1}{7} \end{bmatrix}$

44. (a) $(AB)^{-1} = B^{-1}A^{-1}$

$$= \begin{bmatrix} 6 & 5 & -3 \\ -2 & 4 & -1 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -25 & 24 \\ -6 & 10 & 7 \\ 17 & 7 & 15 \end{bmatrix}$$

(b) $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 4 \\ -4 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

(c) $(2A)^{-1} = \frac{1}{2}A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -2 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 2 & 1 & \frac{1}{2} \end{bmatrix}$

46. The coefficient matrix for each system is

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

and the formula for the inverse of a 2×2 matrix produces

$$A^{-1} = \frac{1}{2+2}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(a) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

The solution is: $x = 1$ and $y = 5$.

(b) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

The solution is: $x = -1$ and $y = -1$.

48. The coefficient matrix for each system is

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Using the algorithm to invert a matrix, you find that the inverse is

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ \frac{1}{3} & \frac{2}{3} & -1 \end{bmatrix}.$$

(a) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ \frac{1}{3} & \frac{2}{3} & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The solution is: $x_1 = 1$, $x_2 = 1$, and $x_3 = 1$.

(b) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ \frac{1}{3} & \frac{2}{3} & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

The solution is: $x_1 = 1$, $x_2 = 0$, and $x_3 = 1$.

50. Using a graphing utility or software program, you have

$A\mathbf{x} = \mathbf{b}$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 1 & -1 & 3 & -1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 2 & -1 \\ 2 & 1 & 4 & 1 & -1 \\ 3 & 1 & 1 & -2 & 1 \end{bmatrix},\ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix},\text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 3 \\ -1 \\ 5 \end{bmatrix}.$$

The solution is: $x_1 = 1$, $x_2 = 2$, $x_3 = -1$, $x_4 = 0$, and $x_5 = 1$.

52. Using a graphing utility or software program, you have

$$A\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 4 & -2 & 4 & 2 & -5 & -1 \\ 3 & 6 & -5 & -6 & 3 & 3 \\ 2 & -3 & 1 & 3 & -1 & -2 \\ -1 & 4 & -4 & -6 & 2 & 4 \\ 3 & -1 & 5 & 2 & -3 & -5 \\ -2 & 3 & -4 & -6 & 1 & 2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \text{ and}$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ -11 \\ 0 \\ -9 \\ 1 \\ -12 \end{bmatrix}.$$

The solution is: $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$, $x_5 = 0$, and $x_6 = 1$.

54. The inverse of A is given by

$$A^{-1} = \frac{1}{x-4}\begin{bmatrix} -2 & -x \\ 1 & 2 \end{bmatrix}.$$

Letting $A^{-1} = A$, you find that $\dfrac{1}{x-4} = -1$.

So, $x = 3$.

56. The matrix $\begin{bmatrix} x & 2 \\ -3 & 4 \end{bmatrix}$ will be singular if

$ad - bc = (x)(4) - (-3)(2) = 0$, which implies that

$4x = -6$ or $x = -\frac{3}{2}$.

58. First, find $4A$.

$$4A = \left[(4A)^{-1}\right]^{-1} = \frac{1}{4+12}\begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{4} \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix}$$

Then, multiply by $\frac{1}{4}$ to obtain

$$A = \tfrac{1}{4}(4A) = \tfrac{1}{4}\begin{bmatrix} \frac{1}{8} & -\frac{1}{4} \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix} = \begin{bmatrix} \frac{1}{32} & -\frac{1}{16} \\ \frac{3}{64} & \frac{1}{32} \end{bmatrix}.$$

60. Using the formula for the inverse of a 2×2 matrix, you have

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{\sec^2\theta - \tan^2\theta}\begin{bmatrix} \sec\theta & -\tan\theta \\ -\tan\theta & \sec\theta \end{bmatrix} \\ &= \begin{bmatrix} \sec\theta & -\tan\theta \\ -\tan\theta & \sec\theta \end{bmatrix}. \end{aligned}$$

62. Adjoin the identity matrix to form

$$\begin{bmatrix} F & I \end{bmatrix} = \begin{bmatrix} 0.017 & 0.010 & 0.008 & 1 & 0 & 0 \\ 0.010 & 0.012 & 0.010 & 0 & 1 & 0 \\ 0.008 & 0.010 & 0.017 & 0 & 0 & 1 \end{bmatrix}.$$

Using elementary row operations, reduce the matrix as follows.

$$\begin{bmatrix} I & F^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 115.56 & -100 & 4.44 \\ 0 & 1 & 0 & -100 & 250 & -100 \\ 0 & 0 & 1 & 4.44 & -100 & 115.56 \end{bmatrix}$$

So, $F^{-1} = \begin{bmatrix} 115.56 & -100 & 4.44 \\ -100 & 250 & -100 \\ 4.44 & -100 & 115.56 \end{bmatrix}$ and

$$\mathbf{w} = F^{-1}\mathbf{d} = \begin{bmatrix} 115.56 & -100 & 4.44 \\ -100 & 250 & -100 \\ 4.44 & -100 & 115.56 \end{bmatrix}\begin{bmatrix} 0 \\ 0.15 \\ 0 \end{bmatrix} = \begin{bmatrix} -15 \\ 37.5 \\ -15 \end{bmatrix}.$$

64. $A^T\left(A^{-1}\right)^T = \left(A^{-1}A\right)^T = I_n^T = I_n$ and

$\left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T = I_n^T = I_n$

So, $\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$.

66.

$$\begin{aligned} (I - 2A)(I - 2A) &= I^2 - 2IA - 2AI + 4A^2 \\ &= I - 4A + 4A^2 \\ &= I - 4A + 4A \quad \left(\text{because } A = A^2\right) \\ &= I \end{aligned}$$

So, $(I - 2A)^{-1} = I - 2A$.

68. Because $ABC = I$, A is invertible and $A^{-1} = BC$.

So, $ABC\ A = A$ and $BC\ A = I$.

So, $B^{-1} = CA$.

70. Let $A^2 = A$ and suppose A is nonsingular. Then, A^{-1} exists, and you have the following.

$$\begin{aligned} A^{-1}\left(A^2\right) &= A^{-1}A \\ \left(A^{-1}A\right)A &= I \\ A &= I \end{aligned}$$

72. (a) True. See Theorem 2.8, part 1 on page 67.

(b) False. For example, consider the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, which is not invertible, but $1 \cdot 1 - 0 \cdot 0 = 1 \neq 0$.

(c) False. If A is a square matrix then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if A is a *nonsingular* matrix.

74. A has an inverse if $a_{ii} \neq 0$ for all $i = 1 \dots n$ and

$$A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_{22}} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}.$$

76. $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

(a) $A^2 - 2A + 5I = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $A\left(\frac{1}{5}(2I - A)\right) = \frac{1}{5}\left(2A - A^2\right) = \frac{1}{5}(5I) = I$

Similarly, $\left(\frac{1}{5}(2I - A)\right)A = I$. Or, $\frac{1}{5}(2I - A) = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = A^{-1}$ directly.

(c) The calculation in part (b) did not depend on the entries of A.

78. Let C be the inverse of $(I - AB)$, that is $C = (I - AB)^{-1}$. Then $C(I - AB) = (I - AB)C = I$.

Consider the matrix $I + BCA$. Claim that this matrix is the inverse of $I - BA$. To check this claim, show that $(I + BCA)(I - BA) = (I - BA)(I + BCA) = I$.

First, show
$$\begin{aligned} (I - BA)(I + BCA) &= I - BA + BCA - BABCA \\ &= I - BA + B(C - ABC)A \\ &= I - BA + B\underbrace{((I - AB)C)}_{I}A \\ &= 1 - BA + BA = 1 \end{aligned}$$

Similarly, show $(I + BCA)(I - BA) = I$.

80. Answers will vary. *Sample answer*:

$A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ or $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

82. $$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad - bc}\right)\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Section 2.4 Elementary Matrices

2. This matrix is *not* elementary, because it is not square.

4. This matrix is elementary. It can be obtained by interchanging the two rows of I_2.

6. This matrix is elementary. It can be obtained by multiplying the first row of I_3 by 2, and adding the result to the third row.

8. This matrix is *not* elementary, because two elementary row operations are required to obtain it from I_4.

10. C is obtained by adding the third row of A to the first row. So,

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

12. A is obtained by adding -1 times the third row of C to the first row. So,

$$E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

14. Answers will vary. Sample answer:

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 3 & -3 & 6 \\ 0 & 0 & 2 & 2 \end{bmatrix}$	$R_1 \leftrightarrow R_2$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$	$\left(\frac{1}{3}\right)R_2 \rightarrow R_2$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$	$\left(\frac{1}{2}\right)R_3 \rightarrow R_3$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

So, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 3 & -3 & 6 \\ 1 & -1 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$

16. Answers will vary. Sample answer:

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & -1 \\ 3 & -2 & -4 \end{bmatrix}$	$(-2)R_1 + R_2 \rightarrow R_2$	$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & -1 \\ 0 & -11 & -4 \end{bmatrix}$	$(-3)R_1 + R_3 \rightarrow R_3$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & -11 & -4 \end{bmatrix}$	$(-1)R_2 \rightarrow R_2$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$	$(11)R_2 + R_3 \rightarrow R_3$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 11 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\left(\frac{1}{7}\right)R_3 \rightarrow R_3$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$

So, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 11 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & -1 \\ 3 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$

18.

Matrix	Elementary Row Operations	Elementary Matrix
$\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & -3 & 3 & 9 \\ 0 & 17 & -1 & -3 \\ 4 & 8 & -5 & 1 \end{bmatrix}$	$R_3 + (-2)R_1 \to R_3$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & -3 & 3 & 9 \\ 0 & 17 & -1 & -3 \\ 0 & 32 & -5 & -7 \end{bmatrix}$	$R_4 + (-4)R_2 \to R_4$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 17 & -1 & -3 \\ 0 & 32 & -5 & -7 \end{bmatrix}$	$\left(-\frac{1}{3}\right)R_2 \to R_2$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 16 & 48 \\ 0 & 32 & -5 & -7 \end{bmatrix}$	$R_3 + (-17)R_2 \to R_3$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -17 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 16 & 48 \\ 0 & 0 & 27 & 89 \end{bmatrix}$	$R_4 + (-32)R_2 \to R_2$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -32 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 27 & 89 \end{bmatrix}$	$\left(\frac{1}{16}\right)R_3 \to R_3$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 8 \end{bmatrix}$	$R_4 + (-27)R_3 \to R_4$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -27 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\left(\frac{1}{8}\right)R_4 \to R_4$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix}$

So, $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -27 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -32 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -17 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & -3 & 3 & 9 \\ 2 & 5 & -1 & 1 \\ 4 & 8 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

20. To obtain the inverse matrix, reverse the elementary row operation that produced it. So, multiply the first row by $\frac{1}{25}$ to obtain

$$E^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix}.$$

22. To obtain the inverse matrix, reverse the elementary row operation that produced it. So, add 3 times the second row to the third row to obtain

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

24. To obtain the inverse matrix, reverse the elementary row operation that produced it. So, add $-k$ times the third row to the second row to obtain

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -k & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

26. Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \left(\tfrac{1}{2}\right)R_1 \to R_1 \qquad E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_2 - R_1 \to R_2 \qquad E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Use the elementary matrices to find the inverse.

$$A^{-1} = E_2E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

28. Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \left(\tfrac{1}{2}\right)R_2 \to R_2 \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} R_1 + 2R_3 \to R_1 \qquad E_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} R_2 - \left(\tfrac{1}{2}\right)R_3 \to R_2 \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Use the elementary matrices to find the inverse.

$$A^{-1} = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

For Exercises 30–36, answers will vary. Sample answers are shown below.

30. The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is itself an elementary matrix, so the factorization is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

32. Reduce the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ as follows.

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$	Add -2 times row one to row two.	$E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	Multiply row two by -1.	$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	Add -1 times row two to row one.	$E_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

So, one way to factor A is

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

34. Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$ as follows.

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}$	Add -2 times row one to row two.	$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	Add -1 times row one to row three.	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Add -1 times row two to row three.	$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Add -3 times row three to row one.	$E_4 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Add -2 times row two to row one.	$E_5 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So, one way to factor A is

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

36. Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & -2 \end{bmatrix} \quad \left(\tfrac{1}{4}\right)R_1 \to R_1 \qquad E_1 = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -\frac{5}{2} \end{bmatrix} \quad R_4 - R_1 \to R_4 \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \left(-\tfrac{2}{5}\right)R_4 \to R_4 \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{2}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -R_3 \to R_3 \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 - \left(\tfrac{1}{2}\right)R_4 \to R_1 \qquad E_5 = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_2 - R_4 \to R_2 \qquad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_3 + 2R_4 \to R_3 \qquad E_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So, one way to factor A is

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}$$

$$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{5}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

38. (a) EA has the same rows as A except the two rows that are interchanged in E will be interchanged in EA.

(b) Multiplying a matrix on the left by E interchanges the same two rows that are interchanged from I_n in E. So, multiplying E by itself interchanges the rows twice and $E^2 = I_n$.

40. $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -a & 0 \\ -b & 1+ab & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}.$

42. (a) False. It is impossible to obtain the zero matrix by applying any elementary row operation to the identity matrix.

(b) True. If $A = E_1E_2 \ldots E_k$, where each E_i is an elementary matrix, then A is invertible (because every elementary matrix is) and $A^{-1} = E_k^{-1} \ldots E_2^{-1}E_1^{-1}$.

(c) True. See equivalent conditions (2) and (3) of Theorem 2.15.

44. Matrix — Elementary Matrix

$\begin{bmatrix} -2 & 1 \\ -6 & 4 \end{bmatrix} = A$

$\begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} = U \qquad E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$E_1A = U \Rightarrow A = E_1^{-1}U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} = LU$

46. Matrix — Elementary Matrix

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 10 & 12 & 3 \end{bmatrix} = A$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 12 & 3 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 7 \end{bmatrix} = U \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

$$E_2E_1A = U \Rightarrow A = E_1^{-1}E_2^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 7 \end{bmatrix} = LU$$

48. Matrix — Elementary Matrix

$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 1 & -1 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = A$

$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = U \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$E_3E_2E_1A = U \Rightarrow A = E_1^{-1}E_2^{-1}E_3^{-1}U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

$$L\mathbf{y} = \mathbf{b}: \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 15 \\ -1 \end{bmatrix}$$

$y_1 = 4, -y_1 + y_2 = -4 \Rightarrow y_2 = 0,$
$3y_1 + 2y_2 + y_3 = 15 \Rightarrow y_3 = 3,$ and $y_4 = -1.$

$$U\mathbf{x} = \mathbf{y}: \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ -1 \end{bmatrix}$$

$x_4 = 1,\ x_3 = 1,\ x_2 - x_3 = 0 \Rightarrow x_2 = 1,$ and $x_1 = 2.$

So, the solution to the system $A\mathbf{x} = \mathbf{b}$ is: $x_1 = 2,$ $x_2 = x_3 = x_4 = 1.$

50. $A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq A.$

Because $A^2 \neq A$, A is *not* idempotent.

52. $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Because $A^2 \neq A$, A is *not* idempotent.

54. Assume A is idempotent. Then

$$\begin{aligned} A^2 &= A \\ \left(A^2\right)^T &= A^T \\ \left(A^T A^T\right) &= A^T \end{aligned}$$

which means that A^T is idempotent.

Now assume A^T is idempotent. Then

$$\begin{aligned} A^T A^T &= A^T \\ \left(A^T A^T\right)^T &= \left(A^T\right)^T \\ AA &= A \end{aligned}$$

which means that A is idempotent.

56. $$\begin{aligned} (AB)^2 &= (AB)(AB) \\ &= A(BA)B \\ &= A(AB)B \\ &= (AA)(BB) \\ &= AB \end{aligned}$$

So, $(AB)^2 = AB$, and AB is idempotent.

58. If A is row-equivalent to B, then

$A = E_k \cdots E_2 E_1 B,$

where $E_1, \ldots, E_k$ are elementary matrices.

So,

$B = E_1^{-1} E_2^{-1} \cdots E_k^{-1} A,$

which shows that B is row equivalent to A.

60. (a) When an elementary row operation is performed on a matrix A, perform the same operation on I to obtain the matrix E.

(b) Keep track of the row operations used to reduce A to an upper triangular matrix U. If A row reduces to U using only the row operation of adding a multiple of one row to another row below it, then the inverse of the product of the elementary matrices is the matrix L, and $A = LU$.

(c) For the system $A\mathbf{x} = \mathbf{b}$, find an LU factorization of A. Then solve the system $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ and $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

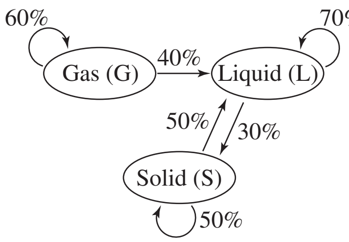
Section 2.5 Markov Chains

2. The matrix is *not* stochastic because every entry of a stochastic matrix satisfies the inequality $0 \leq a_{ij} \leq 1$.

4. The matrix is *not* stochastic because the sum of entries in a column of a stochastic matrix is 1.

6. The matrix is stochastic because each entry is between 0 and 1, and each column adds up to 1.

8.



The matrix of transition probabilities is shown.

$$P = \begin{array}{c} \overbrace{\begin{array}{ccc} \text{G} & \text{L} & \text{S} \end{array}}^{\text{From}} \\ \begin{bmatrix} 0.60 & 0 & 0 \\ 0.40 & 0.70 & 0.50 \\ 0 & 0.30 & 0.50 \end{bmatrix} \end{array} \left.\begin{array}{c} \text{G} \\ \text{L} \\ \text{S} \end{array}\right\} \text{To}$$

The initial state matrix represents the amounts of the physical states is shown.

$$X_0 = \begin{bmatrix} 0.20(10{,}000) \\ 0.60(10{,}000) \\ 0.20(10{,}000) \end{bmatrix} = \begin{bmatrix} 2000 \\ 6000 \\ 2000 \end{bmatrix}$$

To represent the amount of each physical state after the catalyst is added, multiply P by X_0 to obtain

$$PX_0 = \begin{bmatrix} 0.60 & 0 & 0 \\ 0.40 & 0.70 & 0.50 \\ 0 & 0.30 & 0.50 \end{bmatrix}\begin{bmatrix} 2000 \\ 6000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 1200 \\ 6000 \\ 2800 \end{bmatrix}.$$

So, after the catalyst is added there are 1200 molecules in a gas state, 6000 molecules in a liquid state, and 2800 molecules in a solid state.

10.

$$X_1 = PX_0 = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.9 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{15} \\ \frac{1}{3} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 0.2\overline{6} \\ 0.\overline{3} \\ 0.4 \end{bmatrix}$$

$$X_2 = PX_1 = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.9 \end{bmatrix}\begin{bmatrix} \frac{4}{15} \\ \frac{1}{3} \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{17}{75} \\ \frac{49}{150} \\ \frac{67}{150} \end{bmatrix} = \begin{bmatrix} 0.22\overline{6} \\ 0.32\overline{6} \\ 0.44\overline{6} \end{bmatrix}$$

$$X_3 = PX_2 = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.9 \end{bmatrix}\begin{bmatrix} \frac{17}{75} \\ \frac{49}{150} \\ \frac{67}{150} \end{bmatrix} = \begin{bmatrix} \frac{151}{750} \\ \frac{239}{750} \\ \frac{12}{25} \end{bmatrix} = \begin{bmatrix} 0.201\overline{3} \\ 0.318\overline{6} \\ 0.48 \end{bmatrix}$$

12. Form the matrix representing the given transition probabilities. A represents infected mice and B noninfected.

$$P = \begin{array}{c} \overbrace{\begin{array}{cc} A & B \end{array}}^{\text{From}} \\ \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \end{array} \left.\begin{array}{c} A \\ B \end{array}\right\} \text{To}$$

The state matrix representing the current population is

$$X_0 = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} \begin{array}{c} A \\ B \end{array}.$$

(a) The state matrix for next week is

$$X_1 = PX_0 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.13 \\ 0.87 \end{bmatrix}.$$

So, next week $0.13(1000) = 130$ mice will be infected.

(b) $$X_2 = PX_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}\begin{bmatrix} 0.13 \\ 0.87 \end{bmatrix} = \begin{bmatrix} 0.113 \\ 0.887 \end{bmatrix}$$

$$X_3 = PX_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}\begin{bmatrix} 0.113 \\ 0.887 \end{bmatrix} = \begin{bmatrix} 0.1113 \\ 0.8887 \end{bmatrix}$$

In 3 weeks, $0.1113(1000) \approx 111$ mice will be infected.

14. Form the matrix representing the given transition probabilities. Let S represent those who swim and B represent those who play basketball.

$$P = \begin{array}{c} \overbrace{\begin{array}{cc} S & B \end{array}}^{\text{From}} \\ \begin{bmatrix} 0.30 & 0.40 \\ 0.70 & 0.60 \end{bmatrix} \end{array} \begin{array}{c} \\ \left.\begin{array}{c} S \\ B \end{array}\right\} \text{To} \end{array}$$

The state matrix representing the students is

$$X_0 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \begin{array}{c} S \\ B \end{array}.$$

(a) The state matrix for tomorrow is

$$X_1 = PX_0 = \begin{bmatrix} 0.30 & 0.40 \\ 0.70 & 0.60 \end{bmatrix}\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.36 \\ 0.64 \end{bmatrix}.$$

So, tomorrow $0.36(250) = 90$ students will swim and $0.64(250) = 160$ students will play basketball.

(b) The state matrix for two days from now is

$$X_2 = P^2X_0 = \begin{bmatrix} 0.37 & 0.36 \\ 0.63 & 0.64 \end{bmatrix}\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.364 \\ 0.636 \end{bmatrix}.$$

So, two days from now $0.364(250) = 91$ students will swim and $0.636(250) = 159$ students will play basketball.

(c) The state matrix for four days from now is

$$X_4 = P^4X_0 = \begin{bmatrix} 0.363637 & 0.363637 \\ 0.636363 & 0.636363 \end{bmatrix}\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.36364 \\ 0.63636 \end{bmatrix}.$$

So, four days from now, $0.36364(250) \approx 91$ students will swim and $0.63636(250) \approx 159$ students will play basketball.

16. Form the matrix representing the given transition probabilities. Let A represent users of Brand A, B users of Brand B, and N users of neither brands.

$$P = \begin{array}{c} \overbrace{\begin{array}{ccc} A & B & N \end{array}}^{\text{From}} \\ \begin{bmatrix} 0.75 & 0.15 & 0.10 \\ 0.20 & 0.75 & 0.15 \\ 0.05 & 0.10 & 0.75 \end{bmatrix} \end{array} \begin{array}{c} \\ \left.\begin{array}{c} A \\ B \\ N \end{array}\right\} \text{To} \end{array}$$

The state matrix representing the current product usage is

$$X_0 = \begin{bmatrix} \frac{2}{11} \\ \frac{3}{11} \\ \frac{5}{11} \end{bmatrix} \begin{array}{c} A \\ B \\ N \end{array}$$

(a) The state matrix for next month is

$$X_1 = P^1X_0 = \begin{bmatrix} 0.75 & 0.15 & 0.10 \\ 0.20 & 0.75 & 0.15 \\ 0.05 & 0.10 & 0.75 \end{bmatrix}\begin{bmatrix} \frac{2}{11} \\ \frac{3}{11} \\ \frac{5}{11} \end{bmatrix} = \begin{bmatrix} 0.222\overline{7} \\ 0.30\overline{9} \\ 0.37\overline{2} \end{bmatrix}.$$

So, next month the distribution of users will be

$0.222\overline{7} \cdot 110{,}000 = 24{,}500$ for Brand A,

$0.30\overline{9} \cdot 110{,}000 = 34{,}000$ for Brand B, and

$0.37\overline{2} \cdot 110{,}000 = 41{,}500$ for neither.

(b) $$X_2 = P^2X_0 \approx \begin{bmatrix} 0.2511 \\ 0.3330 \\ 0.325 \end{bmatrix}$$

In 2 months, the distribution of users will be

$0.2511 \cdot 110{,}000 = 27{,}625$ for Brand A,

$0.3330 \cdot 110{,}000 = 36{,}625$ for Brand B, and

$0.325 \cdot 110{,}000 = 35{,}750$ for neither.

(c) $$X_{18} = P^{18}X_0 \approx \begin{bmatrix} 0.3139 \\ 0.3801 \\ 0.2151 \end{bmatrix}$$

In 18 months, the distribution of users will be

$0.3139 \cdot 110{,}000 \approx 34{,}530$ for Brand A,

$0.3801 \cdot 110{,}000 \approx 41{,}808$ for Brand B, and

$0.2151 \cdot 110{,}000 \approx 23{,}662$ for neither.

18. The stochastic matrix

$$P = \begin{bmatrix} 0 & 0.3 \\ 1 & 0.7 \end{bmatrix}$$

is regular because P^2 has only positive entries.

$$P\overline{X} = \overline{X} \Rightarrow \begin{bmatrix} 0 & 0.3 \\ 1 & 0.7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 0.3x_2 &= x_1 \\ x_1 + 0.7x_2 &= x_2 \end{aligned}$$

Because $x_1 + x_2 = 1$, the system of linear equations is as follows.

$$\begin{aligned} -x_1 + 0.3x_2 &= 0 \\ x_1 - 0.3x_2 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

The solution to the system is $x_2 = \frac{10}{13}$ and

$x_1 = 1 - \frac{10}{13} = \frac{3}{13}$.

So, $\overline{X} = \begin{bmatrix} \frac{3}{13} \\ \frac{10}{13} \end{bmatrix}$.

20. The stochastic matrix

$$P = \begin{bmatrix} 0.2 & 0 \\ 0.8 & 1 \end{bmatrix}$$

is not regular because every power of P has a zero in the second column.

$$P\overline{X} = \overline{X} \Rightarrow \begin{bmatrix} 0.2 & 0 \\ 0.8 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 0.2x_1 &= x_1 \\ 0.8x_1 + x_2 &= x_2 \end{aligned}$$

Because $x_1 + x_2 = 1$, the system of linear equations is as follows.

$$\begin{aligned} -0.8x_1 &= 0 \\ 0.8x_1 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

The solution of the system is $x_1 = 0$ and $x_2 = 1$.

So, $\overline{X} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

22. The stochastic matrix

$$P = \begin{bmatrix} \frac{2}{5} & \frac{7}{10} \\ \frac{3}{5} & \frac{3}{10} \end{bmatrix}$$

is regular because P^1 has only positive entries.

$$P\overline{X} = \overline{X} \Rightarrow \begin{bmatrix} \frac{2}{5} & \frac{7}{10} \\ \frac{3}{5} & \frac{3}{10} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} \tfrac{2}{5}x_1 + \tfrac{7}{10}x_2 &= x_1 \\ \tfrac{3}{5}x_1 + \tfrac{3}{10}x_2 &= x_2 \end{aligned}$$

Because $x_1 + x_2 = 1$, the system of linear equations is as follows.

$$\begin{aligned} -\tfrac{3}{5}x_1 + \tfrac{7}{10}x_2 &= 0 \\ \tfrac{3}{5}x_1 - \tfrac{7}{10}x_2 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

The solution of the system is $x_2 = \frac{6}{13}$ and

$x_1 = 1 - \frac{6}{13} = \frac{7}{13}$.

So, $\overline{X} = \begin{bmatrix} \frac{7}{13} \\ \frac{6}{13} \end{bmatrix}$.

24. The stochastic matrix

$$P = \begin{bmatrix} \frac{2}{9} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{4}{9} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$$

is regular because P^1 has only positive entries.

$$P\overline{X} = \overline{X} \Rightarrow \begin{bmatrix} \frac{2}{9} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{4}{9} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} \tfrac{2}{9}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{3}x_3 &= x_1 \\ \tfrac{1}{3}x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 &= x_2 \\ \tfrac{4}{9}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{3}x_3 &= x_3 \end{aligned}$$

Because $x_1 + x_2 + x_3 = 1$, the system of linear equations is as follows.

$$\begin{aligned} -\tfrac{7}{9}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{3}x_3 &= 0 \\ \tfrac{1}{3}x_1 - \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 &= 0 \\ \tfrac{4}{9}x_1 + \tfrac{1}{4}x_2 + \tfrac{2}{3}x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

The solution of the system is $x_3 = 0.33$, $x_2 = 0.4$, and $x_1 = 1 - 0.4 - 0.33 = 0.27$.

So, $\overline{X} = \begin{bmatrix} 0.27 \\ 0.4 \\ 0.33 \end{bmatrix}$.

26. The stochastic matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & 1 \\ \frac{1}{3} & \frac{1}{5} & 0 \\ \frac{1}{6} & \frac{3}{5} & 0 \end{bmatrix}$$

is regular because P^2 has only positive entries.

$$P\overline{X} = \overline{X} \Rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & 1 \\ \frac{1}{3} & \frac{1}{5} & 0 \\ \frac{1}{6} & \frac{3}{5} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} \tfrac{1}{2}x_1 + \tfrac{1}{5}x_2 + x_3 &= x_1 \\ \tfrac{1}{3}x_1 + \tfrac{1}{5}x_2 &= x_2 \\ \tfrac{1}{6}x_1 + \tfrac{3}{5}x_2 &= x_3 \end{aligned}$$

Because $x_1 + x_2 + x_3 = 1$, the system of linear equations is as follows.

$$\begin{aligned} -\tfrac{1}{2}x_1 + \tfrac{1}{5}x_2 + x_3 &= 0 \\ \tfrac{1}{3}x_1 - \tfrac{4}{5}x_2 &= 0 \\ \tfrac{1}{6}x_1 + \tfrac{3}{5}x_2 - x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

The solution of the system is

$x_3 = \frac{5}{22}$, $x_2 = \frac{5}{17} - \frac{5}{17}\left(\frac{5}{22}\right) = \frac{5}{22}$, and

$x_1 = 1 - \frac{5}{22} - \frac{5}{22} = \frac{6}{11}$.

So, $\overline{X} = \begin{bmatrix} \frac{6}{11} \\ \frac{5}{22} \\ \frac{5}{22} \end{bmatrix} \approx \begin{bmatrix} 0.54 \\ 0.22\overline{7} \\ 0.22\overline{7} \end{bmatrix}$.

28. The stochastic matrix

$$P = \begin{bmatrix} 0.1 & 0 & 0.3 \\ 0.7 & 1 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}$$

is not regular because every power of P has two zeros in the second column.

$$P\overline{X} = \overline{X} \Rightarrow \begin{bmatrix} 0.1 & 0 & 0.3 \\ 0.7 & 1 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} 0.1x_1 + 0.3x_3 &= x_1 \\ 0.7x_1 + x_2 + 0.3x_3 &= x_2 \\ 0.2x_1 + 0.4x_3 &= x_3 \end{aligned}$$

Because $x_1 + x_2 + x_3 = 1$, the system of linear equations is as follows.

$$\begin{aligned} -0.9x_1 + 0.3x_3 &= 0 \\ 0.7x_1 + 0.3x_3 &= 0 \\ 0.2x_1 - 0.6x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

The solution of the system is $x_3 = 0$, $x_2 = 1 - 0 = 1$, and $x_1 = 1 - 1 - 0 = 0$.

So, $\overline{X} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

30. The stochastic matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is not regular because every power of P has three zeros in the first column.

$$P\overline{X} = \overline{X} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{aligned} x_1 &= x_1 \\ x_3 &= x_2 \\ x_2 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

Because $x_1 + x_2 + x_3 + x_4 = 1$, the system of linear equations is as follows.

$$\begin{aligned} 0 &= 0 \\ -x_2 + x_3 &= 0 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 1 \end{aligned}$$

Let $x_3 = s$ and $x_4 = t$. The solution of the system is $x_4 = t$, $x_3 = s$, $x_2 = s$, and $x_1 = 1 - 2s - t$, where $0 \le s \le 1$, $0 \le t \le 1$, and $2s + t \le 1$.

32. Exercise 3: To find $\overline{X}$, let $\overline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then use the matrix equation $P\overline{X} = \overline{X}$ to obtain

$$\begin{bmatrix} 0.\overline{3} & 0.1\overline{6} & 0.25 \\ 0.\overline{3} & 0.\overline{6} & 0.25 \\ 0.\overline{3} & 0.1\overline{6} & 0.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\begin{aligned} 0.\overline{3}x_1 + 0.1\overline{6}x_2 + 0.25x_3 &= x_1 \\ 0.\overline{3}x_1 + 0.\overline{6}x_2 + 0.25x_3 &= x_2 \\ 0.\overline{3}x_1 + 0.1\overline{6}x_2 + 0.5x_3 &= x_3 \end{aligned}$$

Use these equations and the fact that $x_1 + x_2 + x_3 = 1$ to write the system of linear equations shown.

$$\begin{aligned} -0.\overline{6}x_1 + 0.1\overline{6}x_2 + 0.25x_3 &= 0 \\ 0.\overline{3}x_1 - 0.\overline{3}x_2 + 0.25x_3 &= 0 \\ 0.\overline{3}x_1 + 0.1\overline{6}x_2 + 0.5x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

The solution of the system is

$x_1 = \frac{3}{13}$, $x_2 = \frac{6}{13}$, and $x_3 = \frac{4}{13}$.

So, the steady state matrix is

$$\overline{X} = \begin{bmatrix} \frac{3}{13} \\ \frac{6}{13} \\ \frac{4}{13} \end{bmatrix}.$$

Exercise 5: To find $\overline{X}$, let $\overline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Then use the matrix equation $P\overline{X} = \overline{X}$ to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

or

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= x_2 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}.$$

Use these equations and the fact that $x_1 + x_2 + x_3 + x_4 = 1$ to write the system of linear equations shown.

$$x_1 + x_2 + x_3 + x_4 = 1$$

Let $x_2 = r$, $x_3 = s$, and $x_4 = t$, where r, s, and t are real numbers between 0 and 1.

The solution of the system is

$x_1 = 1 - r - s - t$, $x_2 = r$, $x_3 = s$, and $x_4 = t$, where r, s, and t are real numbers such that $0 \le r \le 1$, $0 \le s \le 1$, $0 \le t \le 1$, and $r + s + t \le 1$.

So, the steady state matrix is

$$\overline{X} = \begin{bmatrix} 1 - r - s - t \\ r \\ s \\ t \end{bmatrix}.$$

Exercise 6: To find $\overline{X}$, let $\overline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Then use the matrix equation $P\overline{X} = \overline{X}$ to obtain

$$\begin{bmatrix} \frac{1}{2} & \frac{2}{9} & \frac{1}{4} & \frac{4}{15} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & \frac{4}{15} \\ \frac{1}{6} & \frac{2}{9} & \frac{1}{4} & \frac{4}{15} \\ \frac{1}{6} & \frac{2}{9} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

or

$$\begin{aligned} \tfrac{1}{2}x_1 + \tfrac{2}{9}x_2 + \tfrac{1}{4}x_3 + \tfrac{4}{15}x_4 &= x_1 \\ \tfrac{1}{6}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{4}x_3 + \tfrac{4}{15}x_4 &= x_2 \\ \tfrac{1}{6}x_1 + \tfrac{2}{9}x_2 + \tfrac{1}{4}x_3 + \tfrac{4}{15}x_4 &= x_3 \\ \tfrac{1}{6}x_1 + \tfrac{2}{9}x_2 + \tfrac{1}{4}x_3 + \tfrac{1}{5}x_4 &= x_4 \end{aligned}.$$

Use these equations and the fact that $x_1 + x_2 + x_3 + x_4 = 1$ to write the system of equations shown.

$$\begin{aligned} -\tfrac{1}{2}x_1 + \tfrac{2}{9}x_2 + \tfrac{1}{4}x_3 + \tfrac{4}{15}x_4 &= 0 \\ \tfrac{1}{6}x_1 - \tfrac{2}{3}x_2 + \tfrac{1}{4}x_3 + \tfrac{4}{15}x_4 &= 0 \\ \tfrac{1}{6}x_1 - \tfrac{2}{9}x_2 - \tfrac{3}{4}x_3 + \tfrac{4}{15}x_4 &= 0 \\ \tfrac{1}{6}x_1 + \tfrac{2}{9}x_2 + \tfrac{1}{4}x_3 - \tfrac{4}{5}x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 1 \end{aligned}$$

The solution of the system is

$x_1 = \frac{24}{73}$, $x_2 = \frac{18}{73}$, $x_3 = \frac{16}{73}$, and $x_4 = \frac{15}{73}$.

So, the steady state matrix is

$$\overline{X} = \begin{bmatrix} \frac{24}{73} \\ \frac{18}{73} \\ \frac{16}{73} \\ \frac{15}{73} \end{bmatrix} \approx \begin{bmatrix} 0.3288 \\ 0.2466 \\ 0.2192 \\ 0.2055 \end{bmatrix}.$$

34. Form the matrix representing the given transition probabilities. Let A represent those who received an "A" and let N represent those who did not.

$$\begin{array}{c} \text{From} \\ \overbrace{\begin{array}{cc} A & N \end{array}} \end{array}$$

$$P = \begin{array}{c} \left[\begin{array}{cc} 0.70 & 0.10 \\ 0.30 & 0.90 \end{array}\right] \end{array} \left.\begin{array}{c} A \\ N \end{array}\right\} \text{To}$$

To find the steady state matrix, solve the equation $P\overline{X} = \overline{X}$, where $\overline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and use the fact that $x_1 + x_2 = 1$ to write a system of equations.

$$\begin{aligned} 0.70x_1 + 0.10x_2 &= x_1 & \quad -0.3x_1 + 0.1x_2 &= 0 \\ 0.30x_1 + 0.90x_2 &= x_2 \Rightarrow & 0.3x_1 - 0.1x_2 &= 0 \\ x_1 + x_2 &= 1 & x_1 + x_2 &= 1 \end{aligned}$$

The solution of the system is $x_1 = \frac{1}{4}$ and $x_2 = \frac{3}{4}$. So, the steady state matrix is $\overline{X} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$. This indicates that eventually $\frac{1}{4}$ of the students will receive assignment grades of "A" and $\frac{3}{4}$ of the students will not.

36. Form the matrix representing transition probabilities. Let A represent Theatre A, let B represent Theatre B, and let N represent neither theatre.

$$\begin{array}{c} \text{From} \\ \overbrace{\begin{array}{ccc} A & B & N \end{array}} \end{array}$$

$$P = \left[\begin{array}{ccc} 0.10 & 0.06 & 0.03 \\ 0.05 & 0.08 & 0.04 \\ 0.85 & 0.86 & 0.97 \end{array}\right] \left.\begin{array}{c} A \\ B \\ B \end{array}\right\} \text{To}$$

To find the steady state matrix, solve the equation $P\overline{X} = \overline{X}$ where $\overline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and use the fact that $x_1 + x_2 + x_3 = 1$ to write a system of equations.

$$\begin{aligned} 0.10x_1 + 0.06x_2 + 0.03x_3 &= x_1 & \quad -0.90x_1 + 0.06x_2 + 0.03x_3 &= 0 \\ 0.05x_1 + 0.08x_2 + 0.04x_3 &= x_2 \Rightarrow & 0.05x_1 - 0.92x_2 + 0.04x_3 &= 0 \\ 0.85x_1 + 0.86x_2 + 0.97x_3 &= x_3 & 0.85x_1 + 0.86x_2 - 0.03x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 & x_1 + x_2 + x_3 &= 1 \end{aligned}$$

The solution of the system is $x_1 = \frac{4}{119}$, $x_2 = \frac{5}{119}$, and $x_3 = \frac{110}{119}$. So, the steady state matrix is $\overline{X} = \begin{bmatrix} \frac{4}{119} \\ \frac{5}{119} \\ \frac{110}{119} \end{bmatrix}$. This indicates that eventually $\frac{4}{119} \approx 3.4\%$ of the people will attend Theatre A, $\frac{5}{119} \approx 4.2\%$ of the people will attend Theatre B, and $\frac{110}{119} \approx 92.4\%$ of the people will attend neither theatre on any given night.

38. The matrix is not absorbing; The first state S_1 is absorbing, however the corresponding Markov chain is not absorbing because there is no way to move from S_2 or S_3 to S_1.

40. The matrix is absorbing; The fourth state S_4 is absorbing and it is possible to move from any of the states to S_4 in one transition.

42. Use the matrix equation $P\overline{X} = \overline{X}$, or

$$\begin{bmatrix} 0.1 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.7 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

along with the equation $x_1 + x_2 + x_3 = 1$ to write the linear system

$$\begin{aligned} -0.9x_1 &= 0 \\ 0.2x_1 &= 0 \\ 0.7x_1 &= 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}.$$

The solution of this system is $x_1 = 0$, $x_2 = 1 - t$, and $x_3 = t$, where t is a real number such that $0 \le t \le 1$.

So, the steady state matrix is $\overline{X} = \begin{bmatrix} 0 \\ 1 - t \\ t \end{bmatrix}$, where $0 \le t \le 1$.

44. Use the matrix equation $P\overline{X} = \overline{X}$ or

$$\begin{bmatrix} 0.7 & 0 & 0.2 & 0.1 \\ 0.1 & 1 & 0.5 & 0.6 \\ 0 & 0 & 0.2 & 0.2 \\ 0.2 & 0 & 0.1 & 0.1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

along with the equation $x_1 + x_2 + x_3 + x_4 = 1$ to write the linear system

$$\begin{aligned} -0.3x_1 + 0.2x_3 + 0.1x_4 &= 0 \\ 0.1x_1 + 0.5x_3 + 0.6x_4 &= 0 \\ - 0.8x_3 + 0.2x_4 &= 0 \\ 0.2x_1 + 0.1x_3 - 0.9x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 1 \end{aligned}.$$

The solution of this system is $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, and $x_4 = 0$. So, the steady state matrix is $\overline{X} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

46. Let S_n be the state that Player 1 has n chips.

$$P = \overbrace{\begin{array}{ccccc} S_0 & S_1 & S_2 & S_3 & S_4 \end{array}}^{\text{From}}$$

$$P = \left.\begin{bmatrix} 1 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0.3 & 0 & 0.7 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 1 \end{bmatrix}\begin{matrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{matrix}\right\}\text{To} \text{ and } X_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

So,

$$P^n X_0 \to \overline{P}X_0 = \begin{bmatrix} \frac{49}{58} \\ 0 \\ 0 \\ 0 \\ \frac{9}{58} \end{bmatrix}.$$

So, the probability that Player 1 reaches S_4 and wins the tournament is $\frac{9}{58} \approx 0.155$.

48. (a) To find the nth state matrix of a Markov chain, compute $X_n = P^n X_0$, where X_0 is the initial state matrix.

(b) To find the steady state matrix of a Markov chain, determine the limit of $P^n X_0$, as $n \to \infty$, where X_0 is the initial state matrix.

(c) The regular Markov chain is $PX_0, P^2X_0, P^3X_0, \ldots$, where P is a regular stochastic matrix and X_0 is the initial state matrix.

(d) An absorbing Markov chain is a Markov chain with at least one absorbing state and it is possible for a member of the population to move from any nonabsorbing state to an absorbing state in a finite number of transitions.

(e) An absorbing Markov chain is concerned with having an entry of 1 and the rest 0 in a column, whereas a regular Markov chain is concerned with the repeated multiplication of the regular stochastic matrix.

50. (a) When the chain reaches S_1 or S_4, it is certain in the next step to transition to an adjacent state, S_2 or S_3, respectively, so S_1 and S_4 reflect to S_2 or S_3.

(b)
$$P = \begin{bmatrix} 0 & 0.4 & 0 & 0 \\ 1 & 0 & 0.3 & 0 \\ 0 & 0.6 & 0 & 1 \\ 0 & 0 & 0.7 & 0 \end{bmatrix}$$

(c)
$$P^{30} \approx \begin{bmatrix} \frac{1}{6} & 0 & \frac{1}{6} & 0 \\ 0 & \frac{5}{12} & 0 & \frac{5}{12} \\ \frac{5}{6} & 0 & \frac{5}{6} & 0 \\ 0 & \frac{7}{12} & 0 & \frac{7}{12} \end{bmatrix}$$

$$P^{30} \approx \begin{bmatrix} 0 & \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{5}{6} & 0 & \frac{5}{12} & 0 \\ 0 & \frac{5}{6} & 0 & \frac{5}{6} \\ \frac{7}{12} & 0 & \frac{7}{12} & 0 \end{bmatrix}$$

Other high even or odd powers of P give similar results where the columns alternate.

(d) $\bar{X} = \begin{bmatrix} \frac{1}{12} \\ \frac{5}{24} \\ \frac{5}{12} \\ \frac{7}{24} \end{bmatrix}$

Half the sum entries in the corresponding columns of P^n and P^{n+1} approach the corresponding entries in $\bar{X}$.

52. (a) Yes, it is possible.

(b) Yes, it is possible.

Both matrices X satisfy $P^1X = X$. The steady state matrix depends on the initial state matrix. In general, the steady state matrix is $\bar{X} = \begin{bmatrix} \frac{6}{11} - t \\ \frac{5}{11} - \frac{5}{6}t \\ \frac{5}{6}t \\ t \end{bmatrix}$,

where t is any real number such that $0 \le t \le \frac{6}{11}$. In part (a) $t = 0$ and in part (b), $t = \frac{6}{11}$.

54. Let

$$P = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$$

be a 2×2 stochastic matrix, and consider the system of equations $PX = X$.

$$\begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

You have

$$\begin{aligned} ax_1 + \quad bx_2 &= x_1 \\ (1-a)x_1 + (1-b)x_2 &= x_2 \end{aligned}$$

or

$$\begin{aligned} (a-1)x_1 + bx_2 &= 0 \\ (1-a)x_1 - bx_2 &= 0. \end{aligned}$$

Letting $x_1 = b$ and $x_2 = 1 - a$, you have the 2×1 state matrix X satisfying $PX = X$

$$X = \begin{bmatrix} b \\ 1-a \end{bmatrix}.$$

56. Let P be a regular stochastic matrix and X_0 be the initial state matrix.

$$\begin{aligned} \lim_{n\to\infty} P^nX_0 &= \lim_{n\to\infty} P^n(x_1 + x_2 + \cdots + x_k) \\ &= \lim_{n\to\infty} P^n \cdot x_1 + \lim_{n\to\infty} P^n \cdot x_2 + \cdots + \lim_{n\to\infty} P^n \cdot x_k \\ &= \bar{P}x_1 + \bar{P}x_2 + \cdots + \bar{P}x_k \\ &= \bar{P}(x_1 + x_2 + \cdots + x_k) \\ &= \bar{P}X_0 \\ &= \bar{X}, \text{ where } \bar{X} \text{ is a unique steady state matrix.} \end{aligned}$$

Section 2.6 More Applications of Matrix Operations

2. Divide the message into groups of four and form the uncoded matrices.

H E L P _ I S _ C O M I N G _ _

$[8\ \ 5\ \ 12\ \ 16]\ \ [0\ \ 9\ \ 19\ \ 0]\ \ [3\ \ 15\ \ 13\ \ 9]\ \ [14\ \ 7\ \ 0\ \ 0]$

Multiplying each uncoded row matrix on the right by A yields the coded row matrices

$$[8\ \ 5\ \ 12\ \ 16]A = [8\ \ 5\ \ 12\ \ 16]\begin{bmatrix} -2 & 3 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 2 \\ 3 & 1 & -2 & -4 \end{bmatrix}$$

$$= [15\ \ 33\ \ -23\ \ -43]$$

$$[0\ \ 9\ \ 19\ \ 0]A = [-28\ \ -10\ \ 28\ \ 47]$$

$$[3\ \ 15\ \ 13\ \ 9]A = [-7\ \ 20\ \ 7\ \ 2]$$

$$[14\ \ 7\ \ 0\ \ 0]A = [-35\ \ 49\ \ -7\ \ -7].$$

So, the coded message is 15, 33, -23, -43, -28, -10, 28, 47, -7, 20, 7, 2, -35, 49, -7, -7.

4. Find $A^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$, and multiply each coded row matrix on the right by A^{-1} to find the associated uncoded row matrix.

$$[85\ \ 120]\begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} = [20\ \ 15] \Rightarrow \text{T}, O$$

$$[6\ \ 8]A^{-1} = [0\ \ 2] \Rightarrow \_,\ \text{B}$$

$$[10\ \ 15]A^{-1} = [5\ \ 0] \Rightarrow \text{E},\ \_$$

$$[84\ \ 117]A^{-1} = [15\ \ 18] \Rightarrow \text{O, R}$$

$$[42\ \ 56]A^{-1} = [0\ \ 14] \Rightarrow \_,\ \text{N}$$

$$[90\ \ 125]A^{-1} = [15\ \ 20] \Rightarrow \text{O, T}$$

$$[60\ \ 80]A^{-1} = [0\ \ 20] \Rightarrow \_,\ \text{T}$$

$$[30\ \ 45]A^{-1} = [15\ \ 0] \Rightarrow \text{O},\ \_$$

$$[19\ \ 26]A^{-1} = [2\ \ 5] \Rightarrow \text{B, E}$$

So, the message is TO_BE_OR_NOT_TO_BE.

6. Find $A^{-1} = \begin{bmatrix} 11 & 2 & -8 \\ 4 & 1 & -3 \\ -8 & -1 & 6 \end{bmatrix}$, and multiply each coded row matrix on the right by A^{-1} to find the associated uncoded row matrix.

$$[112\ \ -140\ \ 83]A^{-1} = [112\ \ -140\ \ 83]\begin{bmatrix} 11 & 2 & -8 \\ 4 & 1 & -3 \\ -8 & -1 & 6 \end{bmatrix} = [8\ \ 1\ \ 22] \Rightarrow \text{H, A, V}$$

$$[19\ \ -25\ \ 13]A^{-1} = [5\ \ 0\ \ 1] \Rightarrow \text{E},\ \_,\ \text{A}$$

$$[72\ \ -76\ \ 61]A^{-1} = [0\ \ 7\ \ 18] \Rightarrow \_,\ \text{G, R}$$

$$[95\ \ -118\ \ 71]A^{-1} = [5\ \ 1\ \ 20] \Rightarrow \text{E, A, T}$$

$$[20\ \ 21\ \ 38]A^{-1} = [0\ \ 23\ \ 5] \Rightarrow \_,\ \text{W, E}$$

$$[35\ \ -23\ \ 36]A^{-1} = [5\ \ 11\ \ 5] \Rightarrow \text{E, K, E}$$

$$[42\ \ -48\ \ 32]A^{-1} = [14\ \ 4\ \ 0] \Rightarrow \text{N, D},\ \_$$

The message is HAVE_A_GREAT_WEEKEND_.

8. Let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and find that

$$\overset{\text{\_}\quad\text{S}}{\begin{bmatrix} -19 & -19 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 19 \end{bmatrix}}$$

$$\overset{\text{U}\quad\text{E}}{\begin{bmatrix} 37 & 16 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 21 & 5 \end{bmatrix}}.$$

This produces a system of 4 equations.

$$\begin{aligned} -19a \quad\quad - 19c \quad\quad &= 0 \\ - 19b \quad\quad - 19d &= 19 \\ 37a \quad\quad + 16c \quad\quad &= 21 \\ 37b \quad\quad + 16d &= 5. \end{aligned}$$

Solving this system, you find $a = 1$, $b = 1$, $c = -1$, and $d = -2$. So,

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}.$$

Multiply each coded row matrix on the right by A^{-1} to yield the uncoded row matrices.

$[3 \quad 1]$, $[14 \quad 3]$, $[5 \quad 12]$, $[0 \quad 15]$, $[18 \quad 4]$,

$[5 \quad 18]$, $[19 \quad 0]$, $[0 \quad 19]$, $[21 \quad 5]$.

This corresponds to the message

CANCEL_ORDERS_SUE.

10. You have

$$\begin{bmatrix} 45 & -35 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 10 & 15 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} 38 & -30 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 8 & 14 \end{bmatrix}.$$

So, $45w - 35y = 10$ and $45x - 35z = 15$
$38w - 30y = 8$ $\quad 38x - 30z = 14.$

Solving these two systems gives $w = y = 1$, $x = -2$, and $z = -3$. So,

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}.$$

(b) Decoding, you have:

$$\begin{aligned} \begin{bmatrix} 45 & -35 \end{bmatrix}A^{-1} &= \begin{bmatrix} 10 & 15 \end{bmatrix} &\Rightarrow&\ \text{J, O} \\ \begin{bmatrix} 38 & -30 \end{bmatrix}A^{-1} &= \begin{bmatrix} 8 & 14 \end{bmatrix} &\Rightarrow&\ \text{H, N} \\ \begin{bmatrix} 18 & -18 \end{bmatrix}A^{-1} &= \begin{bmatrix} 0 & 18 \end{bmatrix} &\Rightarrow&\ \text{\_, R} \\ \begin{bmatrix} 35 & -30 \end{bmatrix}A^{-1} &= \begin{bmatrix} 5 & 20 \end{bmatrix} &\Rightarrow&\ \text{E, T} \\ \begin{bmatrix} 81 & -60 \end{bmatrix}A^{-1} &= \begin{bmatrix} 21 & 18 \end{bmatrix} &\Rightarrow&\ \text{U, R} \\ \begin{bmatrix} 42 & -28 \end{bmatrix}A^{-1} &= \begin{bmatrix} 14 & 0 \end{bmatrix} &\Rightarrow&\ \text{N, \_} \\ \begin{bmatrix} 75 & -55 \end{bmatrix}A^{-1} &= \begin{bmatrix} 20 & 15 \end{bmatrix} &\Rightarrow&\ \text{T, O} \\ \begin{bmatrix} 2 & -2 \end{bmatrix}A^{-1} &= \begin{bmatrix} 0 & 2 \end{bmatrix} &\Rightarrow&\ \text{\_, B} \\ \begin{bmatrix} 22 & -21 \end{bmatrix}A^{-1} &= \begin{bmatrix} 1 & 19 \end{bmatrix} &\Rightarrow&\ \text{A, S} \\ \begin{bmatrix} 15 & -10 \end{bmatrix}A^{-1} &= \begin{bmatrix} 5 & 0 \end{bmatrix} &\Rightarrow&\ \text{E, \_} \end{aligned}$$

The message is JOHN_RETURN_TO_BASE_.

12. Use the given information to find D.

$$D = \begin{array}{c} \overbrace{\text{A} \qquad \text{B}}^{\text{User}} \\ \begin{bmatrix} 0.30 & 0.20 \\ 0.40 & 0.40 \end{bmatrix} \end{array} \begin{array}{c} \\ \left.\begin{matrix} \text{A} \\ \text{B} \end{matrix}\right\} \text{Supplier} \end{array}$$

The equation $X = DX + E$ may be rewritten in the form $(I - D)X = E$, that is

$$\begin{bmatrix} 0.7 & -0.2 \\ -0.4 & 0.6 \end{bmatrix}X = \begin{bmatrix} 10{,}000 \\ 20{,}000 \end{bmatrix}.$$

Solve this system by using Gauss-Jordan elimination to obtain

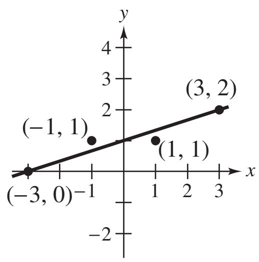
$$x \approx \begin{bmatrix} 29{,}412 \\ 52{,}941 \end{bmatrix}.$$

14. From the given matrix D, form the linear system $X = DX + E$, which can be written as $(I - D)X = E$, that is

$$\begin{bmatrix} 0.8 & -0.4 & -0.4 \\ -0.4 & 0.8 & -0.2 \\ 0 & -0.2 & 0.8 \end{bmatrix} X = \begin{bmatrix} 5000 \\ 2000 \\ 8000 \end{bmatrix}.$$

Solving this system, $X = \begin{bmatrix} 21{,}875 \\ 17{,}000 \\ 14{,}250 \end{bmatrix}$.

16. (a) The line that best fits the given points is shown in the graph.



(b) Using the matrices

$$X = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

you have $X^T X = \begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix}$, $X^T Y = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, and

$$A = \left(X^T X\right)^{-1} X^T Y = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{20} \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{10} \end{bmatrix}.$$

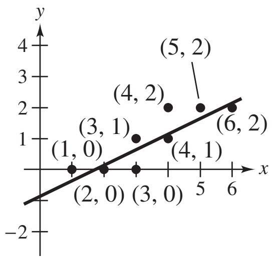
So, the least squares regression line is $y = \frac{3}{10}x + 1$.

(c) Solving $Y = XA + E$ for E, you have

$$E = Y - XA = \begin{bmatrix} -0.1 \\ 0.3 \\ -0.3 \\ 0.1 \end{bmatrix}.$$

So, the sum of the squares error is $E^T E = 0.2$.

18. (a) The line that best fits the given points is shown in the graph.



(b) Using the matrices

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 8 & 28 \\ 28 & 116 \end{bmatrix},\ X^T Y = \begin{bmatrix} 8 \\ 37 \end{bmatrix}, \text{ and}$$

$$A = \left(X^T X\right)^{-1}\left(X^T Y\right) = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix}.$$

So, the least squares regression line is $y = \frac{1}{2}x - \frac{3}{4}$.

(c) Solving $Y = XA + E$ for E, you have

$$E = Y - XA = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}^T$$

and the sum of the squares error is $E^T E = 1.5$.

20. Using the matrices

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix},$$

$$X^T Y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 9 \\ 39 \end{bmatrix}, \text{ and}$$

$$A = \left(X^T X\right)^{-1}\left(X^T Y\right) = \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix}.$$

So, the least squares regression line is $y = \frac{3}{2}x - \frac{3}{2}$.

22. Using matrices

$$X = \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \text{ and } Y = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 5 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 4 & 0 \\ 0 & 40 \end{bmatrix}, \quad X^T Y = \begin{bmatrix} 8 \\ 32 \end{bmatrix}, \text{ and}$$

$$A = \left(X^T X\right)^{-1}\left(X^T Y\right) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{40} \end{bmatrix}\begin{bmatrix} 8 \\ 32 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.8 \end{bmatrix}.$$

So, the least squares regression line is $y = 0.8x + 2$.

24. Using matrices

$$X = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix}, \quad X^T Y = \begin{bmatrix} 7 \\ -13 \end{bmatrix}, \text{ and}$$

$$A = \left(X^T X\right)^{-1}\left(X^T Y\right) = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{20} \end{bmatrix}\begin{bmatrix} 7 \\ -13 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ -\frac{13}{20} \end{bmatrix}.$$

So, the least squares regression line is
$y = -0.65x + 1.75$.

26. Using matrices

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 5 \\ 1 & 8 \\ 1 & 10 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 6 \\ 3 \\ 0 \\ -4 \\ -5 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 5 & 8 & 10 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 5 \\ 1 & 8 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 27 \\ 27 & 205 \end{bmatrix},$$

$$X^T Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 5 & 8 & 10 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 0 \\ -4 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -70 \end{bmatrix}, \text{ and}$$

$$A = \left(X^T X\right)^{-1}\left(X^T Y\right) = \frac{1}{296}\begin{bmatrix} 205 & -27 \\ -27 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ -70 \end{bmatrix}$$

$$= \frac{1}{296}\begin{bmatrix} 1890 \\ -350 \end{bmatrix}.$$

So, the least squares regression line is
$y = -\frac{175}{148}x + \frac{945}{148}$.

28. Using matrices

$$X = \begin{bmatrix} 1 & 9 \\ 1 & 10 \\ 1 & 11 \\ 1 & 12 \\ 1 & 13 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0.72 \\ 0.92 \\ 1.17 \\ 1.34 \\ 1.60 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 5 & 55 \\ 55 & 615 \end{bmatrix} \text{ and } X^T Y = \begin{bmatrix} 5.75 \\ 65.43 \end{bmatrix}.$$

$$A = \left(X^T X\right)^{-1} X^T Y = \begin{bmatrix} -1.248 \\ 0.218 \end{bmatrix}$$

So, the least squares regression line is
$y = 0.218x - 1.248$.

30. (a) To encode a message, convert the message to numbers and partition it into uncoded row matrices of size $1 \times n$. Then multiply on the right by an invertible $n \times n$ matrix A to obtain coded row matrices. To decode a message, multiply the coded row matrices on the right by A^{-1} and convert the numbers back to letters.

(b) A Leontief input-output model uses an $n \times n$ matrix to represent the input needs of an economic system, and an $n \times 1$ matrix to represent any external demands on the system.

(c) The coefficients of the least squares regression line are given by $A = \left(X^T X\right)^{-1} X^T Y$.

Review Exercises for Chapter 2

2. $-2\begin{bmatrix}1 & 2\\5 & -4\\6 & 0\end{bmatrix} + 8\begin{bmatrix}7 & 1\\1 & 2\\1 & 4\end{bmatrix} = \begin{bmatrix}-2 & -4\\-10 & 8\\-12 & 0\end{bmatrix} + \begin{bmatrix}56 & 8\\8 & 16\\8 & 32\end{bmatrix} = \begin{bmatrix}54 & 4\\-2 & 24\\-4 & 32\end{bmatrix}$

4. $\begin{bmatrix}1 & 5\\2 & -4\end{bmatrix}\begin{bmatrix}6 & -2 & 8\\4 & 0 & 0\end{bmatrix} = \begin{bmatrix}1(6)+5(4) & 1(-2)+5(0) & 1(8)+5(0)\\2(6)-4(4) & 2(-2)-4(0) & 2(8)-4(0)\end{bmatrix} = \begin{bmatrix}26 & -2 & 8\\-4 & -4 & 16\end{bmatrix}$

6. $\begin{bmatrix}2 & 1\\6 & 0\end{bmatrix}\begin{bmatrix}4 & 2\\-3 & 1\end{bmatrix} + \begin{bmatrix}-2 & 4\\0 & 4\end{bmatrix} = \begin{bmatrix}5 & 5\\24 & 12\end{bmatrix} + \begin{bmatrix}-2 & 4\\0 & 4\end{bmatrix} = \begin{bmatrix}3 & 9\\24 & 16\end{bmatrix}$

8. Letting $A = \begin{bmatrix}2 & -1\\3 & 2\end{bmatrix}$, $\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix}5\\-4\end{bmatrix}$, the system can be written as

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix}2 & -1\\3 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}5\\-4\end{bmatrix}.$$

Using Gaussian elimination, the solution of the system is

$$\mathbf{x} = \begin{bmatrix}\frac{6}{7}\\-\frac{23}{7}\end{bmatrix}.$$

10. Letting $A = \begin{bmatrix}2 & 3 & 1\\2 & -3 & -3\\4 & -2 & 3\end{bmatrix}$, $\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix}10\\22\\-2\end{bmatrix}$, the system can be written as

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix}2 & 3 & 1\\2 & -3 & -3\\4 & -2 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}10\\22\\-2\end{bmatrix}.$$

Using Gaussian elimination, the solution of the system is

$$\mathbf{x} = \begin{bmatrix}5\\2\\-6\end{bmatrix}.$$

12. $A^T = \begin{bmatrix}3 & 2\\-1 & 0\end{bmatrix}$

$$A^TA = \begin{bmatrix}3 & 2\\-1 & 0\end{bmatrix}\begin{bmatrix}3 & -1\\2 & 0\end{bmatrix} = \begin{bmatrix}13 & -3\\-3 & 1\end{bmatrix}$$

$$AA^T = \begin{bmatrix}3 & -1\\2 & 0\end{bmatrix}\begin{bmatrix}3 & 2\\-1 & 0\end{bmatrix} = \begin{bmatrix}10 & 6\\6 & 4\end{bmatrix}$$

14. $A^T = \begin{bmatrix}1\\-2\\-3\end{bmatrix}$

$$A^TA = \begin{bmatrix}1\\-2\\-3\end{bmatrix}\begin{bmatrix}1 & -2 & -3\end{bmatrix} = \begin{bmatrix}1 & -2 & -3\\-2 & 4 & 6\\-3 & 6 & 9\end{bmatrix}$$

$$AA^T = \begin{bmatrix}1 & -2 & -3\end{bmatrix}\begin{bmatrix}1\\-2\\-3\end{bmatrix} = \begin{bmatrix}14\end{bmatrix}$$

16. From the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix},$$

you see that $ad - bc = 4(2) - (-1)(-8) = 0$, and so the matrix has no inverse.

18. Begin by adjoining the identity matrix to the given matrix.

$$\begin{bmatrix}A & I\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1 & 0 & 0\\0 & 1 & 1 & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{bmatrix}$$

This matrix reduces to

$$\begin{bmatrix}I & A^{-1}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1 & -1 & 0\\0 & 1 & 0 & 0 & 1 & -1\\0 & 0 & 1 & 0 & 0 & 1\end{bmatrix}.$$

So, the inverse matrix is

$$A^{-1} = \begin{bmatrix}1 & -1 & 0\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}.$$

20.

$$\begin{array}{ccc} A & \mathbf{x} & \mathbf{b} \\ \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = & \begin{bmatrix} 1 \\ -3 \end{bmatrix} \end{array}$$

Because $A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{bmatrix}$, solve the equation $A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

22.

$$\begin{array}{ccc} A & \mathbf{x} & \mathbf{b} \\ \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = & \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \end{array}$$

Using Gauss-Jordan elimination, you find that

$$A^{-1} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{3}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}.$$

Solve the equation $A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{3}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

24.

$$\begin{array}{ccc} A & \mathbf{x} & \mathbf{b} \\ \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} = & \begin{bmatrix} 5 \\ -2 \end{bmatrix} \end{array}$$

Because $A^{-1} = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ -\frac{3}{11} & \frac{2}{11} \end{bmatrix}$, solve the equation

$A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ -\frac{3}{11} & \frac{2}{11} \end{bmatrix}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{18}{11} \\ -\frac{19}{11} \end{bmatrix}$$

26.

$$\begin{array}{ccc} A & \mathbf{x} & \mathbf{b} \\ \begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & -3 & -4 \end{bmatrix} & \begin{bmatrix} x \\ y \\ z \end{bmatrix} = & \begin{bmatrix} 0 \\ -1 \\ -7 \end{bmatrix} \end{array}$$

Using Gauss-Jordan elimination, you find that

$$A^{-1} = \begin{bmatrix} \frac{5}{18} & \frac{1}{9} & \frac{1}{6} \\ -\frac{8}{9} & \frac{4}{9} & -\frac{1}{3} \\ \frac{17}{18} & -\frac{2}{9} & \frac{1}{6} \end{bmatrix}$$

Solve the equation $A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{5}{18} & \frac{1}{9} & \frac{1}{6} \\ -\frac{8}{9} & \frac{4}{9} & -\frac{1}{3} \\ \frac{17}{18} & -\frac{2}{9} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ -7 \end{bmatrix} = \begin{bmatrix} -\frac{23}{18} \\ \frac{17}{9} \\ -\frac{17}{18} \end{bmatrix}$$

28. Because $(2A)^{-1} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$, you can use the formula for the inverse of a 2×2 matrix to obtain

$$2A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{2-0}\begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}.$$

So, $A = \frac{1}{4}\begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -1 \\ 0 & \frac{1}{2} \end{bmatrix}.$

30. The matrix $\begin{bmatrix} 2 & x \\ 1 & 4 \end{bmatrix}$ will be nonsingular if

$ad - bc = (2)(4) - (1)(x) \neq 0$, which implies that $x \neq 8$.

32. Because the given matrix represents 6 times the second row, the inverse will be $\frac{1}{6}$ times the second row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For Exercises 34 and 36, answers will vary. Sample answers are shown below.

34. Begin by finding a sequence of elementary row operations to write A in reduced row-echelon form.

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & -4 \\ -3 & 13 \end{bmatrix}$	Interchange the rows.	$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$	Add 3 times row 1 to row 2.	$E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	Add 4 times row 2 to row 1.	$E_3 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

Then, you can factor A as follows.

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$$

36. Begin by finding a sequence of elementary row operations to write A in reduced row-echelon form.

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$	Multiply row one by $\frac{1}{3}$.	$E_1 = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Add -1 times row one to row three.	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Add -2 times row three to row one.	$E_3 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Multiply row two by $\frac{1}{2}$.	$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So, you can factor A as follows.

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

38. Letting $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, you have

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & cb + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So, many answers are possible.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, etc.

40. There are many possible answers.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

But, $BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0.$

42. Because $(A^{-1} + B^{-1})(A^{-1} + B^{-1}) = I$, if $(A^{-1} + B^{-1})^{-1}$ exists, it is sufficient to show that $(A^{-1} + B^{-1})\left(A(A + B)^{-1}B\right) = I$ for equality of the second factors in each equation.

$$\begin{aligned}
(A^{-1} + B^{-1})\left(A(A + B)^{-1}B\right) &= A^{-1}\left(A(A + B)^{-1}B\right) + B^{-1}\left(A(A + B)^{-1}B\right) \\
&= A^{-1}A(A + B)^{-1}B + B^{-1}A(A + B)^{-1}B \\
&= I(A + B)^{-1}B + B^{-1}A(A + B)^{-1}B \\
&= (I + B^{-1}A)\left((A + B)^{-1}B\right) \\
&= (B^{-1}B + B^{-1}A)\left((A + B)^{-1}B\right) \\
&= B^{-1}(B + A)(A + B)^{-1}B \\
&= B^{-1}(A + B)(A + B)^{-1}B \\
&= B^{-1}IB \\
&= B^{-1}B \\
&= I
\end{aligned}$$

Therefore, $(A^{-1} + B^{-1})^{-1} = A(A + B)^{-1}B.$

44. Answers will vary. Sample answer:

Matrix	Elementary Matrix
$\begin{bmatrix} -3 & 1 \\ 12 & 0 \end{bmatrix} = A$	
$\begin{bmatrix} -3 & 1 \\ 0 & 4 \end{bmatrix} = U$	$E = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

$EA = U$

$$A = E^{-1}U = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 0 & 4 \end{bmatrix} = LU$$

46.

Matrix	Elementary Matrix
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = A$	
$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$	$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$	$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$E_3E_2E_1A = U \Rightarrow A = E_1^{-1}E_2^{-1}E_3^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

48. Matrix — Elementary Matrix

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 2 & 1 & 1 & -2 \end{bmatrix} = A$$

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = U \qquad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$EA = U \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

$$L\mathbf{y} = \mathbf{b}: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 2 \\ 8 \end{bmatrix} \Rightarrow \mathbf{y} = \begin{bmatrix} 7 \\ -3 \\ 2 \\ 1 \end{bmatrix}$$

$$U\mathbf{x} = \mathbf{y}: \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} 4 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

50. $$1.1\begin{bmatrix} 100 & 90 & 70 & 30 \\ 40 & 20 & 60 & 60 \end{bmatrix} = \begin{bmatrix} 110 & 99 & 77 & 33 \\ 44 & 22 & 66 & 66 \end{bmatrix}$$

52. (a) In matrix B, grading system 1 counts each midterm as 25% of the grade and the final exam as 50% of the grade.
Grading system 2 counts each midterm as 20% of the grade and the final exam as 60% of the grade.

(b) $$AB = \begin{bmatrix} 78 & 82 & 80 \\ 84 & 88 & 85 \\ 92 & 93 & 90 \\ 88 & 86 & 90 \\ 74 & 78 & 80 \\ 96 & 95 & 98 \end{bmatrix}\begin{bmatrix} 0.25 & 0.20 \\ 0.25 & 0.20 \\ 0.50 & 0.60 \end{bmatrix} = \begin{bmatrix} 80 & 80 \\ 85.5 & 85.4 \\ 91.25 & 91 \\ 88.5 & 88.8 \\ 78 & 78.4 \\ 96.75 & 97 \end{bmatrix}$$

(c) Two students received an "A" in each grading system.

54. $$f(A) = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^3 - 3\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix} - \begin{bmatrix} 6 & 3 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

56. The matrix is not stochastic because the sum of entries in columns 1 and 2 do not add up to 1.

58. This matrix is stochastic because each entry is between 0 and 1, and each column adds up to 1.

60. $$X_1 = PX_0 = \begin{bmatrix} 0.307 \\ 0.693 \end{bmatrix}$$
$$X_2 = PX_1 = \begin{bmatrix} 0.38246 \\ 0.61754 \end{bmatrix}$$
$$X_3 = PX_2 = \begin{bmatrix} 0.3659 \\ 0.6341 \end{bmatrix}$$

62. $$X_1 = PX_0 = \begin{bmatrix} \frac{4}{9} \\ \frac{5}{27} \\ \frac{10}{27} \end{bmatrix} \approx \begin{bmatrix} 0.\overline{4} \\ 0.\overline{185} \\ 0.\overline{370} \end{bmatrix}$$
$$X_2 = PX_1 = \begin{bmatrix} \frac{37}{81} \\ \frac{22}{81} \\ \frac{22}{81} \end{bmatrix} \approx \begin{bmatrix} 0.4568 \\ 0.2716 \\ 0.2716 \end{bmatrix}$$
$$X_3 = PX_2 = \begin{bmatrix} \frac{103}{243} \\ \frac{59}{243} \\ \frac{1}{3} \end{bmatrix} \approx \begin{bmatrix} 0.4239 \\ 0.2428 \\ 0.\overline{3} \end{bmatrix}$$

64. Begin by forming the matrix of transition probabilities.

$$\begin{array}{ccc} & \overbrace{\begin{array}{ccc}\;1\; & \;\;2\;\; & \;\;3\;\end{array}}^{\text{From Region}} & \\ P = & \begin{bmatrix}0.85 & 0.15 & 0.10\\ 0.10 & 0.80 & 0.10\\ 0.05 & 0.05 & 0.80\end{bmatrix} & \left.\begin{array}{c}1\\2\\3\end{array}\right\}\text{To Region}\end{array}$$

(a) The population in each region after 1 year is given by

$$PX = \begin{bmatrix}0.85 & 0.15 & 0.10\\ 0.10 & 0.80 & 0.10\\ 0.05 & 0.05 & 0.80\end{bmatrix}\begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{bmatrix} = \begin{bmatrix}0.3\overline{6}\\ 0.\overline{3}\\ 0.3\end{bmatrix}.$$

$$\text{So, } 300{,}000\begin{bmatrix}0.3\overline{6}\\ 0.\overline{3}\\ 0.3\end{bmatrix} = \begin{bmatrix}110{,}000\\ 100{,}000\\ 90{,}000\end{bmatrix}\begin{array}{l}\text{Region 1}\\ \text{Region 2.}\\ \text{Region 3}\end{array}$$

(b) The population in each region after 3 years is given by

$$P^3X = \begin{bmatrix}0.665375 & 0.322375 & 0.2435\\ 0.219 & 0.562 & 0.219\\ 0.115625 & 0.115625 & 0.5375\end{bmatrix}\begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{bmatrix} = \begin{bmatrix}0.4104\\ 0.\overline{3}\\ 0.25625\end{bmatrix}.$$

$$\text{So, } 300{,}000\begin{bmatrix}0.4104\\ 0.\overline{3}\\ 0.25625\end{bmatrix} = \begin{bmatrix}123{,}125\\ 100{,}000\\ 76{,}875\end{bmatrix}\begin{array}{l}\text{Region 1}\\ \text{Region 2.}\\ \text{Region 3}\end{array}$$

66. The stochastic matrix

$$P = \begin{bmatrix}1 & \frac{4}{7}\\ 0 & \frac{3}{7}\end{bmatrix}$$

is not regular because P^n has a zero in the first column for all powers.

To find $\overline{X}$, begin by letting $\overline{X} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$. Then use the matrix equation $P\overline{X} = \overline{X}$ to obtain

$$\begin{bmatrix}1 & \frac{4}{7}\\ 0 & \frac{3}{7}\end{bmatrix}\begin{bmatrix}x_1\\ x_1\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

Use these matrices and the fact that $x_1 + x_2 = 1$ to write the system of linear equations shown.

$$\begin{aligned}\tfrac{4}{7}x_2 &= 0\\ -\tfrac{4}{7}x_2 &= 0\\ x_1 + x_2 &= 1\end{aligned}$$

The solution of the system is $x_1 = 1$ and $x_2 = 0$

So, the steady state matrix is $\overline{X} = \begin{bmatrix}1\\ 0\end{bmatrix}$.

68. The stochastic matrix

$$P = \begin{bmatrix}0 & 0 & 0.2\\ 0.5 & 0.9 & 0\\ 0.5 & 0.1 & 0.8\end{bmatrix}$$

is regular because P^2 has only positive entries.

To find $\overline{X}$, let $\overline{X} = [x_1\ x_2\ x_3]^T$. Then use the matrix equation $P\overline{X} = \overline{X}$ to obtain.

$$\begin{bmatrix}0 & 0 & 0.2\\ 0.5 & 0.9 & 0\\ 0.5 & 0.1 & 0.8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}.$$

Use these matrices and the fact that $x_1 + x_2 + x_3 = 1$ to write the system of linear equations shown.

$$\begin{array}{rcrcrcl}-x_1 & & & + & 0.2x_3 & = & 0\\ 0.5x_1 & - & 0.1x_2 & & & = & 0\\ 0.5x_1 & + & 0.1x_2 & - & 0.2x_3 & = & 0\\ x_1 & + & x_2 & + & x_3 & = & 1\end{array}$$

The solution of the system is $x_1 = \frac{1}{11}$, $x_2 = \frac{5}{11}$, and $x_3 = \frac{5}{11}$.

So, the steady state matrix is $\overline{X} = \begin{bmatrix}\frac{1}{11}\\ \frac{5}{11}\\ \frac{5}{11}\end{bmatrix}$.

70. Form the matrix representing the given probabilities. Let C represent the classified documents, D represent the declassified documents, and S represent the shredded documents.

$$P = \begin{array}{c} \overbrace{\begin{array}{ccc} C & D & S \end{array}}^{\text{From}} \\ \begin{bmatrix} 0.70 & 0.20 & 0 \\ 0.10 & 0.75 & 0 \\ 0.20 & 0.05 & 1 \end{bmatrix} \end{array} \left.\begin{array}{c} C \\ D \\ S \end{array}\right\} \text{To}$$

Solve the equation $P\overline{X} = \overline{X}$, where $\overline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and use the fact that $x_1 + x_2 + x_3 = 1$ to write a system of equations.

$$\begin{aligned} 0.70x_1 + 0.20x_2 &= x_1 \\ 0.10x_1 + 0.75x_2 &= x_2 \\ 0.20x_1 + 0.05x_2 + x_3 &= x_3 \\ x_1 + x_2 + x_3 &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -0.3x_1 + 0.2x_2 &= 0 \\ 0.1x_1 - 0.25x_2 &= 0 \\ 0.2x_1 + 0.05x_2 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

So, the steady state matrix is $\overline{X} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

This indicates that eventually all of the documents will be shredded.

72. The matrix

$$P = \begin{bmatrix} 1 & 0 & 0.38 \\ 0 & 0.30 & 0 \\ 0 & 0.70 & 0.62 \end{bmatrix}$$

is absorbing. The first state S_1 is absorbing and it is possible to move from S_2 to S_1 in two transitions and to move from S_3 to S_1 in one transition.

74. (a) False. See Exercise 65, page 61.

(b) False. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Then $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$A + B$ is a *singular* matrix, while both A and B are *nonsingular* matrices.

76. (a) True. See Section 2.5, Example 4(b).

(b) False. See Section 2.5, Example 7(a).

78. The uncoded row matrices are

B E A	M _ M	E _ U	P _ S	C O T	T Y _
$[2 \;\; 5 \;\; 1]$	$[13 \;\; 0 \;\; 13]$	$[5 \;\; 0 \;\; 21]$	$[16 \;\; 0 \;\; 19]$	$[3 \;\; 15 \;\; 20]$	$[20 \;\; 25 \;\; 0]$

Multiplying each 1×3 matrix on the right by A yields the coded row matrices.

$[17 \;\; 6 \;\; 20]$ $[0 \;\; 0 \;\; 13]$ $[-32 \;\; -16 \;\; -43]$ $[-6 \;\; -3 \;\; 7]$ $[11 \;\; -2 \;\; -3]$ $[115 \;\; 45 \;\; 155]$

So, the coded message is

17, 6, 20, 0, 0, 13, −32, −16, −43, −6, −3, 7, 11, −2, −3, 115, 45, 155.

80. Find A^{-1} to be

$$A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

and the coded row matrices are

$[11 \quad 52], [-8 \quad -9], [-13 \quad -39], [5 \quad 20], [12 \quad 56], [5 \quad 20], [-2 \quad 7], [9 \quad 41], [25 \quad 100]$.

Multiplying each coded row matrix on the right by A^{-1} yields the uncoded row matrices.

S H	O W	_ M	E _	T H	E _	M O	N E	Y _
[19 8]	[15 23]	[0 13]	[5 0]	[20 8]	[5 0]	[13 15]	[14 5]	[25 0]

So, the message is SHOW_ME_THE_MONEY_.

82. Find A^{-1} to be

$$A^{-1} = \begin{bmatrix} \frac{4}{13} & \frac{2}{13} & \frac{1}{13} \\ \frac{8}{13} & -\frac{9}{13} & \frac{2}{13} \\ \frac{5}{13} & -\frac{4}{13} & -\frac{2}{13} \end{bmatrix},$$

and multiply each coded row matrix on the right by A^{-1} to find the associated uncoded row matrix.

$$[66 \quad 27 \quad -31]A^{-1} = [66 \quad 27 \quad -31]\begin{bmatrix} \frac{4}{13} & \frac{2}{13} & \frac{1}{13} \\ \frac{8}{13} & -\frac{9}{13} & \frac{2}{13} \\ \frac{5}{13} & -\frac{4}{13} & -\frac{2}{13} \end{bmatrix} = [25 \quad 1 \quad 14] \Rightarrow \text{Y, A, N}$$

$$\begin{aligned}
[37 \quad 5 \quad -9]A^{-1} &= [11 \quad 5 \quad 5] \Rightarrow \text{K, E, E} \\
[61 \quad 46 \quad -73]A^{-1} &= [19 \quad 0 \quad 23] \Rightarrow \text{S, _, W} \\
[46 \quad -14 \quad 9]A^{-1} &= [9 \quad 14 \quad 0] \Rightarrow \text{I, N, _} \\
[94 \quad 21 \quad -49]A^{-1} &= [23 \quad 15 \quad 18] \Rightarrow \text{W, O, R} \\
[32 \quad -4 \quad 12]A^{-1} &= [12 \quad 4 \quad 0] \Rightarrow \text{L, D, _} \\
[66 \quad 31 \quad -53]A^{-1} &= [19 \quad 5 \quad 18] \Rightarrow \text{S, E, R} \\
[47 \quad 33 \quad -67]A^{-1} &= [9 \quad 5 \quad 19] \Rightarrow \text{I, E, S}
\end{aligned}$$

$$[32 \quad 19 \quad -56]A^{-1} = [0 \quad 9 \quad 14] \Rightarrow \text{\_, I, N}$$

$$[43 \quad -9 \quad -20]A^{-1} = [0 \quad 19 \quad 5] \Rightarrow \text{\_, S, E}$$

$$[68 \quad 23 \quad -34]A^{-1} = [22 \quad 5 \quad 14] \Rightarrow \text{V, E, N}$$

The message is YANKEES_WIN_WORLD_SERIES_IN_SEVEN.

84. Solve the equation $X = DX + E$ for X to obtain $(I - D)X = E$, which corresponds to solving the augmented matrix.

$$\begin{bmatrix} 0.9 & -0.3 & -0.2 & 3000 \\ 0 & 0.8 & -0.3 & 3500 \\ -0.4 & -0.1 & 0.9 & 8500 \end{bmatrix}$$

The solution to this system is

$$X = \begin{bmatrix} 10{,}000 \\ 10{,}000 \\ 15{,}000 \end{bmatrix}.$$

86. Using the matrices

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 4 \end{bmatrix},$$

you have

$$X^T X = \begin{bmatrix} 5 & 20 \\ 20 & 90 \end{bmatrix}, \; X^T Y = \begin{bmatrix} 14 \\ 63 \end{bmatrix}, \text{ and}$$

$$A = \left(X^T X\right)^{-1} X^T Y = \begin{bmatrix} 1.8 & -0.4 \\ -0.4 & 0.1 \end{bmatrix}\begin{bmatrix} 14 \\ 63 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7 \end{bmatrix}.$$

So, the least squares regression line is $y = 0.7x$.

88. Using the matrices

$$X = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 4 \\ 2 \\ 1 \\ -2 \\ -3 \end{bmatrix}, \text{ you have}$$

$$X^T X = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix},\ X^T Y = \begin{bmatrix} 2 \\ -18 \end{bmatrix}, \text{ and } A = \left(X^T X\right)^{-1} X^T Y = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{10} \end{bmatrix}\begin{bmatrix} 2 \\ -18 \end{bmatrix}\begin{bmatrix} 0.4 \\ -1.8 \end{bmatrix}.$$

So, the least squares regression line is $y = -1.8x + 0.4$, or $y = -\frac{9}{5}x + \frac{2}{5}$.

90. (a) Using the matrices $X = \begin{bmatrix} 1 & 8 \\ 1 & 9 \\ 1 & 10 \\ 1 & 11 \\ 1 & 12 \\ 1 & 13 \end{bmatrix}$ and $Y = \begin{bmatrix} 2.93 \\ 3.00 \\ 3.01 \\ 3.10 \\ 3.21 \\ 3.39 \end{bmatrix}$, you have

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 8 & 9 & 10 & 11 & 12 & 13 \end{bmatrix}\begin{bmatrix} 1 & 8 \\ 1 & 9 \\ 1 & 10 \\ 1 & 11 \\ 1 & 12 \\ 1 & 13 \end{bmatrix} = \begin{bmatrix} 6 & 63 \\ 63 & 679 \end{bmatrix}$$

and

$$X^T Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 8 & 9 & 10 & 11 & 12 & 13 \end{bmatrix}\begin{bmatrix} 2.93 \\ 3.00 \\ 3.01 \\ 3.10 \\ 3.21 \\ 3.39 \end{bmatrix} = \begin{bmatrix} 18.64 \\ 197.23 \end{bmatrix}.$$

Now, using $\left(X^T X\right)^{-1}$ to find the coefficient matrix A, you have

$$A = \left(X^T X\right)^{-1} X^T Y = \begin{bmatrix} \frac{97}{15} & \frac{-3}{5} \\ \frac{-3}{5} & \frac{2}{35} \end{bmatrix}\begin{bmatrix} 18.64 \\ 197.23 \end{bmatrix} \approx \begin{bmatrix} 2.2007 \\ 0.0863 \end{bmatrix}.$$

So, the least squares regression line is $y = 0.0863x + 2.2007$.

(b) Using a graphing utility, the regression line is $y = 0.0863x + 2.2007$.

(c)

Year	2008	2009	2009	2010	2011	2012	2013
Actual	2.93	2.93	3.00	3.01	3.10	3.21	3.39
Estimated	2.89	2.89	2.98	3.06	3.15	3.24	3.32

The estimated values are close to the actual values.

Project Solutions for Chapter 2

1 Exploring Matrix Multiplication

1. Test 1 seems to be the more difficult test. The averages were:

Test 1 average $= 75$

Test 2 average $= 85.5$

2. Anna, David, Chris, Bruce

3. Answers will vary. Sample answer:

$M\begin{bmatrix}1\\0\end{bmatrix}$ represents scores on the first test.

$M\begin{bmatrix}0\\1\end{bmatrix}$ represents scores on the second test.

4. Answers will vary. Sample answer:

$\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}M$ represents Anna's scores.

$\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}M$ represents Chris's scores.

5. Answers will vary. Sample answer:

$M\begin{bmatrix}1\\1\end{bmatrix}$ represents the sum of the test scores for each student, and $\frac{1}{2}M\begin{bmatrix}1\\1\end{bmatrix}$ represents each students' average.

6. $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}M$ represents the sum of scores on each test; $\frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}M$ represents the average on each test.

7. $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}M\begin{bmatrix}1\\1\end{bmatrix}$ represents the overall points total for all students on all tests.

8. $\frac{1}{8}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}M\begin{bmatrix}1\\1\end{bmatrix} = 80.25$

9. $M\begin{bmatrix}1.1\\1.0\end{bmatrix}$

2 Nilpotent Matrices

1. $A^2 \neq 0$ and $A^3 = 0$, so the index is 3.

2. (a) Nilpotent of index 2

(b) Not nilpotent

(c) Nilpotent of index 2

(d) Not nilpotent

(e) Nilpotent of index 2

(f) Nilpotent of index 3

3. $\begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ index 2; $\begin{bmatrix}0 & 1 & 1\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$ index 3

4. $\begin{bmatrix}0 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$ index 2; $\begin{bmatrix}0 & 0 & 1 & 1\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$ index 3;

$\begin{bmatrix}0 & 1 & 1 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$ index 4

5. $\begin{bmatrix}0 & 1 & 1 & 1 & 1\\0 & 0 & 1 & 1 & 1\\0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$

6. No. If A is nilpotent and invertible, then $A^k = O$ for some k and $A^{k-1} \neq O$. So,

$$A^{-1}A = I \Rightarrow O = A^{-1}A^k = \left(A^{-1}A\right)A^{k-1} = IA^{k-1} \neq O,$$

which is impossible.

7. If A is nilpotent, then $\left(A^k\right)^T = \left(A^T\right)^k = O$. But $\left(A^T\right)^{k-1} = \left(A^{k-1}\right)^T \neq O$, which shows that A^T is nilpotent with the same index.

8. Let A be nilpotent of index k. Then

$$(I - A)\left(A^{k-1} + A^{k-2} + \cdots + A^2 + A + I\right) = I - A^k = I,$$

which shows that

$$\left(A^{k-1} + A^{k-2} + \cdots + A^2 + A + I\right)$$

is the inverse of $I - A$.