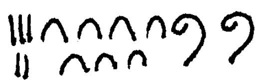
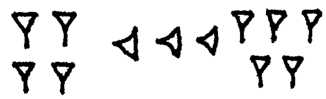
SOLUTIONS TO PROBLEMS

CHAPTER ONE

1. We write 275 as follows in Egyptian hieroglyphics (on the left) and Babylonian cuneiform (on the right):





2.

1	5	
$'10$	50	(multiply by 10)
2	10	(double first line)
4	20	(double third line)
$'8$	40	(double fourth line)
$'\overline{2}$	$2\ \overline{2}$	(halve first line)
$\underline{'\overline{10}}$	$\underline{\overline{2}}$	(invert third line)
$18\ \overline{2}\ \overline{10}$	93	

3.

1	$7\ \overline{2}\ \overline{4}\ \overline{8}$
2	$15\ \overline{2}\ \overline{4}$
$'4$	$31\ \overline{2}$
$'8$	63
$\underline{'\overline{\overline{3}}}$	$\underline{4\ \overline{\overline{3}}\ \overline{3}\ \overline{6}\ \overline{12}}$
$12\ \overline{\overline{3}}$	$98\ \overline{2}\ \overline{\overline{3}}\ \overline{3}\ \overline{6}\ \overline{12}$ $99\ \overline{2}\ \overline{4}$

4.

$2 \div 11$	1	11	$2 \div 23$	1	23
	$\overline{\overline{3}}$	$7\ \overline{3}$		$\overline{\overline{3}}$	$15\ \overline{3}$
	$\overline{3}$	$3\ \overline{\overline{3}}$		$\overline{3}$	$7\ \overline{\overline{3}}$
	$\overline{6}$	$1\ \overline{\overline{3}}\ \overline{6}'$		$\overline{6}$	$3\ \overline{2}\ \overline{3}$
	$\overline{66}$	$\overline{6}'$		$\overline{12}$	$1\ \overline{2}\ \overline{4}\ \overline{6}'$
				$\overline{276}$	$\overline{12}'$
	$\overline{6}\ \overline{\overline{66}}$	2			
				$\overline{12}\ \overline{276}$	2

5.

$$5 \div 13 = (2 \div 13) + (3 \div 13) = \overline{8}\ \overline{52}\ \overline{104} + \overline{8}\ \overline{13}\ \overline{52}\ \overline{104} = \overline{4}\ \overline{13}\ \overline{26}\ \overline{52}$$

$$6 \div 13 = 2(3 \div 13) = \overline{4}\ \overline{8}\ \overline{52}\ \overline{104}\ \overline{26}\ \overline{52} = \overline{4}\ \overline{8}\ \overline{13}\ \overline{104}$$

$$8 \div 13 = 2(4 \div 13) = \overline{2}\ \overline{13}\ \overline{26}$$

6. $x + \frac{1}{7}x = 19$. Choose $x = 7$; then $7 + \frac{1}{7} \cdot 7 = 8$. Since $19 \div 8 = 2\frac{3}{8}$, the correct answer is $2\frac{3}{8} \times 7 = 16\frac{5}{8}$.

7. $(x + \frac{2}{3}x) - \frac{1}{3}(x + \frac{2}{3}x) = 10$. In this case, the "obvious" choice for x is $x = 9$. Then 9 added to 2/3 of itself is 15, while 1/3 of 15 is 5. When you subtract 5 from 15, you get 10. So in this case our "guess" is correct.

8. The equation here is $(1 + \frac{1}{3} + \frac{1}{4})x = 2$. Therefore. we can find the solution by dividing 2 by $1 + \frac{1}{3} + \frac{1}{4}$. We set up that problem:

1	$1\ \overline{2}\ \overline{4}$
$\overline{\overline{3}}$	$1\ \overline{18}$
$\overline{3}$	$\overline{2}\ \overline{36}$
$\overline{6}$	$\overline{4}\ \overline{72}$
$\overline{12}$	$\overline{8}\ \overline{144}$

The sum of the numbers in the right hand column beneath the initial line is $1\frac{141}{144}$. So we need to find multipliers giving us $\frac{3}{144} = \overline{144}\ \overline{72}$. But $1\ \overline{3}\ \overline{4}$ times 144 is 228. It follows that multiplying $1\ \overline{3}\ \overline{4}$ by $\overline{228}$ gives $\overline{144}$ and multiplying by $\overline{114}$ gives $\overline{72}$. Thus, the answer is $1\ \overline{6}\ \overline{12}\ \overline{114}\ \overline{228}$.

9. Since x must satisfy $100 : 10 = x : 45$, we would get that $x = \frac{45\times100}{10}$; the scribe breaks this up into a sum of two parts, $\frac{35\times100}{10}$ and $\frac{10\times100}{10}$.

10. The ratio of the cross section area of a log of 5 handbreadths in diameter to one of 4 handbreadths diameter is $5^2 : 4^2 = 25 : 16 = 1\frac{9}{16}$. Thus, 100 logs of 5 handbreadths diameter are equivalent to $1\frac{9}{16} \times 100 = 156\frac{1}{4}$ logs of 4 handbreadths diameter.

12.

$$
\begin{array}{rrrrrrr}
 & & 8 & 34 & 17 & 8 & \cdots \\
7) & 1 & 00 & 00 & 00 & 00 & 00 \\
 & & \underline{56} & & & & \\
 & & 4 & 00 & & & \\
 & & \underline{3} & \underline{58} & & & \\
 & & & 2 & 00 & & \\
 & & & \underline{1} & \underline{59} & & \\
 & & & & 1 & 00 & \\
 & & & & & \underline{56} & \\
 & & & & & 4 & \cdots
\end{array}
$$

13. Since $3 \times 18 = 54$, which is 6 less than 60, it follows that the reciprocal of 18 is $3\frac{1}{3}$, or, putting this in sexagesimal notation, 3,20. Since 60 is $(1\frac{7}{8}) \times 32$, and $\frac{7}{8}$ can be expressed as 52,30, the reciprocal of 32 is 1,52,30. Since $60 = 1\frac{1}{9} \times 54$, and $\frac{1}{9}$ can be expressed as $\frac{1}{10} + \frac{1}{90} = \frac{6}{60} + \frac{40}{3600} = 0;06,40$, the reciprocal of 54 is $1,06,40$. Also, because $60 = \frac{15}{16} \times 64$, the reciprocal of 64 is $\frac{15}{16}$. Since $\frac{1}{16} = 3,45$, we get that $\frac{15}{16} = 56,15$. If the only prime divisors of n are 2, 3, 5, then n is a regular sexagesimal.

14. $25 \times 1,04 = 1,40 + 25,00 = 26,40$. $18 \times 1,21 = 6,18 + 18,00 = 24,18$. $50 \div 18 = 50 \times 0;3,20 = 2;30 + 0;16,40 = 2;46,40$. $1,21 \div 32 = 1,21 \times 0;01,52,30 = 1;21 + 1;10,12 + 0;00,40,30 = 2;31,52,30$.

15. Since the length of the circumference C is given by $C = 4a$, and because $C = 6r$, it follows that $r = \frac{2}{3}a$. The length T of the long transversal is then $T = r\sqrt{2} = (\frac{2}{3}a)(\frac{17}{12}) = \frac{17}{18}a$. The length t of the short transversal is $t = 2(r - \frac{t}{2}) = 2a(\frac{2}{3} - \frac{17}{36}) = \frac{7}{18}a$. The area A of the barge is twice the difference between the area of a quarter circle and the area of the right triangle formed by the long transversal and two perpendicular radii drawn from the two ends of that line. Thus

$$A = 2\left(\frac{C^2}{48} - \frac{r^2}{2}\right) = 2\left(\frac{a^2}{3} - \frac{2a^2}{9}\right) = \frac{2}{9}a^2.$$

16. Since the length of the circumference C is given by $C = 3a$, and because $C = 6r$, it follows that $r = \frac{a}{2}$. The length T of the long transversal is then $T = r\sqrt{3} = (\frac{a}{2})(\frac{7}{4}) = \frac{7}{8}a$. The length t of the short transversal is twice the distance from the midpoint of the arc to

the center of the long transversal. If we set up our circle so that it is centered on the origin, the midpoint of the arc has coordinates $(\frac{r}{2}, \frac{\sqrt{3}r}{2})$ while the midpoint of the long transversal has coordinates $(\frac{r}{4}, \frac{\sqrt{3}r}{4})$. Thus the length of half of the short transversal is $\frac{r}{2}$ and then $t = r = \frac{a}{2}$. The area A of the bull's eye is twice the difference between the area of a third of a circle and the area of the triangle formed by the long transversal and radii drawn from the two ends of that line. Thus

$$A = 2\left(\frac{C^2}{36} - \frac{1}{2}\frac{r}{2}T\right) = 2\left(\frac{9a^2}{36} - \frac{1}{2}\frac{a}{4}\frac{7a}{8}\right) = 2a^2\left(\frac{1}{4} - \frac{7}{64}\right) = \frac{9}{32}a^2.$$

17. The correct formula in the first case gives $V = 56$, while the Babylonian version gives $V = \frac{1}{2}(2^2+4^2)6 = 60$ for a percentage error of 7%. In the second case, the correct formula gives $488/3 = 162\frac{2}{3}$, while the Babylonian formula gives $V = \frac{1}{2}(8^2 + 10^2)2 = 164$, for an error of 0.8%.

18. $1;24,51,10 = 1+\frac{24}{60}+\frac{51}{3600}+\frac{10}{216000} = 1+0.4+0.0141666666+0.0000462962 = 1.414212963$. On the other hand, $\sqrt{2} = 1.414213562$. Thus the Babylonian value differs from the true value by approximately 0.00004%.

19. Because $(1;25)^2 = 2;00,25$, we have

$$\sqrt{2} = \sqrt{2;00,25 - 0;00,25} \approx 1;25 - (0;30)(0;00,25)(1/1;25).$$

An approximation to the reciprocal of 1;25 is 0;42,21,11. The product of 0;30 by 0;00,25 by 0;42,21,11 is 0;00,08,49,25. The the approximation to $\sqrt{2}$ is $1;25 - 0;00,08,49,25 = 1;24,51,10,35$, which, with the last term truncated, is the Babylonian value.

20. $\sqrt{3} = \sqrt{2^2 - 1} \approx 2 - \frac{1}{2} \cdot 1 \cdot \frac{1}{2} = 2 - 0;15 = 1;45$. Since an approximate reciprocal of 1;45 is 0;34,17.09, we get further that $\sqrt{3} = \sqrt{(1;45)^2 - 0;03,45} = 1;45 - (0;30)(0;03,45)(0;34,17.09) = 1;45 - 0;01,04,17,09 = 1;43,55,42,51$, which we truncate to 1;43,55,42 because we know this value is a slight over-approximation.

21. $12\ \overline{\overline{3}}\ \overline{15}\ \overline{24}\ \overline{32} = 12\frac{129}{160}$. $(12\frac{129}{160})^2 = (12.80625)^2 = 164.0000391\ldots$

22. $v + u = 1;48 = 1\frac{4}{5}$ and $v - u = 0;33,20 = \frac{5}{9}$. So $2v = 2;21,20$ and $v = 1;10,40 = \frac{106}{90}$. Similarly, $2u = 1;14,40$ and $u = 0;37,20 = \frac{56}{90}$. Multiplying by 90 gives $x = 56$, $d = 106$. In the second part, $v + u = 2;05 = 2\frac{1}{12}$ and $v - u = 0;28,48 = \frac{12}{25}$. So $2v = 2;33,48$ and $v = 1;16,54 = \frac{769}{600}$. Similarly, $2u = 1;36,12$ and $u = 0;48,06 = \frac{481}{600}$. Multiplying by 600 gives $x = 481$, $d = 769$. Next, if $v = \frac{481}{360}$ and $u = \frac{319}{360}$, then $v + u = 2\frac{2}{9} = 2;13,20$. Finally, if $v = \frac{289}{240}$ and $u = \frac{161}{240}$, then $v + u = 1\frac{7}{8} = 1;52,30$.

23. The equations for u and v can be solved to give $v = 1;22,08,27 = \frac{295707}{216000} = \frac{98569}{72000}$ and $u = 0;56,05,57 = \frac{201957}{216000} = \frac{67319}{72000}$. Thus the associated Pythagorean triple is 67319, 72000, 98569.

24. The two equations are $x^2 + y^2 = 1525$; $y = \frac{2}{3}x + 5$. If we substitute the second equation into the first and simplify, we get $13x^2 + 60x = 13500$. The solution is then $x = 30$, $y = 25$.

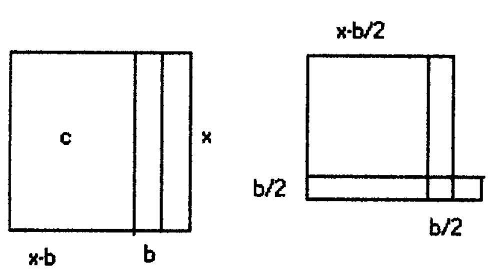
25. If we guess that the length of the rectangle is 60, then the width is 45 and the diagonal is $\sqrt{60^2 + 45^2} = 75$. Since this value is $1\frac{7}{8}$ times the given value of 40, the correct length of the rectangle should be $60 \div 1\frac{7}{8} = 32$. Then the width is 24.

26. One way to solve this is to let x and $x - 600$ be the areas of the two fields. Then the equation is $\frac{2}{3}x + \frac{1}{2}(x - 600) = 1100$. This reduces to $\frac{7}{6}x = 1400$, so $x = 1200$. The second field then has area 600.

27. Let x be the weight of the stone. The equation to solve is then $x - \frac{1}{7}x - \frac{1}{13}(x - \frac{1}{7}x) = 60$. We do this using false position twice. First, set $y = x - \frac{1}{7}x$. The equation in y is then $y - \frac{1}{13}y = 60$. We guess $y = 13$. Since $13 - \frac{1}{13}13 = 12$, instead of 60, we multiply our guess by 5 to get $y = 65$. We then solve $x - \frac{1}{7}x = 65$. Here we guess $x = 7$ and calculate the value of the left side as 6. To get 65, we need to multiply our guess by $\frac{65}{6} = 10\frac{1}{6}$. So our answer is $x = 7 \times \frac{65}{6} = 75\frac{5}{6}$ *gin*, or 1 *mina* $15\frac{5}{6}$ *gin*.

28. We do this in three steps, each using false position. First, set $z = x - \frac{1}{7}x + \frac{1}{11}(x - \frac{1}{7}x)$. The equation for z is then $z - \frac{1}{13}z = 60$. We guess 13 for z and calculate the value of the left side to be 12, instead of 60. Thus we must multiply our original guess by 5 and put $z = 65$. Then set $y = x - \frac{1}{7}x$. The equation for y is $y + \frac{1}{11}y = 65$. If we now guess $y = 11$, the result on the left side is 12, instead of 65. So we must multiply our guess by $\frac{65}{12}$ to get $y = \frac{715}{12} = 59\frac{7}{12}$. We now solve $x - \frac{1}{7}x = 59\frac{7}{12}$. If we guess $x = 7$, the left side becomes 6 instead of $59\frac{7}{12}$. So to get the correct value, we must multiply 7 by $\frac{715}{12}/6 = \frac{715}{72}$. Therefore, $x = 7 \times \frac{715}{72} = \frac{5005}{72} = 69\frac{37}{72}$ *gin* $= 1$ *mina* $9\frac{37}{72}$ *gin*.

29.



30. If we substitute the first equation into the second, the result is $30y - (30-y)^2 = 500$ or $y^2 + 1400 = 90y$. This equation has the two positive roots 20 and 70. If we subtract the second equation from the square of the first equation, we get $(x^2 = 900) - (xy - (x-y)^2 = 500)$, or $(x-y)^2 + x(x-y) = 400$, or finally $(x-y)^2 + 30(x-y) = 400$. This latter equation has $x - y = 10$ as its only positive solution. Since we know that $x = 30$, we also get that $y = 20$.

31. The equations can be rewritten in the form $x + y = 5\frac{5}{6}$; $x + y + xy = 14$. By subtracting the first equation from the second, we get the new equation $xy = 8\frac{1}{6}$. The standard method then gives

$$x = \frac{5\frac{5}{6}}{2} + \sqrt{\left(\frac{5\frac{5}{6}}{2}\right)^2 - 8\frac{1}{6}} = 2\frac{11}{2} + \sqrt{8\frac{73}{144} - 8\frac{1}{6}} = 2\frac{11}{12} = \sqrt{\frac{49}{144}} = 2\frac{11}{12} + \frac{7}{12} = 3\frac{1}{2}.$$

Similarly, $y = 2\frac{1}{3}$.

CHAPTER TWO

1. Since $AB = BC$; since the two angles at B are equal; and since the angles at A and C are both right angles, it follows by the angle-side-angle theorem that $\triangle EBC$ is congruent to $\triangle SBA$ and therefore that $SA = EC$.

2. Because both angles at E are right angles; because AE is common to the two triangles; and because the two angles CAE are equal to one another, it follows by the angle-side-angle theorem that $\triangle AET$ is congruent to $\triangle AES$. Therefore $SE = ET$.

3. $T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Therefore the oblong number $n(n+1)$ is double the triangular number T_n.

4. $n^2 = \frac{(n-1)n}{2} + \frac{n(n+1)}{2}$, and the summands are the triangular numbers T_{n-1} and T_n.

5. $\frac{8n(n+1)}{2} + 1 = 4n^2 + 4n + 1 = (2n+1)^2$.

6. Examples using the first formula are (3,4,5), (5,12,13), (7,24,25), (9,40,41), (11,60,61). Examples using the second formula are (8,15,17), (12,35,37), (16,63,65), (20,99,101), (24,143,145).

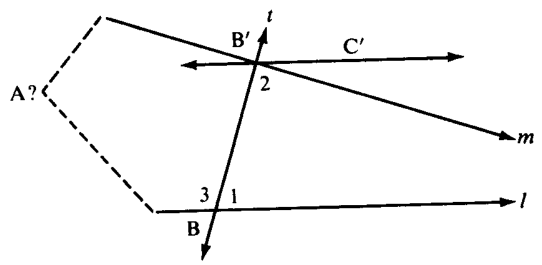
7. Consider the right triangle ABC where AB has unit length and the hypotenuse BC has length 2. Then the square on the leg AC is three times the square on the leg AB. Assume the legs AB and AC are commensurable, so that each is represented by the number of times it is measured by their greatest common measure, and assume further that these numbers are relatively prime, for otherwise there would be a larger common measure. Thus the squares on AC and AB are represented by square numbers, where the former is three times the latter. It follows that leg AC is divisible by three and therefore its square is divisible by nine. Since the square on AB is one third that on AC, it is divisible by three, and hence the side AB itself is divisible by three, contradicting the assumption that the numbers measuring the two legs are relatively prime.

9. Let ABC be the given triangle. Extend BC to D and draw CE parallel to AB. By I–29, angles BAC and ACE are equal, as are angles ABC and ECD. Therefore angle ACD equals the sum of the angles ABC and BAC. If we add angle ACB to each of these, we get that the sum of the three interior angles of the triangle is equal to the straight angle BCD. Because this latter angle equals two right angles, the theorem is proved.

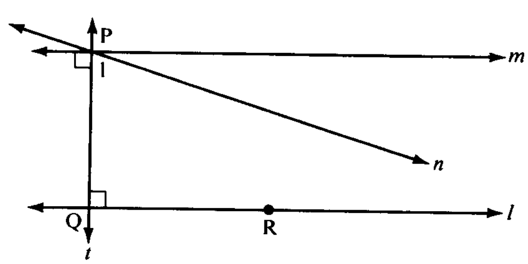
10. Place the given rectangle $BEFG$ so that BE is in a straight line with AB. Extend FG to H so that AH is parallel to BG. Connect HB and extend it until it meets the extension of FE at D. Through D draw DL parallel to FH and extend GB and HA so they meet DL in M and L respectively. Then HD is the diagonal of the rectangle $FDLH$ and

so divides it into two equal triangles HFD and HLD. Because triangle BED is equal to triangle BMD and also triangle BGH is equal to triangle BAH, it follows that the remainders, namely rectangles $BEFG$ and $ABML$ are equal. Thus $ABML$ has been applied to AB and is equal to the given rectangle $BEFG$.

11. In this proof, we shall refer to certain propositions in Euclid's Book I, all of which are proved before Euclid first uses Postulate 5. (That occurs in proposition 29.) First, assume Playfair's axiom. Suppose line t crosses lines m and l and that the sum of the two interior angles (angles 1 and 2 in the diagram) is less than two right angles. We know that the sum of angles 1 and 3 is equal to two right angles. Therefore $\angle 2 < \angle 3$. Now on line BB' and point B' construct line $B'C'$ such that $\angle C'B'B = \angle 3$ (Proposition 23). Therefore, line $B'C'$ is parallel to line l (Proposition 27). Therefore, by Playfair's axiom, line m is not parallel to line l. It therefore meets l. We must show that the two lines meet on the same side as C'. If the meeting point A is on the opposite side, then $\angle 2$ is an exterior angle to triangle ABB', yet it is smaller than $\angle 3$, one of the interior angles, contradicting proposition 16. We have therefore derived Euclid's postulate 5.



Second, assume Euclid's postulate 5. Let l be a given line and P a point outside the line. Construct the line t perpendicular to l through P (Proposition 12). Next, construct the line m perpendicular to line t at P (Proposition 11). Since the alternate interior angles formed by line t crossing lines m and l are both right and therefore are equal, it follows from Proposition 27 that m is parallel to l. Now suppose n is any other line through P. We will show that n meets l and is therefore not parallel to l. Let $\angle 1$ be the acute angle that n makes with t. Then the sum of angle 1 and angle PQR is less than two right angles. By postulate 5, the lines meet.



Note that in this proof, we have actually proved the equivalence of Euclid's Postulate 5 to the statement that given a line l and a point P not on l, there is at most one line through P which is parallel to l. The other part of Playfair's Axiom was proved (in the second part above) without use of postulate 5 and was not used at all in the first part.

12. One possibility: If the line has length a and is cut at a point with coordinate x, then $4ax + (a - x)^2 = (a + x)^2$. This is a valid identity.

13. In the circle ABC, let the angle BEC be an angle at the center and the angle BAC be an angle at the circumference which cuts off the same arc BC. Connect AE and continue the line to F. Since $EA = EB$, $\angle EAB = \angle EBA$. Since $\angle BEF$ equals the sum of those two angles, $\angle BEF$ is double $\angle EAB$. Similarly, $\angle FEC$ is double $\angle EAC$. Therefore the entire $\angle BEC$ is double the entire $\angle BAC$. Note that this argument holds as long as line EF is within $\angle BEC$. If it is not, an analogous argument by subtraction holds.

14. Let $\angle BAC$ be an angle cutting off the diameter BC of the circle. Connect A to the center E of the circle. Since $EB = EA$, it follows that $\angle EBA = \angle EAB$. Similarly, $\angle ECA = \angle EAC$. Therefore the sum of $\angle EBA$ and $\angle ECA$ is equal to $\angle BAC$. But the sum of all three angles equals two right angles. Therefore, twice $\angle BAC$ is equal to two right angles, and angle BAC is itself a right angle.

15. In the circle, inscribe a side AC of an equilateral triangle and a side AB of an equilateral pentagon. Then arc BC is the difference between one-third and one-fifth of the circumference of the circle. That is, arc $BC = \frac{2}{15}$ of the circumference. Thus, if we bisect that arc at E, then lines BE and EC will each be a side of a regular 15-gon.

16. Let the triangle be ABC and draw DE parallel to BC cutting the side AB at D and the side AC at E. Connect BE and CD. Then triangles BDE and CDE are equal in area, having the same base and in the same parallels. Therefore, triangle BDE is to triangle ADE and triangle CDE is to triangle ADE. But triangles withe the same altitude are to one another as their bases. Thus triangle BDE is to triangle ADE as BD is to AD, and triangle CDE is to triangle ADE and CE is to AE. It follows that BD is to AD as CE is to AE, as desired.

17. Let ABC be the triangle, and let the angle at A be bisected by AD, where D lies on the side BC. Now draw CE parallel to AD, meeting BA extended at E. Now angle CAD is equal to angle BAD by hypothesis. But also angle CAD equals angle ACE and angle BAD equals angle AEC, since in both cases we have a transversal falling across parallel lines. It follows that angle AEC equals angle ACE, and therefore that $AC = AE$. By proposition VI-2, we know that BD is to DC as BA is to AE. Therefore BD is to DC as BA is to AC, as claimed.

18. Let $a = s_1 b + r_1$, $b = s_2 r_1 + r_2$, $\ldots$, $r_{k-1} = s_{k+1} r_k$. Then r_k divides r_{k-1} and therefore also $r_{k-2}, \ldots, b, a$. If there were a greater common divisor of a and b, it would divide r_1, r_2, $\ldots$, r_k. Since it is impossible for a greater number to divide a smaller, we have shown that r_k is in fact the greatest common divisor of a and b.

19.

$$
\begin{aligned}
963 &= 1 \cdot 657 + 306 \\
657 &= 2 \cdot 306 + 45 \\
306 &= 6 \cdot 45 + 36 \\
45 &= 1 \cdot 36 + 9 \\
36 &= 4 \cdot 9 + 0
\end{aligned}
$$

Therefore, the greatest common divisor of 963 and 657 is 9.

20. Since $1 - x = x^2$, we have

$$1 = 1 \cdot x + (1 - x) = 1 \cdot x + x^2$$

$$x = 1 \cdot x^2 + (x - x^2) = 1 \cdot x^2 + x(1 - x) = 1 \cdot x^2 + x^3$$
$$x^2 = 1 \cdot x^3 + (x^2 - x^3) = 1 \cdot x^3 + x^2(1 - x) = 1 \cdot x^3 + x^4$$
$$\cdots$$

Thus $1 : x$ can be expressed in the form $(1, 1, 1, \ldots)$.

21.

$$
\begin{array}{rclcl}
46 & = & 7 \cdot 6 & + & 4 \\
6 & = & 1 \cdot 4 & + & 2 \\
4 & = & 2 \cdot 2 & &
\end{array}
\qquad
\begin{array}{rclcl}
23 & = & 7 \cdot 3 & + & 2 \\
3 & = & 1 \cdot 2 & + & 1 \\
2 & = & 2 \cdot 1 & &
\end{array}
$$

Note that the multiples 7, 1, 2 in the first example equal the multiples 7, 1, 2 in the second.

22. In Figure 2.16 (left), let $AB = 7$ and the area of the given figure be 10. The construction described on p. 72 then determines x to be BS. This value is $\frac{7}{2} - \sqrt{\frac{49}{4} - 10} = \frac{7}{2} - \sqrt{\frac{9}{4}} = \frac{7}{2} - \frac{3}{2} = 2$. The second solution is $BE + ES = AE + ES = AS$. This value is $\frac{7}{2} + \sqrt{\frac{49}{4} - 10} = \frac{7}{2} + \sqrt{\frac{9}{4}} = \frac{7}{2} + \frac{3}{2} = 5$.

23. In Figure 2.16 (right), let $AB = 10$ and the area of the given figure be 39. The construction described on p. 72 then determines x to be BS. This value is $\sqrt{5^2 + 39} - 5 = \sqrt{64} - 5 = 8 - 5 = 3$.

24. Suppose m factors two different ways as a product of primes: $m = pqr \cdots s = p'q'r' \cdots s'$. Since p divides $pqr \cdots s$, it must also divide $p'q'r' \cdots s'$. By VII–30, p must divide one of the prime factors, say p'. But since both p and p' are prime, we must have $p = p'$. After canceling these two factors from their respective products, we can then repeat the argument to show that each prime factor on the left is equal to a prime factor on the right and conversely.

25. One standard modern proof is as follows. Assume there are only finitely many prime numbers p_1, p_2, p_3, …, p_n. Let $N = p_1 p_2 p_3 \cdots p_n + 1$. There are then two possibilities. Either N is prime or N is divisible by a prime other than the given ones, since division by any of those leaves remainder 1. Both cases contradict the original hypothesis, which therefore cannot be true.

26. We begin with a square inscribed in a circle of radius 1. If we divide the square into four isosceles triangles, each with vertex angle a right angle, then the base of each triangle has length $b_1 = \sqrt{2}$ and height $h_1 = \frac{\sqrt{2}}{2}$. Then the area A_1 of the square is equal to $4 \cdot \frac{1}{2} b_1 h_1 = 2 b_1 h_1 = 2$. If we next construct an octagon by bisecting the vertex angles of each of these triangles and connecting the points on the circumference, the octagon is formed of eight isosceles triangles. The base of each triangle has length

$$b_2 = \sqrt{\left(\frac{b_1}{2}\right)^2 + (1-h_1)^2} = \sqrt{\left(\frac{b_1}{2}\right)^2 + h_1^2 - 2h_1 + 1} = \sqrt{2 - 2h_1} = \sqrt{2 - \sqrt{2}}$$

and height

$$h_2 = \sqrt{1 - \left(\frac{b_2}{2}\right)^2} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

Thus the octagon has area $A_2 = 8 \cdot \frac{1}{2} b_2 h_2 = 4 b_2 h_2 = 2\sqrt{2} = 2.828427$. If we continue in this way by always bisecting the vertex angles of the triangles to construct a new polygon, we get that the area A_n of the nth polygon is given by the formula $A_n = 2^{n+1} \cdot \frac{1}{2} b_n h_n = 2^n b_n h_n$, where

$$b_n = \sqrt{\left(\frac{b_{n-1}}{2}\right)^2 + (1-h_{n-1})^2} = \sqrt{\left(\frac{b_{n-1}}{2}\right)^2 + h_{n-1}^2 - 2h_{n-1} + 1} = \sqrt{2 - 2h_{n-1}}$$

and

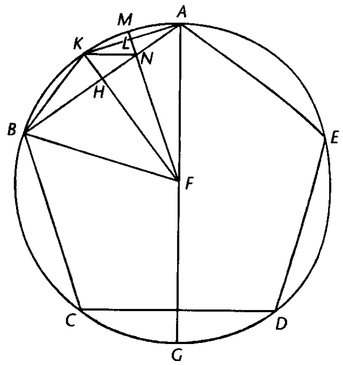
$$h_n = \sqrt{1 - \left(\frac{b_n}{2}\right)^2}.$$

The next two results using this formula are $A_3 = 3.061467$ and $A_4 = 3.121445$.

27. Since BC is the side of a decagon, triangle EBC is a 36-72-72 triangle. Thus $\angle ECD = 108°$. Since CD, the side of a hexagon, is equal to the radius CE, it follows that triangle ECD is an isosceles triangle with base angles equal to 36°. Thus triangle EBD is a 36-72-72 triangle and is similar to triangle EBC. Therefore $BD : EC = EC : BC$ or $BD : CD = CD : BC$ and the point C divides the line segment BD in extreme and mean ratio.

28. Let $ABCDE$ be the pentagon inscribed in the circle with center F. Connect AF and extend it to meet the circle at G. Draw FH perpendicular to AB and extend it to

meet the circle at K. Connect AK. Then AK is a side of the decagon inscribed in the circle, while $BF = AF$ is the side of the hexagon inscribed in the circle. Draw FL perpendicular to AK; let N be its intersection with AB and M be its intersection with the circle. Connect KN. Now triangles ANK and AKB are isosceles triangles with a common base angle at A. Therefore, the triangles are similar. So $BA : AK = AK : AN$, or $AK^2 = BA \cdot AN$. Further, note that arc BKM has measure 54°, while arc BCG has measure 108°. It follows that $\angle BFN = \angle BAF$. Since triangles BFN and BAF also have angle FBA in common, the triangles are similar. Therefore, $BA : BF = BF : BN$, or $BF^2 = BA \cdot BN$. We therefore have $AK^2 + BF^2 = BA \cdot AN + BA \cdot BN = BA \cdot (AN + BN) = BA^2$. That is, the sum of the squares on the side of the decagon and the side of the hexagon is equal to the square on the side of the pentagon.



29. $C = \frac{360}{7\frac{1}{5}} \cdot 5000 = 250,000$ stades. This value equals 129,175,000 feet or 24,465 miles. The diameter then equals 7,787 miles.

CHAPTER THREE

1. Lemma 1: $DA/DC = OA/OC$ by *Elements* VI–3. Therefore $DA/OA = DC/OC = (DC+DA)/(OC+OA) = AC/(CO+OA)$. Also, $DO^2 = OA^2+DA^2$ by the Pythagorean Theorem.

   Lemma 2: $AD/DB = BD/DE = AC/CE = AB/BE = (AB + AC)/(CE + BE) = (AB + AC)/BC$. Therefore, $AD^2/BD^2 = (AB + AC)^2/BC^2$. But $AD^2 = AB^2 - BD^2$. So $(AB^2 - BD^2)/BD^2 = (AB + AC)^2/BC^2$ and $AB^2/BD^2 = 1 + (AB + AC)^2/BC^2$.

2. Set $r = 1$, t_i and u_i as in the text, and P_i the perimeter of the ith circumscribed polygon. Then the first ten iterations of the algorithm give the following:

$t_1 = .577350269$	$u_1 = 1.154700538$	$P_1 = 3.464101615$
$t_2 = .267949192$	$u_2 = 1.03527618$	$P_2 = 3.21539031$
$t_3 = .131652497$	$u_3 = 1.008628961$	$P_3 = 3.159659943$
$t_4 = .065543462$	$u_4 = 1.002145671$	$P_4 = 3.146086215$
$t_5 = .03273661$	$u_5 = 1.0005357$	$P_5 = 3.1427146$
$t_6 = .016363922$	$u_6 = 1.00013388$	$P_6 = 3.141873049$
$t_7 = .0081814134$	$u_7 = 1.000033467$	$P_7 = 3.141662746$
$t_8 = .004090638249$	$u_8 = 1.000008367$	$P_8 = 3.141610175$
$t_9 = .002045310568$	$u_9 = 1.000002092$	$P_9 = 3.141597032$
$t_{10} = .001022654214$	$u_{10} = 1.000000523$	$P_{10} = 3.141593746$

3. Let d be the diameter of the circle, t_i the length of one side of the regular inscribed polygon of $3 \cdot 2^i$ sides, and u_i the length of the other leg of the right triangle formed from the diameter and the side of the polygon. Then

$$\frac{{t_{i+1}}^2}{d^2} = \frac{t_i^2}{t_i^2 + (d + u_i)^2}$$

or

$$t_{i+1} = \frac{dt_i}{\sqrt{t_i^2 + (d + u_i)^2}} \qquad u_{i+1} = \sqrt{d^2 - {t_{i+1}}^2}.$$

If P_i is the perimeter of the ith inscribed polygon, then $\frac{P_i}{d} = \frac{3 \cdot 2^i t_i}{d}$. So let $d = 1$. Then $t_1 = \frac{d}{2} = 0.5$ and $u_1 = \frac{\sqrt{3}d}{2} = 0.8660254$. Then repeated use of the algorithm gives us:

$t_1 = 0.500000000$	$u_1 = 0.866025403$	$P_1 = 3.000000000$
$t_2 = 0.258819045$	$u_2 = 0.965925826$	$P_2 = 3.105828542$
$t_3 = 0.130526194$	$u_3 = 0.991444861$	$P_3 = 3.132628656$
$t_4 = 0.06540313$	$u_4 = 0.997858923$	$P_4 = 3.13935025$
$t_5 = 0.032719083$	$u_5 = 0.999464587$	$P_5 = 3.141031999$
$t_6 = 0.016361731$	$u_6 = 0.999866137$	$P_6 = 3.141452521$
$t_7 = 0.008181140$	$u_7 = 0.999966533$	$P_7 = 3.141557658$
$t_8 = 0.004090604$	$u_8 = 0.999991633$	$P_8 = 3.141583943$

$$t_9 = 0.002045306 \qquad u_9 = 0.999997908 \qquad P_9 = 3.141590016$$

4. We can prove the inequality simply by squaring each side and noting that $b < 2a+1$. To find the approximands to $\sqrt{3}$, begin with $2-\frac{1}{4} > \sqrt{2^2-1} > 2-\frac{1}{3}$, or $\frac{7}{4} > \sqrt{3} > \frac{5}{3}$. Since $\sqrt{3} = \frac{1}{3}\sqrt{5^2+2}$, we continue with $\frac{1}{3}(5+\frac{1}{5}) > \frac{1}{3}\sqrt{5^2+2} > \frac{1}{3}(5+\frac{2}{11})$, or $\frac{26}{15} > \sqrt{3} > \frac{57}{33}$. Again, since $\sqrt{3} = \frac{1}{15}\sqrt{26^2-1}$, we get $\frac{1}{15}(26-\frac{1}{52}) > \frac{1}{15}\sqrt{26^2-1} > \frac{1}{15}(26-\frac{1}{51})$, or $\frac{1351}{780} > \sqrt{3} > \frac{1325}{765} = \frac{265}{153}$.

5. Let the equation of the parabola be $y = -x^2+1$. Then the tangent line at $C = (1,0)$ has the equation $y = -2x+2$. Let the point O have coordinates $(-a,0)$. Then $MO = 2a+2$, $OP = -a^2+1$, $CA = 2$, $AO = -a+1$. So $MO : OP = (2a+2) : (1-a^2) = 2 : (1-a) = CA : AO$.

6. a. Draw line AO. Then $MS \cdot SQ = CA \cdot AS = AO^2 = OS^2 + AS^2 = OS^2 + SQ^2$.
   b. Since $HA = AC$, we have $HA : AS = MS : SQ = MS^2 : MS \cdot SQ = MS^2 : (OS^2+SQ^2) = MN^2 : (OP^2+QR^2)$. Since circles are to one another as the squares on their diameters, the latter ratio equals that of the circle with diameter MN to the sum of the circle with diameter OP and that with diameter QR.
   c. Since then $HA : AS =$ (circle in cylinder):(circle in sphere + circle in cone), it follows that the circle placed where it is is in equilibrium about A with the circle in the sphere together with the circle in the cone if the latter circles have their centers at H.
   d. Since the above result is true whatever line MN is taken, and since the circles make up the three solids involved, Archimedes can conclude that the cylinder placed where it is is in equilibrium about A with the sphere and cone together, if both of them are placed with their center of gravity at H. Since K is the center of gravity of the cylinder, it follows that $HA : AK =$ (cylinder):(sphere + cone).
   e. Since $HA = 2AK$, it follows that the cylinder is twice the sphere plus the cone AEF. But we know that the cylinder is three times the cone AEF. Therefore the cone AEF is twice the sphere. But the cone AEF is eight times the cone ABD, because each of the dimensions of the former are double that of the latter. Therefore, the sphere is four times the cone ABD.

7. Since $BOAPC$ is a parabola, we have $DA : AS = BD^2 : OS^2$, or $HA : AS = MS^2 : OS^2$. Thus $HA : AS =$ (circle in cylinder):(circle in paraboloid). Thus the circle in the cylinder, placed where it is, balances the circle in the paraboloid placed with its center of gravity at H. Since the same is true whatever cross section line MN is taken, Archimedes can conclude that the cylinder, placed where it is, balances the paraboloid, placed with its center of gravity at H. If we let K be the midpoint of AD, then K is the center of gravity of the cylinder. Thus $HA : AK =$ cylinder:paraboloid. But $HA = 2AK$. So the cylinder is double the paraboloid. But the cylinder is also triple the volume of the cone ABC. Therefore, the volume of the paraboloid is 3/2 the volume of the cone ABC which has the same base and same height.

8. Let the parabola be given by $y = a - bx^2$. Then the area A of the segment cut off by the x axis is given by

$$A = 2\int_0^{\sqrt{a/b}} (a - bx^2)\,dx = 2\left(ax - \frac{1}{3}bx^3\right)\Big|_0^{\sqrt{a/b}}$$
$$= 2a\sqrt{\frac{a}{b}} - \frac{2a}{3}\sqrt{\frac{a}{b}} = \frac{4a}{3}\sqrt{\frac{a}{b}}.$$

Since the area of the inscribed triangle is $a\sqrt{\frac{a}{b}}$, the result is established.

9. Let the equation of the parabola be $y = x^2$, and let the straight line defining the segment be the line through the points $(-a, a^2)$ and (b, b^2). Thus the equation of this line is $(a - b)x + y = ab$, and its normal vector is $N = (a - b, 1)$. Also, since the midpoint of that line segment is $B = (\frac{b-a}{2}, \frac{b^2+a^2}{2})$, the x-coordinate of the vertex of the segment is $\frac{b-a}{2}$. If $S = (x, x^2)$ is an arbitrary point on the parabola, then the vector M from $(-a, a^2)$ to S is given by $(x + a, x^2 - a^2)$. The perpendicular distance from S to the line is then the dot product of M with N, divided by the length of N. Since the length of N is a constant, to maximize the distance it is only necessary to maximize this dot product. The dot product is $(x+a, x^2-a^2)\cdot(a-b, 1) = ax-bx+a^2-ab+x^2-a^2 = ax-bx+x^2-ab$. The maximum of this function occurs when $a - b + 2x = 0$, or when $x = \frac{b-a}{2}$. And, as we have already noted, the point on the parabola with that x-coordinate is the vertex of the segment. So the vertex is the point whose perpendicular distance to the base of the segment is the greatest.

10. Let r be the radius of the sphere. Then we know from calculus that the volume of the sphere is $V_S = \frac{4}{3}\pi r^3$ and the surface area of the sphere is $A_S = 4\pi r^2$. The volume of the cylinder whose base is a great circle in the sphere and whose height equals the diameter has volume is $V_C = \pi r^2(2r) = 2\pi r^3$, while the total surface area of the cylinder is $A_C = (2\pi r)(2r) + 2\pi r^2 = 6\pi r^2$. Therefore, $V_C = \frac{3}{2}V_S$ and $A_C = \frac{3}{2}A_S$, as desired.

11. Suppose the cylinder P has diameter d and height h, and suppose the cylinder Q is constructed with the same volume but with its height and diameter both equal to f. It follows that $d^2 : f^2 = f : h$, or that $f^3 = d^2h$. It follows that one needs to construct the cube root of the quantity d^2h, and this can be done by finding two mean proportionals between 1 and d^2h, or, alternatively, two mean proportionals between d and h (where the first one will be the desired diameter f).

12. The two equations are $x^2 = 4ay$ and $y(3a - x) = ab$. Pick easy values for a and b, say $a = 1$, $b = 1$, and then the parabola and hyperbola may be sketched.

13. The focus of $y^2 = px$ is at $(\frac{p}{4}, 0)$. The length of the latus rectum is $2\sqrt{p\frac{p}{4}} = p$.