Chapter 2

SYSTEMS OF LINEAR EQUATIONS AND MATRICES

2.1 Solution of Linear Systems by the Echelon Method

Your Turn 1

$$\begin{aligned} 2x + 3y &= 12 \quad (1) \\ 3x - 4y &= 1 \quad (2) \end{aligned}$$

Use row transformations to eliminate x in equation (2).

$$\begin{aligned} & 2x + 3y = 12 \\ 3R_1 + (-2)R_2 \rightarrow R_2 \quad & 17y = 34 \end{aligned}$$

Now make the coefficient of the first term in each equation equal to 1.

$$\begin{aligned} \tfrac{1}{2}R_1 \rightarrow R_1 \quad & x + \frac{3}{2}y = 6 \\ \tfrac{1}{17}R_2 \rightarrow R_2 \quad & y = 2 \end{aligned}$$

Back-substitute to solve for x.

$$\begin{aligned} x + \frac{3}{2}(2) &= 6 \\ x + 3 &= 6 \\ x &= 3 \end{aligned}$$

The solution of the system is $(3, 2)$.

Your Turn 2

Let $x =$ number of shares of Kohl's stock

$y =$ number of shares of Best Buy stock

$$\begin{aligned} x + y &= 188 \quad (1) \\ 52x + 27y &= 8601 \quad (2) \end{aligned}$$

$$\begin{aligned} & x + y = 188 \\ R_2 + (-52)R_1 \rightarrow R_2 \quad & -25y = -1175 \\ & y = 7 \end{aligned}$$

$$\begin{aligned} & x + y = 188 \\ \left(-\frac{1}{25}\right) \rightarrow R_2 \quad & y = 47 \end{aligned}$$

Substitue $y = 47$ in equation (1).

$$\begin{aligned} x + 47 &= 188 \\ x &= 141 \end{aligned}$$

Olinda owns 141 shares of Kohl's and 47 shares of Best Buy.

Your Turn 3

Since there are 60 utensils with a total weight of 189.6 pounds, the equations of Example 5 become

$$\begin{aligned} x + y + z &= 60 \quad (1) \\ 3.9x + 3.6y + 3.0z &= 189.6 \quad (2) \end{aligned}$$

where $x =$ the number of knives, $y =$ the number of forks, and $z =$ the number of spoons.

$$\begin{aligned} & x + y + z = 60 \\ 3.9\,R_1 + (-1)\ R_2 \rightarrow R_2 \quad & 0.3y + 0.9z = 44.4 \end{aligned}$$

$$\begin{aligned} & x + y + z = 60 \\ \frac{1}{0.3}\ R_2 \rightarrow R_2 \quad & y + 3z = 148 \end{aligned}$$

Solve for y in terms of the parameter z: $y = 148 - 3z$

Substitute this expression into equation (1).

$$x + (148 - 3z) + z = 60$$

Then solve for x.

$$x = 2z - 88$$

Since both x and y must be nonnegative, we have:

$$3z \le 148 \quad \text{or} \quad z \le \frac{148}{3}, \quad \text{so} \quad z \le 49$$

$$2z \ge 88 \quad \text{or} \quad z \ge 44$$

The largest number of spoons is thus 49; the corresponding number of knives is $(2)(49) - 88 = 10$, and the number of forks is $148 - (3)(49) = 1$.

2.1 Exercises

In Exercises 1–16 and 19–28, check each solution by substituting it in the original equations of the system.

1. $$\begin{aligned} x + y &= 5 \quad (1) \\ 2x - 2y &= 2 \quad (2) \end{aligned}$$

To eliminate x in equation (2), multiply equation (1) by -2 and add the result to equation (2). The new system is

$$\begin{aligned} & x + y = 5 \quad (1) \\ -2R_1 + R_2 \rightarrow R_2 \quad & -4y = -8. \quad (3) \end{aligned}$$

Now make the coefficient of the first term in each row equal 1. To accomplish this, multiply equation (3) by $-\frac{1}{4}$.

$$\begin{aligned} & x + y = 5 \quad (1) \\ -\tfrac{1}{4}R_2 \to R_2 \quad & y = 2 \quad (4) \end{aligned}$$

Substitute 2 for y in equation (1).

$$\begin{aligned} x + 2 &= 5 \\ y &= 3 \end{aligned}$$

The solution is (3, 2).

2. $4x + y = 9 \quad (1)$
$3x - y = 5 \quad (2)$

First use transformation 3 to eliminate the x-term from equation (2). Multiply equation (1) by 3 and add the result to -4 times equation (2).

$$\begin{aligned} & 4x + y = 9 \quad (1) \\ 3R_1 + (-4)R_2 \to R_2 \quad & 7y = 7 \quad (3) \end{aligned}$$

Now use transformation 2 to make the coefficient of the first term in each equation equal to 1.

$$\begin{aligned} \tfrac{1}{4}R_1 \to R_1 \quad & x + \frac{1}{4}y = \frac{9}{4} \quad (4) \\ \tfrac{1}{7}R_2 \to R_2 \quad & y = 1 \quad (5) \end{aligned}$$

Complete the solution by back-substitution. Substitute 1 for y in equation (4) to get

$$\begin{aligned} x + \frac{1}{4}(1) &= \frac{9}{4} \\ x &= \frac{8}{4} = 2. \end{aligned}$$

The solution is (2, 1).

3. $3x - 2y = -3 \quad (1)$
$5x - y = 2 \quad (2)$

To eliminate x in equation (2), multiply equation (1) by -5 and equation (2) by 3. Add the results. The new system is

$$\begin{aligned} & 3x - 2y = -3 \quad (1) \\ -5R_1 + 3R_2 \to R_2 \quad & 7y = 21. \quad (3) \end{aligned}$$

Now make the coefficient of the first term in each row equal 1. To accomplish this, multiply equation (1) by $\frac{1}{3}$ and equation (3) by $\frac{1}{7}$.

$$\begin{aligned} \tfrac{1}{3}R_1 \to R_1 \quad & x - \frac{2}{3}y = -1 \quad (4) \\ \tfrac{1}{7}R_2 \to R_2 \quad & y = 3 \quad (5) \end{aligned}$$

Back-substitution of 3 for y in equation (4) gives

$$\begin{aligned} x - \frac{2}{3}(3) &= -1 \\ x - 2 &= -1 \\ x &= 1. \end{aligned}$$

The solution is (1, 3).

4. $2x + 7y = -8 \quad (1)$
$-2x + 3y = -12 \quad (2)$

$$\begin{aligned} & 2x + 7y = -8 \quad (1) \\ R_1 + R_2 \to R_2 \quad & 10y = -20 \quad (3) \end{aligned}$$

Make each leading coefficient equal 1.

$$\begin{aligned} \tfrac{1}{2}R_1 \to R_1 \quad & x + \frac{7}{2}y = -4 \quad (4) \\ \tfrac{1}{10}R_2 \to R_2 \quad & y = -2 \quad (5) \end{aligned}$$

Substitute -2 for y in equation (4).

$$\begin{aligned} x + \frac{7}{2}(-2) &= -4 \\ x - 7 &= -4 \\ x &= 3 \end{aligned}$$

The solution is $(3, -2)$.

5. $3x + 2y = -6 \quad (1)$
$5x - 2y = -10 \quad (2)$

Eliminate x in equation (2) to get the system

$$\begin{aligned} & 3x + 2y = -6 \quad (1) \\ 5R_1 + (-3)R_2 \to R_2 \quad & 16y = 0. \quad (3) \end{aligned}$$

Make the coefficient of the first term in each equation equal 1.

$$\begin{aligned} \tfrac{1}{3}R_1 \to R_1 \quad & x + \frac{2}{3}y = -2 \quad (4) \\ \tfrac{1}{16}R_2 \to R_2 \quad & y = 0 \quad (5) \end{aligned}$$

Substitute 0 for y in equation (4) to get $x = -2$. The solution is $(-2, 0)$.

6. $-3x + y = 4 \quad (1)$
$2x - 2y = -4 \quad (2)$

Eliminate x in equation (2).

$$\begin{aligned} & -3x + y = 4 \quad (1) \\ 2R_1 + 3R_2 \to R_2 \quad & -4y = -4 \quad (3) \end{aligned}$$

Make the coefficient of the first term in each row equal 1.

$$\begin{aligned} -\tfrac{1}{3}R_1 \to R_1 \quad & x - \frac{1}{3}y = -\frac{4}{3} \quad (4) \\ -\tfrac{1}{4}R_2 \to R_2 \quad & y = 1 \quad (5) \end{aligned}$$

Back-substitution of 1 for y in equation (4) gives

$$x - \frac{1}{3}(1) = -\frac{4}{3}$$
$$x - \frac{1}{3} = -\frac{4}{3}$$
$$x = -1.$$

The solution is $(-1, 1)$.

7. $6x - 2y = -4 \quad (1)$
$3x + 4y = 8 \quad (2)$

Eliminate x in equation (2).

$$\begin{aligned} & & 6x - 2y &= -4 & (1) \\ -1R_1 + 2R_2 &\to R_2 & 10y &= 20 & (3) \end{aligned}$$

Make the coefficient of the first term in each row equal 1.

$$\begin{aligned} \tfrac{1}{6}R_1 &\to R_1 & x - \frac{1}{3}y &= -\frac{2}{3} & (4) \\ \tfrac{1}{10}R_2 &\to R_2 & y &= 2 & (5) \end{aligned}$$

Substitute 2 for y in equation (4) to get $x = 0$. The solution is $(0, 2)$.

8. $4m + 3n = -1$
$2m + 5n = 3$

$$\begin{aligned} & & 4m + 3n &= -1 \\ R_1 + (-2)R_2 &\to R_2 & -7n &= -7 \end{aligned}$$

Make each leading coefficient equal 1.

$$\begin{aligned} \tfrac{1}{4}R_1 &\to R_1 & m + \frac{3}{4}n &= -\frac{1}{4} \\ -\tfrac{1}{7}R_2 &\to R_2 & n &= 1 \end{aligned}$$

Back-substitution gives

$$m + \frac{3}{4}(1) = -\frac{1}{4}$$
$$m = -\frac{4}{4} = -1.$$

The solution is $(-1, 1)$.

9. $5p + 11q = -7 \quad (1)$
$3p - 8q = 25 \quad (2)$

Eliminate p in equation (2).

$$\begin{aligned} & & 5p + 11q &= -7 & (1) \\ -3R_1 + 5R_2 &\to R_2 & -73q &= 146 & (3) \end{aligned}$$

Make the coefficient of the first term in each row equal 1.

$$\begin{aligned} \tfrac{1}{5}R_1 &\to R_1 & p + \frac{11}{5}q &= -\frac{7}{5} & (4) \\ -\tfrac{1}{73}R_2 &\to R_2 & q &= -1 & (5) \end{aligned}$$

Substitute -2 for q in equation (4) to get $p = 3$. The solution is $(3, -2)$.

10. $12s - 5t = 9$
$3s - 8t = -18$

$$\begin{aligned} & & 12s - 5t &= 9 \\ R_1 + (-4)R_2 &\to R_2 & 27t &= 81 \\ \tfrac{1}{12}R_1 &\to R_1 & s - \frac{5}{12}t &= \frac{3}{4} \\ \tfrac{1}{27}R_2 &\to R_2 & t &= 3 \end{aligned}$$

Back substitution gives

$$s - \frac{5}{12}(3) = \frac{3}{4}$$
$$s - \frac{5}{4} = \frac{3}{4}$$
$$s = \frac{8}{4} = 2.$$

The solution is $(2, 3)$.

11. $6x + 7y = -2 \quad (1)$
$7x - 6y = 26 \quad (2)$

Eliminate x in equation (2).

$$\begin{aligned} & & 6x + 7y &= -2 & (1) \\ 7R_1 + (-6)R_2 &\to R_2 & 85y &= -170 & (3) \end{aligned}$$

Make the coefficient of the first term in each equation equal 1.

$$\begin{aligned} \tfrac{1}{6}R_1 &\to R_1 & x + \frac{7}{6}y &= -\frac{1}{3} & (4) \\ \tfrac{1}{85}R_2 &\to R_2 & y &= -2 & (5) \end{aligned}$$

Substitute -2 for y in equation (4) to get $x = 2$. The solution is $(2, -2)$.

12. $3a - 8b = 14 \quad (1)$
$a - 2b = 2 \quad (2)$

Eliminate a in equation (2).

$$\begin{aligned} & & 3a - 8b &= 14 & (1) \\ R_1 + (-3)R_2 &\to R_2 & -2b &= 8 & (3) \end{aligned}$$

Make the coefficient of the first term in each row equal 1.

$$\begin{aligned} \tfrac{1}{3}R_1 &\to R_1 & a - \frac{8}{3}b &= \frac{14}{3} & (4) \\ -\tfrac{1}{2}R_2 &\to R_2 & b &= -4 & (5) \end{aligned}$$

Back substitute -4 for b in equation (4) to get $a = -6$. The solution is $(-6, -4)$.

13. $3x + 2y = 5 \quad (1)$
$6x + 4y = 8 \quad (2)$

Eliminate x in equation (2).

$$\begin{aligned} 3x + 2y &= 5 \quad (1) \\ -2R_1 + R_2 \to R_2 \qquad 0 &= -2 \quad (3) \end{aligned}$$

Equation (3) is a false statement.

The system is inconsistent and has no solution.

14. $9x - 5y = 1$

$-18x + 10y = 1$

$$\begin{aligned} 9x - 5y &= 1 \\ 2R_1 + R_2 \to R_2 \qquad 0 &= 3 \end{aligned}$$

The equation $0 = 3$ is a false statement, which indicates that the system is inconsistent and has no solution.

15. $3x - 2y = -4 \quad (1)$

$-6x + 4y = 8 \quad (2)$

Eliminate x in equation (2).

$$\begin{aligned} 3x - 2y &= -4 \quad (1) \\ 2R_1 + R_2 \to R_2 \qquad 0 &= 0 \quad (3) \end{aligned}$$

The true statement in equation (3) indicates that there are an infinite number of solutions for the system. Solve equation (1) for x.

$$\begin{aligned} 3x - 2y &= -4 \quad (1) \\ 3x &= 2y - 4 \\ x &= \frac{2y-4}{3} \quad (4) \end{aligned}$$

For each value of y, equation (4) indicates that $x = \frac{2y-4}{3}$, and all ordered pairs of the form $\left(\frac{2y-4}{3}, y\right)$ are solutions.

16. $3x + 5y + 2 = 0$

$9x + 15y + 6 = 0$

Begin by rewriting the equations in standard form,

$$\begin{aligned} 3x + 5y &= -2 \\ 9x + 15y &= -6 \\ 3x + 5y &= -2 \\ 3R_1 + (-1)R_2 \to R_2 \qquad 0 &= 0 \end{aligned}$$

The true statement, $0 = 0$, shows that the two equations have the same graph, which means that there are an infinite number of solutions for the system. All ordered pairs that satisfy the equation $3x + 5y = -2$ are solutions. Solve this equation for x.

$$\begin{aligned} 3x &= -5y - 2 \\ x &= \frac{-5y-2}{3} \end{aligned}$$

The general solution is the set of all ordered pairs of the form

$$\left(\frac{-5y-2}{3}, y\right),$$

where y is any real number.

17. $x - \frac{3y}{2} = \frac{5}{2} \quad (1)$

$\frac{4x}{3} + \frac{2y}{3} = 6 \quad (2)$

Rewrite the equations without fractions.

$$\begin{aligned} 2R_1 \to R_1 \quad 2x - 3y &= 5 \quad (3) \\ 3R_2 \to R_2 \quad 4x + 2y &= 18 \quad (4) \end{aligned}$$

Eliminate x in equation (4).

$$\begin{aligned} 2x - 3y &= 5 \quad (3) \\ -2R_1 + R_2 \to R_2 \qquad 8y &= 8 \quad (5) \end{aligned}$$

Make the coefficient of the first term in each equation equal 1.

$$\begin{aligned} \tfrac{1}{2}R_1 \to R_1 \quad x - \frac{3}{2}y &= \frac{5}{2} \quad (6) \\ \tfrac{1}{8}R_2 \to R_2 \qquad y &= 1 \quad (7) \end{aligned}$$

Substitute 1 for y in equation (6) to get $x = 4$.

The solution is (4, 1).

18. $\frac{x}{5} + 3y = 31$

$2x - \frac{y}{5} = 8$

Multiply each equation by 5 to eliminate fractions.

$$\begin{aligned} x + 15y &= 155 \\ 10x - y &= 40 \\ x + 15y &= 155 \\ 10R_1 + (-1)R_2 \to R_2 \qquad 151y &= 1510 \\ x + 15y &= 155 \\ \tfrac{1}{151}R_2 \to R_2 \qquad y &= 10 \end{aligned}$$

Back-substitution gives

$$\begin{aligned} x + 15(10) &= 155 \\ x &= 5. \end{aligned}$$

The solution is (5, 10).

19. $\frac{x}{2} + y = \frac{3}{2} \quad (1)$

$\frac{x}{3} + y = \frac{1}{3} \quad (2)$

Rewrite the equations without fractions.

$$\begin{aligned} 2R_1 \to R_1 \quad x + 2y &= 3 \quad (3) \\ 3R_2 \to R_2 \quad x + 3y &= 1 \quad (4) \end{aligned}$$

Eliminate x in equation (4).

$$\begin{aligned} x + 2y &= 3 \quad (3) \\ -1R_1 + R_2 \to R_2 \qquad y &= -2 \quad (5) \end{aligned}$$

Substitute -2 for y in equation (3) to get $x = 7$. The solution is $(7, -2)$.

20. $\frac{x}{9} + \frac{y}{6} = \frac{1}{3}$ (*1*)

$2x + \frac{8y}{5} = \frac{2}{5}$ (*2*)

Rewrite the equations without fractions.

$18R_1 \to R_1 \quad 2x + 3y = 6 \quad (3)$

$5R_2 \to R_2 \quad 10x + 8y = 2 \quad (4)$

Eliminate x in equation (4).

$2x + 3y = 6 \quad (3)$

$-5R_1 + R_2 \to R_2 \quad -7y = -28 \quad (5)$

Make the coefficient of the first term in each row equal 1.

$\frac{1}{2}R_1 \to R_1 \quad x + \frac{3}{2}y = 3 \quad (6)$

$-\frac{1}{7}R_2 \to R_2 \quad y = 4 \quad (7)$

Substitute 4 for y in equation (6) to get $x = -3$. The solution is $(-3, 4)$.

21. An inconsistent system has *no* solutions.

22. The solution of a system with two dependent equations in two variables is *an infinite set of ordered pairs.*

25. $2x + 3y - z = 1$ (*1*)

$3x + 5y + z = 3$ (*2*)

Eliminate x in equation (2).

$2x + 3y - z = 1 \quad (1)$

$-3R_1 + 2R_2 \to R_2 \quad y + 5z = 3 \quad (3)$

Since there are only two equations, it is not possible to continue with the echelon method as in the previous exercises involving systems with three equations and three variables. To complete the solution, make the coefficient of the first term in the each equation equal 1.

$\frac{1}{2}R_1 \to R_1 \quad x + \frac{3}{2}y + \frac{1}{2}z = \frac{1}{2} \quad (4)$

$y + 5z = 3 \quad (3)$

Solve equation (3) for y in terms of the parameter z.

$$y + 5z = 3$$
$$y = 3 - 5z$$

Substitute this expression for y in equation (4) to solve for x in terms of the parameter z.

$$x + \frac{3}{2}(3 - 5z) - \frac{1}{2}z = \frac{1}{2}$$
$$x + \frac{9}{2} - \frac{15}{2}z - \frac{1}{2}z = \frac{1}{2}$$
$$x - 8z = -4$$
$$x = 8z - 4$$

The solution is $(8z - 4, 3 - 5z, z)$.

26. $3x + y - z = 0$ (*1*)

$2x - y + 3z = -7$ (*2*)

Eliminate x in equation (2).

$3x + y - z = 0 \quad (1)$

$2R_1 + (-3)R_2 \to R_2 \quad 5y - 11z = 21 \quad (3)$

Make the coefficient of the first term in each equation equal 1.

$\frac{1}{3}R_1 \to R_1 \quad x + \frac{1}{3}y - \frac{1}{3}z = 0 \quad (4)$

$\frac{1}{5}R_2 \to R_2 \quad y - \frac{11}{5}z = \frac{21}{5} \quad (5)$

Solve equation (5) for y in terms of z.

$$y = \frac{11}{5}z + \frac{21}{5}$$

Substitute this expression for y in equation (4), and solve the equation for x.

$$x + \frac{1}{3}\left(\frac{11}{5}z + \frac{21}{5}\right) - \frac{1}{3}z = 0$$
$$x + \frac{11}{5}z + \frac{7}{5} - \frac{1}{3}z = 0$$
$$x + \frac{2}{5}z = -\frac{7}{5}$$
$$x = -\frac{2}{5}z - \frac{7}{5}$$

The solution is

$$\left(-\frac{2}{5}z - \frac{7}{5}, \frac{11}{5}z + \frac{21}{5}, z\right) \text{ or}$$
$$\left(\frac{-2z - 7}{5}, \frac{11z + 21}{5}, z\right).$$

27. $x + 2y + 3z = 11$ (*1*)

$2x - y + z = 2$ (*2*)

$x + 2y + 3z = 11 \quad (1)$

$-2R_1 + R_2 \to R_2 \quad -5y - 5z = -20 \quad (3)$

$x + 2y + 3z = 11 \quad (1)$

$-\frac{1}{5}R_2 \to R_2 \quad y + z = 4 \quad (4)$

Since there are only two equations, it is not possible to continue with the echelon method. To complete the solution, solve equation (4) for y in terms of the parameter z.

$$y = 4 - z$$

Now substitute $4 - z$ for y in equation (1) and solve for x in terms of z.

$$x + 2(4 - z) + 3z = 11$$
$$x + 8 - 2z + 3z = 11$$
$$x = 3 - z$$

The solution is $(3 - z,\ 4 - z,\ z)$.

28. $-x + y - z = -7 \quad (1)$
$2x + 3y + z = 7 \quad (2)$

Eliminate x in equation (2).

$$-x + y - z = -7 \quad (1)$$
$$2R_1 + R_2 \to R_2 \qquad 5y - z = -7 \quad (3)$$

Make the coefficient of the first term in each equation equal 1.

$$-1R_1 \to R_1 \quad x - y + z = 7 \quad (4)$$
$$\tfrac{1}{5}R_2 \to R_2 \qquad y - \frac{1}{5}z = -\frac{7}{5} \quad (5)$$

Solve equation (5) for y in terms of z.

$$y = \frac{1}{5}z - \frac{7}{5}$$

Substitute this expression for y in equation (4), and solve the equation for x.

$$x - \left(\frac{1}{5}z - \frac{7}{5}\right) + z = 7$$
$$x - \frac{1}{5}z + \frac{7}{5} + z = 7$$
$$x + \frac{4}{5}z = \frac{28}{5}$$
$$x = -\frac{4}{5}z + \frac{28}{5}$$

The solution of the system is

$$\left(-\frac{4}{5}z + \frac{28}{5}, \frac{1}{5}z - \frac{7}{5}, z\right) \text{ or}$$
$$\left(\frac{-4z + 28}{5}, \frac{z - 7}{5}, z\right).$$

29. $x + 2y + 3z = 90 \quad (1)$
$3y + 4z = 36 \quad (2)$

Let z be the parameter and solve equation (2) for y in terms of z.

$$3y + 4z = 36$$
$$3y = 36 - 4z$$
$$y = 12 - \frac{4}{3}z$$

Substitute this expression for y in equation (1) to solve for x in terms of z.

$$x + 2\left(12 - \frac{4}{3}z\right) + 3z = 90$$
$$x + 24 - \frac{8}{3}z + 3z = 90$$
$$x + \frac{1}{3}z = 66$$
$$x = 66 - \frac{1}{3}z$$

Thus the solutions are $\left(66 - \frac{1}{3}z, 12 - \frac{4}{3}z, z\right)$, where z is any real number. Since the solutions have to be nonnegative integers, set $66 - \frac{1}{3}z \ge 0$. Solving for z gives $z \le 198$.

Since y must be nonnegative, we have $12 - \frac{4}{3}z \ge 0$. Solving for z gives $z \le 9$.

Since z must be a multiple of 3 for x and y to be integers, the permissible values of z are 0, 3, 6, and 9, which gives 4 solutions.

30. $x - 7y + 4z = 75 \quad (1)$
$2y + 7z = 60 \quad (2)$

Let z be the parameter and solve equation (2) for y in terms of z.

$$2y + 7z = 60$$
$$2y = 60 - 7z$$
$$y = 30 - \frac{7}{2}z$$

Substitute this expression for y in equation (1) to solve for x in terms of z.

$$x - 7\left(30 - \frac{7}{2}z\right) + 4z = 75$$
$$x - 210 + \frac{49}{2}z + 4z = 75$$
$$x + \frac{57}{2}z = 285$$
$$x = 285 - \frac{57}{2}z$$

Thus the solutions are $\left(285 - \frac{57}{2}z, 30 - \frac{7}{2}z,\ z\right)$, where z is any real number. Since the solutions have to be nonnegative integers, set $285 - \frac{57}{2}z \ge 0$. Solving for z gives $z \le 10$.

Since y must be nonnegative, we have $30 - \frac{7}{2}z \ge 0$. Solving for z gives $z \le 8.57$.

Since z must be a multiple of 2 for x and y to be integers, the permissible values of z are 0, 2, 4, 6, and 8, which gives 5 solutions.

31. $3x + 2y + 4z = 80 \quad (1)$

$y - 3z = 10 \quad (2)$

Let z be the parameter and solve equation (2) for y in terms of z.

$$y - 3z = 10$$
$$y = 3z + 10$$

Substitute this expression for y in equation (1) to solve for x in terms of z.

$$3x + 2(3z + 10) + 4z = 80$$
$$3x + 6z + 20 + 4z = 80$$
$$3x + 10z = 60$$
$$3x = 60 - 10z$$
$$x = 20 - \frac{10}{3}z$$

Thus the solutions are $\left(20 - \frac{10}{3}z, 3z + 10, z\right)$, where z is any real number. Since the solutions have to be nonnegative integers, set $20 - \frac{10}{3}z \geq 0$. Solving for z gives $z \leq 6$.

Since y must be nonnegative, we have $3z + 10 \geq 0$. Solving for z gives $z \geq -10\frac{1}{3}$.

Since z must be a multiple of 3 for x to be an integer, the permissible values of z are 0, 3, and, 6, which gives 3 solutions.

32. $4x + 2y + 3z = 72 \quad (1)$

$2y - 3z = 12 \quad (2)$

Let z be the parameter and solve equation (2) for y in terms of z.

$$2y - 3z = 12$$
$$2y = 3z + 12$$
$$y = \frac{3}{2}z + 6$$

Substitute this expression for y in equation (1) to solve for x in terms of z.

$$4x + 2\left(\frac{3}{2}z + 6\right) + 3z = 72$$
$$4x + 3z + 12 + 3z = 72$$
$$4x + 6z = 60$$
$$4x = 60 - 6z$$
$$x = 15 - \frac{3}{2}z$$

Thus the solutions are $\left(15 - \frac{3}{2}z, \frac{3}{2}z + 6, z\right)$, where z is any real number. Since the solutions have to be nonnegative integers, set $15 - \frac{3}{2}z \geq 0$. Solving for z gives $z \leq 10$.

Since y must be nonnegative, we have $\frac{3}{2}z + 6 \geq 0$. Solving for z gives $z \leq -4$.

Since $-4 \leq z \leq 10$ and z must be a multiple of 2 for x and y to be integers, the permissible values of z are 0, 2, 4, 6, 8, and 10, which gives 6 solutions.

33. $nb + (\sum x)m = \sum y \quad (1)$

$(\sum x)b + (\sum x^2)m = \sum xy \quad (2)$

Multiply equation (1) by $\frac{1}{n}$.

$$b + \frac{\sum x}{n}m = \frac{\sum y}{n} \quad (3)$$
$$(\sum x)b + (\sum x^2)m = \sum xy \quad (2)$$

Eliminate b from equation (2).

$$b + \frac{\sum x}{n}m = \frac{\sum y}{n} \quad (3)$$

$(-\sum x)\text{R}_1 + \text{R}_2 \rightarrow \text{R}_2$

$$\left[-\frac{(\sum x)^2}{n} + \sum x^2\right]m = \frac{-(\sum x)(\sum y)}{n} + \sum xy \quad (4)$$

Multiply equation (4) by $\dfrac{1}{-\frac{(\sum x)^2}{n} + \sum x^2}$.

$$b + \frac{\sum x}{n}m = \frac{\sum y}{n} \quad (3)$$

$$m = \left[\frac{-(\sum x)(\sum y)}{n} + \sum xy\right]\left[\frac{1}{-\frac{(\sum x)^2}{n} + \sum x^2}\right] \quad (5)$$

Simplify the right side of equation (5).

$$m = \left[\frac{-(\sum x)(\sum y) + n\sum xy}{n}\right]\left[\frac{n}{-(\sum x)^2 + n(\sum x^2)}\right]$$
$$m = \frac{n\sum xy - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

From equation (3) we have

$$b = \frac{\sum y}{n} - \frac{\sum x}{n}m$$
$$b = \frac{\sum y - m(\sum x)}{n}.$$

35. Let x = the cost per pound of rice, and
y = the cost per pound of potatoes.

The system to be solved is

$$20x + 10y = 16.20 \quad (1)$$
$$30x + 12y = 23.04. \quad (2)$$

Multiply equation (1) by $\frac{1}{20}$.

$$\tfrac{1}{20}R_1 \to R_1 \qquad x + 0.5y = 0.81 \quad (3)$$
$$30x + 12y = 23.04 \quad (2)$$

Eliminate x in equation (2).

$$x + 0.5y = 0.81 \quad (3)$$
$$-30R_1 + R_2 \to R_2 \qquad -3y = -1.26 \quad (4)$$

Multiply equation (4) by $-\frac{1}{3}$.

$$x + 0.5y = 0.81 \quad (3)$$
$$-\tfrac{1}{3}R_2 \to R_2 \qquad y = 0.42 \quad (5)$$

Substitute 0.42 for y in equation (3).

$$x + 0.5(0.42) = 0.81$$
$$x + 0.21 = 0.81$$
$$x = 0.60$$

The cost of 10 pounds of rice and 50 pounds of potatoes is

$$10(0.60) + 50(0.42) = 27,$$

that is, \$27.

36. Let $x =$ the number of new releases Blake downloaded
$y =$ the number of older songs Blake downloaded.

The system to be solved is

$$x + y = 33 \quad (1)$$
$$1.29x + 0.99y = 35.97 \quad (2)$$

Eliminate x in eqatuon (2).

$$x + y = 33 \quad (1)$$
$$R_2 + (-1.29)R_1 \to R_2 \qquad -0.3y = -6.67 \quad (3)$$

Solve equation (3) for y.

$$y = \frac{-6.6}{-0.3} = 22$$

Substitute 22 for y in equation (1) and solve for x.

$$x + 22 = 33$$
$$x = 11$$

Blake bought 11 new releses and 22 older songs.

37. Let $x =$ the number of seats on the main floor, and
$y =$ the number of seats in the balcony.

The system to be solved is

$$8x + 5y = 4200 \quad (1)$$
$$0.25(8x) + 0.40(5y) = 1200. \quad (2)$$

Make the coefficient of the first term in equation (l) equal 1.

$$\tfrac{1}{8}R_1 \to R_1 \qquad x + \frac{5}{8}y = 525 \quad (3)$$
$$2x + 2y = 1200 \quad (2)$$

Eliminate x in equation (2).

$$x + \frac{5}{8}y = 525 \quad (3)$$
$$-2R_1 + R_2 \to R_2 \qquad \frac{6}{8}y = 150 \quad (4)$$

Make the coefficient of the first term in equation (4) equal 1.

$$x + \frac{5}{8}y = 525 \quad (3)$$
$$\tfrac{8}{6}R_2 \to R_2 \qquad y = 200 \quad (5)$$

Substitute 200 for y in equation (3).

$$x + \frac{5}{8}(200) = 525$$
$$x + 125 = 525$$
$$x = 400$$

There are 400 main floor seats and 200 balcony seats.

38. Let $x =$ the number of skirts originally in the store, and
$y =$ the number of blouses originally in the store.

The system to be solved is

$$45x + 35y = 51{,}750 \quad (1)$$
$$45\left(\frac{1}{2}x\right) + 35\left(\frac{2}{3}y\right) = 30{,}600. \quad (2)$$

Simplify each equation. Multiply equation (1) by $\frac{1}{5}$ and equation (2) by $\frac{6}{5}$.

$$9x + 7y = 10{,}350 \quad (3)$$
$$27x + 28y = 36{,}720 \quad (4)$$

Eliminate x from equation (4).

$$9x + 7y = 10{,}350 \quad (3)$$
$$-3R_1 + R_2 \to R_2 \qquad 7y = 5670 \quad (5)$$

Make each leading coefficient equal 1.

$$\tfrac{1}{9}R_1 \to R_1 \qquad x + \frac{7}{9}y = 1150 \quad (6)$$
$$\tfrac{1}{7}R_2 \to R_2 \qquad y = 810 \quad (7)$$

Substitute 810 for y in equation (6).

$$x + \frac{7}{9}(810) = 1150$$
$$x + 630 = 1150$$
$$x = 520$$

Half of the skirts are sold, leaving half in the store, so

$$\frac{1}{2}x = \frac{1}{2}(520) = 260.$$

Two-thirds of the blouses are sold, leaving one-third in the store, so

$$\frac{1}{3}y = \frac{1}{3}(810) = 270.$$

There are 260 skirts and 270 blouses left in the store.

39. Let x = the number of model 201 to make each day, and
y = the number of model 301 to make each day.

The system to be solved is

$$2x + 3y = 34 \quad (1)$$
$$18x + 27y = 335. \quad (2)$$

Make the coefficient of the first term in equation (1) equal 1.

$$\tfrac{1}{2}R_1 \to R_1 \qquad x + \frac{3}{2}y = 17 \quad (3)$$
$$18x + 27y = 335 \quad (2)$$

Eliminate x in equation (2).

$$x + \frac{3}{2}y = 17 \quad (3)$$
$$-18R_1 + R_2 \to R_2 \qquad 0 = 29 \quad (4)$$

Since equation (4) is false, the system is inconsistent. Therefore, this situation is impossible.

40. Let x = the number of shares of Disney stock, and
y = the number of shares of Intel stock.

$$30x + 70y = 16{,}000 \quad (1)$$
$$45x + 105y = 25{,}500 \quad (2)$$

Simplify each equation. Multiply equation (1) by $\frac{1}{10}$ and equation (2) by $\frac{1}{15}$.

$$3x + 7y = 1600 \quad (3)$$
$$3x + 7y = 1700 \quad (4)$$

Since $3x + 7y$ cannot equal both 1600 and 1700 for one point (x, y), we have an inconsistent system. Therefore, this situation is not possible.

41. **(a)** Let x = the number of fives
y = the number of tens
z = the number of twenties

We want to solve the following system.

$$x + y + z = 70 \quad (1)$$
$$5x + 10y + 20z = 960 \quad (2)$$

Eliminate x in equation (2).

$$x + y + z = 70$$
$$-5R_1 + R_2 \to R_2 \qquad 5y + 15z = 610$$

Let z be the parameter. Solve for y and for x in terms of z.

$$5y + 15z = 610$$
$$5y = 610 - 15z$$
$$y = 122 - 3z$$

$$x + (122 - 3z) + z = 70$$
$$x - 2z = -52$$
$$x = 2z - 52$$

The solutions are $(2z - 52, 122 - 3z, z)$, where z is any real number. All variables must represent nonnegative integers.

$$122 - 3z \geq 0 \qquad 2z - 52 \geq 0$$
$$-3z \geq -122 \qquad 2z \geq 52$$
$$z \leq 40.67 \qquad z \geq 26$$

Therefore, z must be an integer such that $26 \leq z \leq 40.$ Thus, there are a total of 15 solutions.

(b) The smallest value of z is 26.

$$x = 2z - 52 = 2(26) - 52 = 0 \qquad y = 122 - 3z = 122 - 3(26) = 44$$

The solutions with the smallest number of five-dollar bills is no fives, 44 tens, and 26 twenties.

(c) The largest value of z is 40.

$$x = 2z - 52 = 2(40) - 52 = 28 \qquad y = 122 - 3z = 122 - 3(40) = 2$$

The solutions with the largest number of five-dollar bills is 28 fives, 2 tens, and 40 twenties.

42. Let x = the number of EZ models,
y = the number of compact models, and
z = the number of commercial models.

Make a table.

	EZ	Compact	Commercial	Totals
Weight	10	20	60	440
Space	10	8	28	248

$$10x + 20y + 60z = 440 \quad (1)$$
$$10x + 8y + 28z = 248 \quad (2)$$

Eliminate x from equation (2).

$$10x + 20y + 60z = 440 \quad (1)$$
$$R_1 + (-1)R_2 \to R_2 \quad 12y + 32z = 192 \quad (3)$$

Make the leading coefficients equal 1.

$$\tfrac{1}{10}R_1 \to R_1 \quad x + 2y + 6z = 44 \quad (4)$$
$$\tfrac{1}{12}R_2 \to R_2 \quad y + \frac{8}{3}z = 16 \quad (5)$$

Solve equation (5) for y.

$$y = 16 - \frac{8}{3}z$$
$$y = \frac{48 - 8z}{3}$$
$$y = \frac{8(6 - z)}{3}$$

Substitute this expression for y into equation (4) and solve for x.

$$x + 2\left[\frac{8(6 - z)}{3}\right] + 6z = 44$$
$$x = 44 - 6z - \frac{16(6 - z)}{3}$$
$$x = \frac{132 - 18z - 96 + 16z}{3}$$
$$x = \frac{36 - 2z}{3}$$
$$x = \frac{2(18 - z)}{3}$$

The solution of the system is

$$\left(\frac{2(18 - z)}{3}, \frac{8(6 - z)}{3}, z\right).$$

The solutions must be nonnegative integers. Therefore, $0 \le z \le 6$. (Any larger values of z would cause y to be negative, which would make no sense in the problem.)

Values of z	Solutions
0	(12, 16, 0)
1	$\left(\frac{34}{3}, \frac{40}{3}, 1\right)$
2	$\left(\frac{32}{3}, \frac{32}{3}, 2\right)$
3	(10, 8, 3)
4	$\left(\frac{28}{3}, \frac{16}{3}, 4\right)$
5	$\left(\frac{26}{3}, \frac{8}{3}, 5\right)$
6	(8, 0, 6)

Ignore solutions containing values that are not integers. There are three possible solutions:

1. 12 EZ models, 16 compact models, and 0 commercial models;
2. 10 EZ models, 8 compact models, and 3 commercial models; or
3. 8 EZ models, 0 compact models, and 6 commercial models.

43. Let x = the number of buffets produced each week,
y = the number of chairs produced each week.
z = the number of tables produced each week.

Make a table.

	Buffet	Chair	Table	Totals
Construction	30	10	10	350
Finishing	10	10	30	150

The system to be solved is

$$30x + 10y + 10z = 350 \quad (1)$$
$$10x + 10y + 30z = 150. \quad (2)$$

Make the coefficient of the first term in equation (1) equal 1.

$$\tfrac{1}{30}R_1 \to R_1 \quad x + \frac{1}{3}y + \frac{1}{3}z = \frac{35}{3} \quad (3)$$
$$10x + 10y + 30z = 150 \quad (2)$$

Eliminate x from equation (2).

$$x + \frac{1}{3}y + \frac{1}{3}z = \frac{35}{3} \quad (3)$$
$$-10R_1 + R_2 \to R_2 \quad \frac{20}{3}y + \frac{80}{3}z = \frac{100}{3} \quad (4)$$

Solve equation (4) for y. Multiply by 3.

$$20y + 80z = 100$$
$$y + 4z = 5$$
$$y = 5 - 4z$$

Substitute $5 - 4z$ for y in equation (1) and solve for x.

$$30x + 10(5 - 4z) + 10z = 350$$
$$30x + 50 - 40z + 10z = 350$$
$$30x = 300 + 30z$$
$$x = 10 + z$$

The solution is $(10 + z, 5 - 4z, z)$. All variables must be nonnegative integers. Therefore,

$$5 - 4z \ge 0$$
$$5 \ge 4z$$
$$z \le \frac{5}{4},$$

so $z = 0$ or $z = 1$. (Any larger value of z would cause y to be negative, which would make no sense in the problem.) If $z = 0$, then the solution is (10, 5, 0). If $z = 1$, then the solution is (11, 1, 1).

Therefore, the company should make either 10 buffets, 5 chairs, and no tables or 11 buffets, 1 chair, and 1 table each week.

44. **(a)** The system to be solved is

$$\begin{aligned} 43{,}500x - y &= 1{,}295{,}000 \quad (1) \\ 27{,}000x - y &= 440{,}000 \quad (2) \end{aligned}$$

Eliminate x in equation (2).

$$\begin{aligned} & 43{,}500x - y = 1{,}295{,}000 \quad (1) \\ -\tfrac{27{,}000}{43{,}500}R_1 + R_2 \to R_2 \quad & -\frac{11}{29}y = -\frac{10{,}550{,}000}{29} \quad (3) \end{aligned}$$

Make the coefficient of the first term in equation (3) equal 1.

$$\begin{aligned} & 43{,}500x - y = 1{,}295{,}000 \quad (1) \\ -\tfrac{29}{11}R_2 \to R_2 \quad & y = \frac{10{,}550{,}000}{11} \quad (4) \end{aligned}$$

Substitute $\frac{10{,}550{,}000}{11}$ for y in equation (1).

$$\begin{aligned} 43{,}500x - \frac{10{,}550{,}000}{11} &= 1{,}295{,}000 \\ 43{,}500x &= \frac{24{,}795{,}000}{11} \\ x &= \frac{570}{11} \end{aligned}$$

The solution is $\left(\frac{570}{11}, \frac{10{,}550{,}000}{11}\right)$.

The profit/loss will be equal after $\frac{570}{11}$ weeks or about 51.8 weeks. At that point, the profit will be $\frac{10{,}550{,}000}{11}$ or about \$959,091.

(b) If the show lasts longer than 51.8 weeks, Broadway is a more profitable venue. If it lasts less than 51.8 weeks, off Broadway is a more profitable venue.

45. **(a)** For the first equation, the first sighting in 2000 was on day $y = 759 - 0.338(2000) = 83$, or during the eighty-third day of the year. Since 2000 was a leap year, the eighty-third day fell on March 23.

For the second equation, the first sighting in 2000 was on day $y = 1637 - 0.779(2000) = 79$, or during the seventy-ninth day of the year. Since 2000 was a leap year, the seventy-ninth day fell on March 19.

(b)

$$\begin{aligned} y &= 759 - 0.338x \quad (1) \\ y &= 1637 - 0.779x \quad (2) \end{aligned}$$

Rewrite equations so that variables are on the left side and constant term is on the right side.

$$\begin{aligned} 0.338x + y &= 759 \quad (3) \\ 0.779x + y &= 1637 \quad (4) \end{aligned}$$

Eliminate y from equation (4).

$$\begin{aligned} & 0.338x + y = 759 \quad (3) \\ -1R_1 + R_2 \to R_2 \quad & 0.441x = 878 \quad (5) \end{aligned}$$

Make leading coefficient for equation (5) equal 1.

The two estimates agree in the year closest to $x = \frac{878}{0.441} \approx 1990.93$, so they agree in 1991.

The estimated number of days into the year when a robin can be expected is

$$\begin{aligned} 0.338\left(\frac{878}{0.441}\right) + y &= 759 \\ y &\approx 86. \end{aligned}$$

46. **(a)** We are given the equation

$$y = ax^2 + bx + c.$$

Since a car traveling at 0 mph has a stopping distance of 0 feet, then $y = 0$ when $x = 0$. Substituting these values into $y = ax^2 + bx + c$ yields

$$\begin{aligned} 0 &= a(0)^2 + b(0) + c, \text{ so} \\ c &= 0. \end{aligned}$$

Therefore, we have

$$y = ax^2 + bx.$$

After substituting the given values for the stopping distances (y) and speeds (x) in mph, the system to be solved is

$$\begin{aligned} 61.7 &= a(25)^2 + b(25) \quad (1) \\ 106 &= a(35)^2 + b(35). \quad (2) \end{aligned}$$

These equations can be written as

$$\begin{aligned} 625a + 25b &= 61.7 \quad (1) \\ 1225a + 35b &= 106. \quad (2) \end{aligned}$$

Multiply equation (1) by $\frac{1}{625}$; also eliminate the decimal in 61.7 by multiplying the numerator and denominator of the fraction by 10.

$$\begin{aligned} \tfrac{1}{625}R_1 \to R_1 \quad & a + \frac{1}{25}b = \frac{617}{6250} \quad (3) \\ & 1225a + 35b = 106 \quad (2) \end{aligned}$$

Eliminate a in equation (2).

$$a + \frac{1}{25}b = \frac{617}{6250} \quad (3)$$

$$-1225R_1 + R_2 \rightarrow R_2 \quad -14b = -\frac{3733}{250} \quad (4)$$

Multiply equation (4) by $-\frac{1}{14}$.

$$a + \frac{1}{25}b = \frac{617}{6250} \quad (3)$$

$$-\frac{1}{14}R_2 \rightarrow R_2 \quad b = \frac{3733}{3500} \quad (5)$$

Substitute $\frac{3733}{3500}$ for b in equation (3).

$$a + \frac{1}{25}\left(\frac{3733}{3500}\right) = \frac{617}{6250}$$

$$a = \frac{4905}{87{,}500}$$

Therefore,

$$a = \frac{4905}{87{,}500} \approx 0.056057,$$

and $b = \dfrac{3733}{3500} \approx 1.06657.$

(b) Substitute the values from part (a) for a and b and 55 for x in the equation $y = ax^2 + bx$. Solve for y.

$$y = 0.056057(55)^2 + 1.06657(55)$$
$$y \approx 228$$

The stopping distance of a car traveling 55 mph is approximately 228 ft.

47. Let x = number of field goals, and y = number of foul shots.

Then
$$x + y = 64 \quad (1)$$
$$2x + y = 100 \quad (2).$$

Eliminate x in equation (2).

$$x + y = 64 \quad (1)$$
$$-2R_1 + R_2 \rightarrow R_2 \quad -y = -28 \quad (3)$$

Make the coefficients of the first term of each equation equal 1.

$$x + y = 64 \quad (1)$$
$$-1R_2 \rightarrow R_2 \quad y = 28 \quad (4)$$

Substitute 28 for y in equation (1) to get $x = 36$. Wilt Chamberlain made 36 field goals and 28 foul shots.

48. Let x = flight time eastward

y = difference in time zones

The flight time westward is one hour longer than the time eastward, so flight time westward is $x+1$. The system to be solved is

$$x + y = 13$$
$$(x+1) - y = 2$$

or

$$x - y = 1 \quad (1)$$
$$x + y = 13 \quad (2)$$

$$x - y = 1$$
$$(-1)R_1 + R_2 \rightarrow R_2 \quad 2y = 12$$

$$x - y = 1$$
$$\frac{1}{2}R_2 \rightarrow R_2 \quad y = 6$$

Substitute $y = 6$ into (1).

$$x - 6 = 1$$
$$x = 7$$

The flight time is 7 hours, and the difference in time zones is 6 hours.

49. **(a)** Since 8 and 9 must be two of the four numbers combined using addition, subtraction, multiplication, and/or division to get 24, begin by finding two numbers to use with 8 and 9. One possibility is 8 and 3 since

$$(9 - 8) \cdot 8 \cdot 3 = 24.$$

If we can find values of x and y such that either $x + y = 8$ and $3x + 2y = 3$, or $x + y = 3$ and $3x + 2y = 8$, we will have found a solution. Solving the first system gives $x = -13$ and $y = 21$. This, however, does not satisfy the condition that x and y be single-digit positive integers. Solving the second system gives $x = 2$ and $y = 1$. Since both of these values are single-digit positive integers, we have one possible system. Thus, one system is

$$\begin{cases} x + y = 3 \\ 3x + 2y = 8 \end{cases}$$

Its solution is (2, 1). These values of x and y give the numbers 8, 9, 8, and 3 on the game card. These numbers can be combined as $(9 - 8) \cdot 8 \cdot 3$ to make 24.

2.2 Solution of Linear Systems by the Gauss-Jordan Method

Your Turn 1

$$\begin{aligned} 4x + 5y &= 10 \\ 7x + 8y &= 19 \end{aligned}$$

Write the augmented matrix and transform the matrix.

$$\left[\begin{array}{cc|c} 4 & 5 & 10 \\ 7 & 8 & 19 \end{array}\right]$$

$$\tfrac{1}{4}R_1 \to R_1 \left[\begin{array}{cc|c} 1 & \frac{5}{4} & \frac{5}{2} \\ 7 & 8 & 19 \end{array}\right]$$

$$-7R_1 + R_2 \to R_2 \left[\begin{array}{cc|c} 1 & \frac{5}{4} & \frac{5}{2} \\ 0 & -\frac{3}{4} & \frac{3}{2} \end{array}\right]$$

$$-\tfrac{4}{3}R_2 \to R_2 \left[\begin{array}{cc|c} 1 & \frac{5}{4} & \frac{5}{2} \\ 0 & 1 & -2 \end{array}\right]$$

$$-\tfrac{5}{4}R_2 + R_1 \to R_1 \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \end{array}\right]$$

The solution is $x = 5$ and $y = -2$, or $(5, -2)$.

Your Turn 2

$$\begin{aligned} x + 2y + 3z &= 2 \\ 2x + 2y - 3z &= 27 \\ 3x + 2y + 5z &= 10 \end{aligned}$$

Write the augmented matrix and transform the matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 2 & 2 & -3 & 27 \\ 3 & 2 & 5 & 10 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -2 & -9 & 23 \\ 0 & -4 & -4 & 4 \end{array}\right]$$

$$-\tfrac{1}{2}R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & \frac{9}{2} & -\frac{23}{2} \\ 0 & -4 & -4 & 4 \end{array}\right]$$

$$\begin{array}{r} -2R_2 + R_1 \to R_1 \\ \\ 4R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -6 & 25 \\ 0 & 1 & \frac{9}{2} & -\frac{23}{2} \\ 0 & 0 & 14 & -42 \end{array}\right]$$

$$\tfrac{1}{14}R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 0 & -6 & 25 \\ 0 & 1 & \frac{9}{2} & -\frac{23}{2} \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$$\begin{array}{r} 6R_3 + R_1 \to R_1 \\ -\frac{9}{2}R_3 + R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

The solution is $x = 7, y = 2,$ and $z = -3,$ or $(7, 2, -3)$.

Your Turn 3

$$\begin{aligned} 2x - 2y + 3z - 4w &= 6 \\ 3x + 2y + 5z - 3w &= 7 \\ 4x + y + 2z - 2w &= 8 \end{aligned}$$

Write the augmented matrix and transform the matrix.

$$\left[\begin{array}{cccc|c} 2 & -2 & 3 & -4 & 6 \\ 3 & 2 & 5 & -3 & 7 \\ 4 & 1 & 2 & -2 & 8 \end{array}\right]$$

$$\tfrac{1}{2}R_1 \to R_1 \left[\begin{array}{cccc|c} 1 & -1 & \frac{3}{2} & -2 & 3 \\ 3 & 2 & 5 & -3 & 7 \\ 4 & 1 & 2 & -2 & 8 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1 + R_2 \to R_2 \\ -4R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & \frac{3}{2} & -2 & 3 \\ 0 & 5 & \frac{1}{2} & 3 & -2 \\ 0 & 5 & -4 & 6 & -4 \end{array}\right]$$

$$\tfrac{1}{5}R_2 \to R_2 \left[\begin{array}{cccc|c} 1 & -1 & \frac{3}{2} & -2 & 3 \\ 0 & 1 & \frac{1}{10} & \frac{3}{5} & -\frac{2}{5} \\ 0 & 5 & -4 & 6 & -4 \end{array}\right]$$

$$\begin{array}{r} R_2 + R_1 \to R_1 \\ \\ -5R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & \frac{8}{5} & -\frac{7}{5} & \frac{13}{5} \\ 0 & 1 & \frac{1}{10} & \frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & -\frac{9}{2} & 3 & -2 \end{array}\right]$$

$$-\tfrac{2}{9}R_3 \to R_3 \left[\begin{array}{cccc|c} 1 & 0 & \frac{8}{5} & -\frac{7}{5} & \frac{13}{5} \\ 0 & 1 & \frac{1}{10} & \frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{4}{9} \end{array}\right]$$

$$\begin{array}{r} -\frac{8}{5}R_3 + R_1 \to R_1 \\ -\frac{1}{10}R_3 + R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{3} & \frac{17}{9} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{4}{9} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{4}{9} \end{array}\right]$$

We cannot change the values in column 4 without changing the form of the other three columns. So, let w be the parameter. The last matrix gives these equations.

$$x - \frac{1}{3}w = \frac{17}{9}, \quad \text{or} \quad x = \frac{17}{9} + \frac{1}{3}w$$
$$y + \frac{2}{3}w = -\frac{4}{9}, \quad \text{or} \quad y = -\frac{4}{9} - \frac{2}{3}w$$
$$z - \frac{2}{3}w = \frac{4}{9}, \quad \text{or} \quad z = \frac{4}{9} + \frac{2}{3}w$$

The solution is $\left(\frac{17}{9} + \frac{1}{3}w, -\frac{4}{9} - \frac{2}{3}w, \frac{4}{9} + \frac{2}{3}w, w\right)$, where w is a real number.

2.2 Warmup Exercises

W1. Using z as the parameter, the general solution is

$(24+3z, 10-2z, z)$.

For nonnegative solutions we require

$24+3z \geq 0,\ 10-2z \geq 0$, and $z \geq 0$.

Solving each inequality for z, we have

$z \geq -8,\ z \leq 5$, and $z \geq 0$.

Thus, $0 \leq z \leq 5$ and there are 6 solutions in nonegative intergers.

W2. Using z as the parameter, the general solution is

$(25-2z,\ -14+3z,\ z)$.

For nonnegative solutions we require

$25-2z \geq 0,\ -14+3z \geq 0$, and $z \geq 0$.

Solving each inequality for z, we have

$z \leq \frac{25}{2},\ z \geq \frac{14}{3}$, and $z \geq 0$.

Thus, $\frac{14}{3} \leq z \leq \frac{25}{2}$, and for interger z, $5 \leq z \leq 12$, so there are 8 solutions in nonnegative intergers.

2.2 Exercises

1.
$$\begin{aligned} 3x + y &= 6 \\ 2x + 5y &= 15 \end{aligned}$$

The equations are already in proper form. The augmented matrix obtained from the coefficients and the constants is

$$\left[\begin{array}{cc|c} 3 & 1 & 6 \\ 2 & 5 & 15 \end{array}\right].$$

2.
$$\begin{aligned} 4x - 2y &= 8 \\ -7y &= -12 \end{aligned}$$

The equations are already in proper form. The augmented matrix obtained from the coefficients and the constants is

$$\left[\begin{array}{cc|c} 4 & -2 & 8 \\ 0 & -7 & -12 \end{array}\right].$$

3.
$$\begin{aligned} 2x + y + z &= 3 \\ 3x - 4y + 2z &= -7 \\ x + y + z &= 2 \end{aligned}$$

leads to the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \\ 1 & 1 & 1 & 2 \end{array}\right].$$

4.
$$\begin{aligned} 2x - 5y + 3z &= 4 \\ -4x + 2y - 7z &= -5 \\ 3x - y &= 8 \end{aligned}$$

The equations are already in proper form. The augmented matrix obtained from the coefficients and the constants is

$$\left[\begin{array}{ccc|c} 2 & -5 & 3 & 4 \\ -4 & 2 & -7 & -5 \\ 3 & -1 & 0 & 8 \end{array}\right].$$

5. We are given the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right].$$

This is equivalent to the system of equations

$$\begin{aligned} x &= 2 \\ y &= 3, \end{aligned}$$

or $x = 2, y = 3$.

6. $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \end{array}\right]$

is equivalent to the system

$$\begin{aligned} x &= 5 \\ y &= -3. \end{aligned}$$

7. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$

The system associated with this matrix is

$$\begin{aligned} x &= 4 \\ y &= -5 \\ z &= 1, \end{aligned}$$

or $x = 4,\ y = -5,\ z = 1$.

8. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

is equivalent to the system

$$\begin{aligned} x &= 4 \\ y &= 2 \\ z &= 3. \end{aligned}$$

or $x = 4,\ y = 2,\ z = 3$.

9. *Row operations* on a matrix correspond to transformations of a system of equations.

11. $\left[\begin{array}{ccc|c} 3 & 7 & 4 & 10 \\ 1 & 2 & 3 & 6 \\ 0 & 4 & 5 & 11 \end{array}\right]$

Find $R_1 + (-3)R_2$.

In row 2, column 1,

$$3 + (-3)1 = 0.$$

In row 2, column 2,

$$7 + (-3)2 = 1.$$

In row 2, column 3,

$$4 + (-3)3 = -5.$$

In row 2, column 4,

$$10 + (-3)6 = -8.$$

Replace R_2 with these values. The new matrix is

$$\left[\begin{array}{ccc|c} 3 & 7 & 4 & 10 \\ 0 & 1 & -5 & -8 \\ 0 & 4 & 5 & 11 \end{array}\right].$$

12. Replace R_3 by $-1R_1 + 3R_3$.

The original matrix is

$$\left[\begin{array}{ccc|c} 3 & 2 & 6 & 18 \\ 2 & -2 & 5 & 7 \\ 1 & 0 & 5 & 20 \end{array}\right].$$

The resulting matrix is

$$\left[\begin{array}{ccc|c} 3 & 2 & 6 & 18 \\ 2 & -2 & 5 & 7 \\ 0 & -2 & 9 & 42 \end{array}\right].$$

13. $\left[\begin{array}{ccc|c} 1 & 6 & 4 & 7 \\ 0 & 3 & 2 & 5 \\ 0 & 5 & 3 & 7 \end{array}\right]$

Find $(-2)R_2 + R_1 \rightarrow R_1$

$$\left[\begin{array}{ccc|c} (-2)0+1 & (-2)3+6 & (-2)2+4 & (-2)5+7 \\ 0 & 3 & 2 & 5 \\ 0 & 5 & 3 & 7 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 3 & 2 & 5 \\ 0 & 5 & 3 & 7 \end{array}\right]$$

14. Replace R_1 by $R_3 + (-3)R_1$.

The original matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 21 \\ 0 & 6 & 5 & 30 \\ 0 & 0 & 12 & 15 \end{array}\right].$$

The resulting matrix is

$$\left[\begin{array}{ccc|c} -3 & 0 & 0 & -48 \\ 0 & 6 & 5 & 30 \\ 0 & 0 & 12 & 15 \end{array}\right].$$

15. $\left[\begin{array}{ccc|c} 3 & 0 & 0 & 18 \\ 0 & 5 & 0 & 9 \\ 0 & 0 & 4 & 8 \end{array}\right]$

$$\tfrac{1}{3}R_1 \rightarrow R_1 \left[\begin{array}{ccc|c} \frac{1}{3}(3) & \frac{1}{3}(0) & \frac{1}{3}(0) & \frac{1}{3}(18) \\ 0 & 5 & 0 & 9 \\ 0 & 0 & 4 & 8 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 5 & 0 & 9 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

16. Replace R_3 by $\frac{1}{6}R_3$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 30 \\ 0 & 1 & 0 & 17 \\ 0 & 0 & 6 & 162 \end{array}\right].$$

The resulting matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 30 \\ 0 & 1 & 0 & 17 \\ 0 & 0 & 1 & 27 \end{array}\right].$$

17. $x + y = 5$
$3x + 2y = 12$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 3 & 2 & 12 \end{array}\right]$$

$$-3R_1 + R_2 \to R_2 \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & -1 & -3 \end{array}\right]$$

$$-1R_2 \to R_2 \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \end{array}\right]$$

$$-1R_2 + R_1 \to R_1 \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

The solution is (2, 3).

18. $x + 2y = 5$
$2x + y = -2$

To begin, write the augmented matrix for the given system.

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 1 & -2 \end{array}\right]$$

The third row operation is used to change the 2 in row 2 to 0.

$$-2R_1 + R_2 \to R_2 \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -12 \end{array}\right]$$

Next, change the 2 in row 1 to 0.

$$2R_2 + 3R_1 \to R_1 \left[\begin{array}{cc|c} 3 & 0 & -9 \\ 0 & -3 & -12 \end{array}\right]$$

Finally, change the first nonzero number in each row to 1.

$$\begin{array}{l} \frac{1}{3}R_1 \to R_1 \\ -\frac{1}{3}R_2 \to R_2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 4 \end{array}\right]$$

The final matrix is equivalent to the system

$$x = -3$$
$$y = 4,$$

so the solution of the original system is $(-3, 4)$.

19. $x + y = 7$
$4x + 3y = 22$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 4 & 3 & 22 \end{array}\right]$$

$$-4R_1 + R_2 \to R_2 \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & -1 & -6 \end{array}\right]$$

$$-1R_2 \to R_2 \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 6 \end{array}\right]$$

$$-1R_2 + R_1 \to R_1 \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 6 \end{array}\right]$$

The solution is (1, 6).

20. $4x - 2y = 3$
$-2x + 3y = 1$

The augmented matrix for the system is

$$\left[\begin{array}{cc|c} 4 & -2 & 3 \\ -2 & 3 & 1 \end{array}\right].$$

$$R_1 + 2R_2 \to R_2 \left[\begin{array}{cc|c} 4 & -2 & 3 \\ 0 & 4 & 5 \end{array}\right]$$

$$R_2 + 2R_1 \to R_1 \left[\begin{array}{cc|c} 8 & 0 & 11 \\ 0 & 4 & 5 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{8}R_1 \to R_1 \\ \frac{1}{4}R_2 \to R_2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & \frac{11}{8} \\ 0 & 1 & \frac{5}{4} \end{array}\right]$$

The solution is $\left(\frac{11}{8}, \frac{5}{4}\right)$.

21. $2x - 3y = 2$
$4x - 6y = 1$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{cc|c} 2 & -3 & 2 \\ 4 & -6 & 1 \end{array}\right]$$

$$-2R_1 + R_2 \to R_2 \left[\begin{array}{cc|c} 2 & -3 & 2 \\ 0 & 0 & -3 \end{array}\right]$$

The system associated with the last matrix is

$$2x - 3y = 2$$
$$0x + 0y = -3.$$

Since the second equation, $0 = -3$, is false, the system is inconsistent and therefore has no solution.

22. $2x + 3y = 9$
$4x + 6y = 7$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{cc|c} 2 & 3 & 9 \\ 4 & 6 & 7 \end{array}\right]$$

$$-2R_1 + R_2 \to R_2 \left[\begin{array}{cc|c} 2 & 3 & 9 \\ 0 & 0 & -11 \end{array}\right]$$

The system associated with the last matrix is

$$\begin{aligned} 2x + 3y &= 9 \\ 0x + 0y &= -11. \end{aligned}$$

Since the second equation $0 = -11$, is false, the system is inconsistent and therefore has no solution.

23. $6x - 3y = 1$
$-12x + 6y = -2$

Write the augmented matrix of the system and use row operations.

$$\left[\begin{array}{cc|c} 6 & -3 & 1 \\ -12 & 6 & -2 \end{array}\right]$$

$$2R_1 + R_2 \to R_2 \left[\begin{array}{cc|c} 6 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\tfrac{1}{6}R_1 \to R_1 \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 0 \end{array}\right]$$

This is as far as we can go with the Gauss-Jordan method. To complete the solution, write the equation that corresponds to the first row of the matrix.

$$x - \frac{1}{2}y = \frac{1}{6}$$

Solve this equation for x in terms of y.

$$x = \frac{1}{2}y + \frac{1}{6} = \frac{3y+1}{6}$$

The solution is $\left(\frac{3y+1}{6}, y\right)$, where y is any real number.

24. $x - y = 1$
$-x + y = -1$

The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ -1 & 1 & -1 \end{array}\right]$$

$$R_1 + R_2 \to R_2 \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

The row of zeros indicates dependent equations. (Both equations have the same line as their graph.) The remaining equation is $x - y = 1$. Solving for x gives $x = y + 1$. There are an infinite number of solutions, each of the form $(y + 1, y)$, for any real number y.

25. $y = x - 3$
$y = 1 + z$
$z = 4 - x$

First write the system in proper form.

$$\begin{aligned} -x + y \quad &= -3 \\ y - z &= 1 \\ x \quad + z &= 4 \end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & -3 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 4 \end{array}\right]$$

$$-1R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 4 \end{array}\right]$$

$$-1R_1 + R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{r} R_2 + R_1 \to R_1 \\ \\ -1R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\begin{array}{r} R_3 + 2R_1 \to R_1 \\ R_3 + 2R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 8 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{2}R_1 \to R_1 \\ \frac{1}{2}R_2 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The solution is (4, 1, 0).

26. $x = 1 - y$
$2x = z$
$2z = -2 - y$

Put the equations in proper form to obtain the system

$$\begin{aligned} x + y \quad\quad &= 1 \\ 2x \quad\; - z &= 0 \\ y + 2z &= -2. \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ 0 & 1 & 2 & -2 \end{array}\right]$$

$$-2R_1 + R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & -1 & -2 \\ 0 & 1 & 2 & -2 \end{array}\right]$$

$$\begin{array}{r} R_2 + 2R_1 \to R_1 \\ \\ R_2 + 2R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 3 & -6 \end{array}\right]$$

$$\begin{array}{r} R_3 + 3R_1 \to R_1 \\ R_3 + 3R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 6 & 0 & 0 & -6 \\ 0 & -6 & 0 & -12 \\ 0 & 0 & 3 & -6 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{6}R_1 \to R_1 \\ -\frac{1}{6}R_2 \to R_2 \\ \frac{1}{3}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The solution is $(-1, 2, -2)$.

27. $2x - 2y = -5$
$2y + z = 0$
$2x + z = -7$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & -5 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & -7 \end{array}\right]$$

$$-1R_1 + R_3 \to R_3 \left[\begin{array}{ccc|c} 2 & -2 & 0 & -5 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & -2 \end{array}\right]$$

$$\begin{array}{r} R_2 + R_1 \to R_1 \\ \\ -1R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 0 & 1 & -5 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

This matrix corresponds to the system of equations

$$\begin{aligned} 2x + z &= -5 \\ 2y + z &= 0 \\ 0 &= -2. \end{aligned}$$

This false statement $0 = -2$ indicates that the system is inconsistent and therefore has no solution.

28. $x - z = -3$
$y + z = 9$
$-2x + 3y + 5z = 33$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 9 \\ -2 & 3 & 5 & 33 \end{array}\right]$$

$$2R_1 + R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 9 \\ 0 & 3 & 3 & 27 \end{array}\right]$$

$$-3R_2 + R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last row indicates an infinite number of solutions. The remaining equations are

$$x - z = -3 \text{ and } y + z = 9.$$

Solve these for x and y, the solutions are

$$(z - 3, -z + 9, z)$$

for any real number z.

29. $4x + 4y - 4z = 24$
$2x - y + z = -9$
$x - 2y + 3z = 1$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{ccc|c} 4 & 4 & -4 & 24 \\ 2 & -1 & 1 & -9 \\ 1 & -2 & 3 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ R_1 + (-2)R_2 \to R_2 \\ R_1 + (-4)R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 4 & 4 & -4 & 24 \\ 0 & 6 & -6 & 42 \\ 0 & 12 & -16 & 20 \end{array}\right]$$

$$\begin{array}{r} 2R_2 + (-3)R_1 \to R_1 \\ \\ -2R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} -12 & 0 & 0 & 12 \\ 0 & 6 & -6 & 42 \\ 0 & 0 & -4 & -64 \end{array}\right]$$

$$-3R_3 + 2R_2 \to R_2 \left[\begin{array}{ccc|c} -12 & 0 & 0 & 12 \\ 0 & 12 & 0 & 276 \\ 0 & 0 & -4 & -64 \end{array}\right]$$

$$\begin{array}{r} -\frac{1}{12}R_1 \to R_1 \\ \frac{1}{12}R_2 \to R_2 \\ -\frac{1}{4}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 23 \\ 0 & 0 & 1 & 16 \end{array}\right]$$

The solution is $(-1, 23, 16)$.

30. $x + 2y - 7z = -2$
$-2x - 5y + 2z = 1$
$3x + 5y + 4z = -9$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 1 & 2 & -7 & -2 \\ -2 & -5 & 2 & 1 \\ 3 & 5 & 4 & -9 \end{array}\right]$$

$$\begin{array}{r} \\ 2R_1 + R_2 \rightarrow R_2 \\ -3R_1 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & -7 & -2 \\ 0 & -1 & -12 & -3 \\ 0 & -1 & 25 & -3 \end{array}\right]$$

$$\begin{array}{r} 2R_2 + R_1 \rightarrow R_1 \\ \\ -1R_2 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & -31 & -8 \\ 0 & -1 & -12 & -3 \\ 0 & 0 & 37 & 0 \end{array}\right]$$

$$\begin{array}{r} 31R_3 + 37R_1 \rightarrow R_1 \\ 12R_3 + 37R_2 \rightarrow R_2 \\ \\ \end{array} \left[\begin{array}{rrr|r} 37 & 0 & 0 & -296 \\ 0 & -37 & 0 & -111 \\ 0 & 0 & 37 & 0 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{37}R_1 \rightarrow R_1 \\ -\frac{1}{37}R_2 \rightarrow R_2 \\ \frac{1}{37}R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The solution is $(-8, 3, 0)$.

31. $3x + 5y - z = 0$
$4x - y + 2z = 1$
$7x + 4y + z = 1$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 3 & 5 & -1 & 0 \\ 4 & -1 & 2 & 1 \\ 7 & 4 & 1 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ 4R_1 + (-3)R_2 \rightarrow R_2 \\ 7R_1 + (-3)R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrr|r} 3 & 5 & -1 & 0 \\ 0 & 23 & -10 & -3 \\ 0 & 23 & -10 & -3 \end{array}\right]$$

$$\begin{array}{r} 23R_1 + (-5)R_2 \rightarrow R_1 \\ \\ R_2 + (-1)R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrr|r} 69 & 0 & 27 & 15 \\ 0 & 23 & -10 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{69}R_1 \rightarrow R_1 \\ \frac{1}{23}R_2 \rightarrow R_2 \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & \frac{9}{23} & \frac{5}{23} \\ 0 & 1 & -\frac{10}{23} & -\frac{3}{23} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The row of zeros indicates dependent equations. Solve the first two equations respectively for x and y in terms of z to obtain

$$x = -\frac{9}{23}z + \frac{5}{23} = \frac{-9z + 5}{23}$$

and

$$y = \frac{10}{23}z - \frac{3}{23} = \frac{10z - 3}{23}.$$

The solution is $\left(\frac{-9z + 5}{23}, \frac{10z - 3}{23}, z\right)$.

32. $3x - 6y + 3z = 11$
$2x + y - z = 2$
$5x - 5y + 2z = 6$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 3 & -6 & 3 & 11 \\ 2 & 1 & -1 & 2 \\ 5 & -5 & 2 & 6 \end{array}\right].$$

$$\begin{array}{r} \\ -2R_1 + 3R_2 \rightarrow R_2 \\ -5R_1 + 3R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrr|r} 3 & -6 & 3 & 11 \\ 0 & 15 & -9 & -16 \\ 0 & 15 & -9 & -37 \end{array}\right]$$

$$\begin{array}{r} 2R_2 + 5R_1 \rightarrow R_1 \\ \\ -R_2 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrr|r} 15 & 0 & -3 & 23 \\ 0 & 15 & -9 & -16 \\ 0 & 0 & 0 & -21 \end{array}\right]$$

The last row indicates inconsistent equations. There is no solution to the system.

33. $5x - 4y + 2z = 6$
$5x + 3y - z = 11$
$15x - 5y + 3z = 23$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 5 & -4 & 2 & 6 \\ 5 & 3 & -1 & 11 \\ 15 & -5 & 3 & 23 \end{array}\right]$$

$$\begin{array}{r} \\ -1R_1 + R_2 \rightarrow R_2 \\ -3R_1 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrr|r} 5 & -4 & 2 & 6 \\ 0 & 7 & -3 & 5 \\ 0 & 7 & -3 & 5 \end{array}\right]$$

$$\begin{array}{r} 4R_2 + 7R_1 \rightarrow R_1 \\ \\ -1R_2 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrr|r} 35 & 0 & 2 & 62 \\ 0 & 7 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{35}R_1 \rightarrow R_1 \\ \frac{1}{7}R_2 \rightarrow R_2 \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & \frac{2}{35} & \frac{62}{35} \\ 0 & 1 & -\frac{3}{7} & \frac{5}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The row of zeros indicates dependent equations. Solve the first two equations respectively for x and y in terms of z to obtain

$$x = -\frac{2}{35}z + \frac{62}{35} = \frac{-2z + 62}{35}$$

and

$$y = \frac{3}{7}z + \frac{5}{7} = \frac{3z + 5}{7}.$$

The solution is $\left(\frac{-2z+62}{35}, \frac{3z+5}{7}, z\right)$.

34.
$$\begin{aligned} 3x + 2y - z &= -16 \\ 6x - 4y + 3z &= 12 \\ 5x - 2y + 2z &= 4 \end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 3 & 2 & -1 & -16 \\ 6 & -4 & 3 & 12 \\ 5 & -2 & 2 & 4 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ -5R_1 + 3R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 3 & 2 & -1 & -16 \\ 0 & -8 & 5 & 44 \\ 0 & -16 & 11 & 92 \end{array}\right]$$

$$\begin{array}{r} R_2 + 4R_1 \to R_1 \\ \\ -2R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 12 & 0 & 1 & -20 \\ 0 & -8 & 5 & 44 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{r} -1R_3 + R_1 \to R_1 \\ -5R_3 + R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{rrr|r} 12 & 0 & 0 & -24 \\ 0 & -8 & 0 & 24 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{12}R_1 \to R_1 \\ -\frac{1}{8}R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Read the solution from the last column of the matrix. The solution is $(-2, -3, 4)$.

35.
$$\begin{aligned} 2x + 3y + z &= 9 \\ 4x + 6y + 2z &= 18 \\ -\frac{1}{2}x - \frac{3}{4}y - \frac{1}{4}z &= -\frac{9}{4} \end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 2 & 3 & 1 & 9 \\ 4 & 6 & 2 & 18 \\ -\frac{1}{2} & -\frac{3}{4} & -\frac{1}{4} & -\frac{9}{4} \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ \frac{1}{4}R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 2 & 3 & 1 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The rows of zeros indicate dependent equations. Since the equation involves *x, y,* and *z,* let *y* and *z* be parameters. Solve the equation for *x* to obtain $x = \frac{9 - 3y - z}{2}$.

The solution is $\left(\frac{9z - 3y - z}{2}, y, z\right)$, where y and z are any real numbers.

36.
$$\begin{aligned} 3x - 5y - 2z &= -9 \\ -4x + 3y + z &= 11 \\ 8x - 5y + 4z &= 6 \end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 3 & -5 & -2 & -9 \\ -4 & 3 & 1 & 11 \\ 8 & -5 & 4 & 6 \end{array}\right]$$

$$\begin{array}{r} \\ 4R_1 + 3R_2 \to R_2 \\ -8R_1 + 3R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 3 & -5 & -2 & -9 \\ 0 & -11 & -5 & -3 \\ 0 & 25 & 28 & 90 \end{array}\right]$$

$$\begin{array}{r} -5R_2 + 11R_1 \to R_1 \\ \\ 25R_2 + 11R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 33 & 0 & 3 & -84 \\ 0 & -11 & -5 & -3 \\ 0 & 0 & 183 & 915 \end{array}\right]$$

$$\begin{array}{r} -R_3 + 61R_1 \to R_1 \\ 5R_3 + 183R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{rrr|r} 2013 & 0 & 0 & -6039 \\ 0 & -2013 & 0 & 4026 \\ 0 & 0 & 183 & 915 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{2013}R_1 \to R_1 \\ -\frac{1}{2013}R_2 \to R_2 \\ \frac{1}{183}R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Read the solution from the last column of the matrix. The solution is $(-3, -2, 5)$.

37.
$$\begin{aligned} x + 2y \qquad - w &= 3 \\ 2x + \qquad 4z + 2w &= -6 \\ x + 2y - z \qquad &= 6 \\ 2x - y + z + w &= -3 \end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrrr|r} 1 & 2 & 0 & -1 & 3 \\ 2 & 0 & 4 & 2 & -6 \\ 1 & 2 & -1 & 0 & 6 \\ 2 & -1 & 1 & 1 & -3 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ -1R_1 + R_3 \to R_3 \\ -2R_1 + R_4 \to R_4 \end{array} \left[\begin{array}{rrrr|r} 1 & 2 & 0 & -1 & 3 \\ 0 & -4 & 4 & 4 & -12 \\ 0 & 0 & -1 & 1 & 3 \\ 0 & -5 & 1 & 3 & -9 \end{array}\right]$$

$$\begin{array}{r} R_2 + 2R_1 \to R_1 \\ \\ \\ -5R_2 + 4R_4 \to R_4 \end{array} \left[\begin{array}{rrrr|r} 2 & 0 & 4 & 2 & -6 \\ 0 & -4 & 4 & 4 & -12 \\ 0 & 0 & -1 & 1 & 3 \\ 0 & 0 & -16 & -8 & 24 \end{array}\right]$$

$$\begin{array}{r} 4R_3 + R_1 \to R_1 \\ 4R_3 + R_2 \to R_2 \\ \\ 16R_3 + (-1)R_4 \to R_4 \end{array} \left[\begin{array}{rrrr|r} 2 & 0 & 0 & 6 & 6 \\ 0 & -4 & 0 & 8 & 0 \\ 0 & 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 24 & 24 \end{array}\right]$$

$$\begin{array}{r} R_4+(-4)R_1 \to R_1 \\ R_4+(-3)R_2 \to R_2 \\ R_4+(-24)R_3 \to R_3 \\ \\ \end{array}\left[\begin{array}{cccc|c} -8 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 24 \\ 0 & 0 & 24 & 0 & -48 \\ 0 & 0 & 0 & 24 & 24 \end{array}\right]$$

$$\begin{array}{r} -\frac{1}{8}R_1 \to R_1 \\ \frac{1}{12}R_2 \to R_2 \\ \frac{1}{24}R_3 \to R_3 \\ \frac{1}{24}R_4 \to R_4 \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

The solution is $x = 0$, $y = 2$, $z = -2$, $w = 1$, or $(0, 2, -2, 1)$.

38.
$$\begin{aligned} x + 3y - 2z - w &= 9 \\ 2x + 4y \qquad + 2w &= 10 \\ -3x - 5y + 2z - w &= -15 \\ x - y - 3z + 2w &= 6 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 9 \\ 2 & 4 & 0 & 2 & 10 \\ -3 & -5 & 2 & -1 & -15 \\ 1 & -1 & -3 & 2 & 6 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1+R_2 \to R_2 \\ 3R_1+R_3 \to R_3 \\ -1R_1+R_4 \to R_4 \end{array}\left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 9 \\ 0 & -2 & 4 & 4 & -8 \\ 0 & 4 & -4 & -4 & 12 \\ 0 & -4 & -1 & 3 & -3 \end{array}\right]$$

$$\begin{array}{r} 3R_2+2R_1 \to R_1 \\ \\ 2R_2+R_3 \to R_3 \\ -2R_2+R_4 \to R_4 \end{array}\left[\begin{array}{cccc|c} 2 & 0 & 8 & 10 & -6 \\ 0 & -2 & 4 & 4 & -8 \\ 0 & 0 & 4 & 4 & -4 \\ 0 & 0 & -9 & -5 & 13 \end{array}\right]$$

$$\begin{array}{r} -2R_3+R_1 \to R_1 \\ -1R_3+R_2 \to R_2 \\ \\ 9R_3+4R_4 \to R_4 \end{array}\left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 2 \\ 0 & -2 & 0 & 0 & -4 \\ 0 & 0 & 4 & 4 & -4 \\ 0 & 0 & 0 & 16 & 16 \end{array}\right]$$

$$\begin{array}{r} R_4+(-8)R_1 \to R_1 \\ \\ R_4+(-4)R_3 \to R_3 \\ \\ \end{array}\left[\begin{array}{cccc|c} -16 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -4 \\ 0 & 0 & -16 & 0 & 32 \\ 0 & 0 & 0 & 16 & 16 \end{array}\right]$$

$$\begin{array}{r} -\frac{1}{16}R_1 \to R_1 \\ -\frac{1}{2}R_2 \to R_2 \\ -\frac{1}{16}R_3 \to R_3 \\ \frac{1}{16}R_4 \to R_4 \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

The solution is $(0, 2, -2, 1)$.

39.
$$\begin{aligned} x + y - z + 2w &= -20 \\ 2x - y + z + w &= 11 \\ 3x - 2y + z - 2w &= 27 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & -20 \\ 2 & -1 & 1 & 1 & 11 \\ 3 & -2 & 1 & -2 & 27 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3 \end{array}\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & -20 \\ 0 & -3 & 3 & -3 & 51 \\ 0 & -5 & 4 & -8 & 87 \end{array}\right]$$

$$\begin{array}{r} \\ -\frac{1}{3}R_2 \to R_2 \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & -20 \\ 0 & 1 & -1 & 1 & -17 \\ 0 & -5 & 4 & -8 & 87 \end{array}\right]$$

$$\begin{array}{r} -1R_2+R_1 \to R_1 \\ \\ 5R_2+R_3 \to R_3 \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -3 \\ 0 & 1 & -1 & 1 & -17 \\ 0 & 0 & -1 & -3 & 2 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_3 \to R_3 \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -3 \\ 0 & 1 & -1 & 1 & -17 \\ 0 & 0 & 1 & 3 & -2 \end{array}\right]$$

$$\begin{array}{r} \\ R_3+R_2 \to R_2 \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 4 & -19 \\ 0 & 0 & 1 & 3 & -2 \end{array}\right]$$

This is as far as we can go using row operations. To complete the solution, write the equations that correspond to the matrix.

$$\begin{aligned} x + w &= -3 \\ y + 4w &= -19 \\ z + 3w &= -2 \end{aligned}$$

Let w be the parameter and express x, y, and z in terms of w. From the equations above, $x = -w - 3$, $y = -4w - 19$, and $z = -3w - 2$.

The solution is $(-w - 3,\ -4w - 19,\ -3w - 2,\ w)$, where w is any real number.

40.
$$\begin{aligned} 4x - 3y + z + w &= 21 \\ -2x - y + 2z + 7w &= 2 \\ 10x \qquad - 5z - 20w &= 15 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 4 & -3 & 1 & 1 & 21 \\ -2 & -1 & 2 & 7 & 2 \\ 10 & 0 & -5 & -20 & 15 \end{array}\right]$$

Interchange rows 1 and 2.

$$\left[\begin{array}{rrrr|r} -2 & -1 & 2 & 7 & 2 \\ 4 & -3 & 1 & 1 & 21 \\ 10 & 0 & -5 & -20 & 15 \end{array}\right]$$

$$\begin{array}{l} \\ 2R_1 + R_2 \rightarrow R_2 \\ 5R_1 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrrr|r} -2 & -1 & 2 & 7 & 2 \\ 0 & -5 & 5 & 15 & 25 \\ 0 & -5 & 5 & 15 & 25 \end{array}\right]$$

$$\begin{array}{l} \\ -\frac{1}{5}R_2 \rightarrow R_2 \\ \frac{1}{5}R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrrr|r} -2 & -1 & 2 & 7 & 2 \\ 0 & 1 & -1 & -3 & -5 \\ 0 & -1 & 1 & 3 & 5 \end{array}\right]$$

$$\begin{array}{l} R_2 + R_1 \rightarrow R_1 \\ \\ R_2 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrrr|r} -2 & 0 & 1 & 4 & -3 \\ 0 & 1 & -1 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last row indicates that there are an infinite number of solutions. The remaining equations are

$$-2x + z + 4w = -3 \text{ and } y - z - 3w = -5$$

Solve these for x and y to obtain

$$x = \frac{z + 4w + 3}{2} \text{ and } y = z + 3w - 5.$$

There are an infinite number of solutions, each of the form

$$\left(\frac{z + 4w + 3}{2}, z + 3w - 5, z, w\right),$$

or

$$(1.5 + 0.5z + 2w, -5 + z + 3w, z, w),$$

for any real numbers z and w.

41.
$$\begin{aligned} 10.47x + 3.52y + 2.58z - 6.42w &= 218.65 \\ 8.62x - 4.93y - 1.75z + 2.83w &= 157.03 \\ 4.92x + 6.83y - 2.97z + 2.65w &= 462.3 \\ 2.86x + 19.10y - 6.24z - 8.73w &= 398.4 \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{rrrr|r} 10.47 & 3.52 & 2.58 & -6.42 & 218.65 \\ 8.62 & -4.93 & -1.75 & 2.83 & 157.03 \\ 4.92 & 6.83 & -2.97 & 2.65 & 462.3 \\ 2.86 & 19.10 & -6.24 & -8.73 & 398.4 \end{array}\right]$$

This exercise should be solved by graphing calculator or computer methods. The solution, which may vary slightly, is $x \approx 28.9436$, $y \approx 36.6326$, $z \approx 9.6390$, and $w \approx 37.1036$, or

$$(28.9436, 36.6326, 9.6390, 37.1036).$$

42.
$$\begin{aligned} 28.6x + 94.5y + 16.0z - 2.94w &= 198.3 \\ 16.7x + 44.3y - 27.3z + 8.9w &= 254.7 \\ 12.5x - 38.7y + 92.5z + 22.4w &= 562.7 \\ 40.1x - 28.3y + 17.5z - 10.2w &= 375.4 \end{aligned}$$

This exercise should be solved by graphing calculator or computer methods. The solution, which may vary slightly, is

$$(11.844, -1.153, 0.609, 14.004).$$

43. Insert the given values, introduce variables, and the table is as follows.

$\frac{3}{8}$	a	b
c	d	$\frac{1}{4}$
e	f	g

From this, we obtain the following system of equations.

$$\begin{aligned} a + b + \tfrac{3}{8} &= 1 \\ c + d + \tfrac{1}{4} &= 1 \\ e + f + g &= 1 \\ c + e + \tfrac{3}{8} &= 1 \\ a + d + f &= 1 \\ b + g + \tfrac{1}{4} &= 1 \\ d + g + \tfrac{3}{8} &= 1 \\ b + d + e &= 1 \end{aligned}$$

The augmented matrix and the final form after row operations are as follows.

$$\left[\begin{array}{ccccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{5}{8} \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & \frac{5}{8} \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & \frac{5}{8} \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{11}{24} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{5}{12} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{5}{24} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{7}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The solution to the system is read from the last column.

$$a = \frac{1}{6}, b = \frac{11}{24}, c = \frac{5}{12}, d = \frac{1}{3},$$
$$e = \frac{5}{24}, f = \frac{1}{2}, \text{ and } g = \frac{7}{24}$$

So the magic square is:

$\frac{3}{8}$	$\frac{1}{6}$	$\frac{11}{24}$
$\frac{5}{12}$	$\frac{1}{3}$	$\frac{1}{4}$
$\frac{5}{24}$	$\frac{1}{2}$	$\frac{7}{24}$

44. Let $x =$ the number of hours to hire the Garcia firm, and
$y =$ the number of hours to hire the Wong firm.

The system to be solved is

$$\begin{aligned} 10x + 20y &= 500 \quad (1) \\ 30x + 10y &= 750 \quad (2) \\ 5x + 10y &= 250. \quad (3) \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{cc|c} 10 & 20 & 500 \\ 30 & 10 & 750 \\ 5 & 10 & 250 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{10}R_1 \to R_1 \\ \frac{1}{10}R_2 \to R_2 \\ \frac{1}{5}R_3 \to R_3 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 50 \\ 3 & 1 & 75 \\ 1 & 2 & 50 \end{array}\right]$$

$$\begin{array}{l} \\ -3R_1 + R_2 \to R_2 \\ -1R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 50 \\ 0 & -5 & -75 \\ 0 & 0 & 0 \end{array}\right]$$

$$-\frac{1}{5}R_2 \to R_2 \left[\begin{array}{cc|c} 1 & 2 & 50 \\ 0 & 1 & 15 \\ 0 & 0 & 0 \end{array}\right]$$

$$-2R_2 + R_1 \to R_1 \left[\begin{array}{cc|c} 1 & 0 & 20 \\ 0 & 1 & 15 \\ 0 & 0 & 0 \end{array}\right]$$

The solution is (20, 15). Hire the Garcia firm for 20 hr and the Wong firm for 15 hr.

45. Let $x =$ amount invested in U.S. savings bonds,
$y =$ amount invested in mutual funds, and
$z =$ amount invested in a money market account.

Since the total amount invested was \$10,000, $x + y + z = 10{,}000.$

Katherine invested twice as much in mutual funds as in savings bonds, so $y = 2x.$

The total return on her investments was \$470, so $0.025x + 0.06y + 0.045z = 470.$

The system to be solved is

$$\begin{aligned} x + y + z &= 10{,}000 \quad (1) \\ 2x - y &= 0 \quad (2) \\ 0.025x + 0.06y + 0.045z &= 470 \quad (3). \end{aligned}$$

Simplify the system by multiplying equation (3) by 1000.

$$\begin{array}{rrcl} & x + y + z &=& 10{,}000 \quad (1) \\ & 2x - y &=& 0 \quad (2) \\ 1000R_3 \to R_3 & 25x + 60y + 45z &=& 470{,}000 \quad (4) \end{array}$$

Eliminate x in equations (2) and (4).

$$\begin{array}{rrcl} & x + y + z &=& 10{,}000 \quad (1) \\ -2R_1 + R_2 \to R_2 & -3y - 2z &=& -20{,}000 \quad (5) \\ -25R_1 + R_3 \to R_3 & 35y + 20z &=& 220{,}000 \quad (6) \end{array}$$

Eliminate y in equation (6).

$$\begin{array}{rrcl} & x + y + z &=& 10{,}000 \quad (1) \\ & -3y - 2z &=& -20{,}000 \quad (5) \\ 35R_2 + 3R_3 \to R_3 & -10z &=& -40{,}000 \quad (7) \end{array}$$

Make each leading coefficient equal 1.

$$\begin{array}{rrcl} & x + y + z &=& 10{,}000 \quad (1) \\ -\frac{1}{3}R_2 \to R_2 & y + \frac{2}{3}z &=& \frac{20{,}000}{3} \quad (8) \\ -\frac{1}{10}R_3 \to R_3 & z &=& 4000 \quad (9) \end{array}$$

Substitute 4000 for z in equation (8) to get $y = 4000$. Finally, substitute 4000 for 2 and 4000 for y in equation (1) to get $x = 2000$. Ms. Chong invested \$2000 in U.S. savings bonds, \$4000 in mutual funds, and \$4000 in a money market account.

46. Let $x =$ the number of inkjet printers purchased
$y =$ the number of LCD monitors purchased
$z =$ the number of memory chips purchased

We want to solve the following system.

$$\begin{aligned} x + y + z &= 46 \\ 109x + 129y + 89z &= 4774 \\ z &= 2y \end{aligned}$$

Write the augmented matrix and transform the matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 46 \\ 109 & 129 & 89 & 4774 \\ 0 & -2 & 1 & 0 \end{array}\right]$$

$$-109R_1 + R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 46 \\ 0 & 20 & -20 & -240 \\ 0 & -2 & 1 & 0 \end{array}\right]$$

$$\frac{1}{20}R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 46 \\ 0 & 1 & -1 & -12 \\ 0 & -2 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} -1R_2 + R_1 \to R_1 \\ \\ 2R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 58 \\ 0 & 1 & -1 & -12 \\ 0 & 0 & -1 & -24 \end{array}\right]$$

$$\begin{array}{r} 2R_3 + R_1 \to R_1 \\ -1R_3 + R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & -1 & -24 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 24 \end{array}\right]$$

The solution is (10, 12, 24). Pyro-Tech, Inc. purchases 10 inkjet printers, 12 LCD monitors, and 24 memory chips.

47. Let x = the number of chairs produced each week,
y = the number of cabinets produced each week, and
z = the number of buffets produced each week.

Make a table to organize the information.

	Chair	Cabinet	Buffet	Totals
Cutting	0.2	0.5	0.3	1950
Assembly	0.3	0.4	0.1	1490
Finishing	0.1	0.6	0.4	2160

The system to be solved is

$$\begin{aligned} 0.2x + 0.5y + 0.3z &= 1950 \\ 0.3x + 0.4y + 0.1z &= 1490 \\ 0.1x + 0.6y + 0.4z &= 2160. \end{aligned}$$

Write the augmented matrix of the system

$$\left[\begin{array}{ccc|c} 0.2 & 0.5 & 0.3 & 1950 \\ 0.3 & 0.4 & 0.1 & 1490 \\ 0.1 & 0.6 & 0.4 & 2160 \end{array}\right]$$

$$\begin{array}{r} 10R_1 \to R_1 \\ 10R_2 \to R_2 \\ 10R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 5 & 3 & 19{,}500 \\ 3 & 4 & 1 & 14{,}900 \\ 1 & 6 & 4 & 21{,}600 \end{array}\right]$$

Interchange rows 1 and 3.

$$\left[\begin{array}{ccc|c} 1 & 6 & 4 & 21{,}600 \\ 3 & 4 & 1 & 14{,}900 \\ 2 & 5 & 3 & 19{,}500 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 6 & 4 & 21{,}600 \\ 0 & -14 & -11 & -49{,}900 \\ 0 & -7 & -5 & -23{,}700 \end{array}\right]$$

$$\begin{array}{r} \\ -\frac{1}{14}R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 6 & 4 & 21{,}600 \\ 0 & 1 & \frac{11}{14} & \frac{24{,}950}{7} \\ 0 & -7 & -5 & -23{,}700 \end{array}\right]$$

$$\begin{array}{r} -6R_2 + R_1 \to R_1 \\ \\ 7R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{7} & \frac{1500}{7} \\ 0 & 1 & \frac{11}{14} & \frac{24{,}950}{7} \\ 0 & 0 & \frac{1}{2} & 1250 \end{array}\right]$$

$$\begin{array}{r} \\ \\ 2R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{7} & \frac{1500}{7} \\ 0 & 1 & \frac{11}{14} & \frac{24{,}950}{7} \\ 0 & 0 & 1 & 2500 \end{array}\right]$$

$$\begin{array}{r} \frac{5}{7}R_3 + R_1 \to R_1 \\ -\frac{11}{14}R_3 + R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2000 \\ 0 & 1 & 0 & 1600 \\ 0 & 0 & 1 & 2500 \end{array}\right]$$

The solution is (2000, 1600, 2500). Therefore, 2000 chairs, 1600 cabinets, and 2500 buffets should be produced.

48. Let x = the number of vans to be purchased,
y = the number of small trucks to be purchased, and
z = the number of large trucks to be purchased.

The system to be solved is

$$\begin{aligned} x + y + z &= 200 \\ 35{,}000x + 30{,}000y + 50{,}000z &= 7{,}000{,}000 \\ x &= 2y. \end{aligned}$$

To simplify the system, divide the second equation by 1000. Write the system in proper form, obtain the augmented matrix, and use row operations to solve.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 35 & 30 & 50 & 7000 \\ 1 & -2 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -35R_1 + R_2 \to R_2 \\ -1R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 0 & -5 & 15 & 0 \\ 0 & -3 & -1 & -200 \end{array}\right]$$

$$\begin{array}{r} R_2 + 5R_1 \to R_1 \\ \\ -3R_2 + 5R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 5 & 0 & 20 & 1000 \\ 0 & -5 & 15 & 0 \\ 0 & 0 & -50 & -1000 \end{array}\right]$$

$$\begin{array}{r} 2R_3 + 5R_1 \to R_1 \\ 3R_3 + 10R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 25 & 0 & 0 & 3000 \\ 0 & -50 & 0 & -3000 \\ 0 & 0 & -50 & -1000 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{25}R_1 \to R_1 \\ -\frac{1}{50}R_2 \to R_2 \\ -\frac{1}{50}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 120 \\ 0 & 1 & 0 & 60 \\ 0 & 0 & 1 & 20 \end{array}\right]$$

The solution is (120, 60, 20). U-Drive Rent-A-Truck should buy 120 vans, 60 small trucks, and 20 large trucks.

49. Let x = the amount borrowed at 8%,
y = the amount borrowed at 9%, and
z = the amount borrowed at 10%.

(a) The system to be solved is

$$\begin{aligned} x + y + z &= 25{,}000 \\ 0.08x + 0.09y + 0.10z &= 2190 \\ y &= z + 1000 \end{aligned}$$

Multiply the second equation by 100 and rewrite the equations in standard form.

$$\begin{aligned} x + y + z &= 25{,}000 \\ 8x + 9y + 10z &= 219{,}000 \\ y - z &= 1000. \end{aligned}$$

Write the augmented matrix and use row operations to solve

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25{,}000 \\ 8 & 9 & 10 & 219{,}000 \\ 0 & 1 & -1 & 1000 \end{array}\right]$$

$$-8R_1 + R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25{,}000 \\ 0 & 1 & 2 & 19{,}000 \\ 0 & 1 & -1 & 1000 \end{array}\right]$$

$$\begin{array}{l} -1R_2 + R_1 \to R_1 \\ \\ -1R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 6000 \\ 0 & 1 & 2 & 19{,}000 \\ 0 & 0 & -3 & -18{,}000 \end{array}\right]$$

$$\begin{array}{l} -1R_3 + 3R_1 \to R_1 \\ 2R_3 + 3R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 3 & 0 & 0 & 36{,}000 \\ 0 & 3 & 0 & 21{,}000 \\ 0 & 0 & -3 & -18{,}000 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{3}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \\ \frac{1}{3}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12{,}000 \\ 0 & 1 & 0 & 7000 \\ 0 & 0 & 1 & 6000 \end{array}\right]$$

The solution is (12,000, 7000, 6000). The company borrowed \$12,000 at 8%, \$7000 at 9%, and \$6000 at 10%.

(b) If the condition is dropped, the initial augmented matrix and solution is found as before.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25{,}000 \\ 8 & 9 & 10 & 219{,}000 \end{array}\right]$$

$$-8R_1 + R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25{,}000 \\ 0 & 1 & 2 & 19{,}000 \end{array}\right]$$

$$-1R_2 + R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & 0 & -1 & 6000 \\ 0 & 1 & 2 & 19{,}000 \end{array}\right]$$

This gives the system of equations

$$\begin{aligned} x &= z + 6000 \\ y &= -2x + 19{,}000 \end{aligned}$$

Since all values must be nonnegative,

$$\begin{array}{lcl} z + 6000 \geq 0 & \text{and} & -2z + 19{,}000 \geq 0 \\ z \geq -6000 & & z \leq 9500. \end{array}$$

The second inequality produces the condition that the amount borrowed at 10% must be less than or equal to \$9500. If \$5000 is borrowed at 10%, $z = 5000$, and

$$\begin{aligned} x &= 500 + 6000 = 11{,}000 \\ y &= -2(5000) + 19{,}000 = 9000. \end{aligned}$$

This means \$11,000 is borrowed at 8% and \$9000 is borrowed at 9%.

(c) The original conditions resulted in \$12,000 borrowed at 8%. So, if the bank sets a maximum of \$10,000 at the 8% rate, no solution is possible.

(d) The total interest would be

$$\begin{aligned} &0.08(10{,}000) + 0.09(8000) + 0.10(7000) \\ &= 800 + 720 + 700 \\ &= 2220 \end{aligned}$$

or \$2220, which is not the \$2190 interest as specified as one of the conditions of the problem.

50. **(a)** Let x = the number of deluxe models
y = the number of super-deluxe models
z = the number of ultra models

Make a table to organize the information.

	Deluxe	Super-Deluxe	Ultra	Totals
Electronic	2	1	2	54
Assembly	3	3	2	72
Finishing	5	2	6	148

We want to solve the following system.

$$\begin{aligned} 2x + y + 2z &= 54 \\ 3x + 3y + 2z &= 72 \\ 5x + 2y + 6z &= 148 \end{aligned}$$

Write the augmented matrix and transform the matrix.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 54 \\ 3 & 3 & 2 & 72 \\ 5 & 2 & 6 & 148 \end{array}\right]$$

$$\tfrac{1}{2}R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 27 \\ 3 & 3 & 2 & 72 \\ 5 & 2 & 6 & 148 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1 + R_2 \to R_2 \\ -5R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 27 \\ 0 & \frac{3}{2} & -1 & -9 \\ 0 & -\frac{1}{2} & 1 & 13 \end{array}\right]$$

$$\tfrac{2}{3}R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 27 \\ 0 & 1 & -\frac{2}{3} & -6 \\ 0 & -\frac{1}{2} & 1 & 13 \end{array}\right]$$

$$\begin{array}{r} -\frac{1}{2}R_2 + R_1 \to R_1 \\ \\ \frac{1}{2}R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 30 \\ 0 & 1 & -\frac{2}{3} & -6 \\ 0 & 0 & \frac{2}{3} & 10 \end{array}\right]$$

$$\tfrac{3}{2}R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 30 \\ 0 & 1 & -\frac{2}{3} & -6 \\ 0 & 0 & 1 & 15 \end{array}\right]$$

$$\begin{array}{r} -\frac{4}{3}R_3 + R_1 \to R_1 \\ \frac{2}{3}R_3 + R_2 \to R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 15 \end{array}\right]$$

The solution is (10, 4, 15). Each week 10 deluxe models, 4 super-deluxe models, and 15 ultra models should be produced.

(b) We want to solve the following system.

$$\begin{aligned} 2x + y + 2z &= 54 \\ 3x + 3y + 2z &= 72 \\ 5x + y + 6z &= 148 \end{aligned}$$

Write the augmented matrix and transform the matrix.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 54 \\ 3 & 3 & 2 & 72 \\ 5 & 1 & 6 & 148 \end{array}\right]$$

$$\tfrac{1}{2}R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 27 \\ 3 & 3 & 2 & 72 \\ 5 & 1 & 6 & 148 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1 + R_2 \to R_2 \\ -5R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 27 \\ 0 & \frac{3}{2} & -1 & -9 \\ 0 & -\frac{3}{2} & 1 & 13 \end{array}\right]$$

$$\tfrac{2}{3}R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 27 \\ 0 & 1 & -\frac{2}{3} & -6 \\ 0 & -\frac{3}{2} & 1 & 13 \end{array}\right]$$

$$\begin{array}{r} -\frac{1}{2}R_2 + R_1 \to R_1 \\ \\ \frac{3}{2}R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 30 \\ 0 & 1 & -\frac{2}{3} & -6 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

The last row indicates that there is no solution to the system.

(c) We want to solve the following system.

$$\begin{aligned} 2x + y + 2z &= 54 \\ 3x + 3y + 2z &= 72 \\ 5x + y + 6z &= 144 \end{aligned}$$

Write the augmented matrix and transform the matrix.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 54 \\ 3 & 3 & 2 & 72 \\ 5 & 1 & 6 & 148 \end{array}\right]$$

$$\tfrac{1}{2}R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 27 \\ 3 & 3 & 2 & 72 \\ 5 & 1 & 6 & 144 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1 + R_2 \to R_2 \\ -5R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 27 \\ 0 & \frac{3}{2} & -1 & -9 \\ 0 & -\frac{3}{2} & 1 & 9 \end{array}\right]$$

$$\tfrac{2}{3}R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 27 \\ 0 & 1 & -\frac{2}{3} & -6 \\ 0 & -\frac{3}{2} & 1 & 9 \end{array}\right]$$

$$\begin{array}{l} -\frac{1}{2}R_2 + R_1 \to R_1 \\ \\ \frac{3}{2}R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 30 \\ 0 & 1 & -\frac{2}{3} & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is dependent. Let z be the parameter and solve the first two equations for x and y, which gives

$$x = 30 - \frac{4}{3}z \text{ and } y = \frac{2}{3}z - 6.$$

Since x, y, and z must be nonnegative integers, we have

$$\begin{array}{rcl} 30 - \frac{4}{3}z \ge 0 & \quad & \frac{2}{3}z - 6 \ge 0 \\ 30 \ge \frac{4}{3}z & & \frac{2}{3}z \ge 6 \\ 22.5 \le z & & z \ge 9 \end{array}$$

Thus, z is an integer such that $9 \le z \le 22.5$, and z must be a multiple of 3 so that x and y are integers. The permissible values of z are 9, 12, 15, 18, and 21. There are 5 solutions.

51. **(a)** Let x be the number of trucks used, y be the number of vans, and z be the number of SUVs. We first obtain the equations given here.

$$\begin{aligned} 2x + 3y + 3z &= 25 \\ 2x + 4y + 5z &= 33 \\ 3x + 2y + z &= 22 \end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{ccc|c} 2 & 3 & 3 & 25 \\ 2 & 4 & 5 & 33 \\ 3 & 2 & 1 & 22 \end{array}\right]$$

$$\begin{array}{r} \\ -1R_1 + R_2 \to R_2 \\ -3R_1 + 2R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 3 & 3 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & -5 & -7 & -31 \end{array}\right]$$

$$\begin{array}{r} -3R_2 + R_1 \to R_1 \\ \\ 5R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 3 & 9 \end{array}\right]$$

$$\begin{array}{r} R_3 + R_1 \to R_1 \\ -2R_3 + 3R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 10 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 3 & 9 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{2}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \\ \frac{1}{3}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Read the solution from the last column of the matrix. The solution is 5 trucks, 2 vans, and 3 SUVs.

(b) The system of equations is now

$$\begin{aligned} 2x + 3y + 3z &= 25 \\ 2x + 4y + 5z &= 33. \end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{ccc|c} 2 & 3 & 3 & 25 \\ 2 & 4 & 5 & 33 \end{array}\right]$$

$$\begin{array}{r} \\ -R_1 + R_2 \to R_2 \end{array} \left[\begin{array}{ccc|c} 2 & 3 & 3 & 25 \\ 0 & 1 & 2 & 8 \end{array}\right]$$

$$-3R_2 + R_1 \to R_1 \left[\begin{array}{ccc|c} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 8 \end{array}\right]$$

Obtain a one in row 1, column 1.

$$\frac{1}{2}R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 2 & 8 \end{array}\right]$$

The last row indicates multiple solutions are possible. The remaining equations are

$$x - \frac{3}{2}z = \frac{1}{2} \quad \text{and} \quad y + 2z = 8.$$

Solving these for x and y, we have

$$x = \frac{3}{2}z + \frac{1}{2} \text{ and } y = -2z + 8.$$

The form of the solution is

$$\left(\frac{3}{2}z + \frac{1}{2},\ -2z + 8,\ z\right).$$

Since the solutions must be whole numbers,

$$\begin{array}{rcr} \frac{3}{2}z + \frac{1}{2} \ge 0 & \text{and} & -2z + 8 \ge 0 \\ & & -2z \ge -8 \\ \frac{3}{2}z \ge -\frac{1}{2} & & z \le 4 \\ z \ge -\frac{1}{3} & & \end{array}$$

Thus, there are 4 possible solutions but each must be checked to determine if they produce whole numbers for x and y.

When $z = 0, \left(\frac{1}{2}, 8, 0\right)$ which is not realistic.

When $z = 1, (2, 6, 1)$.

When $z = 2, \left(\frac{7}{2}, 4, 2\right)$ which is not realistic.

When $z = 3, (5, 2, 3)$.

When $z = 4$, $\left(\frac{13}{2}, 0, 4\right)$ which is not realistic.

The company has 2 options. Either use 2 trucks, 6 vans, and 1 SUV or use 5 trucks, 2 vans, and 3 SUVs.

52. Let $x =$ the number of two-person tents,
$y =$ the number of four-person tents, and
$z =$ the number of six-person tents that were ordered.

(a) The problem is to solve the following system of equations.

$$\begin{aligned} 2x + 4y + 6z &= 200 \\ 40x + 64y + 88z &= 3200 \\ 129x + 179y + 229z &= 8950 \end{aligned}$$

Write the augmented matrix and use row operations to solve.

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 200 \\ 40 & 64 & 88 & 3200 \\ 129 & 179 & 229 & 8950 \end{array}\right]$$

$$\begin{array}{r} \\ 20R_1 + (-1)R_2 \to R_2 \\ 129R_1 + (-2)R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 4 & 6 & 200 \\ 0 & 16 & 32 & 800 \\ 0 & 158 & 316 & 7900 \end{array}\right]$$

$$\begin{array}{r} R_2 + (-4)R_1 \to R_1 \\ \\ 79R_2 + (-8)R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} -8 & 0 & 8 & 0 \\ 0 & 16 & 32 & 800 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} -\frac{1}{8}R_1 \to R_1 \\ \\ \frac{1}{16}R_2 \to R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 50 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the last row is all zeros, there is more than one solution. Let z be the parameter. The matrix gives

$$\begin{aligned} x - z &= 0 \\ y + 2z &= 50. \end{aligned}$$

Solving these equations for x and y, the solution is $(z, -2z + 50, z)$. The numbers in the solution must be nonnegative integers. Therefore,

$$\begin{aligned} y &\geq 0 \\ -2z + 50 &\geq 0 \\ z &\leq 25. \end{aligned}$$

Thus, $z \in \{0, 1, 2, 3, \ldots, 25\}$ In other words, depending on the number of six-person tents, there are 26 solutions to this problem.

(b) The number of four-person tents is given by the value of the variable y. Since $y = -2z + 50$, the most four-person tents will result when z is as small as possible, or 0. When this occurs, $y = -2(0) + 50 = 50$. And since $x = z$, the solution with the most four-person tents is 0 two-person tents, 50 four-person tents, and 0 six-person tents.

(c) The number of two-person tents is given by the value of the variable x. Since $x = z$, the most two-person tents will result when y is as small as possible, or 0. When this occurs,

$$\begin{aligned} 2x + 4y + 6z &= 200 \\ 2(z) + 4(0) + 6(z) &= 200 \\ 8z &= 200 \\ z &= 25. \end{aligned}$$

The solution with the most two-person tents is 25 two-person tents, 0 four-person tents, and 25 six-person tents.

53. Let $x_1 =$ the number of units from first supplier for Roseville,
$x_2 =$ the number of units from first supplier for Akron,
$x_3 =$ the number of units from second supplier for Roseville, and
$x_4 =$ the number of units from second supplier for Akron.

Roseville needs 40 units so

$$x_1 + x_3 = 40.$$

Akron needs 75 units so

$$x_2 + x_4 = 75.$$

The manufacturer orders 75 units from the first supplier so

$$x_1 + x_2 = 75.$$

The total cost is \$10,750 so

$$70x_1 + 90x_2 + 80x_3 + 120x_4 = 10{,}750.$$

The system to be solved is

$$\begin{aligned} x_1 + x_3 &= 40 \\ x_2 + x_4 &= 75 \\ x_1 + x_2 &= 75 \\ 70x_1 + 90x_2 + 80x_3 + 120x_4 &= 10{,}750. \end{aligned}$$

Write augmented matrix and use row operations.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40 \\ 0 & 1 & 0 & 1 & 75 \\ 1 & 1 & 0 & 0 & 75 \\ 70 & 90 & 80 & 120 & 10{,}750 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_1 + R_3 \to R_3 \\ -70R_1 + R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40 \\ 0 & 1 & 0 & 1 & 75 \\ 0 & 1 & -1 & 0 & 35 \\ 0 & 90 & 10 & 120 & 7950 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_2 + R_3 \to R_3 \\ -90R_2 + R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40 \\ 0 & 1 & 0 & 1 & 75 \\ 0 & 0 & -1 & -1 & -40 \\ 0 & 0 & 10 & 30 & 1200 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_3 \to R_3 \\ 10R_3 + R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40 \\ 0 & 1 & 0 & 1 & 75 \\ 0 & 0 & 1 & 1 & 40 \\ 0 & 0 & 0 & 20 & 800 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ \frac{1}{20}R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40 \\ 0 & 1 & 0 & 1 & 75 \\ 0 & 0 & 1 & 1 & 40 \\ 0 & 0 & 0 & 1 & 40 \end{array}\right]$$

$$\begin{array}{r} \\ -1R_4 + R_2 \to R_2 \\ -1R_4 + R_3 \to R_3 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 40 \\ 0 & 1 & 0 & 0 & 35 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 40 \end{array}\right]$$

$$\begin{array}{r} -1R_3 + R_1 \to R_1 \\ \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 40 \\ 0 & 1 & 0 & 0 & 35 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 40 \end{array}\right]$$

The solution of the system is $x_1 = 40$, $x_2 = 35$, $x_3 = 0$, $x_4 = 40$, or (40, 35, 0, 40). The first supplier should send 40 units to Roseville and 35 units to Akron. The second supplier should send 0 units to Roseville and 40 units to Akron.

54. Let $x_1 =$ the number of cars sent from I to A,

$x_2 =$ the number of cars sent from II to A,

$x_3 =$ the number of cars sent from I to B, and

$x_4 =$ the number of cars sent from II to B.

	A	B
I	x_1	x_3
II	x_2	x_4

Plant I has 28 cars, so

$$x_1 + x_3 = 28.$$

Plant II has 8 cars, so

$$x_2 + x_4 = 8.$$

Dealer A needs 20 cars, so

$$x_1 + x_2 = 20$$

Dealer B needs 16 cars, so

$$x_3 + x_4 = 16.$$

The total transportation cost is \$10,640, so

$$220x_1 + 400x_2 + 300x_3 + 180x_4 = 10{,}640.$$

The system to be solved is

$$\begin{aligned} x_1 \qquad + x_3 \qquad &= 28 \\ x_2 \qquad + x_4 &= 8 \\ x_1 + x_2 \qquad\qquad &= 20 \\ x_3 + x_4 &= 16 \\ 220x_1 + 400x_2 + 300x_3 + 180x_4 &= 10{,}640. \end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 28 \\ 0 & 1 & 0 & 1 & 8 \\ 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & 1 & 16 \\ 220 & 400 & 300 & 180 & 10{,}640 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_1 + R_3 \to R_3 \\ \\ -220R_1 + R_5 \to R_5 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 28 \\ 0 & 1 & 0 & 1 & 8 \\ 0 & 1 & -1 & 0 & -8 \\ 0 & 0 & 1 & 1 & 16 \\ 0 & 400 & 80 & 180 & 4480 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_2 + R_3 \to R_3 \\ \\ -400R_2 + R_5 \to R_5 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 28 \\ 0 & 1 & 0 & 1 & 8 \\ 0 & 0 & -1 & -1 & -16 \\ 0 & 0 & 1 & 1 & 16 \\ 0 & 0 & 80 & -220 & 1280 \end{array}\right]$$

$$\begin{array}{r} R_1 + R_3 \to R_1 \\ \\ \\ R_3 + R_4 \to R_4 \\ 80R_3 + R_5 \to R_5 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 12 \\ 0 & 1 & 0 & 1 & 8 \\ 0 & 0 & -1 & -1 & -16 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -300 & 0 \end{array}\right]$$

There is a 0 now in row 4, column 4, where we would like to get a 1. To proceed, interchange the fourth and fifth rows.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 12 \\ 0 & 1 & 0 & 1 & 8 \\ 0 & 0 & -1 & -1 & -16 \\ 0 & 0 & 0 & -300 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} -1R_4+300R_1\to R_1\\ R_4+300R_2\to R_2\\ -1R_4+300R_3\to R_3\\ \\ \\ \end{array}\left[\begin{array}{rrrr|r} 300&0&0&0&3600\\ 0&300&0&0&2400\\ 0&0&-300&0&-4800\\ 0&0&0&-300&0\\ 0&0&0&0&0 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{300}R_1\to R_1\\ \frac{1}{300}R_2\to R_2\\ -\frac{1}{300}R_3\to R_3\\ -\frac{1}{300}R_4\to R_4\\ \\ \end{array}\left[\begin{array}{rrrr|r} 1&0&0&0&12\\ 0&1&0&0&8\\ 0&0&1&0&16\\ 0&0&0&1&0\\ 0&0&0&0&0 \end{array}\right]$$

Each of the original variables has a value, so the last row of all zeros may be ignored. The solution of the system is $x_1 = 12$, $x_2 = 8$, $x_3 = 16$, $x_4 = 0$. Therefore, 12 cars should be sent from I to A, 8 cars from II to A, 16 cars from I to B, and no cars from II to B.

55. Let $w =$ the number of Italian style vegetable packages
$x =$ the number of French style vegetable packages
$y =$ the number of Oriental style vegetable packages
$z =$ the number of Combo style vegetable packages

The system to be solved is

zucchini	$0.3w + 0.2y = 16,200$
broccoli	$0.3w + 0.5x + 0.3y + 0.6z = 42,300$
carrots	$0.4w + 0.2x + 0.2y = 21,000$
cauliflower	$0.3x + 0.3y + 0.4z = 29,500$

Expressing the amount of each vegetable in thousands and multiplying each equation by 10 and forming the augmented matrix we get

$$\left[\begin{array}{rrrr|r} 3&0&2&0&162\\ 3&5&3&6&423\\ 4&2&2&0&210\\ 0&3&3&4&295 \end{array}\right]$$

$$(-1)R_1 + R_2 \to R_2 \quad \left[\begin{array}{rrrr|r} 3&0&2&0&162\\ 0&5&1&6&261\\ 4&2&2&0&210\\ 0&3&3&4&295 \end{array}\right]$$

$$(-\frac{4}{3})R_1 + R_3 \to R_3 \quad \left[\begin{array}{rrrr|r} 3&0&2&0&162\\ 0&5&1&6&261\\ 0&2&-\frac{2}{3}&0&-6\\ 0&3&3&4&295 \end{array}\right]$$

$$-\frac{1}{5}R_2 \to R_2 \quad \left[\begin{array}{rrrr|r} 3&0&2&0&162\\ 0&1&\frac{1}{5}&\frac{6}{5}&\frac{261}{5}\\ 0&2&-\frac{2}{3}&0&-6\\ 0&3&3&4&295 \end{array}\right]$$

$$(-2)R_2 + R_3 \to R_3 \quad \left[\begin{array}{rrrr|r} 3&0&2&0&162\\ 0&1&\frac{1}{5}&\frac{6}{5}&\frac{261}{5}\\ 0&0&-\frac{16}{15}&-\frac{12}{5}&-\frac{552}{5}\\ 0&3&3&4&295 \end{array}\right]$$

$$(-3)R_2 + R_4 \to R_4 \quad \left[\begin{array}{rrrr|r} 3&0&2&0&162\\ 0&1&\frac{1}{5}&\frac{6}{5}&\frac{261}{5}\\ 0&0&-\frac{16}{15}&-\frac{12}{5}&-\frac{552}{5}\\ 0&0&\frac{12}{5}&\frac{2}{5}&295 \end{array}\right]$$

$$\left(\frac{9}{4}\right)R_3 + R_4 \to R_4 \quad \left[\begin{array}{rrrr|r} 3&0&2&0&162\\ 0&1&\frac{1}{5}&\frac{6}{5}&\frac{261}{5}\\ 0&0&-\frac{16}{15}&-\frac{12}{5}&-\frac{552}{5}\\ 0&0&0&-5&-110 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{3}R_1\to R_1\\ \\ -\frac{15}{16}R_3\to R_3\\ -\frac{1}{5}R_4\to R_4 \end{array} \quad \left[\begin{array}{rrrr|r} 1&0&\frac{2}{3}&0&54\\ 0&1&\frac{1}{5}&\frac{6}{5}&\frac{261}{5}\\ 0&0&1&\frac{9}{4}&\frac{207}{2}\\ 0&0&0&1&22 \end{array}\right]$$

$$\frac{9}{4}R_3 + R_1 \to R_1 \quad \left[\begin{array}{rrrr|r} 1&0&0&-\frac{3}{2}&-15\\ 0&1&\frac{1}{5}&\frac{6}{5}&\frac{261}{5}\\ 0&0&1&\frac{9}{4}&\frac{207}{2}\\ 0&0&0&1&22 \end{array}\right]$$

$$\left(-\frac{1}{5}\right)R_3 + R_2 \to R_2 \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{3}{2} & -15 \\ 0 & 1 & 0 & \frac{3}{4} & \frac{63}{2} \\ 0 & 0 & 1 & \frac{9}{4} & \frac{207}{2} \\ 0 & 0 & 0 & 1 & 22 \end{array}\right]$$

$$\begin{array}{l} \left(\frac{3}{2}\right)R_4 + R_1 \to R_1 \\ \left(-\frac{3}{4}\right)R_4 + R_2 \to R_2 \\ \left(-\frac{9}{4}\right)R_4 + R_3 \to R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 18 \\ 0 & 1 & 0 & 0 & 15 \\ 0 & 0 & 1 & 0 & 54 \\ 0 & 0 & 0 & 1 & 22 \end{array}\right]$$

Therefore, the company should produce 18,000 packages of Italian style, 15,000 packges of French style, 54,000 packages of Oriental style, and 22,000 packages of Combo style.

56. Let $w =$ the number of Triple Berry packages
$x =$ the number of Summer Blend packages
$y =$ Berry Banana packages

$z =$ Strawberry Banana Chunk packages

The system to be solved is

strawberries $6w + 4x + 4y + 10z = 302,400$
raspberries $5w + 4x + 3y = 153,700$
blueberries $5w + 3y = 108,100$
bananas $8x + 6y + 6z = 255,000$

The augmented matrix of this system is

$$\left[\begin{array}{cccc|c} 6 & 4 & 4 & 10 & 302,400 \\ 5 & 4 & 3 & 0 & 153,700 \\ 5 & 0 & 3 & 0 & 108,100 \\ 0 & 8 & 6 & 6 & 255,000 \end{array}\right]$$

$$(-1)R_3 + R_2 \to R_2 \quad \left[\begin{array}{cccc|c} 6 & 4 & 4 & 10 & 302,400 \\ 0 & 4 & 0 & 0 & 45,600 \\ 5 & 0 & 3 & 0 & 108,100 \\ 0 & 8 & 6 & 6 & 255,000 \end{array}\right]$$

$$\begin{array}{l} (-1)R_2 + R_1 \to R_1 \\ (-2)R_2 + R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} 6 & 0 & 4 & 0 & 256,800 \\ 0 & 4 & 0 & 0 & 45,600 \\ 5 & 0 & 3 & 0 & 108,100 \\ 0 & 0 & 6 & 6 & 163,800 \end{array}\right]$$

$$\left(-\frac{5}{6}\right)R_1 + R_3 \to R_3 \quad \left[\begin{array}{cccc|c} 6 & 0 & 4 & 10 & 256,800 \\ 0 & 4 & 0 & 0 & 45,600 \\ 0 & 0 & -\frac{1}{3} & -\frac{25}{3} & 108,100 \\ 0 & 0 & 6 & 6 & 163,800 \end{array}\right]$$

$$\begin{array}{l} \left(\frac{1}{6}\right)R_1 \to R_1 \\ \left(\frac{1}{4}\right)R_2 \to R_2 \\ (-3)R_3 \to R_3 \\ \left(\frac{1}{6}\right)R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & \frac{5}{3} & 42,800 \\ 0 & 1 & 0 & 0 & 11,400 \\ 0 & 0 & 1 & 25 & 317,700 \\ 0 & 0 & 1 & 1 & 27,300 \end{array}\right]$$

$$\begin{array}{l} \left(-\frac{2}{3}\right)R_4 + R_1 \to R_1 \\ (-1)R_4 + R_2 \to R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 24,600 \\ 0 & 1 & 0 & 0 & 11,400 \\ 0 & 0 & 0 & 1 & 290,400 \\ 0 & 0 & 1 & 1 & 27,300 \end{array}\right]$$

$$\left(\frac{1}{24}\right)R_3 \to R_3 \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 24,600 \\ 0 & 1 & 0 & 0 & 11,400 \\ 0 & 0 & 0 & 1 & 12,100 \\ 0 & 0 & 1 & 1 & 27,300 \end{array}\right]$$

$$\begin{array}{l} (-1)R_3 + R_1 \to R_1 \\ (-1)R_3 + R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 12,500 \\ 0 & 1 & 0 & 0 & 11,400 \\ 0 & 0 & 0 & 1 & 12,100 \\ 0 & 0 & 1 & 0 & 15,200 \end{array}\right]$$

$$R_3 \leftrightarrow R_4 \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 12,500 \\ 0 & 1 & 0 & 0 & 11,400 \\ 0 & 0 & 1 & 0 & 15,200 \\ 0 & 0 & 0 & 1 & 12,100 \end{array}\right]$$

The company should prepare 12,500 packages of Triple Berry, 11,400 packages of Summer Blend, 15,200 packages of Berry Banana, and 12,100 packages of Strawberry Banana Chunk.

57. Let $x =$ the number of grams of group A,
$y =$ the number of grams of group B, and
$z =$ the number of grams of group C.

(a) The system to be solved is

$$\begin{aligned} x + y + z &= 400 \quad (1) \\ x &= \frac{1}{3}y \quad (2) \\ x + z &= 2y. \quad (3) \end{aligned}$$

Rewrite equations (2) and (3) in proper form and multiply both sides of equation (2) by 3.

$$\begin{aligned} x + \quad y + z &= 400 \\ 3x - \quad y \qquad &= 0 \\ x - 2y + z &= 0 \end{aligned}$$

Write the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 400 \\ 3 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ -3R_1 + R_2 \to R_2 \\ -1R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 400 \\ 0 & -4 & -3 & -1200 \\ 0 & -3 & 0 & -400 \end{array}\right]$$

$$-\tfrac{1}{3}R_3 \to R_3 \left[\begin{array}{rrr|r} 1 & 1 & 1 & 400 \\ 0 & -4 & -3 & -1200 \\ 0 & 1 & 0 & \frac{400}{3} \end{array}\right]$$

Interchange rows 2 and 3.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 400 \\ 0 & 1 & 0 & \frac{400}{3} \\ 0 & -4 & -3 & -1200 \end{array}\right]$$

$$\begin{array}{l} -1R_2 + R_1 \to R_1 \\ \\ 4R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 1 & \frac{800}{3} \\ 0 & 1 & 0 & \frac{400}{3} \\ 0 & 0 & -3 & -\frac{2000}{3} \end{array}\right]$$

$$-\tfrac{1}{3}R_3 \to R_3 \left[\begin{array}{rrr|r} 1 & 0 & 1 & \frac{800}{3} \\ 0 & 1 & 0 & \frac{400}{3} \\ 0 & 0 & 1 & \frac{2000}{9} \end{array}\right]$$

$$-1R_3 + R_1 \to R_1 \left[\begin{array}{rrr|r} 1 & 0 & 0 & \frac{400}{9} \\ 0 & 1 & 0 & \frac{400}{3} \\ 0 & 0 & 1 & \frac{2000}{9} \end{array}\right]$$

The solution is $\left(\frac{400}{9}, \frac{400}{3}, \frac{2000}{9}\right)$. Include $\frac{400}{9}$ g of group A, $\frac{400}{3}$ g of group B, and $\frac{2000}{9}$ g of group C.

(b) If the requirement that the diet include one-third as much of A as of B is dropped, refer to the first two rows of the fifth augmented matrix in part (a).

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & \frac{800}{3} \\ 0 & 1 & 0 & \frac{400}{3} \end{array}\right]$$

This gives

$$x = \frac{800}{3} - z$$
$$y = \frac{400}{3}.$$

Therefore, for any positive number z of grams of group C, there should be z grams less than $\frac{800}{3}$ g of group A and $\frac{400}{3}$ g of group B.

(c) Since there was a unique solution for the original problem, by adding an additional condition. the only possible solution would be the one from part (a). However, by substituting those values of A, B, and C for x, y, and z in the equation for the additional condition, $0.02x + 0.02y + 0.03z = 8.00$, the values do not work. Therefore, a solution is not possible.

58. Let x_1 = the number of cases of Brand A,
x_2 = the number of cases of Brand B,
x_3 = the number of cases of Brand C, and
x_4 = the number of cases of Brand D.

$$\begin{aligned} 25x_1 + 50x_2 + 75x_3 + 100x_4 &= 1200 \\ 30x_1 + 30x_2 + 30x_3 + 60x_4 &= 600 \\ 30x_1 + 20x_2 + 20x_3 + 30x_4 &= 400 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{rrrr|r} 25 & 50 & 75 & 100 & 1200 \\ 30 & 30 & 30 & 60 & 600 \\ 30 & 20 & 20 & 30 & 400 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{5}R_1 \to R_1 \\ \frac{1}{30}R_2 \to R_2 \\ \frac{1}{10}R_3 \to R_3 \end{array} \left[\begin{array}{rrrr|r} 5 & 10 & 15 & 20 & 240 \\ 1 & 1 & 1 & 2 & 20 \\ 3 & 2 & 2 & 3 & 20 \end{array}\right]$$

Interchange rows 1 and 2.

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 2 & 20 \\ 5 & 10 & 15 & 20 & 240 \\ 3 & 2 & 2 & 3 & 40 \end{array}\right]$$

$$\begin{array}{l} \\ -5R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 2 & 20 \\ 0 & 5 & 10 & 10 & 140 \\ 0 & -1 & -1 & -3 & -20 \end{array}\right]$$

$$\tfrac{1}{5}R_2 \to R_2 \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 2 & 20 \\ 0 & 1 & 2 & 2 & 28 \\ 0 & -1 & -1 & -3 & -20 \end{array}\right]$$

$$\begin{array}{l} -1R_2 + R_1 \to R_1 \\ \\ R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & -8 \\ 0 & 1 & 2 & 2 & 28 \\ 0 & 0 & 1 & -1 & 8 \end{array}\right]$$

$$\begin{array}{r} R_3 + R_1 \to R_1 \\ -2R_3 + R_2 \to R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 4 & 12 \\ 0 & 0 & 1 & -1 & 8 \end{array}\right]$$

We cannot change the values in column 4 further without changing the form of the other three columns. Therefore, let x_4 be arbitrary. This matrix gives the equations

$$\begin{aligned} x_1 - x_4 &= 0 \quad \text{or} \quad x_1 = x_4, \\ x_2 + 4x_4 &= 12 \quad \text{or} \quad x_2 = 12 - 4x_4, \\ x_3 - x_4 &= 8 \quad \text{or} \quad x_3 = 8 + x_4. \end{aligned}$$

The solution is $(x_4, 12 - 4x_4, 8 + x_4, x_4)$. Since all solutions must be nonnegative,

$$\begin{aligned} 12 - 4x_4 &\geq 0 \\ x_4 &\leq 3. \end{aligned}$$

If $x_4 = 0$, then $x_1 = 0$, $x_2 = 12$, and $x_3 = 8$.
If $x_4 = 1$, then $x_1 = 1$, $x_2 = 8$, and $x_3 = 9$.
If $x_4 = 2$, then $x_1 = 2$, $x_2 = 4$, and $x_3 = 10$.
If $x_4 = 3$, then $x_1 = 3$, $x_2 = 0$, and $x_3 = 11$.

Therefore, there are four possible solutions. The breeder should mix

1. 0 cases of A, 12 cases of B, 8 cases of C, and 0 cases of D;
2. 1 case of A, 8 cases of B, 9 cases of C, and 1 case of D;
3. 2 cases of A, 4 cases of B, 10 cases of C, and 2 cases of D; or
4. 3 cases of A, 0 cases of B, 11 cases of C, and 3 cases of D.

59. Let $x =$ the number of species A,
$y =$ the number of species B, and
$z =$ the number of species C.

Use a chart to organize the information.

		Species A	Species B	Species C	Totals
Food	I	1.32	2.1	0.86	490
	II	2.9	0.95	1.52	897
	III	1.75	0.6	2.01	653

The system to be solved is

$$\begin{aligned} 1.32x + 2.1y + 0.86z &= 490 \\ 2.9x + 0.95y + 1.52z &= 897 \\ 1.75x + 0.6y + 2.01z &= 653. \end{aligned}$$

Use graphing calculator or computer methods to solve this system. The solution, which may vary slightly, is to stock about 244 fish of species A, 39 fish of species B, and 101 fish of species C.

60. Let $x =$ the number of the first species,
$y =$ the number of the second species, and
$z =$ the number of the third species.

$$\begin{aligned} 1.3x + 1.1y + 8.1z &= 16{,}000 \\ 1.3x + 2.4y + 2.9z &= 28{,}000 \\ 2.3x + 3.7y + 5.1z &= 44{,}000 \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 1.3 & 1.1 & 8.1 & 16{,}000 \\ 1.3 & 2.4 & 2.9 & 28{,}000 \\ 2.3 & 3.7 & 5.1 & 44{,}000 \end{array}\right]$$

This exercise should be solved by graphing calculator or computer methods. The solution, which may vary slightly, is 2340 of the first species, 10,128 of the second species, and 224 of the third species. (All of these are rounded to the nearest whole number.)

61. Let $x =$ the number of acres for honeydews,
$y =$ the number of acres for yellow onions, and
$z =$ the number of acres for lettuce.

(a)
$$\begin{aligned} x + y + z &= 220 \\ 120x + 150y + 180z &= 29{,}100 \\ 180x + 80y + 80z &= 32{,}600 \\ 4.97x + 4.45y + 4.65z &= 480 \end{aligned}$$

Write the augmented matrix for this system.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 220 \\ 120 & 150 & 180 & 29{,}100 \\ 180 & 80 & 80 & 32{,}600 \\ 4.97 & 4.45 & 4.65 & 480 \end{array}\right]$$

Using graphing calculator or computer methods, we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 150 \\ 0 & 1 & 0 & 50 \\ 0 & 0 & 1 & 20 \\ 0 & 0 & 0 & -581 \end{array}\right].$$

There is no solution to the system. Therefore, it is not possible to utilize all resources completely.

(b) If 1061 hr of labor are available, the augmented matrix becomes,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 220 \\ 120 & 150 & 180 & 29{,}100 \\ 180 & 80 & 80 & 32{,}600 \\ 4.97 & 4.45 & 4.65 & 1061 \end{array}\right].$$

Again, using graphing calculator or computer methods we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 150 \\ 0 & 1 & 0 & 50 \\ 0 & 0 & 1 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The solution is (150, 50, 20). Therefore, allot 150 acres for honeydews, 50 acres for onions, and 20 acres for lettuce.

62. **(a)** In 1980, $t = 0$ and $R = 183.9$.

$$183.9 = a(0)^2 + b(0) + c$$
$$183.9 = c$$

In 1990, $t = 10$ and $R = 203.3$.

$$203.3 = a(10)^2 + b(10) + c$$
$$203.3 = 100a + 10b + c$$

In 2000, $t = 20$ and $R = 196.5$.

$$196.5 = a(20)^2 + b(20) + c$$
$$196.5 = 400a + 20b + c$$

The linear system to be solved is

$$\begin{aligned} 400a + 20b + c &= 196.5 \\ 100a + 10b + c &= 203.3 \\ c &= 183.9 \end{aligned}$$

Write the augmented matrix and use row operations to solve.

$$\left[\begin{array}{ccc|c} 400 & 20 & 1 & 196.5 \\ 100 & 10 & 1 & 203.3 \\ 0 & 0 & 1 & 183.9 \end{array}\right]$$

$$-1R_1 + 4R_2 \to R_2 \quad \left[\begin{array}{ccc|c} 400 & 20 & 1 & 196.5 \\ 0 & 20 & 3 & 616.7 \\ 0 & 0 & 1 & 183.9 \end{array}\right]$$

$$-1R_2 + R_1 \to R_1 \quad \left[\begin{array}{ccc|c} 400 & 0 & -2 & -420.2 \\ 0 & 20 & 3 & 616.7 \\ 0 & 0 & 1 & 183.9 \end{array}\right]$$

$$\begin{array}{c} 2R_3 + R_1 \to R_1 \\ -3R_3 + R_2 \to R_2 \end{array} \quad \left[\begin{array}{ccc|c} 400 & 0 & 0 & -52.4 \\ 0 & 20 & 0 & 65 \\ 0 & 0 & 1 & 183.9 \end{array}\right]$$

$$\begin{array}{c} \frac{1}{400}R_1 \to R_1 \\ \frac{1}{20}R_2 \to R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -0.131 \\ 0 & 1 & 0 & 3.25 \\ 0 & 0 & 1 & 183.9 \end{array}\right]$$

The solution is $a = -0.131$, $b = 3.25$, and $c = 183.9$.

(b) In 2010, $t = 30$.

$$\begin{aligned} R &= -0.131t^2 + 3.25t + 183.9 \\ &= -0.131(30)^2 + 3.25(30) + 183.9 \\ &\approx 163.5 \end{aligned}$$

The actual value is 186.2 deaths per 100,000.

(c) Let t equal 0, 10, 20, and 30 to obtain four equations involving the coefficients a, b, c, and d. The linear system to be solved is

$$\begin{aligned} 27,000a + 900b + 30c + d &= 186.2 \\ 800a + 400b + 20c + d &= 196.5 \\ 1000a + 100b + 10c + d &= 203.3 \\ d &= 183.9 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 27,000 & 900 & 30 & 1 & 186.2 \\ 8000 & 400 & 20 & 1 & 196.5 \\ 1000 & 100 & 10 & 1 & 203.3 \\ 0 & 0 & 0 & 1 & 183.9 \end{array}\right]$$

Use a graphing calculator or computer methods to obtain the solution $a = 0.003783$ $b = -0.2445$ $c = 4.007$, and $d = 183.9$.

63. **(a)** Bulls:

The number of white ones was one half plus one third the number of black greater than the brown.

$$X = \left(\frac{1}{2} + \frac{1}{3}\right)Y + T$$
$$X = \frac{5}{6}Y + T$$
$$6X - 5Y = 6T$$

The number of the black, one quarter plus one fifth the number of the spotted greater than the brown.

$$Y = \left(\frac{1}{4} + \frac{1}{5}\right)Z + T$$
$$Y = \frac{9}{20}Z + T$$
$$20Y = 9Z + 20T$$
$$20Y - 9Z = 20T$$

The number of the spotted, one sixth and one seventh the number of the white greater than the brown.

$$Z = \left(\frac{1}{6} + \frac{1}{7}\right)X + T$$

$$Z = \frac{13}{42}X + T$$

$$42Z = 13X + 42T$$

$$42Z - 13X = 42T$$

So the system of equations for the bulls is

$$\begin{aligned} 6X - 5Y &= 6T \\ 20Y - 9Z &= 20T \\ 42Z - 13X &= 42T. \end{aligned}$$

Cows:
The number of white ones was one third plus one quarter of the total black cattle.

$$x = \left(\frac{1}{3} + \frac{1}{4}\right)(Y + y)$$

$$x = \frac{7}{12}(Y + y)$$

$$12x = 7Y + 7y$$

$$12x - 7y = 7Y$$

The number of the black, one quarter plus one fifth the total of the spotted cattle.

$$y = \left(\frac{1}{4} + \frac{1}{5}\right)(Z + z)$$

$$y = \frac{9}{20}(Z + z)$$

$$20y = 9Z + 9z$$

$$20y - 9z = 9Z$$

The number of the spotted, one fifth plus one sixth the total of the brown cattle.

$$z = \left(\frac{1}{5} + \frac{1}{6}\right)(T + t)$$

$$z = \frac{11}{30}(T + t)$$

$$30z = 11T + 11t$$

$$30z - 11t = 11T$$

The number of the brown, one sixth plus one seventh the total of the white cattle.

$$t = \left(\frac{1}{6} + \frac{1}{7}\right)(X + x)$$

$$t = \frac{13}{42}(X + x)$$

$$42t = 13X + 13x$$

$$42t - 13x = 13X$$

So the system of equations for the cows is

$$\begin{aligned} 12x - 7y &= 7Y \\ 20y - 9z &= 9Z \\ 30z - 11t &= 11T \\ -13x + 42t &= 13X \end{aligned}$$

(b) For $T = 4{,}149{,}387$, the 3×3 system to be solved is

$$\begin{aligned} 6X - 5Y \qquad &= 24{,}896{,}322 \\ 20Y - 9Z &= 82{,}987{,}740 \\ -13X \qquad + 42Z &= 174{,}274{,}254 \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{rrr|r} 6 & -5 & 0 & 24{,}896{,}322 \\ 0 & 20 & -9 & 82{,}987{,}740 \\ -13 & 0 & 42 & 174{,}274{,}254 \end{array}\right]$$

This exercise should be solved by graphing calculator or computer methods. The solution is $X = 10{,}366{,}482$ white bulls, $Y = 7{,}460{,}514$ black bulls, and $Z = 7{,}358{,}060$ spotted bulls.

For $X = 10{,}366{,}482$, $Y = 7{,}460{,}514$, and $Z = 7{,}358{,}060$, the 4×4 system to be solved is

$$\begin{aligned} 12x - 7y &= 52{,}223{,}598 \\ 20y - 9z &= 66{,}222{,}540 \\ 30z - 11t &= 45{,}643{,}257 \\ -13x + 42t &= 134{,}764{,}266 \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{rrrr|r} 12 & -7 & 0 & 0 & 52{,}223{,}598 \\ 0 & 20 & -9 & 0 & 66{,}222{,}540 \\ 0 & 0 & 30 & -11 & 45{,}643{,}257 \\ -13 & 0 & 0 & 42 & 134{,}764{,}266 \end{array}\right]$$

This exercise should be solved by graphing calculator or computer methods. The solution is $x = 7{,}206{,}360$ white cows, $y = 4{,}893{,}246$ black cows, $z = 3{,}515{,}820$ spotted cows, and $t = 5{,}439{,}213$ brown cows.

64. (a) The other two equations are

$$\begin{aligned} x_2 + x_3 &= 700 \\ x_3 + x_4 &= 600. \end{aligned}$$

(b) The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 1 & 1 & 0 & 0 & 1100 \\ 0 & 1 & 1 & 0 & 700 \\ 0 & 0 & 1 & 1 & 600 \end{array}\right].$$

$$-1R_1 + R_2 \rightarrow R_2 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 1 & 1 & 0 & 700 \\ 0 & 0 & 1 & 1 & 600 \end{array}\right]$$

$$-1R_2 + R_3 \rightarrow R_3 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & 600 \\ 0 & 0 & 1 & 1 & 600 \end{array}\right]$$

$$-1R_3 + R_4 \rightarrow R_4 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & 600 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let x_4 be arbitrary. Solve the first three equations for x_1, x_2, and x_3.

$$x_1 = 1000 - x_4$$
$$x_2 = 100 + x_4$$
$$x_3 = 600 - x_4$$

The solution is

$$(1000 - x_4, 100 + x_4, 600 - x_4, x_4).$$

(c) For x_4, we see that $x_4 \geq 0$ and $x_4 \leq 600$ since $600 - x_4$ must be nonnegative. Therefore, $0 \leq x_4 \leq 600$.

(d) x_1: If $x_4 = 0$, then $x_1 = 1000$.

If $x_4 = 600$, then
$x_1 = 1000 - 600 = 400$.

Therefore, $400 \leq x_1 \leq 1000$.

x_2: If $x_4 = 0$, then $x_2 = 100$.

If $x_4 = 600$, then
$x_2 = 100 + 600 = 700$.

Therefore, $100 \leq x_2 \leq 700$.

x_3: If $x_4 = 0$, then $x_3 = 600$.

If $x_4 = 600$, then
$x_3 = 600 - 600 = 0$.

Therefore, $0 \leq x_3 \leq 600$.

(e) If you know the number of cars entering or leaving three of the intersections, then the number entering or leaving the fourth is automatically determined because the number leaving must equal the number entering.

65. **(a)** The system to be solved is

$$0 = 200{,}000 - 0.5r - 0.3b$$
$$0 = 350{,}000 - 0.5r - 0.7b.$$

First, write the system in proper form.

$$0.5r + 0.3b = 200{,}000$$
$$0.5r + 0.7b = 350{,}000$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{cc|c} 0.5 & 0.3 & 200{,}000 \\ 0.5 & 0.7 & 350{,}000 \end{array}\right]$$

$$\begin{array}{c} 10R_1 \rightarrow R_1 \\ 10R_2 \rightarrow R_2 \end{array} \left[\begin{array}{cc|c} 5 & 3 & 2{,}000{,}000 \\ 5 & 7 & 3{,}500{,}000 \end{array}\right]$$

$$-1R_1 + R_2 \rightarrow R_2 \left[\begin{array}{cc|c} 5 & 3 & 2{,}000{,}000 \\ 0 & 4 & 1{,}500{,}000 \end{array}\right]$$

$$-\tfrac{3}{4}R_2 + R_1 \rightarrow R_1 \left[\begin{array}{cc|c} 5 & 0 & 875{,}000 \\ 0 & 4 & 1{,}500{,}000 \end{array}\right]$$

$$\begin{array}{c} \tfrac{1}{5}R_1 \rightarrow R_1 \\ \tfrac{1}{4}R_2 \rightarrow R_2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & 175{,}000 \\ 0 & 1 & 375{,}000 \end{array}\right]$$

The solution is (175,000, 375,000). When the rate of increase for each is zero, there are 175,000 soldiers in the Red Army and 375,000 soldiers in the Blue Army.

66. Let x = number of foul shots,
y = number of field goals, and
z = number of three pointers.

Since Bryant made a total of 46 baskets,

$$x + y + z = 46.$$

Since the number of field goals is equal to three times the number of three pointers, $y = 3z$, or

$$y - 3z = 0.$$

And since the total number of points was 81,

$$x + 2y + 3z = 81.$$

The system to be solved is

$$\begin{aligned} x + y + z &= 46 \quad (1) \\ y - 3z &= 0 \quad (2) \\ x + 2y + 3z &= 81 \quad (3). \end{aligned}$$

Eliminate x in equation (3).

$$\begin{aligned} x + y + z &= 46 \quad (1) \\ y - 3z &= 0 \quad (2) \\ -1R_1 + R_3 \rightarrow R_3 \qquad y + 2z &= 35 \quad (4) \end{aligned}$$

Eliminate y in equation (4).

$$\begin{aligned} x + y + z &= 46 \quad (1) \\ y - 3z &= 0 \quad (2) \\ (-1)R_2 + R_3 \rightarrow R_3 \qquad 5z &= 35 \quad (5) \end{aligned}$$

Make each leading coefficient equal 1.

$$\begin{aligned} x + y + z &= 46 \quad (1) \\ y - 3z &= 0 \quad (2) \\ \tfrac{1}{5}R_3 \rightarrow R_3 \qquad z &= 7 \quad (6) \end{aligned}$$

Substitute 7 for z in equation (2) to get $y = 21$. Finally, substitute 7 for z and 21 for y in equation (1) to get $x = 18$. Bryant made 18 foul shots, 21 field goals, and 7 three pointers.

67. Let $x =$ the number of singles,
$y =$ the number of doubles,
$z =$ the number of triples, and
$w =$ the number of home runs hit by Ichiro Suzuki.

The system to be solved is

$$\begin{aligned} x + y + z + w &= 262 \\ z &= w - 3 \\ y &= 3w \\ x &= 45z \end{aligned}$$

Write the equations in proper form, obtain the augmented matrix, and use row operations to solve.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 262 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 1 & 0 & -3 & 0 \\ 1 & 0 & -45 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ -1R_1 + R_4 \rightarrow R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 262 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & -1 & -46 & -1 & -262 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 262 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & -1 & -46 & -1 & -262 \end{array}\right]$$

$$\begin{array}{r} -1R_2 + R_1 \rightarrow R_1 \\ \\ \\ R_2 + R_4 \rightarrow R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 4 & 262 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & -46 & -4 & -262 \end{array}\right]$$

$$\begin{array}{r} -1R_3 + R_1 \rightarrow R_1 \\ \\ \\ 46R_3 + R_4 \rightarrow R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 265 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -50 & -400 \end{array}\right]$$

$$\begin{array}{r} R_4 + 10R_1 \rightarrow R_1 \\ -3R_4 + 50R_2 \rightarrow R_2 \\ -1R_4 + 50R_3 \rightarrow R_3 \\ \\ \end{array} \left[\begin{array}{cccc|c} 10 & 0 & 0 & 0 & 2250 \\ 0 & 50 & 0 & 0 & 1200 \\ 0 & 0 & 50 & 0 & 250 \\ 0 & 0 & 0 & -50 & -400 \end{array}\right]$$

$$\begin{array}{r} \tfrac{1}{10}R_1 \rightarrow R_1 \\ \tfrac{1}{50}R_2 \rightarrow R_2 \\ \tfrac{1}{50}R_3 \rightarrow R_3 \\ -\tfrac{1}{50}R_4 \rightarrow R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 225 \\ 0 & 1 & 0 & 0 & 24 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right]$$

Ichiro Suzuki hit 225 singles, 24 doubles, 5 triples, and 8 home runs during the 2004 season.

68. **(a)** $5.4 = a(8)^2 + b(8) + c$
$5.4 = 64a + 8b + c$
$6.3 = a(13)^2 + b(13) + c$
$6.3 = 169a + 13b + c$
$5.6 = a(18)^2 + b(18) + c$
$5.6 = 324a + 18b + c$

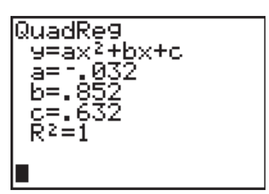
The linear system to be solved is

$$\begin{aligned} 64a + 8b + c &= 5.4 \\ 169a + 13b + c &= 6.3 \\ 324a + 18b + c &= 5.6. \end{aligned}$$

Use a graphing calculator or computer methods to solve this system. The solution is $a = -0.032$, $b = 0.852$, and $c = 0.632$. Thus, the equation is

$$y = -0.032x^2 + 0.852x + 0.632.$$

(b)



The answer obtained using Gauss-Jordan elimination is the same as the answer obtained using the quadratic regression feature on a graphing calculator.

69. Let $x =$ the number of balls,
$y =$ the number of dolls, and
$z =$ the number of cars.

(a) The system to be solved is

$$\begin{aligned} x + \quad y \qquad z &= 100 \\ 2x + \ 3y + \ 4z &= 295 \\ 12x + 16y + 18z &= 1542. \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 2 & 3 & 4 & 295 \\ 12 & 16 & 18 & 1542 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ -12R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 0 & 1 & 2 & 95 \\ 0 & 4 & 6 & 342 \end{array}\right]$$

$$\begin{array}{r} -1R_2 + R_1 \to R_1 \\ \\ -4R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 2 & 95 \\ 0 & 0 & -2 & -38 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -\frac{1}{2}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 2 & 95 \\ 0 & 0 & 1 & 19 \end{array}\right]$$

$$\begin{array}{r} R_3 + R_1 \to R_1 \\ -2R_3 + R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 24 \\ 0 & 1 & 0 & 57 \\ 0 & 0 & 1 & 19 \end{array}\right]$$

The solution is (24, 57, 19). There were 24 balls, 57 dolls, and 19 cars.

(b) The augmented matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 2 & 3 & 4 & 295 \\ 11 & 15 & 19 & 1542 \end{array}\right].$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ -11R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 0 & 1 & 2 & 95 \\ 0 & 4 & 8 & 442 \end{array}\right]$$

$$\begin{array}{r} -1R_2 + R_1 \to R_1 \\ \\ -4R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 2 & 95 \\ 0 & 0 & 0 & 62 \end{array}\right]$$

Since row 3 yields a false statement, $0 = 62$, there is no solution.

(c) The augmented matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 2 & 3 & 4 & 295 \\ 11 & 15 & 19 & 1480 \end{array}\right].$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ -11R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 0 & 1 & 2 & 95 \\ 0 & 4 & 8 & 380 \end{array}\right]$$

$$\begin{array}{r} -1R_2 + R_1 \to R_1 \\ \\ -4R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 2 & 95 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the last row is all zeros, there are infinitely many solutions Let z be the parameter. The matrix gives

$$\begin{aligned} x - z &= 5 \\ y + 2z &= 95. \end{aligned}$$

Solving these equations for x and y, the solution is $(5 + z, 95 - 2z, z)$. The numbers in the solution must be nonnegative integers. Therefore,

$$\begin{aligned} 95 - 2z &\geq 0 \\ -2z &\geq -95 \\ z &\leq 47.5. \end{aligned}$$

Thus, $z \in \{0, 1, 2, 3, ..., 47\}$. There are 48 possible solutions.

(d) For the smallest number of cars, $z = 0$, the solution is (5, 95, 0). This means 5 balls, 95 dolls, and no cars.

(e) For the largest number of cars, $z = 47$, the solution is (52, 1, 47). This means 52 balls, 1 doll, and 47 cars.

70. Let $x =$ the number of calories in each gram of fat
$y =$ the number of calories in each gram of carbohydrates
$z =$ the number of calories in each gram of protein

We want to solve the following system.

$$\begin{aligned} 10x + 36y + \ 2z &= 240 \\ 14x + 37y + \ 3z &= 280 \\ 20x + 23y + 11z &= 295 \end{aligned}$$

Write the augmented matrix and transform the matrix.

$$\left[\begin{array}{ccc|c} 10 & 36 & 2 & 240 \\ 14 & 37 & 3 & 280 \\ 20 & 23 & 11 & 295 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{10}R_1 \to R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 3.6 & 0.2 & 24 \\ 14 & 37 & 3 & 280 \\ 20 & 23 & 11 & 295 \end{array}\right]$$

$$\begin{array}{r} \\ -14R_1 + R_2 \to R_2 \\ -20R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3.6 & 0.2 & 24 \\ 0 & -13.4 & 0.2 & -56 \\ 0 & -49 & 7 & -185 \end{array}\right]$$

$$-\frac{1}{13.4}R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & 3.6 & 0.2 & 24 \\ 0 & 1 & -0.014925 & 4.179104 \\ 0 & -49 & 7 & -185 \end{array}\right]$$

$$\begin{array}{r} -3.6R_2+R_1 \to R_1 \\ \\ 49R_2+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0.25373 & 8.95522 \\ 0 & 1 & -0.014925 & 4.179104 \\ 0 & 0 & 6.268675 & 19.7761 \end{array}\right]$$

$$\frac{1}{6.268675}R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 0 & 0.25373 & 8.95522 \\ 0 & 1 & -0.014925 & 4.179104 \\ 0 & 0 & 1 & 3.15475 \end{array}\right]$$

$$\begin{array}{r} -0.25373R_3 + R_1 \to R_1 \\ 0.014925R_3 + R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8.15476 \\ 0 & 1 & 0 & 4.22619 \\ 0 & 0 & 1 & 3.15475 \end{array}\right]$$

The solution is $(8.15, 4.23, 3.15)$. There are 8.15 calories in a gram of fat, 4.23 calories in a gram of carbohydrates, and 3.15 calories in a gram of protein.

71. **(a)** $x_{11} + x_{12} + x_{21} = 1$
$x_{11} + x_{12} + x_{22} = 1$
$x_{11} + x_{21} + x_{22} = 1$
$x_{12} + x_{21} + x_{22} = 1$

Write the augmented matrix of the system.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_1 + R_2 \to R_2 \\ -1R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

Since $-1 = 1$ modulo 2, replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

Interchange rows 2 and 3.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{r} -1R_2 + R_1 \to R_1 \\ \\ \\ -R_2 + R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Again, replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} -1R_3 + R_1 \to R_1 \\ \\ \\ -1R_3 + R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

Replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -1R_4 + R_2 \to R_2 \\ -1R_4 + R_3 \to R_3 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Finally, replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

The solution (1, 1, 1, 1) corresponds to $x_{11} = 1$, $x_{12} = 1$, $x_{21} = 1$, and $x_{22} = 1$. Since 1 indicates that a button is pushed, the strategy required to turn all the lights out is to push every button one time.

(b) $x_{11} + x_{12} + x_{21} = 0$
$x_{11} + x_{12} + x_{22} = 1$
$x_{11} + x_{21} + x_{22} = 1$
$x_{12} + x_{21} + x_{22} = 0$

Write the augmented matrix of the system.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -1R_1 + R_2 \to R_2 \\ -1R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Interchange rows 2 and 3.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} -1R_2 + R_1 \to R_1 \\ \\ \\ -1R_2 + R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right]$$

Replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} -1R_3 + R_1 \to R_1 \\ \\ \\ -1R_3 + R_4 \to R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

Replace -1 with 1.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -1R_4 + R_2 \to R_2 \\ -1R_4 + R_3 \to R_3 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The solution (0, 1, 1, 0) corresponds to $x_{11} = 0$, $x_{12} = 1$, $x_{21} = 1$, and $x_{22} = 0$. Since 1 indicates that a button is pushed and 0 indicates that it is not, the strategy required to turn all the lights out is to push the button in the first row, second column, and push the button in the second row first column.

72. **(a)** Let $x =$ amount of paddy in one top grade bundle

$y =$ amount of paddy in one medium grade bundle

$z =$ amount of paddy in one low grade bundle

The system to be solved is

$$3x + 2y + z = 39$$
$$2x + 3y + z = 34$$
$$x + 2y + 3z = 26$$

(b) The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array}\right]$$

$$R_3 \leftrightarrow R_1 \left[\begin{array}{ccc|c} 1 & 2 & 3 & 26 \\ 2 & 3 & 1 & 34 \\ 3 & 2 & 1 & 39 \end{array}\right]$$

$$\begin{array}{r} \\ (-2)R_1 + R_2 \to R_2 \\ (-3)R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 26 \\ 0 & -1 & -5 & -18 \\ 0 & -4 & -8 & -39 \end{array}\right]$$

$$(-1)R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & 2 & 3 & 26 \\ 0 & 1 & 5 & 18 \\ 0 & -4 & -8 & -39 \end{array}\right]$$

$$\begin{array}{r} (-2)R_2 + R_1 \to R_1 \\ \\ 4R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -7 & -10 \\ 0 & 1 & 5 & 18 \\ 0 & 0 & 12 & 33 \end{array}\right]$$

$$\left(\frac{1}{12}\right)R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 0 & -7 & -10 \\ 0 & 1 & 5 & 18 \\ 0 & 0 & 1 & \frac{11}{4} \end{array}\right]$$

$$\begin{array}{r} 7R_3 + R_1 \to R_1 \\ (-5)R_3 + R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 9.25 \\ 0 & 1 & 0 & 4.25 \\ 0 & 0 & 1 & 2.75 \end{array}\right]$$

The bundle of top grade paddy yields 9.25 dou, one bundle of medium grade paddy yields 4.25 dou, and one bundle of low grade paddy yields 2.75 dou.

73. **(a)** Let $x =$ the cost of a sheep

$y =$ the cost of a dog

$z =$ the coast of a hen

$w =$ the cost of a rabbit

The system to be solved is

$$\begin{aligned} 5x+4y+3z+2w&=1496\\ 4x+2y+6z+3w&=1175\\ x+y+7z+5w&=958\\ x+3y+5z+w&=861 \end{aligned}$$

(b) Using a graphing calculator or computer methods, we find the following solution: A sheep costs 177 coins, a dog costs 121 coins, a hen costs 23 coins, and a rabbit costs 29 coins.

2.3 Addition and Subtraction of Matrices

Your Turn 1

(a) It is not possible to add a 2×4 matrix and a 2×3 matrix.

(b)

$$\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 4 \\ -2 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 9 \\ -1 & -2 & 11 \end{bmatrix}$$

Your Turn 2

$$\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 4 \\ -2 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 1 \\ 3 & 6 & -5 \end{bmatrix}$$

2.3 Exercises

1. $\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$

This statement is false, since not all corresponding elements are equal.

2. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

This statement is false. For two matrices to be equal, they must be the same size, and each pair of corresponding elements must be equal. These two matrices are different sizes.

3. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ if $x = -2$ and $y = 8$.

This statement is true. The matrices are the same size and corresponding elements are equal.

4. $\begin{bmatrix} 3 & 5 & 2 & 8 \\ 1 & -1 & 4 & 0 \end{bmatrix}$ is a 4×2 matrix.

This statement is false. Since the matrix has 2 rows and 4 columns, it is a 2×4 matrix.

5. $\begin{bmatrix} 1 & 9 & -4 \\ 3 & 7 & 2 \\ -1 & 1 & 0 \end{bmatrix}$ is a square matrix.

This statement is true. The matrix has 3 rows and 3 columns.

6. $\begin{bmatrix} 2 & 4 & -1 \\ 3 & 7 & 5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 7 & 5 \end{bmatrix}$

This statement is false since the matrices are different sizes.

7. $\begin{bmatrix} -4 & 8 \\ 2 & 3 \end{bmatrix}$ is a 2×2 square matrix.

8. $\begin{bmatrix} 2 & -3 & 7 \\ 1 & 0 & 4 \end{bmatrix}$

This matrix has 2 rows and 3 columns, so it is a 2×3 matrix.

9. $\begin{bmatrix} -6 & 8 & 0 & 0 \\ 4 & 1 & 9 & 2 \\ 3 & -5 & 7 & 1 \end{bmatrix}$ is a 3×4 matrix.

10. $\begin{bmatrix} 8 & -2 & 4 & 6 & 3 \end{bmatrix}$

The matrix has 1 row and 5 columns, so it is a 1×5 matrix. It is a row matrix since it has only 1 row.

11. $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$ is a 2×1 column matrix.

12. $\begin{bmatrix} -9 \end{bmatrix}$

This matrix has 1 row and 1 column, so it is a 1×1 square matrix. It is also a row matrix since it has only 1 row, and a column matrix because it has only 1 column.

13. Undefined

14. Since A is a 5×2 matrix, and since A and K can be added, we know that K is also a 5×2 matrix. Also, since $A + K = A$, all entries of K must be 0.

15. $\begin{bmatrix} 3 & 4 \\ -8 & 1 \end{bmatrix} = \begin{bmatrix} 3 & x \\ y & z \end{bmatrix}$

Corresponding elements must be equal for the matrices to be equal. Therefore, $x = 4$, $y = -8$, and $z = 1$.

16. $\begin{bmatrix} -5 \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

Two matrices can be equal only if they are the same size and corresponding elements are equal. These matrices will be equal if $y = 8$.

17. $\begin{bmatrix} s-4 & t+2 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -5 & r \end{bmatrix}$

Corresponding elements must be equal

$s - 4 = 6 \quad t + 2 = 2 \quad r = 7.$
$s = 10 \quad t = 0$

Thus, $s = 10$, $t = 0$, and $r = 7$.

18. $\begin{bmatrix} 9 & 7 \\ r & 0 \end{bmatrix} = \begin{bmatrix} m-3 & n+5 \\ 8 & 0 \end{bmatrix}$

The matrices are the same size, so they will be equal if corresponding elements are equal.

$9 = m - 3 \quad 7 = n + 5 \quad r = 8$
$12 = m \quad 2 = n$

Thus, $m = 12$, $n = 2$, and $r = 8$.

19. $\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$

Add the two matrices on the left side to obtain

$$\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (a+2)+(-7) & 3b+2b & 4c+6 \\ d+(-3d) & 7f+(-6) & 8+(-2) \end{bmatrix}$$

$$= \begin{bmatrix} a-5 & 5b & 4c+6 \\ -2d & 7f-6 & 6 \end{bmatrix}$$

Corresponding elements of this matrix and the matrix on the right side of the original equation must be equal.

$a - 5 = 15 \quad 5b = 25 \quad 4c + 6 = 6$
$a = 20 \quad b = 5 \quad c = 0$

$-2d = -8 \quad 7f - 6 = 1$
$d = 4 \quad f = 1$

Thus, $a = 20$, $b = 5$, $c = 0$, $d = 4$, and $f = 1$.

20. $\begin{bmatrix} a+2 & 3z+1 & 5m \\ 4k & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2z & 5m \\ 2k & 5 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 10 & -14 & 80 \\ 10 & 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 4a+2 & 5z+1 & 10m \\ 6k & 5 & 9 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 5 & 9 \end{bmatrix}$$

$4a + 2 = 10 \quad 5z + 1 = -14$
$a = 2 \quad z = -3$

$10m = 80 \quad 6k = 10$
$m = 8 \quad k = \frac{5}{3}$

Thus, $a = 2$, $z = -3$, $m = 8$, and $k = \frac{5}{3}$.

21. $\begin{bmatrix} 2 & 4 & 5 & -7 \\ 6 & -3 & 12 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 & -10 & 1 \\ -2 & 8 & -9 & 11 \end{bmatrix}$

$$= \begin{bmatrix} 2+8 & 4+0 & 5+(-10) & -7+1 \\ 6+(-2) & -3+8 & 12+(-9) & 0+11 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 4 & -5 & -6 \\ 4 & 5 & 3 & 11 \end{bmatrix}$$

22. $\begin{bmatrix} 1 & 5 \\ 2 & -3 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 8 & 5 \\ -1 & 9 \end{bmatrix} = \begin{bmatrix} 1+2 & 5+3 \\ 2+8 & -3+5 \\ 3+(-1) & 7+9 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 8 \\ 10 & 2 \\ 2 & 16 \end{bmatrix}$$

23. $\begin{bmatrix} 1 & 3 & -2 \\ 4 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 6 & 4 \\ -5 & 2 \end{bmatrix}$

These matrices cannot be added since the first matrix has size 2×3, while the second has size 3×2. Only matrices that are the same size can be added.

24. $\begin{bmatrix} 8 & 0 & -3 \\ 1 & 19 & -5 \end{bmatrix} - \begin{bmatrix} 1 & -5 & 2 \\ 3 & 9 & -8 \end{bmatrix}$

$$= \begin{bmatrix} 8-1 & 0-(-5) & -3-2 \\ 1-3 & 19-9 & -5-(-8) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 5 & -5 \\ -2 & 10 & 3 \end{bmatrix}$$

25. The matrices have the same size, so the subtraction can be done. Let A and B represent the given matrices.

$A - B =$

$$= \begin{bmatrix} 2-1 & 8-3 & 12-6 & 0-9 \\ 7-2 & 4-(-3) & -1-(-3) & 5-4 \\ 1-8 & 2-0 & 0-(-2) & 10-17 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 6 & -9 \\ 5 & 7 & 2 & 1 \\ -7 & 2 & 2 & -7 \end{bmatrix}$$

26. $\begin{bmatrix} 2 & 1 \\ 5 & -3 \\ -7 & 2 \\ 9 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -8 & 0 \\ 5 & 3 & 2 \\ -6 & 7 & -5 \\ 2 & -1 & 0 \end{bmatrix}$

This operation is not possible because the matrices are different sizes. Only matrices of the same size can be added.

27. $\begin{bmatrix} 2 & 3 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 2+4-3 & 3+3-2 \\ -2+7-1 & 4+8-4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 8 \end{bmatrix}$$

28. $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 4-1+1 & 3-1+1 \\ 1-1+1 & 2-0+4 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 6 \end{bmatrix}$$

29. $\begin{bmatrix} 2 & -1 \\ 0 & 13 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ -5 & 7 \end{bmatrix} + \begin{bmatrix} 12 & 7 \\ 5 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2-4+12 & -1-8+7 \\ 0-(-5)+5 & 13-7+3 \end{bmatrix} = \begin{bmatrix} 10 & -2 \\ 10 & 9 \end{bmatrix}$$

30. $\begin{bmatrix} 5 & 8 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} -5 & -8 \\ 6 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 5+0+(-5) & 8+1+(-8) \\ -3+(-2)+6 & 1+(-2)+1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

31. $\begin{bmatrix} -4x+2y & -3x+y \\ 6x-3y & 2x-5y \end{bmatrix} + \begin{bmatrix} -8x+6y & 2x \\ 3y-5x & 6x+4y \end{bmatrix}$

$$= \begin{bmatrix} (-4x+2y)+(-8x+6y) & (-3x+y)+2x \\ (6x-3y)+(3y-5x) & (2x-5y)+(6x+4y) \end{bmatrix}$$

$$= \begin{bmatrix} -12x+8y & -x+y \\ x & 8x-y \end{bmatrix}$$

32. $\begin{bmatrix} 4k-8y \\ 6z-3x \\ 2k+5a \\ -4m+2n \end{bmatrix} - \begin{bmatrix} 5k+6y \\ 2z+5x \\ 4k+6a \\ 4m-2n \end{bmatrix}$

$$= \begin{bmatrix} 4k-8y-(5k+6y) \\ 6z-3x-(2z+5x) \\ 2k+5a-(4k+6a) \\ -4m+2n-(4m-2n) \end{bmatrix} = \begin{bmatrix} -k-14y \\ 4z-8x \\ -2k-a \\ -8m+4n \end{bmatrix}$$

33. $O - X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$$= \begin{bmatrix} 0-x & 0-y \\ 0-z & 0-w \end{bmatrix} = \begin{bmatrix} -x & -y \\ -z & -w \end{bmatrix}$$

34. Verify that $X + T = T + X$.

$$X + T = \begin{bmatrix} x+r & y+s \\ z+t & w+u \end{bmatrix}$$

$$T + X = \begin{bmatrix} r+x & s+y \\ t+z & u+w \end{bmatrix}$$

Because of the commutative property for addition of real numbers, $x + r = r + x$. This also applies to the other corresponding elements, so we conclude that $T + X = X + T$.

35. Show that $X + (T + P) = (X + T) + P$.

On the left side, the sum $T + P$ is obtained first, and then

$$X + (T + P).$$

This gives the matrix

$$\begin{bmatrix} x+(r+m) & y+(s+n) \\ z+(t+p) & w+(u+q) \end{bmatrix}.$$

For the right side, first the sum $X + T$ is obtained, and then

$$(X + T) + P.$$

This gives the matrix

$$\begin{bmatrix} (x+r)+m & (y+s)+n \\ (z+t)+p & (w+u)+q \end{bmatrix}.$$

Comparing corresponding elements, we see that they are equal by the associative property of addition of real numbers. Thus,

$$X + \left(T + P\right) = \left(X + T\right) + P.$$

36. Verify that $(X - X) = O$.

$$X - X = \begin{bmatrix} x-x & y-y \\ z-z & w-w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

37. Show that $P + O = P$.

$$P + O = \begin{bmatrix} m & n \\ p & q \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m+0 & n+0 \\ p+0 & q+0 \end{bmatrix} = \begin{bmatrix} m & n \\ p & q \end{bmatrix} = P$$

Thus, $P + O = P$.

38. All of these properties are valid for matrices that are not square, as long as all necessary sums exist. The sizes of all matrices in each equation must be the same.

39. **(a)**

$$\begin{matrix} & \text{Cars} & \text{Trucks} \\ \text{New} & 15 & 15 \\ \text{Used} & 36 & 25 \end{matrix}$$

(b) $$\begin{bmatrix} 17 & 12 \\ 24 & 18 \end{bmatrix} + \begin{bmatrix} 15 & 15 \\ 36 & 25 \end{bmatrix} = \begin{bmatrix} 32 & 27 \\ 60 & 43 \end{bmatrix}$$

(c) $$\begin{bmatrix} 15 & 15 \\ 36 & 25 \end{bmatrix} - \begin{bmatrix} 17 & 12 \\ 24 & 18 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 12 & 7 \end{bmatrix}$$

40. **(a)**

$$\begin{matrix} & \text{I} & \text{II} & \text{III} \\ \text{Bread} & 88 & 105 & 60 \\ \text{Milk} & 48 & 72 & 40 \\ \text{PB} & 16 & 21 & 0 \\ \text{Cold cuts} & 112 & 147 & 50 \end{matrix}$$

(b)

$$\begin{bmatrix} 88 + 0.25(88) & 105 + \frac{1}{3}(105) & 60 + 0.1(60) \\ 48 + 0.25(48) & 72 + \frac{1}{3}(72) & 40 + 0.1(40) \\ 16 + 0.25(16) & 21 + \frac{1}{3}(21) & 0 + 0.1(0) \\ 112 + 0.25(112) & 147 + \frac{1}{3}(147) & 50 + 0.1(50) \end{bmatrix} = \begin{bmatrix} 110 & 140 & 66 \\ 60 & 96 & 44 \\ 20 & 28 & 0 \\ 140 & 196 & 55 \end{bmatrix}$$

(c) Add the final matrices from parts (a) and (b).

$$\begin{bmatrix} 88 & 105 & 60 \\ 48 & 72 & 40 \\ 16 & 21 & 0 \\ 112 & 147 & 50 \end{bmatrix} + \begin{bmatrix} 110 & 140 & 66 \\ 60 & 96 & 44 \\ 20 & 28 & 0 \\ 140 & 196 & 55 \end{bmatrix} = \begin{bmatrix} 198 & 245 & 126 \\ 108 & 168 & 84 \\ 36 & 49 & 0 \\ 252 & 343 & 105 \end{bmatrix}$$

41. **(a)** $$C = \begin{bmatrix} 21.0 & 19.1 \\ 1.5 & 0.9 \\ 7.6 & 12.1 \\ 18.3 & 105.6 \\ 32.3 & 27.3 \\ 39.8 & 39.8 \\ 130.1 & 92.8 \end{bmatrix}$$ **(b)** $$M = \begin{bmatrix} 14.2 & 14.8 \\ 0.6 & 3.0 \\ 7.4 & 1.7 \\ 23.7 & 40.3 \\ 26.0 & 5.2 \\ 28.0 & 20.0 \\ 93.9 & 154.5 \end{bmatrix}$$

(c) $$C + M = \begin{bmatrix} 35.2 & 33.9 \\ 2.1 & 3.9 \\ 15.0 & 13.8 \\ 42.0 & 145.9 \\ 58.3 & 32.5 \\ 67.8 & 59.8 \\ 224.0 & 247.3 \end{bmatrix}$$

(d) $$C - M = \begin{bmatrix} -6.8 & -4.3 \\ -0.9 & 2.1 \\ -0.2 & -10.4 \\ 5.4 & -65.3 \\ -6.3 & -22.1 \\ -11.8 & -19.8 \\ -36.2 & 61.7 \end{bmatrix}$$

42. **(a)** $$A = \begin{bmatrix} 178.9 & 230.8 \\ 16.2 & 100.0 \\ 111.3 & 135.9 \\ 64.9 & 146.5 \\ 29.4 & 58.5 \end{bmatrix}$$ **(b)** $$B = \begin{bmatrix} 300.0 & 332.1 \\ 122.0 & 440.4 \\ 226.2 & 280.5 \\ 65.1 & 138.5 \\ 47.4 & 114.6 \end{bmatrix}$$

(c) $$B - A = \begin{bmatrix} 121.4 & 101.3 \\ 105.8 & 340.4 \\ 114.9 & 144.6 \\ 0.2 & -8.0 \\ 18.0 & 56.1 \end{bmatrix}$$

All trade amounts increased from 2000 to 2013 except for U.S. imports from Japan.

43. **(a)** There are four food groups and three meals. To represent the data by a 3×4 matrix, we must use the rows to correspond to the meals, breakfast, lunch, and dinner, and the columns to correspond to the four food groups. Thus, we obtain the matrix

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

(b) There are four food groups. These will correspond to the four rows. There are three components in each food group: fat, carbohydrates, and protein. These will correspond to the three columns. The matrix is

$$\begin{bmatrix} 5 & 0 & 7 \\ 0 & 10 & 1 \\ 0 & 15 & 2 \\ 10 & 12 & 8 \end{bmatrix}.$$

(c) The matrix is

$$\begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix}.$$

44. (a)

$$\begin{matrix} \text{Length} \\ \text{Weight} \end{matrix} \begin{bmatrix} 5.6 & 6.4 & 6.9 & 7.6 & 6.1 \\ 144 & 138 & 149 & 152 & 146 \end{bmatrix}$$

(b)

$$\begin{matrix} \text{Length} \\ \text{Weight} \end{matrix} \begin{bmatrix} 10.2 & 11.4 & 11.4 & 12.7 & 10.8 \\ 196 & 196 & 225 & 250 & 230 \end{bmatrix}$$

(c) Subtract the matrix in part (a) from the one in part (b).

$$\begin{bmatrix} 10.2 & 11.4 & 11.4 & 12.7 & 10.8 \\ 196 & 196 & 225 & 250 & 230 \end{bmatrix} - \begin{bmatrix} 5.6 & 6.4 & 6.9 & 7.6 & 6.1 \\ 144 & 138 & 149 & 152 & 146 \end{bmatrix}$$
$$= \begin{bmatrix} 4.6 & 5.0 & 4.5 & 5.1 & 4.7 \\ 52 & 58 & 76 & 98 & 84 \end{bmatrix} \begin{matrix} \text{Change in length} \\ \text{Change in weight} \end{matrix}$$

(d) Add the new matrix to the one from part (b).

$$\begin{bmatrix} 10.2 & 11.4 & 11.4 & 12.7 & 10.8 \\ 196 & 196 & 225 & 250 & 230 \end{bmatrix} + \begin{bmatrix} 1.8 & 1.5 & 2.3 & 1.8 & 2.0 \\ 25 & 22 & 29 & 33 & 20 \end{bmatrix}$$
$$= \begin{bmatrix} 12.0 & 12.9 & 13.7 & 14.5 & 12.8 \\ 221 & 218 & 254 & 283 & 250 \end{bmatrix} \begin{matrix} \text{Final length} \\ \text{Final weight} \end{matrix}$$

45.

	Obtained Pain Relief	
	Yes	No
Painfree	22	3
Placebo	8	17

(a) Of the 25 patients who took the placebo, 8 got relief.

(b) Of the 25 patients who took Painfree, 3 got no relief.

(c)

$$\begin{bmatrix} 22 & 3 \\ 8 & 17 \end{bmatrix} + \begin{bmatrix} 21 & 4 \\ 6 & 19 \end{bmatrix} + \begin{bmatrix} 19 & 6 \\ 10 & 15 \end{bmatrix} + \begin{bmatrix} 23 & 2 \\ 3 & 22 \end{bmatrix} = \begin{bmatrix} 85 & 15 \\ 27 & 73 \end{bmatrix}$$

(d) Yes, it appears that Painfree is effective. Of the 100 patients who took the medication, 85% got relief.

46. (a) The matrix for motorcycle helmet usage in 2012 is

$$A = \begin{matrix} \text{Northeast} \\ \text{Midwest} \\ \text{South} \\ \text{West} \end{matrix} \overset{\begin{matrix}\text{Compliant} & \text{Noncompliant}\end{matrix}}{\begin{bmatrix} 60 & 6 \\ 49 & 9 \\ 61 & 16 \\ 82 & 4 \end{bmatrix}}$$

The matrix for motorcycle helmet usage in 2013 is

$$B = \begin{bmatrix} 52 & 10 \\ 42 & 5 \\ 65 & 11 \\ 92 & 3 \end{bmatrix}$$

(b) The matrix showing the change in helmet usage from 2012 to 2013 is

$$B - A = \begin{bmatrix} 52 & 10 \\ 42 & 5 \\ 65 & 11 \\ 92 & 3 \end{bmatrix} - \begin{bmatrix} 60 & 6 \\ 49 & 9 \\ 61 & 16 \\ 82 & 4 \end{bmatrix} = \begin{bmatrix} -8 & 4 \\ -7 & -4 \\ 4 & -5 \\ 10 & -1 \end{bmatrix}$$

(c) The percentage of compliant helmets incressed in the South and West but decreased elsewhere. The percentage of noncompliant helmets decreased everywhere except the Northeast.

47. (a) The matrix for the life expectancy of African Americans is

$$\begin{matrix} 1970 \\ 1980 \\ 1990 \\ 2000 \\ 2010 \end{matrix} \overset{\begin{matrix}\text{M} & \text{F}\end{matrix}}{\begin{bmatrix} 60.0 & 68.3 \\ 63.8 & 72.5 \\ 64.5 & 73.6 \\ 68.2 & 75.1 \\ 71.8 & 78.0 \end{bmatrix}}$$

(b) The matrix for the life expectancy of White Americans is

$$\begin{matrix} 1970 \\ 1980 \\ 1990 \\ 2000 \\ 2010 \end{matrix} \overset{\begin{matrix}\text{M} & \text{F}\end{matrix}}{\begin{bmatrix} 68.0 & 75.6 \\ 70.7 & 78.1 \\ 72.7 & 79.4 \\ 74.7 & 79.9 \\ 76.5 & 81.3 \end{bmatrix}}$$

(c) The matrix showing the difference between the life expectancy between the two groups is

$$\begin{bmatrix} 60.0 & 68.3 \\ 63.8 & 72.5 \\ 64.5 & 73.6 \\ 68.2 & 75.1 \\ 71.8 & 78.0 \end{bmatrix} - \begin{bmatrix} 68.0 & 75.6 \\ 70.7 & 78.1 \\ 72.7 & 79.4 \\ 74.7 & 79.9 \\ 76.5 & 81.3 \end{bmatrix} = \begin{bmatrix} -8.0 & -7.3 \\ -6.9 & -5.6 \\ -8.2 & -5.8 \\ -6.5 & -4.8 \\ -4.7 & -3.3 \end{bmatrix}$$

48. **(a)** The matrix for the educational attainment of males is

	4 Years of High School or More	4 Years of College or More
1970	55.0	14.1
1980	69.2	20.9
1990	77.7	24.4
2000	84.2	27.8
2010	86.0	30.3

(b) The matrix for the educational attainment of females is

	4 Years of High School or More	4 Years of College or More
1970	55.4	8.2
1980	68.1	13.6
1990	77.5	18.4
2000	84.0	23.6
2010	87.6	29.6

(c) The matrix showing how much more (or less) education males have attained than females is

$$\begin{bmatrix} 55.0 & 14.1 \\ 69.2 & 20.9 \\ 77.7 & 24.4 \\ 84.2 & 27.8 \\ 86.6 & 30.3 \end{bmatrix} - \begin{bmatrix} 55.4 & 8.2 \\ 68.1 & 13.6 \\ 77.5 & 18.4 \\ 84.0 & 23.6 \\ 87.6 & 29.6 \end{bmatrix} = \begin{bmatrix} -0.4 & 5.9 \\ 1.1 & 7.3 \\ 0.2 & 6.0 \\ 0.2 & 4.2 \\ -1.0 & 0.7 \end{bmatrix}$$

49. **(a)** The matrix for the educational attainment of African Americans is

	4 Years of High School or More	4 Years of College or More
1980	51.2	7.9
1985	59.8	11.1
1990	66.2	11.3
1995	73.8	13.2
2000	78.5	16.5
2005	81.3	17.6
2010	84.2	19.8

(b) The matrix for the educational attainment of Hispanic Americans is

	4 Years of High School or More	4 Years of College or More
1980	45.3	7.9
1985	47.9	8.5
1990	50.8	9.2
1995	53.4	9.3
2000	57.0	10.6
2005	58.5	12.0
2010	62.9	13.9

(c) The matrix showing the difference in the educational attainment between African and Hispanic Americans is

$$\begin{bmatrix} 51.2 & 7.9 \\ 59.8 & 11.1 \\ 66.2 & 11.3 \\ 73.8 & 13.2 \\ 78.5 & 16.5 \\ 81.3 & 17.6 \\ 84.2 & 19.8 \end{bmatrix} - \begin{bmatrix} 45.3 & 7.9 \\ 47.9 & 8.5 \\ 50.8 & 9.2 \\ 53.4 & 9.3 \\ 57.0 & 10.6 \\ 58.5 & 12.0 \\ 62.9 & 13.9 \end{bmatrix} = \begin{bmatrix} 5.9 & 0.0 \\ 11.9 & 2.6 \\ 15.4 & 2.1 \\ 20.4 & 3.9 \\ 21.5 & 5.9 \\ 22.8 & 5.6 \\ 21.3 & 5.9 \end{bmatrix}$$

50. **(a)**

	M	J	Ca	Cl
M	1	1	1	1
J	0	1	0	0
Ca	0	1	1	1
Cl	0	1	1	1

(b) Rows 1 and 2 will stay the same. Since the cats now like Musk, the zeros in rows 3 and 4 change to ones.

	M	J	Ca	Cl
M	1	1	1	1
J	0	1	0	0
Ca	1	1	1	1
Cl	1	1	1	1

2.4 Multiplication of Matrices

Your Turn 1

$$AB = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1) + 4(-2) & 3(-2) + 4(-4) \\ 1(1) + 2(-2) & 1(-2) + 2(-4) \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -22 \\ -3 & -10 \end{bmatrix}$$

Your Turn 2

$$AB = \begin{bmatrix} 3 & 5 & -1 \\ 2 & 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -5 & -3 \end{bmatrix}$$

AB does not exist because a 2×3 matrix cannot be multiplied by a 2×2 matrix.

$$BA = \begin{bmatrix} 3 & -4 \\ -5 & -3 \end{bmatrix}\begin{bmatrix} 3 & 5 & -1 \\ 2 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3) - 4(2) & 3(5) - 4(4) & 3(-1) - 4(-2) \\ -5(3) - 3(2) & -5(5) - 3(4) & -5(-1) - 3(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 5 \\ -21 & -37 & 11 \end{bmatrix}$$

2.4 Exercises

In Exercises 1-6, let

$$A = \begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix}.$$

1. $2A = 2\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -4 & 8 \\ 0 & 6 \end{bmatrix}$

2. $-3B = -3\begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 18 & -6 \\ -12 & 0 \end{bmatrix}$

3. $-6A = -6\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -24 \\ 0 & -18 \end{bmatrix}$

4. $5B = 5\begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -30 & 10 \\ 20 & 0 \end{bmatrix}$

5. $-4A + 5B = -4\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} + 5\begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 8 & -16 \\ 0 & -12 \end{bmatrix} + \begin{bmatrix} -30 & 10 \\ 20 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -22 & -6 \\ 20 & -12 \end{bmatrix}$$

6. $7B - 3A = 7\begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix} - 3\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -42 & 14 \\ 28 & 0 \end{bmatrix} - \begin{bmatrix} -6 & 12 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -36 & 2 \\ 28 & -9 \end{bmatrix}$$

7. Matrix A size Matrix B size

$2 \times \underline{\mathbf{2}}$ $\quad$ $\underline{\mathbf{2}} \times 2$

The number of columns of A is the same as the number of rows of B, so the product AB exists. The size of the matrix AB is 2×2.

Matrix B size Matrix A size

$2 \times \underline{\mathbf{2}}$ $\quad$ $\underline{\mathbf{2}} \times 2$

Since the number of columns of B is the same as the number of rows of A, the product BA also exists and has size 2×2.

8. A is 3×3, and B is 3×3.

The number of columns of A is the same as the number of rows of B, so the product AB exists; its size is 3×3. The number of columns of B is the same as the number of rows of A, so the product BA also exists; its size is 3×3.

9. Matrix A size Matrix B size

$3 \times \underline{\mathbf{4}}$ $\quad$ $\underline{\mathbf{4}} \times 4$

Since matrix A has 4 columns and matrix B has 4 rows, the product AB exists and has size 3×4.

Matrix B size Matrix A size

$4 \times \underline{\mathbf{4}}$ $\quad$ $\underline{\mathbf{3}} \times 4$

Since B has 4 columns and A has 3 rows, the product BA does not exist.

10. Matrix A size Matrix B size

$4 \times \underline{\mathbf{3}}$ $\quad$ $\underline{\mathbf{3}} \times 6$

Since the number of columns of A, 3, is the same as the number of rows of B, 3, the product AB exists. Its size is 4×6.

Matrix B size Matrix A size

$3 \times \underline{\mathbf{6}}$ $\quad$ $\underline{\mathbf{4}} \times 3$

The product BA does not exist since the number of columns of B, 6, is not the same as the number of rows of A, 4.

11. Matrix A size Matrix B size

$4 \times \underline{\mathbf{2}}$ $\quad$ $\underline{\mathbf{3}} \times 4$

The number of columns of A is not the same as the number of rows of B, so the product AB does not exist.

Matrix B size Matrix A size

$3 \times \underline{\mathbf{4}}$ $\quad$ $\underline{\mathbf{4}} \times 2$

The number of columns of B is the same as the number of rows of A, so the product BA exists and has size 3×2.

12. A is 3×2 and B is 1×3.
The product AB does not exist, since the number of columns of A is not the same as the number of rows of B.

The product BA exists; its size is 1×2.

13. To find the product matrix AB, the number of *columns* of A must be the same as the number of *rows* of B.

14. The product matrix AB has the same number of *rows* as A and the same number of *columns* as B.

15. Call the first matrix A and the second matrix B. The product matrix AB will have size 2×1.

Step 1: Multiply the elements of the first row of A by the corresponding elements of the column of B and add.

$$\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ 5 & 8 \end{bmatrix}\begin{bmatrix} \mathbf{3} \\ \mathbf{-2} \end{bmatrix} \qquad 2(3) + (-1)(-2) = 8$$

Therefore, 8 is the first row entry of the product matrix AB.

Step 2: Multiply the elements of the second row of A by the corresponding elements of the column of B and add.

$$\begin{bmatrix} 2 & -1 \\ \mathbf{5} & \mathbf{8} \end{bmatrix}\begin{bmatrix} \mathbf{3} \\ \mathbf{-2} \end{bmatrix} \qquad 5(3) + 8(-2) = -1$$

The second row entry of the product is -1.

Step 3: Write the product using the two entries found above.

$$AB = \begin{bmatrix} 2 & -1 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

16. $$\begin{bmatrix} -1 & 5 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \cdot 6 + 5 \cdot 2 \\ 7 \cdot 6 + 0 \cdot 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 42 \end{bmatrix}$$

17. $$\begin{bmatrix} 2 & -1 & 7 \\ -3 & 0 & -4 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 5 + (-1) \cdot 10 + 7 \cdot 2 \\ (-3) \cdot 5 + 0 \cdot 10 + (-4) \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ -23 \end{bmatrix}$$

18. $$\begin{bmatrix} 5 & 2 \\ 7 & 6 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ 2 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \cdot 1 + 2 \cdot 2 & 5 \cdot 4 + 2(-1) & 5 \cdot 0 + 2 \cdot 2 \\ 7 \cdot 1 + 6 \cdot 2 & 7 \cdot 4 + 6(-1) & 7 \cdot 0 + 6 \cdot 2 \\ 1 \cdot 1 + 0 \cdot 2 & 1 \cdot 4 + 0(-1) & 1 \cdot 0 + 0 \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 18 & 4 \\ 19 & 22 & 12 \\ 1 & 4 & 0 \end{bmatrix}$$

19. $$\begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} -1 & 0 & 4 \\ 5 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot (-1) + (-1) \cdot 5 & 2 \cdot 0 + (-1) \cdot (-2) & 2 \cdot 4 + (-1) \cdot 0 \\ 3 \cdot (-1) + 6 \cdot 5 & 3 \cdot 0 + 6 \cdot (-2) & 3 \cdot 4 + 6 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 2 & 8 \\ 27 & -12 & 12 \end{bmatrix}$$

20. $$\begin{bmatrix} 6 & 0 & -4 \\ 1 & 2 & 5 \\ 10 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \cdot 1 + 0 \cdot 2 + (-4)0 \\ 1 \cdot 1 + 2 \cdot 2 + 5 \cdot 0 \\ 10 \cdot 1 + (-1)2 + 3 \cdot 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$$

21. $$\begin{bmatrix} 2 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -1 & 4 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 0 + 2(-1) + (-1)0 & 2 \cdot 2 + 2 \cdot 4 + (-1)2 \\ 3 \cdot 0 + 0(-1) + 1(0) & 3 \cdot 2 + 0 \cdot 4 + 1 \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 10 \\ 0 & 8 \end{bmatrix}$$

22. $$\begin{bmatrix} -3 & 1 & 0 \\ 6 & 0 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} (-3) \cdot 3 + 1 \cdot (-1) + 0 \cdot (-2) \\ 6 \cdot 3 + 0 \cdot (-1) + 8 \cdot (-2) \end{bmatrix}$$

$$= \begin{bmatrix} -10 \\ 2 \end{bmatrix}$$

23. $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 7 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-1) + 2 \cdot 7 & 1 \cdot 5 + 2 \cdot 0 \\ 3(-1) + 4 \cdot 7 & 3 \cdot 5 + 4 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 5 \\ 25 & 15 \end{bmatrix}$$

24. $$\begin{bmatrix} 2 & 8 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 8\cdot 0 & 2\cdot 0 + 8\cdot 1 \\ (-7)\cdot 1 + 5\cdot 0 & (-7)\cdot 0 + 5\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ -7 & 5 \end{bmatrix}$$

25. $$\begin{bmatrix} -2 & -3 & 7 \\ 1 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2(1) + (-3)2 + 7\cdot 3 \\ 1\cdot 1 + 5\cdot 2 + 6\cdot 3 \end{bmatrix} = \begin{bmatrix} 13 \\ 29 \end{bmatrix}$$

26. $$\begin{bmatrix} 2 \\ -9 \\ 12 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & 2\cdot 0 & 2\cdot(-1) \\ (-9)\cdot 1 & (-9)\cdot 0 & (-9)\cdot(-1) \\ 12\cdot 1 & 12\cdot 0 & 12\cdot(-1) \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ -9 & 0 & 9 \\ 12 & 0 & -12 \end{bmatrix}$$

27. $$\left(\begin{bmatrix} 2 & 1 \\ -3 & -6 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\right)\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ -15 & 12 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -33 \\ 4 \end{bmatrix}$$

28. $$\begin{bmatrix} 2 & 1 \\ -3 & -6 \\ 4 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ -3 & -6 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ -33 \\ 4 \end{bmatrix}$$

29. $$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\left(\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ -1 & 5 \end{bmatrix}\right) = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 11 & 3 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 22 & -8 \\ 11 & -4 \end{bmatrix}$$

30. $$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 7 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 16 & -10 \\ 8 & -5 \end{bmatrix} = \begin{bmatrix} 22 & -8 \\ 11 & -4 \end{bmatrix}$$

31. $$A = \begin{bmatrix} 2 & -5 \\ -3 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} (2)(2) + (-5)(-3) & (2)(-5) + (-5)(0) \\ (-3)(2) + (0)(2) & (-3)(-5) + (0)(0) \end{bmatrix} = \begin{bmatrix} 19 & -10 \\ -6 & 15 \end{bmatrix}$$

32. $$A = \begin{bmatrix} 0.5 & 0.2 \\ 0.7 & -0.1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} (0.5)(0.5) + (0.2)(0.7) & (0.5)(0.2) + (0.2)(-0.1) \\ (0.7)(0.5) + (-0.1)(0.7) & (0.7)(0.2) + (-0.1)(-0.1) \end{bmatrix} = \begin{bmatrix} 0.39 & 0.08 \\ 0.28 & 0.15 \end{bmatrix}$$

33. Use a graphing calculator or computer.

$$A = \begin{bmatrix} 2 & -3 & 6 \\ -9 & 4 & -5 \\ 1 & -8 & 7 \end{bmatrix} \quad A^2 = \begin{bmatrix} 37 & -66 & 69 \\ -59 & 83 & -109 \\ 81 & -91 & 95 \end{bmatrix}$$

34. Use a graphing calculator or computer.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -1 & 2 & 0 \\ 3 & -1 & 2 \end{bmatrix} \quad A^2 = \begin{bmatrix} -8 & 3 & -9 \\ -3 & 4 & 3 \\ 10 & -4 & -5 \end{bmatrix}$$

35. **(a)** $AB = \begin{bmatrix} -2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 16 & 22 \\ 7 & 19 \end{bmatrix}$

(b) $BA = \begin{bmatrix} -2 & 1 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 0 & 30 \end{bmatrix}$

(c) No, AB and BA are not equal here.

(d) No, AB does not always equal BA.

36. Verify that $(PX)T = P(XT)$.

$$(PX)T = \left(\begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} mx + nz & my + nw \\ px + qz & py + qw \end{bmatrix}\begin{bmatrix} r & s \\ t & u \end{bmatrix}$$
$$= \begin{bmatrix} (mx + nz)r + (my + nw)t & (mx + nz)s + (my + nw)u \\ (px + qz)r + (py + qw)t & (px + qz)s + (py + qw)u \end{bmatrix}$$
$$= \begin{bmatrix} mxr + nzr + myt + nwt & mxs + nzs + myu + nwu \\ pxr + qzr + pyt + qwt & pxs + qzs + pyu + qwu \end{bmatrix}$$

$$P(XT) = \begin{bmatrix} m & n \\ p & q \end{bmatrix}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} r & s \\ t & u \end{bmatrix}\right) = \begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} xr + yt & xs + yu \\ zr + wt & zs + wu \end{bmatrix}$$
$$= \begin{bmatrix} m(xr + yt) + n(zr + wt) & m(xs + yu) + n(zs + wu) \\ p(xr + yt) + q(zr + wt) & p(xs + yu) + q(zs + wu) \end{bmatrix}$$
$$= \begin{bmatrix} mxr + myt + nzr + nwt & mxs + myu + nzs + nwu \\ pxr + pyt + qzr + qwt & pxs + pyu + qzs + qwu \end{bmatrix}$$
$$= \begin{bmatrix} mxr + nzr + myt + nwt & mxs + nzs + myu + nwu \\ pxr + qzr + pyt + qwt & pxs + qzs + pyu + qwu \end{bmatrix}$$

Thus, $(PX)T = P(XT)$.

37. Verify that $P(X + T) = PX + PT$.

Find $P(X + T)$ and $PX + PT$ separately and compare their values to see if they are the same.

$$P(X + T) = \begin{bmatrix} m & n \\ p & q \end{bmatrix}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} r & s \\ t & u \end{bmatrix}\right) = \begin{bmatrix} m & n \\ p & q \end{bmatrix}\left(\begin{bmatrix} x + r & y + s \\ z + t & w + u \end{bmatrix}\right)$$
$$= \begin{bmatrix} m(x + r) + n(z + t) & m(y + s) + n(w + u) \\ p(x + r) + q(z + t) & p(y + s) + q(w + u) \end{bmatrix} = \begin{bmatrix} mx + mr + nz + nt & my + ms + nw + nu \\ px + pr + qz + qt & py + ps + qw + qu \end{bmatrix}$$

$$PX + PT = \begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} mx + nz & my + nw \\ px + qz & py + qw \end{bmatrix} + \begin{bmatrix} mr + nt & ms + nu \\ pr + qt & ps + qu \end{bmatrix}$$
$$= \begin{bmatrix} (mx + nz) + (mr + nt) & (my + nw) + (ms + nu) \\ (px + qz) + (pr + qt) & (py + qw) + (ps + qu) \end{bmatrix} = \begin{bmatrix} mx + nz + mr + nt & my + nw + ms + nu \\ px + qz + pr + qt & py + qw + ps + qu \end{bmatrix}$$
$$= \begin{bmatrix} mx + mr + nz + nt & my + ms + nw + nu \\ px + pr + qz + qt & py + ps + qw + qu \end{bmatrix}$$

Observe that the two results are identical. Thus, $P(X + T) = PX + PT$.

38. Prove that $k(X + T) = kX + kT$ for any real number k.

$k(X+T)$

$$= k\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} r & s \\ t & u \end{bmatrix}\right)$$

$$= k\left(\begin{bmatrix} x+r & y+s \\ z+t & w+u \end{bmatrix}\right)$$

$$= \begin{bmatrix} k(x+r) & k(y+s) \\ k(z+t) & k(w+u) \end{bmatrix}$$

$$= \begin{bmatrix} kx+kr & ky+ks \\ kz+kt & kw+ku \end{bmatrix}$$ Distributive property for real numbers

$$= \begin{bmatrix} kx & ky \\ kz & kw \end{bmatrix} + \begin{bmatrix} kr & ks \\ kt & ku \end{bmatrix}$$

$$= k\begin{bmatrix} x & y \\ z & w \end{bmatrix} + k\begin{bmatrix} r & s \\ t & u \end{bmatrix} = kX + kT$$

39. Verify that $(k+h)P = kP + hP$ for any real numbers k and h.

$$(k+h)P = (k+h)\begin{bmatrix} m & n \\ p & q \end{bmatrix}$$

$$= \begin{bmatrix} (k+h)m & (k+h)n \\ (k+h)p & (k+h)q \end{bmatrix}$$

$$= \begin{bmatrix} km+hm & kn+hn \\ kp+hp & kq+hq \end{bmatrix}$$

$$= \begin{bmatrix} km & kn \\ kp & kq \end{bmatrix} + \begin{bmatrix} hm & hn \\ hp & hq \end{bmatrix}$$

$$= k\begin{bmatrix} m & n \\ p & q \end{bmatrix} + h\begin{bmatrix} m & n \\ p & q \end{bmatrix}$$

$$= kP + hP$$

Thus, $(k+h)P = kP + hP$ for any real numbers k and h.

40. (a) $IP = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m & n \\ p & q \end{bmatrix} = \begin{bmatrix} m & n \\ p & q \end{bmatrix} = P$

Thus, $IP = P$.

$$PI = \begin{bmatrix} m & n \\ p & q \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m & n \\ p & q \end{bmatrix} = P$$

Thus, $PI = P$.

$$IX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = X$$

Thus, $IX = X$.

(b) $IT = T$

The matrix I is called an identity matrix because it acts like the multiplicative identity for real numbers, which is 1.

If x is a real number,

$$1 \cdot x = x \cdot 1 = x.$$

If X is a 2×2 matrix,

$$IX = XI = X.$$

(c) I is called an identity matrix, because if a matrix is multiplied by I (assuming that these matrices are 2×2), the resulting matrix is the matrix you originally started with. Hence, I maintains the identity of any 2×2 matrix under multiplication.

41. $\begin{bmatrix} 2 & 3 & 1 \\ 1 & -4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + x_3 \\ x_1 - 4x_2 + 5x_3 \end{bmatrix}$,

and $\begin{bmatrix} 2x_1 + 3x_2 + x_3 \\ x_1 - 4x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$.

This is equivalent to

$$2x_1 + 3x_2 + x_3 = 5$$
$$x_1 - 4x_2 + 5x_3 = 8$$

since corresponding elements of equal matrices must be equal. Reversing this, observe that the given system of linear equations can be written as the matrix equation

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

42. $A = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, B = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$

If $AX = B$, then

$$\begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}.$$

Using the definition of matrix multiplication,

$$\begin{bmatrix} 1x_1 + 2x_2 \\ -3x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}.$$

By the definition of equality of matrices, corresponding elements must be equal, so

$$x_1 + 2x_2 = -4 \quad (1)$$
$$-3x_1 + 5x_2 = 12. \quad (2)$$

This is a linear system of two equations in two variables, x_1 and x_2.

Solve this system by the elimination (addition) method. Multiply equation (1) by 3 and add the result to equation (2).

$$\begin{aligned} 3x_1 + 6x_2 &= -12 \\ -3x_1 + 5x_2 &= 12 \\ \hline 11x_2 &= 0 \\ x_2 &= 0 \end{aligned}$$

Substitute 0 for x_2 in equation (1) to find x_1.

$$\begin{aligned} x_1 + 2(0) &= -4 \\ x_1 &= -4 \end{aligned}$$

The solution of the system is (–4, 0).

Substitute -4 for x_1 and 0 for x_2 in the matrix equation to check this result.

$$\begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(-4) + 2(0) \\ -3(-4) + 5(0) \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}.$$

43. **(a)** Use a graphing calculator or a computer to find the product matrix. The answer is

$$AC = \begin{bmatrix} 6 & 106 & 158 & 222 & 28 \\ 120 & 139 & 64 & 75 & 115 \\ -146 & -2 & 184 & 144 & -129 \\ 106 & 94 & 24 & 116 & 110 \end{bmatrix}.$$

(b) CA does not exist.

(c) AC and CA are clearly not equal, since CA does not even exist.

44. This exercise should be solved by graphing calculator or computer methods. The answers are as follows:

(a) $$CD = \begin{bmatrix} 44 & 75 & -60 & -33 & 11 \\ 20 & 169 & -164 & 18 & 105 \\ 113 & -82 & 239 & 218 & -55 \\ 119 & 83 & 7 & 82 & 106 \\ 162 & 20 & 175 & 143 & 74 \end{bmatrix};$$

(b) $$DC = \begin{bmatrix} 110 & 96 & 30 & 226 & 37 \\ -94 & 127 & 134 & -87 & -33 \\ -52 & 126 & 193 & 153 & 22 \\ 117 & 56 & -55 & 147 & 57 \\ 54 & 69 & 58 & 37 & 31 \end{bmatrix};$$

(c) No, $CD \neq DC$.

45. Use a graphing calculator or computer to find the matrix products and sums. The answers are as follows.

(a) $$C + D = \begin{bmatrix} -1 & 5 & 9 & 13 & -1 \\ 7 & 17 & 2 & -10 & 6 \\ 18 & 9 & -12 & 12 & 22 \\ 9 & 4 & 18 & 10 & -3 \\ 1 & 6 & 10 & 28 & 5 \end{bmatrix}$$

(b) $$(C + D)B = \begin{bmatrix} -2 & -9 & 90 & 77 \\ -42 & -63 & 127 & 62 \\ 413 & 76 & 180 & -56 \\ -29 & -44 & 198 & 85 \\ 137 & 20 & 162 & 103 \end{bmatrix}$$

(c) $$CB = \begin{bmatrix} -56 & -1 & 1 & 45 \\ -156 & -119 & 76 & 122 \\ 315 & 86 & 118 & -91 \\ -17 & -17 & 116 & 51 \\ 118 & 19 & 125 & 77 \end{bmatrix}$$

(d) $$DB = \begin{bmatrix} 54 & -8 & 89 & 32 \\ 114 & 56 & 51 & -60 \\ 98 & -10 & 62 & 35 \\ -12 & -27 & 82 & 34 \\ 19 & 1 & 37 & 26 \end{bmatrix}$$

(e) $$CB + DB = \begin{bmatrix} -2 & -9 & 90 & 77 \\ -42 & -63 & 127 & 62 \\ 413 & 76 & 180 & -56 \\ -29 & -44 & 198 & 85 \\ 137 & 20 & 162 & 103 \end{bmatrix}$$

(f) Yes, $(C + D)B$ and $CB + DB$ are equal, as can be seen by observing that the answers to parts (b) and (e) are identical.

46. Exercise 45 illustrates the distributive property.

47. **(a)** $$\begin{bmatrix} 10 & 4 & 3 & 5 & 6 \\ 7 & 2 & 2 & 3 & 8 \\ 4 & 5 & 1 & 0 & 10 \\ 0 & 3 & 4 & 5 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 4 & 3 \\ 3 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{array}{r} \\ \text{Dept. 1} \\ \text{Dept. 2} \\ \text{Dept. 3} \\ \text{Dept. 4} \end{array} \begin{array}{c} \begin{array}{cc} \text{A} & \text{B} \end{array} \\ \begin{bmatrix} 57 & 70 \\ 41 & 54 \\ 27 & 40 \\ 39 & 40 \end{bmatrix} \end{array}$$

(b) The total cost to buy from supplier A is $57 + 41 + 27 + 39 = \$164$, and the total cost to buy from supplier B is $70 + 54 + 40 + 40 = \$204$. The company should make the purchase from supplier A, since \$164 is a lower total cost than \$204.

48. (a)
$$\begin{array}{c} \\ S \\ C \end{array}\begin{array}{c} \begin{array}{ccc} CC & MM & AD \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.3 \end{bmatrix} \end{array}$$

(b)
$$\begin{array}{c} \\ SD \\ MC \\ M \end{array}\begin{array}{c} \begin{array}{cc} S & C \end{array} \\ \begin{bmatrix} 4 & 3 \\ 2 & 5 \\ 1 & 7 \end{bmatrix} \end{array}$$

(c)
$$\begin{bmatrix} 4 & 3 \\ 2 & 5 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} 0.5 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.3 \end{bmatrix}$$

$$= \begin{array}{c} \\ SD \\ MC \\ M \end{array}\begin{array}{c} \begin{array}{ccc} CC & MM & AD \end{array} \\ \begin{bmatrix} 2.6 & 2.5 & 2.1 \\ 2 & 2.3 & 2.1 \\ 1.9 & 2.5 & 2.4 \end{bmatrix} \end{array}$$

(d) Look at the entry in row 3, column 2 of the last matrix. The cost is \$2.50.

(e)
$$\begin{bmatrix} 2.6 & 2.5 & 2.1 \\ 2 & 2.3 & 2.1 \\ 1.9 & 2.5 & 2.4 \end{bmatrix}\begin{bmatrix} 100 \\ 200 \\ 500 \end{bmatrix} = \begin{bmatrix} 1810 \\ 1710 \\ 1890 \end{bmatrix}$$

The total sugar and chocolate cost is \$1810 in San Diego, \$1710 in Mexico City, and \$1890 in Managua, so the order can be produced for the lowest cost in Mexico City.

49. (a)
$$P = \begin{array}{c} \\ \text{Sal's} \\ \text{Fred's} \end{array}\begin{array}{c} \begin{array}{ccc} Sh & Sa & B \end{array} \\ \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \end{array}$$

(b)
$$F = \begin{array}{c} \\ Sh \\ Sa \\ B \end{array}\begin{array}{c} \begin{array}{cc} CA & AR \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix} \end{array}$$

(c)
$$PF = \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 80\left(\frac{1}{2}\right) + 40\left(\frac{1}{4}\right) + 120\left(\frac{1}{4}\right) & 80\left(\frac{1}{5}\right) + 40\left(\frac{1}{5}\right) + 120\left(\frac{3}{5}\right) \\ 60\left(\frac{1}{2}\right) + 30\left(\frac{1}{4}\right) + 150\left(\frac{1}{4}\right) & 60\left(\frac{1}{5}\right) + 30\left(\frac{1}{5}\right) + 150\left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 96 \\ 75 & 108 \end{bmatrix}$$

The rows give the average price per pair of footwear sold by each store, and the columns give the state.

(d) The average price of footwear at a Fred's outlet in Arizona is \$108.

50.
$$P = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix}, \quad Q = \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix},$$

$$R = \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

(a)
$$QR = \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix}\begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 10(20) + 2(180) + 0(60) + 2(25) \\ 50(20) + 1(180) + 20(60) + 2(25) \end{bmatrix}$$

$$= \begin{bmatrix} 610 \\ 2430 \end{bmatrix}$$

The rows represent the cost of materials for each type of house.

(b) From part (a), $QR = \begin{bmatrix} 610 \\ 2430 \end{bmatrix}$.

$$P(QR) = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix}\begin{bmatrix} 610 \\ 2430 \end{bmatrix} = \begin{bmatrix} 72{,}900 \\ 54{,}700 \\ 60{,}800 \end{bmatrix}$$

$$(PQ)R = \left(\begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix}\begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix}\right)\begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 1100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 72{,}900 \\ 54{,}700 \\ 60{,}800 \end{bmatrix}$$

Therefore, $P(QR) = (PQ)R$.

51. $\frac{1}{3}\left[\begin{bmatrix} 17 & 12 \\ 24 & 18 \end{bmatrix} + \begin{bmatrix} 15 & 15 \\ 36 & 25 \end{bmatrix} + \begin{bmatrix} 19 & 12 \\ 33 & 20 \end{bmatrix}\right] = \begin{bmatrix} 17 & 13 \\ 31 & 21 \end{bmatrix}$

52. **(a)** To increase by 25%, multiply by 1.25 or $\frac{5}{4}$. To increase by $\frac{1}{3}$, multiply by $1\frac{1}{3}$ or $\frac{4}{3}$. To increase by 10% multiply by 1.10 or $\frac{11}{10}$. The matrix is

$$\begin{bmatrix} \frac{5}{4} \\ \frac{4}{3} \\ \frac{11}{10} \end{bmatrix}.$$

(b) $\begin{bmatrix} 88 & 105 & 60 \\ 48 & 72 & 40 \\ 16 & 21 & 0 \\ 112 & 147 & 50 \end{bmatrix} \begin{bmatrix} \frac{5}{4} \\ \frac{4}{3} \\ \frac{11}{10} \end{bmatrix} = \begin{bmatrix} 316 \\ 200 \\ 48 \\ 391 \end{bmatrix}$

53. **(a)**

$$XY = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 7 \\ 0 & 10 & 1 \\ 0 & 15 & 2 \\ 10 & 12 & 8 \end{bmatrix} = \begin{bmatrix} 20 & 52 & 27 \\ 25 & 62 & 35 \\ 30 & 72 & 43 \end{bmatrix}$$

The rows give the amounts of fat, carbohydrates, and protein, respectively, in each of the daily meals.

(b) $YZ = \begin{bmatrix} 5 & 0 & 7 \\ 0 & 10 & 1 \\ 0 & 15 & 2 \\ 10 & 12 & 8 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 75 \\ 45 \\ 70 \\ 168 \end{bmatrix}$

The rows give the number of calories in one exchange of each of the food groups.

(c) Use the matrices found for XY and YZ from parts (a) and (b).

$$(XY)Z = \begin{bmatrix} 20 & 52 & 27 \\ 25 & 62 & 35 \\ 30 & 72 & 43 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 503 \\ 623 \\ 743 \end{bmatrix}$$

$$X(YZ) = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 75 \\ 45 \\ 70 \\ 168 \end{bmatrix} = \begin{bmatrix} 503 \\ 623 \\ 743 \end{bmatrix}$$

The rows give the number of calories in each meal.

54. $\frac{1}{2}\left[\begin{bmatrix} 60 & 6 \\ 49 & 9 \\ 61 & 16 \\ 82 & 4 \end{bmatrix} + \begin{bmatrix} 52 & 10 \\ 42 & 5 \\ 65 & 11 \\ 92 & 3 \end{bmatrix}\right] = \begin{bmatrix} 56 & 8 \\ 45.5 & 7 \\ 63 & 13.5 \\ 87 & 3.5 \end{bmatrix}$

55. $\frac{1}{6}\begin{bmatrix} 60.0 & 68.3 \\ 63.8 & 72.5 \\ 64.5 & 73.6 \\ 68.2 & 75.1 \\ 71.8 & 78.0 \end{bmatrix} + \frac{5}{6}\begin{bmatrix} 68.0 & 75.6 \\ 70.7 & 78.1 \\ 72.7 & 79.4 \\ 74.7 & 79.9 \\ 76.5 & 81.3 \end{bmatrix} = \begin{bmatrix} 66.7 & 74.4 \\ 69.6 & 77.2 \\ 71.3 & 78.4 \\ 73.6 & 79.1 \\ 75.7 & 80.8 \end{bmatrix}$

56. **(a)** Let n represent the present year. Then $j_n = 900,\ s_n = 500,\ a_n = 2600.$

For the year, $n + 1$, we have

$$\begin{bmatrix} j_n + 1 \\ s_n + 1 \\ a_n + 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_n \\ s_n \\ a_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} 900 \\ 500 \\ 2600 \end{bmatrix}$$

$$= \begin{bmatrix} 858 \\ 162 \\ 2799 \end{bmatrix}.$$

For the year $n + 1$, there is a total of $858 + 162 + 2799 = 3819$ female owls.

For the next year, $n + 2$, we have

$$\begin{bmatrix} j_{n+2} \\ s_{n+2} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_{n+1} \\ s_{n+1} \\ a_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} 858 \\ 162 \\ 2799 \end{bmatrix} \approx \begin{bmatrix} 924 \\ 154 \\ 2746 \end{bmatrix}.$$

For the year $n + 2$, there is a total of approximately $924 + 154 + 2746 = 3824$ female owls.

For the next year, $n + 3$, we have

$$\begin{bmatrix} j_{n+3} \\ s_{n+3} \\ a_{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_{n+2} \\ s_{n+2} \\ a_{n+2} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} 924 \\ 154 \\ 2746 \end{bmatrix} \approx \begin{bmatrix} 906 \\ 166 \\ 2691 \end{bmatrix}.$$

For the year $n + 3$, there is a total of approximately $906 + 166 + 2691 = 3763$ female owls.

For the next year, $n + 4$, we have

$$\begin{bmatrix} j_{n+4} \\ s_{n+4} \\ a_{n+4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_{n+3} \\ s_{n+3} \\ a_{n+3} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} 906 \\ 166 \\ 2691 \end{bmatrix} \approx \begin{bmatrix} 888 \\ 163 \\ 2647 \end{bmatrix}.$$

For the year $n + 4$, there is a total of approximately $888 + 163 + 2647 = 3698$ female owls.

For the year $n + 5$, we have

$$\begin{bmatrix} j_{n+5} \\ s_{n+5} \\ a_{n+5} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_{n+4} \\ s_{n+4} \\ a_{n+4} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} 888 \\ 163 \\ 2647 \end{bmatrix}$$
$$\approx \begin{bmatrix} 874 \\ 160 \\ 2604 \end{bmatrix}.$$

For the year $n + 5$, there is a total of approximately $874 + 160 + 2604 = 3638$ female owls.

(b) Each year, the population is about 98 percent of the population of the previous year. In the long run, the northern spotted owl will become extinct.

(c) Change 0.18 in the original matrix equation to 0.40.

For the next year, $n + 1$, we have

$$\begin{bmatrix} j_{n+1} \\ s_{n+1} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.40 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_{n} \\ s_{n} \\ a_{n} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0.33 \\ 0.40 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} 900 \\ 500 \\ 2600 \end{bmatrix}$$
$$= \begin{bmatrix} 858 \\ 360 \\ 2799 \end{bmatrix}.$$

For the year $n + 1$, there would be a total of $858 + 360 + 2799 = 4017$ female owls.

For the next year, $n + 2$, we have

$$\begin{bmatrix} j_{n+2} \\ s_{n+2} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.40 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_{n+1} \\ s_{n+1} \\ a_{n+1} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0.33 \\ 0.40 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} 858 \\ 360 \\ 2799 \end{bmatrix} \approx \begin{bmatrix} 924 \\ 343 \\ 2887 \end{bmatrix}.$$

For the year $n + 2$, there would be a total of approximately $924 + 343 + 2887 = 4154$ female owls. For the next year, $n + 3$, we have

$$\begin{bmatrix} j_{n+3} \\ s_{n+3} \\ a_{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.40 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_{n+2} \\ s_{n+2} \\ a_{n+2} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0.33 \\ 0.40 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} 924 \\ 343 \\ 2887 \end{bmatrix} = \begin{bmatrix} 953 \\ 370 \\ 2957 \end{bmatrix}.$$

For the year $n + 3$, there would be a total of approximately $953 + 370 + 2957 = 4280$ female owls.

For the next year, $n + 4$, we have

$$\begin{bmatrix} j_{n+4} \\ s_{n+4} \\ a_{n+4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.40 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_{n+3} \\ s_{n+3} \\ a_{n+3} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0.33 \\ 0.40 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} 953 \\ 370 \\ 2957 \end{bmatrix}$$
$$\approx \begin{bmatrix} 976 \\ 381 \\ 3042 \end{bmatrix}.$$

For the year $n + 4$, there would be a total of approximately $976 + 381 + 3042 = 4399$ female owls.

For the next year, $n + 5$, we have

$$\begin{bmatrix} j_{n+5} \\ s_{n+5} \\ a_{n+5} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.40 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_{n+4} \\ s_{n+4} \\ a_{n+4} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0.33 \\ 0.40 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} 976 \\ 381 \\ 3042 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1004 \\ 390 \\ 3130 \end{bmatrix}.$$

For the year $n + 5$, there would be a total of approximately $1004 + 390 + 3130 = 4524$ female owls.

Assuming that better habitat management could increase the survival rate of juvenile female spotted owls from 18 percent to 40 percent, the overall population would increase each year and would, therefore, not become extinct.

57. (a) The matrices are

$$A = \begin{bmatrix} 0.0346 & 0.0118 \\ 0.0174 & 0.0073 \\ 0.0189 & 0.0059 \\ 0.0135 & 0.0083 \\ 0.0099 & 0.0103 \end{bmatrix}$$

$$B = \begin{bmatrix} 361 & 2038 & 286 & 227 & 460 \\ 473 & 2494 & 362 & 252 & 484 \\ 627 & 2978 & 443 & 278 & 499 \\ 803 & 3435 & 524 & 314 & 511 \\ 1013 & 3824 & 591 & 344 & 522 \end{bmatrix}$$

(b) The total number of births and deaths each year is found by multiplying matrix B by matrix A.

$$BA = \begin{bmatrix} 361 & 2038 & 286 & 227 & 460 \\ 473 & 2494 & 362 & 252 & 484 \\ 627 & 2978 & 443 & 278 & 499 \\ 803 & 3435 & 524 & 314 & 511 \\ 1013 & 3824 & 591 & 344 & 522 \end{bmatrix} \begin{bmatrix} 0.0346 & 0.0118 \\ 0.0174 & 0.0073 \\ 0.0189 & 0.0059 \\ 0.0135 & 0.0083 \\ 0.0099 & 0.0103 \end{bmatrix}$$

$$= \begin{array}{c} \\ 1970 \\ 1980 \\ 1990 \\ 2000 \\ 2010 \end{array} \begin{array}{c} \text{Births} \quad \text{Deaths} \\ \begin{bmatrix} 60.98 & 27.45 \\ 74.80 & 33.00 \\ 90.58 & 39.20 \\ 106.75 & 45.51 \\ 122.57 & 51.59 \end{bmatrix} \end{array}$$

2.5 Matrix Inverses

Your Turn 1

Use row operations to transform the augmented matrix so that the identity matrix is the first three columns.

$$A = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 3 & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 7 & 3 & 1 & -2 & 0 \\ 0 & 9 & 5 & 0 & -3 & 1 \end{array}\right]$$

$$\tfrac{1}{7}R_2 \to R_2 \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{3}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 9 & 5 & 0 & -3 & 1 \end{array}\right]$$

$$\begin{array}{r} 2R_2 + R_1 \to R_1 \\ \\ -9R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & \frac{3}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & \frac{8}{7} & -\frac{9}{7} & -\frac{3}{7} & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \frac{7}{8}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & \frac{3}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & 1 & -\frac{9}{8} & -\frac{3}{8} & \frac{7}{8} \end{array}\right]$$

$$\begin{array}{r} \frac{1}{7}R_3 + R_1 \to R_1 \\ -\frac{3}{7}R_3 + R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & 1 & 0 & \frac{5}{8} & -\frac{1}{8} & -\frac{3}{8} \\ 0 & 0 & 1 & -\frac{9}{8} & -\frac{3}{8} & \frac{7}{8} \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{3}{8} \\ -\frac{9}{8} & -\frac{3}{8} & \frac{7}{8} \end{bmatrix}.$

Your Turn 2

Solve the linear system

$$5x + 4y = 23$$
$$4x - 3y = 6 \ .$$

Let $A = \begin{bmatrix} 5 & 4 \\ 4 & -3 \end{bmatrix}$ be the coefficient matrix,

$$B = \begin{bmatrix} 23 \\ 6 \end{bmatrix}, \text{ and } X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

First find A^{-1}.

$$\left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 4 & -3 & 0 & 1 \end{array}\right]$$

$$\tfrac{1}{5}R_1 \to R_1 \left[\begin{array}{cc|cc} 1 & \frac{4}{5} & \frac{1}{5} & 0 \\ 4 & -3 & 0 & 1 \end{array}\right]$$

$$-4R_1 + R_2 \to R_2 \left[\begin{array}{cc|cc} 1 & \frac{4}{5} & \frac{1}{5} & 0 \\ 0 & -\frac{31}{5} & -\frac{4}{5} & 1 \end{array}\right]$$

$$-\tfrac{5}{31}R_2 \to R_2 \left[\begin{array}{cc|cc} 1 & \frac{4}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{4}{31} & -\frac{5}{31} \end{array}\right]$$

$$-\tfrac{4}{5}R_2 + R_1 \to R_1 \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{31} & \frac{4}{31} \\ 0 & 1 & \frac{4}{31} & -\frac{5}{31} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{31} & \frac{4}{31} \\ \frac{4}{31} & -\frac{5}{31} \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} \frac{3}{31} & \frac{4}{31} \\ \frac{4}{31} & -\frac{5}{31} \end{bmatrix} \begin{bmatrix} 23 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The solution to the system is $(3, 2)$.

Your Turn 3

(a) The word "behold" gives two 3×1 matrices:

$$\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \text{ and } \begin{bmatrix} 15 \\ 12 \\ 4 \end{bmatrix}$$

Use the coding matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$ to find the product of A with each column matrix.

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 49 \\ 33 \\ 26 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 15 \\ 12 \\ 4 \end{bmatrix} = \begin{bmatrix} 67 \\ 54 \\ 88 \end{bmatrix}$$

The coded message is 49, 33, 26, 67, 54, 88.

(b) Use the inverse of the coding matrix to decode the message 96, 87, 74, 141, 117, 114.

$$A^{-1} = \begin{bmatrix} -0.2 & 0.2 & 0.2 \\ 0.4 & -0.6 & 0.2 \\ 0 & 0.4 & -0.2 \end{bmatrix}, \begin{bmatrix} 96 \\ 87 \\ 74 \end{bmatrix}, \text{ and } \begin{bmatrix} 141 \\ 117 \\ 114 \end{bmatrix}$$

$$\begin{bmatrix} -0.2 & 0.2 & 0.2 \\ 0.4 & -0.6 & 0.2 \\ 0 & 0.4 & -0.2 \end{bmatrix}\begin{bmatrix} 96 \\ 87 \\ 74 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} -0.2 & 0.2 & 0.2 \\ 0.4 & -0.6 & 0.2 \\ 0 & 0.4 & -0.2 \end{bmatrix}\begin{bmatrix} 141 \\ 117 \\ 114 \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \\ 24 \end{bmatrix}$$

The message is the word "matrix."

2.5 Warmup Exercises

W1.

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 3 & -2 & 8 \end{array}\right]$$

$$-\tfrac{3}{2}R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & -\frac{7}{2} & \frac{7}{2} \end{array}\right]$$

$$\begin{array}{l} \tfrac{1}{2}R_1 \to R_1 \\ -\tfrac{2}{7}R_2 \to R_2 \end{array} \quad \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -1 \end{array}\right]$$

$$-\tfrac{1}{2}R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

The solution is $(2, -1)$.

W2.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -4 \\ 2 & -1 & -2 & 9 \\ 1 & 2 & 3 & -5 \end{array}\right]$$

$$\begin{array}{l} (-2)R_1 + R_2 \to R_2 \\ (-1)R_1 + R_3 \to R_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & -4 \\ 0 & -1 & -6 & 17 \\ 0 & 2 & 1 & -1 \end{array}\right]$$

$2R_2 + R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -4 \\ 0 & -1 & -6 & 17 \\ 0 & 0 & -11 & 33 \end{array}\right]$

$(-1)R_2 \to R_2$, $\left(\frac{1}{11}\right)R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -4 \\ 0 & 1 & 6 & -17 \\ 0 & 0 & 1 & -3 \end{array}\right]$

$(-2)R_3 + R_1 \to R_1$, $(-6)R_2 \to R_2$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right]$

The solution is $(2, 1, -3)$.

W3.

$$\begin{bmatrix} 0.2 & 0.5 \\ -0.4 & 1.1 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} (0.2)(4) + (0.5)(6) \\ (-0.4)(4) + (1.1)(6) \end{bmatrix} = \begin{bmatrix} 3.8 \\ 5.0 \end{bmatrix}$$

W4.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & -5 \\ -2 & 4 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (1)(-1) + (-1)(2) + (0)(3) \\ (0)(-1) + (3)(2) + (-5)(3) \\ (-2)(-1) + (4)(2) + (2)(3) \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \\ 16 \end{bmatrix}$$

2.5 Exercises

1. $$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & -2+2 \\ 15-15 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & 3-3 \\ -10+10 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since the products obtained by multiplying the matrices in either order are both the 2×2 identity matrix, the given matrices are inverses of each other.

2. $$\begin{bmatrix} 1 & -4 \\ 2 & -7 \end{bmatrix}\begin{bmatrix} -7 & 4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -7+8 & 4-4 \\ -14+14 & 8-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\begin{bmatrix} -7 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2 & -7 \end{bmatrix} = \begin{bmatrix} -7+8 & 28-28 \\ -2+2 & 8-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since the products obtained by multiplying the matrices in either order are both the 2×2 identity matrix, the given matrices are inverses of each other.

3. $$\begin{bmatrix} 2 & 6 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 10 & -20 \\ 6 & -12 \end{bmatrix} \neq I$$

No, the matrices are not inverses of each other since their product matrix is not *I*.

4. $$\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \neq I$$

Since this product is not the 2×2 identity matrix, the given matrices are not inverses of each other.

5. $$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2+0-1 & 2+0-2 & -2+0+2 \\ 1+0-2 & 1+1-4 & -1+0+4 \\ 0+0+0 & 0+1+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -2 & 3 \\ 0 & 1 & 0 \end{bmatrix} \neq I$$

No, the matrices are not inverses of each other since their product matrix is not *I*.

6. $$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq I$$

Since this product is not the 3×3 identity matrix, the given matrices are not inverses of each other.

7. $\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}\begin{bmatrix}7&-3&-3\\-1&1&0\\-1&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$

$\begin{bmatrix}7&-3&-3\\-1&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$

Yes, these matrices are inverses of each other.

8. $\begin{bmatrix}1&0&0\\-1&-2&3\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\\frac{1}{3}&\frac{1}{3}&\frac{2}{3}\end{bmatrix}$

$=\begin{bmatrix}1+0+0&0+0+0&0+0+0\\-1+0+1&0+0+1&0-2+2\\0+0+0&0+0+0&0+1+0\end{bmatrix}$

$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$

$\begin{bmatrix}1&0&0\\0&0&1\\\frac{1}{3}&\frac{1}{3}&\frac{2}{3}\end{bmatrix}\begin{bmatrix}1&0&0\\-1&-2&3\\0&1&0\end{bmatrix}$

$=\begin{bmatrix}1+0+0&0+0+0&0+0+0\\0+0+0&0+0+1&0+0+0\\\frac{1}{3}-\frac{1}{3}+0&0-\frac{2}{3}+\frac{2}{3}&0+1+0\end{bmatrix}$

$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$

Since the products obtained by multiplying the matrices in either order are both the 3×3 identity matrix, the given matrices are inverses of each other.

9. No, a matrix with a row of all zeros does not have an inverse; the row of all zeros makes it impossible to get all the 1's in the main diagonal of the identity matrix.

10. Since the inverse of A is $A^{-1}, (A^{-1})^{-1}$ would be the inverse of the inverse of A, which is A.

11. Let $A=\begin{bmatrix}1&-1\\2&0\end{bmatrix}$.

Form the augmented matrix $[A|I]$.

$$[A|I]=\left[\begin{array}{cc|cc}1&-1&1&0\\2&0&0&1\end{array}\right]$$

Perform row operations on $[A|I]$ to get a matrix of the form $[I|B]$.

$$\left[\begin{array}{cc|cc}1&-1&1&0\\2&0&0&1\end{array}\right]$$

$$-2R_1+R_2\to R_2\quad\left[\begin{array}{cc|cc}1&-1&1&0\\0&2&-2&1\end{array}\right]$$

$$2R_1+R_2\to R_1\quad\left[\begin{array}{cc|cc}2&0&0&1\\0&2&-2&1\end{array}\right]$$

$$\begin{array}{l}\frac{1}{2}R_1\to R_1\\ \frac{1}{2}R_2\to R_2\end{array}\quad\left[\begin{array}{cc|cc}1&0&0&\frac{1}{2}\\0&1&-1&\frac{1}{2}\end{array}\right]=[I|B]$$

The matrix B in the last transformation is the desired multiplicative inverse.

$$A^{-1}=\begin{bmatrix}0&\frac{1}{2}\\-1&\frac{1}{2}\end{bmatrix}$$

This answer may be checked by showing that $AA^{-1}=I$ and $A^{-1}A=I$.

12. Find the inverse of $A=\begin{bmatrix}1&1\\2&3\end{bmatrix}$, if it exists.

Write the augmented matrix $[A|I]$.

$$[A|I]=\left[\begin{array}{cc|cc}1&1&1&0\\2&3&0&1\end{array}\right]$$

Perform row operations on $[A|I]$ to get a matrix of the form $[I|B]$.

$$-2R_1+R_2\to R_2\quad\left[\begin{array}{cc|cc}1&1&1&0\\0&1&-2&1\end{array}\right]$$

$$-1R_2+R_1\to R_1\quad\left[\begin{array}{cc|cc}1&0&3&-1\\0&1&-2&1\end{array}\right]$$

The last augmented matrix is of the form $[I|B]$, so the desired inverse is

$$A^{-1}=\begin{bmatrix}3&-1\\-2&1\end{bmatrix}.$$

13. Let $A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}.$

$$[A|I] = \left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ -5 & 2 & 0 & 1 \end{array}\right]$$

$$5R_1 + 3R_2 \to R_2 \quad \left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 0 & 1 & 5 & 3 \end{array}\right]$$

$$R_1 + R_2 \to R_1 \quad \left[\begin{array}{cc|cc} 3 & 0 & 6 & 3 \\ 0 & 1 & 5 & 3 \end{array}\right]$$

$$\tfrac{1}{3}R_1 \to R_1 \quad \left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 5 & 3 \end{array}\right] = [I|B]$$

The desired inverse is

$$A^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}.$$

14. Find the inverse of $A = \begin{bmatrix} -3 & -8 \\ 1 & 3 \end{bmatrix}$, if it exists.

Write the augmented matrix $[A|I]$.

$$[A|I] = \left[\begin{array}{cc|cc} -3 & -8 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

Perform row operations on $[A|I]$ to get a matrix of the form $[I|B]$.

$$R_1 + 3R_2 \to R_2 \quad \left[\begin{array}{cc|cc} -3 & -8 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$8R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|cc} -3 & 0 & 9 & 24 \\ 0 & -1 & 1 & 3 \end{array}\right]$$

$$-\tfrac{1}{3}R_1 \to R_1 \quad \left[\begin{array}{cc|cc} 1 & 0 & -3 & -8 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

The last augmented matrix is of the form $[I|B]$, so the desired inverse is $A^{-1} = \begin{bmatrix} -3 & -8 \\ 1 & 3 \end{bmatrix}.$

Note that matrix A is its own inverse: $AA = I$.

15. Let $A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}.$

$$[A|I] = \left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{array}\right]$$

$$2R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

Because the last row has all zeros to the left of the vertical bar, there is no way to complete the desired transformation. A has no inverse.

16. Find the inverse of $A = \begin{bmatrix} 5 & 10 \\ -3 & -6 \end{bmatrix}$, if it exists.

$$[A|I] = \left[\begin{array}{cc|cc} 5 & 10 & 1 & 0 \\ -3 & -6 & 0 & 1 \end{array}\right]$$

$$3R_1 + 5R_2 \to R_2 \quad \left[\begin{array}{cc|cc} 5 & 10 & 1 & 0 \\ 0 & 0 & 3 & 5 \end{array}\right]$$

At this point, the matrix should be changed so that the first row, second column element will be 0. Since this cannot be done using row operations, the inverse of the given matrix does not exist.

17. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$-1R_1 + R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$-1R_2 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

18. Find the inverse of $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$, if it exists.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$-1R_1 + R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -3 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} -3R_2 + 2R_1 \to R_1 \\ \\ 3R_2 + 2R_3 \to R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 2 & 0 & 3 & 2 & -3 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 3 & 2 \end{array}\right]$$

$$\begin{matrix} -3R_3 + R_1 \to R_1 \\ R_3 + R_2 \to R_2 \\ \end{matrix} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 8 & -12 & -6 \\ 0 & 2 & 0 & -2 & 4 & 2 \\ 0 & 0 & 1 & -2 & 3 & 2 \end{array}\right]$$

$$\begin{matrix} \frac{1}{2}R_1 \to R_1 \\ \frac{1}{2}R_2 \to R_2 \\ \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -6 & -3 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 3 & 2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 4 & -6 & -3 \\ -1 & 2 & 1 \\ -2 & 3 & 2 \end{bmatrix}$$

19. Let $A = \begin{bmatrix} -1 & -1 & -1 \\ 4 & 5 & 0 \\ 0 & 1 & -3 \end{bmatrix}.$

$$[A|I] = \left[\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ 4 & 5 & 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$4R_1 + R_2 \to R_2 \left[\begin{array}{ccc|ccc} -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -4 & 4 & 1 & 0 \\ 0 & 1 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} R_2 + R_1 \to R_1 \\ \\ -1R_2 + R_3 \to R_3 \end{matrix} \left[\begin{array}{ccc|ccc} -1 & 0 & -5 & 5 & 1 & 0 \\ 0 & 1 & -4 & 4 & 1 & 0 \\ 0 & 0 & 1 & -4 & -1 & 1 \end{array}\right]$$

$$\begin{matrix} 5R_3 + R_1 \to R_1 \\ 4R_3 + R_2 \to R_2 \\ \end{matrix} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -15 & -4 & 5 \\ 0 & 1 & 0 & -12 & -3 & 4 \\ 0 & 0 & 1 & -4 & -1 & 1 \end{array}\right]$$

$$-1R_1 \to R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 15 & 4 & -5 \\ 0 & 1 & 0 & -12 & -3 & 4 \\ 0 & 0 & 1 & -4 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 15 & 4 & -5 \\ -12 & -3 & 4 \\ -4 & -1 & 1 \end{bmatrix}$$

20. Find the inverse of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 4 & -1 & -3 \end{bmatrix}$, if it exists.

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 4 & -1 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} \\ \\ -2R_1 + R_3 \to R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & -3 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} -1R_2 + 3R_1 \to R_1 \\ \\ R_2 + R_3 \to R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 6 & 0 & -1 & 3 & -1 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -2 & 1 & 1 \end{array}\right]$$

$$\begin{matrix} -1R_3 + 2R_1 \to R_1 \\ R_3 + 2R_2 \to R_2 \\ \end{matrix} \left[\begin{array}{ccc|ccc} 12 & 0 & 0 & 8 & -3 & -1 \\ 0 & 1 & 0 & -2 & 3 & 1 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{array}\right]$$

$$\begin{matrix} \frac{1}{12}R_1 \to R_1 \\ \frac{1}{6}R_2 \to R_2 \\ -\frac{1}{2}R_3 \to R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{4} & -\frac{1}{12} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{12} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

21. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -3 & -2 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} \\ 3R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 8 & 3 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} R_2 + (-2R_1) \to R_1 \\ \\ R_2 + (-2R_3) \to R_3 \end{matrix} \left[\begin{array}{ccc|ccc} -2 & 0 & 2 & 1 & 1 & 0 \\ 0 & 4 & 8 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right]$$

Because the last row has all zeros to the left of the vertical bar, there is no way to complete the desired transformation. A has no inverse.

22. Find the inverse of $A = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 0 & -1 \\ 0 & 0 & -2 \end{bmatrix}$, if it exists.

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 4 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 3 & 0 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_1+(-2)R_2\to R_2 \\ 3R_1+(-2)R_3\to R_3 \end{array}\left[\begin{array}{ccc|ccc} 2 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 6 & 1 & -2 & 0 \\ 0 & 0 & 16 & 3 & 0 & -2 \end{array}\right]$$

Since the second column is all zeros, it will not be possible to get a 1 in the second row, second column position. Therefore, the inverse of the given matrix does not exist.

23. Find the inverse of $A=\begin{bmatrix} 1 & 3 & -2 \\ 2 & 7 & -3 \\ 3 & 8 & -5 \end{bmatrix}$, if it exists.

$$[A|I]=\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 7 & -3 & 0 & 1 & 0 \\ 3 & 8 & -5 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ -2R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3 \end{array}\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} -3R_2+R_1\to R_1 \\ \\ R_2+R_3\to R_3 \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 7 & -3 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & -5 & 1 & 1 \end{array}\right]$$

$$\begin{array}{l} 5R_3+2R_1\to R_1 \\ -1R_3+2R_2\to R_2 \\ \\ \end{array}\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -11 & -1 & 5 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -5 & 1 & 1 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{2}R_1\to R_1 \\ \frac{1}{2}R_2\to R_2 \\ \frac{1}{2}R_3\to R_3 \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{11}{2} & -\frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} -\frac{11}{2} & -\frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{5}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

24. Find the inverse of $A=\begin{bmatrix} 4 & 1 & -4 \\ 2 & 1 & -1 \\ -2 & -4 & 5 \end{bmatrix}$, if it exists.

$$[A|I]=\left[\begin{array}{ccc|ccc} 4 & 1 & -4 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ -2 & -4 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ -R_1+2R_2\to R_2 \\ R_1+2R_3\to R_3 \end{array}\left[\begin{array}{ccc|ccc} 4 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 2 & 0 \\ 0 & -7 & 6 & 1 & 0 & 2 \end{array}\right]$$

$$\begin{array}{l} -1R_2+R_1\to R_1 \\ \\ 7R_2+R_3\to R_3 \end{array}\left[\begin{array}{ccc|ccc} 4 & 0 & -6 & 2 & -2 & 0 \\ 0 & 1 & 2 & -1 & 2 & 0 \\ 0 & 0 & 20 & -6 & 14 & 2 \end{array}\right]$$

$$\begin{array}{l} 3R_3+10R_1\to R_1 \\ -1R_3+10R_2\to R_2 \\ \\ \end{array}\left[\begin{array}{ccc|ccc} 40 & 0 & 0 & 2 & 22 & 6 \\ 0 & 10 & 0 & -4 & 6 & -2 \\ 0 & 0 & 20 & -6 & 12 & 2 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{40}R_1\to R_1 \\ \frac{1}{10}R_2\to R_2 \\ \frac{1}{20}R_3\to R_3 \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{20} & \frac{11}{20} & \frac{3}{20} \\ 0 & 1 & 0 & -\frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{3}{10} & \frac{7}{10} & \frac{1}{10} \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} \frac{1}{20} & \frac{11}{20} & \frac{3}{20} \\ -\frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{3}{10} & \frac{7}{10} & \frac{1}{10} \end{bmatrix}$$

25. Let $A=\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ -2 & 2 & -2 & 4 \\ 0 & 2 & -3 & 1 \end{bmatrix}$.

$$[A|I]=\left[\begin{array}{cccc|cccc} 1 & -2 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ -2 & 2 & -2 & 4 & 0 & 0 & 1 & 0 \\ 0 & 2 & -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ \\ \\ 2R_1+R_3\to R_3 \end{array}\left[\begin{array}{cccc|cccc} 1 & -2 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 4 & 4 & 2 & 0 & 1 & 0 \\ 0 & 2 & -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} 2R_2+R_1\to R_1 \\ \\ 2R_2+R_3\to R_3 \\ -2R_2+R_4\to R_4 \end{array}\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 6 & 2 & 2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_3+(-2)R_1\to R_1 \\ R_3+2R_2\to R_2 \\ \\ R_3+2R_4\to R_4 \end{array}\left[\begin{array}{cccc|cccc} -2 & 0 & 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 2 & 0 & 8 & 2 & 4 & 1 & 0 \\ 0 & 0 & 2 & 6 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 4 & 2 & -2 & 1 & 2 \end{array}\right]$$

$$\begin{array}{l} R_4+(-2)R_1\to R_1 \\ -2R_4+R_2\to R_2 \\ -3R_4+2R_3\to R_3 \\ \\ \end{array}\left[\begin{array}{cccc|cccc} 4 & 0 & 0 & 0 & 2 & 2 & -1 & 2 \\ 0 & 2 & 0 & 0 & -2 & 8 & -1 & -4 \\ 0 & 0 & 4 & 0 & -2 & 10 & -1 & -6 \\ 0 & 0 & 0 & 4 & 2 & -2 & 1 & 2 \end{array}\right]$$

$$\begin{matrix} \frac{1}{4}R_1 \to R_1 \\ \frac{1}{2}R_2 \to R_2 \\ \frac{1}{4}R_3 \to R_3 \\ \frac{1}{4}R_4 \to R_4 \end{matrix} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -1 & 4 & -\frac{1}{2} & -2 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{5}{2} & -\frac{1}{4} & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \\ -1 & 4 & -\frac{1}{2} & -2 \\ -\frac{1}{2} & \frac{5}{2} & -\frac{1}{4} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

26. Find the inverse of $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & -1 & 1 & -1 \\ 3 & 3 & 2 & -2 \\ 1 & 2 & 1 & 1 \end{bmatrix}$, if it exists.

$$[A|I] = \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 3 & 3 & 2 & -2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Carry out these row operations to get the next matrix: $-2R_1 + R_2 \to R_2$, $-3R_1 + R_3 \to R_3$, $-1R_1 + R_4 \to R_4$.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & -5 & -2 & 1 & 0 & 0 \\ 0 & 0 & 2 & -8 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & -1 & 0 & 0 & 1 \end{array}\right]$$

Carry out the row operations $R_2 + 3R_1 \to R_1$, $R_2 + 3R_4 \to R_4$

$$\left[\begin{array}{cccc|cccc} 3 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & -5 & -2 & 1 & 0 & 0 \\ 0 & 0 & 2 & -8 & -3 & 0 & 1 & 0 \\ 0 & 0 & 4 & -11 & -5 & 1 & 0 & 3 \end{array}\right]$$

$R_3 + (-2)R_1 \to R_1$, $R_3 + (-2)R_2 \to R_2$,

$-2R_3 + R_4 \to R_4$

$$\left[\begin{array}{cccc|cccc} -6 & 0 & 0 & -10 & -5 & -2 & 1 & 0 \\ 0 & 6 & 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & -8 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 & 1 & -2 & 3 \end{array}\right]$$

$2R_4 + R_1 \to R_1$, $-2R_4 + 5R_2 \to R_2$, $8R_4 + 5R_3 \to R_3$

$$\left[\begin{array}{cccc|cccc} -6 & 0 & 0 & 0 & -3 & 0 & -3 & 6 \\ 0 & 30 & 0 & 0 & 3 & -12 & 9 & -6 \\ 0 & 0 & 10 & 0 & -7 & 8 & -11 & 24 \\ 0 & 0 & 0 & 5 & 1 & 1 & -2 & 3 \end{array}\right]$$

$-\frac{1}{6}R_1 \to R_1$, $\frac{1}{30}R_2 \to R_2$, $\frac{1}{10}R_3 \to R_3$, $\frac{1}{5}R_4 \to R_4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{10} & -\frac{2}{5} & \frac{3}{10} & -\frac{1}{5} \\ 0 & 0 & 1 & 0 & -\frac{7}{10} & \frac{4}{5} & -\frac{11}{10} & \frac{12}{5} \\ 0 & 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & -1 \\ \frac{1}{10} & -\frac{2}{5} & \frac{3}{10} & -\frac{1}{5} \\ -\frac{7}{10} & \frac{4}{5} & -\frac{11}{10} & \frac{12}{5} \\ \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

27. $2x + 5y = 15$
$x + 4y = 9$

First, write the system in matrix form.

$$\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 15 \\ 9 \end{bmatrix}$.

The system in matrix form is $AX = B$. We wish to find $X = A^{-1}AX = A^{-1}B$. Use row operations to find A^{-1}.

$$[A|I] = \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$$

$$-1R_1 + 2R_2 \to R_2 \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & 3 & -1 & 2 \end{array}\right]$$

$$-5R_2 + 3R_1 \to R_1 \left[\begin{array}{cc|cc} 6 & 0 & 8 & -10 \\ 0 & 3 & -1 & 2 \end{array}\right]$$

$$\begin{matrix} \frac{1}{6}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \end{matrix} \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{3} & -\frac{5}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{4}{3} & -\frac{5}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & -5 \\ -1 & 2 \end{bmatrix}$$

Next find the product $A^{-1}B$.

$$X = A^{-1}B = \begin{bmatrix} \frac{4}{3} & -\frac{5}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 15 \\ 9 \end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} 4 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 15 \\ 9 \end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} 15 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Thus, the solution is (5, 1).

28. $-x + 2y = 15$
$-2x - y = 20$

This system may be written as the matrix equation

$$\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix}.$$

In Exercise 12, it was calculated that

$$\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix},$$

and we now know that $X = A^{-1}B$ is the solution to $AX = B$.

Thus,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 15 \\ 20 \end{bmatrix} = \begin{bmatrix} -11 \\ 2 \end{bmatrix},$$

and $(-11, 2)$ is the solution of the system.

29. $2x + y = 5$
$5x + 3y = 13$

Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$.

Use row operations to obtain

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}.$$

$$X = A^{-1}B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 13 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The solution is (2, 1).

30. $-x - 2y = 8$
$3x + 4y = 24$

This system may be written as the matrix equation

$$\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 24 \end{bmatrix}.$$

In Exercise 14, it was calculated that

$$\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix},$$

and since $X = A^{-1}B$, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 8 \\ 24 \end{bmatrix} = \begin{bmatrix} 40 \\ -24 \end{bmatrix}.$$

Therefore, the solution is $(40, -24)$.

31. $3x - 2y = 3$
$7x - 5y = 0$

First, write the system in matrix form.

$$\begin{bmatrix} 3 & -2 \\ 7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Let $A = \begin{bmatrix} 3 & -2 \\ 7 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

The system is in matrix form $AX = B$. We wish to find $X = A^{-1}\ AX = A^{-1}B$. Use row operations to find A^{-1}.

$$[A|I] = \left[\begin{array}{cc|cc} 3 & -2 & 1 & 0 \\ 7 & -5 & 0 & 1 \end{array}\right]$$

$$-7R_1 + 3R_2 \rightarrow R_2 \quad \left[\begin{array}{cc|cc} 3 & -2 & 1 & 0 \\ 0 & -1 & -7 & 3 \end{array}\right]$$

$$-2R_2 + R_1 \rightarrow R_1 \quad \left[\begin{array}{cc|cc} 3 & 0 & 15 & -6 \\ 0 & -1 & -7 & 3 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{3}R_1 \rightarrow R_1 \\ -1R_2 \rightarrow R_2 \end{array} \quad \left[\begin{array}{cc|cc} 1 & 0 & 5 & -2 \\ 0 & 1 & 7 & -3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ 7 & -3 \end{bmatrix}$$

Next find the product $A^{-1}B$.

$$X = A^{-1}B = \begin{bmatrix} 5 & -2 \\ 7 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 21 \end{bmatrix}$$

Thus, the solution is (15, 21).

32. $3x - 6y = 1$
$-5x + 9y = -1$

This system may be written as the matrix equation

$$\begin{bmatrix} 3 & -6 \\ -5 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Calculate the inverse of

$$A = \begin{bmatrix} 3 & -6 \\ -5 & 9 \end{bmatrix}.$$

$$A^{-1} = \begin{bmatrix} -3 & -2 \\ -\frac{5}{3} & -1 \end{bmatrix},$$

so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -\frac{5}{3} & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{2}{3} \end{bmatrix}.$

The solution is $\left(-1, -\frac{2}{3}\right)$.

33. $-x - 8y = 12$
$3x + 24y = -36$

Let $A = \begin{bmatrix} -1 & -8 \\ 3 & 24 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 12 \\ -36 \end{bmatrix}$.

Using row operations on $[A|I]$ leads to the matrix

$$\left[\begin{array}{cc|cc} 1 & 8 & -1 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right],$$

but the zeros in the second row indicate that matrix A does not have an inverse. We cannot complete the solution by this method.

Since the second equation is a multiple of the first, the equations are dependent. Solve the first equation of the system for x.

$$\begin{aligned} -x - 8y &= 12 \\ -x &= 8y + 12 \\ x &= -8y - 12 \end{aligned}$$

The solution is $(-8y - 12, y)$, where y is any real number.

34. $2x + 7y = 14$
$4x + 14y = 28$

First, write the system in matrix form.

$$\begin{bmatrix} 2 & 7 \\ 4 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 28 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & 7 \\ 4 & 14 \end{bmatrix}$. Use row operations to find A^{-1}.

$$[A|I] = \left[\begin{array}{cc|cc} 2 & 7 & 1 & 0 \\ 4 & 14 & 0 & 1 \end{array}\right]$$

$$-2R_1 + R_2 \to R_2 \left[\begin{array}{cc|cc} 2 & 7 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

The zeros in the second row indicate that matrix A does not have an inverse. Since the second equation is a multiple of the first, the equations are dependent. Solve the first equation for x.

$$\begin{aligned} 2x + 7y &= 14 \\ 2x &= 14 - 7y \\ x &= \frac{14 - 7y}{2} \end{aligned}$$

The solution is $\left(\frac{14 - 7y}{2}, y\right)$, where y is any real number.

35. $-x - y - z = 1$
$4x + 5y = -2$
$y - 3z = 3$

has coefficient matrix

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 4 & 5 & 0 \\ 0 & 1 & -3 \end{bmatrix}.$$

In Exercise 19, it was found that

$$A^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ 4 & 5 & 0 \\ 0 & 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 15 & 4 & -5 \\ -12 & -3 & 4 \\ -4 & -1 & 1 \end{bmatrix}.$$

Since $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 & 4 & -5 \\ -12 & -3 & 4 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \\ 1 \end{bmatrix}.$$

The solution is $(-8, 6, 1)$.

36. $2x + y = 1$
$3y + z = 8$
$4x - y - 3z = 8$

has coefficient matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 4 & -1 & -3 \end{bmatrix}.$$

In Exercise 20, it was calculated that

$$A^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 4 & -1 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{12} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Since $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{12} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ -7 \end{bmatrix}.$$

Thus, the solution is $(-2, 5, -7)$.

37.
$$\begin{aligned} x + 3y - 2z &= 4 \\ 2x + 7y - 3z &= 8 \\ 3x + 8y - 5z &= -4 \end{aligned}$$

has coefficient matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 7 & -3 \\ 3 & 8 & -5 \end{bmatrix}.$$

In Exercise 23, it was calculated that

$$A^{-1} = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 7 & -3 \\ 3 & 8 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{11}{2} & -\frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{5}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -11 & -1 & 5 \\ 1 & 1 & -1 \\ -5 & 1 & 1 \end{bmatrix}.$$

Since $X = A^{-1}B$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -11 & -1 & 5 \\ 1 & 1 & -1 \\ -5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ -4 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -72 \\ 16 \\ -16 \end{bmatrix} = \begin{bmatrix} -36 \\ 8 \\ -8 \end{bmatrix}$$

Thus, the solution is $(-36, 8, -8)$.

38.
$$\begin{aligned} 4x + y - 4z &= 17 \\ 2x + y - z &= 12 \\ -2x - 4y + 5z &= 17 \end{aligned}$$

has coefficient matrix

$$A = \begin{bmatrix} 4 & 1 & -4 \\ 2 & 1 & -1 \\ -2 & -4 & 5 \end{bmatrix}.$$

In Exercise 24, it was calculated that

$$A^{-1} = \begin{bmatrix} 4 & 1 & -4 \\ 2 & 1 & -1 \\ -2 & -4 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{20} & \frac{11}{20} & \frac{3}{20} \\ -\frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{3}{10} & \frac{7}{10} & \frac{1}{10} \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 1 & 11 & 3 \\ -8 & 12 & -4 \\ -6 & 14 & 2 \end{bmatrix}.$$

Since $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 1 & 11 & 3 \\ -8 & 12 & -4 \\ -6 & 14 & 2 \end{bmatrix} \begin{bmatrix} 17 \\ 12 \\ 17 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 200 \\ -60 \\ 100 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ 5 \end{bmatrix}.$$

Thus, the solution is $(10, -3, 5)$.

39.
$$\begin{aligned} 2x - 2y &= 5 \\ 4y + 8z &= 7 \\ x + 2z &= 1 \end{aligned}$$

has coefficient matrix

$$A = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & 8 \\ 1 & 0 & 2 \end{bmatrix}.$$

However, using row operations on $[A|I]$ shows that A does not have an inverse, so another method must be used.

Try the Gauss-Jordan method. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 5 \\ 0 & 4 & 8 & 7 \\ 1 & 0 & 2 & 1 \end{array}\right].$$

After several row operations, we obtain the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{17}{4} \\ 0 & 1 & 2 & \frac{7}{4} \\ 0 & 0 & 0 & 13 \end{array}\right].$$

The bottom row of this matrix shows that the system has no solution, since $0 = 13$ is a false statement.

40.
$$\begin{aligned} x \quad\quad + 2z &= -1 \\ y - z &= 5 \\ x + 2y \quad\quad &= 7 \end{aligned}$$

The system can be written as the matrix equation

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix}$$

First calculate A^{-1} using row operations.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$-R_1 + R_3 \to R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$-2R_2 + R_3 \to R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$$

The zeros in the third row indicate that matrix A does not have an inverse. Since the equations are not multiples of each other, the system has no solution.

41.
$$\begin{aligned} x - 2y + 3z \quad\quad &= 4 \\ y - z + w &= -8 \\ -2x + 2y - 2z + 4w &= 12 \\ 2y - 3z + w &= -4 \end{aligned}$$

has coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ -2 & 2 & -2 & 4 \\ 0 & 2 & -3 & 1 \end{bmatrix}.$$

In Exercise 25, it was found that

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \\ -1 & 4 & -\frac{1}{2} & -2 \\ -\frac{1}{2} & \frac{5}{2} & -\frac{1}{4} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

Since $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \\ -1 & 4 & -\frac{1}{2} & -2 \\ -\frac{1}{2} & \frac{5}{2} & -\frac{1}{4} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 4 \\ -8 \\ 12 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ -34 \\ -19 \\ 7 \end{bmatrix}.$$

The solution is $(-7, -34, -19, 7)$.

42.
$$\begin{aligned} x + y + \quad\quad 2w &= 3 \\ 2x - y + z - w &= 3 \\ 3x + 3y + 2z - 2w &= 5 \\ x + 2y + z \quad\quad &= 3 \end{aligned}$$

This system may be written as the matrix equation

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & -1 & 1 & -1 \\ 3 & 3 & 2 & -2 \\ 1 & 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \\ 3 \end{bmatrix}.$$

The inverse of this coefficient matrix was calculated in Exercise 26. Use that result to obtain

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & -1 \\ \frac{1}{10} & -\frac{2}{5} & \frac{3}{10} & -\frac{1}{5} \\ -\frac{7}{10} & \frac{4}{5} & -\frac{11}{10} & \frac{12}{5} \\ \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

The solution is (1, 0, 2, 1).

In Exercises 43–48, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

43. $IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

Thus, $IA = A$.

44. $AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} a \cdot 1 + b \cdot 0 & a \cdot 0 + b \cdot 1 \\ c \cdot 1 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{bmatrix}$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

45. $A \cdot O = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Thus, $A \cdot O = O$.

46. $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$

$-cR_1 + aR_2 \to R_2 \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad - bc & -c & a \end{array}\right]$

$-\frac{b}{ad-bc}R_2 + R_1 \to R_1 \left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & ad - bc & -c & a \end{array}\right]$

$\begin{array}{l} \frac{1}{a}R_1 \to R_1 \\ \frac{1}{ad-bc}R_2 \to R_2 \end{array} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$

Thus, $A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$, if $ad - bc \neq 0$.

47. In Exercise 46, it was found that

$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$A^{-1}A = \left(\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \frac{1}{ad - bc} \left(\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

$= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Thus, $A^{-1}A = I$.

48. Show that $AA^{-1} = I$.

From Exercise 46,

$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$.

$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$

$= \begin{bmatrix} \frac{ad-bc}{ad-bc} & \frac{-ab+ab}{ad-bc} \\ \frac{cd-cd}{ad-bc} & \frac{-bc+ad}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

49.
$$\begin{aligned} AB &= O \\ A^{-1}(AB) &= A^{-1} \cdot O \\ (A^{-1}A)B &= O \\ I \cdot B &= O \\ B &= O \end{aligned}$$

Thus, if $AB = O$ and A^{-1} exists, then $B = O$.

50. Assume that matrix A has two inverses B and C. Then

$AB = BA = I \quad (1)$

and $\quad AC = CA = I. \quad (2)$

Multiply equation (1) by C.

$C(AB) = C(BA) = CI$

Since matrix multiplication is associative,

$C(AB) = (CA)B.$

Since I is an identity matrix,

$CI = C.$

Combining these results, we have

$C(AB) = C$

$(CA)B = C.$

From equation (2), $CA = I$, giving

$IB = C$

$B = C.$

Thus, if it exists, the inverse of a matrix is unique.

51. This exercise should be solved by graphing calculator or computer methods. The solution, which may vary slightly, is

$$C^{-1} = \begin{bmatrix} -0.0477 & -0.0230 & 0.0292 & 0.0895 & -0.0402 \\ 0.0921 & 0.0150 & 0.0321 & 0.0209 & -0.0276 \\ -0.0678 & 0.0315 & -0.0404 & 0.0326 & 0.0373 \\ 0.0171 & -0.0248 & 0.0069 & -0.0003 & 0.0246 \\ -0.0208 & 0.0740 & 0.0096 & -0.1018 & 0.0646 \end{bmatrix}.$$

(Entries are rounded to 4 places.)

52. Use a graphing calculator or a computer to perform this calculation. With entries rounded to 6 places, the answer is

$$(CD)^{-1} = \begin{bmatrix} 0.010146 & -0.011883 & 0.002772 & 0.020724 & -0.012273 \\ 0.006353 & 0.014233 & -0.001861 & -0.029146 & 0.019225 \\ -0.000638 & 0.006782 & -0.004823 & -0.022658 & 0.019344 \\ -0.005261 & 0.003781 & 0.006192 & 0.004837 & -0.006910 \\ -0.012252 & -0.001177 & -0.006126 & 0.006744 & 0.002792 \end{bmatrix}.$$

53. This exercise should be solved by graphing calculator or computer methods. The solution, which may vary slightly, is

$$D^{-1} = \begin{bmatrix} 0.0394 & 0.0880 & 0.0033 & 0.0530 & -0.1499 \\ -0.1492 & 0.0289 & 0.0187 & 0.1033 & 0.1668 \\ -0.1330 & -0.0543 & 0.0356 & 0.1768 & 0.1055 \\ 0.1407 & 0.0175 & -0.0453 & -0.1344 & 0.0655 \\ 0.0102 & -0.0653 & 0.0993 & 0.0085 & -0.0388 \end{bmatrix}.$$

(Entries are rounded to 4 places.)

54. Use a graphing calculator or a computer to determine that, no, $C^{-1}D^{-1}$ and $(CD)^{-1}$ are not equal.

55. This exercise should be solved by graphing calculator or computer methods. The solution may vary slightly. The answer is, yes, $D^{-1}C^{-1} = (CD)^{-1}$.

56. Use a graphing calculator to obtain

$$A^{-1} = \begin{bmatrix} -\frac{2}{27} & \frac{4}{27} & \frac{1}{27} \\ \frac{2}{99} & -\frac{31}{99} & \frac{8}{99} \\ \frac{53}{297} & -\frac{79}{297} & \frac{14}{297} \end{bmatrix}.$$

Use a graphing calculator again to solve the matrix equation $X = A^{-1}B$.

$$X = \begin{bmatrix} 1.18519 \\ -0.95960 \\ -1.30976 \end{bmatrix}$$

57. This exercise should be solved by graphing calculator or computer methods. The solution, which may vary slightly, is

$$\begin{bmatrix} 1.51482 \\ 0.053479 \\ -0.637242 \\ 0.462629 \end{bmatrix}.$$

58. Use a graphing calculator or a computer to obtain

$$X = \begin{bmatrix} 0.489558 \\ 1.00104 \\ 2.11853 \\ -1.20793 \\ -0.961346 \end{bmatrix}.$$

59. **(a)** The matrix is $B = \begin{bmatrix} 72 \\ 48 \\ 60 \end{bmatrix}$.

(b) The matrix equation is

$$\begin{bmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 72 \\ 48 \\ 60 \end{bmatrix}.$$

(c) To solve the system, begin by using row operations to find A^{-1}.

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 4 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ R_1 + (-1)R_2 \rightarrow R_2 \\ R_1 + (-1)R_3 \rightarrow R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & 4 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & -1 & 0 \\ 0 & 3 & -1 & 1 & 0 & -1 \end{array}\right]$$

$$\begin{array}{r} -4R_2 + 3R_1 \rightarrow R_1 \\ \\ R_2 + (-1)R_3 \rightarrow R_3 \end{array} \left[\begin{array}{ccc|ccc} 6 & 0 & 6 & -1 & 4 & 0 \\ 0 & 3 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\begin{array}{r} -6R_3 + R_1 \rightarrow R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 6 & 0 & 0 & -1 & 10 & -6 \\ 0 & 3 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{6}R_1 \rightarrow R_1 \\ \frac{1}{3}R_2 \rightarrow R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{6} & \frac{5}{3} & -1 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

The inverse matrix is

$$A^{-1} = \begin{bmatrix} -\frac{1}{6} & \frac{5}{3} & -1 \\ \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Since $X = A^{-1}B$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{5}{3} & -1 \\ \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 72 \\ 48 \\ 60 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 12 \end{bmatrix}.$$

There are 8 daily orders for type I, 8 for type II, and 12 for type III.

60. Let $x =$ the number of transistors,
$y =$ the number of resistors, and
$z =$ the number of computer chips.

Solve the following system:

$$\begin{aligned} 3x + 3y + 2z &= \text{amount of copper available;} \\ x + 2y + z &= \text{amount of zinc available;} \\ 2x + y + 2z &= \text{amount of glass available.} \end{aligned}$$

First, find the inverse of the coefficient matrix

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 3 & 3 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ R_1 + (-3)R_2 \to R_2 \\ 2R_1 + (-3)R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & -3 & -1 & 1 & -3 & 0 \\ 0 & 3 & -2 & 2 & 0 & -3 \end{array}\right]$$

$$\begin{array}{r} R_2 + R_1 \to R_1 \\ \\ R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 2 & -3 & 0 \\ 0 & -3 & -1 & 1 & -3 & 0 \\ 0 & 0 & -3 & 3 & -3 & -3 \end{array}\right]$$

$$\begin{array}{r} R_3 + 3R_1 \to R_1 \\ R_3 + (-3)R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 9 & 0 & 0 & 9 & -12 & -3 \\ 0 & 9 & 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 3 & -3 & -3 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{9}R_1 \to R_1 \\ \frac{1}{9}R_2 \to R_2 \\ -\frac{1}{3}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 & 1 \end{bmatrix}$$

(a) 810 units of copper, 410 units of zinc, and 490 units of glass

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 810 \\ 410 \\ 490 \end{bmatrix} = \begin{bmatrix} 100 \\ 110 \\ 90 \end{bmatrix}$$

100 transistors, 110 resistors, and 90 computer chips can be made.

(b) 765 units of copper, 385 units of zinc, and 470 units of glass

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 765 \\ 385 \\ 470 \end{bmatrix} = \begin{bmatrix} 95 \\ 100 \\ 90 \end{bmatrix}$$

95 transistors, 100 resistors, and 90 computer chips can be made.

(c) 1010 units of copper, 500 units of zinc, and 610 units of glass

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1010 \\ 500 \\ 610 \end{bmatrix} = \begin{bmatrix} 140 \\ 130 \\ 100 \end{bmatrix}$$

140 transistors, 130 resistors, and 100 computer chips can be made.

61. Let $x =$ the amount invested in AAA bonds,
$y =$ the amount invested in A bonds, and
$z =$ amount invested in B bonds.

(a) The total investment is $x + y + z = 25{,}000$. The annual return is $0.06x + 0.065y + 0.08z = 1650$. Since twice as much is invested in AAA bonds as in B bonds, $x = 2z$.

The system to be solved is

$$\begin{aligned} x + y + z &= 25{,}000 \\ 0.06x + 0.065y + 0.08z &= 1650 \\ x \quad\quad - 2z &= 0 \end{aligned}$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0.06 & 0.065 & 0.08 \\ 1 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 25{,}000 \\ 1650 \\ 0 \end{bmatrix}$,

and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Use a graphing calculator to obtain

$$A^{-1} = \begin{bmatrix} -26 & 400 & 3 \\ 40 & -600 & -4 \\ -13 & 200 & 1 \end{bmatrix}.$$

Use a graphing calculator again to solve the matrix equation $X = A^{-1}B$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -26 & 400 & 3 \\ 40 & -600 & -4 \\ -13 & 200 & 1 \end{bmatrix} \begin{bmatrix} 25{,}000 \\ 1650 \\ 0 \end{bmatrix} = \begin{bmatrix} 10{,}000 \\ 10{,}000 \\ 5000 \end{bmatrix}$$

\$10,000 should be invested at 6% in AAA bonds, \$10,000 at 6.5% in A bonds, and \$5000 at 8% in B bonds.

(b) The matrix of constants is changed to

$$B = \begin{bmatrix} 30{,}000 \\ 1985 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -26 & 400 & 3 \\ 40 & -600 & -4 \\ -13 & 200 & 1 \end{bmatrix} \begin{bmatrix} 30{,}000 \\ 1985 \\ 0 \end{bmatrix} = \begin{bmatrix} 14{,}000 \\ 9{,}000 \\ 7000 \end{bmatrix}$$

\$14,000 should be invested at 6% in AAA bonds, \$9000 at 6.5% in A bonds, and \$7000 at 8% in B bonds.

(c) The matrix of constants is changed to

$$B = \begin{bmatrix} 40{,}000 \\ 2660 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -26 & 400 & 3 \\ 40 & -600 & -4 \\ -13 & 200 & 1 \end{bmatrix} \begin{bmatrix} 40{,}000 \\ 2660 \\ 0 \end{bmatrix} = \begin{bmatrix} 24{,}000 \\ 4000 \\ 12{,}000 \end{bmatrix}$$

\$24,000 should be invested at 6% in AAA bonds, \$4000 at 6.5% in A bonds, and \$12,000 at 8% in B bonds.

62. Let $x =$ the number of pounds of pretzels,
$y =$ the number of pounds of dried fruit, and
$z =$ the number of pounds of nuts.

The total amount of trail mix is $x + y + z$. The cost of the trail mix is $4x + 5y + 9z$. Since twice the weight of pretzels as dried fruit is to be used, $x = 2y$.

(a) For a total of 140 pounds at \$6 per pound, a total value of $140 \times \$6 = \$840$, the system to be solved is

$$\begin{aligned} x + y + z &= 140 \\ 4x + 5y + 9z &= 840 \\ x - 2y &= 0 \end{aligned}$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 9 \\ 1 & -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 140 \\ 840 \\ 0 \end{bmatrix}$,

and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Use row operations to obtain the inverse of the coefficient matrix.

$$A^{-1} = \begin{bmatrix} \frac{9}{7} & -\frac{1}{7} & \frac{2}{7} \\ \frac{9}{14} & -\frac{1}{14} & -\frac{5}{14} \\ -\frac{13}{14} & \frac{3}{14} & \frac{1}{14} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 18 & -2 & 4 \\ 9 & -1 & -5 \\ -13 & 3 & 1 \end{bmatrix}$$

Since $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 18 & -2 & 4 \\ 9 & -1 & -5 \\ -13 & 3 & 1 \end{bmatrix} \begin{bmatrix} 140 \\ 480 \\ 0 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 840 \\ 420 \\ 700 \end{bmatrix} = \begin{bmatrix} 60 \\ 30 \\ 50 \end{bmatrix}.$$

Use 60 pounds of pretzels, 30 pounds of dried fruits, and 50 pounds of nuts.

(b) For a total of 100 pounds at \$7.60 per pound, a total value of $100 \times \$7.60 = \$760$, the system to be solved is

$$\begin{aligned} x + y + z &= 100 \\ 4x + 5y + 9z &= 760 \\ x - 2y &= 0 \end{aligned}$$

The matrix of constants is changed to $B = \begin{bmatrix} 100 \\ 760 \\ 0 \end{bmatrix}$.

Since $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 18 & -2 & 4 \\ 9 & -1 & -5 \\ -13 & 3 & 1 \end{bmatrix}\begin{bmatrix} 100 \\ 760 \\ 0 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 280 \\ 140 \\ 980 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \\ 70 \end{bmatrix}.$$

Use 20 pounds of pretzels, 10 pounds of dried fruits, and 70 pounds of nuts.

(c) For a total of 125 lb at \$6.20 per pound, a total value of $125 \times \$6.20 = \$775$, .the system to be solved is

$$\begin{aligned} x + \ \ y + \ \ z &= 125 \\ 4x + 5y + 9z &= 775 \\ x - 2y \qquad &= \ \ 0 \end{aligned}$$

The matrix of constants is changed to $B = \begin{bmatrix} 125 \\ 775 \\ 0 \end{bmatrix}$.

Since $X = A^{-1}B$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 18 & -2 & 4 \\ 9 & -1 & -5 \\ -13 & 3 & 1 \end{bmatrix}\begin{bmatrix} 125 \\ 775 \\ 0 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 700 \\ 350 \\ 700 \end{bmatrix} = \begin{bmatrix} 50 \\ 25 \\ 50 \end{bmatrix}.$$

Use 50 pounds of pretzels, 25 pounds of dried fruits, and 50 pounds of nuts.

63. Let $x =$ the number of Super Vim tablets,
$y =$ the number of Multitab tablets, and
$z =$ the number of Mighty Mix tablets.

The total number of vitamins is

$$x + y + z.$$

The total amount of niacin is

$$15x + 20y + 25z.$$

The total amount of Vitamin E is

$$12x + 15y + 35z.$$

(a) The system to be solved is

$$\begin{aligned} x + \ \ \ y + \ \ \ z &= 225 \\ 15x + 20y + 25z &= 4750 \\ 12x + 15y + 35z &= 5225. \end{aligned}$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 15 & 20 & 25 \\ 20 & 15 & 35 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 225 \\ 4750 \\ 5225 \end{bmatrix}$.

Thus, $AX = B$ and

$$\begin{bmatrix} 1 & 1 & 1 \\ 15 & 20 & 25 \\ 12 & 15 & 35 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 225 \\ 4750 \\ 5225 \end{bmatrix}.$$

Use row operations to obtain the inverse of the coefficient matrix.

$$A^{-1} = \begin{bmatrix} \frac{65}{17} & -\frac{4}{17} & \frac{1}{17} \\ -\frac{45}{17} & \frac{23}{85} & -\frac{2}{17} \\ -\frac{3}{17} & -\frac{3}{85} & \frac{1}{17} \end{bmatrix}$$

Since $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{65}{17} & -\frac{4}{17} & \frac{1}{17} \\ -\frac{45}{17} & \frac{23}{85} & -\frac{2}{17} \\ -\frac{3}{17} & -\frac{3}{85} & \frac{1}{17} \end{bmatrix}\begin{bmatrix} 225 \\ 4750 \\ 5225 \end{bmatrix} = \begin{bmatrix} 50 \\ 75 \\ 100 \end{bmatrix}.$$

There are 50 Super Vim tablets, 75 Multitab tablets, and 100 Mighty Mix tablets.

(b) The matrix of constants is changed to

$$B = \begin{bmatrix} 185 \\ 3625 \\ 3750 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{65}{17} & -\frac{4}{17} & \frac{1}{17} \\ -\frac{45}{17} & \frac{23}{85} & -\frac{2}{17} \\ -\frac{3}{17} & -\frac{3}{85} & \frac{1}{17} \end{bmatrix}\begin{bmatrix} 185 \\ 3625 \\ 3750 \end{bmatrix} = \begin{bmatrix} 75 \\ 50 \\ 60 \end{bmatrix}$$

There are 75 Super Vim tablets, 50 Multitab tablets, and 60 Mighty Mix tablets.

(c) The matrix of constants is changed to

$$B = \begin{bmatrix} 230 \\ 4450 \\ 4210 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{65}{17} & -\frac{4}{17} & \frac{1}{17} \\ -\frac{45}{17} & \frac{23}{85} & -\frac{2}{17} \\ -\frac{3}{17} & -\frac{3}{85} & \frac{1}{17} \end{bmatrix}\begin{bmatrix} 230 \\ 4450 \\ 4210 \end{bmatrix} = \begin{bmatrix} 80 \\ 100 \\ 50 \end{bmatrix}$$

There are 80 Super Vim tablets, 100 Multitab tablets, and 50 Mighty Mix tablets.

64. **(a)** First, divide the letters and spaces of the sentence into groups of 3, writing each group as a column vector.

$$\begin{bmatrix} A \\ l \\ l \end{bmatrix}, \begin{bmatrix} \text{(space)} \\ i \\ s \end{bmatrix}, \begin{bmatrix} \text{(space)} \\ f \\ a \end{bmatrix}, \begin{bmatrix} i \\ r \\ \text{(space)} \end{bmatrix},$$
$$\begin{bmatrix} i \\ n \\ \text{(space)} \end{bmatrix}, \begin{bmatrix} l \\ o \\ v \end{bmatrix}, \begin{bmatrix} e \\ \text{(space)} \\ a \end{bmatrix}, \begin{bmatrix} n \\ d \\ \text{(space)} \end{bmatrix}, \begin{bmatrix} w \\ a \\ r \end{bmatrix}$$

Next, convert each letter into a number, assigning 1 to A, 2 to B, and so on, with the number 27 used to represent each space between words.

$$\begin{bmatrix} 1 \\ 12 \\ 12 \end{bmatrix}, \begin{bmatrix} 27 \\ 9 \\ 19 \end{bmatrix}, \begin{bmatrix} 27 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix}, \begin{bmatrix} 9 \\ 14 \\ 27 \end{bmatrix},$$
$$\begin{bmatrix} 12 \\ 15 \\ 22 \end{bmatrix}, \begin{bmatrix} 5 \\ 27 \\ 1 \end{bmatrix}, \begin{bmatrix} 14 \\ 4 \\ 27 \end{bmatrix}, \begin{bmatrix} 23 \\ 1 \\ 18 \end{bmatrix}$$

Now find the product of the coding matrix presented in Example 7, $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$, and each column vector above. This produces a new set of vectors, which represents the coded message.

$$\begin{bmatrix} 85 \\ 50 \\ 40 \end{bmatrix}, \begin{bmatrix} 130 \\ 120 \\ 145 \end{bmatrix}, \begin{bmatrix} 49 \\ 63 \\ 121 \end{bmatrix}, \begin{bmatrix} 171 \\ 117 \\ 99 \end{bmatrix}, \begin{bmatrix} 159 \\ 113 \\ 91 \end{bmatrix},$$
$$\begin{bmatrix} 145 \\ 105 \\ 100 \end{bmatrix}, \begin{bmatrix} 90 \\ 40 \\ 75 \end{bmatrix}, \begin{bmatrix} 134 \\ 113 \\ 91 \end{bmatrix}, \begin{bmatrix} 98 \\ 101 \\ 112 \end{bmatrix}$$

The message will be transmitted as 85, 50, 40, 130, 120, 145, 49, 63, 121, 171, 117, 99, 159, 113, 91, 145, 105, 100, 90, 40, 75, 134, 113, 91, 98, 101, 112.

(b) First, divide the coded message into groups of three numbers and form each group into a column vector.

$$\begin{bmatrix} 138 \\ 81 \\ 102 \end{bmatrix}, \begin{bmatrix} 101 \\ 67 \\ 109 \end{bmatrix}, \begin{bmatrix} 162 \\ 124 \\ 173 \end{bmatrix}, \begin{bmatrix} 210 \\ 150 \\ 165 \end{bmatrix}$$

Next, find the product of the decoding matrix presented in Example 7, the inverse of matrix A in part (a) above,

$A^{-1} = \begin{bmatrix} -0.2 & 0.2 & 0.2 \\ 0.4 & -0.6 & 0.2 \\ 0 & 0.4 & -0.2 \end{bmatrix}$, and each of the column vectors above. This produces a new set of vectors, which represents the decoded message.

$$\begin{bmatrix} 9 \\ 27 \\ 12 \end{bmatrix}, \begin{bmatrix} 15 \\ 22 \\ 5 \end{bmatrix}, \begin{bmatrix} 27 \\ 25 \\ 15 \end{bmatrix}, \begin{bmatrix} 21 \\ 27 \\ 27 \end{bmatrix}$$

Lastly, convert each number into a letter, assigning A to 1, B to 2, and so on, with the number 27 used to represent each space between words.

Decoded message: I love you

65. **(a)** First, divide the letters and spaces of the message into groups of three, writing each group as a column vector.

$$\begin{bmatrix} T \\ o \\ \text{(space)} \end{bmatrix}, \begin{bmatrix} b \\ e \\ \text{(space)} \end{bmatrix}, \begin{bmatrix} o \\ r \\ \text{(space)} \end{bmatrix}, \begin{bmatrix} n \\ o \\ t \end{bmatrix}, \begin{bmatrix} \text{(space)} \\ t \\ o \end{bmatrix}, \begin{bmatrix} \text{(space)} \\ b \\ e \end{bmatrix}$$

Next, convert each letter into a number, assigning 1 to A, 2 to B, and so on, with the number 27 used to represent each space between words.

$$\begin{bmatrix} 20 \\ 15 \\ 27 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 27 \end{bmatrix}, \begin{bmatrix} 15 \\ 18 \\ 27 \end{bmatrix}, \begin{bmatrix} 14 \\ 15 \\ 20 \end{bmatrix}, \begin{bmatrix} 27 \\ 20 \\ 15 \end{bmatrix}, \begin{bmatrix} 27 \\ 2 \\ 5 \end{bmatrix}$$

Now, find the product of the coding matrix B and each column vector. This produces a new set of vectors, which represents the coded message.

$$\begin{bmatrix} 262 \\ -161 \\ -12 \end{bmatrix}, \begin{bmatrix} 186 \\ -103 \\ -22 \end{bmatrix}, \begin{bmatrix} 264 \\ -168 \\ -9 \end{bmatrix}, \begin{bmatrix} 208 \\ -134 \\ -5 \end{bmatrix}, \begin{bmatrix} 224 \\ -152 \\ 5 \end{bmatrix}, \begin{bmatrix} 92 \\ -50 \\ -3 \end{bmatrix}$$

This message will be transmitted as 262, −161, −12, 186, −103, −22, 264, −168, −9, 208, −134, −5, 224, −152, 5, 92, −50, −3.

(b) Use row operations or a graphing calculator to find the inverse of the coding matrix B.

$$B^{-1} = \begin{bmatrix} 1.75 & 2.5 & 3 \\ -0.25 & -0.5 & 0 \\ -0.25 & -0.5 & -1 \end{bmatrix}$$

(c) First, divide the coded message into groups of three numbers and form each group into a column vector.

$$\begin{bmatrix}116\\-60\\-15\end{bmatrix},\ \begin{bmatrix}294\\-197\\-2\end{bmatrix},\ \begin{bmatrix}148\\-92\\-9\end{bmatrix},\ \begin{bmatrix}96\\-64\\-4\end{bmatrix},\ \begin{bmatrix}264\\-182\\-2\end{bmatrix}$$

Next, find the product of the decoding matric B^{-1} and each of the column vectors. This produces a new set of vectors, which represents the decoded message.

$$\begin{bmatrix}8\\1\\16\end{bmatrix},\ \begin{bmatrix}16\\25\\27\end{bmatrix},\ \begin{bmatrix}2\\9\\18\end{bmatrix},\ \begin{bmatrix}20\\8\\4\end{bmatrix},\ \begin{bmatrix}1\\25\\27\end{bmatrix}$$

Last, convert each number into a letter, assigning A to 1, B to 2, and so on, with the number 27 used to represent a space between words. The decoded message is HAPPY BIRTHDAY.

66. **(a)** $$[B|I] = \left[\begin{array}{ccc|ccc}50 & 50 & 45 & 1 & 0 & 0\\ 0 & 15 & 20 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & 0 & 1\end{array}\right]$$

Interchange rows 1 and 3.

$$\left[\begin{array}{ccc|ccc}1 & 1 & 1 & 0 & 0 & 1\\ 0 & 15 & 20 & 0 & 1 & 0\\ 50 & 50 & 45 & 1 & 0 & 0\end{array}\right]$$

$$-50R_1 + R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc}1 & 1 & 1 & 0 & 0 & 1\\ 0 & 15 & 20 & 0 & 1 & 0\\ 0 & 0 & -5 & 1 & 0 & -50\end{array}\right]$$

$$\begin{array}{l}R_3 + R_1 \to R_1\\ 4R_3 + R_2 \to R_2\end{array} \quad \left[\begin{array}{ccc|ccc}-15 & 0 & 0 & 1 & 1 & -65\\ 0 & 15 & 0 & 4 & 1 & -200\\ 0 & 0 & -5 & 1 & 0 & -50\end{array}\right]$$

$$\begin{array}{l}-\frac{1}{15}R_1 \to R_1\\ \frac{1}{15}R_2 \to R_2\\ -\frac{1}{5}R_3 \to R_3\end{array} \quad \left[\begin{array}{ccc|ccc}1 & 0 & 0 & -\frac{1}{15} & -\frac{1}{15} & \frac{13}{3}\\ 0 & 1 & 0 & \frac{4}{15} & \frac{1}{15} & -\frac{40}{3}\\ 0 & 0 & 1 & -\frac{1}{5} & 0 & 10\end{array}\right]$$

$$R_2 + (-15)R_1 \to R_1 \quad \left[\begin{array}{ccc|ccc}-15 & 0 & 5 & 0 & 1 & -15\\ 0 & 15 & 20 & 0 & 1 & 0\\ 0 & 0 & -5 & 1 & 0 & -50\end{array}\right]$$

$$B^{-1} = \begin{bmatrix}-\frac{1}{15} & -\frac{1}{15} & \frac{13}{3}\\ \frac{4}{15} & \frac{1}{15} & -\frac{40}{3}\\ -\frac{1}{5} & 0 & 10\end{bmatrix}$$

(b) $$A = \begin{bmatrix}40 & 55 & 60\\ 10 & 10 & 15\\ 1 & 1 & 1\end{bmatrix} B^{-1}$$

$$= \begin{bmatrix}40 & 55 & 60\\ 10 & 10 & 15\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}-\frac{1}{15} & -\frac{1}{15} & \frac{13}{3}\\ \frac{4}{15} & \frac{1}{15} & -\frac{40}{3}\\ -\frac{1}{5} & 0 & 10\end{bmatrix}$$

$$= \begin{bmatrix}0 & 1 & 40\\ -1 & 0 & 60\\ 0 & 0 & 1\end{bmatrix}$$

(c) Denoting the original positions of the band members as x_1, x_2, and so on, the original shape was

25								
20			x_1	x_2	x_3	x_4	x_5	
15					x_6			
10					x_7			
5					x_8			
0					x_9			
	30	35	40	45	50	55	60	65

We are given that band member x_9, originally positioned at (50, 0), moved to (40, 10); band member x_6, originally at (50, 15), moved to (55, 10); band member x_2, originally at (45, 20), moved to (60, 15).

To find the new position of x_1, originally at (40, 20), multiply A by the vector $\begin{bmatrix}40\\20\\1\end{bmatrix}$.

$$A\begin{bmatrix}40\\20\\1\end{bmatrix} = \begin{bmatrix}0 & 1 & 40\\ -1 & 0 & 60\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}40\\20\\1\end{bmatrix} = \begin{bmatrix}60\\20\\1\end{bmatrix}$$

The new position of band member x_1 is (60, 20).

To find the new position of x_3, originally at (50, 20), multiply A by the vector $\begin{bmatrix}50\\20\\1\end{bmatrix}$.

$$A\begin{bmatrix}50\\20\\1\end{bmatrix} = \begin{bmatrix}0 & 1 & 40\\ -1 & 0 & 60\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}50\\20\\1\end{bmatrix} = \begin{bmatrix}60\\10\\1\end{bmatrix}$$

The new position of band member x_3 is (60, 10).

To find the new position of x_4, originally at (55, 20), multiply A by the vector $\begin{bmatrix}55\\20\\1\end{bmatrix}$.

$$A\begin{bmatrix}55\\20\\1\end{bmatrix}=\begin{bmatrix}0&1&40\\-1&0&60\\0&0&1\end{bmatrix}\begin{bmatrix}55\\20\\1\end{bmatrix}=\begin{bmatrix}60\\5\\1\end{bmatrix}$$

The new position of band member x_4 is (60, 5).

To find the new position of x_5, originally at (60, 20), multiply A by the vector $\begin{bmatrix}60\\20\\1\end{bmatrix}$.

$$A\begin{bmatrix}60\\20\\1\end{bmatrix}=\begin{bmatrix}0&1&40\\-1&0&60\\0&0&1\end{bmatrix}\begin{bmatrix}60\\20\\1\end{bmatrix}=\begin{bmatrix}60\\0\\1\end{bmatrix}$$

The new position of band member x_5 is (60, 0).

To find the new position of x_7, originally at (50, 10), multiply A by the vector $\begin{bmatrix}50\\10\\1\end{bmatrix}$.

$$A\begin{bmatrix}50\\10\\1\end{bmatrix}=\begin{bmatrix}0&1&40\\-1&0&60\\0&0&1\end{bmatrix}\begin{bmatrix}50\\10\\1\end{bmatrix}=\begin{bmatrix}50\\10\\1\end{bmatrix}$$

The new position of band member x_7 is (50, 10).

To find the new position of x_8, originally at (50, 5), multiply A by the vector $\begin{bmatrix}50\\5\\1\end{bmatrix}$.

$$A\begin{bmatrix}50\\5\\1\end{bmatrix}=\begin{bmatrix}0&1&40\\-1&0&60\\0&0&1\end{bmatrix}\begin{bmatrix}50\\5\\1\end{bmatrix}=\begin{bmatrix}45\\10\\1\end{bmatrix}$$

The new position of band member x_8 is (45, 10). The new position of the band is

25								
20							x_1	
15							x_2	
10			x_9	x_8	x_7	x_6	x_3	
5							x_4	
0							x_5	
	30	35	40	45	50	55	60	65

Thus, the new shape is a sideways T whose vertical and horizontal intersection is at mark (60, 10).

2.6 Input-Output Models

Your Turn 1

$$X=(I-A)^{-1}D$$

$$X=\begin{bmatrix}1.395&0.496&0.589\\0.837&1.364&0.620\\0.558&0.465&1.302\end{bmatrix}\begin{bmatrix}322\\447\\133\end{bmatrix}=\begin{bmatrix}749.239\\961.682\\560.697\end{bmatrix}$$

The productions of 749 units of agriculture, 962 units of manufacturing, and 561 units of transportation are required to satisfy the demands of 322, 447, and 133 units, respectively.

Your Turn 2

$$A=\begin{bmatrix}\frac{1}{2}&\frac{1}{4}&\frac{1}{6}\\0&\frac{1}{4}&\frac{1}{6}\\\frac{1}{2}&\frac{1}{2}&\frac{2}{3}\end{bmatrix}\qquad I-A=\begin{bmatrix}\frac{1}{2}&-\frac{1}{4}&-\frac{1}{6}\\0&\frac{3}{4}&-\frac{1}{6}\\-\frac{1}{2}&-\frac{1}{2}&\frac{1}{3}\end{bmatrix}$$

$$(I-A)X=\begin{bmatrix}\frac{1}{2}x_1&-\frac{1}{4}x_2&-\frac{1}{6}x_3\\0x_1&\frac{3}{4}x_2&-\frac{1}{6}x_3\\-\frac{1}{2}x_1&-\frac{1}{2}x_2&\frac{1}{3}x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

We get the following system.

$$\begin{aligned}\frac{1}{2}x_1-\frac{1}{4}x_2-\frac{1}{6}x_3&=0\\\frac{3}{4}x_2-\frac{1}{6}x_3&=0\\-\frac{1}{2}x_1-\frac{1}{2}x_2+\frac{1}{3}x_3&=0\end{aligned}$$

Clearing fractions gives the following system.

$$\begin{aligned}6x_1-3x_2-2x_3&=0\\9x_2-2x_3&=0\\-3x_1-3x_2+2x_3&=0\end{aligned}$$

$$\begin{aligned}9x_2-2x_3&=0\\9x_2&=2x_3\\x_2&=\frac{2}{9}x_3\end{aligned}$$

$$\begin{aligned}6x_1-3\left(\frac{2}{9}x_3\right)-2x_3&=0\\6x_1-\frac{2}{3}x_3-2x_3&=0\\6x_1-\frac{8}{3}x_3&=0\\6x_1&=\frac{8}{3}x_3\\x_1&=\frac{4}{9}x_3\end{aligned}$$

The solution is $\left(\frac{4}{9}x_3, \frac{2}{9}x_3, x_3\right)$, where x_3 is any real number. For $x_3 = 9$, the solution is $(4, 2, 9)$. So, the production of the three commodities should be in the ratio 4:2:9.

2.6 Warmup Exercises

W1.

$$\left[\begin{array}{cc|cc} 3 & -2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cc|cc} -1 & 1 & 0 & 1 \\ 3 & -2 & 1 & 0 \end{array}\right]$$

$$3R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|cc} -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$(-1)R_1 \to R_1 \quad \left[\begin{array}{cc|cc} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

The inverse is $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

W2.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 4 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 + R_2 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & -2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} 2R_2 + R_1 \to R_1 \\ \\ (-1)R_2 + R_2 \to R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 12 & 3 & 2 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$\begin{array}{l} (-12)R_3 + R_1 \to R_1 \\ (-4)R_3 + R_2 \to R_2 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 15 & 14 & -12 \\ 0 & 1 & 0 & 5 & 5 & -4 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

The inverse is $\begin{bmatrix} 15 & 14 & -12 \\ 5 & 5 & -4 \\ -1 & -1 & 1 \end{bmatrix}$.

2.6 Exercises

1. $A = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.7 \end{bmatrix}, D = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

To find the production matrix, first calculate $I - A$.

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.3 \end{bmatrix}$$

Using row operations, find the inverse of $I - A$.

$$[I - A|I] = \left[\begin{array}{cc|cc} 0.2 & -0.2 & 1 & 0 \\ -0.2 & 0.3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} 10R_1 \to R_1 \\ 10R_2 \to R_2 \end{array} \left[\begin{array}{cc|cc} 2 & -2 & 10 & 0 \\ -2 & 3 & 0 & 10 \end{array}\right]$$

$$R_1 + R_2 \to R_2 \left[\begin{array}{cc|cc} 2 & -2 & 10 & 0 \\ 0 & 1 & 10 & 10 \end{array}\right]$$

$$2R_2 + R_1 \to R_1 \left[\begin{array}{cc|cc} 2 & 0 & 30 & 20 \\ 0 & 1 & 10 & 10 \end{array}\right]$$

$$\tfrac{1}{2}R_1 \to R_1 \left[\begin{array}{cc|cc} 1 & 0 & 15 & 10 \\ 0 & 1 & 10 & 10 \end{array}\right]$$

$$(I - A)^{-1} = \begin{bmatrix} 15 & 10 \\ 10 & 10 \end{bmatrix}$$

Since $X = (I - A)^{-1}D$, the product matrix is

$$X = \begin{bmatrix} 15 & 10 \\ 10 & 10 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 60 \\ 50 \end{bmatrix}.$$

2. $A = \begin{bmatrix} 0.2 & 0.04 \\ 0.6 & 0.05 \end{bmatrix}, D = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$

To find the production matrix X, first calculate $I - A$.

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.04 \\ 0.6 & 0.05 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.04 \\ -0.6 & 0.95 \end{bmatrix}$$

Next, use row operations to find the inverse of $I - A$.

$$(I - A)^{-1} = \begin{bmatrix} 1.29 & 0.054 \\ 0.815 & 1.087 \end{bmatrix}$$

Since $X = (I - A)^{-1}D$,

$$X = \begin{bmatrix} 1.29 & 0.054 \\ 0.815 & 1.087 \end{bmatrix} \begin{bmatrix} 3 \\ 10 \end{bmatrix} = \begin{bmatrix} 4.4 \\ 13.3 \end{bmatrix}.$$

3. $A = \begin{bmatrix} 0.1 & 0.03 \\ 0.07 & 0.6 \end{bmatrix}, D = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

First, calculate $I - A$.

$$I - A = \begin{bmatrix} 0.9 & -0.03 \\ -0.07 & 0.4 \end{bmatrix}$$

Use row operations to find the inverse of $I - A$, which is

$$(I - A)^{-1} \approx \begin{bmatrix} 1.118 & 0.084 \\ 0.196 & 2.515 \end{bmatrix}.$$

Since $X = (I - A)^{-1}D$, the production matrix is

$$X = \begin{bmatrix} 1.118 & 0.084 \\ 0.196 & 2.515 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 6.43 \\ 26.12 \end{bmatrix}.$$

4. $A = \begin{bmatrix} 0.02 & 0.03 \\ 0.06 & 0.08 \end{bmatrix}, D = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$

To find the production matrix, first calculate $I - A$.

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.02 & 0.03 \\ 0.06 & 0.08 \end{bmatrix} = \begin{bmatrix} 0.98 & -0.03 \\ -0.06 & 0.92 \end{bmatrix}$$

Using row operations, find the inverse of $I - A$.

$$[I - A|I] = \left[\begin{array}{cc|cc} 0.98 & -0.03 & 1 & 0 \\ -0.06 & 0.92 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} 100R_1 \to R_1 \\ 100R_2 \to R_2 \end{array}\left[\begin{array}{cc|cc} 98 & -3 & 100 & 0 \\ -6 & 92 & 0 & 100 \end{array}\right]$$

$$3R_1 + 49R_2 \to R_2 \left[\begin{array}{cc|cc} 98 & -3 & 100 & 0 \\ 0 & 4499 & 300 & 4900 \end{array}\right]$$

$$3R_2 + 4499R_1 \to R_1 \left[\begin{array}{cc|cc} 440{,}902 & 0 & 450{,}800 & 14{,}700 \\ 0 & 4499 & 300 & 4900 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{440{,}902}R_1 \to R_1 \\ \frac{1}{4499}R_2 \to R_2 \end{array}\left[\begin{array}{cc|cc} 1 & 0 & 1.022449 & 0.033341 \\ 0 & 1 & 0.066681 & 1.089131 \end{array}\right]$$

$$(I - A)^{-1} = \begin{bmatrix} 1.022449 & 0.033341 \\ 0.066681 & 1.089131 \end{bmatrix}$$

Since $X = (I - A)^{-1}D$, the production matrix is

$$X = \begin{bmatrix} 1.022449 & 0.033341 \\ 0.066681 & 1.089131 \end{bmatrix}\begin{bmatrix} 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 108.91 \\ 224.49 \end{bmatrix} \text{(rounded)}.$$

5. $A = \begin{bmatrix} 0.8 & 0 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0 & 0.2 & 0.7 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$

To find the production matrix, first calculate $I - A$.

$$I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8 & 0 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0 & 0.2 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.2 & 0 & -0.1 \\ -0.1 & 0.5 & -0.2 \\ 0 & 0.2 & 0.3 \end{bmatrix}$$

Using row operations, find the inverse of $I - A$.

$$[I - A \mid I] = \left[\begin{array}{ccc|ccc} 0.2 & 0 & -0.1 & 1 & 0 & 0 \\ -0.1 & 0.5 & -0.2 & 0 & 1 & 0 \\ 0 & 0.2 & 0.3 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} 10R_1 \to R_1 \\ 10R_2 \to R_2 \\ 10R_3 \to R_3 \end{array}\left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 10 & 0 & 0 \\ -1 & 5 & -2 & 0 & 10 & 0 \\ 0 & 0.2 & 3 & 0 & 0 & 10 \end{array}\right]$$

$$R_1 + 2R_2 \to R_2 \left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 10 & 0 & 0 \\ 0 & 10 & -5 & 10 & 20 & 0 \\ 0 & 0.2 & 3 & 0 & 0 & 10 \end{array}\right]$$

$$\frac{1}{5}R_2 + R_3 \to R_3 \left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 10 & 0 & 0 \\ 0 & 10 & -5 & 10 & 20 & 0 \\ 0 & 0 & 2 & 2 & 4 & 10 \end{array}\right]$$

$$\frac{1}{2}R_3 \to R_3 \left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 10 & 0 & 0 \\ 0 & 10 & -5 & 10 & 20 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{array}\right]$$

$$\begin{array}{l} R_3 + R_1 \to R_1 \\ 5R_3 + R_2 \to R_2 \end{array}\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 11 & 2 & 5 \\ 0 & 10 & 0 & 15 & 30 & 25 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{2}R_1 \to R_1 \\ \frac{1}{10}R_2 \to R_2 \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{2} & 1 & \frac{5}{2} \\ 0 & 1 & 0 & \frac{3}{2} & 3 & \frac{5}{2} \\ 0 & 0 & 1 & 1 & 2 & 5 \end{array}\right]$$

$$(I - A)^{-1} = \begin{bmatrix} \frac{11}{2} & 1 & \frac{5}{2} \\ \frac{3}{2} & 3 & \frac{5}{2} \\ 1 & 2 & 5 \end{bmatrix}$$

Since $X = (I - A)^{-1}D,$ the production matrix is

$$\begin{bmatrix} \frac{11}{2} & 1 & \frac{5}{2} \\ \frac{3}{2} & 3 & \frac{5}{2} \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 \\ 27 \\ 28 \end{bmatrix}.$$

6. $A = \begin{bmatrix} 0.1 & 0.5 & 0 \\ 0 & 0.3 & 0.4 \\ 0.1 & 0.2 & 0.1 \end{bmatrix}, D = \begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix}$

$$I - A = \begin{bmatrix} 0.9 & -0.5 & 0 \\ 0 & 0.7 & -0.4 \\ -0.1 & -0.2 & 0.9 \end{bmatrix}$$

$$(I - A)^{-1} = \begin{bmatrix} 1.158 & 0.947 & 0.421 \\ 0.084 & 1.705 & 0.758 \\ 0.147 & 0.484 & 1.33 \end{bmatrix}$$

$$X = (I - A)^{-1}D = \begin{bmatrix} 1.158 & 0.947 & 0.421 \\ 0.084 & 1.705 & 0.758 \\ 0.147 & 0.484 & 1.33 \end{bmatrix}\begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 16.21 \\ 9.18 \\ 6.06 \end{bmatrix}$$

7.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.3 & 0.1 & 0.8 \\ 0.5 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.1 \end{bmatrix} \end{array} = A$$

$$I - A = \begin{bmatrix} 0.7 & -0.1 & -0.8 \\ -0.5 & 0.4 & -0.1 \\ -0.2 & -0.3 & 0.9 \end{bmatrix}$$

Set $(I - A)X = O$ to obtain the following.

$$\begin{bmatrix} 0.7 & -0.1 & -0.8 \\ -0.5 & 0.4 & -0.1 \\ -0.2 & -0.3 & 0.9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.7x_1 - 0.1x_2 - 0.8x_3 \\ -0.5x_1 + 0.4x_2 - 0.1x_3 \\ -0.2x_1 - 0.3x_2 + 0.9x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rewrite this matrix equation as a system of equations.

$$\begin{aligned} 0.7x_1 - 0.1x_2 - 0.8x_3 &= 0 \\ -0.5x_1 + 0.4x_2 - 0.1x_3 &= 0 \\ -0.2x_1 - 0.3x_2 + 0.9x_3 &= 0 \end{aligned}$$

Rewrite the equations without decimals.

$$\begin{aligned} 7x_1 - x_2 - 8x_3 &= 0 \quad (1) \\ -5x_1 + 4x_2 - x_3 &= 0 \quad (2) \\ -2x_1 - 3x_2 + 9x_3 &= 0 \quad (3) \end{aligned}$$

Use row operations to solve this system of equations. Begin by eliminating x_1 in equations (2) and (3)

$$\begin{array}{lr} & 7x_1 - x_2 - 8x_3 = 0 \quad (1) \\ 5R_1 + 7R_2 \to R_1 & 23x_2 - 47x_3 = 0 \quad (4) \\ 2R_1 + 7R_3 \to R_3 & -23x_2 + 47x_3 = 0 \quad (5) \end{array}$$

Eliminate x_2 in equations (1) and (5).

$$\begin{array}{lr} 23R_1 + R_2 \to R_1 & 161x_1 \qquad - 231x_3 = 0 \quad (6) \\ & 23x_2 - 47x_3 = 0 \quad (4) \\ R_2 + R_3 \to R_3 & 0 = 0 \quad (7) \end{array}$$

The true statement in equation (7) indicates that the equations are dependent. Solve equation (6) for x_1 and equation (4) for x_2, each in terms of x_3.

$$x_1 = \frac{231}{161}x_3 = \frac{33}{23}x_3$$

$$x_2 = \frac{47}{23}x_3$$

The solution of the system is

$$\left(\frac{33}{23}x_3, \frac{47}{23}x_3, x_3\right).$$

If $x_3 = 23,$ then $x_1 = 33$ and $x_2 = 47,$ so the production of the three commodities should be in the ratio 33:47:23.

8.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.1 & 0.5 & 0.4 \\ 0.6 & 0.3 & 0.3 \end{bmatrix} \end{array} = A$$

Calculate $I - A,$ and then set $(I - A)X = O$ to find X.

$$I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.1 & 0.5 & 0.4 \\ 0.6 & 0.3 & 0.3 \end{bmatrix}$$
$$= \begin{bmatrix} 0.7 & -0.2 & -0.3 \\ -0.1 & 0.5 & -0.4 \\ -0.6 & -0.3 & 0.7 \end{bmatrix}$$
$$(I - A)X = \begin{bmatrix} 0.7 & -0.2 & -0.3 \\ -0.1 & 0.5 & -0.4 \\ -0.6 & -0.3 & 0.7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0.7x_1 - 0.2x_2 - 0.3x_3 = 0$$
$$-0.1x_1 + 0.5x_2 - 0.4x_3 = 0$$
$$0.6x_1 - 0.3x_2 + 0.7x_3 = 0$$

Solving this system with x_3 as the parameter will give the solution

$$\left(\frac{23}{33}x_3, \frac{31}{33}x_3, x_3\right).$$

If $x_3 = 33$, then $x_1 = 23$ and $x_2 = 31$, so the production of A, B, and C should be in the ratio 23:31:33.

9. Use a graphing calculator or a computer to find the production matrix $X = (I - A)^{-1}D$. The answer is

$$X = \begin{bmatrix} 7697 \\ 4205 \\ 6345 \\ 4106 \end{bmatrix}.$$

Values have been rounded.

10. This exercise should be solved using a graphing calculator or a computer, to find $X = (1 - A)^{-1} D$. The answer, rounded to the nearest whole number, is

$$X = \begin{bmatrix} 7022 \\ 4845 \\ 5116 \\ 4647 \end{bmatrix}.$$

11. In Example 4, it was found that

$$(I - A)^{-1} \approx \begin{bmatrix} 1.3882 & 0.1248 \\ 0.5147 & 1.1699 \end{bmatrix}.$$

Since $X = (I - A)^{-1}D$, the production matrix is

$$X = \begin{bmatrix} 1.3882 & 0.1248 \\ 0.5147 & 1.1699 \end{bmatrix}\begin{bmatrix} 925 \\ 1250 \end{bmatrix} = \begin{bmatrix} 1440.085 \\ 1938.473 \end{bmatrix}.$$

Thus, about 1440. metric tons of wheat and 1938 metric tons of oil should be produced.

12. $A = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{5} & 0 \end{bmatrix}, \; D = \begin{bmatrix} 500 \\ 1000 \end{bmatrix}$

$$X = (I - A)^{-1}D$$
$$= \begin{bmatrix} 1 & -\frac{1}{3} \\ -\frac{1}{5} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 500 \\ 1000 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{15}{14} & \frac{5}{14} \\ \frac{3}{14} & \frac{15}{14} \end{bmatrix}\begin{bmatrix} 500 \\ 1000 \end{bmatrix}$$
$$= \begin{bmatrix} 892.9 \\ 1178.6 \end{bmatrix}$$

Produce about 893 metric tons of wheat and about 1179 metric tons of oil.

13. In Example 3, it was found that

$$(I - A)^{-1} \approx \begin{bmatrix} 1.3953 & 0.4961 & 0.5891 \\ 0.8372 & 1.3643 & 0.6202 \\ 0.5581 & 0.4651 & 1.3023 \end{bmatrix}.$$

Since $X = (I - A)^{-1}D$, the production matrix is

$$X = \begin{bmatrix} 1.3953 & 0.4961 & 0.5891 \\ 0.8372 & 1.3643 & 0.6202 \\ 0.5581 & 0.4651 & 1.3023 \end{bmatrix}\begin{bmatrix} 607 \\ 607 \\ 607 \end{bmatrix} = \begin{bmatrix} 1505.66 \\ 1712.77 \\ 1411.58 \end{bmatrix}.$$

Thus, about 1506 units of agriculture, 1713 units of manufacturing, and 1412 units of transportation should be produced.

14. $A = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}, \; D = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}$

$$I - A = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ -\frac{1}{3} & 1 & 0 \\ 0 & -\frac{1}{4} & 1 \end{bmatrix}$$

$$(I - A)^{-1} = \begin{bmatrix} 1.043478 & 0.130435 & 0.521739 \\ 0.347826 & 1.043478 & 0.173913 \\ 0.086957 & 0.260870 & 1.043418 \end{bmatrix}$$

$$X = (I - A)^{-1} D = \begin{bmatrix} 1695.6 \\ 1565.2 \\ 1391.3 \end{bmatrix}$$

Produce about 1696 units of agriculture, about 1565 units of manufacturing, and about 1391 units of transportation.

15. From the given data, we get the input-output matrix

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & 0 \end{bmatrix}.$$

$$I - A = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{4} & 1 \end{bmatrix}$$

Use row operations to find the inverse of $I - A$, which is

$$(I - A)^{-1} \approx \begin{bmatrix} 1.538 & 0.923 & 0.615 \\ 0.615 & 1.436 & 0.513 \\ 0.923 & 0.821 & 1.436 \end{bmatrix}.$$

Since $X = (I - A)^{-1}D,$ the production matrix is

$$X = \begin{bmatrix} 1.538 & 0.923 & 0.615 \\ 0.615 & 1.436 & 0.513 \\ 0.923 & 0.821 & 1.436 \end{bmatrix} \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} \approx \begin{bmatrix} 3077 \\ 2564 \\ 3179 \end{bmatrix}.$$

Thus, the production should be about 3077 units of agriculture, 2564 units of manufacturing, and 3179 units of transportation.

16. $A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & 0 \end{bmatrix}, D = \begin{bmatrix} 500 \\ 500 \\ 500 \end{bmatrix}$

$$I - A = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{4} & 1 \end{bmatrix}$$

$$(I - A)^{-1} = \begin{bmatrix} 1.538 & 0.9231 & 0.6154 \\ 0.6154 & 1.436 & 0.5128 \\ 0.9231 & 0.8205 & 1.436 \end{bmatrix}$$

$$X = (I - A)^{-1}D = \begin{bmatrix} 1538.3 \\ 1282.1 \\ 1589.8 \end{bmatrix}$$

Produce about 1538 units of agriculture, about 1282 units of manufacturing, and about 1590 units of transportation.

17. From the given data, we get the input-output matrix

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{6} \\ \frac{1}{2} & 0 \end{bmatrix}.$$

$$I - A = \begin{bmatrix} \frac{3}{4} & -\frac{1}{6} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Use row operations to find the inverse of $I - A$, which is

$$(I - A)^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{1}{4} \\ \frac{3}{4} & \frac{9}{8} \end{bmatrix}.$$

(a) The production matrix is

$$X = (I - A)^{-1}D = \begin{bmatrix} \frac{3}{2} & \frac{1}{4} \\ \frac{3}{4} & \frac{9}{8} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ \frac{15}{8} \end{bmatrix}.$$

Thus, $\frac{7}{4}$ bushels of yams and $\frac{15}{8} \approx 2$ pigs should be produced.

(b) The production matrix is

$$X = (I - A)^{-1}D = \begin{bmatrix} \frac{3}{2} & \frac{1}{4} \\ \frac{3}{4} & \frac{9}{8} \end{bmatrix} \begin{bmatrix} 100 \\ 70 \end{bmatrix} = \begin{bmatrix} 167.5 \\ 153.75 \end{bmatrix}.$$

Thus, 167.5 bushels of yams and $153.75 \approx 154$ pigs should be produced.

18. For this economy,

$$A = \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0.4 & 0.2 & 0.1 \\ 0 & 0.1 & 0.2 \end{bmatrix}.$$

Use a graphing calculator or a computer to find $(I - A)^{-1}D$ where

$$D = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}.$$

The solution, which may vary slightly, is

$$\begin{bmatrix} 3179.35 \\ 3043.48 \\ 1630.43 \end{bmatrix}.$$

Produce about 3179 units of oil, 3043 units of corn, and 1630 units of coffee.

19. Use a graphing calculator or a computer to find the production matrix $X = (I - A)^{-1}D.$ The answer is

$$\begin{bmatrix} 848 \\ 516 \\ 2970 \end{bmatrix}.$$

Values have been rounded.

Produce 848 units of agriculture, 516 units of manufacturing, and 2970 units of households.

20. Use a graphing calculator or a computer to find $(I - A)^{-1}D.$ The solution, which may vary slightly, is

$$\begin{bmatrix} 18.2 \\ 73.2 \\ 66.7 \end{bmatrix}.$$

Values have been rounded.

Produce \$18.2 billion of agriculture, \$73.2 billion of manufacturing, and \$66.7 billion of households.

21. Use a graphing calculator or a computer to find the production matrix $X = (I - A)^{-1}D.$ The answer is

$$\begin{bmatrix} 195{,}492 \\ 25{,}933 \\ 13{,}580 \end{bmatrix}.$$

Values have been rounded. Change from thousands of pounds to millions of pounds.

Produce about 195 million Israeli pounds of agriculture, 26 million Israeli pounds of manufacturing, and 13.6 million Israeli pounds of energy.

22. **(a)** Use a graphing calculator or a computer to find $(I - A)^{-1}D.$ The solution, which may vary slightly, is

$$\begin{bmatrix} 183{,}464 \\ 304{,}005 \\ 42{,}037 \end{bmatrix}.$$

Values have been rounded.

In 100,000 RMB, produce about 183,000 for agriculture, 304,000 for industry/construction, and 42,000 for transportation/commerce.

(b) The entries in matrix $(I - A)^{-1}$ are called multipliers, and they give the desired economic values.

$$(I - A)^{-1} \approx \begin{bmatrix} 1.24 & 0.34 & 0.04 \\ 0.30 & 1.85 & 0.14 \\ 0.03 & 0.08 & 1.02 \end{bmatrix}$$

(Each entry has been rounded to two decimal places.) Since we are interested in the result when there is a 1 RMB increase in demand for agricultural exports, we are interested in the first column of this matrix. Interpreting the multipliers shown, we find that an increase of 1 RMB in demand for agricultural exports will result in a 1.24 RMB increase in production of agricultural commodities, a 0.30 RMB increase in production of industrial/ construction commodities, and a 0.03 RMB increase in production of transportation/ commercial commodities.

23. Use a graphing calculator or a computer to find the production matrix $X = (I - A)^{-1}D.$ The answer is

$$\begin{bmatrix} 532 \\ 481 \\ 805 \\ 1185 \end{bmatrix}.$$

Values have been rounded.

Produce about 532 units of natural resources, 481 manufacturing units, 805 trade and service units, and 1185 personal consumption units. Units are millions of dollars.

24. **(a)** Use a graphing calculator or a computer to find $(I - B)^{-1}C.$ The solution, which may vary slightly, is

$$\begin{bmatrix} 3 \\ 60 \\ 27 \\ 42 \\ 1002 \end{bmatrix}.$$

Values have been rounded.

A \$50 million increase in manufacturing demand will result in a \$3 million production increase in natural resources, a \$60 million production increase in manufacturing, a \$27 million production increase in trade and services, a \$42 million production increase in personal consumption, and 1002 new jobs.

(b) The matrix $(I - B)^{-1}$ is

$$\begin{bmatrix} 1.1 & 0.1 & 0 & 0 & 0 \\ 0.2 & 1.2 & 0.2 & 0.1 & 0 \\ 0.7 & 0.5 & 1.9 & 0.7 & 0 \\ 1.3 & 0.8 & 1.3 & 1.6 & 0 \\ 40.9 & 20.0 & 39.2 & 16.0 & 1 \end{bmatrix}$$

The bottom row of this matrix indicates the total employment requirement, per million dollars, of a sector. For example, the total employment requirement, per million dollars, of natural resource output is 40.9 employees.

25. **(a)** Use a graphing calculator or a computer to find the matrix $(I - A)^{-1}$. The answer is

$$\begin{bmatrix} 1.67 & 0.56 & 0.56 \\ 0.19 & 1.17 & 0.06 \\ 3.15 & 3.27 & 4.38 \end{bmatrix}.$$

Values have been rounded.

(b) These multipliers imply that if the demand for one community's output increases by \$1 then the output in the other community will increase by the amount in the row and column of this matrix. For example, if the demand for Hermitage's output increases by \$1, then output from Sharon will increase \$0.56, from Farrell by \$0.06, and from Hermitage by \$4.38.

26. Find the value of $I - A$, then set $(I - A)X = O$.

$$(I - A)X = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{3}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4}x_1 - \frac{1}{2}x_2 \\ -\frac{3}{4}x_1 + \frac{1}{2}x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus,

$$\frac{3}{4}x_1 - \frac{1}{2}x_2 = 0$$
$$\frac{3}{4}x_1 = \frac{1}{2}x_2$$
$$x_1 = \frac{2}{3}x_2.$$

If $x_2 = 3$, then $x_1 = 2$. Therefore, produce 2 units of yams for every 3 units of pigs.

27. Calculate $I - A$, and then set $(I - A)X = O$ to find X.

$$(I - A)X = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{2}{3} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & -\frac{1}{3} \\ -\frac{1}{4} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}x_1 - \frac{1}{3}x_2 \\ -\frac{1}{4}x_1 + \frac{1}{3}x_2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus,

$$\frac{1}{4}x_1 - \frac{1}{3}x_2 = 0$$
$$\frac{1}{4}x_1 = \frac{1}{3}x_2$$
$$x_1 = \frac{4}{3}x_2$$

If $x_2 = 3, x_1 = 4$. Therefore, produce 4 units of steel for every 3 units of coal.

28. Find the value of $I - A$, then set $(I - A)X = O.$

$$(I - A)X = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} & 0 \\ -\frac{1}{3} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{3} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3}x_1 - \frac{1}{2}x_2 \\ -\frac{1}{3}x_1 + \frac{3}{4}x_2 - \frac{1}{4}x_3 \\ -\frac{1}{3}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system to be solved is

$$\frac{2}{3}x_1 - \frac{1}{2}x_2 = 0$$
$$-\frac{1}{3}x_1 + \frac{3}{4}x_2 - \frac{1}{4}x_3 = 0$$
$$-\frac{1}{3}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 = 0.$$

Write the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} \frac{2}{3} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{3} & \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{3} & -\frac{1}{4} & \frac{1}{4} & 0 \end{array}\right]$$

$$\begin{array}{l} \frac{3}{2}R_1 \to R_1 \\ 12R_2 \to R_2 \\ 12R_3 \to R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -\frac{3}{4} & 0 & 0 \\ -4 & 9 & -3 & 0 \\ -4 & -3 & 3 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ 4R_1 + R_2 \to R_2 \\ 4R_1 + R_3 \to R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 6 & -3 & 0 \\ 0 & -6 & 3 & 0 \end{array}\right]$$

$$\frac{1}{6}R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -6 & 3 & 0 \end{array}\right]$$

$$\begin{array}{l} \frac{3}{4}R_2 + R_1 \to R_1 \\ \\ 6R_2 + R_3 \to R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{8} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Use x_3 as the parameter. Therefore, $x_1 = \frac{3}{8}x_3$ and $x_2 = \frac{1}{2}x_3$, and the solution is $\left(\frac{3}{8}x_3, \frac{1}{2}x_3, x_3\right)$. If $x_3 = 8$, then $x_1 = 3$ and $x_2 = 4$.

Produce 3 units of agriculture to every 4 units of manufacturing and 8 units of transportation.

29. For this economy,

$$A = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}.$$

Find the value of $I - A$, then set $(I - A)X = O$.

$$(I - A)X = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ -\frac{2}{5} & \frac{4}{5} & -\frac{4}{5} \\ -\frac{2}{5} & -\frac{1}{5} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{5}x_1 - \frac{3}{5}x_2 \\ -\frac{2}{5}x_1 + \frac{4}{5}x_2 - \frac{4}{5}x_3 \\ -\frac{2}{5}x_1 - \frac{1}{5}x_2 + \frac{4}{5}x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system to be solved is

$$\begin{aligned} \frac{4}{5}x_1 - \frac{3}{5}x_2 \qquad\quad &= 0 \\ -\frac{2}{5}x_1 + \frac{4}{5}x_2 - \frac{4}{5}x_3 &= 0 \\ -\frac{2}{5}x_1 - \frac{1}{5}x_2 + \frac{4}{5}x_3 &= 0. \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} \frac{4}{5} & -\frac{3}{5} & 0 & 0 \\ -\frac{2}{5} & \frac{4}{5} & -\frac{4}{5} & 0 \\ -\frac{2}{5} & -\frac{1}{5} & \frac{4}{5} & 0 \end{array}\right]$$

$$\begin{array}{l} \frac{5}{4}R_1 \to R_1 \\ 5R_2 \to R_2 \\ 5R_3 \to R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -\frac{3}{4} & 0 & 0 \\ -2 & 4 & -4 & 0 \\ -2 & -1 & 4 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ 2R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & \frac{5}{2} & -4 & 0 \\ 0 & -\frac{5}{2} & 4 & 0 \end{array}\right]$$

$$\frac{2}{5}R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 1 & -\frac{8}{5} & 0 \\ 0 & -\frac{5}{2} & 4 & 0 \end{array}\right]$$

$$\begin{array}{l} \frac{3}{4}R_2 + R_1 \to R_1 \\ \\ \frac{5}{2}R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{6}{5} & 0 \\ 0 & 1 & -\frac{8}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Use x_3 as the parameter. Therefore, $x_1 = \frac{6}{5}x_3$ and $x_2 = \frac{8}{5}x_3$, and the solution is $\left(\frac{6}{5}x_3, \frac{8}{5}x_3, x_3\right)$. If $x_3 = 5$, then $x_1 = 6$ and $x_2 = 8$.

Produce 6 units of mining for every 8 units of manufacturing and 5 units of communication.

Chapter 2 Review Exercises

1. True

2. True

3. False; a system with three equations and four unknowns has an infinite number of solutions.

4. False; only row operations can be used.

5. True

6. False; matrix A is a 2×2 matrix and matrix B is a 3×2 matrix. Only matrices having the same dimension can be added.

7. False; only matrices having the same dimension can be added.

8. True

9. False; in general, matrix multiplication is not commutative.

10. False; only square matrices can have inverses.

11. True; for example, $0 \cdot A = A \cdot 0 = 0$

12. False; any $n \times n$ zero matrix does not have an inverse, and the matrix $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$ is an example of another square matrix that doesn't have an inverse.

13. False; if $AB = C$ and A has an inverse, then $B = A^{-1}C$.

14. True

15. False; $AB = CB$ implies $A = C$ only if B is the identity matrix or B has an inverse.

16. True

17. For a system of m linear equations in n unknowns and $m = n$, there could be one, none, or an infinite number of solutions. If $m < n$, there are an infinite number of solutions. If $m > n$, there could be one, none, or an infinite number of solutions.

19.
$$\begin{aligned} 2x - 3y &= 14 \quad (1) \\ 3x + 2y &= -5 \quad (2) \end{aligned}$$

Eliminate x in equation (2).

$$\begin{array}{lrl} & 2x - 3y &= 14 \quad (1) \\ -3R_1 + 2R_2 \to R_2 & 13y &= -52 \quad (3) \end{array}$$

Make each leading coefficient equal 1.

$$\begin{array}{lrl} \frac{1}{2}R_1 \to R_1 & x - \frac{3}{2}y &= 7 \quad (4) \\ \frac{1}{13}R_2 \to R_2 & y &= -4 \quad (5) \end{array}$$

Substitute –4 for y in equation (4) to get $x = 1$. The solution is $(1, -4)$.

20.
$$\begin{aligned} \frac{x}{2} + \frac{y}{4} &= 3 \\ \frac{x}{4} - \frac{y}{2} &= 4 \end{aligned}$$

First, multiply both equations by 4 to clear fractions.

$$\begin{aligned} 2x + y &= 12 \\ x - 2y &= 16 \end{aligned}$$

Now proceed by the echelon method.

$$\begin{array}{lrl} & 2x + y &= 12 \\ R_1 + (-2)R_2 \to R_2 & 5y &= -20 \\ \frac{1}{2}R_1 \to R_1 & x + \frac{1}{2}y &= 6 \\ \frac{1}{5}R_2 \to R_2 & y &= -4 \end{array}$$

Back-substitution gives

$$\begin{aligned} x + \frac{1}{2}(-4) &= 6 \\ x &= 8. \end{aligned}$$

The solution is $(8, -4)$.

21.
$$\begin{aligned} 2x - 3y + z &= -5 \quad (1) \\ 5x + 5y + 3z &= 14 \quad (2) \end{aligned}$$

Eliminate x in equation (2).

$$\begin{array}{lrl} & 2x - 3y + z &= -5 \\ 5R_1 + (-2)R_2 \to R_2 & -25y - z &= -53 \end{array}$$

Let z be the parameter. Solve for y and for x in terms of z.

$$\begin{aligned}-25y - z &= -53\\ -25y &= -53 + z\\ y &= \frac{53-z}{25}\\ 2x - 3\left(\frac{53-z}{25}\right) + z &= -5\\ 2x - \frac{159}{25} + \frac{3z}{25} + z &= -5\\ 2x + \frac{28}{25}z &= \frac{34}{25}\\ 2x &= \frac{34}{25} - \frac{28}{25}z\\ x &= \frac{34-28z}{50}\end{aligned}$$

The solutions are $\left(\frac{34-28z}{50}, \frac{53-z}{25}, z\right)$, where z is any real number.

22. $2x - 3y + 4z = 5$ (1)
$3x + 4y + 5z = 6$ (2)

Eliminate x in equation (2).

$$\begin{aligned}& 2x - 3y + 4z = 5\\ 3R_1 + (-2)R_2 \to R_2 \quad & 25y + 2z = 3\end{aligned}$$

Let z be the parameter. Solve for y and for x in terms of z.

$$\begin{aligned}y + 2z &= 3\\ y &= 3 - 2z\end{aligned}$$

$$\begin{aligned}2x - 3(3 - 2z) + 4z &= 5\\ 2x + 9 - 6z + 4z &= 5\\ 2x - 2z &= -4\\ 2x &= 2z - 4\\ x &= z - 2\end{aligned}$$

The solutions are $(z-2, 3-2z, z)$, where z is any real number.

23. $2x + 4y = -6$
$-3x - 5y = 12$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rr|r}2 & 4 & -6\\ -3 & -5 & 12\end{array}\right]$$

$$3R_1 + 2R_2 \to R_2 \quad \left[\begin{array}{rr|r}2 & 4 & -6\\ 0 & 2 & 6\end{array}\right]$$

$$-2R_2 + R_1 \to R_1 \quad \left[\begin{array}{rr|r}2 & 0 & -18\\ 0 & 2 & 6\end{array}\right]$$

$$\begin{array}{l}\frac{1}{2}R_1 \to R_1\\ \frac{1}{2}R_2 \to R_2\end{array} \quad \left[\begin{array}{rr|r}1 & 0 & -9\\ 0 & 1 & 3\end{array}\right]$$

The solution is $(-9, 3)$.

24. $x - 4y = 10$
$5x + 3y = 119$

Write the system in augmented matrix form and use row operations to solve.

$$\left[\begin{array}{rr|r}1 & -4 & 10\\ 5 & 3 & 119\end{array}\right]$$

$$-5R_1 + R_2 \to R_2 \quad \left[\begin{array}{rr|r}1 & -4 & 10\\ 0 & 23 & 69\end{array}\right]$$

$$4R_2 + 23R_1 \to R_1 \quad \left[\begin{array}{rr|r}23 & 0 & 506\\ 0 & 23 & 69\end{array}\right]$$

$$\begin{array}{l}\frac{1}{23}R_1 \to R_1\\ \frac{1}{23}R_2 \to R_2\end{array} \quad \left[\begin{array}{rr|r}1 & 0 & 22\\ 0 & 1 & 3\end{array}\right]$$

The solution is (22, 3).

25. $x - y + 3z = 13$
$4x + y + 2z = 17$
$3x + 2y + 2z = 1$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 13 \\ 4 & 1 & 2 & 17 \\ 3 & 2 & 2 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ -4R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 3 & 13 \\ 0 & 5 & -10 & -35 \\ 0 & 5 & -7 & -38 \end{array}\right]$$

$$\begin{array}{l} R_2 + 5R_1 \to R_1 \\ \\ -1R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 5 & 0 & 5 & 30 \\ 0 & 5 & -10 & -35 \\ 0 & 0 & 3 & -3 \end{array}\right]$$

$$\begin{array}{l} 5R_3 + (-3R_1) \to R_1 \\ 10R_3 + 3R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{rrr|r} -15 & 0 & 0 & -105 \\ 0 & 15 & 0 & -135 \\ 0 & 0 & 3 & -3 \end{array}\right]$$

$$\begin{array}{l} -\frac{1}{15}R_1 \to R_1 \\ \frac{1}{15}R_2 \to R_2 \\ \frac{1}{3}R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The solution is $(7, -9, -1)$.

26.
$$\begin{aligned} x + 2y + 3z &= 9 \quad (1) \\ x - 2y &= 4 \quad (2) \\ 3x + 2z &= 12 \quad (3) \end{aligned}$$

Eliminate x in equations (2) and (3).

$$\begin{array}{lrl} & x + 2y + 3z = 9 & (1) \\ -1R_1 + R_2 \to R_2 & -4y - 3z = -5 & (4) \\ -3R_1 + R_3 \to R_3 & -6y - 7z = -15 & (5) \end{array}$$

Eliminate y in equation (5).

$$\begin{array}{lrl} & x + 2y + 3z = 9 & (1) \\ & -4y - 3z = -5 & (4) \\ 3R_2 + (-2)R_3 \to R_3 & 5z = 15 & (6) \end{array}$$

Make each leading coefficient equal 1.

$$\begin{array}{lrl} & x + 2y + 3z = 9 & (1) \\ -\frac{1}{4}R_2 \to R_2 & y + \frac{3}{4}z = \frac{5}{4} & (7) \\ \frac{1}{5}R_3 \to R_3 & z = 3 & (8) \end{array}$$

Back-substitution gives

$$y + \frac{3}{4}(3) = \frac{5}{4}$$
$$y = \frac{5}{4} - \frac{9}{4} = -1$$

and

$$x + 2(-1) + 3(3) = 9$$
$$x = 2.$$

The solution is $(2, -1, 3)$.

27.
$$\begin{aligned} 3x - 6y + 9z &= 12 \\ -x + 2y - 3z &= -4 \\ x + y + 2z &= 7 \end{aligned}$$

Write the augmented matrix and use row operations.

$$\left[\begin{array}{rrr|r} 3 & -6 & 9 & 12 \\ -1 & 2 & -3 & -4 \\ 1 & 1 & 2 & 7 \end{array}\right]$$

$$\begin{array}{l} \\ R_1 + 3R_2 \to R_2 \\ -1R_1 + 3R_3 \to R_3 \end{array} \left[\begin{array}{rrr|r} 3 & -6 & 9 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 9 & -3 & 9 \end{array}\right]$$

The zero in row 2, column 2 is an obstacle. To proceed, interchange the second and third rows.

$$\left[\begin{array}{rrr|r} 3 & -6 & 9 & 12 \\ 0 & 9 & -3 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$3R_1 + 2R_2 \to R_1 \left[\begin{array}{rrr|r} 9 & 0 & 21 & 54 \\ 0 & 9 & -3 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{9}R_1 \to R_1 \\ \frac{1}{9}R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & 0 & \frac{7}{3} & 6 \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The row of zeros indicates dependent equations.

Solve the first two equations respectively for x and y in terms of z to obtain

$$x = 6 - \frac{7}{3}z \text{ and } y = 1 + \frac{1}{3}z$$

The solution of the system is

$$\left(6 - \frac{7}{3}z,\ 1 + \frac{1}{3}z,\ z\right),$$

where z is any real number.

28.
$$\begin{aligned} x - 2z &= 5 \\ 3x + 2y &= 8 \\ -x + 2z &= 10 \end{aligned}$$

Write the system in augmented matrix form and apply row operations.

$$\left[\begin{array}{rrr|r} 1 & 0 & -2 & 5 \\ 3 & 2 & 0 & 8 \\ -1 & 0 & 2 & 10 \end{array}\right]$$

$$\begin{array}{r} -3R_1 + R_2 \rightarrow R_2 \\ R_1 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & -2 & 5 \\ 0 & 2 & 6 & -7 \\ 0 & 0 & 0 & 15 \end{array}\right]$$

The last row says that $0 = 15$, which is false, so the system is inconsistent and there is no solution.

In Exercises 29–32, corresponding elements must be equal.

29. $\begin{bmatrix} 2 & 3 \\ 5 & q \end{bmatrix} = \begin{bmatrix} a & b \\ c & 9 \end{bmatrix}$

Size: 2×2; $a = 2, b = 3, c = 5, q = 9$; square matrix

30. $\begin{bmatrix} 2 & x \\ y & 6 \\ 5 & z \end{bmatrix} = \begin{bmatrix} a & -1 \\ 4 & 6 \\ p & 7 \end{bmatrix}$

The size of these matrices is 3×2. For matrices to be equal, corresponding elements must be equal, so $a = 2$, $x = -1$, $y = 4$, $p = 5$, and $z = 7$.

31. $\begin{bmatrix} 2m & 4 & 3z & -12 \end{bmatrix}$
$= \begin{bmatrix} 12 & k+1 & -9 & r-3 \end{bmatrix}$

Size: 1×4; $m = 6, k = 3, z = -3, r = -9$; row matrix

32. $\begin{bmatrix} a+5 & 3b & 6 \\ 4c & 2+d & -3 \\ -1 & 4p & q-1 \end{bmatrix} = \begin{bmatrix} -7 & b+2 & 2k-3 \\ 3 & 2d-1 & 4\ell \\ m & 12 & 8 \end{bmatrix}$

These are 3×3 square matrices. Since corresponding elements must be equal,

$a + 5 = -7$, so $a = -12$;

$3b = b + 2$, so $b = 1$;

$6 = 2k - 3$, so $k = \frac{9}{2}$;

$4c = 3$, so $c = \frac{3}{4}$;

$2 + d = 2d - 1$, so $d = 3$;

$-3 = 4\ell$, so $\ell = -\frac{3}{4}$;

$m = -1$;

$4p = 12$, so $p = 3$; and

$q - 1 = 8$, so $q = 9$

33. $A + C = \begin{bmatrix} 4 & 10 \\ -2 & -3 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 9 & 10 \\ -3 & 0 \\ 10 & 16 \end{bmatrix}$

34. $2G - 4F = 2\begin{bmatrix} -2 & 0 \\ 1 & 5 \end{bmatrix} - 4\begin{bmatrix} -1 & 4 \\ 3 & 7 \end{bmatrix}$

$= \begin{bmatrix} -4 & 0 \\ 2 & 10 \end{bmatrix} - \begin{bmatrix} -4 & 16 \\ 12 & 28 \end{bmatrix}$

$= \begin{bmatrix} 0 & -16 \\ -10 & -18 \end{bmatrix}$

35. $3C + 2A = 3\begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix} + 2\begin{bmatrix} 4 & 10 \\ -2 & -3 \\ 6 & 9 \end{bmatrix}$

$= \begin{bmatrix} 15 & 0 \\ -3 & 9 \\ 12 & 21 \end{bmatrix} + \begin{bmatrix} 8 & 20 \\ -4 & -6 \\ 12 & 18 \end{bmatrix}$

$= \begin{bmatrix} 23 & 20 \\ -7 & 3 \\ 24 & 39 \end{bmatrix}$

36. Since B is a 3×3 matrix, and C is a 3×2 matrix, the calculation of $B - C$ is not possible.

37. $2A - 5C = 2\begin{bmatrix} 4 & 10 \\ -2 & -3 \\ 6 & 9 \end{bmatrix} - 5\begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 8 & 20 \\ -4 & -6 \\ 12 & 18 \end{bmatrix} - \begin{bmatrix} 25 & 0 \\ -5 & 15 \\ 20 & 35 \end{bmatrix}$

$= \begin{bmatrix} -17 & 20 \\ 1 & -21 \\ -8 & -17 \end{bmatrix}$

38. A has size 3×2 and G has 2×2, so AG will have size 3×2.

$$AG = \begin{bmatrix} 4 & 10 \\ -2 & -3 \\ 6 & 9 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 50 \\ 1 & -15 \\ -3 & 45 \end{bmatrix}$$

39. A is 3×2 and C is 3×2, so finding the product AC is not possible.

$$\begin{matrix} A & C \\ 3 \times \mathbf{2} & \mathbf{3} \times 2 \end{matrix}$$

(The inner two numbers must match.)

40. D has size 3×1 and E has size 1×3, so DE will have size 3×3.

$$DE = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 18 & -24 \\ 1 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

41. $ED = [1 \quad 3 \quad -4] \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$

$= [1.6 + 3.1 + (-4)0] = [9]$

42. B has size 3×3 and D has size 3×1, so BD will have size 3×1.

$$BD = \begin{bmatrix} 2 & 3 & -2 \\ 2 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 16 \\ 1 \end{bmatrix}$$

43. $EC = \begin{bmatrix} 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix}$

$= [1 \cdot 5 + 3(-1) + (-4) \cdot 4 \quad 1 \cdot 0$
$\quad + 3 \cdot 3 + (-4) \cdot 7]$
$= \begin{bmatrix} -14 & -19 \end{bmatrix}$

44. $F = \begin{bmatrix} -1 & 4 \\ 3 & 7 \end{bmatrix}$

$$[F|I] = \left[\begin{array}{cc|cc} -1 & 4 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right]$$

$$3R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|cc} -1 & 4 & 1 & 0 \\ 0 & 19 & 3 & 1 \end{array}\right]$$

$$4R_2 + (-19R_1) \to R_1 \quad \left[\begin{array}{cc|cc} 19 & 0 & -7 & 4 \\ 0 & 19 & 3 & 1 \end{array}\right]$$

$$\begin{matrix} \frac{1}{19}R_1 \to R_1 \\ \frac{1}{19}R_2 \to R_2 \end{matrix} \quad \left[\begin{array}{cc|cc} 1 & 0 & -\frac{7}{19} & \frac{4}{19} \\ 0 & 1 & \frac{3}{19} & \frac{1}{19} \end{array}\right]$$

$$F^{-1} = \begin{bmatrix} -\frac{7}{19} & \frac{4}{19} \\ \frac{3}{19} & \frac{1}{19} \end{bmatrix}$$

45. Find the inverse of $B = \begin{bmatrix} 2 & 3 & -2 \\ 2 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, if it exists.

Write the augmented matrix to obtain

$$[B|I] = \left[\begin{array}{ccc|ccc} 2 & 3 & -2 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right].$$

$$-1R_1 + R_2 \to R_2 \quad \left[\begin{array}{ccc|ccc} 2 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} -3R_2 + R_1 \to R_1 \\ \\ -1R_2 + R_3 \to R_3 \end{matrix} \quad \left[\begin{array}{ccc|ccc} 2 & 0 & -8 & 4 & -3 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

No inverse exists, since the third row is all zeros to the left of the vertical bar.

46. A and C are 3×2 matrices, so their sum $A + C$ is a 3×2 matrix. Only square matrices have inverses. Therefore, $(A + C)^{-1}$ does not exist.

47. Find the inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$, if it exists.

Write the augmented matrix $[A|I]$.

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

Perform row operations on $[A|I]$ to get a matrix of the form $[I|B]$.

$$-2R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$-3R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

The last augmented matrix is of the form $[I|B]$, so the desired inverse is

$$A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}.$$

48. Let $A = \begin{bmatrix} -4 & 2 \\ 0 & 3 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{cc|cc} -4 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right]$$

$$2R_2 + (-3R_1) \rightarrow R_1 \quad \left[\begin{array}{cc|cc} 12 & 0 & -3 & 2 \\ 0 & 3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{12}R_1 \rightarrow R_1 \\ \frac{1}{3}R_2 \rightarrow R_2 \end{array} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & \frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{6} \\ 0 & \frac{1}{3} \end{bmatrix}$$

49. Find the inverse of $A = \begin{bmatrix} 3 & -6 \\ -4 & 8 \end{bmatrix}$, if it exists.

Write the augmented matrix $[A|I]$.

$$[A|I] = \left[\begin{array}{cc|cc} 3 & -6 & 1 & 0 \\ -4 & 8 & 0 & 1 \end{array}\right]$$

Perform row operations on $[A|I]$ to get a matrix of the form $[I|B]$.

$$4R_1 + 3R_2 \rightarrow R_2 \quad \left[\begin{array}{cc|cc} 3 & -6 & 1 & 0 \\ 0 & 0 & 4 & 3 \end{array}\right]$$

Since the entries left of the vertical bar in the second row are zeros, no inverse exists.

50. Let $A = \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$.

$$[A|I] = \left[\begin{array}{cc|cc} 6 & 4 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right]$$

$$R_1 + (-2)R_2 \rightarrow R_2 \left[\begin{array}{cc|cc} 6 & 4 & 1 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The zeros in the second row indicate that the original matrix has no inverse.

51. Find the inverse of $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}$, if it exists.

The augmented matrix is

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right].$$

$$\begin{array}{l} \\ R_1 + (-2)R_2 \rightarrow R_2 \\ R_1 + (-2)R_3 \rightarrow R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & -2 & 0 \\ 0 & 3 & 0 & 1 & 0 & -2 \end{array}\right]$$

$$\begin{array}{l} -1R_2 + R_1 \rightarrow R_1 \\ \\ 3R_2 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 1 & -2 & 0 \\ 0 & 0 & -6 & 4 & -6 & -2 \end{array}\right]$$

$$\begin{array}{l} R_3 + 3R_1 \rightarrow R_1 \\ R_3 + (-3)R_2 \rightarrow R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 6 & 0 & 0 & 4 & 0 & -2 \\ 0 & 3 & 0 & 1 & 0 & -2 \\ 0 & 0 & -6 & 4 & -6 & -2 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{6}R_1 \rightarrow R_1 \\ \frac{1}{3}R_2 \rightarrow R_2 \\ -\frac{1}{6}R_3 \rightarrow R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & 1 & \frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{2}{3} \\ -\frac{2}{3} & 1 & \frac{1}{3} \end{bmatrix}$$

52. Let $A = \begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$.

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 4 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$-2R_2 + R_1 \rightarrow R_2 \left[\begin{array}{ccc|ccc} 2 & 0 & 4 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$-2R_3 + R_2 \rightarrow R_3 \left[\begin{array}{ccc|ccc} 2 & 0 & 4 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 & -2 & 0 \\ 0 & 0 & 8 & 1 & -2 & -2 \end{array}\right]$$

$$\begin{array}{l} -1R_3 + 2R_1 \rightarrow R_1 \\ -1R_3 + 2R_2 \rightarrow R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & 1 & 2 & 2 \\ 0 & 4 & 0 & 1 & -2 & 2 \\ 0 & 0 & 8 & 1 & -2 & -2 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{4}R_1 \to R_1 \\ \frac{1}{4}R_2 \to R_2 \\ \frac{1}{8}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

53. Find the inverse of $A = \begin{bmatrix} 1 & 3 & 6 \\ 4 & 0 & 9 \\ 5 & 15 & 30 \end{bmatrix}$, if it exists

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 0 \\ 4 & 0 & 9 & 0 & 1 & 0 \\ 5 & 15 & 30 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -4R_1 + R_2 \to R_2 \\ -5R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 0 \\ 0 & -12 & -15 & -4 & 1 & 0 \\ 0 & 0 & 0 & -5 & 0 & 1 \end{array}\right]$$

The last row is all zeros to the left of the bar, so no inverse exists.

54. Find the inverse of $A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 5 & 7 \\ -4 & 6 & -8 \end{bmatrix}$, if it exists.

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & -3 & 4 & 1 & 0 & 0 \\ 1 & 5 & 7 & 0 & 1 & 0 \\ -4 & 6 & -8 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -1R_1 + 2R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & -3 & 4 & 1 & 0 & 0 \\ 0 & 13 & 10 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \end{array}\right]$$

The zeros in the third row to the left of the vertical bar indicate that the original matrix has no inverse.

55. $A = \begin{bmatrix} 5 & 1 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} -8 \\ 24 \end{bmatrix}$

The matrix equation to be solved is $AX = B$, or

$$\begin{bmatrix} 5 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 24 \end{bmatrix}.$$

Calculate the inverse of the coefficient matrix A to obtain

$$\begin{bmatrix} 5 & 1 \\ -1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ -\frac{1}{4} & -\frac{5}{8} \end{bmatrix}.$$

Now $X = A^{-1}B$, so

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ -\frac{1}{4} & -\frac{5}{8} \end{bmatrix}\begin{bmatrix} -8 \\ 24 \end{bmatrix} = \begin{bmatrix} 1 \\ -13 \end{bmatrix}.$$

56. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

Row operations may be used to see that matrix A has no inverse. The matrix equation $AX = B$ may be written as the system of equations

$$\begin{aligned} x + 2y &= 5 \quad (1) \\ 2x + 4y &= 10. \quad (2) \end{aligned}$$

Use the elimination method to solve this system. Begin by eliminating x in equation (2).

$$\begin{array}{rr} & x + 2y = 5 \quad (1) \\ -2R_1 + R_2 \to R_2 & 0 = 0 \quad (3) \end{array}$$

The true statement in equation (3) indicates that the equations are dependent. Solve equation (1) for x in terms of y.

$$x = -2y + 5$$

The solution is $(-2y + 5, y)$, where y is any real number.

57. $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 8 \\ 4 \\ -6 \end{bmatrix}$

By the usual method, we find that the inverse of the coefficient matrix is

$$A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ -2 & 1 & 1 \\ \frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

Since $X = A^{-1}B$,

$$X = \begin{bmatrix} -2 & 0 & 1 \\ -2 & 1 & 1 \\ \frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 8 \\ 4 \\ -6 \end{bmatrix} = \begin{bmatrix} -22 \\ -18 \\ 15 \end{bmatrix}.$$

58. $A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 72 \\ -24 \\ 48 \end{bmatrix}$

Use row operations to find the inverse of *A,* which is

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{8} & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Since $X = A^{-1}B$,

$$X = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{8} & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 72 \\ -24 \\ 48 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 16 \end{bmatrix}.$$

59. $x + 2y = 4$
$2x - 3y = 1$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}.$$

Calculate the inverse of *A*.

$$A^{-1} = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix}$$

Use $X = A^{-1}B$ to solve.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The solution is (2, 1).

60. $5x + 10y = 80$
$3x - 2y = 120$

Let $A = \begin{bmatrix} 5 & 10 \\ 3 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 80 \\ 120 \end{bmatrix}.$

Use row operations to find the inverse of *A,* which is

$$A^{-1} = \begin{bmatrix} \frac{1}{20} & \frac{1}{4} \\ \frac{3}{40} & -\frac{1}{8} \end{bmatrix}.$$

Since $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{20} & \frac{1}{4} \\ \frac{3}{40} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 80 \\ 120 \end{bmatrix} = \begin{bmatrix} 34 \\ -9 \end{bmatrix}$$

The solution is $(34, -9)$.

61. $x + y + z = 1$
$2x + y = -2$
$3y + z = 2$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

Find that the inverse of *A* is

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{6}{5} & -\frac{3}{5} & -\frac{1}{5} \end{bmatrix}.$$

Now $X = A^{-1}B$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{6}{5} & -\frac{3}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

The solution is $(-1, 0, 2)$.

62. $x - 4y + 2z = -1$
$-2x + y - 3z = -9$
$3x + 5y - 2z = 7$

Let $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 1 & -3 \\ 3 & 5 & -2 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} -1 \\ -9 \\ 7 \end{bmatrix}.$

Use row operations to find the inverse of *A,* which is

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{39} & \frac{10}{39} \\ -\frac{1}{3} & -\frac{8}{39} & -\frac{1}{39} \\ -\frac{1}{3} & -\frac{17}{39} & -\frac{7}{39} \end{bmatrix} = \frac{1}{39}\begin{bmatrix} 13 & 2 & 10 \\ -13 & -8 & -1 \\ -13 & -17 & -7 \end{bmatrix}.$$

Since $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{39}\begin{bmatrix} 13 & 2 & 10 \\ -13 & -8 & -1 \\ -13 & -17 & -7 \end{bmatrix}\begin{bmatrix} -1 \\ -9 \\ 7 \end{bmatrix}$$

$$= \frac{1}{39}\begin{bmatrix} 39 \\ 78 \\ 117 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The solution is (1, 2, 3).

63. $A = \begin{bmatrix} 0.01 & 0.05 \\ 0.04 & 0.03 \end{bmatrix}, \ D = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$

$$X = (I - A)^{-1}D$$

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.01 & 0.05 \\ 0.04 & 0.03 \end{bmatrix} = \begin{bmatrix} 0.99 & -0.05 \\ -0.04 & 0.97 \end{bmatrix}$$

Use row operations to find the inverse of $I - A$, which is

$$(I - A)^{-1} = \begin{bmatrix} 1.0122 & 0.0522 \\ 0.0417 & 1.0331 \end{bmatrix}.$$

Since $X = (I - A)^{-1}D$, the production matrix is

$$X = \begin{bmatrix} 1.0122 & 0.0522 \\ 0.0417 & 1.0331 \end{bmatrix}\begin{bmatrix} 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 218.1 \\ 318.3 \end{bmatrix}.$$

64. $A = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.1 & 0 & 0.2 \\ 0 & 0 & 0.4 \end{bmatrix}, D = \begin{bmatrix} 500 \\ 200 \\ 100 \end{bmatrix}$

$$X = (I - A)^{-1}D$$

$$I - A = \begin{bmatrix} 0.8 & -0.1 & -0.3 \\ -0.1 & 1 & -0.2 \\ 0 & 0 & 0.6 \end{bmatrix}$$

$$(I - A)^{-1} \approx \begin{bmatrix} 1.266 & 0.1266 & 0.6751 \\ 0.1266 & 1.0127 & 0.40084 \\ 0 & 0 & 1.6667 \end{bmatrix}$$

Since $X = (I - A)^{-1}D$,

$$X = \begin{bmatrix} 1.266 & 0.1266 & 0.6751 \\ 0.1266 & 1.0127 & 0.40084 \\ 0 & 0 & 1.6667 \end{bmatrix}\begin{bmatrix} 500 \\ 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 725.7 \\ 305.9 \\ 166.7 \end{bmatrix}.$$

65.
$$\begin{aligned} x + 2y + z &= 7 \quad (1) \\ 2x - y - z &= 2 \quad (2) \\ 3x - 3y + 2z &= -5 \quad (3) \end{aligned}$$

(a) To solve the system by the echelon method, begin by eliminating x in equations (2) and (3).

$$\begin{aligned} & x + 2y + z = 7 && (1) \\ -2R_1 + R_2 \to R_2 \quad & -5y - 3z = -12 && (4) \\ -3R_1 + R_3 \to R_3 \quad & -9y - z = -26 && (5) \end{aligned}$$

Eliminate y in equation (5).

$$\begin{aligned} & x + 2y + z = 7 && (1) \\ & -5y - 3z = -12 && (4) \\ -9R_2 + 5R_3 \to R_3 \quad & 22z = -22 && (6) \end{aligned}$$

Make each leading coefficient equal 1.

$$\begin{aligned} & x + 2y + z = 7 && (1) \\ -\tfrac{1}{5}R_2 \to R_2 \quad & y + \tfrac{3}{5}z = \tfrac{12}{5} && (7) \\ \tfrac{1}{22}R_3 \to R_3 \quad & z = -1 && (8) \end{aligned}$$

Substitute -1 for z in equation (7) to get $y = 3$. Substitute -1 for z and 3 for y in equation (1) to get $x = 2$.

The solution is $(2, 3, -1)$.

(b) The same system is to be solved using the Gauss-Jordan method. Write the augmented matrix and use row operations.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 2 & -1 & -1 & 2 \\ 3 & -3 & 2 & -5 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -5 & -3 & -12 \\ 0 & -9 & -1 & -26 \end{array}\right]$$

$$\begin{array}{r} 2R_2 + 5R_1 \to R_1 \\ \\ -9R_2 + 5R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 5 & 0 & -1 & 11 \\ 0 & -5 & -3 & -12 \\ 0 & 0 & 22 & -22 \end{array}\right]$$

$$\begin{array}{r} R_3 + 22R_1 \to R_1 \\ 3R_3 + 22R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 110 & 0 & 0 & 220 \\ 0 & -110 & 0 & -330 \\ 0 & 0 & 22 & -22 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{110}R_1 + R_1 \\ -\frac{1}{110}R_2 \to R_2 \\ \frac{1}{22}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The corresponding system is
$$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= -1. \end{aligned}$$

The solution is $(2, 3, -1)$

(c) The system can be written as a matrix equation $AX = B$ by writing

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 3 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ -5 \end{bmatrix}.$$

(d) The inverse of the coefficient matrix A can be found by using row operations.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & -1 & -1 & 0 & 1 & 0 \\ 3 & -3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -9 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} 2R_2 + 5R_1 \to R_1 \\ \\ -9R_2 + 5R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 5 & 0 & -1 & 1 & 2 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & 0 & 22 & 3 & -9 & 5 \end{array}\right]$$

$$\begin{array}{r} R_3 + 22R_1 \to R_1 \\ 3R_3 + 22R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 110 & 0 & 0 & 25 & 35 & 5 \\ 0 & -110 & 0 & -35 & -5 & 15 \\ 0 & 0 & 22 & 3 & -9 & 5 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{110}R_1 + R_1 \\ -\frac{1}{110}R_2 \to R_2 \\ \frac{1}{22}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{22} & \frac{7}{22} & \frac{1}{22} \\ 0 & 1 & 0 & \frac{7}{22} & \frac{1}{22} & -\frac{3}{22} \\ 0 & 0 & 1 & \frac{3}{22} & -\frac{9}{22} & \frac{5}{22} \end{array}\right]$$

The inverse of matrix A is

$$A^{-1} = \begin{bmatrix} \frac{5}{22} & \frac{7}{22} & \frac{1}{22} \\ \frac{7}{22} & \frac{1}{22} & -\frac{3}{22} \\ \frac{3}{22} & -\frac{9}{22} & \frac{5}{22} \end{bmatrix} \approx \begin{bmatrix} 0.23 & 0.32 & 0.05 \\ 0.32 & 0.05 & -0.14 \\ 0.14 & -0.41 & 0.23 \end{bmatrix}.$$

(e) Since $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{5}{22} & \frac{7}{22} & \frac{1}{22} \\ \frac{7}{22} & \frac{1}{22} & -\frac{3}{22} \\ \frac{3}{22} & -\frac{9}{22} & \frac{5}{22} \end{bmatrix}\begin{bmatrix} 7 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}.$$

Once again, the solution is $(2, 3, -1)$.

66. Use a table to organize the information.

	Standard	**Extra Large**	**Time Available**
Hours Cutting	$\frac{1}{4}$	$\frac{1}{3}$	4
Hours Shaping	$\frac{1}{2}$	$\frac{1}{3}$	6

Let $x =$ the number of standard paper clips (in thousands),

and $y =$ the number of extra large paper clips (in thousands).

The given information leads to the system

$$\begin{aligned} \frac{1}{4}x + \frac{1}{3}y &= 4 \\ \frac{1}{2}x + \frac{1}{3}y &= 6. \end{aligned}$$

Solve this system by any method to get $x = 8$, $y = 6$. The manufacturer can make 8 thousand (8000) standard and 6 thousand (6000) extra large paper clips.

67. Let $x_1 =$ the number of blankets,

$x_2 =$ the number of rugs, and

$x_3 =$ the number of skirts.

The given information leads to the system

$$\begin{aligned} 24x_1 + 30x_2 + 12x_3 &= 306 \quad (1) \\ 4x_1 + 5x_2 + 3x_3 &= 59 \quad (2) \\ 15x_1 + 18x_2 + 9x_3 &= 201. \quad (3) \end{aligned}$$

Simplify equations (1) and (3).

$$\frac{1}{6}R_1 \to R_1 \quad 4x_1 + 5x_2 + 2x_3 = 51 \qquad (4)$$
$$4x_1 + 5x_2 + 3x_3 = 59 \qquad (2)$$
$$\frac{1}{3}R_3 \to R_3 \quad 5x_1 + 6x_2 + 3x_3 = 67 \qquad (5)$$

Solve this system by the Gauss-Jordan method. Write the augmented matrix and use row operations.

$$\left[\begin{array}{ccc|c} 4 & 5 & 2 & 51 \\ 4 & 5 & 3 & 59 \\ 5 & 6 & 3 & 67 \end{array}\right]$$

$$\begin{array}{r} \\ -1R_1 + R_2 \to R_2 \\ -4R_3 + 5R_1 \to R_3 \end{array} \left[\begin{array}{ccc|c} 4 & 5 & 2 & 51 \\ 0 & 0 & 1 & 8 \\ 0 & 1 & -2 & -13 \end{array}\right]$$

Interchange the second and third rows.

$$\left[\begin{array}{ccc|c} 4 & 5 & 2 & 51 \\ 0 & 1 & -2 & -13 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

$$-5R_2 + R_1 \to R_1 \left[\begin{array}{ccc|c} 4 & 0 & 12 & 116 \\ 0 & 1 & -2 & -13 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

$$\begin{array}{r} -12R_3 + R_1 \to R_1 \\ 2R_3 + R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 4 & 0 & 0 & 20 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

$$\frac{1}{4}R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

The solution of the system is $x = 5, y = 3, z = 8$. So, 5 blankets, 3 rugs, and 8 skirts can be made.

68. Let $x =$ Tulsa's number of gallons,
$y =$ New Orleans' number of gallons, and
$z =$ Ardmore's number of gallons.

The system that may be written is

$$0.5x + 0.4y + 0.3z = 219{,}000 \qquad \textit{Chicago}$$
$$0.2x + 0.4y + 0.4z = 192{,}000 \qquad \textit{Dallas}$$
$$0.3x + 0.2y + 0.3z = 144{,}000. \qquad \textit{Atlanta}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 0.5 & 0.4 & 0.3 & 219{,}000 \\ 0.2 & 0.4 & 0.4 & 192{,}000 \\ 0.3 & 0.2 & 0.3 & 144{,}000 \end{array}\right].$$

$$\begin{array}{r} \\ 2R_1 + (-5)R_2 \to R_2 \\ 3R_1 + (-5)R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 0.5 & 0.4 & 0.3 & 219{,}000 \\ 0 & -1.2 & -1.4 & -522{,}000 \\ 0 & 0.2 & -0.6 & -63{,}000 \end{array}\right]$$

$$\begin{array}{r} -2R_3 + R_1 \to R_1 \\ \\ R_2 + 6R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 0.5 & 0 & 1.5 & 345{,}000 \\ 0 & -1.2 & -1.4 & -522{,}000 \\ 0 & 0 & -5 & -900{,}000 \end{array}\right]$$

$$\begin{array}{r} 0.3R_3 + R_1 \to R_1 \\ -14R_3 + 50R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 0.5 & 0 & 0 & 75{,}000 \\ 0 & -60 & 0 & -13{,}500{,}000 \\ 0 & 0 & -5 & -900{,}000 \end{array}\right]$$

$$\begin{array}{r} 2R_1 \to R_1 \\ -\frac{1}{60}R_2 \to R_2 \\ -\frac{1}{5}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 150{,}000 \\ 0 & 1 & 0 & 225{,}000 \\ 0 & 0 & 1 & 180{,}000 \end{array}\right]$$

Thus, 150,000 gal were produced at Tulsa, 225,000 gal at New Orleans, and 180,000 gal at Ardmore.

69. **(a)** Let $x =$ the number of football jerseys
$y =$ the number of basketball jerseys
$z =$ the number of baseball jerseys

The system to be solved is

Cutting $1.5x + 0.5y + z = 380$
Sewing $1.2x + 0.6y + 0.9z = 330$

(b) Adding 4 times the first equation to -5 times the second yields

$$-y - 0.5z = 130$$

so

$$y = 130 - 0.5z.$$

Substitute this value for y in the original cutting equation and solve for x in terms of z.

$$1.5x + 0.5(130 - 0.5z) + z = 380$$
$$x = 210 - 0.5z$$

The general solution is $(210 - 0.5z,\ 130 - 0.5z,\ z)$.

(c) For all quantities to be nonnegative we need

$130 - 0.5z \geq 0$ or $0 \leq z \leq 260$

So the possible values for z are 0, 1, 2, … , 260.

(d) The greatest number of baseball jerseys is 260, which yields the solution

$(210 - 0.5(260),\ 130 - 0.5(260),\ 260)$

or $(80,\ 0,\ 260)$

The production will be 80 football jerseys, no basketball jerseys, and 260 baseball jerseys.

70. (a) Let x = the number of reserved seats

y = the number of box seats

z = the number of infield seats

The system to be solved is

$$\begin{aligned} 8x + 10y + 12z &= 47,000 \\ + y + z &= 5000 \\ x &= 2y \end{aligned}$$

Substitute $2y$ for x in the first two equations.

$$\begin{aligned} 26y + 12z &= 47,000 \\ y + z &= 5000 \end{aligned}$$

Add -12 times the second equation to the first equation.

$$\begin{aligned} -10y &= -13,000 \\ y &= 1300 \\ x &= 2(1300) = 2600 \\ z &= 5000 - 1300 - 2600 = 1100 \end{aligned}$$

The stadium has 2600 reserved seats, 1300 box seats, and 1100 infield seats.

(b) The system to be solved is

$$\begin{aligned} 8x + 10y + 12z &= 22,264 \\ x + y + z &= 3608 \\ x &= 3z \end{aligned}$$

Substitute $3z$ for x in the first two equations.

$$\begin{aligned} 10y + 36z &= 22,264 \\ y + 4z &= 3608 \end{aligned}$$

Add -9 times the second equation to the first equation.

$$y = 22,264 - (9)(3608) = -10,208$$

Since the solution yields a negative number of box seats, the outcome is impossible.

71. (a)

$$\begin{array}{c} \begin{matrix} \text{NE} & \text{MW} & \text{S} & \text{W} \end{matrix} \\ \begin{bmatrix} 31 & 56 & 180 & 102 \\ 38 & 67 & 209 & 134 \\ 33 & 61 & 237 & 109 \end{bmatrix} \begin{matrix} \text{Jan} \\ \text{Feb} \\ \text{Mar} \end{matrix} \end{array}$$

$$\begin{array}{c} \begin{matrix} \text{NE} & \text{MW} & \text{S} & \text{W} \end{matrix} \\ \begin{bmatrix} 30 & 51 & 256 & 120 \\ 24 & 57 & 261 & 95 \\ 30 & 57 & 228 & 92 \end{bmatrix} \begin{matrix} \text{Jan} \\ \text{Feb} \\ \text{Mar} \end{matrix} \end{array}$$

(b)

$$\begin{bmatrix} 30 & 51 & 256 & 120 \\ 24 & 57 & 261 & 95 \\ 30 & 57 & 228 & 92 \end{bmatrix} - \begin{bmatrix} 31 & 56 & 180 & 102 \\ 38 & 67 & 209 & 134 \\ 33 & 61 & 237 & 109 \end{bmatrix} =$$

$$\begin{bmatrix} -1 & -5 & 76 & 18 \\ -14 & -10 & 52 & -39 \\ -3 & -4 & -9 & -17 \end{bmatrix}$$

(c) The general trend is down, with only the South showing strongly improved sales averaged over the three months.

72. (a)

$$\begin{matrix} \text{High} \\ \text{Medium} \\ \text{Coated} \end{matrix} \begin{bmatrix} 3170 \\ 2360 \\ 1800 \end{bmatrix}$$

(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(c) $\begin{bmatrix} 10 & 5 & 8 \\ 12 & 0 & 4 \\ 0 & 10 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3170 \\ 2360 \\ 1800 \end{bmatrix}$

(d) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 & 5 & 8 \\ 12 & 0 & 4 \\ 0 & 10 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 3170 \\ 2360 \\ 1800 \end{bmatrix}$

$$= \begin{bmatrix} -0.154 & 0.212 & 0.0769 \\ -0.231 & 0.192 & 0.2154 \\ 0.462 & -0.385 & -0.231 \end{bmatrix} \begin{bmatrix} 3170 \\ 2360 \\ 1800 \end{bmatrix} = \begin{bmatrix} 150 \\ 110 \\ 140 \end{bmatrix}$$

73. (a) The input-output matrix is

$$A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{2}{3} & 0 \end{bmatrix}.$$

(b) $I - A = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{2}{3} & 1 \end{bmatrix},\ D = \begin{bmatrix} 400 \\ 800 \end{bmatrix}$

Use row operations to find the inverse of $I - A$, which is

$$(I - A)^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{3}{4} \\ 1 & \frac{3}{2} \end{bmatrix}.$$

Since $X = (I - A)^{-1}D$,

$$X = \begin{bmatrix} \frac{3}{2} & \frac{3}{4} \\ 1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 400 \\ 800 \end{bmatrix} = \begin{bmatrix} 1200 \\ 1600 \end{bmatrix}.$$

The production required is 1200 units of cheese and 1600 units of goats.

74. **(a)** Use a graphing calculator or a computer to find $(I - A)^{-1}$. The solution, which may vary slightly, is

$$\begin{bmatrix} 1.30 & 0.045 & 0.567 & 0.012 & 0.068 & 0.020 \\ 0.204 & 1.030 & 0.183 & 0.004 & 0.022 & 0.006 \\ 0.155 & 0.038 & 1.120 & 0.020 & 0.114 & 0.034 \\ 0.018 & 0.021 & 0.028 & 1.080 & 0.016 & 0.033 \\ 0.537 & 0.525 & 0.483 & 0.279 & 1.740 & 0.419 \\ 0.573 & 0.346 & 0.497 & 0.536 & 0.087 & 1.940 \end{bmatrix}$$

Values have been rounded.
The value in row 2, column 1 of this matrix, 0.204, indicates that every \$1 of increased demand for livestock will result in an increase of production demand of \$0.204 in crops.

(b) Use a graphing calculator or computer to find $(I - A)^{-1}D$. The solution, which may vary slightly, is

$$\begin{bmatrix} 3855 \\ 1476 \\ 2726 \\ 1338 \\ 8439 \\ 10,256 \end{bmatrix}.$$

Values have been rounded.
In millions of dollars, produce \$3855 in livestock, \$1476 in crops, \$2726 in food products, \$1338 in mining and manufacturing, \$8439 in households, and \$10,256 in other business sectors.

75. The given information can be written as the following 4×3 matrix.

$$\begin{bmatrix} 8 & 8 & 8 \\ 10 & 5 & 9 \\ 7 & 10 & 7 \\ 8 & 9 & 7 \end{bmatrix}$$

76. **(a)** The X-ray passes through cells B and C, so the attenuation value for beam 3 is $b + c$.

(b) Beam 1: $a + b = 0.8$
Beam 2: $a + c = 0.55$
Beam 3: $b + c = 0.65$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.55 \\ 0.65 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0.8 \\ 0.55 \\ 0.65 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0.8 \\ 0.55 \\ 0.65 \end{bmatrix}$$

$$= \begin{bmatrix} 0.35 \\ 0.45 \\ 0.2 \end{bmatrix}$$

The solution is (0.35,0.45,0.2), so A is tumorous, B is bone, and C is healthy.

(c) For patient X,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0.54 \\ 0.40 \\ 0.52 \end{bmatrix} = \begin{bmatrix} 0.21 \\ 0.33 \\ 0.19 \end{bmatrix}.$$

A and C are healthy; B is tumorous.

For patient Y,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0.65 \\ 0.80 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 0.35 \\ 0.3 \\ 0.45 \end{bmatrix}.$$

A and B are tumorous; C is bone.

For patient Z,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0.51 \\ 0.49 \\ 0.44 \end{bmatrix} = \begin{bmatrix} 0.28 \\ 0.23 \\ 0.21 \end{bmatrix}.$$

A could be healthy or tumorous; B and C are healthy.

77. **(a)** $a + b = 0.60 \quad (1)$
$c + d = 0.75 \quad (2)$
$a + c = 0.65 \quad (3)$
$b + d = 0.70 \quad (4)$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0.60 \\ 0 & 0 & 1 & 1 & 0.75 \\ 1 & 0 & 1 & 0 & 0.65 \\ 0 & 1 & 0 & 1 & 0.70 \end{array}\right].$$

$$-1R_1 + R_3 \to R_3 \quad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0.60 \\ 0 & 0 & 1 & 1 & 0.75 \\ 0 & -1 & 1 & 0 & 0.05 \\ 0 & 1 & 0 & 1 & 0.70 \end{array}\right]$$

Interchange rows 2 and 4.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0.60 \\ 0 & 1 & 0 & 1 & 0.70 \\ 0 & -1 & 1 & 0 & 0.05 \\ 0 & 0 & 1 & 1 & 0.75 \end{array}\right]$$

$$\begin{array}{r} -1R_2 + R_1 \to R_1 \\ \\ R_2 + R_3 \to R_3 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -0.10 \\ 0 & 1 & 0 & 1 & 0.70 \\ 0 & 0 & 1 & 1 & 0.75 \\ 0 & 0 & 1 & 1 & 0.75 \end{array}\right]$$

Since R_3 and R_4 are identical, there will be infinitely many solutions. We do not have enough information to determine the values of *a, b, c*, and *d*.

(b) i. If $d = 0.33$, the system of equations in part (a) becomes

$a + b = 0.60 \quad (1)$
$c + 0.33 = 0.75 \quad (2)$
$a + c = 0.65 \quad (3)$
$b + 0.33 = 0.70. \quad (4)$

Equation (2) gives $c = 0.42$, and equation (4) gives $b = 0.37$. Substituting $c = 0.42$ into equation (3) gives $a = 0.23$. Therefore, $a = 0.23$, $b = 0.37$, $c = 0.42$, and $d = 0.33$.

Thus, A is healthy, B and D are tumorous, and *C* is bone.

ii. If $d = 0.43$, the system of equations in part (a) becomes

$a + b = 0.60 \quad (1)$
$c + 0.43 = 0.75 \quad (2)$
$a + c = 0.65 \quad (3)$
$b + 0.43 = 0.70. \quad (4)$

Equation (2) gives $c = 0.32$, and equation (4) gives $b = 0.27$. Substituting $c = 0.32$ into equation (3) gives $a = 0.33$. Therefore, $a = 0.33$, $b = 0.27$, $c = 0.32$, and $d = 0.43$.

Thus, A and C are tumorous, B could be healthy or tumorous, and D is bone.

(c) The original system now has two additional equations.

$a + b = 0.60 \quad (1)$
$c + d = 0.75 \quad (2)$
$a + c = 0.65 \quad (3)$
$b + d = 0.70 \quad (4)$
$b + c = 0.85 \quad (5)$
$a + d = 0.50 \quad (6)$

The augmented matrix of this system is

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0.60 \\ 0 & 0 & 1 & 1 & 0.75 \\ 1 & 0 & 1 & 0 & 0.65 \\ 0 & 1 & 0 & 1 & 0.70 \\ 0 & 1 & 1 & 0 & 0.85 \\ 1 & 0 & 0 & 1 & 0.50 \end{array}\right].$$

Using the Gauss-Jordan method we obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0.20 \\ 0 & 1 & 0 & 0 & 0.40 \\ 0 & 0 & 1 & 0 & 0.45 \\ 0 & 0 & 0 & 1 & 0.30 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore, $a = 0.20$, $b = 0.40$, $c = 0.45$, and $d = 0.30$. Thus, A is healthy, B and C are bone, and D is tumorous.

(d) As we saw in part (c), the six equations reduced to four independent equations. We need only four beams, correctly chosen, to obtain a solution. The four beams must pass through all four cells and must lead to independent equations. One such choice would be beams 1, 2, 3, and 6. Another choice would be beams 1, 2, 4, and 5.

78. The matrix representing the rates per 1000 athlete-exposures for specific injuries that caused a player wearing either shield to miss one or more events is

$$\begin{bmatrix} 3.54 & 1.41 \\ 1.53 & 1.57 \\ 0.34 & 0.29 \\ 7.53 & 6.21 \end{bmatrix}.$$

Since an equal number of players wear each type of shield and the total number of athlete-exposures for the league in a season is 8000, each type of shield is worn by 4000 players. Since the rates are given per 1000 athletic-exposures, the matrix representing the number of 1000 athlete-exposures for each type of shield is

$$\begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

The product of these matrices is

$$\begin{bmatrix} 20 \\ 12 \\ 3 \\ 55 \end{bmatrix}.$$

Values have been rounded.

There would be about 20 head and face injuries, 12 concussions, 3 neck injuries, and 55 other injuries.

79. $\frac{\sqrt{3}}{2}(W_1 + W_2) = 100 \quad (1)$

$W_1 - W_2 = 0 \quad (2)$

Equation (2) gives $W_1 = W_2$. Substitute W_1 for W_2 in equation (1).

$$\frac{\sqrt{3}}{2}(W_1 + W_1) = 100$$
$$\frac{\sqrt{3}}{2}(2W_1) = 100$$
$$\sqrt{3}W_1 = 100$$
$$W_1 = \frac{100}{\sqrt{3}} = \frac{100\sqrt{3}}{3} \approx 58$$

Therefore, $W_1 = W_2 \approx 58$ lb.

80. $\frac{1}{2}W_1 + \frac{\sqrt{2}}{2}W_2 = 150 \quad (1)$

$\frac{\sqrt{3}}{2}W_1 - \frac{\sqrt{2}}{2}W_2 = 0 \quad (2)$

Adding equations (1) and (2) gives

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)W_1 = 150.$$

Multiply by 2.

$$(1 + \sqrt{3})W_1 = 300$$
$$W_1 = \frac{300}{1 + \sqrt{3}} \approx 110$$

From equation (2),

$$\frac{\sqrt{3}}{2}W_1 = \frac{\sqrt{2}}{2}W_2$$
$$W_2 = \frac{\sqrt{3}}{\sqrt{2}}W_1.$$

Substitute $\frac{300}{1+\sqrt{3}}$ from above for W_1.

$$W_2 = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{300}{1 + \sqrt{3}} = \frac{300\sqrt{3}}{(1 + \sqrt{3})\sqrt{2}} \approx 134$$

Therefore, $W_1 \approx 110$ lb and $W_2 \approx 134$ lb.

81. $C = at^2 + by + c$

Use the values for C from the table.

(a) Letting $t = 0$ in 1960, we get three equations for the coefficients a, b, and c.

$$c = 317$$
$$400a + 20b + c = 339$$
$$2500a + 50b + c = 390$$

Using the value for c in the first equation in the other equations we have the system

$$400a + 20b = 22$$
$$2500a + 50b = 73$$

Adding -5 times the first equation to twice the second equation gives

$$3000a = 146 - 110 = 36.$$

So $a = \frac{36}{3000} = 0.012$

$$b = \frac{22 - (400)(0.012)}{20} = 0.86$$

Thus, $C = 0.012t^2 + 0.86t + 317$

(b) The concentration will have doubled when $C = 634$.

Solve

$$0.012t^2 + 0.86t + 317 = 634$$
$$0.012t^2 + 0.86t - 317 = 0$$

The roots are given by the quadratic formula:

$$\frac{-0.86+\sqrt{(0.86)^2-(4)(0.012)(-317)}}{(2)(0.012)}=130.602$$

$$\frac{-0.86-\sqrt{(0.86)^2-(4)(0.012)(-317)}}{(2)(0.012)}=202.269$$

The positive root represents about 131 years after 1960, or 2091.

82. **(a)** $\begin{bmatrix}1\\1\end{bmatrix}x+\begin{bmatrix}0\\2\end{bmatrix}y=\begin{bmatrix}1\\2\end{bmatrix}$

$\begin{bmatrix}x\\x\end{bmatrix}+\begin{bmatrix}0\\2y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

$\begin{bmatrix}x\\x+2y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

Since corresponding elements must be equal, $x=1$ and $x+2y=2$. Substituting $x=1$ in the second equation gives $y=\frac{1}{2}$. Note that $x=1$ and $y=\frac{1}{2}$ are the values that balance the equation.

(b) $x\,CO_2+y\,H_2+z\,CO=H_2O$

$\begin{bmatrix}1\\0\\2\end{bmatrix}x+\begin{bmatrix}0\\2\\0\end{bmatrix}y+\begin{bmatrix}1\\0\\1\end{bmatrix}z=\begin{bmatrix}0\\2\\1\end{bmatrix}$

$\begin{bmatrix}x\\0\\2x\end{bmatrix}+\begin{bmatrix}0\\2y\\0\end{bmatrix}+\begin{bmatrix}z\\0\\z\end{bmatrix}=\begin{bmatrix}0\\2\\1\end{bmatrix}$

$\begin{bmatrix}x+z\\2y\\2x+z\end{bmatrix}=\begin{bmatrix}0\\2\\1\end{bmatrix}$

Since corresponding elements must be equal, $x+z=0$, $2y=2$, and $2x+z=1$. Solving $2y=2$ gives $y=1$. Solving the system

$$\begin{cases}x+z=0\\2x+z=1\end{cases}$$

gives $x=1$ and $z=-1$. Thus, the values that balance the equation are $x=1$, $y=1$, and $z=-1$.

83. Let $x=$ the number of boys and $y=$ the number of girls.

$$0.2x+0.3y=500 \quad (1)$$
$$0.6x+0.9y=1500 \quad (2)$$

The augmented matrix is

$$\left[\begin{array}{cc|c}0.2&0.3&500\\0.6&0.9&1500\end{array}\right].$$

$$5R_1\to R_1\quad\left[\begin{array}{cc|c}1&1.5&2500\\0.6&0.9&1500\end{array}\right]$$

$$-0.6R_1+R_2\to R_2\quad\left[\begin{array}{cc|c}1&1.5&2500\\0&0&0\end{array}\right]$$

Thus,

$$x+1.5y=2500$$
$$x=2500-1.5y.$$

There are y girls and $2500-1.5y$ boys, where y is any even integer between 0 and 1666 since $y\ge 0$ and

$$2500-1.5y\ge 0$$
$$-1.5y\ge -2500$$
$$y\le 1666.\overline{6}.$$

84. Let $x=$ the number of singles,
$y=$ the number of doubles,
$z=$ the number of triples, and
$w=$ the number of home runs hit by Ichiro Suzuki.

If the number of singles he hit was 11 more than four times the total of doubles and home runs, then $x=4(y+w)+11$, or $x-4y-4w=11$.

If the number of doubles he hit was 1 more than twice this total of triples and home runs, then $y=2(z+w)+1$, or $-2z+y-2w=1$.

If the total of singles and home runs he hit was 15 more than five times the total of doubles and triples, then $x+w=5(y+z)+15$, or $x-5y-5z+w=15$.

Write the augmented matrix and use row operations to solve the system.

$$\left[\begin{array}{cccc|c}1&1&1&1&225\\1&-4&0&-4&11\\0&1&-2&-2&1\\1&-5&-5&1&15\end{array}\right]$$

$$R_2\leftrightarrow R_3\left[\begin{array}{cccc|c}1&1&1&225&0\\0&1&-2&-2&1\\1&-4&1&-4&11\\1&-5&-5&1&15\end{array}\right]$$

$$\begin{array}{r} \\ \\ -R_1+R_3\to R_3 \\ -R_1+R_4\to R_4 \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 1 & 225 & 0 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & -5 & -1 & -5 & -214 \\ 0 & -6 & -6 & 0 & -210 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -R_3\to R_3 \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 225 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 5 & 1 & 5 & 214 \\ 0 & -6 & -6 & 0 & -210 \end{array}\right]$$

$$\begin{array}{r} -R_2+R_1\to R_1 \\ \\ -5R_2+R_3\to R_3 \\ 6R_2+R_4\to R_4 \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 3 & 3 & 224 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 11 & 15 & 209 \\ 0 & 0 & -18 & -12 & -204 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ R_3\leftrightarrow R_4 \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 3 & 3 & 224 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & -18 & -12 & -204 \\ 0 & 0 & 11 & 15 & 209 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -\frac{1}{18}R_3\to R_3 \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 3 & 3 & 224 \\ 0 & 1 & -2 & -2 & 1 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{34}{3} \\ 0 & 0 & 11 & 15 & 209 \end{array}\right]$$

$$\begin{array}{r} -3R_3+R_1\to R_1 \\ 2R_3+R_2\to R_2 \\ \\ -11R_3+R_4\to R_4 \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 190 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{71}{3} \\ 0 & 0 & 1 & \frac{2}{3} & \frac{34}{3} \\ 0 & 0 & 0 & \frac{23}{3} & \frac{253}{3} \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ \frac{3}{23}R_4\to R_4 \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 190 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{71}{3} \\ 0 & 0 & 1 & \frac{2}{3} & \frac{34}{3} \\ 0 & 0 & 0 & 1 & 11 \end{array}\right]$$

$$\begin{array}{r} -1R_4+R_1\to R_1 \\ \frac{2}{3}R_4+R_2\to R_2 \\ -\frac{2}{3}R_4+R_3\to R_3 \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 179 \\ 0 & 1 & 0 & 0 & 31 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 11 \end{array}\right]$$

Ichiro Suzuki hit 179 singles, 31 doubles, 4 triples, and 11 home runs.

85. Let x = the weight of a single chocolate wafer and
y = the weight of a single layer of vanilla creme.

A serving of regular Oreo cookies is three cookies so that $3(2x + y) = 34$.

A serving of Double Stuf is two cookies so that $2(2x + 2y) = 29$.

Write the equations in proper form, obtain the augmented matrix, and use row operations to solve.

$$\left[\begin{array}{cc|c} 6 & 3 & 34 \\ 4 & 4 & 29 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + 3R_2 \to R_2 \end{array}\left[\begin{array}{cc|c} 6 & 3 & 34 \\ 0 & 6 & 19 \end{array}\right]$$

$$\begin{array}{r} -1R_2 + 2R_1 \to R_1 \\ \\ \end{array}\left[\begin{array}{cc|c} 12 & 0 & 49 \\ 0 & 6 & 19 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{12}R_1 \to R_1 \\ \frac{1}{6}R_2 \to R_2 \end{array}\left[\begin{array}{cc|c} 1 & 0 & \frac{49}{12} \\ 0 & 1 & \frac{19}{6} \end{array}\right]$$

The solution is $\left(\frac{49}{12}, \frac{19}{6}\right)$, or about (4.08,3.17).

A chocolate wafer weighs 4.08 g and a single layer of vanilla creme weighs 3.17g.

Extended Application: Contagion

1. $$PQ = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 2 & 1 \end{bmatrix}$$

2. In the product PQ, $a_{23} = 0$, so there were no contacts.

3. In the product PQ, column 3 has all zeros. The third person had no contacts with the first group.

4. In the product PQ, the entries in columns 2 and 4 both have a sum of 4 while the entries in column 5 have a sum of 3. In Q, columns 2, 4, and 5 have a sum of 2, 2, and 3, respectively. Therefore, the second, fourth, and fifth persons in the third group each had a total of 6 first- and second-order contacts.