Chapter 2

Solving Linear Systems

Section 2.1, p. 94

2. (a) Possible answer: $\begin{array}{l} -1\mathbf{r}_1 \to \mathbf{r}_1 \\ 3\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2 \\ -4\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3 \\ 2\mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3 \end{array}$ $\begin{bmatrix} 1 & -1 & 1 & 0 & -3 \\ 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   (b) Possible answer: $\begin{array}{l} 2\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2 \\ -4\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3 \\ \mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3 \\ \frac{1}{6}\mathbf{r}_3 \to \mathbf{r}_3 \end{array}$ $\begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

4. (a) $\begin{array}{l} 3\mathbf{r}_3 + \mathbf{r}_1 \to \mathbf{r}_1 \\ -\mathbf{r}_3 + \mathbf{r}_2 \to \mathbf{r}_2 \end{array}$ $\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $-3\mathbf{r}_2 + \mathbf{r}_1 \to \mathbf{r}_1$ $\begin{bmatrix} 1 & 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$

6. (a) $\begin{array}{l} -\mathbf{r}_1 \to \mathbf{r}_1 \\ -2\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2 \\ -2\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3 \\ \frac{1}{2}\mathbf{r}_2 \to \mathbf{r}_2 \\ -3\mathbf{r}_3 \to \mathbf{r}_3 \\ \frac{4}{3}\mathbf{r}_3 + \mathbf{r}_2 \to \mathbf{r}_2 \\ -5\mathbf{r}_3 + \mathbf{r}_1 \to \mathbf{r}_1 \\ 2\mathbf{r}_2 + \mathbf{r}_1 \to \mathbf{r}_1 \end{array}$ I_3 (b) $\begin{array}{l} -3\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}_2 \\ -5\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}_3 \\ 2\mathbf{r}_1 + \mathbf{r}_4 \to \mathbf{r}_4 \\ -\mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}_3 \\ -\mathbf{r}_2 + \mathbf{r}_1 \to \mathbf{r}_1 \end{array}$ $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8. (a) REF (b) RREF (c) N

9. Consider the columns of A which contain leading entries of nonzero rows of A. If this set of columns is the entire set of n columns, then $A = I_n$. Otherwise there are fewer than n leading entries, and hence fewer than n nonzero rows of A.

10. (a) A is row equivalent to itself: the sequence of operations is the empty sequence.

    (b) Each elementary row operation of types I, II or III has a corresponding inverse operation of the same type which "undoes" the effect of the original operation. For example, the inverse of the operation "add d times row r of A to row s of A" is "subtract d times row r of A from row s of A." Since B is assumed row equivalent to A, there is a sequence of elementary row operations which gets from A to B. Take those operations in the reverse order, and for each operation do its inverse, and that takes B to A. Thus A is row equivalent to B.

    (c) Follow the operations which take A to B with those which take B to C.

12. (a) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & \frac{5}{3} & 1 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

Section 2.2, p. 113

2. (a) $x = -6 - s - t$, $y = s$, $z = t$, $w = 5$.
   (b) $x = -3$, $y = -2$, $z = 1$.
4. (a) $x = 5 + 2t$, $y = 2 - t$, $z = t$.
   (b) $x = 1$, $y = 2$, $z = 4 + t$, $w = t$.
6. (a) $x = -2 + r$, $y = -1$, $z = 8 - 2r$, $x_4 = r$, where r is any real number.
   (b) $x = 1$, $y = \frac{2}{3}$, $z = -\frac{2}{3}$.
   (c) No solution.
8. (a) $x = 1 - r$, $y = 2$, $z = 1$, $x_4 = r$, where r is any real number.
   (b) $x = 1 - r$, $y = 2 + r$, $z = -1 + r$, $x_4 = r$, where r is any real number.

10. $\mathbf{x} = \begin{bmatrix} r \\ 0 \end{bmatrix}$, where $r \neq 0$.

12. $\mathbf{x} = \begin{bmatrix} -\frac{1}{4}r \\ \frac{1}{4}r \\ r \end{bmatrix}$, where $r \neq 0$.

14. (a) $a = -2$. (b) $a \neq \pm 2$. (c) $a = 2$.
16. (a) $a = \pm\sqrt{6}$. (b) $a \neq \pm\sqrt{6}$.

18. The augmented matrix is $\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \end{array}\right]$. If we reduce this matrix to reduced row echelon form, we see that the linear system has only the trivial solution if and only if A is row equivalent to I_2. Now show that this occurs if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$ then at least one of a or c is $\neq 0$, and it is a routine matter to show that A is row equivalent to I_2. If $ad - bc = 0$, then by case considerations we find that A is row equivalent to a matrix that has a row or column consisting entirely of zeros, so that A is not row equivalent to I_2.

    Alternate proof: If $ad - bc \neq 0$, then A is nonsingular, so the only solution is the trivial one. If $ad - bc = 0$, then $ad = bc$. If $ad = 0$ then either a or $d = 0$, say $a = 0$. Then $bc = 0$, and either b or $c = 0$. In any of these cases we get a nontrivial solution. If $ad \neq 0$, then $\frac{a}{c} = \frac{b}{d}$, and the second equation is a multiple of the first one so we again have a nontrivial solution.

19. This had to be shown in the first proof of Exercise 18 above. If the alternate proof of Exercise 18 was given, then Exercise 19 follows from the former by noting that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if A is row equivalent to I_2 and this occurs if and only if $ad - bc \neq 0$.

20. $\begin{bmatrix} \frac{3}{2} \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} t$, where t is any number.

22. $-a + b + c = 0$.
24. (a) Change "row" to "column."
    (b) Proceed as in the proof of Theorem 2.1, changing "row" to "column."

25. Using Exercise 24(b) we can assume that every $m \times n$ matrix A is column equivalent to a matrix in column echelon form. That is, A is column equivalent to a matrix B that satisfies the following:

    (a) All columns consisting entirely of zeros, if any, are at the right side of the matrix.

    (b) The first nonzero entry in each column that is not all zeros is a 1, called the leading entry of the column.

    (c) If the columns j and $j+1$ are two successive columns that are not all zeros, then the leading entry of column $j+1$ is below the leading entry of column j.

    We start with matrix B and show that it is possible to find a matrix C that is column equivalent to B that satisfies

    (d) If a row contains a leading entry of some column then all other entries in that row are zero.

    If column j of B contains a nonzero element, then its first (counting top to bottom) nonzero element is a 1. Suppose the 1 appears in row r_j. We can perform column operations of the form $ac_j + c_k$ for each of the nonzero columns c_k of B such that the resulting matrix has row r_j with a 1 in the (r_j, j) entry and zeros everywhere else. This can be done for each column that contains a nonzero entry hence we can produce a matrix C satisfying (d). It follows that C is the unique matrix in reduced column echelon form and column equivalent to the original matrix A.

26. $-3a - b + c = 0$.

28. Apply Exercise 18 to the linear system given here. The coefficient matrix is

$$\begin{bmatrix} a-r & d \\ c & b-r \end{bmatrix}.$$

    Hence from Exercise 18, we have a nontrivial solution if and only if $(a-r)(b-r) - cd = 0$.

29. (a) $A(\mathbf{x}_p + \mathbf{x}_h) = A\mathbf{x}_p + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

    (b) Let $\mathbf{x}_p$ be a particular solution to $A\mathbf{x} = \mathbf{b}$ and let $\mathbf{x}$ be any solution to $A\mathbf{x} = \mathbf{b}$. Let $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p$. Then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h = \mathbf{x}_p + (\mathbf{x} - \mathbf{x}_p)$ and $A\mathbf{x}_h = A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}$. Thus $\mathbf{x}_h$ is in fact a solution to $A\mathbf{x} = \mathbf{0}$.

30. (a) $3x^2 + 2$ (b) $2x^2 - x - 1$

32. $\frac{3}{2}x^2 - x + \frac{1}{2}$.

34. (a) $x = 0$, $y = 0$ (b) $x = 5$, $y = -7$

36. $r = 5$, $r_2 = 5$.

37. The GPS receiver is located at the tangent point where the two circles intersect.

38. $4\mathrm{Fe} + 3\mathrm{O}_2 \rightarrow 2\mathrm{Fe}_2\mathrm{O}_3$

40. $\mathbf{x} = \begin{bmatrix} 0 \\ \frac{1}{4} - \frac{1}{4}i \end{bmatrix}$.

42. No solution.

Section 2.3, p. 124

1. The elementary matrix E which results from I_n by a type I interchange of the ith and jth row differs from I_n by having 1's in the (i,j) and (j,i) positions and 0's in the (i,i) and (j,j) positions. For that E, EA has as its ith row the jth row of A and for its jth row the ith row of A.

   The elementary matrix E which results from I_n by a type II operation differs from I_n by having $c \neq 0$ in the (i,i) position. Then EA has as its ith row c times the ith row of A.

   The elementary matrix E which results from I_n by a type III operation differs from I_n by having c in the (j,i) position. Then EA has as jth row the sum of the jth row of A and c times the ith row of A.

2. (a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

4. (a) Add 2 times row 1 to row 3: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C$

   (b) Add 2 times row 1 to row 3: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = B$

   (c) $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   Therefore B is the inverse of A.

6. If E_1 is an elementary matrix of type I then $E_1^{-1} = E_1$. Let E_2 be obtained from I_n by multiplying the ith row of I_n by $c \neq 0$. Let E_2^* be obtained from I_n by multiplying the ith row of I_n by $\frac{1}{c}$. Then $E_2E_2^* = I_n$. Let E_3 be obtained from I_n by adding c times the ith row of I_n to the jth row of I_n. Let E_3^* be obtained from I_n by adding $-c$ times the ith row of I_n to the jth row of I_n. Then $E_3E_3^* = I_n$.

8. $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ -1 & 0 & 1 \end{bmatrix}$.

10. (a) Singular. (b) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ -\frac{3}{2} & \frac{5}{2} & -\frac{1}{2} \end{bmatrix}$. (c) $\begin{bmatrix} -1 & \frac{3}{2} & \frac{1}{2} \\ 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. (d) $\begin{bmatrix} \frac{3}{5} & -\frac{3}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{3}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$.

12. (a) $A^{-1} = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{5} & 1 & \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{1}{2} & -\frac{2}{5} & -\frac{1}{5} \end{bmatrix}$. (b) Singular.

14. A is row equivalent to I_3; a possible answer is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

16. $A = \begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -1 & 1 & 0 \end{bmatrix}.$

18. (b) and (c).

20. For $a = -1$ or $a = 3$.

21. This follows directly from Exercise 19 of Section 2.1 and Corollary 2.2. To show that

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

we proceed as follows:

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} ad - bc & db - bd \\ -ca + ac & -bc + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

22. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

23. The matrices A and B are row equivalent if and only if $B = E_k E_{k-1} \cdots E_2 E_1 A$.
Let $P = E_k E_{k-1} \cdots E_2 E_1$.

24. If A and B are row equivalent then $B = PA$, where P is nonsingular, and $A = P^{-1}B$ (Exercise 23). If A is nonsingular then B is nonsingular, and conversely.

25. Suppose B is singular. Then by Theorem 2.9 there exists $\mathbf{x} \neq \mathbf{0}$ such that $B\mathbf{x} = \mathbf{0}$. Then $(AB)\mathbf{x} = A\mathbf{0} = \mathbf{0}$, which means that the homogeneous system $(AB)\mathbf{x} = \mathbf{0}$ has a nontrivial solution. Theorem 2.9 implies that AB is singular, a contradiction. Hence, B is nonsingular. Since $A = (AB)B^{-1}$ is a product of nonsingular matrices, it follows that A is nonsingular.

    Alternate Proof: If AB is nonsingular it follows that AB is row equivalent to I_n, so $P(AB) = I_n$. Since P is nonsingular, $P = E_k E_{k-1} \cdots E_2 E_1$. Then $(PA)B = I_n$ or $(E_k E_{k-1} \cdots E_2 E_1 A)B = I_n$. Letting $E_k E_{k-1} \cdots E_2 E_1 A = C$, we have $CB = I_n$, which implies that B is nonsingular. Since $PAB = I_n$, $A = P^{-1}B^{-1}$, so A is nonsingular.

26. The matrix A is row equivalent to O if and only if $A = PO = O$ where P is nonsingular.

27. The matrix A is row equivalent to B if and only if $B = PA$, where P is a nonsingular matrix. Now $B^T = A^T P^T$, so A is row equivalent to B if and only if A^T is column equivalent to B^T.

28. If A has a row of zeros, then A cannot be row equivalent to I_n, and so by Corollary 2.2, A is singular. If the jth column of A is the zero column, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, the vector $\mathbf{x}$ with 1 in the jth entry and zeros elsewhere. By Theorem 2.9, A is singular.

29. (a) No. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $(A + B)^{-1}$ exists but A^{-1} and B^{-1} do not. Even supposing they all exist, equality need not hold. Let $A = \begin{bmatrix} 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \end{bmatrix}$ so $(A + B)^{-1} = \begin{bmatrix} \frac{1}{3} \end{bmatrix} \neq \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \end{bmatrix} = A^{-1} + B^{-1}$.

(b) Yes, for A nonsingular and $r \neq 0$.

$$(rA)\left[\frac{1}{r}A^{-1}\right] = r\left[\frac{1}{r}\right]A \cdot A^{-1} = 1 \cdot I_n = I_n.$$

30. Suppose that A is nonsingular. Then $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = A^{-1}\mathbf{b}$ for every $n \times 1$ matrix $\mathbf{b}$. Conversely, suppose that $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$. Letting $\mathbf{b}$ be the matrices

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

we see that we have solutions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ to the linear systems

$$A\mathbf{x}_1 = \mathbf{e}_1, \quad A\mathbf{x}_2 = \mathbf{e}_2, \quad \ldots, \quad A\mathbf{x}_n = \mathbf{e}_n. \tag{$*$}$$

Letting C be the matrix whose jth column is $\mathbf{x}_j$, we can write the n systems in $(*)$ as $AC = I_n$, since $I_n = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$. Hence, A is nonsingular.

31. We consider the case that A is nonsingular and upper triangular. A similar argument can be given for A lower triangular.

By Theorem 2.8, A is a product of elementary matrices which are the inverses of the elementary matrices that "reduce" A to I_n. That is,

$$A = E_1^{-1} \cdots E_k^{-1}.$$

The elementary matrix E_i will be upper triangular since it is used to introduce zeros into the upper triangular part of A in the reduction process. The inverse of E_i is an elementary matrix of the same type and also an upper triangular matrix. Since the product of upper triangular matrices is upper triangular and we have $A^{-1} = E_k \cdots E_1$ we conclude that A^{-1} is upper triangular.

Section 2.4, p. 129

1. See the answer to Exercise 4, Section 2.1. Where it mentions only row operations, now read "row and column operations".

2. (a) $\begin{bmatrix} I_4 \\ 0 \end{bmatrix}$. (b) I_3. (c) $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$. (d) I_4.

4. Allowable equivalence operations ("elementary row or elementary column operation") include in particular elementary row operations.

5. A and B are equivalent if and only if $B = E_t \cdots E_2E_1AF_1F_2 \cdots F_s$. Let $E_tE_{t-1} \cdots E_2E_1 = P$ and $F_1F_2 \cdots F_s = Q$.

6. $B = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$; a possible answer is: $B = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

8. Suppose A were nonzero but equivalent to O. Then some ultimate elementary row or column operation must have transformed a nonzero matrix A_r into the zero matrix O. By considering the types of elementary operations we see that this is impossible.

9. Replace "row" by "column" and vice versa in the elementary operations which transform A into B.

10. Possible answers are:

(a) $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & -1 & 4 & 3 \\ 0 & 2 & -5 & -2 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 5 & 5 & 4 & 4 \end{bmatrix}$.

11. If A and B are equivalent then $B = PAQ$ and $A = P^{-1}BQ^{-1}$. If A is nonsingular then B is nonsingular, and conversely.

Section 2.5, p. 136

2. $\mathbf{x} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$.

4. $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 5 \end{bmatrix}$.

6. $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} -3 & 1 & -2 \\ 0 & 6 & 2 \\ 0 & 0 & -4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$.

8. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ -2 & 3 & 2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} -5 & 4 & 0 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 5 \\ -4 \end{bmatrix}$.

10. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 1 & 0 & 0 \\ -0.4 & 0.8 & 1 & 0 \\ 2 & -1.2 & -0.4 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 4 & 1 & 0.25 & -0.5 \\ 0 & 0.4 & 1.2 & -2.5 \\ 0 & 0 & -0.85 & 2 \\ 0 & 0 & 0 & -2.5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -1.5 \\ 4.2 \\ 2.6 \\ -2 \end{bmatrix}$.

Supplementary Exercises for Chapter 2, p. 137

2. (a) $a = -4$ or $a = 2$.
   (b) The system has a solution for each value of a.

4. $c + 2a - 3b = 0$.

5. (a) Multiply the jth row of B by $\frac{1}{k}$.
   (b) Interchange the ith and jth rows of B.
   (c) Add $-k$ times the jth row of B to its ith row.

6. (a) If we transform E_1 to reduced row echelon form, we obtain I_n. Hence E_1 is row equivalent to I_n and thus is nonsingular.
   (b) If we transform E_2 to reduced row echelon form, we obtain I_n. Hence E_2 is row equivalent to I_n and thus is nonsingular.

(c) If we transform E_3 to reduced row echelon form, we obtain I_n. Hence E_3 is row equivalent to I_n and thus is nonsingular.

8. $\begin{bmatrix} 1 & -a & a^2 & -a^3 \\ 0 & 1 & -a & a^2 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

10. (a) $\begin{bmatrix} -41 \\ 47 \\ -35 \end{bmatrix}$. (b) $\begin{bmatrix} 83 \\ -45 \\ -62 \end{bmatrix}$.

12. $s \neq 0, \pm\sqrt{2}$.

13. For any angle θ, $\cos\theta$ and $\sin\theta$ are never simultaneously zero. Thus at least one element in column 1 is not zero. Assume $\cos\theta \neq 0$. (If $\cos\theta = 0$, then interchange rows 1 and 2 and proceed in a similar manner to that described below.) To show that the matrix is nonsingular and determine its inverse, we put

$$\left[\begin{array}{cc|cc} \cos\theta & \sin\theta & 1 & 0 \\ -\sin\theta & \cos\theta & 0 & 1 \end{array}\right]$$

into reduced row echelon form. Apply row operations $\frac{1}{\cos\theta}$ times row 1 and $\sin\theta$ times row 1 added to row 2 to obtain

$$\left[\begin{array}{cc|cc} 1 & \dfrac{\sin\theta}{\cos\theta} & \dfrac{1}{\cos\theta} & 0 \\ 0 & \dfrac{\sin^2\theta}{\cos\theta} + \cos\theta & \dfrac{\sin\theta}{\cos\theta} & 1 \end{array}\right].$$

Since

$$\frac{\sin^2\theta}{\cos\theta} + \cos\theta = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta},$$

the $(2,2)$-element is not zero. Applying row operations $\cos\theta$ times row 2 and $\left(-\frac{\sin\theta}{\cos\theta}\right)$ times row 2 added to row 1 we obtain

$$\left[\begin{array}{cc|cc} 1 & 0 & \cos\theta & -\sin\theta \\ 0 & 1 & \sin\theta & \cos\theta \end{array}\right].$$

It follows that the matrix is nonsingular and its inverse is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

14. (a) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.
(b) $A(\mathbf{u}-\mathbf{v}) = A\mathbf{u} - A\mathbf{v} = \mathbf{0} - \mathbf{0} = \mathbf{0}$.
(c) $A(r\mathbf{u}) = r(A\mathbf{u}) = r\mathbf{0} = \mathbf{0}$.
(d) $A(r\mathbf{u}+s\mathbf{v}) = r(A\mathbf{u}) + s(A\mathbf{v}) = r\mathbf{0} + s\mathbf{0} = \mathbf{0}$.

15. If $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{b}$, then $A(\mathbf{u}-\mathbf{v}) = A\mathbf{u} - A\mathbf{v} = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

16. Suppose at some point in the process of reducing the augmented matrix to reduced row echelon form we encounter a row whose first n entries are zero but whose $(n+1)$st entry is some number $c \neq 0$. The corresponding linear equation is

$$0 \cdot x_1 + \cdots + 0 \cdot x_n = c \quad \text{or} \quad 0 = c.$$

This equation has no solution, thus the linear system is inconsistent.

17. Let $\mathbf{u}$ be one solution to $A\mathbf{x} = \mathbf{b}$. Since A is singular, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{u}_0$. Then for any real number r, $\mathbf{v} = r\mathbf{u}_0$ is also a solution to the homogeneous system. Finally, by Exercise 29, Sec. 2.2, for each of the infinitely many vectors $\mathbf{v}$, the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$ is a solution to the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$.

18. $s = 1$, $t = 1$.

20. If any of the diagonal entries of L or U is zero, there will not be a unique solution.

21. The outer product of X and Y can be written in the form

$$XY^T = \begin{bmatrix} x_1 \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \\ x_2 \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \\ \vdots \\ x_n \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \end{bmatrix}.$$

If either $X = O$ or $Y = O$, then $XY^T = O$. Thus assume that there is at least one nonzero component in X, say x_i, and at least one nonzero component in Y, say y_j. Then $\left(\frac{1}{x_i}\right)\mathrm{Row}_i(XY^T)$ makes the ith row exactly Y^T. Since all the other rows are multiples of Y^T, row operations of the form $-x_k R_i + R_p$, for $p \neq i$, can be performed to zero out everything but the ith row. It follows that either XY^T is row equivalent to O or to a matrix with $n - 1$ zero rows.

Chapter Review for Chapter 2, p. 138

True or False

1. False.	2. True.	3. False.	4. True.	5. True.
6. True.	7. True.	8. True.	9. True.	10. False.

Quiz

1. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

2. (a) No.
   (b) Infinitely many.
   (c) No.
   (d) $\mathbf{x} = \begin{bmatrix} -6 + 2r + 7s \\ r \\ -3s \\ s \end{bmatrix}$, where r and s are any real numbers.

3. $k = 6$.

4. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

5. $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

6. $P = A^{-1}$, $Q = B$.

7. Possible answers: Diagonal, zero, or symmetric.