CONTENTS

CHAPTER 1

PRELIMINARIES

We suggest that this chapter be treated as review and covered quickly, without detailed classroom discussion. For one reason, many of these ideas will be already familiar to the students — at least informally. Further, we believe that, in practice, those notions of importance are best learned in the arena of real analysis, where their use and significance are more apparent. Dwelling on the formal aspect of sets and functions does not contribute very greatly to the students' understanding of real analysis.

If the students have already studied abstract algebra, number theory or combinatorics, they should be familiar with the use of mathematical induction. If not, then some time should be spent on mathematical induction.

The third section deals with finite, infinite and countable sets. These notions are important and should be briefly introduced. However, we believe that it is not necessary to go into the proofs of these results at this time.

Section 1.1

Students are usually familiar with the notations and operations of set algebra, so that a brief review is quite adequate. One item that should be mentioned is that two sets A and B are often proved to be equal by showing that: (i) if $x \in A$, then $x \in B$, and (ii) if $x \in B$, then $x \in A$. This type of element-wise argument is very common in real analysis, since manipulations with set identities is often not suitable when the sets are complicated.

Students are often not familiar with the notions of functions that are injective (= one-one) or surjective (= onto).

Sample Assignment: Exercises 1, 3, 9, 14, 15, 20.

Partial Solutions:

1. (a) $B \cap C = \{5, 11, 17, 23, \ldots\} = \{6k - 1 : k \in \mathbb{N}\}$, $A \cap (B \cap C) = \{5, 11, 17\}$
   (b) $(A \cap B) \setminus C = \{2, 8, 14, 20\}$
   (c) $(A \cap C) \setminus B = \{3, 7, 9, 13, 15, 19\}$
2. The sets are equal to (a) A, (b) $A \cap B$, (c) the empty set.
3. If $A \subseteq B$, then $x \in A$ implies $x \in B$, whence $x \in A \cap B$, so that $A \subseteq A \cap B \subseteq A$. Thus, if $A \subseteq B$, then $A = A \cap B$.
   Conversely, if $A = A \cap B$, then $x \in A$ implies $x \in A \cap B$, whence $x \in B$. Thus if $A = A \cap B$, then $A \subseteq B$.
4. If x is in $A \setminus (B \cap C)$, then x is in A but $x \notin B \cap C$, so that $x \in A$ and x is either not in B or not in C. Therefore either $x \in A \setminus B$ or $x \in A \setminus C$, which implies that $x \in (A \setminus B) \cup (A \setminus C)$. Thus $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Conversely, if x is in $(A \setminus B) \cup (A \setminus C)$, then $x \in A \setminus B$ or $x \in A \setminus C$. Thus $x \in A$ and either $x \notin B$ or $x \notin C$, which implies that $x \in A$ but $x \notin B \cap C$, so that $x \in A \setminus (B \cap C)$. Thus $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$.

Since the sets $A \setminus (B \cap C)$ and $(A \setminus B) \cup (A \setminus C)$ contain the same elements, they are equal.

5. (a) If $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$. Hence we either have (i) $x \in A$ and $x \in B$, or we have (ii) $x \in A$ and $x \in C$. Therefore, either $x \in A \cap B$ or $x \in A \cap C$, so that $x \in (A \cap B) \cup (A \cap C)$. This shows that $A \cap (B \cup C)$ is a subset of $(A \cap B) \cup (A \cap C)$.

   Conversely, let y be an element of $(A \cap B) \cup (A \cap C)$. Then either (j) $y \in A \cap B$, or (jj) $y \in A \cap C$. It follows that $y \in A$ and either $y \in B$ or $y \in C$. Therefore, $y \in A$ and $y \in B \cup C$, so that $y \in A \cap (B \cup C)$. Hence $(A \cap B) \cup (A \cap C)$ is a subset of $A \cap (B \cup C)$.

   In view of Definition 1.1.1, we conclude that the sets $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are equal.

   (b) Similar to (a).

6. The set D is the union of $\{x : x \in A \text{ and } x \notin B\}$ and $\{x : x \notin A \text{ and } x \in B\}$.

7. Here $A_n = \{n+1, 2(n+1), \ldots\}$.
   (a) $A_1 = \{2, 4, 6, 8, \ldots\}$, $A_2 = \{3, 6, 9, 12, \ldots\}$, $A_1 \cap A_2 = \{6, 12, 18, 24, \ldots\} = \{6k : k \in \mathbb{N}\} = A_5$.
   (b) $\bigcup A_n = \mathbb{N} \setminus \{1\}$, because if $n > 1$, then $n \in A_{n-1}$; moreover $1 \notin A_n$. Also $\bigcap A_n = \emptyset$, because $n \notin A_n$ for any $n \in \mathbb{N}$.

8. (a) The graph consists of four horizontal line segments.
   (b) The graph consists of three vertical line segments.

9. No. For example, both $(0, 1)$ and $(0, -1)$ belong to C.

10. (a) $f(E) = \{1/x^2 : 1 \le x \le 2\} = \{y : \frac{1}{4} \le y \le 1\} = [\frac{1}{4}, 1]$.
    (b) $f^{-1}(G) = \{x : 1 \le 1/x^2 \le 4\} = \{x : \frac{1}{4} \le x^2 \le 1\} = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$.

11. (a) $f(E) = \{x + 2 : 0 \le x \le 1\} = [2, 3]$, so $h(E) = g(f(E)) = g([2, 3]) = \{y^2 : 2 \le y \le 3\} = [4, 9]$.
    (b) $g^{-1}(G) = \{y : 0 \le y^2 \le 4\} = [-2, 2]$, so $h^{-1}(G) = f^{-1}(g^{-1}(G)) = f^{-1}([-2, 2]) = \{x : -2 \le x + 2 \le 2\} = [-4, 0]$.

12. If 0 is removed from E and F, then their intersection is empty, but the intersection of the images under f is $\{y : 0 < y \le 1\}$.

13. $E \setminus F = \{x : -1 \le x < 0\}$, $f(E) \setminus f(F)$ is empty, and $f(E \setminus F) = \{y : 0 < y \le 1\}$.

14. If $y \in f(E \cap F)$, then there exists $x \in E \cap F$ such that $y = f(x)$. Since $x \in E$ implies $y \in f(E)$, and $x \in F$ implies $y \in f(F)$, we have $y \in f(E) \cap f(F)$. This proves $f(E \cap F) \subseteq f(E) \cap f(F)$.

15. If $x \in f^{-1}(G) \cap f^{-1}(H)$, then $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$, so that $f(x) \in G$ and $f(x) \in H$. Then $f(x) \in G \cap H$, and hence $x \in f^{-1}(G \cap H)$. This shows

that $f^{-1}(G) \cap f^{-1}(H) \subseteq f^{-1}(G \cap H)$. The opposite inclusion is shown in Example 1.1.8(b). The proof for unions is similar.

16. If $f(a) = f(b)$, then $a/\sqrt{a^2+1} = b/\sqrt{b^2+1}$, from which it follows that $a^2 = b^2$. Since a and b must have the same sign, we get $a = b$, and hence f is injective. If $-1 < y < 1$, then $x := y/\sqrt{1-y^2}$ satisfies $f(x) = y$ (why?), so that f takes $\mathbb{R}$ onto the set $\{y : -1 < y < 1\}$. If $x > 0$, then $x = \sqrt{x^2} < \sqrt{x^2+1}$, so it follows that $f(x) \in \{y : 0 < y < 1\}$.
17. One bijection is the familiar linear function that maps a to 0 and b to 1, namely, $f(x) := (x-a)/(b-a)$. Show that this function works.
18. (a) Let $f(x) = 2x$, $g(x) = 3x$.
    (b) Let $f(x) = x^2$, $g(x) = x$, $h(x) = 1$. (Many examples are possible.)
19. (a) If $x \in f^{-1}(f(E))$, then $f(x) \in f(E)$, so that there exists $x_1 \in E$ such that $f(x_1) = f(x)$. If f is injective, then $x_1 = x$, whence $x \in E$. Therefore, $f^{-1}(f(E)) \subseteq E$. Since $E \subseteq f^{-1}(f(E))$ holds for any f, we have set equality when f is injective. See Example 1.1.8(a) for an example.
    (b) If $y \in H$ and f is surjective, then there exists $x \in A$ such that $f(x) = y$. Then $x \in f^{-1}(H)$ so that $y \in f(f^{-1}(H))$. Therefore $H \subseteq f(f^{-1}(H))$. Since $f(f^{-1}(H)) \subseteq H$ for any f, we have set equality when f is surjective. See Example 1.1.8(a) for an example.
20. (a) Since $y = f(x)$ if and only if $x = f^{-1}(y)$, it follows that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.
    (b) Since f is injective, then f^{-1} is injective on $R(f)$. And since f is surjective, then f^{-1} is defined on $R(f) = B$.
21. If $g(f(x_1)) = g(f(x_2))$, then $f(x_1) = f(x_2)$, so that $x_1 = x_2$, which implies that $g \circ f$ is injective. If $w \in C$, there exists $y \in B$ such that $g(y) = w$, and there exists $x \in A$ such that $f(x) = y$. Then $g(f(x)) = w$, so that $g \circ f$ is surjective. Thus $g \circ f$ is a bijection.
22. (a) If $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$, which implies $x_1 = x_2$, since $g \circ f$ is injective. Thus f is injective.
    (b) Given $w \in C$, since $g \circ f$ is surjective, there exists $x \in A$ such that $g(f(x)) = w$. If $y := f(x)$, then $y \in B$ and $g(y) = w$. Thus g is surjective.
23. We have $x \in f^{-1}(g^{-1}(H)) \iff f(x) \in g^{-1}(H) \iff g(f(x)) \in H \iff x \in (g \circ f)^{-1}(H)$.
24. If $g(f(x)) = x$ for all $x \in D(f)$, then $g \circ f$ is injective, and Exercise 22(a) implies that f is injective on $D(f)$. If $f(g(y)) = y$ for all $y \in D(g)$, then Exercise 22(b) implies that f maps $D(f)$ onto $D(g)$. Thus f is a bijection of $D(f)$ onto $D(g)$, and $g = f^{-1}$.

Section 1.2

The method of proof known as Mathematical Induction is used frequently in real analysis, but in many situations the details follow a routine patterns and are left to the reader by means of a phrase such as: "The proof is by Mathematical Induction". Since may students have only a hazy idea of what is involved, it may be a good idea to spend some time explaining and illustrating what constitutes a proof by induction.

Pains should be taken to emphasize that the induction hypothesis does *not* entail "assuming what is to be proved". The inductive step concerns the validity of going from the assertion for $k \in \mathbb{N}$ to that for $k+1$. The truth of falsity of the individual assertion is not an issue here.

Sample Assignment: Exercises 1, 2, 6, 11, 13, 14, 20.

Partial Solutions:

1. The assertion is true for $n=1$ because $1/(1\cdot 2)=1/(1+1)$. If it is true for $n=k$, then it follows for $k+1$ because $k/(k+1)+1/[(k+1)(k+2)]=(k+1)/(k+2)$.
2. The statement is true for $n=1$ because $[\frac{1}{2}\cdot 1\cdot 2]^2=1=1^3$. For the inductive step, use the fact that
$$\left[\tfrac{1}{2}k(k+1)\right]^2+(k+1)^3=\left[\tfrac{1}{2}(k+1)(k+2)\right]^2.$$
3. It is true for $n=1$ since $3=4-1$. If the equality holds for $n=k$, then add $8(k+1)-5=8k+3$ to both sides and show that $(4k^2-k)+(8k+3)=4(k+1)^2-(k+1)$ to deduce equality for the case $n=k+1$.
4. It is true for $n=1$ since $1=(4-1)/3$. If it is true for $n=k$, then add $(2k+1)^2$ to both sides and use some algebra to show that
$$\tfrac{1}{3}(4k^3-k)+(2k+1)^2=\tfrac{1}{3}[4k^3+12k^2+11k+3]=\tfrac{1}{3}[4(k+1)^3-(k+1)],$$
which establishes the case $n=k+1$.
5. Equality holds for $n=1$ since $1^2=(-1)^2(1\cdot 2)/2$. The proof is completed by showing $(-1)^{k+1}[k(k+1)]/2+(-1)^{k+2}(k+1)^2=(-1)^{k+2}[(k+1)(k+2)]/2$.
6. If $n=1$, then $1^3+5\cdot 1=6$ is divisible by 6. If k^3+5k is divisible by 6, then $(k+1)^3+5(k+1)=(k^3+5k)+3k(k+1)+6$ is also, because $k(k+1)$ is always even (why?) so that $3k(k+1)$ is divisible by 6, and hence the sum is divisible by 6.
7. If $5^{2k}-1$ is divisible by 8, then it follows that $5^{2(k+1)}-1=(5^{2k}-1)+24\cdot 5^{2k}$ is also divisible by 8.
8. $5^{k+1}-4(k+1)-1=5\cdot 5^k-4k-5=(5^k-4k-1)+4(5^k-1)$. Now show that 5^k-1 is always divisible by 4.
9. If $k^3+(k+1)^3+(k+2)^3$ is divisible by 9, then $(k+1)^3+(k+2)^3+(k+3)^3=k^3+(k+1)^3+(k+2)^3+9(k^2+3k+3)$ is also divisible by 9.
10. The sum is equal to $n/(2n+1)$.

11. The sum is $1+3+\cdots+(2n-1)=n^2$. Note that $k^2+(2k+1)=(k+1)^2$.
12. If $n_0>1$, let $S_1:=\{n\in\mathbb{N}: n-n_0+1\in S\}$ Apply 1.2.2 to the set S_1.
13. If $k<2^k$, then $k+1<2^k+1<2^k+2^k=2(2^k)=2^{k+1}$.
14. If $n=4$, then $2^4=16<24=4!$. If $2^k<k!$ and if $k\ge 4$, then $2^{k+1}=2\cdot 2^k<2\cdot k!<(k+1)\cdot k!=(k+1)!$. [Note that the inductive step is valid whenever $2<k+1$, including $k=2,3$, even though the statement is false for these values.]
15. For $n=5$ we have $7\le 2^3$. If $k\ge 5$ and $2k-3\le 2^{k-2}$, then $2(k+1)-3=(2k-3)+2\le 2^{k-2}+2^{k-2}=2^{(k+1)-2}$.
16. It is true for $n=1$ and $n\ge 5$, but false for $n=2,3,4$. The inequality $2k+1<2^k$, wich holds for $k\ge 3$, is needed in the induction argument. [The inductive step is valid for $n=3,4$ even though the inequality $n^2<2^n$ is false for these values.]
17. $m=6$ trivially divides n^3-n for $n=1$, and it is the largest integer to divide $2^3-2=6$. If k^3-k is divisible by 6, then since k^2+k is even (why?), it follows that $(k+1)^3-(k+1)=(k^3-k)+3(k^2+k)$ is also divisible by 6.
18. $\sqrt{k}+1/\sqrt{k+1}=(\sqrt{k}\sqrt{k+1}+1)/\sqrt{k+1}>(k+1)/\sqrt{k+1}=\sqrt{k+1}$.
19. First note that since $2\in S$, then the number $1=2-1$ belongs to S. If $m\notin S$, then $m<2^m\in S$, so $2^m-1\in S$, etc.
20. If $1\le x_{k-1}\le 2$ and $1\le x_k\le 2$, then $2\le x_{k-1}+x_k\le 4$, so that $1\le x_{k+1}=(x_{k-1}+x_k)/2\le 2$.

Section 1.3

Every student of advanced mathematics needs to know the meaning of the words “finite”, “infinite”, “countable” and “uncountable”. For most students at this level it is quite enough to learn the definitions and read the statements of the theorems in this section, but to skip the proofs. Probably every instructor will want to show that $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable (see Section 2.5).

Some students will not be able to comprehend that proofs are necessary for “obvious” statements about finite sets. Others will find the material absolutely fascinating and want to prolong the discussion forever. The teacher must avoid getting bogged down in a protracted discussion of cardinal numbers.

Sample Assignment: Exercises 1, 5, 7, 9, 11.

Partial Solutions:

1. If $T_1\neq\emptyset$ is finite, then the definition of a finite set applies to $T_2=\mathbb{N}_n$ for some n. If f is a bijection of T_1 onto T_2, and if g is a bijection of T_2 onto $\mathbb{N}_n$, then (by Exercise 1.1.21) the composite $g\circ f$ is a bijection of T_1 onto $\mathbb{N}_n$, so that T_1 is finite.

2. Part (b) Let f be a bijection of $\mathbb{N}_m$ onto A and let $C = \{f(k)\}$ for some $k \in \mathbb{N}_m$. Define g on $\mathbb{N}_{m-1}$ by $g(i) := f(i)$ for $i = 1, \ldots, k-1$, and $g(i) := f(i+1)$ for $i = k, \ldots, m-1$. Then g is a bijection of $\mathbb{N}_{m-1}$ onto $A \backslash C$. (Why?)
Part (c) First note that the union of two finite sets is a finite set. Now note that if C/B were finite, then $C = B \cup (C \setminus B)$ would also be finite.
3. (a) The element 1 can be mapped into any of the three elements of T, and 2 can then be mapped into any of the two remaining elements of T, after which the element 3 can be mapped into only one element of T. Hence there are $6 = 3 \cdot 2 \cdot 1$ different injections of S into T.
(b) Suppose a maps into 1. If b also maps into 1, then c must map into 2; if b maps into 2, then c can map into either 1 or 2. Thus there are 3 surjections that map a into 1, and there are 3 other surjections that map a into 2.
4. $f(n) := 2n + 13, n \in \mathbb{N}$.
5. $f(1) := 0$, $f(2n) := n, f(2n+1) := -n$ for $n \in \mathbb{N}$.
6. The bijection of Example 1.3.7(a) is one example. Another is the shift defined by $f(n) := n+1$ that maps $\mathbb{N}$ onto $\mathbb{N} \setminus \{1\}$.
7. If T_1 is denumerable, take $T_2 = \mathbb{N}$. If f is a bijection of T_1 onto T_2, and if g is a bijection of T_2 onto $\mathbb{N}$, then (by Exercise 1.1.21) $g \circ f$ is a bijection of T_1 onto $\mathbb{N}$, so that T_1 is denumerable.
8. Let $A_n := \{n\}$ for $n \in \mathbb{N}$, so $\bigcup A_n = \mathbb{N}$.
9. If $S \cap T = \emptyset$ and $f : \mathbb{N} \to S, g \colon \mathbb{N} \to T$ are bijections onto S and T, respectively, let $h(n) := f((n+1)/2)$ if n is odd and $h(n) := g(n/2)$ if n is even. It is readily seen that h is a bijection of $\mathbb{N}$ onto $S \cup T$; hence $S \cup T$ is denumerable. What if $S \cap T \neq \emptyset$?
10. (a) $m + n - 1 = 9$ and $m = 6$ imply $n = 4$. Then $h(6, 4) = \frac{1}{2} \cdot 8 \cdot 9 + 6 = 42$.
(b) $h(m, 3) = \frac{1}{2}(m+1)(m+2) + m = 19$, so that $m^2 + 5m - 36 = 0$. Thus $m = 4$.
11. (a) $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ has $2^2 = 4$ elements.
(b) $\mathcal{P}(\{1, 2, 3\})$ has $2^3 = 8$ elements.
(c) $\mathcal{P}(\{1, 2, 3, 4\})$ has $2^4 = 16$ elements.
12. Let $S_{n+1} := \{x_1, \ldots, x_n, x_{n+1}\} = S_n \cup \{x_{n+1}\}$ have $n+1$ elements. Then a subset of S_{n+1} either (i) contains x_{n+1}, or (ii) does not contain x_{n+1}. The induction hypothesis implies that there are 2^n subsets of type (i), since each such subset is the union of $\{x_{n+1}\}$ and a subset of S_n. There are also 2^n subsets of type (ii). Thus there is a total of $2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$ subsets of S_{n+1}.
13. For each $m \in \mathbb{N}$, the collection of all subsets of $\mathbb{N}_m$ is finite. (See Exercise 12.) Every finite subset of $\mathbb{N}$ is a subset of $\mathbb{N}_m$ for a sufficiently large m. Therefore Theorem 1.3.12 implies that $\mathcal{F}(\mathbb{N}) = \bigcup_{m=1}^{\infty} \mathcal{P}(\mathbb{N}_m)$ is countable.

CHAPTER 2

THE REAL NUMBERS

Students will be familiar with much of the factual content of the first few sections, but the process of deducing these facts from a basic list of axioms will be new to most of them. The ability to construct proofs usually improves gradually during the course, and there are much more significant topics forthcoming. A few selected theorems should be proved in detail, since some experience in writing formal proofs is important to students at this stage. However, one should not spend too much time on this material.

Sections 2.3 and 2.4 on the Completeness Property form the heart of this chapter. *These sections should be covered thoroughly.* Also the Nested Intervals Property in Section 2.5 should be treated carefully.

Section 2.1

One goal of Section 2.1 is to acquaint students with the idea of deducing consequences from a list of basic axioms. Students who have not encountered this type of formal reasoning may be somewhat uncomfortable at first, since they often regard these results as "obvious". Since there is much more to come, a sampling of results will suffice at this stage, making it clear that it is only a sampling. The classic proof of the irrationality of $\sqrt{2}$ should certainly be included in the discussion, and students should be asked to modify this argument for $\sqrt{3}$, etc.

Sample Assignment: Exercises 1(a,b), 2(a,b), 3(a,b), 6, 13, 16(a,b), 20, 23.

Partial Solutions:

1. (a) Apply appropriate algebraic properties to get $b = 0 + b = (-a + a) + b = -a + (a + b) = -a + 0 = -a$.
   (b) Apply (a) to $(-a) + a = 0$ with $b = a$ to conclude that $a = -(-a)$.
   (c) Apply (a) to the equation $a + (-1)a = a(1 + (-1)) = a \cdot 0 = 0$ to conclude that $(-1)a = -a$.
   (d) Apply (c) with $a = -1$ to get $(-1)(-1) = -(-1)$. Then apply (b) with $a = 1$ to get $(-1)(-1) = 1$.
2. (a) $-(a + b) = (-1)(a + b) = (-1)a + (-1)b = (-a) + (-b)$.
   (b) $(-a) \cdot (-b) = ((-1)a) \cdot ((-1)b) = (-1)(-1)(ab) = ab$.
   (c) Note that $(-a)(-(1/a)) = a(1/a) = 1$.
   (d) $-(a/b) = (-1)(a(1/b)) = ((-1)a)(1/b) = (-a)/b$.
3. (a) Add -5 to both sides of $2x + 5 = 8$ and use (A2),(A4),(A3) to get $2x = 3$. Then multiply both sides by $1/2$ to get $x = 3/2$.
   (b) Write $x^2 - 2x = x(x - 2) = 0$ and apply Theorem 2.1.3(b). Alternatively, note that $x = 0$ satisfies the equation, and if $x \neq 0$, then multiplication by $1/x$ gives $x = 2$.

(c) Add -3 to both sides and factor to get $x^2-4=(x-2)(x+2)=0$. Now apply 2.1.3(b) to get $x=2$ or $x=-2$.
(d) Apply 2.1.3(b) to show that $(x-1)(x+2)=0$ if and only if $x=1$ or $x=-2$.

4. Clearly $a=0$ satisfies $a\cdot a=a$. If $a\neq 0$ and $a\cdot a=a$, then $(a\cdot a)(1/a)=a(1/a)$, so that $a=a(a(1/a))=a(1/a)=1$.
5. If $(1/a)(1/b)$ is multiplied by ab, the result is 1. Therefore, Theorem 2.1.3(a) implies that $1/(ab)=(1/a)(1/b)$.
6. Note that if $q\in\mathbb{Z}$ and if $3q^2$ is even, then q^2 is even, so that q is even. Hence, if $(p/q)^2=6$, then it follows that p is even, say $p=2m$, whence $2m^2=3q^2$, so that q is also even.
7. If $p\in\mathbb{N}$, there are three possibilities: for some $m\in\mathbb{N}\cup\{0\}$, (i) $p=3m$, (ii) $p=3m+1$, or (iii) $p=3m+2$. In either case (ii) or (iii), we have $p^2=3h+1$ for some $h\in\mathbb{N}\cup\{0\}$.
8. (a) Let $x=m/n, y=p/q$, where $m,n\neq 0,p,q\neq 0$ are integers. Then $x+y=(mq+np)/nq$ and $xy=mp/nq$ are rational.
(b) If $s:=x+y\in\mathbb{Q}$, then $y=s-x\in\mathbb{Q}$, a contradiction. If $t:=xy\in\mathbb{Q}$ and $x\neq 0$, then $y=t/x\in\mathbb{Q}$, a contradiction.
9. (a) If $x_1=s_1+t_1\sqrt{2}$ and $x_2=s_2+t_2\sqrt{2}$ are in K, then $x_1+x_2=(s_1+s_2)+(t_1+t_2)\sqrt{2}$ and $x_1x_2=(s_1s_2+2t_1t_2)+(s_1t_2+s_2t_1)\sqrt{2}$ are also in K.
(b) If $x=s+t\sqrt{2}\neq 0$ is in K, then $s-t\sqrt{2}\neq 0$ (why?) and
$$\frac{1}{x}=\frac{s-t\sqrt{2}}{(s+t\sqrt{2})(s-t\sqrt{2})}=\left(\frac{s}{s^2-2t^2}\right)-\left(\frac{t}{s^2-2t^2}\right)\sqrt{2}$$
is in K. (Use Theorem 2.1.4.)
10. (a) If $c=d$, then 2.1.7(b) implies $a+c<b+d$. If $c<d$, then $a+c<b+c<b+d$.
(b) If $c=d=0$, then $ac=bd=0$. If $c>0$, then $0<ac$ by the Trichotomy Property and $ac<bc$ follows from 2.1.7(c). If also $c\le d$, then $ac\le ad<bd$. Thus $0\le ac\le bd$ holds in all cases.
11. (a) If $a>0$, then $a\neq 0$ by the Trichotomy Property, so that $1/a$ exists. If $1/a=0$, then $1=a\cdot(1/a)=a\cdot 0=0$, which contradicts (M3). If $1/a<0$, then 2.1.7(c) implies that $1=a(1/a)<0$, which contradicts 2.1.8(b). Thus $1/a>0$, and 2.1.3(a) implies that $1/(1/a)=a$.
(b) If $a<b$, then $2a=a+a<a+b$, and also $a+b<b+b=2b$. Therefore, $2a<a+b<2b$, which, since $\frac{1}{2}>0$ (by 2.1.8(c) and part (a)), implies that $a<\frac{1}{2}(a+b)<b$.
12. Let $a=1$ and $b=2$. If $c=-3$ and $d=-1$, then $ac<bd$. On the other hand, if $c=-3$ and $d=-2$, then $bd<ac$. (Many other examples are possible.)

13. If $a \neq 0$, then 2.1.8(a) implies that $a^2 > 0$; since $b^2 \geq 0$, it follows that $a^2 + b^2 > 0$.

14. If $0 \leq a < b$, then 2.1.7(c) implies $ab < b^2$. If $a = 0$, then $0 = a^2 = ab < b^2$. If $a > 0$, then $a^2 < ab$ by 2.1.7(c). Thus $a^2 \leq ab < b^2$. If $a = 0, b = 1$, then $0 = a^2 = ab < b = 1$.

15. (a) If $0 < a < b$, then 2.1.7(c) implies that $0 < a^2 < ab < b^2$. Then by Example 2.1.13(a), we infer that $a = \sqrt{a^2} < \sqrt{ab} < \sqrt{b^2} = b$.
(b) If $0 < a < b$, then $ab > 0$ so that $1/ab > 0$, and thus $1/a - 1/b = (1/ab)(b - a) > 0$.

16. (a) To solve $(x-4)(x+1) > 0$, look at two cases. Case 1: $x - 4 > 0$ and $x + 1 > 0$, which gives $x > 4$. Case 2: $x - 4 < 0$ and $x + 1 < 0$, which gives $x < -1$. Thus we have $\{x : x > 4 \text{ or } x < -1\}$.
(b) $1 < x^2 < 4$ has the solution set $\{x : 1 < x < 2 \text{ or } -2 < x < -1\}$.
(c) The inequality is $1/x - x = (1-x)(1+x)/x < 0$. If $x > 0$, this is equivalent to $(1-x)(1+x) < 0$, which is satisfied if $x > 1$. If $x < 0$, then we solve $(1-x)(1+x) > 0$, and get $-1 < x < 0$. Thus we get $\{x : -1 < x < 0 \text{ or } x > 1\}$
(d) the solution set is $\{x : x < 0 \text{ or } x > 1\}$.

17. If $a > 0$, we can take $\varepsilon_0 := a > 0$ and obtain $0 < \varepsilon_0 \leq a$, a contradiction.

18. If $b < a$ and if $\varepsilon_0 := (a - b)/2$, then $\varepsilon_0 > 0$ and $a = b + 2\varepsilon_0 > b + \varepsilon_0$.

19. The inequality is equivalent to $0 \leq a^2 - 2ab + b^2 = (a - b)^2$.

20. (a) If $0 < c < 1$, then 2.1.7(c) implies that $0 < c^2 < c$, whence $0 < c^2 < c < 1$.
(b) Since $c > 0$, then 2.1.7(c) implies that $c < c^2$, whence $1 < c < c^2$.

21. (a) Let $S := \{n \in \mathbb{N} : 0 < n < 1\}$. If S is not empty, the Well-Ordering Property of $\mathbb{N}$ implies there is a least element m in S. However, $0 < m < 1$ implies that $0 < m^2 < m$, and since m^2 is also in S, this is a contradiction to the fact that m is the least element of S.
(b) If $n = 2p = 2q - 1$ for some p, q in $\mathbb{N}$, then $2(q - p) = 1$, so that $0 < q - p < 1$. This contradicts (a).

22. (a) Let $x := c - 1 > 0$ and apply Bernoulli's Inequality 2.1.13(c) to get $c^n = (1 + x)^n \geq 1 + nx \geq 1 + x = c$ for all $n \in \mathbb{N}$, and $c^n > 1 + x = c$ for $n > 1$.
(b) Let $b := 1/c$ and use part (a).

23. If $0 < a < b$ and $a^k < b^k$, then 2.1.7(c) implies that $a^{k+1} < ab^k < b^{k+1}$ so Induction applies. If $a^m < b^m$ for some $m \in \mathbb{N}$, the hypothesis that $0 < b \leq a$ leads to a contradiction.

24. (a) If $m > n$, then $k := m - n \in \mathbb{N}$, so Exercise 22(a) implies that $c^k \geq c > 1$. But since $c^k = c^{m-n}$, this implies that $c^m > c^n$. Conversely, the hypothesis that $c^m > c^n$ and $m \leq n$ lead to a contradiction.
(b) Let $b := 1/c$ and use part (a).

25. Let $b := c^{1/mn}$. We claim that $b > 1$; for if $b \le 1$, then Exercise 22(b) implies that $1 < c = b^{mn} \le b \le 1$, a contradiction. Therefore Exercise 24(a) implies that $c^{1/n} = b^m > b^n = c^{1/m}$ if and only if $m > n$.

26. Fix $m \in \mathbb{N}$ and use Mathematical Induction to prove that $a^{m+n} = a^m a^n$ and $(a^m)^n = a^{mn}$ for all $n \in \mathbb{N}$. Then, for a given $n \in \mathbb{N}$, prove that the equalities are valid for all $m \in \mathbb{N}$.

Section 2.2

The notion of absolute value of a real number is defined in terms of the basic order properties of $\mathbb{R}$. We have put it in a separate section to give it emphasis. Many students need extra work to become comfortable with manipulations involving absolute values, especially when inequalities are involved.

We have also used this section to give students an early introduction to the notion of the ε-neighborhood of a point. As a preview of the role of ε-neighborhoods, we have recast Theorem 2.1.9 in terms of ε-neighborhhoods in Theorem 2.2.8.

Sample Assignment: Exercises 1, 4, 5, 6(a,b), 8(a,b), 9, 12(a,b), 15.

Partial Solutions:

1. (a) If $a \ge 0$, then $|a| = a = \sqrt{a^2}$; if $a < 0$, then $|a| = -a = \sqrt{a^2}$.
(b) It suffices to show that $|1/b| = 1/|b|$ for $b \neq 0$ (why?). If $b > 0$, then $1/b > 0$ (why?), so that $|1/b| = 1/b = 1/|b|$. If $b < 0$, then $1/b < 0$, so that $|1/b| = -(1/b) = 1/(-b) = 1/|b|$.
2. First show that $ab \ge 0$ if an only if $|ab| = ab$. Then show that $(|a| + |b|)^2 = (a+b)^2$ if and only if $|ab| = ab$.
3. If $x \le y \le z$, then $|x-y| + |y-z| = (y-x) + (z-y) = z - x = |z-x|$. To establish the converse, show that $y < x$ and $y > z$ are impossible. For example, if $y < x \le z$, it follows from what we have shown and the given relationship that $|x-y| = 0$, so that $y = x$, a contradiction.
4. $|x-a| < \varepsilon \Longleftrightarrow -\varepsilon < x - a < \varepsilon \Longleftrightarrow a - \varepsilon < x < a + \varepsilon$.
5. If $a < x < b$ and $-b < -y < -a$, it follows that $a - b < x - y < b - a$. Since $a - b = -(b-a)$, the argument in 2.2.2(c) gives the conclusion $|x-y| < b - a$. The distance between x and y is less than or equal to $b - a$.
6. (a) $|4x-5| \le 13 \Longleftrightarrow -13 \le 4x - 5 \le 13 \Longleftrightarrow -8 \le 4x \le 18 \Longleftrightarrow -2 \le x \le 9/2$.
(b) $|x^2-1| \le 3 \Longleftrightarrow -3 \le x^2 - 1 \le 3 \Longleftrightarrow -2 \le x^2 \le 4 \Longleftrightarrow 0 \le x^2 \le 4 \Longleftrightarrow -2 \le x \le 2$.
7. Case 1: $x \ge 2 \Rightarrow (x+1) + (x-2) = 2x - 1 = 7$, so $x = 4$.
Case 2: $-1 < x < 2 \Rightarrow (x+1) + (2-x) = 3 \neq 7$, so no solution.
Case 3: $x \le -1 \Rightarrow (-x-1) + (2-x) = -2x + 1 = 7$, so $x = -3$.
Combining these cases, we get $x = 4$ or $x = -3$.

8. (a) If $x > 1/2$, then $x+1 = 2x-1$, so that $x = 2$. If $x \le 1/2$, then $x+1 = -2x+1$, so that $x = 0$. There are two solutions $\{0, 2\}$.
(b) If $x \ge 5$, the equation implies $x = -4$, so no solutions. If $x < 5$, then $x = 2$.
9. (a) If $x \ge 2$, the inequality becomes $-2 \le 1$. If $x \le 2$, the inequality is $x \ge 1/2$, so this case contributes $1/2 \le x \le 2$. Combining the cases gives us all $x \ge 1/2$.
(b) $x \ge 0$ yields $x \le 1/2$, so that we get $0 \le x \le 1/2$. $x \le 0$ yields $x \ge -1$, so that $-1 \le x \le 0$. Combining cases, we get $-1 \le x \le 1/2$.
10. (a) Either consider the three cases: $x < -1$, $-1 \le x \le 1$ and $1 < x$; or, square both sides to get $-2x > 2x$. Either approach gives $x < 0$.
(b) Consider the three cases $x \ge 0$, $-1 \le x < 0$ and $x < -1$ to get $-3/2 < x < 1/2$.
11. $y = f(x)$ where $f(x) := -1$ for $x < 0$, $f(x) := 2x - 1$ for $0 \le x \le 1$, and $f(x) := 1$ for $x > 1$.
12. Case 1: $x \ge 1 \Rightarrow 4 < (x+2) + (x-1) < 5$, so $3/2 < x < 2$.
Case 2: $-2 < x < 1 \Rightarrow 4 < (x+2) + (1-x) < 5$, so there is no solution.
Case 3: $x < -2 \Rightarrow 4 < (-x-2) + (1-x) < 5$, so $-3 < x < -5/2$.
Thus the solution set is $\{x : -3 < x < -5/2 \text{ or } 3/2 < x < 2\}$.
13. $|2x-3| < 5 \iff -1 < x < 4$, and $|x+1| > 2 \iff x < -3$ or $x > 1$. The two inequalities are satisfied simultaneously by points in the intersection $\{x : 1 < x < 4\}$.
14. (a) $|x| = |y| \iff x^2 = y^2 \iff (x-y)(x+y) = 0 \iff y = x$ or $y = -x$. Thus $\{(x, y) : y = x \text{ or } y = -x\}$.
(b) Consider four cases. If $x \ge 0$, $y \ge 0$, we get the line segment joining the points (0, 1) and (1, 0). If $x \le 0, y \ge 0$, we get the line segment joining $(-1, 0)$ and $(0, 1)$, and so on.
(c) The hyperbolas $y = 2/x$ and $y = -2/x$.
(d) Consider four cases corresponding to the four quadrants. The graph consists of a portion of a line segment in each quadrant. For example, if $x \ge 0, y \ge 0$, we obtain the portion of the line $y = x - 2$ in this quadrant.
15. (a) If $y \ge 0$, then $-y \le x \le y$ and we get the region in the upper half-plane on or between the lines $y = x$ and $y = -x$. If $y \le 0$, then we get the region in the lower half-plane on or between the lines $y = x$ and $y = -x$.
(b) This is the region on and inside the diamond with vertices (1, 0), (0, 1), $(-1, 0)$ and $(0, -1)$.
16. For the intersection, let γ be the smaller of ε and δ. For the union, let γ be the larger of ε and δ.
17. Choose any $\varepsilon > 0$ such that $\varepsilon < |a - b|$.
18. (a) If $a \le b$, then $\max\{a, b\} = b = \frac{1}{2}[a + b + (b-a)]$ and $\min\{a, b\} = a = \frac{1}{2}[a + b - (b-a)]$.
(b) If $a = \min\,\{a, b, c\}$, then $\min\{\min\{a, b\}, c\} = a = \min\{a, b, c\}$. Similarly, if b or c is $\min\{a, b, c\}$.

19. If $a \le b \le c$, then $\text{mid}\{a,b,c\} = b = \min\{b,c,c\} = \min\{\max\{a,b\},\ \max\{b,c\},\ \max\{c,a\}\}$. The other cases are similar.

Section 2.3

This section completes the description of the real number system by introducing the fundamental completeness property in the form of the Supremum Property. This property is vital to real analysis and students should attain a working understanding of it. Effort expended in this section and the one following will be richly rewarded later.

Sample Assignment: Exercises 1, 2, 5, 6, 9, 10, 12, 14.

Partial Solutions:

1. Any negative number or 0 is a lower bound. For any $x \ge 0$, the larger number $x+1$ is in S_1, so that x is not an upper bound of S_1. Since $0 \le x$ for all $x \in S_1$, then $u = 0$ is a lower bound of S_1. If $v > 0$, then v is not a lower bound of S_1 because $v/2 \in S_1$ and $v/2 < v$. Therefore inf $S_1 = 0$.
2. S_2 has lower bounds, so that inf S_2 exists. The argument used for S_1 also shows that inf $S_2 = 0$, but that inf S_2 does not belong to S_2. S_2 does not have upper bounds, so that sup S_2 does not exists.
3. Since $1/n \le 1$ for all $n \in \mathbb{N}$, then 1 is an upper bound for S_3. But 1 is a member of S_3, so that $1 = \sup S_3$. (See Exercise 7 below.)
4. sup $S_4 = 2$ and inf $S_4 = 1/2$. (Note that both are members of S_4.)
5. It is interesting to compare algebraic and geometric approaches to these problems.
   (a) inf $A = -5/2$, sup A does not exist,
   (b) sup $B = 2$, inf $B = -1$,
   (c) sup $C = 1$, inf B does not exist,
   (d) sup $D = 1 + \sqrt{6}$, inf $D = 1 - \sqrt{6}$.
6. If S is bounded below, then $S' := \{-s : s \in S\}$ is bounded above, so that $u := \sup S'$ exists. If $v \le s$ for all $s \in S$, then $-v \ge -s$ for all $s \in S$, so that $-v \ge u$, and hence $v \le -u$. Thus inf $S = -u$.
7. Let $u \in S$ be an upper bound of S. If v is another upper bound of S, then $u \le v$. Hence $u = \sup S$.
8. If $t > u$ and $t \in S$, then u is not an upper bound of S.
9. Let $u :=$ sup S. Since u is an upper bound of S, so is $u + 1/n$ for all $n \in \mathbb{N}$. Since u is the supremum of S and $u - 1/n < u$, then there exists $s_0 \in S$ with $u - 1/n < s_0$, whence $u - 1/n$ is not an upper bound of S.
10. Let $u := \sup A$, $v := \sup B$ and $w := \sup\{u, v\}$. Then w is an upper bound of $A \cup B$, because if $x \in A$, then $x \le u \le w$, and if $x \in B$, then $x \le v \le w$. If z is

any upper bound of $A \cup B$, then z is an upper bound of A and of B, so that $u \le z$ and $v \le z$. Hence $w \le z$. Therefore, $w = \sup(A \cup B)$.

11. Since sup S is an upper bound of S, it is an upper bound of S_0, and hence sup $S_0 \le$ sup S.
12. Consider two cases. If $u \ge s^*$, then $u = \sup(S \cup \{u\})$. If $u < s^*$, then there exists $s \in S$ such that $u < s \le s^*$, so that $s^* = \sup(S \cup \{u\})$.
13. If $S_1 := \{x_1\}$, show that $x_1 = \sup\ S_1$. If $S_k := \{x_1, \ldots, x_k\}$ is such that sup $S_k \in S_k$, then preceding exercise implies that $\sup\{x_1, \ldots, x_k, x_{k+1}\}$ is the larger of sup S_k and x_{k+1} and so is in S_{k+1}.
14. If $w = \inf\ S$ and $\varepsilon > 0$, then $w + \varepsilon$ is not a lower bound so that there exists t in S such that $t < w + \varepsilon$. If w is a lower bound of S that satisfies the stated condition, and if $z > w$, let $\varepsilon = z - w > 0$. Then there is t in S such that $t < w + \varepsilon = z$, so that z is not a lower bound of S. Thus, $w = \inf\ S$.

Section 2.4

This section exhibits how the supremum is used in practice, and contains some important properties of $\mathbb{R}$ that will often be used later. The Archimedean Properties 2.4.3–2.4.6 and the Density Properties 2.4.8 and 2.4.9 are the most significant. The exercises also contain some results that will be used later.

Sample Assignment: Exercises 1, 2, 4(b), 5, 7, 10, 12, 13, 14.

Partial Solutions:

1. Since $1 - 1/n < 1$ for all $n \in \mathbb{N}$, the number 1 is an upper bound. To show that 1 is the supremum, it must be shown that for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $1 - 1/n > 1 - \varepsilon$, which is equivalent to $1/n < \varepsilon$. Apply the Archimedean Property 2.4.3 or 2.4.5.
2. inf $S = -1$ and sup $S = 1$. To see the latter note that $1/n - 1/m \le 1$ for all $m, n \in \mathbb{N}$. On the other hand if $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $1/m < \varepsilon$, so that $1/1 - 1/m > 1 - \varepsilon$.
3. Suppose that $u \in \mathbb{R}$ is not the supremum of S. Then either (i) u is not an upper bound of S (so that there exists $s_1 \in S$ with $u < s_1$, whence we take $n \in \mathbb{N}$ with $1/n < s_1 - u$ to show that $u + 1/n$ is not an upper bound of S), or (ii) there exists an upper bound u_1 of S with $u_1 < u$ (in which case we take $1/n < u - u_1$ to show that $u - 1/n$ is not an upper bound of S).
4. (a) Let $u := \sup S$ and $a > 0$. Then $x \le u$ for all $x \in S$, whence $ax \le au$ for all $x \in S$, whence it follows that au is an upper bound of aS. If v is another upper bound of aS, then $ax \le v$ for all $x \in S$, whence $x \le v/a$ for all $x \in S$, showing that v/a is an upper bound for S so that $u \le v/a$, from which we conclude that $au \le v$. Therefore $au = \sup(aS)$. The statement about the infimum is proved similarly.

(b) Let $u := \sup S$ and $b < 0$. If $x \in S$, then (since $b < 0$) $bu \le bx$ so that bu is a lower bound of bS. If $v \le bx$ for all $x \in S$, then $x \le v/b$ (since $b < 0$), so that v/b is an upper bound for S. Hence $u \le v/b$ whence $v \le bu$. Therefore $bu = \inf(bS)$.

5. If $x \in S$, then $0 \le x \le u$, so that $x^2 \le u^2$ which implies $\sup T \le u^2$. If t is any upper bound of T, then $x \in S$ implies $x^2 \le t$ so that $x \le \sqrt{t}$. It follows that $u \le \sqrt{t}$, so that $u^2 \le t$. Thus $u^2 \le \sup T$.
6. Let $u := \sup f(X)$. Then $f(x) \le u$ for all $x \in X$, so that $a + f(x) \le a + u$ for all $x \in X$, whence $\sup\{a + f(x) : x \in X\} \le a + u$. If $w < a + u$, then $w - a < u$, so that there exists $x_w \in X$ with $w - a < f(x_w)$, whence $w < a + f(x_w)$, and thus w is not an upper bound for $\{a + f(x) : x \in X\}$.
7. Let $u := \sup S$, $v := \sup B$, $w := \sup(A + B)$. If $x \in A$ and $y \in B$, then $x + y \le u + v$, so that $w \le u + v$. Now, fix $y \in B$ and note that $x \le w - y$ for all $x \in A$; thus $w - y$ is an upper bound for A so that $u \le w - y$. Then $y \le w - u$ for all $y \in B$, so $v \le w - u$ and hence $u + v \le w$. Combining these inequalities, we have $w = u + v$.
8. If $u := \sup f(X)$ and $v := \sup g(X)$, then $f(x) \le u$ and $g(x) \le v$ for all $x \in X$, whence $f(x) + g(x) \le u + v$ for all $x \in X$. Thus $u + v$ is an upper bound for the set $\{f(x) + g(x) : x \in X\}$. Therefore $\sup\{f(x) + g(x) : x \in X\} \le u + v$.
9. (a) $f(x) = 2x + 1$, $\inf\{f(x) : x \in X\} = 1$.
   (b) $g(y) = y$, $\sup\{g(y) : y \in Y\} = 1$.
10. (a) $f(x) = 1$ for $x \in X$. (b) $g(y) = 0$ for $y \in Y$.
11. If $x \in X$, $y \in Y$, then $g(y) \le h(x, y) \le f(x)$. If we fix $y \in Y$ and take the infimum over $x \in X$, then we get $g(y) \le \inf\{f(x) : x \in X\}$ for each $y \in Y$. Now take the supremum over $y \in Y$.
12. Let $S := \{h(x, y) : x \in X, y \in Y\}$. We have $h(x, y) \le F(x)$ for all $x \in X, y \in Y$ so that $\sup S \le \sup\{F(x) : x \in X\}$. If $w < \sup\{F(x) : x \in X\}$, then there exists $x_0 \in X$ with $w < F(x_0) = \sup\{h(x_0, y) : y \in Y\}$, whence there exists $y_0 \in Y$ with $w < h(x_0, y_0)$. Thus w is not an upper bound of S, and so $w < \sup S$. Since this is true for any w such that $w < \sup\{F(x) : x \in X\}$, we conclude that $\sup\{F(x) : x \in X\} \le \sup S$.
13. If $x \in \mathbb{Z}$, take $n := x + 1$. If $x \notin \mathbb{Z}$, we have two cases: (i) $x > 0$ (which is covered by Cor. 2.4.6), and (ii) $x < 0$. In case (ii), let $z := -x$ and use 2.4.6. If $n_1 < n_2$ are integers, then $n_1 \le n_2 - 1$ so the sets $\{y : n_1 - 1 \le y < n_1\}$ and $\{y : n_2 - 1 \le y < n_2\}$ are disjoint; thus the integer n such that $n - 1 \le x < n$ is unique.
14. Note that $n < 2^n$ (whence $1/2^n < 1/n$) for any $n \in \mathbb{N}$.
15. Let $S_3 := \{s \in \mathbb{R} : 0 \le s, s^2 < 3\}$. Show that S_3 is nonempty and bounded by 3 and let $y := \sup S_3$. If $y^2 < 3$ and $1/n < (3 - y^2)/(2y + 1)$ show that

$y+1/n \in S_3$. If $y^2>3$ and $1/m<(y^2-3)/2y$ show that $y-1/m \in S_3$. Therefore $y^2=3$.

16. Case 1: If $a>1$, let $S_a := \{s \in \mathbb{R} : 0 \le s, s^2 < a\}$. Show that S_a is nonempty and bounded above by a and let $z := \sup S_a$. Now show that $z^2 = a$.
    Case 2: If $0<a<1$, there exists $k \in \mathbb{N}$ such that $k^2 a > 1$ (why?). If $z^2 = k^2 a$, then $(z/k)^2 = a$.
17. Consider $T := \{t \in \mathbb{R} : 0 \le t, t^3 < 2\}$. If $t>2$, then $t^3>2$ so that $t \notin T$. Hence $y := \sup T$ exists. If $y^3<2$, choose $1/n < (2-y^3)/(3y^2+3y+1)$ and show that $(y+1/n)^3 < 2$, a contradiction, and so on.
18. If $x<0<y$, then we can take $r=0$. If $x<y<0$, we apply 2.4.8 to obtain a rational number between $-y$ and $-x$.
19. There exists $r \in \mathbb{Q}$ such that $x/u < r < y/u$.

Section 2.5

Another important consequence of the Supremum Property of $\mathbb{R}$ is the Nested Intervals Property 2.5.2. It is an interesting fact that if we assume the validity of *both* the Archimedean Property 2.4.3 and the Nested Intervals Property, then we can prove the Supremum Property. Hence these two properties could be taken as the completeness axiom for $\mathbb{R}$. However, establishing this logical equivalence would consume valuable time and not significantly advance the study of real analysis, so we will not do so. (There are other properties that could be taken as the completeness axiom.)

The discussion of binary and decimal representations is included to give the student a concrete illustration of the rather abstract ideas developed to this point. However, this material is not vital for what follows and can be omitted or treated lightly. We have kept this discussion informal to avoid getting buried in technical details that are not central to the course.

Sample Assignment: Exercises 3, 4, 5, 6, 7, 8, 10, 11.

Partial Solutions:

1. Note that $[a,b] \subseteq [a',b']$ if and only if $a' \le a \le b \le b'$.
2. S has an upper bound b and a lower bound a if and only if S is contained in the interval $[a,b]$.
3. Since $\inf S$ is a lower bound for S and $\sup S$ is an upper bound for S, then $S \subseteq I_S$. Moreover, if $S \subseteq [a,b]$, then a is a lower bound for S and b is an upper bound for S, so that $[a,b] \supseteq I_S$.
4. Because z is neither a lower bound or an upper bound of S.
5. If $z \in \mathbb{R}$, then z is not a lower bound of S so there exists $x_z \in S$ such that $x_z \le z$. Also z is not an upper bound of S so there exists $y_z \in S$ such that $z \le y_z$. Since z belongs to $[x_z, y_z]$, it follows from the property (1) that $z \in S$.