Chapter 2

1. $P\{X = 0\} = \binom{7}{2}\Big/\binom{10}{2} = \frac{14}{30}$

2. $-n, -n+2, -n+4, \ldots, n-2, n$

3. $P\{X=-2\} = \frac{1}{4} = P\{X=2\}$

 $P\{X=0\} = \frac{1}{2}$

4. (a) 1, 2, 3, 4, 5, 6
 (b) 1, 2, 3, 4, 5, 6
 (c) 2,3,…,11,12
 (d) $-5, -4, \ldots, 4, 5$

5. $P\{\max = 6\} = \frac{11}{36} = P\{\min = 1\}$

 $P\{\max = 5\} = \frac{1}{4} = P\{\min = 2\}$

 $P\{\max = 4\} = \frac{7}{36} = P\{\min = 3\}$

 $P\{\max = 3\} = \frac{5}{36} = P\{\min = 4\}$

 $P\{\max = 2\} = \frac{1}{12} = P\{\min = 5\}$

 $P\{\max = 1\} = \frac{1}{36} = P\{\min = 6\}$

6. (H, H, H, H, H), p^5 if $p = P\{\text{heads}\}$

7. $p(0) = (.3)^3 = .027$

 $p(1) = 3(.3)^2(.7) = .189$

 $p(2) = 3(.3)(.7)^2 = .441$

 $p(3) = (.7)^3 = .343$

8. $p(0) = \frac{1}{2}, p(1) = \frac{1}{2}$

9. $p(0) = \frac{1}{2}, p(1) = \frac{1}{10}, p(2) = \frac{1}{5}, p(3) = \frac{1}{10}, p(3.5) = \frac{1}{10}$

10. $1 - \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) - \binom{3}{3}\left(\frac{1}{6}\right)^3 = \frac{200}{216}$

11. $\frac{3}{8}$

12. $\binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) + \binom{5}{5}\left(\frac{1}{3}\right)^5 = \frac{10+1}{243} = \frac{11}{243}$

13. $\sum_{i=7}^{10} \binom{10}{i} \left[\frac{1}{2}\right]^{10}$

14. $P\{X = 0\} = P\{X = 6\} = \left[\frac{1}{2}\right]^6 = \frac{1}{64}$

$P\{X = 1\} = P\{X = 5\} = 6\left[\frac{1}{2}\right]^6 = \frac{6}{64}$

$P\{X = 2\} = P\{X = 4\} = \binom{6}{2}\left[\frac{1}{2}\right]^6 = \frac{15}{64}$

$P\{X = 3\} = \binom{6}{3}\left[\frac{1}{2}\right]^6 = \frac{20}{64}$

15. $$\frac{P\{X = k\}}{P\{X = k-1\}} = \frac{\frac{n!}{(n-k)!k!}p^k(1-p)^{n-k}}{\frac{n!}{(n-k+1)!(k-1)!}p^{k-1}(1-p)^{n-k+1}} = \frac{n-k+1}{k}\,\frac{p}{1-p}$$

Hence,

$$\frac{P\{X = k\}}{P\{X = k-1\}} \geq 1 \leftrightarrow (n-k+1)p > k(1-p)$$
$$\leftrightarrow (n+1)p \geq k$$

The result follows.

16. $1 - (.95)^{52} - 52(.95)^{51}(.05)$

17. Follows since there are $\frac{n!}{x_1!\cdots x_r!}$ permutations of n objects of which x_1 are alike, x_2 are alike, …, x_r are alike.

18. (a) $P(X_i = x_i,\ i = 1, \ldots, r-1 \mid X_r = j)$

$$= \frac{P(X_i = x_i, i = 1,\ldots,r-1, X_r = j)}{P(X_r = j)}$$

$$= \frac{\frac{n!}{x_1!\cdots x_{r-1}!j!}p_1^{x_1}\cdots p_{r-1}^{x_{r-1}}p_r^j}{\frac{n!}{j!(n-j)!}p_r^j(1-p_r)^{n-j}}$$

$$= \frac{(n-j)!}{x_1!\cdots x_{r-1}!}\prod_{i=1}^{r-1}\left(\frac{p_i}{1-p_r}\right)^{x_i}$$

(b) The conditional distribution of $X_1, \ldots, X_{r-1}$ given that $X_r = j$ is multinomial with parameters $n-j, \frac{p_i}{1-p_r}, i = 1,\ldots,r-1$.

(c) The preceding is true because given that $X_r = j$, each of the $n-j$ trials that did not result in outcome r resulted in outcome i with probability $\frac{p_i}{1-p_r}$, $i = 1, \ldots, r-1$.

19. $P\{X_1 + \cdots + X_k = m\}$

$$= \binom{n}{m}(p_1 + \cdots + p_k)^m(p_{k+1} + \cdots + p_r)^{n-m}$$

20. $\dfrac{5!}{2!1!2!}\left[\dfrac{1}{5}\right]^2\left[\dfrac{3}{10}\right]^2\left[\dfrac{1}{2}\right]^1 = .054$

21. $1 - \left[\dfrac{3}{10}\right]^5 - 5\left[\dfrac{3}{10}\right]^4\left[\dfrac{7}{10}\right] - \left[\dfrac{5}{2}\right]\left[\dfrac{3}{10}\right]^3\left[\dfrac{7}{10}\right]^2$

22. $\dfrac{1}{32}$

23. In order for X to equal n, the first $n-1$ flips must have $r-1$ heads, and then the nth flip must land heads. By independence the desired probability is thus

$$\left[\begin{matrix} n-1 \\ r-1 \end{matrix}\right] p^{r-1}(1-p)^{n-r}\, xp$$

24. It is the number of tails before heads appears for the r th time.

25. A total of 7 games will be played if the first 6 result in 3 wins and 3 losses. Thus,

$$P\{7 \text{ games}\} = \binom{6}{3} p^3(1-p^3)$$

Differentiation yields

$$\begin{aligned}\frac{d}{dp}P\{7\} &= 20\left[3p^2(1-p)^3 - p^3 3(1-p)^2\right] \\ &= 60\, p^2(1-p)^2[1-2\,p]\end{aligned}$$

Thus, the derivative is zero when $p = 1/2$. Taking the second derivative shows that the maximum is attained at this value.

26. Let X denote the number of games played.

(a) $P\{X=2\} = p^2 + (1-p)^2$

$P\{X=3\} = 2p(1-p)$

$$\begin{aligned}E[X] &= 2\left\{p^2 + (1-p)^2\right\} + 6p(1-p) \\ &= 2 + 2p(1-p)\end{aligned}$$

Since $p(1-p)$ is maximized when $p = 1/2$, we see that $E[X]$ is maximized at that value of p.

(b) $P\{X=3\} = p^3 + (1-p)^3$

$P\{X=4\}$

$= P\{X = 4, \text{I has 2 wins in first 3 games}\}$

$+ P\{X = 4, \text{II has 2 wins in first3 games}\}$

$= 3p^2(1-p)p + 3p(1-p)^2(1-p)$

$P\{X=5\}$

$= P\{\text{each player has 2 wins in the first 4 games}\}$

$= 6p^2(1-p)^2$

$$E[X] = 3\left[p^3 + (1-p)^3\right] + 12\,p(1-p)\left[p^2 + (1-p)^2\right] + 30\,p^2(1-p)^2$$

Differentiating and setting equal to 0 shows that the maximum is attained when $p = 1/2$.

27. $P\{\text{same number of heads}\} = \sum_i P\{A = i, B = i\}$

$$= \sum_i \binom{k}{i}(1/2)^k \binom{n-k}{i}(1/2)^{n-k}$$

$$= \sum_i \binom{k}{i}\binom{n-k}{i}(1/2)^n$$

$$= \sum_i \binom{k}{k-i}\binom{n-k}{i}(1/2)^n$$

$$= \binom{n}{k}(1/2)^n$$

Another argument is as follows:

$P\{\#\text{ heads of } A = \#\text{ heads of } B\}$

$= P\{\#\text{ tails of } A = \#\text{ heads of } B\}$

since coin is fair

$= P\{k - \#\text{ heads of } A = \#\text{ heads of } B\} =$

$P\{k = \text{total \# heads}\}$

28. (a) Consider the first time that the two coins give different results. Then

$$P\{X = 0\} = P\{(t,h) \mid (t,h) \text{ or } (h,t)\} = \frac{p(1-p)}{2p(1-p)} = \frac{1}{2}$$

(b) No, with this procedure

$$P\{X = 0\} = P\,\{\text{first flip is a tail}\} = 1 - p$$

29. Each flip after the first will, independently, result in a changeover with probability 1/2. Therefore,

$$P\{k \text{ changeovers}\} = \binom{n-1}{k}(1/2)^{n-1}$$

30. $$\frac{P\{X = i\}}{P\{X = i-1\}} = \frac{e^{-\lambda}\lambda^i/i!}{e^{-\lambda}\lambda^{i-1}/(i-1)!} = \lambda/i$$

Hence, $P\{X = i\}$ is increasing for $\lambda \geq i$ and decreasing for $\lambda < i$.

32. (a) .394 (b) .303 (c) .091

33. $c\int_{-1}^{1}(1-x^2)\,dx = 1$

$$c\left[x-\frac{x^3}{3}\right]_{-1}^{1}=1$$

$$c=\frac{3}{4}$$

$$F(y)=\frac{3}{4}\int_{-1}^{1}(1-x^2)\,dx$$

$$=\frac{3}{4}\left[y-\frac{y^3}{3}+\frac{2}{3}\right],\quad -1<y<1$$

34. $c\int_{0}^{2}\left(4x-2x^2\right)dx=1$

$$c(2x^2-2x^3/3)=1$$

$$8c/3=1$$

$$c=\frac{3}{8}$$

$$P\left\{\frac{1}{2}<X<\frac{3}{2}\right\}=\frac{3}{8}\int_{1/2}^{3/2}\left(4x-2x^2\right)dx$$

$$=\frac{11}{16}$$

35. $P\{X>20\}=\int_{20}^{\infty}\frac{10}{x^2}\,dx=\frac{1}{2}$

36. $P\{D\le x\}=\dfrac{\text{area of disk of radius } x}{\text{area of disk of radius } 1}$

$$=\frac{\pi x^2}{\pi}=x^2$$

37. $P\{M\le x\}=P\{\max(X_1,\ldots,X_n)\le x\}$

$$=P\{X_1\le x,\ldots,X_n\le x\}$$

$$=\prod_{i=1}^{n}P\{X_i\le x\}$$

$$=x^n$$

$$f_M(x)=\frac{d}{dx}p\{M\le x\}=nx^{n-1}$$

38. $c=2$

39. $E[X]=\dfrac{31}{6}$

40. Let X denote the number of games played.

$$
\begin{aligned}
P\{X = 4\} &= p^4 + (1 - p)^4 \\
P\{X = 5\} &= P\{X = 5, \text{ I wins 3 of first 4}\} \\
&\quad + P\{X = 5, \text{ II wins 3 of first 4}\} \\
&= 4p^3(1 - p)p + 4(1 - p)^3 p(1 - p) \\
P\{X = 6\} &= P\{X = 6, \text{ I wins 3 of first 5}\} \\
&\quad + P\{X = 6, \text{ II wins 3 of first 5}\} \\
&= 10\, p^3(1 - p)^2\, p + 10\, p^2(1 - p)^3(1 - p) \\
P\{X = 7\} &= P\{\text{first 6 games are split}\} \\
&= 20\, p^3(1 - p)^3
\end{aligned}
$$

$$E[X] = \sum_{i=4}^{7} iP\{X = i\}$$

When $p = 1/2$, $E\,[X] = 93/16 = 5.8125$

41. Let X_i equal 1 if a changeover results from the i th flip and let it be 0 otherwise. Then

$$\text{number of changeovers} = \sum_{i=2}^{n} X_i$$

As,

$$
\begin{aligned}
E[X_i] &= P\{X_i = 1\} = P\{\text{flip } i - 1 = \text{flip } i\} \\
&= 2p(1 - p)
\end{aligned}
$$

we see that

$$
\begin{aligned}
E\,[\text{number of changeovers}] &= \sum_{i=2}^{n} E[X_i] \\
&= 2(n - 1)p(1 - p)
\end{aligned}
$$

42. Suppose the coupon collector has i different types. Let X_i denote the number of additional coupons collected until the collector has $i + 1$ types. It is easy to see that the X_i are independent geometric random variables with respective parameters $(n - i)/n$, $i = 0, 1, \ldots, n - 1$. Therefore,

$$
\begin{aligned}
E\left[\sum_{i=0}^{n-1} X_i\right] &= \sum_{i=0}^{n-1} E[X_i] = \sum_{i=0}^{n-1} n/(n - i) \\
&= n \sum_{j=1}^{n} 1/j
\end{aligned}
$$

43. (a) $X = \sum_{i=1}^{n} X_i$

(b) $E[X_i] = P\{X_i = 1\}$

$= P\{\text{red ball } i \text{ is chosen before all } n \text{ black balls}\}$

$= 1/(n+1)$ since each of these $n+1$ balls is equally likely to be the one chosen earliest

Therefore,

$$E[X] = \sum_{i=1}^{n} E[X_i] = n/(n+1)$$

44. (a) Let Y_i equal 1 if red ball i is chosen after the first but before the second black ball, $i = 1, \ldots, n$. Then

$$Y = \sum_{i=1}^{n} Y_i$$

(b) $E[Y_i] = P\{Y_i = 1\}$

$= P\{\text{red ball } i \text{ is the second chosen from a set of } n+1 \text{ balls}\} =$

$1/(n+1)$ since each of the $n+1$ is equally likely to be the second one chosen.

Therefore,

$$E[Y] = n/(n+1)$$

(c) Answer is the same as in Problem 41.

(d) We can let the outcome of this experiment be the vector $(R_1, R_2, \ldots, R_n)$ where R_i is the number of red balls chosen after the $(i-1)$st but before the ith black ball. Since all orderings of the $n+m$ balls are equally likely it follows that all different orderings of $R_1, \ldots, R_n$ will have the same probability distribution. For instance,

$$P\{R_1 = a, R_2 = b\} = P\{R_2 = a, R_1 = b\}$$

From this it follows that all the R_i have the same distribution and thus the same mean.

45. Let N_i denote the number of keys in box i, $i = 1, \ldots, k$. Then, with X equal to the number of collisions we have that $X = \sum_{i=1}^{k} (N_i - 1)^+ = \sum_{i=1}^{k} (N_i - 1 + I\{N_i = 0\})$ where $I\{N_i = 0\}$ is equal to 1 if $N_i = 0$ and is equal to 0 otherwise.

Hence,

$$E[X] = \sum_{i=1}^{k} (r p_i - 1 + (1 - p_i)^r) = r - k + \sum_{i=1}^{k} (1 - p_i)^r$$

Another way to solve this problem is to let Y denote the number of boxes having at least one key, and then use the identity $X = r - Y$, which is true since only the first key put in each box does not result in a collision. Writing $Y = \sum_{i=1}^{k} I\{N_i > 0\}$ and taking expectations yields

$$E[X] = r - E[Y] = r - \sum_{i=1}^{k} [1 - (1 - p_i)^r]$$

$$= r - k + \sum_{i=1}^{k} (1 - p_i)^r$$

46. Using that $X = \sum_{n=1}^{\infty} I_n$, we obtain

$$E[X] = \sum_{n=1}^{\infty} E[I_n] = \sum_{n=1}^{\infty} P\{X \geq n\}$$

Making the change of variables $m = n - 1$ gives

$$E[X] = \sum_{m=0}^{\infty} P\{X \geq m + 1\} = \sum_{m=0}^{\infty} P\{X > m\}$$

(b) Let

$$I_n = \begin{cases} 1, & \text{if } n \leq X \\ 0, & \text{if } n > X \end{cases}$$

$$J_m = \begin{cases} 1, & \text{if } m \leq Y \\ 0, & \text{if } m > Y \end{cases}$$

Then

$$XY = \sum_{n=1}^{\infty} I_n \sum_{m=1}^{\infty} J_m = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} I_n J_m$$

Taking expectations now yields the result

$$E[XY] = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E[I_n J_m]$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P(X \geq n, Y \geq m)$$

47. Let X_i be 1 if trial i is a success and 0 otherwise.

(a) The largest value is .6. If $X_1 = X_2 = X_3$, then $1.8 = E[X] = 3E[X_1] = 3P\{X_1 = 1\}$ and so

$$P\{X = 3\} = P\{X1 = 1\} = .6$$

That this is the largest value is seen by Markov's inequality, which yields

$$P\{X \geq 3\} \leq E[X]/3 = .6$$

(b) The smallest value is 0. To construct a probability scenario for which $P\{X = 3\} = 0$ let U be a uniform random variable on (0, 1), and define

$$X_1 = \begin{cases} 1 & \text{if } U \leq .6 \\ 0 & \text{otherwise} \end{cases}$$

$$X_2 = \begin{cases} 1 & \text{if } U \geq .4 \\ 0 & \text{otherwise} \end{cases}$$

$$X_3 = \begin{cases} 1 & \text{if either } U \leq .3 \text{ or } U \geq .7 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that

$$P\{X_1 = X_2 = X_3 = 1\} = 0$$

49. $E[X^2] - (E[X])^2 = Var(X) = E(X - E[X])^2 \geq 0$. Equality when $Var(X) = 0$, that is, when X is constant.

50.
$$\begin{aligned} Var(cX) &= E[(cX - E[cX])^2] \\ &= E[c^2(X - E(X))^2] \\ &= c^2 Var(X) \end{aligned}$$

$$\begin{aligned} Var(c + X) &= E[(c + X - E[c + X])^2] \\ &= E[(X - E[X])^2] \\ &= Var(X) \end{aligned}$$

51. $N = \sum_{i=1}^{r} X_i$ where X_i is the number of flips between the $(i-1)$st and ith head. Hence, X_i is geometric with mean $1/p$. Thus,

$$E[N] = \sum_{i=1}^{r} E[X_i] = \frac{r}{p}$$

52. (a) $\dfrac{n}{n+1}$

(b) 0

(c) 1

53. $\dfrac{1}{n+1}, \dfrac{1}{2n+1} - \left[\dfrac{1}{n+1}\right]^2$

54. (a) Using the fact that $E[X + Y] = 0$ we see that $0 = 2p(1, 1) - 2p(-1, -1)$, which gives the result.

(b) This follows since

$$0 = E[X - Y] = 2p(1,-1) - 2p(-1,1)$$

(c) $Var(X) = E[X^2] = 1$

(d) $Var(Y) = E[Y^2] = 1$

(e) Since

$$\begin{aligned}1 &= p(1,1) + p(-1,1) + p(1,-1) + p(-1,1)\\ &= 2p(1,1) + 2p(1,-1)\end{aligned}$$

we see that if $p = 2p(1, 1)$ then $1 - p = 2p(1, -1)$

Now,

$$\begin{aligned}Cov(X,Y) &= E[XY]\\ &= p(1,1) + p(-1,-1)\\ &\quad - p(1,-1) - p(-1,1)\\ &= p - (1 - p) = 2p - 1\end{aligned}$$

55. (a) $P(Y = j) = \sum_{i=0}^{j} \binom{j}{i} e^{-3\lambda}\lambda^j/j!$

$$\begin{aligned}&= e^{-2\lambda}\frac{\lambda^j}{j!}\sum_{i=0}^{j}\binom{j}{i} 1^i\, 1^{j-1}\\ &= e^{-2\lambda}\frac{(2\lambda)^j}{j!}\end{aligned}$$

(b) $P(X = i) = \sum_{j=i}^{\infty} \binom{j}{i} e^{-2\lambda}\lambda^j/j!$

$$\begin{aligned}&= \frac{1}{i!}e^{-2\lambda}\sum_{j=i}^{\infty}\frac{1}{(j-i)!}\lambda^j\\ &= \frac{\lambda^i}{i!}e^{-2\lambda}\sum_{k=0}^{\infty}\lambda^k/k!\\ &= e^{-\lambda}\frac{\lambda^i}{i!}\end{aligned}$$