Chapter 4 – Continuity

Section 4.1

2. **(a)** No, f is not continuous at $x = 0$, thus not continuous at every point in the interval.

(b) Yes, f is right continuous at every point in the interval.

(c) Yes, f is continuous at every point in the interval.

3. **(a)** The function $x-1$ is continuous on $\Re$. The function $|x|$ is also continuous on $\Re$ by verifying this or using Exercise 5(a). Then, by Theorem 4.1.9, so is f.

(b) Since $x = 0$ is the only accumulation point of the domain D, f is automatically continuous at all $x = \pm\frac{1}{n}$, $n \in N$. Since $x = 0 \notin D$, f is not continuous at 0.

(c) If $x \neq 0$, then $f(x)$ is a quotient of 2 continuous functions and thus, continuous. Observe that f is continuous at $x = 0$ because $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1 = f(0)$. See Remark 3.2.11. Hence, f is continuous on $\Re$.

(d) If $x \neq 0$, then $f(x)$ is continuous by Theorem 4.1.9. At $x = 0$, f is continuous because $\lim\limits_{x\to 0} f(x) = 0 = f(0)$.

(e) Same as in part (d).

(f) By Remark 4.1.5, f is continuous on $\Re \setminus \{1\}$.

(g) By Remark 4.1.11, part (e), f is discontinuous at every integer.

(h) By Example 4.1.4, f is continuous on $\Re \setminus \{0\}$, and by Remark 4.1.11, part (e), f is discontinuous at $x = 0$.

(i) Since f is a quotient of 2 continuous functions, it is continuous on $(0, \infty)$.

(j) Note that every $a \in \Re$ is an accumulation point of the domain $D = \Re$ for the function f. To show that f is not continuous at any real $x = a$, use Remark 4.1.11, part (c), with x_n being rational numbers tending to a, and t_n being irrational numbers tending to a. Then, $\{f(x_n)\}$ converges to 1, and $\{f(t_n)\}$ converges to -1.

(k) If $x \neq 0$, then f is a composition of continuous functions and thus, f is continuous. Since $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\exp\left(-\frac{1}{x^2}\right) = \lim\limits_{t\to\infty}\exp\left(-t^2\right) = \lim\limits_{t\to\infty}\frac{1}{\exp\left(t^2\right)} = 0 = f(0)$, f is continuous at $x = 0$. Hence, f is continuous on $\Re$.

4. **(a)** The functions f and g are not continuous at $x = 0$ by using Remark 4.1.11, part (c), with $x_n = \frac{1}{n}$ and $t_n = \frac{\sqrt{2}}{n}$.

(b) Using Example 4.1.4, f is continuous at $x = \frac{1}{2}$ because $f(x) = 1 - x$ on $\left(\frac{1}{3}, 1\right) \setminus \left\{\frac{1}{2}\right\}$. Since $\lim\limits_{x\to\frac{1}{2}} g(x) = \frac{1}{2} = g\left(\frac{1}{2}\right)$, g is continuous at $x = \frac{1}{2}$.

5. **(a)** By part (b) of Corollary 1.8.6, $\lim\limits_{x\to a}|x| = |a|$. Therefore, $g(x) = |x| : \Re \to \Re^+$ is continuous, with

$f(D) \subseteq \Re$. Thus, by Theorem 4.1.9, $|f(x)|$ is continuous.

(b) Similar to part (a). If the definition is used, consider cases when $f(a)=0$ and when $f(a) \neq 0$.

(c) $\max\{f,g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$. Now apply Theorem 4.1.8 and part (a) of this exercise.

(d) Similar to part (c).

(e) Note that $\lim_{x \to a} x^n = a^n$, by Example 4.1.4. Thus, $[f(x)]^n$ is continuous functions by Theorem 4.1.9.

6. **(a)** $f(x) = \operatorname{sgn} x$ on $[-1,2]$ is bounded but not continuous at $x=0$.

(b) $f(x) = \frac{1}{x}$ on $(0,1)$ is continuous but not bounded.

(c) $f(x)=1$ for x rational, and $f(x)=-1$ for x irrational, is discontinuous at every point, but $[f(x)]^2 \equiv 1$ and thus, continuous.

(d) $f(x)=1$ for $x \geq 0$ and $f(x)=-1$ for $x<0$, and $g(x)=-f(x)$, are both discontinuous on $(-1,2)$, but $(fg)(x) \equiv -1$, which is continuous on $(-1,2)$.

(e) Suppose $f(x) = \operatorname{sgn} x$ on $(-1,2)$ and $g(x)=-f(x)$. Then, f and g are not continuous on $(-1,2)$ since they are both discontinuous at $x=0$ but, $(f+g)(x) \equiv 0$, therefore, continuous.

(f) Suppose $f(x) = \begin{cases} |x|, \text{ if } x \neq 0 \\ 1, \text{ if } x=0 \end{cases}$ and $g(x) = \begin{cases} 1, \text{ if } x \leq 0 \\ -1, \text{ if } x>0. \end{cases}$ Both, f and g are discontinuous on $(-1,2)$ but, $(f \circ g)(x) \equiv 1$, therefore, continuous.

(g) The statement is true because $g = (f+g) - f$, which is a difference of two continuous functions.

(h) Consider $f(x)=0$ on $(-1,2)$, and $g(x)=1$ for $0 \leq x < 2$, and $g(x)=-1$ for $-1<x<0$. Then, $(fg)(x) \equiv 0$, and therefore, continuous on $(-1,2)$.

(i) If $f(x)=1$ for $0 \leq x < 2$, and $f(x)=-1$ for $-1<x<0$, then $|f(x)| \equiv 1$, which is continuous on $(-1,2)$.

(j) The function $\frac{f(x)}{g(x)} = \sin x$ is continuous on $\Re \setminus \{0\}$. It is not defined at $x=0$ and thus not continuous on $\Re$.

(k) True, here is a proof. (Also, see Theorem 3.2.6, part (b).) Let $\varepsilon > 0$ be given. Since f is continuous at $x=c$, that is, $\lim_{x \to c} f(x) = f(c)$, there exists $\delta > 0$ such that $|f(x)-f(c)| < \boxed{\varepsilon}$ whenever $|x-c| < \delta$. Let $\{x_n\}$ be an arbitrary sequence converging to c with $x_n \in [a,b]$. Then there exists $n_1 \in N$ such that $|x_n - c| < \boxed{\delta}$ if $n \geq n_1$. To prove $\{f(x_n)\}$ converges to $f(c)$, we need to find $n^* \in N$ so that $|f(x_n) - f(c)| < \varepsilon$ if $n \geq n^*$. To this end, choose $n^* = n_1$. Then if $n \geq n^*$ we have $|x_n - c| < \delta$, which in turn gives $|f(x_n) - f(c)| < \varepsilon$. Hence, $\lim_{n \to \infty} f(x_n) = f(c)$.

Note that the continuity of f is essential. Without it the statement is false. See Exercise 31 in Section 3.4 for a counterexample.

(l) True, here is a proof. Suppose f is defined on $[a,b]$ and suppose that for any sequence $\{x_n\}$ in $[a,b]$ converging to $c \in [a,b]$, we have that $\{f(x_n)\}$ converges to $f(c)$. To prove f is continuous at $x=c$, we assume to the contrary. Thus, suppose $\lim_{x \to c} f(x) \neq f(c)$. Using Remark 4.1.11, part (b), there exists $\varepsilon > 0$ and a sequence $\{t_n\}$ in $[a,b]$ converging to c such that $|f(t_n) - f(c)| \geq \varepsilon$. Thus, there exists $\delta > 0$ such that $|f(t_n) - f(c)| \geq \varepsilon$ if $|t_n - c| < \delta$. Hence, $\lim_{n \to \infty} f(t_n) \neq f(c)$, which contradicts the hypothesis.

(m) Consider $f(x)=\frac{1}{x}$ on $D=(0,1]$, and $x_n=\frac{1}{n}$. Then, $\{x_n\}$ converges to 0 but, $f(x_n)=n$ and therefore, $\{f(x_n)\}$ diverges to $+\infty$.

(n) Consider $f(x)=\sqrt{x}$ and $a=0$. Then, f is continuous at $x=0$, $f(0^+)=0$, but $f(0^-)$ does not exist.

7. **(a)** We only need to show that $f(x)=c$ if x is irrational and $x\in(a,b)$. To this end, let x_0 be an arbitrary irrational value in (a,b), and let $\{r_n\}$ be a sequence of rational values in (a,b) which converges to x_0. This is possible because irrationals and rationals are dense in $\Re$, and thus, in (a,b). Since f is continuous, by Exercise 5(k), the sequence $\{f(r_n)\}$ converges to $f(x_0)$. But, $f(r_n)=c$ for all n. Therefore, the sequence $\{f(r_n)\}$ converges to c. Due to the uniqueness of the limit, $f(x_0)=c$. (Note that "rational r in (a,b)" can be changed to "any dense subset of (a,b).")

(b) No, f is not defined at irrational values of $(-2,3)$ and thus, not continuous at these values. However, f is continuous on S. Also, $\lim_{x\to a} f(x)=5$ for any $a\in(-2,3)$.

8. **(a)** Proof of part (a) of Theorem 4.1.7. Suppose that f is continuous at $x=a\in D$. Then there exists $\delta>0$ such that $|f(x)-f(a)|<\boxed{1}$ whenever $|x-a|<\delta$ and $x\in D$. Thus, if $x\in(a-\delta,a+\delta)\cap D$, by Exercise 14(a) of Section 1.8, we have, $-1<f(x)-f(a)<1$, which gives $f(a)-1<f(x)<f(a)+1$. Thus, for $x\in(a-\delta,a+\delta)\cap D$ we have $-|f(a)|-1<f(x)<|f(a)|+1$, or equivalently, $|f(x)|<|f(a)|+1$ and thus bounded "near $x=a$."

Proof of part (d) of Theorem 4.1.7. Since f is continuous at a, there exists $\delta>0$ such that $|f(x)-f(a)|<\boxed{\frac{f(a)}{2}}$ if $|x-a|<\delta$. Then, $-\frac{f(a)}{2}<f(x)-f(a)<\frac{f(a)}{2}$, from which the desired conclusion follows.

(b) Since f is continuous at c, there exists $\delta>0$ such that $|f(x)-f(c)|<\boxed{\frac{f(c)}{2}}$ if $|x-c|<\delta$. We will show that if $|x-c|<\delta$, then $f(x)>0$, by assuming to the contrary. Thus, suppose that there exists x^* such that $|x^*-c|<\delta$ and $f(x^*)\le 0$. But, $f(c)>0$, which gives $f(x^*)\le 0<f(c)$. Since the distance of $f(c)$ to 0 is less than the distance of $f(c)$ to $f(x^*)$, we write $|f(x^*)-f(c)|>|f(c)-0|=f(c)$. But, this contradicts our hypothesis. Hence, if f is continuous and positive at $x=c$, then it is positive on a small neighborhood of c.

Continuity is essential in this problem. If f is not continuous, the statement is false. Choose, for example, $f(x)=1$ for x rational and $f(x)=-1$ for x irrational with $c=2$. Certainly, $f(2)=1>0$ but, every neighborhood of 2 contains irrational values at which the functional values are negative.

9. Define $h=f-g$, which is continuous by Theorem 4.1.8, part (a). Also, $h(c)=f(c)-g(c)>0$. Thus, by part (e) of Theorem 4.1.7, there exists $\delta>0$ such that for $x\in(a,b)$ with $|x-c|<\delta$, we have $h(x)>0$. Hence, $f(x)>g(x)$ for such values.

10. **(a)** $f(x)=1$ for x rational, and $f(x)=-1$ for x irrational

(c) $f:[-2,2]\to\Re$, defined by $f(x)=\begin{cases} x^2-1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational}\end{cases}$

11. **(a)** *Case 1.* Suppose that $b=1$. Then, $f(x)\equiv 1$, and the conclusion follows.

Case 2. Suppose $b>1$ and $a=0$. We will prove that $\lim_{x\to 0} b^x=1$ using Theorem 3.2.6, part (b).

To start, recall Exercise 14 in Section 2.1 where we proved that $\lim_{n\to\infty} b^{\frac{1}{n}} = 1 = b^0$. Now, let $\{x_n\}$ be any sequence that converges to 0 and let $\varepsilon > 0$ be given. We need to find $n^* \in N$ such that $|f(x_n) - L| = |b^{x_n} - 1| < \varepsilon$ for all $n \ge n^*$. Since $\lim_{n\to\infty} b^{\frac{1}{n}} = 1$, there exists $n_1 \in N$ such that $b^{\frac{1}{n_1}} - 1 < \varepsilon$. Now choose $n^* > n_1$. Then, if $n \ge n^*$ we have $|f(x_n) - L| = |b^{x_n} - 1| = b^{x_n} - 1 < b^{\frac{1}{n_1}} - 1 < \varepsilon$, if $x_n > 0$; $|f(x_n) - L| = |b^{x_n} - 1| = 1 - b^{x_n} < b^{-x_n} - 1 < b^{\frac{1}{n_1}} - 1 < \varepsilon$, if $x_n < 0$; and $|f(x_n) - L| = |b^{x_n} - 1| = 0 < \varepsilon$, if $x_n = 0$. Therefore, $\{f(x_n)\}$ converges to 1, and by Theorem 3.2.6, $\lim_{x\to 0} b^x = 1 = b^0$.

Case 3. Suppose $b > 1$ and $a \ne 0$. Suppose $\{x_n\}$ is an arbitrary sequence that converges to a. Then, by Exercise 6 of Section 2.1, $\lim_{n\to\infty}(x_n - a) = 0$. Thus, by case 2, $\{f(x_n - a)\}$ converges to 1. But, $f(x_n - a) = b^{x_n - a}$. Therefore, $\lim_{n\to\infty} b^{x_n - a} = 1$. This gives $\lim_{n\to\infty} b^{x_n} b^{-a} = 1$, thus $b^{-a} \lim_{n\to\infty} b^{x_n} = 1$, by Corollary 2.2.4. This yields $\lim_{n\to\infty} b^{x_n} = b^a$. Hence, Theorem 3.2.6, part (b), yields the desired result.

Case 4. Suppose $0 < b < 1$. Then, $\frac{1}{b} > 1$, and by cases 2 and 3 we have $\lim_{x\to a}\left(\frac{1}{b}\right)^x = \left(\frac{1}{b}\right)^a$. But then, employing Theorem 3.2.5 we have $\lim_{x\to a} b^x = b^a$.

We use the just proven result and Theorem 4.1.9 to obtain $\lim_{x\to 2} \frac{3^{4x-1}}{5^{3x+1}} = \frac{3^{4(2)-1}}{5^{3(2)+1}} = \frac{3^7}{5^7} = \left(\frac{3}{5}\right)^7$.

(b) If $g(x) = \ln x$, then $h(x) \equiv xg(x)$ is continuous by Remark 4.1.5 and Theorem 4.1.8, part (b). If $k(x) = e^x$, then k is continuous by Remark 4.1.5, and $f(x) = (k \circ h)(x)$ is continuous by Theorem 4.1.9. Recall that $\exp(x \ln x) = \exp\left(\ln x^x\right) = x^x$ for $x > 0$.

12. Let a be any real number. We will prove that $f(x) = \sin x$ is continuous at $x = a$ by showing that $\lim_{x\to a} \sin x = \sin a$. To this end, using Exercise 9 of Section 3.2 and Exercise 22 of Section 3.3, we write,

$\lim_{x\to a} \sin x = \lim_{h\to 0} \sin(a+h) = \lim_{h\to 0}(\sin a \cos h + \cos a \sin h) = \left(\lim_{h\to 0} \sin a\right)\left(\lim_{h\to 0} \cos h\right) + \left(\lim_{h\to 0} \cos a\right)\left(\lim_{h\to 0} \sin h\right) = \sin a \cdot 1 + \cos a \cdot 0 = \sin a$. Thus, $f(x) = \sin x$ is continuous on $\Re$.

Of course there are many proofs of the given statement. Another argument would be this. By Exercise 51 in Section 1.9 and the fact that $\sin x$ is an odd function, we have that $|\sin x| \le |x|$ for all x real. Also, we know that $|\cos x| \le 1$ and $\sin x - \sin t = 2\sin\frac{x-t}{2}\cos\frac{x+t}{2}$ for all $x, t \in \Re$. Thus, for any a real we have that $|\sin x - \sin a| = 2\left|\sin\frac{x-a}{2}\right|\left|\cos\frac{x+a}{2}\right| \le 2 \cdot \left|\frac{x-a}{2}\right| \cdot 1 = |x - a|$. From here it follows that $\sin x$ is continuous on $\Re$.

To prove that $\cos x$ is continuous on $\Re$ we can proceed in a similar way to the above argument using the fact that $\cos x - \cos t = -2\sin\frac{x+t}{2}\sin\frac{x-t}{2}$ for all $x, t \in \Re$. Thus, for any a real we have that $|\cos x - \cos a| = 2\left|\sin\frac{x+a}{2}\right|\left|\sin\frac{x-a}{2}\right| \le 2 \cdot 1 \cdot \left|\frac{x-a}{2}\right| = |x - a|$. From here it follows that $\cos x$ is continuous on $\Re$.

13. **(a)** True, here is a proof. By Remark 4.1.5, $\ln x$ is a continuous function. Thus, by Exercise 6(k) since $\{a_n\}$ converges to A, the sequence $\{c_n\}$, with $c_n = \ln a_n$ converges to $\ln A$. By Exercise 36 in Section 2.7, that is, Theorem 2.8.6, the sequence $\{d_n\}$, with $d_n = \frac{1}{n}(\ln a_1 + \ln a_2 + \cdots + \ln a_n)$, converges to

$\ln A$. But now, since $b_n = \sqrt[n]{a_1 \cdot a_2 \cdot \cdots \cdot a_n}$, we have $\ln b_n = \frac{1}{n}(\ln a_1 + \ln a_2 + \cdots + \ln a_n)$. Taking limits as n goes to ∞, we obtain $\lim_{n\to\infty} \ln b_n = \ln A = \lim_{n\to\infty} d_n$. Since $\ln x$ is a continuous function, we can write $\ln\left(\lim_{n\to\infty} b_n\right) = \ln A$. Hence, $\lim_{n\to\infty} b_n = A$, which is what we wanted to prove.

(b) False. Suppose that $a_n = \exp\left[(-1)^n\right]$. The sequence $\{a_n\}$ oscillates and thus, it diverges. But, $b_n = 1$ for n even, and $b_n = \sqrt[n]{e^{-1}}$ for n odd. Thus, $\{b_n\}$ converges to 1.

14. Note that, $\ln x - \ln(x+1) = \ln\frac{x}{x+1}$. Also, by Remark 4.1.5, $\ln x$ is continuous. Therefore, by Exercise 6(k), we have that, $\lim_{x\to\infty}[\ln x - \ln(x+1)] = \lim_{x\to\infty} \ln\frac{x}{x+1} = \ln \lim_{x\to\infty}\frac{x}{x+1} = \ln 1 = 0$. Hence, by Theorem 3.1.6, $\{a_n\}$ converges to 0.

15. We write $\lim_{x\to\infty} \arctan\frac{x}{x+1} = \arctan\left(\lim_{x\to\infty}\frac{x}{x+1}\right) = \arctan 1 = \frac{\pi}{4}$, where the first equality holds due to the continuity of the arctangent function. Now use Theorem 3.1.6 to conclude that $\lim_{n\to\infty} \arctan\frac{n}{n+1} = \frac{\pi}{4}$.

16. Observe that $a_n > 0$ for all n.

(a) Suppose that $b = 1$. Then, $a_n = 1 = a_{n+1}$. Thus $\{a_n\}$ is constant and hence increasing. Next, suppose $b > 1$. We use mathematical induction to prove that the statement $P(n)$, which stands for $a_{n+1} > a_n$, is true for all n. Note that $P(1)$ is true because for $b > 1$ we have $a_2 = b^{a_1} = b^b > b = a_1$. Next, suppose $P(k)$ is true for some $k \in N$, that is, $a_{k+1} > a_k$, and show that $P(k+1)$ is true, that is, $a_{k+2} > a_{k+1}$, meaning $b^{a_{k+1}} > b^{a_k}$, or equivalently, $b^{\frac{a_{k+1}}{a_k}} > 1$. To this end, since $a_{k+1} > a_k$, we have $\frac{a_{k+1}}{a_k} > 1$. Thus, $b^{\frac{a_{k+1}}{a_k}} > b^1 > 1$. Hence, $P(n)$ is true for all n, and so the sequence $\{a_n\}$ is increasing.

(b) We will use mathematical induction to prove that $a_n \le 3$ for all n. Certainly, $a_1 = b \le 3$. Next, suppose $a_k \le 3$ for some $k \in N$. We will show that $a_{k+1} \le 3$. To this end, we write $a_{k+1} = b^{a_k} < b^3 < \left(\sqrt[3]{3}\right)^3 =$ 3. Hence, $a_n \le 3$ for all n.

(c) By Theorem 2.4.4, part (a), the sequence $\{a_n\}$ converges to, say, A. Taking limits of the recursion formula and keeping in mind that $f(x) = b^x$ is a continuous function, we obtain $A = b^A$.

(d) In this case $b = \sqrt{2} < \sqrt[3]{3}$. Therefore, the repeated power is the limit of the sequence $\{a_n\}$ as given in parts (a)–(c). Thus, the sequence converges to A, which satisfies $A = \left(\sqrt{2}\right)^A$. This gives $A^2 = 2^A$, and thus, $A = 2$ or $A = 4$. But, in part (b) we verified that $A \le 3$. Hence, $A = 2$.

Section 4.2

1. **(a)** Since $f(0^+) = 1$ and $f(0^-) = -1$, the discontinuity at $x = 0$ is not removable, it is jump.

(b) f is discontinuous at $x = 0$ because $f(0)$ is undefined. However, $\lim_{x\to 0} f(x) = 1$, a finite number. Therefore, $x = 0$ is a point of removable discontinuity. If we define $f(0) = 1$, f will be continuous at $x = 0$.

(c) f is discontinuous at $x=0$ because $f(0)$ is undefined. However, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin\left(\frac{\sin x}{x}\cdot x\right)}{x}$ $= \lim_{x\to 0} \frac{\sin x}{x} = 1$. Therefore, $x=0$ is a point of removable discontinuity. If we define $f(0)=1$, f will be continuous at $x=0$.

(d) Similar to part (c).

(e) Observe that $f(0)=0$, $f(0^+) = \lim_{x\to 0^+}(0-1) = -1$, and $f(0^-) = \lim_{x\to 0^-}(-1-0) = -1$. Thus, since $\lim_{x\to 0} f(x) = -1 \neq f(0)$, f has a removable discontinuity at $x=0$. If we redefine f at $x=0$ to be -1, the resulting function will be continuous at $x=0$.

(f) Since f is a composition of continuous functions, by Theorem 4.1.9, f is continuous. Therefore, no discontinuity at $x=0$.

(g) Since $f(0)=0$ and $\lim_{x\to 0} f(x) = \lim_{x\to 0}(1-x) = 1$, the discontinuity of f at $x=0$ is removable. Redefining $f(0)$ to be 1 will make f continuous at $x=0$.

(h) Observe that $f(0)=1$. Also, $\lim_{x\to 0} f(x)$ does not exist because if $x_n = \frac{1}{n}$ and $t_n = \frac{2}{2n-1}$, or $\frac{\sqrt{2}}{n}$, or, ... , the sequences $\{x_n\}$ and $\{t_n\}$ converge to 0, but $\{f(x_n)\}$ and $\{f(t_n)\}$ converge to 0 and 1, respectively. See Theorem 3.2.6. Therefore, the discontinuity at $x=0$ is not removable. In fact, it is oscillating.

(i) f is discontinuous at $x=0$ because $f(0)$ is undefined. By Example 3.2.8, $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ do not exist. Therefore, discontinuity is not removable. In fact, it is oscillating.

(j) f is discontinuous at $x=0$ because $f(0)$ is undefined. By Exercise 5 from Section 3.2, $\lim_{x\to 0} f(x) = 0$. Therefore, in order to make f continuous at $x=0$, define $f(0)$ to be 0.

(k) f is discontinuous at $x=0$ because $f(0)$ is undefined. However, $\lim_{x\to 0} f(x) = 0$. Therefore, in order to make f continuous at $x=0$, define $f(0)$ to be 0.

(l) f has infinite discontinuity at $x=0$ since $f(0^+) = +\infty$, and $f(0^-) = 0$.

2. (a) The function $g(t) = \lfloor t \rfloor$ is continuous whenever t is not an integer. Thus, to determine the points of discontinuity of f consider the situation when $2x$, is an integer. This occurs when $x = -1, -\frac{1}{2}, 0, \frac{1}{2}$, and 1. When $a = -\frac{1}{2}$, 0, and $\frac{1}{2}$, we have $f(a^-) = 1$ and $f(a^+) = 0$. Therefore, by Definition 4.2.5, part (a), f has nonremovable jump discontinuities at $a = -\frac{1}{2}, 0$, and $\frac{1}{2}$ with jump values of -1. If $x = a = -1$, then $f(-1^-)$ does not exist. However, $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(2x+2) = 0$. And since $f(-1) = 0$ as well, f is continuous at $x=-1$. If $x = a = 1$, then $f(1^+)$ does not exist. But, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x-1) = 1$. And since $f(1) = 0$, part (b) of Definition 4.2.5 is satisfied resulting in a jump discontinuity at $x=1$ with a jump of -1.

(b) The only accumulation point of the domain D of f is 0. Therefore, a discontinuity can occur only at 0. In fact, since $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0^-} f(x) = -1$, $x=0$ is a point of a jump discontinuity with the jump of 2.

(c) $x = -\frac{3}{2}$ is an accumulation point of f but, $-\frac{3}{2} \notin D$, the domain of f. Therefore, f is discontinuous at $x=-\frac{3}{2}$. This discontinuity is removable because f can be continuously extended by simply defining $f\left(-\frac{3}{2}\right)=-\frac{7}{4}$. The function f has jump discontinuities at $x=-1$ and $x=0$, with jump of $\frac{1}{2}$ at each one of them. In addition, since $f(1)=1$ and $\lim_{x\to 1} f(x)=\lim_{x\to 1^-} f(x)=\frac{1}{2}$, f has a jump of $\frac{1}{2}$ at $x=1$ as well. $x=1$ is also a point of removable discontinuity.

(d) If $n=3,5,\ldots$, $\lim_{x\to n^-} f(x)=1$ and $\lim_{x\to n^+} f(x)=-1$. Thus, f has jump discontinuities with a jump of -2. If $n=2,4,6,\ldots$, $\lim_{x\to n^-} f(x)=-1$ and $\lim_{x\to n^+} f(x)=1$. So, f has jump discontinuities with a jump of 2.

(e) Note that $f\left(\frac{p}{q}\right)=\frac{1}{q}$, but $\lim_{x\to a} f(x)=0$ for every $a\in(0,1)$, in particular for a rational $a=\frac{p}{q}$. Thus, f is discontinuous at all rational values and the discontinuities are removable, and there are countably many of them.

(f) Since for $a=\pm\frac{1}{n}$, $n\in N$, we have $\lim_{x\to a} f(x)=1\neq 0=f(a)$, f has removable discontinuities at these points. Next consider $x=a=0$. Since if $x_n=\frac{1}{n}$ and $t_n=\frac{2}{2n-1}$ or $t_n=\frac{\sqrt{2}}{n}$, $n\in N$, sequences $\{x_n\}$ and $\{t_n\}$ both converge to 0 with $\{f(x_n)\}$ and $\{f(t_n)\}$ converging to 0 and 1, respectively. Therefore, $\lim_{x\to 0} f(x)$ does not exist. Hence, f has an oscillating discontinuity at $x=0$. Also, f has jump discontinuities at the endpoints.

(g) If $x_n=\frac{1}{n}$, then $\{x_n\}$ converges to 0 but $\{f(x_n)\}$ diverges to $+\infty$. If $t_n=\frac{2}{2n-1}$, then $\{t_n\}$ converges to 0 and $\{f(t_n)\}$ converges to 0. Therefore, $\lim_{x\to 0^+} f(x)$ does not exist. Similarly, $\lim_{x\to 0^-} f(x)$ does not exist. Thus, f has oscillating discontinuity at $x=0$. In addition, f has a removable discontinuity at $x=\pm\frac{1}{n}$, for any $n\in N$.

(h) If $a<0$, then f is discontinuous at $x=a$ because $\lim_{x\to a^\pm} f(x)$ do not exist, making it an oscillating discontinuity. Since $\lim_{x\to 0^-} f(x)$ does not exist and $\lim_{x\to 0^+} f(x)=+\infty$, the discontinuity at $x=0$ is also oscillating.

(i) $x=0$ is the only possibility for a discontinuity of f. But, $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\exp\left(-\frac{1}{x}\right)=0=\lim_{x\to 0^-} f(x)$, and $f(0)=0$, f is continuous at $x=0$.

(j) $x=0$ is the only possibility for a discontinuity of f. Since, $\lim_{x\to 0^+} f(x)=1$ and $\lim_{x\to 0^-} f(x)=0$, we have a jump discontinuity at $x=0$ with the jump of 1.

(k) $x=0$ is an accumulation point of the interval $(0,\infty)$, the domain of f, and $\lim_{x\to 0^+} f(x)=+\infty$. Since f is undefined for $x<0$, the discontinuity is infinite.

(l) f has an oscillating discontinuity at $x=0$ since $x=0$ is an accumulation point of the interval $(-\infty,0)$ and $\lim_{x\to 0^-} f(x)$ does not exist.

(m) Same as part (l).

(n) The only possibilities for discontinuity are $x=0$ and $x=\pm\frac{1}{n}$, $n\in N$. Consider $x=0$. We can use definition of a limit to verify that $\lim_{x\to 0} f(x)=0=0^2=f(0)$. Thus, f is continuous at $x=0$. Consider

$x=1$. Since $\lim_{x\to 1} f(x)=f(1)$, f is continuous at $x=1$ as well. If $x=a=\pm\frac{1}{n}$ with $n\in N$ and $x\neq 1$, then $\lim_{x\to a} f(x)=\frac{1}{n^2}\neq\pm\frac{1}{n}=f(a)$. Therefore discontinuities at $x=a$ are removable.

(o) The discontinuity at $x=2$ is infinite.

(p) The only possibilities for discontinuity of f are at $x=1$ and $x=-1$. Consider $x=-1$. Since $\lim_{x\to -1^-} f(x)=1$ and $\lim_{x\to -1^+} f(x)=0$, f has a jump discontinuity at $x=-1$ with the jump of -1. Consider $x=1$. Since $\lim_{x\to 1^-} f(x)=0$ and $\lim_{x\to 1^+} f(x)=1$, f has a jump discontinuity at $x=1$ with the jump of 1.

Section 4.3

1. (a) Suppose f is continuous on a closed interval which is not bounded. Say, $f(x)=x^2$ on $[0,\infty)$. This function is not bounded.

Suppose f is continuous on a bounded interval which is not closed. Say, $f(x)=\frac{1}{x}$ on $(0,1)$. This function is not bounded.

Suppose the interval is closed and bounded but f is not continuous. Say, $f:[0,1]\to\Re$ is defined by $f(x)=n$, if $x=\frac{1}{n}$, $n\in N$, and $f(x)=0$, otherwise. This function is not bounded.

(b) Suppose f is continuous on a closed interval which is not bounded. Say, $f(x)=1$ on $[1,\infty)$. This function is bounded.

Suppose f is continuous on a bounded interval which is not closed. Say, $f(x)=1$ on $(-1,3]$. This function is bounded.

Suppose the interval is closed and bounded but f is not continuous. Say, $f:[0,1]\to\Re$ is defined by $f(x)=1$, if x is rational, and $f(x)=-1$, if x is irrational. This function is bounded.

(c) Say, $f:[0,1]\to\Re$ is defined by $f(x)=x$, if $x\in[0,1)$, and $f(1)=0$. Clearly, f cannot be continuous.

2. (a) By Theorem 4.3.4, $f([a,b])$ is bounded. Next, we will prove that $f([a,b])$ is closed. We assume that $\{y_n\}$ is an arbitrary sequence in $f([a,b])$ which converges to some point z_0, and we will prove that $z_0\in f([a,b])$. To this end, let $\{x_n\}$ be the sequence in $[a,b]$ for which $y_n=f(x_n)$. Now, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to $x_0\in[a,b]$, since $[a,b]$ is closed. Therefore, $\{f(x_{n_k})\}$ converges to $f(x_0)$, since f is continuous. But, $\{f(x_{n_k})\}$ also converges to z_0. Hence, $z_0=f(x_0)\in f([a,b])$.

(b) Since $f([a,b])$ is bounded, $\sup f(x)=M$ is finite. Let $\{y_n\}$ be a sequence in $f([a,b])$ converging to M. Since $f([a,b])$ is closed, $M\in f([a,b])$. Hence, $M=\max f(x)$. Minimum is handled similarly.

3. (a) $f(x)=x$ if $0\le x\le 1$, and $f(x)=x-1$ if $1<x\le 2$.

4. $f:\Re\to\Re$ defined by $f(x)=x^2+1$ has no real roots.

5. Note that $p(0)=a_0<0$. Since by Theorem 3.1.13, part (c), $\lim_{x\to\infty} p(x)=+\infty$, there exists $b>0$ such that $p(b)>0$. Since p is continuous (see Example 4.1.4), by Theorem 4.3.6, there exists $c\in(0,b)$ such that $p(c)=0$. The same way we can prove that there exists at least one negative root. Observe that this result is not true if n is odd. Choose $f(x)=x^3-1$ for all $x\in\Re$. Here $a_0=-1<0$, but f has exactly one real root $x=1$.

6. Suppose a is a given positive real number. We want to show it has a unique positive nth root. Consider $f(x) = x^n - a$ for $x \geq 0$. Note that $f(0) = -a < 0$. Since $\lim_{x \to \infty} f(x) = +\infty$, there exists $b > 0$ such that $f(b) > 0$. By Theorem 4.3.6, there exists $c \in (0, b)$ such that $f(c) = 0$. Thus, $c^n = a$ and c is a positive nth root of a. Next we verify that f is strictly increasing. According to Definition 1.2.10, suppose $0 < x_1 < x_2$. By Theorem 1.8.4, part (b), and the mathematical induction, we have that $f(x_1) < f(x_2)$ and thus, strictly increasing. Therefore, positive nth root of a is unique.

7. Suppose f is not constant on $[a,b]$. Then there exists $x_1, x_2 \in [a,b]$ such that $f(x_1) < f(x_2)$. But then there exists an irrational number r such that $f(x_1) < r < f(x_2)$. By Theorem 4.3.6 there exists $c \in (x_1, x_2)$ $\subseteq [a,b]$ such that $f(c) = r$. This is a contradiction because images of f were assumed to be rational.

8. Suppose $f : D \to \Re$ is continuous, one to one, but not monotone. Then since D is an interval, there exists $a, b, c \in D$ such that $a < b < c$ and, either $f(a) < f(b)$ and $f(b) > f(c)$, or $f(a) > f(b)$ and $f(b) < f(c)$. Without loss of generality, we suppose the first possibility holds. Next, we distinguish further 2 cases: $f(a) < f(c)$ and $f(a) > f(c)$. Note that $f(a) = f(c)$ is not a possibility since f is one to one. We again consider the first case and leave 3 remaining situations to the reader. Thus, $f(a) < f(c) < f(b)$. Now, let $k \in (f(c), f(b))$. Then, $k \in (f(a), f(b))$ and by Theorem 4.3.6, there exists $c_1 \in (a,b)$ such that $f(c_1) = k$. But, $k \in (f(c), f(b))$ and so by Theorem 4.3.6, there exists $c_2 \in (b,c)$ such that $f(c_2) = k$. Since $c_1 \neq c_2$, we have a contradiction to the fact that f is one to one on D.

9. **(a)** If $f(x) = x$ if $0 \leq x < 1$, $f(1) = 0$, and $f(x) = x - 2$ if $1 < x \leq 2$, then the range of f is $(-1, 1)$.

   (b) If $f(x) = \dfrac{1}{x}$ if $0 < x < 1$, and $f(x) = 2$ if $x = 0$ and 1, then the range of f is $(1, \infty)$.

   (c) If $f(x) = \dfrac{1}{x}$ if $0 < x \leq 1$, and $f(0) = 1$, then the range of f is $[1, \infty)$.

10. The continuity is a stronger assumption because f continuous implies that f possesses the intermediate value property, and Exercise 3(a) shows the converse is not true.

11. Suppose $f : [a,b] \to \Re$ is continuous and $f(x) \neq 0$ for any $x \in [a,b]$. We first prove that either $f(x) > 0$ for all $x \in [a,b]$ or $f(x) < 0$ for all $x \in [a,b]$. We argue this by contradiction. Suppose there exists c_1 and c_2 in $[a,b]$ such that $c_1 \neq c_2$ but $f(c_1) < 0 < f(c_2)$. Then, by Theorem 4.3.6, there exists x_0 between c_1 and c_2 such that $f(x_0) = 0$. Contradiction to hypothesis. Without loss of generality, we suppose $f(x) > 0$ for all $x \in [a,b]$. We need to prove that there exists $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ for all $x \in [a,b]$. To this end, observe that by Corollary 4.3.9, the range of f is an interval $[c,d]$. But by the above, $0 \notin [c,d]$. Thus, pick $\varepsilon = c$.

12. **(a)** Since for any x we have $e^x e^{-x} = e^{x-x} = e^0 = 1$, we know that $e^x \neq 0$ for any real x. Also, we observe that $e^x > 0$ for all x, because if it were ever negative, then, since $e^0 = 1$, by Theorem 4.3.6, it would have to vanish somewhere, which we showed is not possible. Recall that, by Problem 2.8.1, part (e), we have $2 \leq e < 3$, and by Problem 2.8.2, part (c) we have $e^x > 1$ for any $x > 0$. Next we let any $s, t \in \Re$ with $s < t$, and we show $e^s < e^t$. Since $s < t$, we have $t - s > 0$, which gives us $e^{t-s} > 1$. Thus, since $e^s > 0$, we can write $e^t = e^{t-s} e^s > 1 \cdot e^s = e^s$. Hence, $f(x) = e^x$ is strictly increasing for all x real.

    (b) Since f is strictly increasing and by Example 2.3.4, $\lim_{n \to \infty} f(n) = +\infty$, the conclusion follows.

    (c) $\lim_{x \to -\infty} e^x = \lim_{x \to -\infty} \dfrac{1}{e^{-x}} = \lim_{t \to \infty} \dfrac{1}{e^t} = 0$.

13. If $m = \min\{f(x_1), f(x_2)\}$ and $M = \max\{f(x_1), f(x_2)\}$, then $m \leq f(x_1) \leq M$ and $m \leq f(x_2) \leq M$. This gives

$k_1 m \le k_1 f(x_1) \le k_1 M$ and $k_2 m \le k_2 f(x_2) \le k_2 M$. Adding these together we obtain $m(k_1 + k_2) \le k_1 f(x_1) + k_2 f(x_2) \le M(k_1 + k_2)$. This gives us that $m \le \frac{k_1 f(x_1) + k_2 f(x_2)}{k_1 + k_2} \le M$. The existence of the desired c follows from Theorem 4.3.6.

14. **(a)** Note that $x^2 + x - 2 \le 0 \Leftrightarrow (x+2)(x-1) \le 0$. Since $f(x) \equiv (x+2)(x-1) = 0$ if $x = -2$ and 1, the intervals to be tested are $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$. Since $f(-3) > 0$, none of the values in $(-\infty, -2)$ satisfy the inequality. Since $f(0) < 0$, every value in $(-2,1)$ satisfies the inequality. Finally, since $f(2) > 0$, none of the values in $(1, \infty)$ satisfy the inequality.

(f) Note that $x^3 - 2x + 1 = (x-1)(x^2 + x - 1)$.

(g) First solve $f(x) \equiv x^3 + 2x^2 - 5x - 6 = 0$. Note that $f(-1) = 0$. Therefore, $x + 1$ divides $f(x)$. Thus, $f(x) = (x+1)(x^2 + x - 6) = (x+1)(x+3)(x-2)$. So, $f(x) = 0$ if $x = -3, -1$, and 2. Therefore, intervals to be tested are $(-\infty, -3)$, $(-3, -1)$, $(-1, 2)$, and $(2, \infty)$. Since $f(-4) < 0$, every point in $(-\infty, -3)$ satisfies the given inequality. Since $f(-2) > 0$, none of the points in $(-3, -1)$ satisfy the given inequality. Since $f(0) < 0$, every point in $(-1, 2)$ satisfies the inequality. Lastly, since $f(3) > 0$, none of the points in $(2, \infty)$ satisfy the inequality. In addition, $x = -3, -1$, and 2 do not satisfy $f(x) < 0$. Hence, the desired inequality is satisfied if $x \in (-\infty, -3) \cup (-1, 2)$.

(h) Do not multiply by $x - 5$ unless you decide on its sign. We resort to writing $\frac{2x+1}{x-5} - 3 \le 0$, which is equivalent to $\frac{x-16}{x-5} \ge 0$. Since $x - 16 = 0$ when $x = 16$, and $x - 5 = 0$ when $x = 5$, the intervals to be tested are $(-\infty, 5)$, $(5, 16)$, and $(16, \infty)$. Since $x = 0$ satisfies the above inequality, all the points in $(-\infty, 5)$ will satisfy it. Since $x = 10$ does not satisfy the above inequality, none of the points in $(5,16)$ will. Finally, since 20 satisfies the inequality, then all the points in $(16, \infty)$ will. Also, note that $x = 16$ satisfies the inequality but $x = 5$ does not. Hence, the desired inequality is satisfied if $x \in (-\infty, 5) \cup [16, \infty)$.

(i) Consider two inequalities separately.

15. **(a)** Define $f(x) = 2^x - 3x$. Since $f(0) = 1 > 0$ and $f(1) = -1 < 0$, and f is continuous on $[0,1]$, by Theorem 4.3.6, there exists $c \in (0,1)$ such that $f(c) = 0$. Therefore, $2^c = 3c$.

(b) Define $g(x) = \frac{1}{3}(2^x)$. Since $g : [0,1] \to [0,1]$ and g is continuous, by Theorem 4.3.10, there exists $c \in [0,1]$ such that $g(c) = c$, which yields $\frac{1}{3}(2^c) = c$, and thus, $2^c = 3c$. Note however that $c \ne 0$ and $c \ne 1$. Thus, $c \in (0,1)$.

16. Even if g is continuous with a fixed point, the sequence $\{x_n\}$ might not converge. For example, pick $g(x) = x^2$ and $x_0 = 2$. Then, g is continuous with fixed points $x = 0$, 1 but $\lim_{n\to\infty} x_n = +\infty$. Note that $x_0 = \frac{1}{2}$ would create a sequence that converges to the fixed point $x = 0$. The given argument in this exercise proves that the sequence converges to a fixed point only if we start with a converging sequence $\{x_n\}$.

17. If $D = (0,1) \cup (2,3]$, consider the function $f : D \to \Re$ defined by $f(x) = x$ if $x \in (0,1)$, and $f(x) = 4 - x$ if $x \in (2,3]$. Note that f is a continuous injection on D. However, $f^{-1} : (0,2) \to \Re$ is not continuous, since it has a discontinuity at $x = 1$.

19. **(a)** Use Exercise 18(c). The given function is the inverse of $\tan x$ on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

(b) Use Exercise 18(c). The given function is the inverses of e^x on $\Re$.

20. ($\Rightarrow$) Suppose E is closed and $a \in \Re \setminus E$. We will show there exists a neighborhood I of a such that I is entirely contained in $\Re \setminus E$. To this end, since E contains all accumulation points of E, a is not an accumulation point of E. Thus, there exists a neighborhood I of a that contains no points of E. Therefore, $I \subseteq \Re \setminus E$. Hence, by Definition 4.3.2, $\Re \setminus E$ is open.

($\Leftarrow$) Suppose $\Re \setminus E$ is open and a is an accumulation point of E. We need to show that $a \in E$. To this end, suppose $a \in \Re \setminus E$. But, $\Re \setminus E$ is open. Thus, if $a \in \Re \setminus E$, there exists a neighborhood I of a such that $I \subset \Re \setminus E$ or $I \cap E = \phi$. But this contradicts the fact that a is an accumulation point of E. Hence, $a \notin \Re \setminus E$, which implies $a \in E$ and thus, E is closed. Theorem 4.3.3 proves that $\Re$ and ϕ are both open and closed.

Section 4.4

1. **(a)** The function f is not uniformly continuous because if $\varepsilon = \frac{1}{2}$, $x_n = \frac{1}{n}$, and $t_n = \frac{1}{n+1}$, then $|x_n - t_n| = \frac{1}{n(n+1)} \le \frac{1}{n}$, but $|f(x_n) - f(t_n)| = 1 \ge \frac{1}{2}$. Now use Remark 4.4.4.

(b) We will prove that f is uniformly continuous. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x) - f(t)| < \varepsilon$ for all $x,t \in [0,2)$ satisfying $|x-t| < \delta$. To this end, if $x,t \in [0,2)$, then $|f(x)-f(t)| = |x^3 - t^3| = |x-t||x^2+xt+t^2| \le |x-t|(4+4+4) = 12|x-t| < \varepsilon$, provided $|x-t| < \frac{\varepsilon}{12}$. Thus, we choose $\delta = \frac{\varepsilon}{12}$.

(c) We will prove that f is uniformly continuous. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x) - f(t)| < \varepsilon$ for all $x,t \in [0,2)$ satisfying $|x-t| < \delta$. To this end, if $x,t \in [0,2)$, then $|f(x)-f(t)| = \left|\frac{x}{x+4} - \frac{t}{t+4}\right| = \frac{4|x-t|}{(x+4)(t+4)} \le \frac{4|x-t|}{(0+4)(0+4)} = \frac{1}{4}|x-t| < \varepsilon$, provided $|x-t| < 4\varepsilon$. Thus, we choose $\delta = 4\varepsilon$.

(d) We will prove that f is uniformly continuous. Let $\varepsilon > 0$ be given. Pick $\delta = \varepsilon^3$ and consider the 2 possibilities. *Case 1*. Suppose that both $x,t \in \Re$ satisfy $|x|,|t| \in [0,\delta)$. Then, $\sqrt[3]{|x|},\sqrt[3]{|t|} \in \left[0,\sqrt[3]{\delta}\right) = [0,\varepsilon)$, and thus, $|x-t| < \delta$ implies $|f(x)-f(t)| = \left|\sqrt[3]{x} - \sqrt[3]{t}\right| \le \left|\sqrt[3]{x}\right| < \varepsilon$.

Case 2. Suppose either $|x|$ and/or $|t|$ are/is greater than or equal to δ. Then we have, $\left|x^{\frac{2}{3}} + x^{\frac{1}{3}}t^{\frac{1}{3}} + t^{\frac{2}{3}}\right| \ge \delta^{\frac{2}{3}} = \varepsilon^2$, and thus, $|x-t| < \delta$ implies that $|f(x)-f(t)| =$

$$\left|\sqrt[3]{x} - \sqrt[3]{t}\right| \cdot \frac{\left|x^{\frac{2}{3}} + x^{\frac{1}{3}}t^{\frac{1}{3}} + t^{\frac{2}{3}}\right|}{\left|x^{\frac{2}{3}} + x^{\frac{1}{3}}t^{\frac{1}{3}} + t^{\frac{2}{3}}\right|} = \frac{|x-t|}{\left|x^{\frac{2}{3}} + x^{\frac{1}{3}}t^{\frac{1}{3}} + t^{\frac{2}{3}}\right|} \le \frac{|x-t|}{\varepsilon^2} < \frac{\delta}{\varepsilon^2} = \frac{\varepsilon^3}{\varepsilon^2} = \varepsilon.$$

Hence, we found δ which depends only on ε so that whenever $|x-t| < \delta$, we have $|f(x)-f(t)| < \varepsilon$.

(e) We will prove that f is not uniformly continuous. Choose $\varepsilon = \frac{1}{2}$, $x_n = \frac{1}{2n}$, and $t_n = \frac{1}{2n+\frac{1}{2}} = \frac{2}{4n+1}$. Then, $|x_n - t_n| \le \frac{1}{n}$ but, $|f(x_n) - f(t_n)| = |0-1| = 1 \ge \frac{1}{2}$.

(f) We will prove that f is uniformly continuous. Let $\varepsilon > 0$ be given and pick $\delta = \frac{\varepsilon}{2}$. Then, if $x,t \geq 1$ and $|x-t| < \delta$, we have $|f(x) - f(t)| = \left|\frac{1}{x^2} - \frac{1}{t^2}\right| = |x-t|\left(\frac{1}{xt^2} + \frac{1}{x^2 t}\right) \leq 2|x-t| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$.

2. (a) False statement because c may be a or b. For example, choose $f(x) = \frac{1}{x}$ with $x \in (0,2)$. Then the sequence $\{x_n\}$ defined by $x_n = \frac{1}{n}$ is in $(0,2)$, but the sequence $\{f(x_n)\}$ diverges to $+\infty$. Observe that if we pick any sequence converging to a value $c \in (a,b)$, then by Exercise 5(k) from Section 4.1, the sequence $\{f(x_n)\}$ converges to $f(c)$.

(b) True, because a uniformly continuous function cannot run away too fast. The interval (a, b) can be changed to any bounded domain D. To prove, let $\varepsilon > 0$ be given. Since f is uniformly continuous on (a,b), there exists $\delta > 0$ such that $|f(x) - f(t)| < \boxed{\varepsilon}$ if $|x-t| < \delta$ and $x,t \in (a,b)$. Furthermore, since $\{x_n\}$ converges, it is Cauchy. Therefore, there exists $n_1 \in N$ such that $|x_n - x_m| < \boxed{\delta}$ for $m,n \geq n_1$. We need to find $n^* \in N$ such that $|f(x_n) - f(x_m)| < \varepsilon$, whenever $m,n \geq n^*$. Thus, choose $n^* = n_1$. Then, if $m,n \geq n^*$, we have $|x_n - x_m| < \delta$, which in turn gives $|f(x_n) - f(x_m)| < \varepsilon$. Therefore, $\{f(x_n)\}$ is Cauchy (and hence, convergent.)

(c) False. Choose $f(x) = \sin\frac{1}{x}$, with $x \in (0,1)$.

(d) True. This is the contrapositive of Theorem 4.4.7.

(e) False. If $f(x) = x$ for all $x \in \mathfrak{R}$, then f is uniformly continuous on $\mathfrak{R}$ but not bounded.

(f) True. To prove this recall that uniform continuity implies continuity. Thus, f is continuous on (a,b) and moreover, by Theorem 4.4.7, $f(a^+)$ and $f(b^-)$ are both finite. Therefore, f can be continuously extended to the function $g : [a,b] \rightarrow \mathfrak{R}$ by defining it as $g(x) = f(x)$ if $x \in (a,b)$, $g(a) = f(a^+)$, and $g(b) = f(b^-)$. Thus, by Theorem 4.3.4, g is bounded. Hence, f is bounded on (a,b).

(g) False. Choose any bounded discontinuous function, say, $f(x) = \sin\frac{1}{x}$, with $x \in (0,1)$. By Exercise 1(e), f is not uniformly continuous.

(h) True. Proof. Since f is uniformly continuous on $[a,b]$, then f is continuous on $[a,b]$ and $\lim_{x \to b^-} f(x) = f(b)$. Since f is uniformly continuous on $[b,c]$, then f is continuous on $[b,c]$ and $\lim_{x \to b^+} f(x) = f(b)$. Therefore, $\lim_{x \to b} f(x) = f(b)$, and so f is continuous on $[a,c]$. Since $[a,c]$ is compact, by Theorem 4.4.6, f is uniformly continuous on $[a,c]$.

(i) False. Let $f : (0,3) \rightarrow \mathfrak{R}$ be defined by $f(x) = -1$ if $x \in (0,1)$, and $f(x) = 2$ if $x \in [1,3)$.

3. (a) Here is a proof of Theorem 4.4.7. We assume $f : (a,b) \rightarrow \mathfrak{R}$ is uniformly continuous and prove $f(a^+)$ is finite. Proof that $f(b^-)$ is finite is similar. So, let $\varepsilon > 0$ be given. We prove that $f(a^+) \equiv \lim_{x \to a^+} f(x)$ is finite by the use of Theorem 3.2.6. To this end, f uniformly continuous on (a,b) implies that there exists $\delta > 0$ such that $|f(x) - f(t)| < \boxed{\frac{\varepsilon}{2}}$ if $|x-t| < \delta$ and $x,t \in (a,b)$. Now, pick an arbitrary sequence $\{x_n\}$ in (a,b) that converges to a. Then, there exists $n_1 \in N$ such that $n \geq n_1$ we have $a < x_n < a + \delta$. By the uniform continuity of f we have $|f(x) - f(x_n)| < \frac{\varepsilon}{2}$, if $n \geq n_1$. Also, by Exercise 2(b), $\{f(x_n)\}$ converges to, say, L. This means that there exists $n_2 \in N$ such that

$\left|f(x_n)-L\right|<\boxed{\frac{\varepsilon}{2}}$ if $n \ge n_2$. Now, choose $n^* = \max\{n_1, n_2\}$. Then, if $n \ge n^*$ we have $|f(x)-L| \le$ $|f(x)-f(x_n)|+|f(x_n)-L| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. By Theorem 3.2.6, $f(a^+)$ is finite.

(b) Suppose f is continuous on (a,b). Define $g:[a,b]\to\mathfrak{R}$ by $g(a)=f(a^+)$, $g(x)=f(x)$ if $a<x<b$, and $g(b)=f(b^-)$. Since g is continuous $[a,b]$, by Theorem 4.3.4 it follows that g is bounded on $[a,b]$. Thus, f is bounded on (a,b). Converse is false because $f(x)=\sin\frac{1}{x}$, with $x\in(0,1)$, is bounded and continuous, but $f(0^+)$ does not exist.

(c) ($\Rightarrow$) f uniformly continuous on (a,b) implies $f(a^+)$ and $f(b^-)$ are both finite, by Theorem 4.4.7. Thus, the function g from part (b) is a continuous extension of f.

($\Leftarrow$) Let g be the continuous extension of f to $[a,b]$. Then, by Theorem 4.4.6, g is uniformly continuous on $[a,b]$, and thus, f is uniformly continuous on (a,b).

(d) If $f(x)=\sin\frac{1}{x}$, with $x\in(0,1)$, then $f(0^+)$ does not exist. Therefore, f has no continuous extension to $[0,1]$, and therefore, by Corollary 4.4.8, not uniformly continuous on $(0,1)$.

If $g(x)=x\sin\frac{1}{x}$, with $x\in(0,1)$, then since $g(0^+)=0$ and $g(1^-)=\sin 1$, g has a continuous extension to $[0,1]$. Therefore, by Corollary 4.4.8, g is uniformly continuous on $(0,1)$.

4. Since f is periodic on $\mathfrak{R}$, there exists $p>0$ such that $f(x+p)=f(x)$ for all $x\in\mathfrak{R}$. By Theorem 4.4.6, f is uniformly continuous on $[-p,p]$. Therefore, given any $\varepsilon>0$, there exists $\delta>0$, in particular $\delta<\frac{p}{2}$, such that $|f(x)-f(t)|<\varepsilon$ provided $|x-t|<\delta$ and $x,t\in[-p,p]$, Now, suppose $|x-t|<\delta$, where x and t are any real values. Note that there exists an integer n such that $x+np\in\left[-\frac{p}{2},\frac{p}{2}\right]$. Then, $t+np\in[-p,p]$ since $|x-t|<\delta<\frac{p}{2}$. Thus we can write $|f(x)-f(t)|=|f(x+np)-f(t+np)|<\varepsilon$. Hence, f is uniformly continuous.

5. **(a)** Let $\varepsilon>0$ be given. To show $f+g$ is uniformly continuous on D, we need to find $\delta>0$ so that $|(f+g)(x)-(f+g)(t)|<\varepsilon$, for all $x,t\in D$ satisfying $|x-t|<\delta$. To this end, observe that f uniformly continuous implies that there exists $\delta_1>0$ such that $|f(x)-f(t)|<\boxed{\frac{\varepsilon}{2}}$ whenever $|x-t|<\delta_1$ and $x,t\in D$. Also, f uniformly continuous implies that there exists $\delta_2>0$ such that $|g(x)-g(t)|<\boxed{\frac{\varepsilon}{2}}$ whenever $|x-t|<\delta_2$ and $x,t\in D$. Thus, choose $\delta=\min\{\delta_1,\delta_2\}$. Then for $x,t\in D$ with $|x-t|<\delta$, we have $|(f+g)(x)-(f+g)(t)|\le|f(x)-f(t)|+|g(x)-g(t)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. Hence, $f+g$ is uniformly continuous on D.

(b) Let $\varepsilon>0$ be given. Since f is uniformly continuous, there exists $\delta_1>0$ such that $|f(x)-f(t)|<\boxed{\frac{\varepsilon}{|c|+1}}$ whenever $|x-t|<\delta_1$ and $x,t\in D$. (We added 1 in the denominator to avoid division by 0 in case $c=0$.) Now, choose $\delta=\delta_1$. Then for $x,t\in D$ with $|x-t|<\delta$, we have $|(cf)(x)-(cf)(t)|=$ $|c||f(x)-f(t)|<|c|\cdot\frac{\varepsilon}{|c|+1}<\varepsilon$. Hence, cf is uniformly continuous on D.

(c) Proof by contradiction. Suppose f is unbounded. Then there exists a sequence $\{x_n\}$ in D such that

$|f(x_n)| \geq n$ for each $n \in N$. But, by Theorem 2.6.4, there exists $\{x_{n_k}\}$ in D that converges and thus, by Exercise 2(b), with (a, b) replaced by D, the sequence $\{f(x_{n_k})\}$ converges. Contradiction to the hypothesis. Hence, f must be finite.

(d) Let $\varepsilon > 0$ be given. To show fg is uniformly continuous on D we need to find $\delta > 0$ so that $|(fg)(x) - (fg)(t)| < \varepsilon$ for all $x,t \in D$ satisfying $|x-t| < \delta$. To this end, let $K > 0$ be a bound on f and g. Also, f and g uniformly continuous on D implies that there exist δ_1 and δ_2 such that $|f(x) - f(t)| < \boxed{\frac{\varepsilon}{2K}}$ if $|x-t| < \delta_1$, and $|g(x) - g(t)| < \boxed{\frac{\varepsilon}{2K}}$ if $|x-t| < \delta_2$, provided $x,t \in D$. Thus, choose $\delta = \min\{\delta_1, \delta_2\}$. Then for $x,t \in D$ with $|x-t| < \delta$, we have $|(fg)(x) - (fg)(t)| \leq$ $|f(x)||g(x) - g(t)| + |g(t)||f(x) - f(t)| \leq K|g(x) - g(t)| + K|f(x) - f(t)| < K \cdot \frac{\varepsilon}{2K} + K \cdot \frac{\varepsilon}{2K} = \varepsilon$.

(e) By part (c), f and g are bounded on D. The result follows from part (d).

(f) By Theorem 4.1.8, part (c), $\frac{f}{g}$ is continuous on D. Since D is closed and bounded, the result follows from Theorem 4.4.6.

6. (b) $f(x) = \sin(x^2)$. Note that oscillation of f speeds up.

(c) $f(x) = -x$.

7. (a) The function f is Lipschitz because we can write $|f(x) - f(t)| = |x-t||x+t| \leq |x-t|(|x| + |t|) \leq |x-t|(2+2) = 4|x-t|$. Thus, a Lipschitz constant is 4.

(b) The function f is not Lipschitz because its domain is unbounded. To see this, observe that by Example 4.4.5, f is not uniformly continuous. Therefore, using the contrapositive of the statement in Theorem 4.4.11, f is not Lipschitz on $\mathfrak{R}$. In fact, no polynomial of degree greater than 1 is Lipschitz on $\mathfrak{R}$.

(c) The function f is not Lipschitz if the domain includes a neighborhood of $x = 0$. We will prove f is not Lipschitz by contradiction. Thus, suppose there exists a Lipschitz constant $L > 0$. Then, $|f(x) - f(0)| \leq$ $L|x - 0|$, that is, $\sqrt[3]{x} \leq Lx$ for all $x \in \mathfrak{R}$. So, let $x_n = \frac{1}{n^3}$. Then, $\frac{1}{n} \leq \frac{L}{n^3}$, which is equivalent to $n^2 \leq L$. However, since $\lim_{n\to\infty} n^2 = +\infty$, L cannot be a finite number. Contradiction.

(d) Since f is not uniformly continuous, by the contrapositive of the statement in Theorem 4.4.11, f is not Lipschitz.

(e) The function f is not Lipschitz. By contradiction, suppose there exists a Lipschitz constant $L > 0$. Then pick sequences $\{x_n\}$ and $\{t_n\}$, where $x_n = \frac{1}{n\pi}$ and $t_n = \frac{2}{(2n+1)\pi}$. Now, in Definition 4.4.10, let $x = x_n$ and $t = t_n$. Then, for all $n \in N$ we have $|f(x_n) - f(t_n)| \leq L|x_n - t_n|$. But, $|f(x_n) - f(t_n)| =$ $\frac{2}{(2n+1)\pi}$. Therefore, $\frac{2}{(2n+1)\pi} \leq L\left|\frac{1}{n\pi} - \frac{2}{(2n+1)\pi}\right|$ for all n. Hence, $2 \leq \frac{L}{n}$ for all n, which is a contradiction.

(f) The function f is not Lipschitz. By contradiction, suppose there exists a Lipschitz constant $L > 0$. Then pick sequences $\{x_n\}$ and $\{t_n\}$, where $x_n = \frac{1}{n\pi}$ and $t_n = \frac{2}{(2n+1)\pi}$. Now, in Definition 4.4.10, let $x = x_n$ and $t = t_n$. Then, for all $n \in N$ we have $|f(x_n) - f(t_n)| \leq L|x_n - t_n|$. But, $|f(x_n) - f(t_n)| =$ $\sqrt{\frac{2}{(2n+1)\pi}}$. Thus, $\sqrt{\frac{2}{(2n+1)\pi}} \leq L\left|\frac{1}{n\pi} - \frac{2}{(2n+1)\pi}\right| = \frac{L}{n(2n+1)\pi}$ for all n. Hence, $\sqrt{\frac{2}{\pi}} \leq \frac{L}{n\pi}\frac{\sqrt{2n+1}}{2n+1}$, which tends to 0. Hence, a contradiction.

8. Several examples of functions that are uniformly continuous but not Lipschitz are:
$f:\Re\to\Re$ defined by $f(x)=\sqrt[3]{x}$. See Exercise 1(d) and Exercise 7(e).
$f:[0,\infty)\to\Re$ defined by $f(x)=\sqrt{x}$.
$f:(0,1]\to\Re$ defined by $f(x)=\sqrt{x}\sin\frac{1}{x}$. See Exercise 7(e). The uniform continuity follows from Corollary 4.4.8.
$f:(0,1]\to\Re$ defined by $f(x)=x\sin\frac{1}{x}$.

9. Suppose $f:[a,b]\to\Re$ is contractive, that is, f is Lipschitz with the Lipschitz constant $L\in(0,1)$. Therefore, by Theorem 4.4.11, f is uniformly continuous on $[a,b]$, and thus, continuous. Also, $f:[a,b]\to[a,b]$, and thus, by Theorem 4.3.10, f has a fixed point. This proves the existence.

To prove uniqueness, we suppose that x_1 and x_2 are 2 fixed points of f and show they must be equal. Since f is contractive, we have $|f(x_1)-f(x_2)|\le L|x_1-x_2|$, $L\in(0,1)$. Since x_1 and x_2 are fixed points, that is, $f(x_1)=x_1$ and $f(x_2)=x_2$, the previous inequality gives $|x_1-x_2|\le L|x_1-x_2|$. Since $L<1$, we must have $|x_1-x_2|=0$. Therefore, $x_1=x_2$, and thus uniqueness is established.

10. **(a)** Choose $f(x)=x+1$, with $x\in\Re$. Note that $|f(x)-f(t)|=|x-t|$ for all $x,t\in\Re$. Therefore, $L=1$ but f has no fixed points.

(b) Choose $f(x)=x$, with $x\in\Re$. Note that $|f(x)-f(t)|=|x-t|$ for all $x,t\in\Re$. Therefore, $L=1$ and every real number is a fixed point of f.

(c) Choose $f(x)=2x$, with $x\in\Re$. Note that $|f(x)-f(t)|=2|x-t|$ for all $x,t\in\Re$. Therefore, $L=2$ and f has only one fixed point, namely $x=0$.

11. **(a)** Define $g(x)=|x-f(x)|$ and show g attains its minimum value that is 0. Since f is Lipschitz, f is continuous. Therefore, g is continuous on a closed and bounded interval. By the extreme value theorem, Theorem 4.3.5, g attains its minimum value m. Thus, there exists $p\in[a,b]$ such that $g(p)=m$. We prove $m=0$ by assuming to the contrary. So, since $g(x)\ge 0$, we assume that $m>0$ and look for contradiction. Since $|f(x)-f(t)|<|x-t|$ for any $x,t\in[a,b]$ and $f:[a,b]\to[a,b]$, we choose $x=p$ and $t=f(p)$. Then we have $g(f(p))=|f(p)-f(f(p))|<|p-f(p)|=g(p)=m$. But, $g(p)$ is a minimum value of g, thus $g(f(p))\ge g(p)$. Contradiction. Hence, $g(p)=0$, which implies that $f(p)=p$. Thus, at least one fixed point exists.

(b) Suppose x_1 and x_2 are 2 fixed points of f. Then, $f(x_1)=x_1$ and $f(x_2)=x_2$ and $x_1\ne x_2$. Therefore, $|f(x_1)-f(x_2)|=|x_1-x_2|$. But, $|f(x)-f(t)|<|x-t|$ for all $x,t\in D$ with $x\ne t$. Thus, if $x=x_1$ and $t=x_2$ we have $|x_1-x_2|<|x_1-x_2|$, which is not possible. Hence, there exists at most 1 fixed point.

(c) Suppose $f:(0,1)\to(0,1)$, where $f(x)=\frac{x}{2}$. Note that $|f(x)-f(t)|=\frac{1}{2}|x-t|<|x-t|$. But, f has no fixed points, that is, f does not cross the line $y=x$ when $x\in(0,1)$. Observe that $D=(0,1)$ cannot be both closed and bounded, for otherwise, Theorem 4.3.7 will guarantee a fixed point.

(d) Consider $f:\left[0,\frac{\pi}{2}\right]\to\left[0,\frac{\pi}{2}\right]$, where $f(x)=\sin x$. Note that by Exercise 51 of Section 1.9 and by Exercise 4 in this section, $|f(x)-f(t)|\le|x-t|$ for all $x\in\left[0,\frac{\pi}{2}\right]$, but there is no constant $L\in(0,1)$ such that $|f(x)-f(t)|\le L|x-t|$.

12. **(a)** The condition follows since f is increasing, $f(x) > 0$ for all $x \in \left(0, \frac{1}{3}\right]$, and $f\left(\frac{1}{3}\right) = \frac{1}{9} < \frac{1}{3}$.

(b) Since $|f(x) - f(t)| = |x^2 - t^2| = |x + t||x - t| \le \left(\frac{1}{3} + \frac{1}{3}\right)|x - t| = \frac{2}{3}|x - t|$, f is a contraction.

(c) The function f has no fixed points because $f(x) = x \Rightarrow x^2 = x \Rightarrow x = 0,1$, and neither of them is in $\left(0, \frac{1}{3}\right]$.

13. We will show that for all $x,t \ge 1$ we have $|f(x) - f(t)| \le L|x - t|$ with $L \in (0,1)$. To this end, since for all $x,t \ge 1$, we have that, $xt \ge 1$, and thus, $\frac{2}{xt} \le 2$, we can write that, $|f(x) - f(t)| = \left|\left(\frac{x}{2} + \frac{1}{x}\right) - \left(\frac{t}{2} + \frac{1}{t}\right)\right| = \frac{1}{2}\left|(x - t) - \frac{2}{xt}(x - t)\right| = \frac{1}{2}|x - t|\left|\frac{2}{xt} - 1\right| \le \frac{1}{2}|x - t||2 - 1| = \frac{1}{2}|x - t|$. Therefore, f is a contraction.

14. **(a)** False. Consider $f(x) = \cot x$, if $x \ne n\pi$, n an integer, and $f(x) = 0$ otherwise.

(b) False. There exist infinitely many discontinuities.

(c) False. A constant function has no smallest period.

(d) False. It does if $f(x) = c$, c a constant.

15. **(a)** $f(x) = f(x + p) = f[(x + p) + p] = f(x + 2p)$, and so on. Use a proof by induction.

(b) The fundamental period for $\sin nx$ is $\frac{2\pi}{n}$. By part (a), $n \cdot \frac{2\pi}{n}$ is a period for $\sin nx$ as well. Similarly for $\cos nx$.

Section 4.5

1. T	**9.** T	**17.** T	**25.** T	**33.** F	**41.** T	**49.** F
2. T	**10.** F	**18.** T	**26.** F	**34.** F	**42.** T	**50.** T
3. F	**11.** F	**19.** T	**27.** T	**35.** F	**43.** T	
4. F	**12.** F	**20.** T	**28.** F	**36.** T	**44.** F	
5. T	**13.** F	**21.** F	**29.** F	**37.** F	**45.** T	
6. T	**14.** T	**22.** F	**30.** F	**38.** F	**46.** T	
7. T	**15.** F	**23.** T	**31.** T	**39.** F	**47.** T	
8. F	**16.** F	**24.** F	**32.** T	**40.** T	**48.** F	