Chapter 2: Probability, Statistics, and Traffic Theories

P2.1. A random number generator produces numbers between 1 and 99. If the current value of the random variable is 45, then what is the probability that the next randomly generated value for the same random variable, will also be 45. Explain clearly.

[Solution]

Let the random variable be X such that $1 \leq X \leq 99$.
Current value of $X = 45$.
Since the value of generation of random numbers is independent of the previous generation, the probability of generating 45 again is the same as for the first time. Hence,

$$P(X = 45) = \frac{1}{99}.$$

P2.2. A random digit generator on a computer is activated three times consecutively to simulate a random three-digit number.

(a) How many random three-digit numbers are possible?
(b) How many numbers will begin with the digit 2?
(c) How many numbers will end with the digit 9?
(d) How many numbers will begin with the digit 2 and end with the digit 9?
(e) What is the probability that a randomly formed number ends with 9 given that it begins with a 2?

[Solution]

(a) 900
(b) 100
(c) 90
(d) 10
(e) 1/10

P2.3. A snapshot of the traffic pattern in a cell with 10 users of a wireless system is given as follows:

User Number	1	2	3	4	5	6	7	8	9	10
Call Initiation Time	0	2	0	3	1	7	4	2	5	1
Call Holding Time	5	7	4	8	6	2	1	4	3	2

(a) Assuming the call setup/connection and call disconnection time to be zero, what is the average duration of a call?
(b) What is the minimum number of channels required to support this sequence of calls?
(c) Show the allocation of channels to different users for part (b) of the Problem.
(d) Given the number of channels obtained in part (b), for what fraction of time are the channels utilized?

[Solution]

(a) Average duration of a call is

$$\frac{5+7+4+8+6+2+1+4+3+2}{10} = 4.2.$$

(b) By plotting the number of calls by all users, we can determine how many users need to have a channel simultaneously. This gives us the minimum number of channels required to support the sequence of calls as 6.

(c) Allocation of channels to various users is

Channel number	1	2	3	4	5	6
User number	1	3	5	10	2	8
User number (allocated to same channel)	9	7		4		6

(d) Total duration of the calls is 42.

Total amount of time channels are available is $11 \times 6 = 66$.

Therefore fraction of time channels are used is $\frac{42}{66} = 0.6363$.

P2.4. A department survey found that 4 of 10 graduate students use CDMA cell phone service. If 3 graduate students are selected at random, what is the probability that 3 graduate students use CDMA cell phones?

[Solution]

$$\frac{C_4^3}{C_{10}^3} = \frac{2}{60} = 0.0333.$$

P2.5. There are three red balls and seven white balls in box A, and six red balls and four white balls in box B. After throwing a die, if the number on the die is 1 or 6, then pick a ball from box A. Otherwise, if any other number appears (i.e., 2, 3, 4, or 5), then pick a ball from box B. Selected ball has to put it back before proceeding further. Answer the followings:

(a) What is the probability that the selected ball is red?

(b) What is the probability a white ball is picked up in two successive selections?

[Solution]

(a) Probability of selecting the box × probability of selecting a red ball is equal to

$$= \frac{2}{6} \times \frac{3}{10} \times \frac{4}{6} \times \frac{6}{10} = 0.5.$$

(b) Since we are replacing the balls, the occurrence of two successive white balls is independent.

Thus probability of picking two white balls in succession is equal to

$$= \left(\frac{2}{6} \times \frac{7}{10} + \frac{4}{6} \times \frac{4}{10}\right)\left(\frac{2}{6} \times \frac{7}{10} + \frac{4}{6} \times \frac{4}{10}\right) = \frac{1}{4} = 0.25.$$

P2.6. Consider an experiment consisting of tossing two true dice. Let *X*, *Y*, and *Z* be the numbers shown on the first die, the second die, and total of both dice, respectively. Find P ($X \le 1$, $Z \le 2$) and P ($X \le 1$) P ($Z \le 2$) to show that *X* and *Y* are not independent.

[Solution]

$$P\left(X \le 1, Z \le 1\right) = \frac{1}{36} \approx 0.02778.$$

$$\mathrm{P}\left(X \le 1\right) P\left(Z \le 1\right) = \frac{1}{6} \times \frac{1}{36} = 0.00463$$

Since P ($X \le 1$, $Z \le 1$) and P ($X \le 1$) P ($Z \le 1$) are not equal, they are not independent.

P2.7. The following table shows the density of the random variable X.

x	1	2	3	4	5	6	7	8
p(x)	0.03	0.01	0.04	0.3	0.3	0.1	0.07	?

(a) Find *p*(8)
(b) Find the table for CDF *F*(*x*)
(c) Find $P(3 \le X \le 5)$
(d) Find $P(X \le 4)$ and $P(X < 4)$. Are the probabilities the same?
(e) Find $F(-3)$ and $F(10)$

[Solution]

(a) $p(8) = 0.15$

x	1	2	3	4	5	6	7	8
p(x)	0.03	0.01	0.04	0.3	0.3	0.1	0.07	0.15

(b) *F*(*x*) is given by

x	1	2	3	4	5	6	7	8
p(x)	0.03	0.04	0.08	0.38	0.68	0.78	0.85	1.0

(c) P ($3 \le X \le 5$) = 0.04 + 0.3 + 0.3 = 0.68
(d) $P(X \le 4) = 0.38$ and $P(X < 4) = 0.08$
Thus they are not equal.
(e) $F(-3) = 0$ and $F(10) = 1$

P2.8. The density for *X* is given in the table of Problem 7.

(a) Find $E[X]$
(b) Find $E[X^2]$
(c) Find Var[*X*]
(d) Find the standard deviation for *X*.

[Solution]

(a)

$$\begin{aligned} E[X] &= 1 \times 0.03 + 2 \times 0.01 + 3 \times 0.04 + 4 \times 0.3 + 5 \times 0.3 + 6 \times 0.1 \\ &\quad + 7 \times 0.07 + 8 \times 0.15 \\ &= 5.16. \end{aligned}$$

(b) $E\left[X^2\right] = \sum_{x=1}^{8} x^2 p(x) = 29.36.$

(c) Var[X] = 2.7344.

(d) Standard deviation = 1.65.

P2.9. Find the probability when

(a) $k = 2$ and $\lambda = 0.01$ for Poisson distribution.

(b) $p = 0.01$ and $k = 2$ for geometric distribution.

(c) Repeat (b) when binomial distribution is used and $n = 10$.

[Solution]

(a) $p_k = \dfrac{\lambda^k e^{-\lambda}}{k!}$, when $k = 2$, $\lambda = 0.01$, Probability = 0.0000495025.

(b) $p_k = p(1-p)^k$, when $k = 2$, $\lambda = 0.01$, Probability = 0.0000495025.

(c) $p_k = \begin{pmatrix} k \\ n \end{pmatrix} p^k (1-p)^{n-k}$ when $n = 10$, $k = 2$, $\lambda = 0.01$, Probability = 0.00415235.

P2.10. Find the distribution function of the maximum of a finite set of independent random variables $\{X_1, X_2, ..., X_n\}$, where X_i has distribution function F_{X_i}. What is this distribution when X_i is exponential with a mean of $\dfrac{1}{\mu_i}$.

[Solution]

$$Y = \max\{X_1, X_2, \ldots, X_n\}$$

$$\begin{aligned} P(Y \le y) &= P\left(\max\{X_1, X_2, \ldots, X_n\} \le y\right) \\ &= P\left(X_1 \le y, X_2 \le y, \ldots, X_n \le y\right) \\ &= \prod_{i=1}^{n} P\left(X_i \le y\right) \\ &= \prod_{i=1}^{n} F_{X_i}(y). \end{aligned}$$

When X_i is exponential with mean $\dfrac{1}{\mu_i}$,

$$F_{X_i}(y) = 1 - e^{-\mu_i y}.$$

Therefore, the CDF of Y is given by

$$F_Y(y) = \prod_{i=1}^{n} F_{X_i}(y)$$
$$= \prod_{i=1}^{n} \left(1 - e^{-\mu_i y}\right).$$

P2.11. The number of calls arrive under a particular time in a cell has been established to be a Poisson distribution. The average number of calls arriving in a cell in 1 millisecond is 5. What is the probability that 8 calls arrive in a cell in a given millisecond?

[Solution]

Since $\lambda = 5$,

$$p_k = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{5^8 e^{-5}}{8!} = 0.065278.$$

P2.12. Given that the number of arrivals of data packet in the receiver follows a Poisson distribution on which arrival rate is 10 arrivals per second. What is the probability that the number of arrivals is more than 8 but less than 11 during a time of interval of 2 seconds?

[Solution]

The number of arrivals during a time interval of 2 seconds follows a Poisson distribution with mean 10 × 2 = 20 arrivals/sec. Hence,

$$\begin{aligned} P(8 < k < 11) &= P(k = 9, k = 10) \\ &= P(k = 9) + P(k = 10) \\ &= \frac{20^9 e^{-20}}{9!} + \frac{20^{10} e^{-20}}{10!} \\ &= 0.00872446. \end{aligned}$$

P2.13. In a wireless office environment, all calls are made between 8 am and 5 pm over the period of 24 hours. Assuming the number of calls to be uniformly distributed between 8 am and 5 pm, find the pdf of the number of calls over the 24 hour period. Also, determine the CDF and the variance of the call distribution.

[Solution]

Since

$$f(x) = \begin{cases} \frac{1}{9}, & x \in [8, 17] \\ 0, & x \in [0, 8] \cup (17, 24] \end{cases},$$

the CDF is given by

$$F(x)=\begin{cases}0, & x\in[0,8]\\ \frac{x-8}{9}, & x\in[8,17]\\ 1, & x\in[17,24]\end{cases},$$

Therefore,

$$\begin{aligned}E[X]&=\int_8^{17} xf(x)dx\\ &=\int_8^{17} x\frac{1}{9}dx\\ &=\frac{25}{2}\\ &=12.5,\\ E[X^2]&=\int_8^{17} x^2 f(x)dx\\ &=\int_8^{17} x^2\frac{1}{9}dx\\ &=163,\\ Var(X)&=E[X\ \ ]-(E[X])^2\\ &=6.75.\end{aligned}$$

P2.14. A gambler has a regular coin and a two-headed coin in his packet. The probability of selecting the two-head coin is given as p = 2/3. He select a coin and flips it n = 2 times and obtains heads both the times. What is the probability that the two-headed coin is picked both the times?

[Solution]

Probability is equal to

$$\frac{2}{3}\times 1\times\frac{2}{3}\times 1=\frac{4}{9}\approx 0.4444.2$$

P2.15. A Poisson process exhibits a memoryless property and is of great importance in traffic analysis. Prove that this property is exhibited by all Poisson processes. Explain clearly every step of your analytical proof.

[Solution]

In order to prove that Poisson process is memoryless, we need to prove the following equation:

$$P(X>t+\delta|X>\delta)=P(X>t).$$

Thus,

$$\begin{aligned} P\left(X > t+\delta \middle| X > \delta\right) &= \frac{P\left(X > t+\delta \cap X > \delta\right)}{P\left(X > \delta\right)} \\ &= \frac{P\left(X > t+\delta\right)}{P\left(X > \delta\right)} \\ &= \frac{1-P\left(X \le t+\delta\right)}{1-P\left(X \le \delta\right)} \\ &= 1-P\left(X \le t\right) \\ &= P\left(X > t\right). \end{aligned}$$

P2.16. What should be a relationship between call arrival rate and service rate when a cellular system is in a steady state? Explain clearly.

[Solution]

When the system reaches the steady state, the call arrival rate should be less or equal to the service rate, otherwise the system will be unstable.

P2.17. Consider a cellular system with an infinite number of channels. In such a system, all arriving calls begin receiving service immediately. The average call holding time is $\frac{1}{n\mu}$ when there are n calls in the system. Draw a state transition diagram for this system and develop expressions for the following:

(a) Steady-state probability P_n of n calls in the system.
(b) Steady-state probability P_0 of no calls in the system.
(c) Average number of calls in the system, L_S.
(d) Average dwell time, W_S.
(e) Average queue length, L_Q.

[Solution]

The system model is M/M/∞/∞ with average arrival rate of λ and average service rate of μ/n.
Probability P_n for the system with n calls can be found through the system balance equations as follows:

$$\lambda P_i = \left(i+1\right)\mu P_{i+1}, \qquad 0 \le i \le \infty.$$

Thus, we have

$$P_i = \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i P_0, \qquad 0 \le \mathrm{i} \le \infty.$$

Using normalized condition

$$\sum_{i=0}^{\infty} P_i = 1,$$

we have

$$P_0 = \left[\sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i\right]^{-1}$$

$$= e^{-\frac{\lambda}{\mu}}.$$

(a) Steady-state probability P_n of n calls in the system is given by

$$P_n = \frac{1}{\text{n!}}\left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\lambda}{\mu}}, \quad 0 \le n \le \infty.$$

(b) Steady-state probability P_0 of no calls in the system is given by

$$P_0 = e^{-\frac{\lambda}{\mu}}.$$

(c) Average number of calls in the system L_S is given by

$$L_S = \sum_{i=1}^{\infty} iP_i$$

$$= \frac{\lambda}{\mu}.$$

(d) Average dwell time W_S is given by

$$W_S = \frac{L_S}{\lambda}$$

$$= \frac{1}{\mu}.$$

(e) Average queue length is equal to 0.

P2.18. Consider a cellular system in which each cell has only one channel (single server) and an infinite buffer for storage the calls. In this cellular system, call arrival rates are discouraged, that is, the call rate is only $\lambda/(n + 1)$ when there are n calls in the system. The interarrival times of calls are exponentially distributed. The call holding times are exponentially distributed with mean rate μ. Develop expressions for the following:

(a) Steady-state probability P_n of n calls in the system.
(b) Steady-state probability P_0 of no calls in the system.
(c) Average number of calls in the system, L_S.
(d) Average dwell time, W_S.
(e) Average queue length, L_Q.

[Solution]

The system model is M/M/1/∞ with average arrival rate of $\lambda/(n + 1)$, and average service rate of μ.

Traffic intensity $\rho = \dfrac{\frac{\lambda}{n+1}}{\mu} = \dfrac{\lambda}{(n+1)\mu}.$

Probability P_n for the system with n calls can be found through the system balance equations as follows:

$$\lambda P_i = (i+1)\mu P_{i+1}, \quad 0 \le i \le \infty.$$

Thus, we have

$$P_i = \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i P_0, \quad 0 \le \mathrm{i} \le \infty.$$

Using normalized condition

$$\sum_{i=0}^{\infty} P_i = 1,$$

we have

$$P_0 = \left[\sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i\right]^{-1}$$
$$= e^{-\frac{\lambda}{\mu}}.$$

(a) Steady-state probability P_n of n calls in the system is given by

$$P_n = \frac{1}{\mathrm{n}!}\left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\lambda}{\mu}}, \quad 0 \le n \le \infty.$$

(b) Steady-state probability P_0 of no calls in the system is given by

$$P_0 = e^{-\frac{\lambda}{\mu}}.$$

(c) Average number of calls in the system L_S is given by

$$L_S = \sum_{i=1}^{\infty} iP_i$$
$$= \frac{\lambda}{\mu}.$$

(d) Average arrival rate is

$$\overline{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda}{i+1} P_i$$
$$= \mu\left(1 - e^{-\frac{\lambda}{u}}\right).$$

Therefore, average dwell time W_S is given by

$$W_S = \frac{L_S}{\overline{\lambda}}$$
$$= \frac{\lambda}{\mu^2\left(1 - e^{-\frac{\lambda}{\mu}}\right)}.$$

(e) Since average queue waiting time is given by

$$W_Q = W_S - \frac{1}{\mu}$$

$$= \frac{\lambda - \mu + \mu e^{-\frac{\lambda}{\mu}}}{\mu^2\left(1 - e^{-\frac{\lambda}{\mu}}\right)},$$

average queue length is given by

$$L_Q = \bar{\lambda} W_Q$$

$$= \frac{\lambda - \mu + \mu e^{-\frac{\lambda}{\mu}}}{\mu}.$$

P2.19. In a transition diagram of M/M/5 model, write the state transition equations and find a relation for the system to be in each state.

[Solution]

Hint: The system model is M/M/5 with average arrival rate, and average service rate of

$$\mu = \begin{cases} n\mu, & n = 1, 2, 3, 4, \\ 5\mu, & n \geq 5, \end{cases}$$

The probability of n jobs in the system is given by

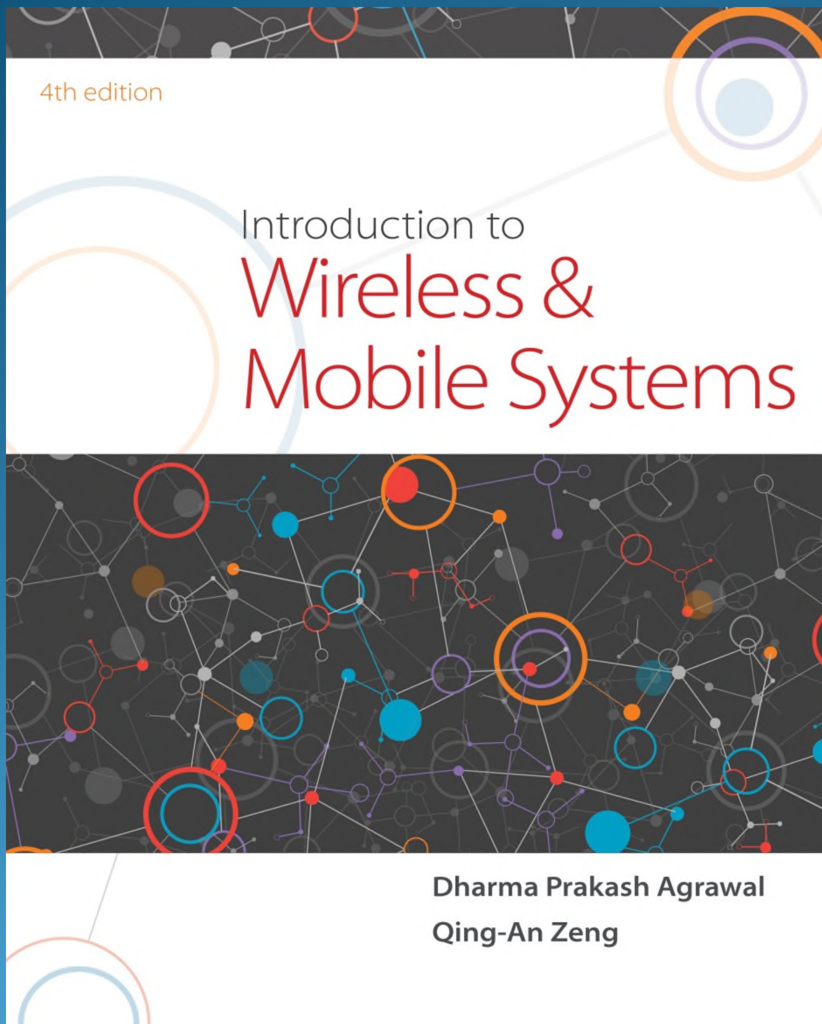
$$P_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, & n = 1, 2, 3, 4, \\ \frac{1}{5!5^{n-5}}\left(\frac{\lambda}{\mu}\right)^n P_0, & n \geq 5, \end{cases}$$

P2.20. In the M/M/1/∞ queuing system, suppose λ and μ are doubled. How are L_s and W_s changed?

[Solution] L_S is the same as M/M/1/∞ and W_S is the helf of M/M/1/∞.

P2.21. If the number of mechanics in Example 2.4 is increased to 3, what is the impact on the performance parameters?

[Solution] It can be seen that the system is M/M/3/∞ queuing system with λ = 3/m, and the service rate $\mu = \frac{60}{10}$ = 6/m. Therfore, the offered load $\rho = \frac{\lambda}{3\mu} = \frac{1}{6}$.



Chapter 2

Probability, Statistics, and Traffic Theories

Outline

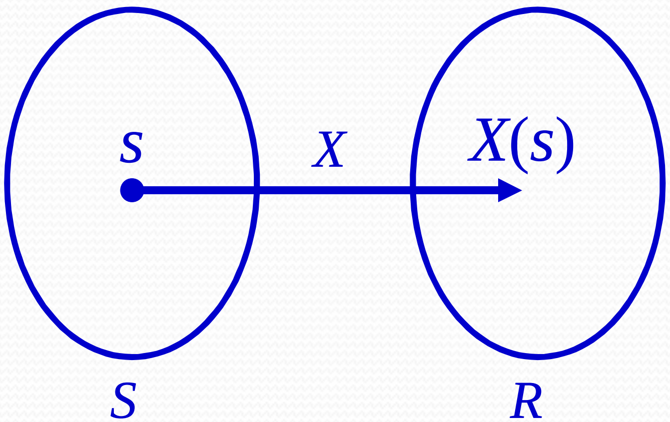
- Introduction
- Probability Theory and Statistics Theory
  - Random variables
  - Probability mass function (pmf)
  - Probability density function (pdf)
  - Cumulative distribution function (CDF)
  - Expected value, nth moment, nth central moment, and variance
  - Some important distributions
- Traffic Theory
  - Poisson arrival model, etc.
- Basic Queuing Systems
  - Little's law
  - Basic queuing models

Introduction

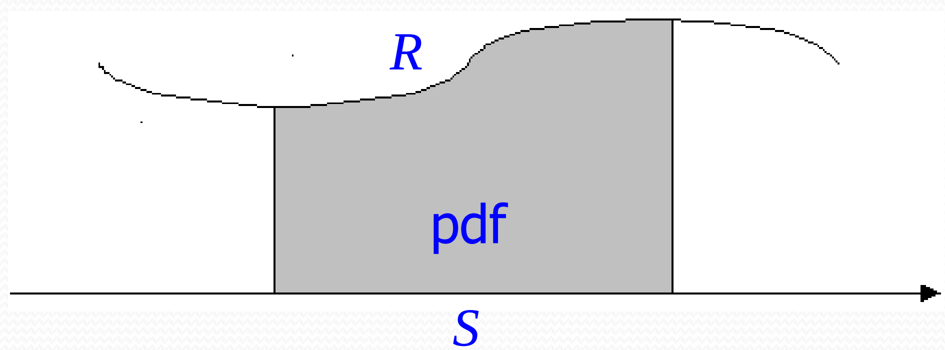
- Several factors influence the performance of wireless systems:
  - Density of mobile users
  - Cell size
  - Moving direction and speed of users (Mobility models)
  - Call rate, call duration
  - Interference, etc.
- Probability, statistics theory and traffic patterns, help make these factors tractable

Probability Theory and Statistics Theory

- Random Variables (RVs)
  - Let *S* be sample associated with experiment *E*
  - *X* is a function that associates a real number to each $s \in S$
  - RVs can be of two types: Discrete or Continuous
  - Discrete random variable => probability mass function (pmf)
  - Continuous random variable => probability density function (pdf)



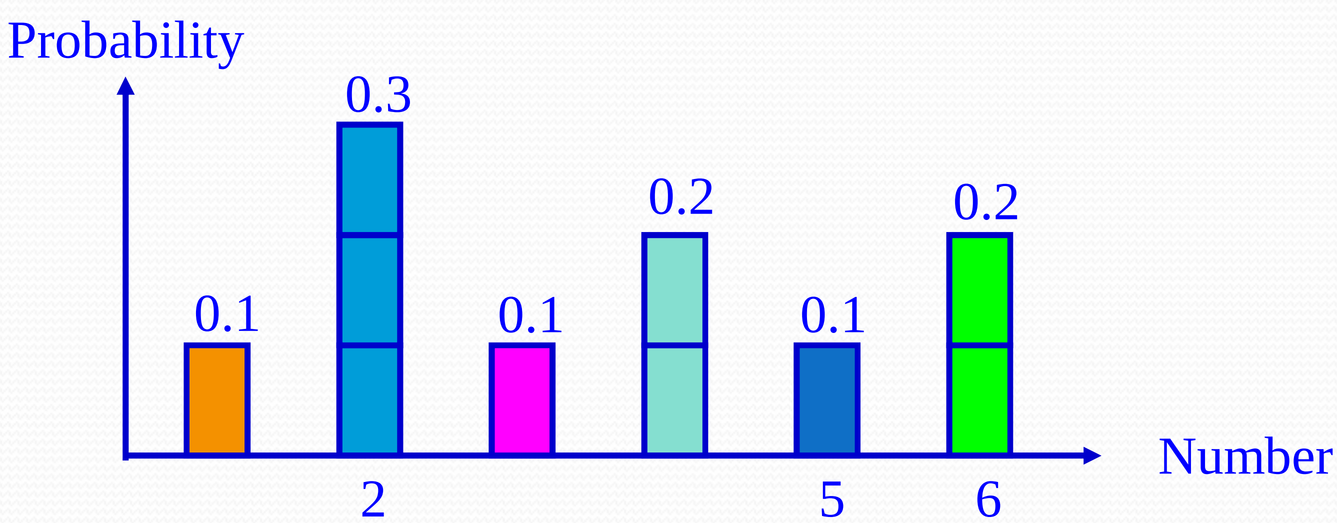
Discrete random variable



Continuous random variable

Discrete Random Variables

- In this case, $X(s)$ contains a finite or infinite number of values
  - The possible values of X can be enumerated
- **For Example:** Throw a 6 sided dice and calculate the probability of a particular number appearing.



Discrete Random Variables

- The probability mass function (pmf) $p(k)$ of X is defined as:

  $p(k) = p(X = k)$, for k = 0, 1, 2, ...

  where

  1. Probability of each state occurring

     $0 \le p(k) \le 1$, for every k;

  2. Sum of all states

     $\sum p(k) = 1$, for all k

Continuous Random Variables

- In this case, X contains an infinite number of values
- Mathematically, X is a continuous random variable if there is a function f, called probability density function (pdf) of X that satisfies the following criteria:
  1. $f(x) \geq 0$, for all x;
  2. $\int f(x)dx = 1$

Cumulative Distribution Function

- Applies to all random variables
- A cumulative distribution function (CDF) is defined as:
  - For discrete random variables:

$$P(k) = P(X \leq k) = \sum_{\text{all} \leq k} P(X = k)$$

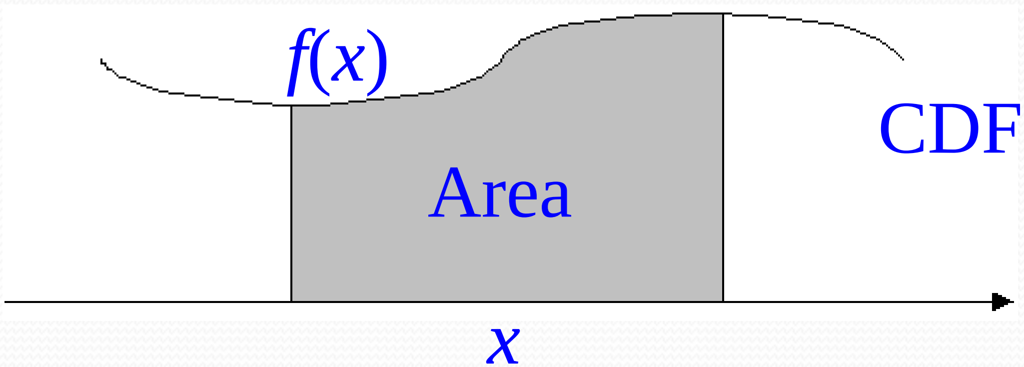
  - For continuous random variables:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

Probability Density Function

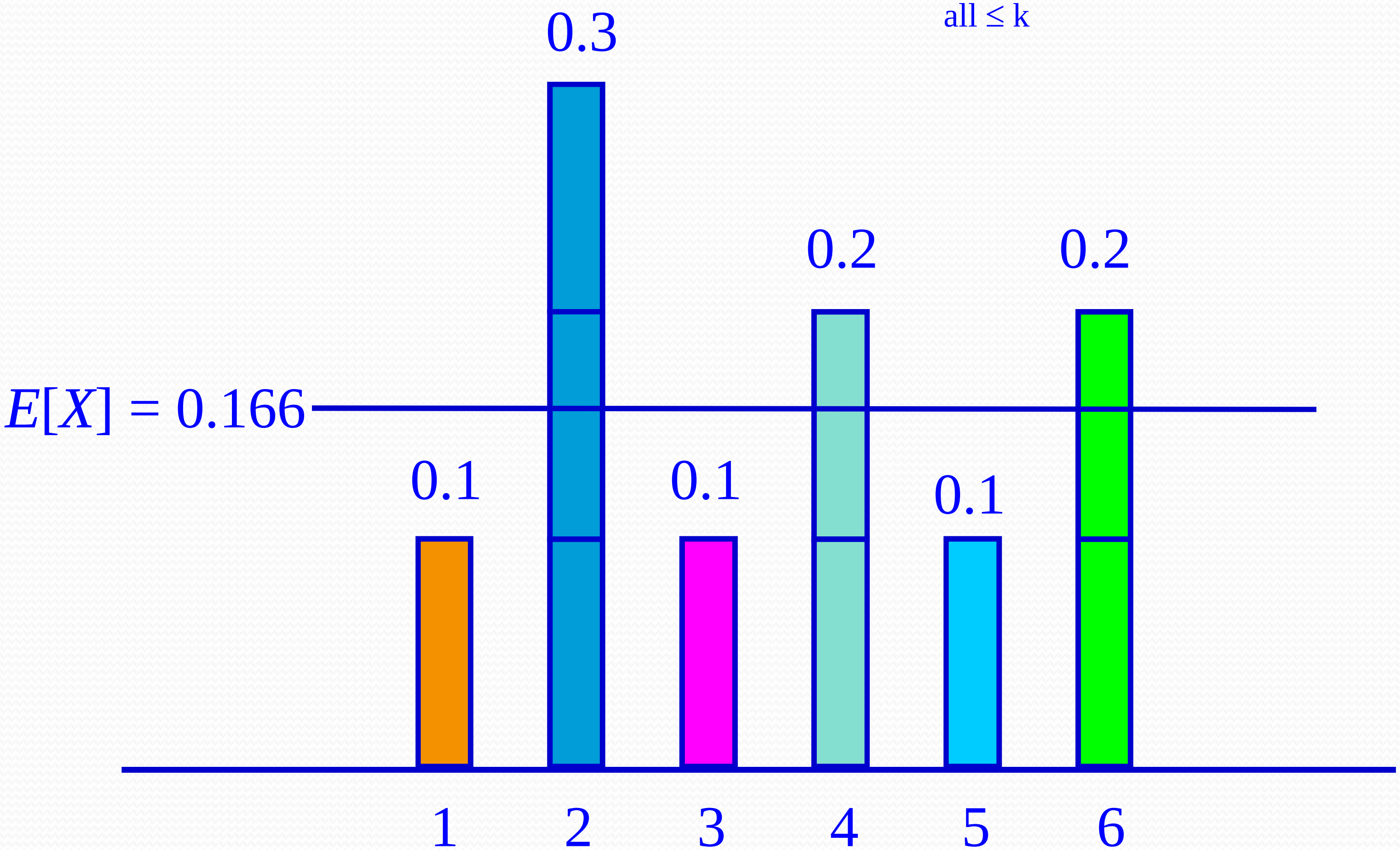
- The pdf $f(x)$ of a continuous random variable X is the derivative of the CDF $F(x)$, i.e.,

$$f(x) = \frac{dF_X(x)}{dx}$$



Expected Value, n^{th} Moment, n^{th} Central Moment, and Variance

Average $E[X] = \sum_{\text{all} \leq k} kP(X = k) = 1/6$



Expected Value, n^{th} Moment, n^{th} Central Moment, and Variance

- Discrete Random Variables
  - Expected value represented by E or average of random variable

$$E[X] = \sum_{\text{all} \le k} kP(X = k)$$

  - Variance or the second central moment

$$\sigma^2 = Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

  - n^{th} moment

$$E[X^n] = \sum_{\text{all} \le k} k^n P(X = k)$$

  - n^{th} central moment

$$E[(X - E[X])^n] = \sum_{\text{all} \le k} (k - E[X])^n P(X = k)$$

Expected Value, n^th Moment, n^th Central Moment, and Variance

- Continuous Random Variable
  - Expected value or mean value

$$E[X] = \int_{-\infty}^{+\infty} x f(x) dx$$

  - Variance or the second central moment

$$\sigma^2 = Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

  - n^th moment

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f(x) dx$$

  - n^th central moment

$$E[(X - E[X])^n] = \int_{-\infty}^{+\infty} (x - E[X])^n f(x) dx$$

Some Important Discrete Random Distributions

- Poisson

$$P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}, k=0, 1, 2, \ldots, \text{ and } \lambda>0$$

  - $E[X] = \lambda$, and $Var(X) = \lambda$

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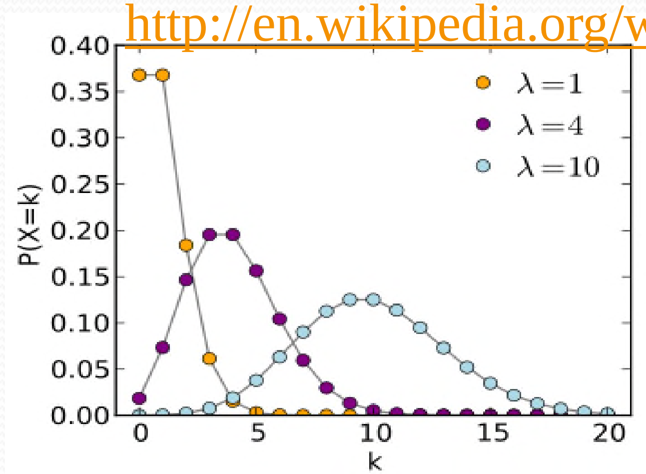
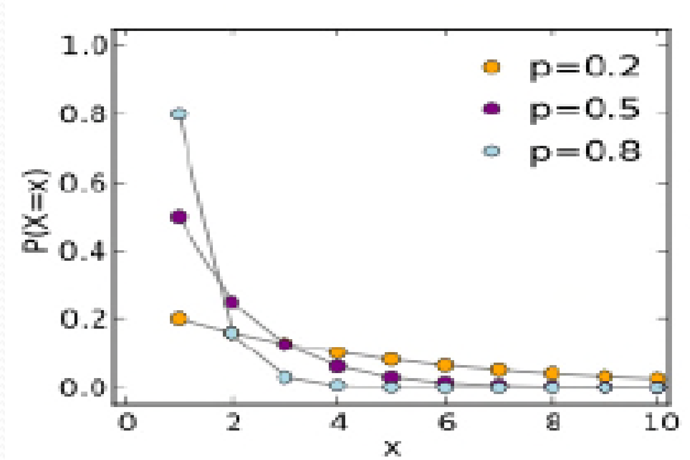


- Geometric

$$P(X = k) = p(1-p)^{k-1} ,$$

where p is success probability

  - $E[X] = 1/(1-p)$, and $Var(X) = p/(1-p)^2$



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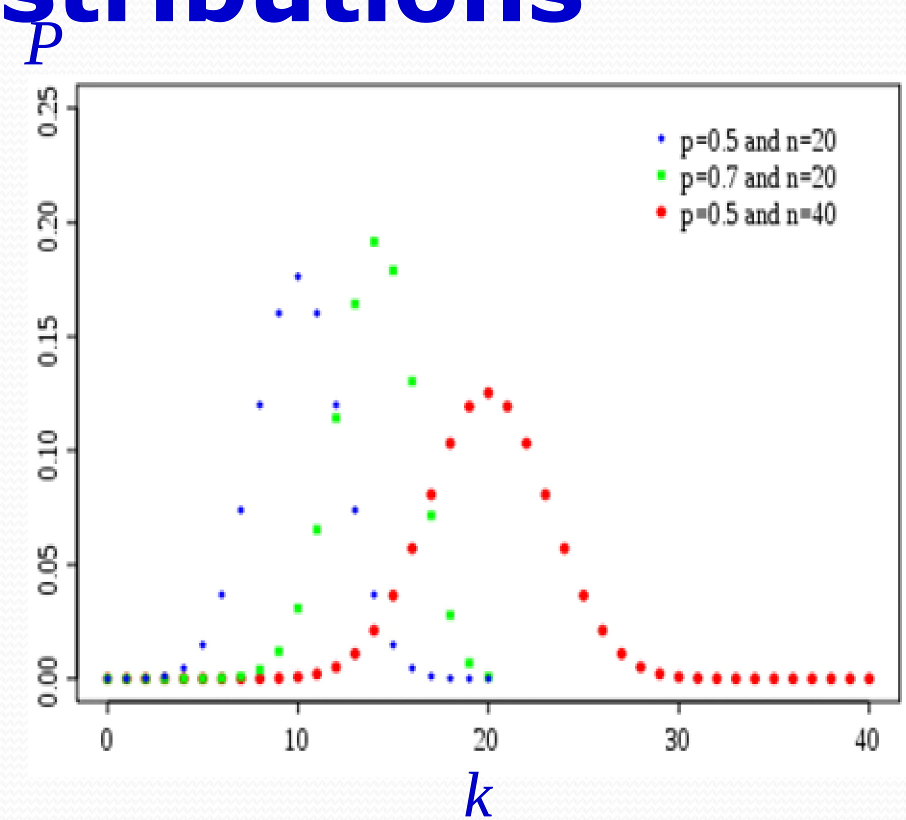
Some Important Discrete Random Distributions

- Binomial

Out of n dice, exactly k dice have the same value: probability p^k and $(n-k)$ dice have different values: probability$(1-p)^{n-k}$.

For any k dice out of n:



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$$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k},$$

where,

$k=0,1,2,\ldots,n$; $n=0,1,2,\ldots$; p is the sucess probability, and

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

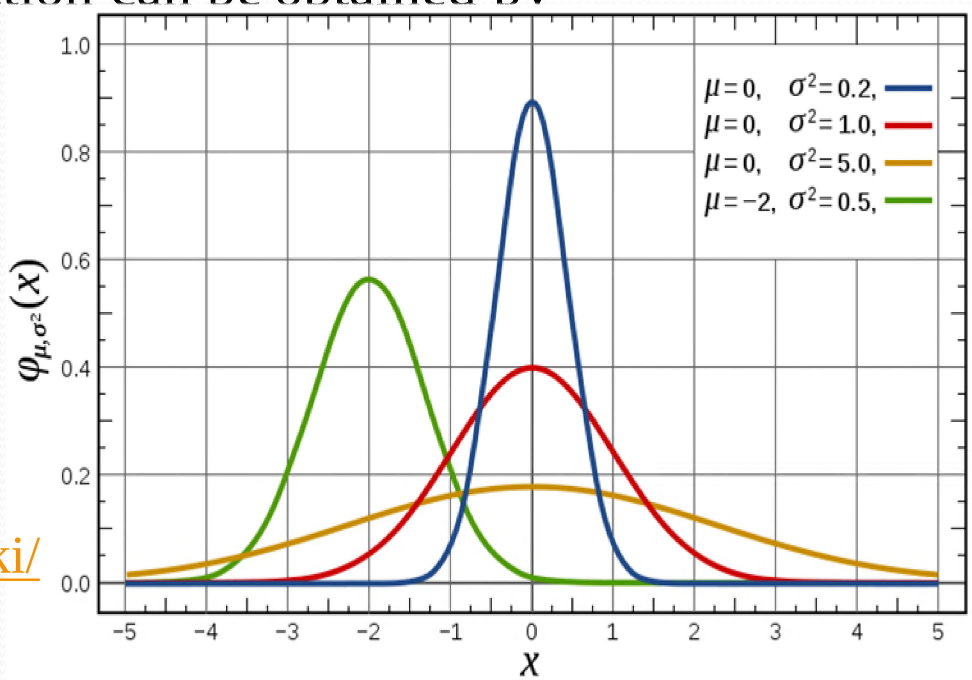
Some Important Continuous Random Distributions

- Normal: $E[X] = \mu$, and $Var(X) = \sigma^2$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \text{ for } -\infty < x < \infty$$

and the cumulative distribution function can be obtained by

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy$$



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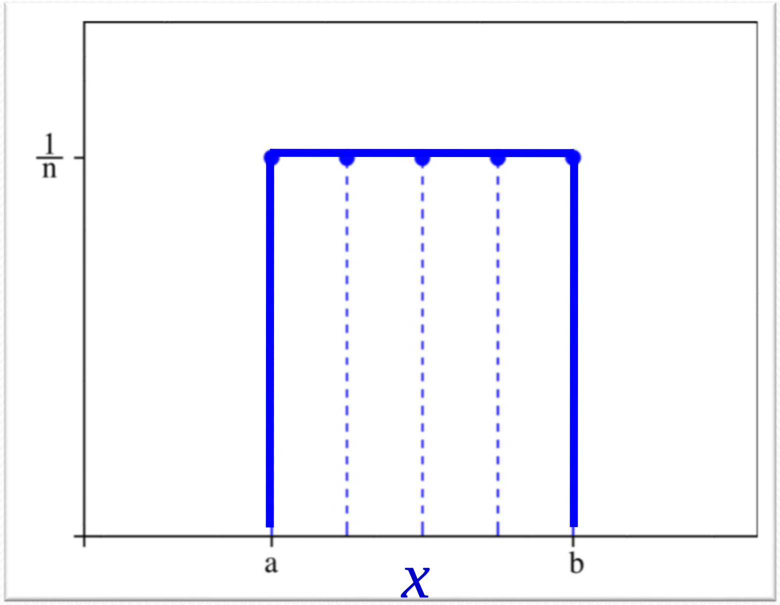
Some Important Continuous Random Distributions

- Uniform

http://en.wikipedia.org/wiki/

$$f_X(x) = \begin{cases} \dfrac{d}{b-a}, & \text{for } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$



and the cumulative distribution function is

$$F_X(x) = \begin{cases} 0, & \text{for } x < a \\ \dfrac{x-a}{b-a}, & \text{for } a \le x \le b \\ 1, & \text{for x > b} \end{cases}$$

- $E[X] = (a+b)/2$, and $Var(X) = (b-a)^2/12$

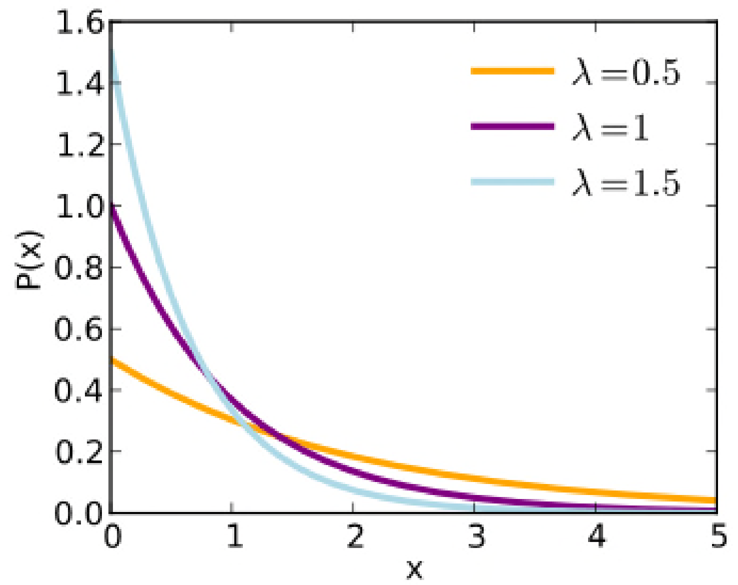
Some Important Continuous Random Distributions

- Exponential

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & \text{for } 0 \le x < \infty \end{cases}$$

and the cumulative distribution function is

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & \text{for } 0 \le x < \infty \end{cases}$$



http://en.wikipedia.org/wiki/

- $E[X] = 1/\lambda$, and $Var(X) = 1/\lambda^2$

Multiple Random Variables

- There are cases where the result of one experiment determines the values of several random variables
- The joint probabilities of these variables are:
  - Discrete variables:

    $p(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n)$
  - Continuous variables:

    CDF: $F_{x1x2\ldots xn}(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n)$

    pdf: $$f_{X_1, X_2, \ldots X_n}(x_1, x_2, \ldots x_n) = \frac{\partial^n F_{X_1, X_2, \ldots X_n}(x_1, x_2, \ldots x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

Independence and Conditional Probability

- Independence: The random variables are said to be independent of each other when the occurrence of one does not affect the other. The pmf for discrete random variables in such a case is given by:

  $p(x_1,x_2,\ldots x_n)=P(X_1=x_1)P(X_2=x_2)\ldots P(X_3=x_3)$ and for continuous random variables as:

  $F_{X1,X2,\ldots Xn} = F_{X1}(x_1)F_{X2}(x_2)\ldots F_{Xn}(x_n)$

- Conditional probability: is the probability that $X_1= x_1$ given that $X_2= x_2$. Then for discrete random variables the probability becomes:

$$P(X_1 = x_1 \mid X_2 = x_2,\ldots, X_n = x_n) = \frac{P(X_1 = x_1, X_2 = x_2,\ldots, X_n = x_n)}{P(X_2 = x_2,\ldots, X_n = x_n)}$$

and for continuous random variables it is:

$$P(X_1 \le x_1 \mid X_2 \le x_2,\ldots, X_n \le x_n) = \frac{P(X_1 \le x_1, X_2 \le x_2,\ldots, X_n \le x_n)}{P(X_2 \le x_2,\ldots X_n \le x_n)}$$

Bayes Theorem

- A theorem concerning conditional probabilities of the form $P(X|Y)$ (read: the probability of X, given Y) is

$$P(X \mid Y) = \frac{P(Y \mid X) P(X)}{P(Y)}$$

where $P(X)$ and $P(Y)$ are the unconditional probabilities of X and Y, respectively

Important Properties of Random Variables

- Sum property of the expected value
  - Expected value of the sum of random variables:

$$E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i]$$

- Product property of the expected value
  - Expected value of product of stochastically independent random variables

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i]$$

Important Properties of Random Variables

- Sum property of the variance
  - Variance of the sum of random variables is

$$Var\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i a_j \operatorname{cov}[X_i, X_j]$$

where $cov[X_i,X_j]$ is the covariance of random variables X_i and X_j and

$$\begin{aligned}\operatorname{cov}[X_i, X_j] &= E[(X_i - E[X_i])(X_j - E[X_j])] \\ &= E[X_i X_j] - E[X_i]E[X_j]\end{aligned}$$

If random variables are independent of each other, i.e., $cov[X_i,X_j]=0$, then

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

Important Properties of Random Variables

- **Distribution of sum -** For continuous random variables with joint pdf $f_{XY}(x, y)$ and if $Z = \Phi(X,Y)$, the distribution of Z may be written as

$$F_Z(z) = P(Z \le z) = \int_{\phi Z} f_{XY}(x, y)dxdy$$

where Φ_Z *is a subset of* Z.

- For a special case $Z= X+Y$

$$F_Z(z) = \iint_{\phi Z} f_{XY}(x, y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)dxdy$$

- If X and Y are independent variables, the $f_{XY}(x,y)= f_X(x)f_Y(y)$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx, \quad \text{for } -\infty \le z < \infty$$

- If both X and Y are non negative random variables, then pdf is the convolution of the individual pdfs, $f_X(x)$ and $f_Y(y)$

$$f_Z(z) = \int_{0}^{z} f_X(x) f_Y(z-x)dx, \quad \text{for } -\infty \le z < \infty$$

Central Limit Theorem

The *Central Limit Theorem* states that whenever a random sample (X_1, X_2,.. X_n) of size n is taken from any distribution with expected value $E[X_i] = \mu$ and variance $Var(X_i) = \sigma^2$, where i =1,2,..,n, then their arithmetic mean is defined by

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Central Limit Theorem

- Mean of a sufficiently large number of independent random variables, each with finite mean and variance, will be approximately normally distributed
- The sample mean is approximated to a normal distribution with
  - $E[S_n] = \mu$, and
  - $Var(S_n) = \sigma^2 / n$
- The larger the value of the sample size n, the better the approximation to the normal
- This is very useful when inference between signals needs to be considered

Poisson Arrival Model

- Events occur continuously and independently of one another
- A Poisson process is a sequence of events "randomly spaced in time"
- For example, customers arriving at a bank and Geiger counter clicks are similar to packets arriving at a buffer
- The rate λ of a Poisson process is the average number of events per unit time (over a long time)

Properties of a Poisson Process

- Properties of a Poisson process
  - For a time interval $[0, t]$, the probability of n arrivals in t units of time is

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

  - For two disjoint (non overlapping) intervals (t_1, t_2) and (t_3, t_4), (i.e. , $t_1 < t_2 < t_3 < t_4$), the number of arrivals in (t_1, t_2) is independent of arrivals in (t_3, t_4)

Interarrival Times of Poisson Process

- Interarrival times of a Poisson process
  - We pick an arbitrary starting point t_0 in time . Let T_1 be the time until the next arrival. We have

    $$P(T_1 > t) = P_0(t) = e^{-\lambda t}$$

  - Thus the cumulative distribution function of T_1 is given by

    $$F_{T1}(t) = P(T_1 \leq t) = 1 - e^{-\lambda t}$$

  - The pdf of T_1 is given by

    $$f_{T1}(t) = \lambda e^{-\lambda t}$$

Therefore, T_1 has an exponential distribution with mean rate λ

Exponential Distribution

- Similarly T_2 is the time between first and second arrivals, we define T_3 as the time between the second and third arrivals, T_4 as the time between the third and fourth arrivals and so on
- The random variables T_1, T_2, T_3… are called the interarrival times of the Poisson process
- T_1, T_2, T_3,… are independent of each other and each has the same exponential distribution with mean arrival rate λ

Memoryless and Merging Properties

- Memoryless property
  - A random variable *X* has the property that "the future is independent of the past" i.e., the fact that it hasn't happened yet, tells us nothing about how much longer it will take before it does happen
- Merging property
  - If we merge *n* Poisson processes with distributions for the inter arrival times

$$1- e^{-\lambda_i t} \quad \text{for } i = 1, 2, \ldots, n$$

into one single process, then the result is a Poisson process for which the inter arrival times have the distribution $1- e^{-\lambda t}$ with mean

$$\lambda = \lambda_1 + \lambda_2 + .. + \lambda_n$$

Basic Queuing Systems

- What is queuing theory?
  - Queuing theory is the study of queues (sometimes called waiting lines)
  - Can be used to describe real world queues, or more abstract queues, found in many branches of computer science, such as operating systems
- Basic queuing theory

  Queuing theory is divided into 3 main sections:
  - Traffic flow
  - Scheduling
  - Facility design and employee allocation

Kendall's Notation

- D.G. Kendall in 1951 proposed a standard notation for classifying queuing systems into different types. Accordingly the systems were described by the notation A/B/C/D/E where:

A	Distribution of inter arrival times of customers
B	Distribution of service times
C	Number of servers
D	Maximum number of customers in the system
E	Calling population size

Kendall's notation

A and B can take any of the following distributions types:

M	Exponential distribution (Markovian)
D	Degenerate (or deterministic) distribution
E_k	Erlang distribution (k = shape parameter)
H_k	Hyper exponential with parameter k

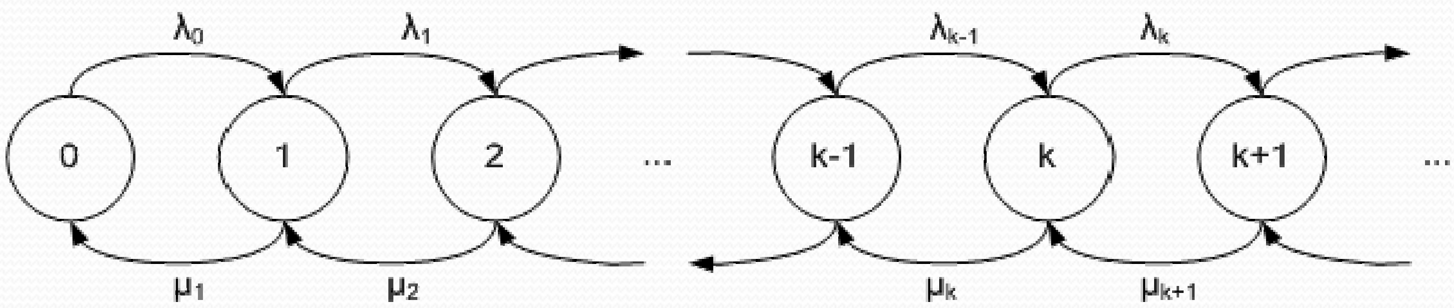
Little's Law

- Assuming a queuing environment to be operating in a stable steady state where all initial transients have vanished, the key parameters characterizing the system are:
  - λ – the mean steady state consumer arrival
  - N – the average no. of customers in the system
  - T – the mean time spent by each customer in the system

which gives

$$N = \lambda T$$

Markov Process

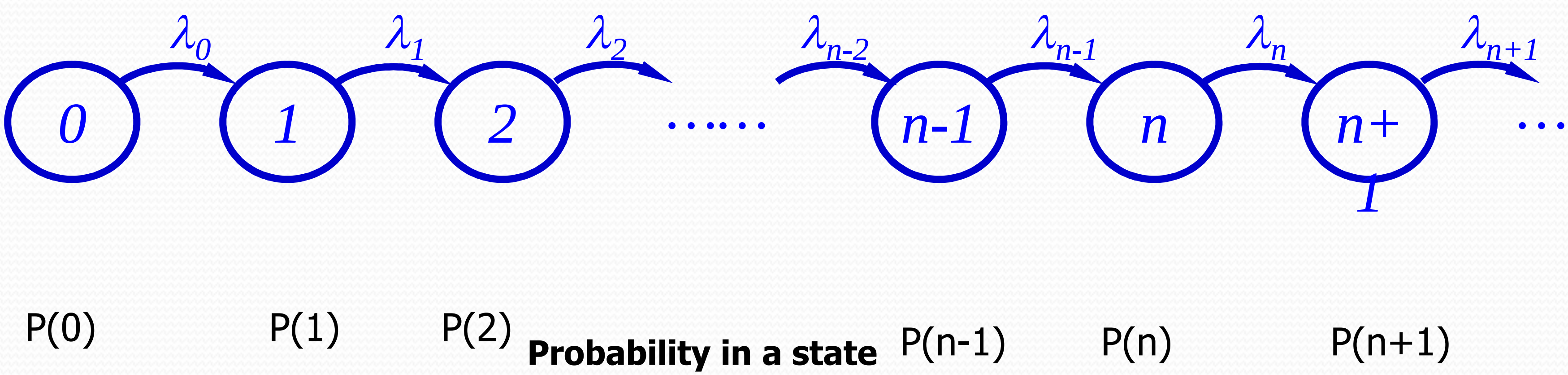


- A Markov process is one in which the next state of the process depends only on the present state, irrespective of any previous states taken by the process
- The knowledge of the current state and the transition probabilities from this state allows us to predict the next state

Birth-Death Process

- Special type of Markov process
- Often used to model a population (or, number of jobs in a queue)
- If, at some time, the population has *n* entities (*n* jobs in a queue), then birth of another entity (arrival of another job) causes the state to change to *n*+1
- On the other hand, a death (a job removed from the queue for service) would cause the state to change to *n*-1
- Any state transitions can be made only to one of the two neighboring states

State Transition Diagram

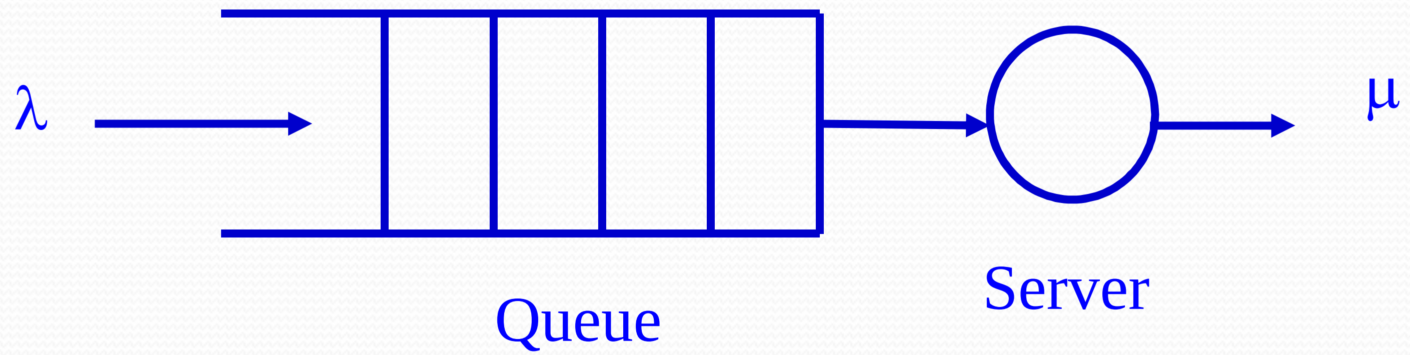


The state transition diagram of the continuous birth-death process

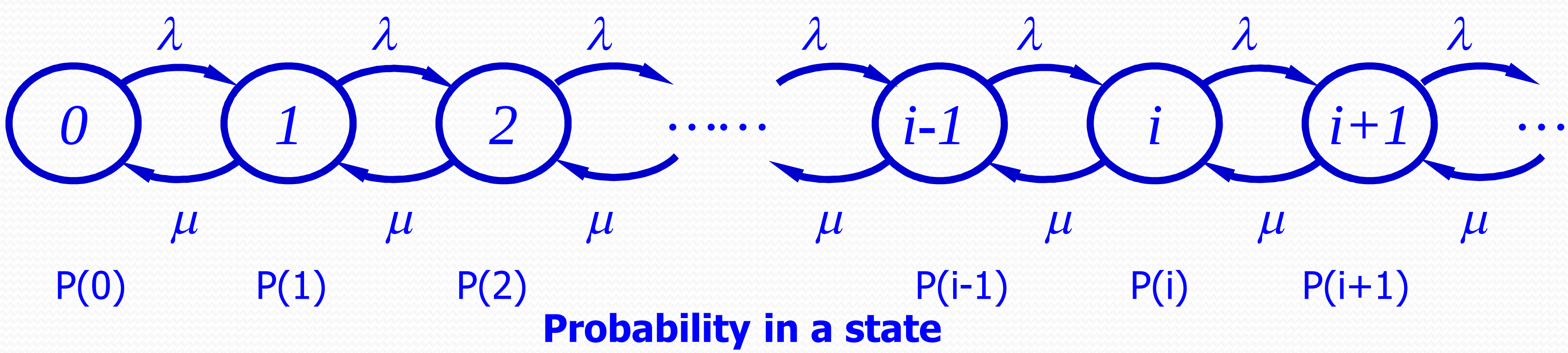
M/M/1/∞ or M/M/1 Queuing System

- Distribution of inter arrival times of customers: M
- Distribution of service times: M
- Number of servers: 1
- Maximum number of customers in the system: ∞
- When a customer arrives in this system it will be served if the server is free, otherwise the customer is queued
- In this system, customers arrive according to a Poisson distribution and compete for the service in a FIFO (first in first out) manner
- Service times are independent identically distributed (IID) random variables, the common distribution being exponential

Queuing Model and State Transition Diagram



The M/M/1/∞ queuing model



The state transition diagram of the M/M/1/∞ queuing system

Equilibrium State Equations

- If mean arrival rate is λ and mean service rate is μ, i = 0, 1, 2 be the number of customers in the system and $P(i)$ be the state probability of the system having i customers
- From the state transition diagram, the equilibrium state equations are given by

$$\lambda P(0) = \mu P(1), \qquad i = 0,$$

$$(\lambda + \mu)P(i) = \lambda P(i-1) + \mu P(i+1), \qquad i \geq 1$$

$$P(i) = (\tfrac{\lambda}{\mu})^i P(0), \qquad i \geq 1$$

Traffic Intensity

- We know that the $P(0)$ is the probability of server being free. Since $P(0) > 0$, the necessary condition for a system being in steady state is,

$$\rho = \frac{\lambda}{\mu} < 1$$

This means that the arrival rate cannot be more than the service rate, otherwise an infinite queue will form and jobs will experience infinite service time

Queuing System Metrics

- $\rho = 1 - P(0)$, is the probability of the server being busy. Therefore, we have

$$P(i) = \rho^{i}(1- \rho)$$

- The average number of customers in the system is

$$L_s = \frac{\lambda}{\mu - \lambda}$$

- The average dwell time of customers is

$$W_s = \frac{1}{\mu - \lambda}$$

Queuing System Metrics

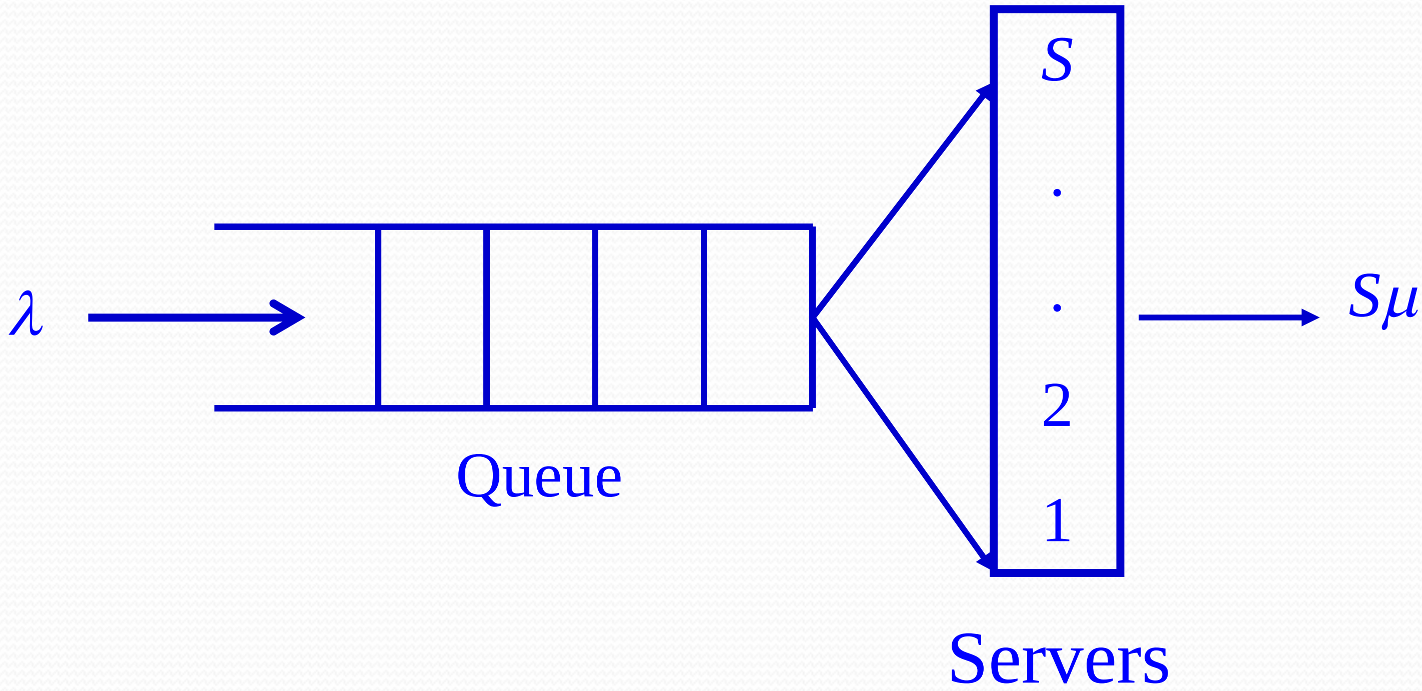
- The average queuing length is

$$Lq = \sum_{i=1}^{\infty} (i-1)P(i) = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

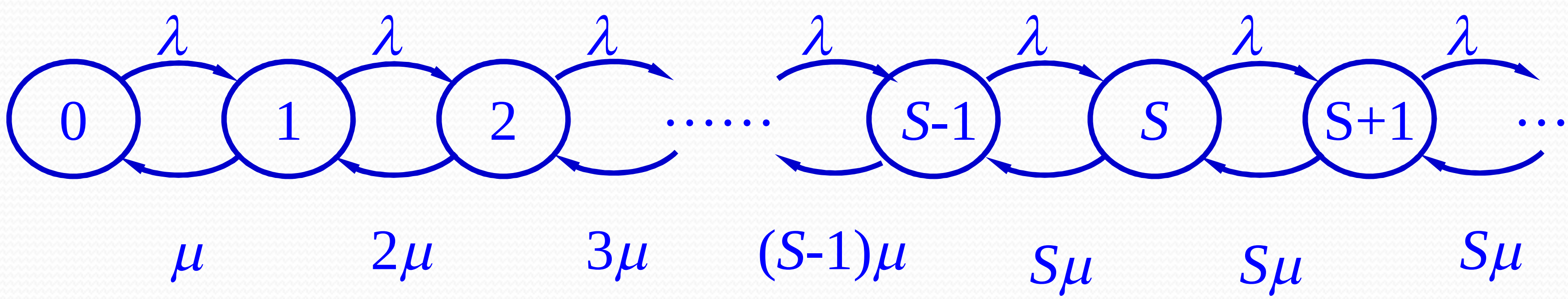
- The average waiting time of customers is

$$W_q = \frac{L_q}{\lambda} = \frac{\rho^2}{\lambda(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

M/M/S/∞ Queuing Model



State Transition Diagram



Queuing System Metrics

- The average number of customers in the system is

$$L_s = \sum_{i=0}^{\infty} iP(i) = \alpha + \frac{\rho\alpha^S P(0)}{S!(1-\rho)^2}$$

- The average dwell time of a customer in the system is given by

$$W_S = \frac{L_S}{\lambda} = \frac{1}{\mu} + \frac{\alpha^S P(0)}{S\mu . S!(1-\rho)^2}$$

Queuing System Metrics

- The average queue length is

$$L_q = \sum_{i=s}^{\infty}(i-S)P(i) = \frac{\alpha^{S+1}P(0)}{(S-1)!(S-\alpha)^2}$$

- The average waiting time of customers is

$$W_q = \frac{L_q}{\lambda} = \frac{\alpha^S P(0)}{S\mu . S!(1-\rho)^2}$$

M/G/1/∞ Queuing Model

- We consider a single server queuing system whose arrival process is Poisson with mean arrival rate λ
- Service times are independent and identically distributed with distribution function F_B and pdf f_b
- Jobs are scheduled for service as FIFO

Basic Queuing Model

- Let $N(t)$ denote the number of jobs in the system (those in queue plus in service) at time t.
- Let t_n (n= 1, 2,..) be the time of departure of the n^{th} job and X_n be the number of jobs in the system at time t_n, so that

$$X_n = N(t_n), \quad \text{for } n = 1, 2, ..$$

- The stochastic process can be modeled as a discrete Markov chain known as imbedded Markov chain, which helps convert a non-Markovian problem into a Markovian one.

Queuing System Metrics

- The average number of jobs in the system, in the steady state is

$$E[N] = \rho + \frac{\lambda^2 E[B^2]}{2(1-\rho)}$$

- The average dwell time of customers in the system is

$$Ws = \frac{E[N]}{\lambda} = \frac{1}{\mu} + \frac{\lambda E[B^2]}{2(1-\rho)}$$

- The average waiting time of customers in the queue is

$$E[N] = \lambda W_q + \rho$$

- Average waiting time of customers in the queue is

$$W_q = \frac{\lambda E[B^2]}{2(1-\rho)}$$

- The average queue length is

$$L_q = \frac{\lambda^2 E[B^2]}{2(1-\rho)}$$