CHAPTER 2

2.1 Limits of Sequences.

2.1.0. a) True. If x_n converges, then there is an $M > 0$ such that $|x_n| \le M$. Choose by Archimedes an $N \in \mathbf{N}$ such that $N > M/\varepsilon$. Then $n \ge N$ implies $|x_n/n| \le M/n \le M/N < \varepsilon$.
b) False. $x_n = \sqrt{n}$ does not converge, but $x_n/n = 1/\sqrt{n} \to 0$ as $n \to \infty$.
c) False. $x_n = 1$ converges and $y_n = (-1)^n$ is bounded, but $x_n y_n = (-1)^n$ does not converge.
d) False. $x_n = 1/n$ converges to 0 and $y_n = n^2 > 0$, but $x_n y_n = n$ does not converge.

2.1.1. a) By the Archimedean Principle, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $N > 1/\varepsilon$. Thus $n \ge N$ implies

$$|(2 - 1/n) - 2| \equiv |1/n| \le 1/N < \varepsilon.$$

b) By the Archimedean Principle, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $N > \pi^2/\varepsilon^2$. Thus $n \ge N$ implies

$$|1 + \pi/\sqrt{n} - 1| \equiv |\pi/\sqrt{n}| \le \pi/\sqrt{N} < \varepsilon.$$

c) By the Archimedean Principle, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $N > 3/\varepsilon$. Thus $n \ge N$ implies

$$|3(1 + 1/n) - 3| \equiv |3/n| \le 3/N < \varepsilon.$$

d) By the Archimedean Principle, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $N > 1/\sqrt{3\varepsilon}$. Thus $n \ge N$ implies

$$|(2n^2 + 1)/(3n^2) - 2/3| \equiv |1/(3n^2)| \le 1/(3N^2) < \varepsilon.$$

2.1.2. a) By hypothesis, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $n \ge N$ implies $|x_n - 1| < \varepsilon/2$. Thus $n \ge N$ implies

$$|1 + 2x_n - 3| \equiv 2\,|x_n - 1| < \varepsilon.$$

b) By hypothesis, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n > 1/2$ and $|x_n - 1| < \varepsilon/4$. In particular, $1/x_n < 2$. Thus $n \ge N$ implies

$$|(\pi x_n - 2)/x_n - (\pi - 2)| \equiv 2\,|(x_n - 1)/x_n| < 4\,|x_n - 1| < \varepsilon.$$

c) By hypothesis, given $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n > 1/2$ and $|x_n - 1| < \varepsilon/(1 + 2e)$. Thus $n \ge N$ and the triangle inequality imply

$$|(x_n^2 - e)/x_n - (1 - e)| \equiv |x_n - 1| \left|1 + \frac{e}{x_n}\right| \le |x_n - 1| \left(1 + \frac{e}{|x_n|}\right) < |x_n - 1|(1 + 2e) < \varepsilon.$$

2.1.3. a) If $n_k = 2k$, then $3 - (-1)^{n_k} \equiv 2$ converges to 2; if $n_k = 2k + 1$, then $3 - (-1)^{n_k} \equiv 4$ converges to 4.
b) If $n_k = 2k$, then $(-1)^{3n_k} + 2 \equiv (-1)^{6k} + 2 = 1 + 2 = 3$ converges to 3; if $n_k = 2k + 1$, then $(-1)^{3n_k} + 2 \equiv (-1)^{6k+3} + 2 = -1 + 2 = 1$ converges to 1.
c) If $n_k = 2k$, then $(n_k - (-1)^{n_k} n_k - 1)/n_k \equiv -1/(2k)$ converges to 0; if $n_k = 2k+1$, then $(n_k - (-1)^{n_k} n_k - 1)/n_k \equiv (2n_k - 1)/n_k = (4k + 1)/(2k + 1)$ converges to 2.

2.1.4. Suppose x_n is bounded. By Definition 2.7, there are numbers M and m such that $m \le x_n \le M$ for all $n \in \mathbf{N}$. Set $C := \max\{1, |M|, |m|\}$. Then $C > 0$, $M \le C$, and $m \ge -C$. Therefore, $-C \le x_n \le C$, i.e., $|x_n| < C$ for all $n \in \mathbf{N}$.
Conversely, if $|x_n| < C$ for all $n \in \mathbf{N}$, then x_n is bounded above by C and below by $-C$.

2.1.5. If $C = 0$, there is nothing to prove. Otherwise, given $\varepsilon > 0$ use Definition 2.1 to choose an $N \in \mathbf{N}$ such that $n \ge N$ implies $|b_n| \equiv b_n < \varepsilon/|C|$. Hence by hypothesis, $n \ge N$ implies

$$|x_n - a| \le |C| b_n < \varepsilon.$$

By definition, $x_n \to a$ as $n \to \infty$.

2.1.6. If $x_n = a$ for all n, then $|x_n - a| = 0$ is less than any positive ε for all $n \in \mathbf{N}$. Thus, by definition, $x_n \to a$ as $n \to \infty$.

2.1.7. a) Let a be the common limit point. Given $\varepsilon > 0$, choose $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - a|$ and $|y_n - a|$ are both $< \varepsilon/2$. By the Triangle Inequality, $n \geq N$ implies

$$|x_n - y_n| \leq |x_n - a| + |y_n - a| < \varepsilon.$$

By definition, $x_n - y_n \to 0$ as $n \to \infty$.

b) If n converges to some a, then given $\varepsilon = 1/2$, $1 = |(n+1) - n| < |(n+1) - a| + |n - a| < 1$ for n sufficiently large, a contradiction.

c) Let $x_n = n$ and $y_n = n + 1/n$. Then $|x_n - y_n| = 1/n \to 0$ as $n \to \infty$, but neither x_n nor y_n converges.

2.1.8. By Theorem 2.6, if $x_n \to a$ then $x_{n_k} \to a$. Conversely, if $x_{n_k} \to a$ for every subsequence, then it converges for the "subsequence" x_n.

2.2 Limit Theorems.

2.2.0. a) False. Let $x_n = n^2$ and $y_n = -n$ and note by Exercise 2.2.2a that $x_n + y_n \to \infty$ as $n \to \infty$.

b) True. Let $\varepsilon > 0$. If $x_n \to -\infty$ as $n \to \infty$, then choose $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n < -1/\varepsilon$. Then $x_n < 0$ so $|x_n| = -x_n > 0$. Multiply $x_n < -1/\varepsilon$ by $\varepsilon/(-x_n)$ which is positive. We obtain $-\varepsilon < 1/x_n$, i.e., $|1/x_n| = -1/x_n < \varepsilon$.

c) False. Let $x_n = (-1)^n/n$. Then $1/x_n = (-1)^n n$ has no limit as $n \to \infty$.

d) True. Since $(2^x - x)' = 2^x \log 2 - 1 > 1$ for all $x \geq 2$, i.e., $2^x - x$ is increasing on $[2, \infty)$. In particular, $2^x - x \geq 2^2 - 2 > 0$, i.e., $2^x > x$ for $x \geq 2$. Thus, since $x_n \to \infty$ as $n \to \infty$, we have $2^{x_n} > x_n$ for n large, hence

$$2^{-x_n} < \frac{1}{x_n} \to 0$$

as $n \to \infty$.

2.2.1. a) $|x_n| \leq 1/n \to 0$ as $n \to \infty$ and we can apply the Squeeze Theorem.

b) $2n/(n^2 + \pi) = (2/n)/(1 + \pi/n^2) \to 0/(1+0) = 0$ by Theorem 2.12.

c) $(\sqrt{2n} + 1)/(n + \sqrt{2}) = ((\sqrt{2}/\sqrt{n}) + (1/n))/(1 + (\sqrt{2}/n)) \to 0/(1+0) = 0$ by Exercise 2.2.5 and Theorem 2.12.

d) An easy induction argument shows that $2n + 1 < 2^n$ for $n = 3, 4, \dots$. We will use this to prove that $n^2 \leq 2^n$ for $n = 4, 5, \dots$. It's surely true for $n = 4$. If it's true for some $n \geq 4$, then the inductive hypothesis and the fact that $2n + 1 < 2^n$ imply

$$(n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1 < 2^n + 2^n = 2^{n+1}$$

so the second inequality has been proved.

Now the second inequality implies $n/2^n < 1/n$ for $n \geq 4$. Hence by the Squeeze Theorem, $n/2^n \to 0$ as $n \to \infty$.

2.2.2. a) Let $M \in \mathbf{R}$ and choose by Archimedes an $N \in \mathbf{N}$ such that $N > \max\{M, 2\}$. Then $n \geq N$ implies $n^2 - n = n(n-1) \geq N(N-1) > M(2-1) = M$.

b) Let $M \in \mathbf{R}$ and choose by Archimedes an $N \in \mathbf{N}$ such that $N > -M/2$. Notice that $n \geq 1$ implies $-3n \leq -3$ so $1 - 3n \leq -2$. Thus $n \geq N$ implies $n - 3n^2 = n(1 - 3n) \leq -2n \leq -2N < M$.

c) Let $M \in \mathbf{R}$ and choose by Archimedes an $N \in \mathbf{N}$ such that $N > M$. Then $n \geq N$ implies $(n^2+1)/n = n + 1/n > N + 0 > M$.

d) Let $M \in \mathbf{R}$ satisfy $M \leq 0$. Then $2 + \sin\theta \geq 2 - 1 = 1$ implies $n^2(2 + \sin(n^3 + n + 1)) \geq n^2 \cdot 1 > 0 \geq M$ for all $n \in \mathbf{N}$. On the other hand, if $M > 0$, then choose by Archimedes an $N \in \mathbf{N}$ such that $N > \sqrt{M}$. Then $n \geq N$ implies $n^2(2 + \sin(n^3 + n + 1)) \geq n^2 \cdot 1 \geq N^2 > M$.

2.2.3. a) Following Example 2.13,

$$\frac{2 + 3n - 4n^2}{1 - 2n + 3n^2} = \frac{(2/n^2) + (3/n) - 4}{(1/n^2) - (2/n) + 3} \to \frac{-4}{3}$$

as $n \to \infty$.

b) Following Example 2.13,

$$\frac{n^3 + n - 2}{2n^3 + n - 2} = \frac{1 + (1/n^2) - (2/n^3)}{2 + (1/n^2) - (2/n^3)} \to \frac{1}{2}$$

as $n \to \infty$.

c) Rationalizing the expression, we obtain

$$\sqrt{3n+2}-\sqrt{n}=\frac{(\sqrt{3n+2}-\sqrt{n})(\sqrt{3n+2}+\sqrt{n})}{\sqrt{3n+2}+\sqrt{n}}=\frac{2n+2}{\sqrt{3n+2}+\sqrt{n}}\to\infty$$

as $n\to\infty$ by the method of Example 2.13. (Multiply top and bottom by $1/\sqrt{n}$.)

d) Multiply top and bottom by $1/\sqrt{n}$ to obtain

$$\frac{\sqrt{4n+1}-\sqrt{n}}{\sqrt{9n+1}-\sqrt{n+2}}=\frac{\sqrt{4+1/n}-\sqrt{1-1/n}}{\sqrt{9+1/n}-\sqrt{1+2/n}}\to\frac{2-1}{3-1}=\frac{1}{2}.$$

2.2.4. a) Clearly,

$$\frac{x_n}{y_n}-\frac{x}{y}=\frac{x_ny-xy_n}{yy_n}=\frac{x_ny-xy+xy-xy_n}{yy_n}.$$

Thus

$$\left|\frac{x_n}{y_n}-\frac{x}{y}\right|\le\frac{1}{|y_n|}|x_n-x|+\frac{|x|}{|yy_n|}|y_n-y|.$$

Since $y\ne 0$, $|y_n|\ge|y|/2$ for large n. Thus

$$\left|\frac{x_n}{y_n}-\frac{x}{y}\right|\le\frac{2}{|y|}|x_n-x|+\frac{2|x|}{|y|^2}|y_n-y|\to 0$$

as $n\to\infty$ by Theorem 2.12i and ii. Hence by the Squeeze Theorem, $x_n/y_n\to x/y$ as $n\to\infty$.

b) By symmetry, we may suppose that $x=y=\infty$. Since $y_n\to\infty$ implies $y_n>0$ for n large, we can apply Theorem 2.15 directly to obtain the conclusions when $\alpha>0$. For the case $\alpha<0$, $x_n>M$ implies $\alpha x_n<\alpha M$. Since any $M_0\in\mathbf{R}$ can be written as αM for some $M\in\mathbf{R}$, we see by definition that $x_n\to-\infty$ as $n\to\infty$.

2.2.5. *Case 1.* $x=0$. Let $\epsilon>0$ and choose N so large that $n\ge N$ implies $|x_n|<\epsilon^2$. By (8) in 1.1, $\sqrt{x_n}<\epsilon$ for $n\ge N$, i.e., $\sqrt{x_n}\to 0$ as $n\to\infty$.

Case 2. $x>0$. Then

$$\sqrt{x_n}-\sqrt{x}=(\sqrt{x_n}-\sqrt{x})\left(\frac{\sqrt{x_n}+\sqrt{x}}{\sqrt{x_n}+\sqrt{x}}\right)=\frac{x_n-x}{\sqrt{x_n}+\sqrt{x}}.$$

Since $\sqrt{x_n}\ge 0$, it follows that

$$|\sqrt{x_n}-\sqrt{x}|\le\frac{|x_n-x|}{\sqrt{x}}.$$

This last quotient converges to 0 by Theorem 2.12. Hence it follows from the Squeeze Theorem that $\sqrt{x_n}\to\sqrt{x}$ as $n\to\infty$.

2.2.6. By the Density of Rationals, there is an r_n between $x+1/n$ and x for each $n\in\mathbf{N}$. Since $|x-r_n|<1/n$, it follows from the Squeeze Theorem that $r_n\to x$ as $n\to\infty$.

2.2.7. a) By Theorem 2.9 we may suppose that $|x|=\infty$. By symmetry, we may suppose that $x=\infty$. By definition, given $M\in\mathbf{R}$, there is an $N\in\mathbf{N}$ such that $n\ge N$ implies $x_n>M$. Since $w_n\ge x_n$, it follows that $w_n>M$ for all $n\ge N$. By definition, then, $w_n\to\infty$ as $n\to\infty$.

b) If x and y are finite, then the result follows from Theorem 2.17. If $x=y=\pm\infty$ or $-x=y=\infty$, there is nothing to prove. It remains to consider the case $x=\infty$ and $y=-\infty$. But by Definition 2.14 (with $M=0$), $x_n>0>y_n$ for n sufficiently large, which contradicts the hypothesis $x_n\le y_n$.

2.2.8. a) Take the limit of $x_{n+1}=1-\sqrt{1-x_n}$, as $n\to\infty$. We obtain $x=1-\sqrt{1-x}$, i.e., $x^2-x=0$. Therefore, $x=0,1$.

b) Take the limit of $x_{n+1}=2+\sqrt{x_n-2}$ as $n\to\infty$. We obtain $x=2+\sqrt{x-2}$, i.e., $x^2-5x+6=0$. Therefore, $x=2,3$. But $x_1>3$ and induction shows that $x_{n+1}=2+\sqrt{x_n-2}>2+\sqrt{3-2}=3$, so the limit must be $x=3$.

c) Take the limit of $x_{n+1}=\sqrt{2+x_n}$ as $n\to\infty$. We obtain $x=\sqrt{2+x}$, i.e., $x^2-x-2=0$. Therefore, $x=2,-1$. But $x_{n+1}=\sqrt{2+x_n}\ge 0$ by definition (all square roots are nonnegative), so the limit must be $x=2$.

This proof doesn't change if $x_1>-2$, so the limit is again $x=2$.

2.2.9. a) Let $E = \{k \in \mathbf{Z} : k \geq 0 \text{ and } k \leq 10^{n+1}y\}$. Since $10^{n+1}y < 10$, $E \subseteq \{0, 1, \ldots, 9\}$. Hence $w := \sup E \in E$. It follows that $w \leq 10^{n+1}y$, i.e., $w/10^{n+1} \leq y$. On the other hand, since $w+1$ is not the supremum of E, $w + 1 > 10^{n+1}y$. Therefore, $y < w/10^{n+1} + 1/10^{n+1}$.

b) Apply a) for $n = 0$ to choose $x_1 = w$ such that $x_1/10 \leq x < x_1/10 + 1/10$. Suppose

$$s_n := \sum_{k=1}^{n} \frac{x_k}{10^k} \leq x < \sum_{k=1}^{n} \frac{x_k}{10^k} + \frac{1}{10^n}.$$

Then $0 < x - s_n < 1/10^n$, so by a) choose x_{n+1} such that $x_{n+1}/10^{n+1} \leq x - s_n < x_{n+1}/10^{n+1} + 1/10^{n+1}$, i.e.,

$$\sum_{k=1}^{n+1} \frac{x_k}{10^k} \leq x < \sum_{k=1}^{n+1} \frac{x_k}{10^k} + \frac{1}{10^{n+1}}.$$

c) Combine b) with the Squeeze Theorem.

d) Since an easy induction proves that $9^n > n$ for all $n \in \mathbf{N}$, we have $9^{-n} < 1/n$. Hence the Squeeze Theorem implies that $9^{-n} \to 0$ as $n \to \infty$. Hence, it follows from Exercise 1.4.4c and definition that

$$.4999\cdots = \frac{4}{10} + \lim_{n\to\infty} \sum_{k=2}^{n} \frac{9}{10^k} = \frac{4}{10} + \lim_{n\to\infty} \frac{1}{10}\left(1 - \frac{1}{9^n}\right) = \frac{4}{10} + \frac{1}{10} = 0.5.$$

Similarly,

$$.999\cdots = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{9}{10^k} = \lim_{n\to\infty} \left(1 - \frac{1}{9^n}\right) = 1.$$

2.3 The Bolzano–Weierstrass Theorem.

2.3.0. a) False. $x_n = 1/4 + 1/(n+4)$ is strictly decreasing and $|x_n| \leq 1/4 + 1/5 < 1/2$, but $x_n \to 1/4$ as $n \to \infty$.

b) True. Since $(n-1)/(2n-1) \to 1/2$ as $n \to \infty$, this factor is bounded. Since $|\cos(n^2 + n + 1)| \leq 1$, it follows that $\{x_n\}$ is bounded. Hence it has a convergent subsequence by the Bolzano–Weierstrass Theorem.

c) False. $x_n = 1/2 - 1/n$ is strictly increasing and $|x_n| \leq 1/2 < 1 + 1/n$, but $x_n \to 1/2$ as $n \to \infty$.

d) False. $x_n = (1 + (-1)^n)n$ satisfies $x_n = 0$ for n odd and $x_n = 2n$ for n even. Thus $x_{2k+1} \to 0$ as $k \to \infty$, but x_n is NOT bounded.

2.3.1. Suppose that $-1 < x_{n-1} < 0$ for some $n \geq 0$. Then $0 < x_{n-1} + 1 < 1$ so $0 < x_{n-1} + 1 < \sqrt{x_{n-1} + 1}$ and it follows that $x_{n-1} < \sqrt{x_{n-1} + 1} - 1 = x_n$. Moreover, $\sqrt{x_{n-1} + 1} - 1 \leq 1 - 1 = 0$. Hence by induction, x_n is increasing and bounded above by 0. It follows from the Monotone Convergence Theorem that $x_n \to a$ as $n \to \infty$. Taking the limit of $\sqrt{x_{n-1} + 1} - 1 = x_n$ we see that $a^2 + a = 0$, i.e., $a = -1, 0$. Since x_n increases from $x_0 > -1$, the limit is 0. If $x_0 = -1$, then $x_n = -1$ for all n. If $x_0 = 0$, then $x_n = 0$ for all n.

Finally, it is easy to verify that if $x_0 = \ell$ for $\ell = -1$ or 0, then $x_n = \ell$ for all n, hence $x_n \to \ell$ as $n \to \infty$.

2.3.2. If $x_1 = 0$ then $x_n = 0$ for all n, hence converges to 0. If $0 < x_1 < 1$, then by 1.4.1c, x_n is decreasing and bounded below. Thus the limit, a, exists by the Monotone Convergence Theorem. Taking the limit of $x_{n+1} = 1 - \sqrt{1 - x_n}$, as $n \to \infty$, we have $a = 1 - \sqrt{1 - a}$, i.e., $a = 0, 1$. Since $x_1 < 1$, the limit must be zero.

Finally,

$$\frac{x_{n+1}}{x_n} = \frac{1 - \sqrt{1 - x_n}}{x_n} = \frac{1 - (1 - x_n)}{x_n(1 + \sqrt{1 - x_n})} \to \frac{1}{1+1} = \frac{1}{2}.$$

2.3.3. *Case 1.* $x_0 = 2$. Then $x_n = 2$ for all n, so the limit is 2.

Case 2. $2 < x_0 < 3$. Suppose that $2 < x_{n-1} \leq 3$ for some $n \geq 1$. Then $0 < x_{n-1} - 2 \leq 1$ so $\sqrt{x_{n-1} - 2} \geq x_{n-1} - 2$, i.e., $x_n = 2 + \sqrt{x_{n-1} - 2} \geq x_{n-1}$. Moreover, $x_n = 2 + \sqrt{x_{n-1} - 2} \leq 2 + 1 = 3$. Hence by induction, x_n is increasing and bounded above by 3. It follows from the Monotone Convergence Theorem that $x_n \to a$ as $n \to \infty$. Taking the limit of $2 + \sqrt{x_{n-1} - 2} = x_n$ we see that $a^2 - 5a + 6 = 0$, i.e., $a = 2, 3$. Since x_n increases from $x_0 > 2$, the limit is 3.

Case 3. $x_0 \geq 3$. Suppose that $x_{n-1} \geq 3$ for some $n \geq 1$. Then $x_{n-1} - 2 \geq 1$ so $\sqrt{x_{n-1} - 2} \leq x_{n-1} - 2$, i.e., $x_n = 2 + \sqrt{x_{n-1} - 2} \leq x_{n-1}$. Moreover, $x_n = 2 + \sqrt{x_{n-1} - 2} \geq 2 + 1 = 3$. Hence by induction, x_n is decreasing and bounded above by 3. By repeating the steps in Case 2, we conclude that x_n decreases from $x_0 \geq 3$ to the limit 3.

2.3.4. *Case 1.* $x_0 < 1$. Suppose $x_{n-1} < 1$. Then

$$x_{n-1} = \frac{2x_{n-1}}{2} < \frac{1 + x_{n-1}}{2} = x_n < \frac{2}{2} = 1.$$

Thus $\{x_n\}$ is increasing and bounded above, so $x_n \to x$. Taking the limit of $x_n = (1 + x_{n-1})/2$ as $n \to \infty$, we see that $x = (1 + x)/2$, i.e., $x = 1$.

Case 2. $x_0 \geq 1$. If $x_{n-1} \geq 1$ then

$$1 = \frac{2}{2} \leq \frac{1 + x_{n-1}}{2} = x_n \leq \frac{2x_{n-1}}{2} = x_{n-1}.$$

Thus $\{x_n\}$ is decreasing and bounded below. Repeating the argument in Case 1, we conclude that $x_n \to 1$ as $n \to \infty$.

2.3.5. The result is obvious when $x = 0$. If $x > 0$ then by Example 2.2 and Theorem 2.6,

$$\lim_{n\to\infty} x^{1/(2n-1)} = \lim_{m\to\infty} x^{1/m} = 1.$$

If $x < 0$ then since $2n - 1$ is odd, we have by the previous case that $x^{1/(2n-1)} = -(-x)^{1/(2n-1)} \to -1$ as $n \to \infty$.

2.3.6. a) Suppose that $\{x_n\}$ is increasing. If $\{x_n\}$ is bounded above, then there is an $x \in \mathbf{R}$ such that $x_n \to x$ (by the Monotone Convergence Theorem). Otherwise, given any $M > 0$ there is an $N \in \mathbf{N}$ such that $x_N > M$. Since $\{x_n\}$ is increasing, $n \geq N$ implies $x_n \geq x_N > M$. Hence $x_n \to \infty$ as $n \to \infty$.

b) If $\{x_n\}$ is decreasing, then $-x_n$ is increasing, so part a) applies.

2.3.7. Choose by the Approximation Property an $x_1 \in E$ such that $\sup E - 1 < x_1 \leq \sup E$. Since $\sup E \notin E$, we also have $x_1 < \sup E$. Suppose $x_1 < x_2 < \cdots < x_n$ in E have been chosen so that $\sup E - 1/n < x_n < \sup E$. Choose by the Approximation Property an $x_{n+1} \in E$ such that $\max\{x_n, \sup E - 1/(n + 1)\} < x_{n+1} \leq \sup E$. Then $\sup E - 1/(n + 1) < x_{n+1} < \sup E$ and $x_n < x_{n+1}$. Thus by induction, $x_1 < x_2 < \dots$ and by the Squeeze Theorem, $x_n \to \sup E$ as $n \to \infty$.

2.3.8. a) This follows immediately from Exercise 1.2.6.

b) By a), $x_{n+1} = (x_n + y_n)/2 < 2x_n/2 = x_n$. Thus $y_{n+1} < x_{n+1} < \cdots < x_1$. Similarly, $y_n = \sqrt{y_n^2} < \sqrt{x_n y_n} = y_{n+1}$ implies $x_{n+1} > y_{n+1} > y_n \cdots > y_1$. Thus $\{x_n\}$ is decreasing and bounded below by y_1 and $\{y_n\}$ is increasing and bounded above by x_1.

c) By b),

$$x_{n+1} - y_{n+1} = \frac{x_n + y_n}{2} - \sqrt{x_n y_n} < \frac{x_n + y_n}{2} - y_n = \frac{x_n - y_n}{2}.$$

Hence by induction and a), $0 < x_{n+1} - y_{n+1} < (x_1 - y_1)/2^n$.

d) By b), there exist $x, y \in \mathbf{R}$ such that $x_n \downarrow x$ and $y_n \uparrow y$ as $n \to \infty$. By c), $|x - y| \leq (x_1 - y_1) \cdot 0 = 0$. Hence $x = y$.

2.3.9. Since $x_0 = 1$ and $y_0 = 0$,

$$\begin{aligned} x_{n+1}^2 - 2y_{n+1}^2 &= (x_n + 2y_n)^2 - 2(x_n + y_n)^2 \\ &= -x_n^2 + 2y_n^2 = \cdots = (-1)^n(x_0 - 2y_0) = (-1)^n. \end{aligned}$$

Notice that $x_1 = 1 = y_1$. If $y_{n-1} \geq n - 1$ and $x_{n-1} \geq 1$ then $y_n = x_{n-1} + y_{n-1} \geq 1 + (n - 1) = n$ and $x_n = x_{n-1} + 2y_{n-1} \geq 1$. Thus $1/y_n \to 0$ as $n \to \infty$ and $x_n \geq 1$ for all $n \in \mathbf{N}$. Since

$$\left|\frac{x_n^2}{y_n^2} - 2\right| = \left|\frac{x_n^2 - 2y_n^2}{y_n^2}\right| = \frac{1}{y_n^2} \to 0$$

as $n \to \infty$, it follows that $x_n/y_n \to \pm\sqrt{2}$ as $n \to \infty$. Since $x_n, y_n > 0$, the limit must be $\sqrt{2}$.

2.3.10. a) Notice $x_0 > y_0 > 1$. If $x_{n-1} > y_{n-1} > 1$ then $y_{n-1}^2 - x_{n-1}y_{n-1} = y_{n-1}(y_{n-1} - x_{n-1}) > 0$ so $y_{n-1}(y_{n-1} + x_{n-1}) < 2x_{n-1}y_{n-1}$. In particular,

$$x_n = \frac{2x_{n-1}y_{n-1}}{x_{n-1} + y_{n-1}} > y_{n-1}.$$

It follows that $\sqrt{x_n} > \sqrt{y_{n-1}} > 1$, so $x_n > \sqrt{x_n y_{n-1}} = y_n > 1 \cdot 1 = 1$. Hence by induction, $x_n > y_n > 1$ for all $n \in \mathbf{N}$.

Now $y_n < x_n$ implies $2y_n < x_n + y_n$. Thus

$$x_{n+1} = \frac{2x_n y_n}{x_n + y_n} < x_n.$$

Hence, $\{x_n\}$ is decreasing and bounded below (by 1). Thus by the Monotone Convergence Theorem, $x_n \to x$ for some $x \in \mathbf{R}$.

On the other hand, y_{n+1} is the geometric mean of x_{n+1} and y_n, so by Exercise 1.2.6, $y_{n+1} \geq y_n$. Since y_n is bounded above (by x_0), we conclude that $y_n \to y$ as $n \to \infty$ for some $y \in \mathbf{R}$.

b) Let $n \to \infty$ in the identity $y_{n+1} = \sqrt{x_{n+1}y_n}$. We obtain, from part a), $y = \sqrt{xy}$, i.e., $x = y$. A direct calculation yields $y_6 > 3.141557494$ and $x_7 < 3.14161012$.

2.4 Cauchy sequences.

2.4.0. a) False. $a_n = 1$ is Cauchy and $b_n = (-1)^n$ is bounded, but $a_n b_n = (-1)^n$ does not converge, hence cannot be Cauchy by Theorem 2.29.

b) False. $a_n = 1$ and $b_n = 1/n$ are Cauchy, but $a_n/b_n = n$ does not converge, hence cannot be Cauchy by Theorem 2.29.

c) True. If $(a_n + b_n)^{-1}$ converged to 0, then given any $M \in \mathbf{R}$, $M \neq 0$, there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $|a_n + b_n|^{-1} < 1/|M|$. It follows that $n \geq N$ implies $|a_n + b_n| > |M| > 0 > M$. In particular, $|a_n + b_n|$ diverges to ∞. But if a_n and b_n are Cauchy, then by Theorem 2.29, $a_n + b_n \to x$ where $x \in \mathbf{R}$. Thus $|a_n + b_n| \to |x|$, NOT ∞.

d) False. If $x_{2^k} = \log k$ and $x_n = 0$ for $n \neq 2^k$, then $x_{2^k} - x_{2^{k-1}} = \log(k/(k-1)) \to 0$ as $k \to \infty$, but x_k does not converge, hence cannot be Cauchy by Theorem 2.29.

2.4.1. Since $(2n^2 + 3)/(n^3 + 5n^2 + 3n + 1) \to 0$ as $n \to \infty$, it follows from the Squeeze Theorem that $x_n \to 0$ as $n \to \infty$. Hence by Theorem 2.29, x_n is Cauchy.

2.4.2. If x_n is Cauchy, then there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - x_N| < 1$. Since $x_n - x_N \in \mathbf{Z}$, it follows that $x_n = x_N$ for all $n \geq N$. Thus set $a := x_N$.

2.4.3. Suppose x_n and y_n are Cauchy and let $\varepsilon > 0$.

a) If $\alpha = 0$, then $\alpha x_n = 0$ for all $n \in \mathbf{N}$, hence is Cauchy. If $\alpha \neq 0$, then there is an $N \in \mathbf{N}$ such that $n, m \geq N$ implies $|x_n - x_m| < \varepsilon/|\alpha|$. Hence

$$|\alpha x_n - \alpha x_m| \leq |\alpha|\, |x_n - x_m| < \varepsilon$$

for $n, m \geq N$.

b) There is an $N \in \mathbf{N}$ such that $n, m \geq N$ implies $|x_n - x_m|$ and $|y_n - y_m|$ are $< \varepsilon/2$. Hence

$$|x_n + y_n - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m| < \varepsilon$$

for $n, m \geq N$.

c) By repeating the proof of Theorem 2.8, we can show that every Cauchy sequence is bounded. Thus choose $M > 0$ such that $|x_n|$ and $|y_n|$ are both $\leq M$ for all $n \in \mathbf{N}$. There is an $N \in \mathbf{N}$ such that $n, m \geq N$ implies $|x_n - x_m|$ and $|y_n - y_m|$ are both $< \varepsilon/(2M)$. Hence

$$|x_n y_n - (x_m y_m)| \leq |x_n - x_m|\, |y_m| + |x_n|\, |y_n - y_m| < \varepsilon$$

for $n, m \geq N$.

2.4.4. Let $s_n = \sum_{k=1}^{n-1} x_k$ for $n = 2, 3, \ldots$. If $m > n$ then $s_{m+1} - s_n = \sum_{k=n}^{m} x_k$. Therefore, s_n is Cauchy by hypothesis. Hence s_n converges by Theorem 2.29.

2.4.5. Let $x_n = \sum_{k=1}^{n}(-1)^k/k$ for $n \in \mathbf{N}$. Suppose n and m are even and $m > n$. Then

$$S := \sum_{k=n}^{m} \frac{(-1)^k}{k} \equiv \frac{1}{n} - \left(\frac{1}{n+1} - \frac{1}{n+2}\right) - \cdots - \left(\frac{1}{m-1} - \frac{1}{m}\right).$$

Each term in parentheses is positive, so the absolute value of S is dominated by $1/n$. Similar arguments prevail for all integers n and m. Since $1/n \to 0$ as $n \to \infty$, it follows that x_n satisfies the hypotheses of Exercise 2.4.4. Hence x_n must converge to a finite real number.

2.4.6. By Exercise 1.4.4c, if $m \ge n$ then

$$|x_{m+1} - x_n| = |\sum_{k=n}^{m}(x_{k+1} - x_k)| \le \sum_{k=n}^{m} \frac{1}{a^k} = \left(1 - \frac{1}{a^m} - (1 - \frac{1}{a^n})\right)\frac{1}{a-1}.$$

Thus $|x_{m+1} - x_n| \le (1/a^n - 1/a^m)/(a-1) \to 0$ as $n, m \to \infty$ since $a > 1$. Hence $\{x_n\}$ is Cauchy and must converge by Theorem 2.29.

2.4.7. a) Suppose a is a cluster point for some set E and let $r > 0$. Since $E \cap (a-r, a+r)$ contains infinitely many points, so does $E \cap (a-r, a+r) \setminus \{a\}$. Hence this set is nonempty. Conversely, if $E \cap (a-s, a+s) \setminus \{a\}$ is always nonempty for all $s > 0$ and $r > 0$ is given, choose $x_1 \in E \cap (a-r, a+r)$. If distinct points $x_1, \dots, x_k$ have been chosen so that $x_k \in E \cap (a-r, a+r)$ and $s := \min\{|x_1 - a|, \dots, |x_k - a|\}$, then by hypothesis there is an $x_{k+1} \in E \cap (a-s, a+s)$. By construction, x_{k+1} does not equal any x_j for $1 \le j \le k$. Hence $x_1, \dots, x_{k+1}$ are distinct points in $E \cap (a-r, a+r)$. By induction, there are infinitely many points in $E \cap (a-r, a+r)$.

b) If E is a bounded infinite set, then it contains distinct points $x_1, x_2, \dots$. Since $\{x_n\} \subseteq E$, it is bounded. It follows from the Bolzano–Weierstrass Theorem that x_n contains a convergent subsequence, i.e., there is an $a \in \mathbf{R}$ such that given $r > 0$ there is an $N \in \mathbf{N}$ such that $k \ge N$ implies $|x_{n_k} - a| < r$. Since there are infinitely many x_{n_k}'s and they all belong to E, a is by definition a cluster point of E.

2.4.8. a) To show $E := [a, b]$ is sequentially compact, let $x_n \in E$. By the Bolzano–Weierstrass Theorem, x_n has a convergent subsequence, i.e., there is an $x_0 \in \mathbf{R}$ and integers n_k such that $x_{n_k} \to x_0$ as $k \to \infty$. Moreover, by the Comparison Theorem, $x_n \in E$ implies $x_0 \in E$. Thus E is sequentially compact by definition.

b) $(0, 1)$ is bounded and $1/n \in (0, 1)$ has no convergent subsequence with limit in $(0, 1)$.

c) $[0, \infty)$ is closed and $n \in [0, \infty)$ is a sequence which has no convergent subsequence.

2.5 Limits supremum and infimum.

2.5.1. a) Since $3 - (-1)^n = 2$ when n is even and 4 when n is odd, $\limsup_{n\to\infty} x_n = 4$ and $\liminf_{n\to\infty} x_n = 2$.

b) Since $\cos(n\pi/2) = 0$ if n is odd, 1 if $n = 4m$ and -1 if $n = 4m+2$, $\limsup_{n\to\infty} x_n = 1$ and $\liminf_{n\to\infty} x_n = -1$.

c) Since $(-1)^{n+1} + (-1)^n/n = -1 + 1/n$ when n is even and $1 - 1/n$ when n is odd, $\limsup_{n\to\infty} x_n = 1$ and $\liminf_{n\to\infty} x_n = -1$.

d) Since $x_n \to 1/2$ as $n \to \infty$, $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = 1/2$ by Theorem 2.36.

e) Since $|y_n| \le M$, $|y_n/n| \le M/n \to 0$ as $n \to \infty$. Therefore, $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = 0$ by Theorem 2.36.

f) Since $n(1 + (-1)^n) + n^{-1}((-1)^n - 1) = 2n$ when n is even and $-2/n$ when n is odd, $\limsup_{n\to\infty} x_n = \infty$ and $\liminf_{n\to\infty} x_n = 0$.

g) Clearly $x_n \to \infty$ as $n \to \infty$. Therefore, $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \infty$ by Theorem 2.36.

2.5.2. By Theorem 1.20,

$$\liminf_{n\to\infty}(-x_n) := \lim_{n\to\infty}(\inf_{k\ge n}(-x_k)) = -\lim_{n\to\infty}(\sup_{k\ge n} x_k) = -\limsup_{n\to\infty} x_n.$$

A similar argument establishes the second identity.

2.5.3. a) Since $\lim_{n\to\infty}(\sup_{k\ge n} x_k) < r$, there is an $N \in \mathbf{N}$ such that $\sup_{k\ge N} x_k < r$, i.e., $x_k < r$ for all $k \ge N$.

b) Since $\lim_{n\to\infty}(\sup_{k\ge n} x_k) > r$, there is an $N \in \mathbf{N}$ such that $\sup_{k\ge N} x_k > r$, i.e., there is a $k_1 \in \mathbf{N}$ such that $x_{k_1} > r$. Suppose $k_\nu \in \mathbf{N}$ have been chosen so that $k_1 < k_2 < \cdots < k_j$ and $x_{k_\nu} > r$ for $\nu = 1, 2, \dots, j$. Choose $N > k_j$ such that $\sup_{k\ge N} x_k > r$. Then there is a $k_{j+1} > N > k_j$ such that $x_{k_{j+1}} > r$. Hence by induction, there are distinct natural numbers $k_1, k_2, \dots$ such that $x_{k_j} > r$ for all $j \in \mathbf{N}$.

2.5.4. a) Since $\inf_{k\ge n} x_k + \inf_{k\ge n} y_k$ is a lower bound of $x_j + y_j$ for any $j \ge n$, we have $\inf_{k\ge n} x_k + \inf_{k\ge n} y_k \le \inf_{j\ge n}(x_j + y_j)$. Taking the limit of this inequality as $n \to \infty$, we obtain

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \le \liminf_{n\to\infty}(x_n + y_n).$$

Note, we used Corollary 1.16 and the fact that the sum on the left is not of the form $\infty - \infty$. Similarly, for each $j \ge n$,

$$\inf_{k\ge n}(x_k + y_k) \le x_j + y_j \le \sup_{k\ge n} x_k \ + y_j.$$

Taking the infimum of this inequality over all $j \ge n$, we obtain $\inf_{k\ge n}(x_k+y_k) \le \sup_{k\ge n} x_k + \inf_{j\ge n} y_j$. Therefore,

$$\liminf_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty} x_n + \liminf_{n\to\infty} y_n.$$

The remaining two inequalities follow from Exercise 2.5.2. For example,

$$\begin{aligned}\limsup_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &= -\liminf_{n\to\infty}(-x_n) - \limsup_{n\to\infty}(-y_n)\\ &\le -\liminf_{n\to\infty}(-x_n - y_n) = \limsup_{n\to\infty}(x_n + y_n).\end{aligned}$$

b) It suffices to prove the first identity. By Theorem 2.36 and a),

$$\lim_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \le \liminf_{n\to\infty}(x_n + y_n).$$

To obtain the reverse inequality, notice by the Approximation Property that for each $n \in \mathbf{N}$ there is a $j_n > n$ such that $\inf_{k\ge n}(x_k + y_k) > x_{j_n} - 1/n + y_{j_n}$. Hence

$$\inf_{k\ge n}(x_k + y_k) > x_{j_n} - \frac{1}{n} + \inf_{k\ge n} y_k$$

for all $n \in \mathbf{N}$. Taking the limit of this inequality as $n \to \infty$, we obtain

$$\liminf_{n\to\infty}(x_n + y_n) \ge \lim_{n\to\infty} x_n + \liminf_{n\to\infty} y_n.$$

c) Let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. Then the limits infimum are both -1, the limits supremum are both 1, but $x_n + y_n = 0 \to 0$ as $n \to \infty$. If $x_n = (-1)^n$ and $y_n = 0$ then

$$\liminf_{n\to\infty}(x_n + y_n) = -1 < 1 = \limsup_{n\to\infty} x_n + \liminf_{n\to\infty} y_n.$$

2.5.5. a) For any $j \ge n$, $x_j \le \sup_{k\ge n} x_k$ and $y_j \le \sup_{k\ge n} y_k$. Multiplying these inequalities, we have $x_j y_j \le (\sup_{k\ge n} x_k)(\sup_{k\ge n} y_k)$, i.e.,

$$\sup_{j\ge n} x_j y_j \le (\sup_{k\ge n} x_k)(\sup_{k\ge n} y_k).$$

Taking the limit of this inequality as $n \to \infty$ establishes a). The inequality can be strict because if

$$x_n = 1 - y_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd}\end{cases}$$

then $\limsup_{n\to\infty}(x_n y_n) = 0 < 1 = (\limsup_{n\to\infty} x_n)(\limsup_{n\to\infty} y_n)$.

b) By a),

$$\liminf_{n\to\infty}(x_n y_n) = -\limsup_{n\to\infty}(-x_n y_n) \ge -\limsup_{n\to\infty}(-x_n)\limsup_{n\to\infty} y_n = \liminf_{n\to\infty} x_n \limsup_{n\to\infty} y_n.$$

2.5.6. *Case 1.* $x = \infty$. By hypothesis, $C := \limsup_{n\to\infty} y_n > 0$. Let $M > 0$ and choose $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n \ge 2M/C$ and $\sup_{n\ge N} y_n > C/2$. Then $\sup_{k\ge N}(x_k y_k) \ge x_n y_n \ge (2M/C) y_n$ for any $n \ge N$ and $\sup_{k\ge N}(x_k y_k) \ge (2M/C)\sup_{n\ge N} y_n > M$. Therefore, $\limsup_{n\to\infty}(x_n y_n) = \infty$.

Case 2. $0 \le x < \infty$. By Exercise 2.5.6a and Theorem 2.36,

$$\limsup_{n\to\infty}(x_n y_n) \le (\limsup_{n\to\infty} x_n)(\limsup_{n\to\infty} y_n) = x \limsup_{n\to\infty} y_n.$$

On the other hand, given $\epsilon > 0$ choose $n \in \mathbf{N}$ so that $x_k > x - \epsilon$ for $k \ge n$. Then $x_k y_k \ge (x-\epsilon) y_k$ for each $k \ge n$, i.e., $\sup_{k\ge n}(x_k y_k) \ge (x-\epsilon)\sup_{k\ge n} y_k$. Taking the limit of this inequality as $n \to \infty$ and as $\epsilon \to 0$, we obtain

$$\limsup_{n\to\infty}(x_n y_n) \ge x \limsup_{n\to\infty} y_n.$$

2.5.7. It suffices to prove the first identity. Let $s = \inf_{n\in\mathbf{N}}(\sup_{k\ge n} x_k)$.
Case 1. $s = \infty$. Then $\sup_{k\ge n} x_k = \infty$ for all $n \in \mathbf{N}$ so by definition,

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}(\sup_{k\ge n} x_k) = \infty = s.$$

Case 2. $s = -\infty$. Let $M > 0$ and choose $N \in \mathbf{N}$ such that $\sup_{k\ge N} x_k \le -M$. Then $\sup_{k\ge n} x_k \le \sup_{k\ge N} x_k \le -M$ for all $n \ge N$, i.e., $\limsup_{n\to\infty} x_n = -\infty$.
Case 3. $-\infty < s < -\infty$. Let $\epsilon > 0$ and use the Approximation Property to choose $N \in \mathbf{N}$ such that $\sup_{k\ge N} x_k < s + \epsilon$. Since $\sup_{k\ge n} x_k \le \sup_{k\ge N} x_k < s+\epsilon$ for all $n \ge N$, it follows that

$$s - \epsilon < s \le \sup_{k\ge n} x_k < s + \epsilon$$

for $n \ge N$, i.e., $\limsup_{n\to\infty} x_n = s$.

2.5.8. It suffices to establish the first identity. Let $s = \liminf_{n\to\infty} x_n$.
Case 1. $s = 0$. Then by Theorem 2.35 there is a subsequence k_j such that $x_{k_j} \to 0$, i.e., $1/x_{k_j} \to \infty$ as $j \to \infty$. In particular, $\sup_{k\ge n}(1/x_k) = \infty$ for all $n \in \mathbf{N}$, i.e., $\limsup_{n\to\infty}(1/x_n) = \infty = 1/s$.
Case 2. $s = \infty$. Then $x_k \to \infty$, i.e., $1/x_k \to 0$, as $k \to \infty$. Thus by Theorem 2.36, $\limsup_{n\to\infty}(1/x_n) = 0 = 1/s$.
Case 3. $0 < s < \infty$. Fix $j \ge n$. Since $1/\inf_{k\ge n} x_k \ge 1/x_j$ implies $1/\inf_{k\ge n} x_k \ge \sup_{j\ge n}(1/x_j)$, it is clear that $1/s \ge \limsup_{n\to\infty}(1/x_n)$. On the other hand, given $\epsilon > 0$ and $n \in \mathbf{N}$, choose $j > N$ such that $\inf_{k\ge n} x_k + \epsilon > x_j$, i.e., $1/(\inf_{k\ge n} x_k + \epsilon) < 1/x_j \le \sup_{k\ge n}(1/x_k)$. Taking the limit of this inequality as $n \to \infty$ and as $\epsilon \to 0$, we conclude that $1/s \le \limsup_{n\to\infty}(1/x_n)$.

2.5.9. If $x_n \to 0$, then $|x_n| \to 0$. Thus by Theorem 2.36, $\limsup_{n\to\infty}|x_n| = 0$. Conversely, if $\limsup_{n\to\infty}|x_n| \le 0$, then

$$0 \le \liminf_{n\to\infty}|x_n| \le \limsup_{n\to\infty}|x_n| \le 0,$$

implies that the limits supremum and infimum of $|x_n|$ are equal (to zero). Hence by Theorem 2.36, the limit exists and equals zero.