Chapter 2

Euclidean Space

2.1 Vectors

1. Determine $3u + v - 2w$,where

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

Ans: $\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$

2. Express the given vector equation as a system of linear equations.

$$x_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Ans: $3x_1 + 2x_2 = 4$

$-2x_1 + 5x_2 = 1$

3. Express the given vector equation as a system of linear equations.

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Ans: $2x_1 - x_2 + 5x_3 = -1$

$4x_2 + 3x_3 = 2$

$5x_1 + 7x_2 = 1$

4. Express the given system of linear equations as a single vector equation.

$x_1 + 2x_2 - 3x_3 = 6$

$-x_1 + x_3 = 3$

Ans: $x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

5. Express the given system of linear equations as a single vector equation.

$2x_1 + x_2 - 2x_3 = 1$

$$-x_1 + x_2 + x_3 = 1$$
$$7x_1 + 3x_2 - x_3 = 1$$

Ans: $x_1\begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

6. The general solution to a linear system is given. Express this solution as a linear combination of vectors.

$$x_1 = 3 - s_1$$
$$x_2 = s_1$$

Ans: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + s_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

7. The general solution to a linear system is given. Express this solution as a linear combination of vectors.

$$x_1 = 3 - s_1 + 3s_2$$
$$x_2 = s_1$$
$$x_3 = 3 + s_2$$
$$x_4 = s_2$$

Ans: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s_1\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2\begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

8. Find the unknowns in the given vector equation.

$$2\begin{bmatrix} 1 \\ a \end{bmatrix} + 4\begin{bmatrix} b \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

Ans: $a = 1,\ b = -1$

9. Find the unknowns in the given vector equation.

$$2\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} - \begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ c \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix}$$

Ans: $a = 2, b = -1, c = 0$

10. Express $\mathbf{b}$ as a linear combination of the other vectors, if possible.

$$\mathbf{a}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 14 \\ -2 \end{bmatrix}$$

Ans: $3a_1 - 2a_2 = b$

11. Express **b** as a linear combination of the other vectors, if possible.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

Ans: $-2a_1 + 2a_2 + a_3 = b$

True or **False**: If $\mathbf{u}$ and $\mathbf{v}$ are vectors, and c and d are scalars, then $c(du + v) = (cd)u + v$.

Ans: False

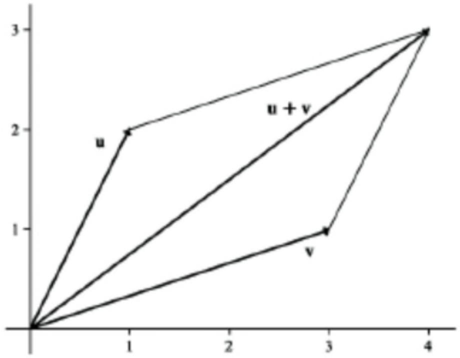
13. **True** or **False**: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, then $u - (v + w) = (u - v) + (u - w)$.

Ans: False

14. **True** or **False**: If $u + v = w$, then $v = w - u$.

Ans: True

15. Sketch the graph of $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and then use the Parallelogram Rule to sketch the graph of $u + v$.

Ans: 

16. Determine how to divide a total mass of 18 kg among the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix} \text{ so that the center of mass is } \begin{bmatrix} 2/9 \\ 2/9 \\ 2/9 \end{bmatrix}.$$

Ans: Place 10 kg at $\mathbf{u}_1$, 1 kg at $\mathbf{u}_2$, and 7 kg at $\mathbf{u}_3$.

17. Find an example of a linear system with two equations and three variables that has

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ as the general solution.}$$

Ans: A possible answer is

$$x_1 + x_2 - 5x_3 = 5$$
$$x_1 - x_2 - x_3 = -1$$

2.2 Span

1. Find four vectors that are in the span of the given vectors.

   $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   Ans: For example, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2. Find five vectors that are in the span of the given vectors.

   $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$

   Ans: For example, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ 10 \\ 6 \end{bmatrix}$

3. Determine if $\mathbf{b}$ is in the span of the other given vectors. If so, write $\mathbf{b}$ as a linear combination of the other vectors.

   $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

   Ans: $\mathrm{b} = \mathrm{a}_1 - \mathrm{a}_2$

4. Determine if $\mathbf{b}$ is in the span of the other given vectors. If so, write $\mathbf{b}$ as a linear combination of the other vectors.

   $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

   Ans: $\mathbf{b}$ is not in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$.

5. Find A, $\mathbf{x}$, and $\mathbf{b}$ such that $\mathrm{Ax} = \mathrm{b}$ corresponds to the given linear system.

   $$\begin{aligned} \mathrm{x}_1 + \mathrm{x}_2 - 2\mathrm{x}_3 &= 1 \\ -\mathrm{x}_1 + 2\mathrm{x}_2 + 4\mathrm{x}_3 &= 8 \end{aligned}$$

   Ans: $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

6. Find A, $\mathbf{x}$, and $\mathbf{b}$ such that $\mathrm{Ax} = \mathrm{b}$ corresponds to the given linear system.

   $$\mathrm{x}_1 - \mathrm{x}_3 = 1$$

$2x_2 + 4x_3 = 3$

$-x_1 + 5x_3 = 1$

Ans: $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ -1 & 0 & 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

7. Express the given system of linear equations as a vector equation.

$2x_1 - x_3 + x_4 = 1$

$x_1 + 2x_2 + 4x_3 - x_4 = 0$

Ans: $x_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 2 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 4 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

8. Determine if the columns of the given matrix span $\mathbb{R}^2$.

$\begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$

Ans: Yes, the columns span $\mathbb{R}^2$.

9. Determine if the columns of the given matrix span $\mathbb{R}^3$.

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

Ans: No, the columns do not span $\mathbb{R}^3$.

10. Determine if the system $Ax = b$ (where $\mathbf{x}$ and $\mathbf{b}$ have the appropriate number of components) has a solution for all choices of $\mathbf{b}$.

$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

Ans: Yes, a solution exists.

11. Find all values of h such that the vectors span $\mathbb{R}^2$.

$\mathbf{a}_1 = \begin{bmatrix} h \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 2 \\ h \end{bmatrix}$

Ans: All real numbers, except $h \neq \pm 2$.

12. For what value(s) of h do the given vectors span $\mathbb{R}^3$?

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ h \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

Ans: All real numbers, except $h \neq 5$.

13. **True** or **False**: Suppose a matrix A has n rows and m columns, with $m < n$. Then the

columns of A do not span $\mathbb{R}^n$.

Ans: True

14. **True** or **False**: Suppose a matrix A has n rows and m columns, with $m > n$. Then the columns of A span $\mathbb{R}^n$.

    Ans: False

15. **True** or **False**: If the columns of a matrix A with n rows and m columns do not span $\mathbb{R}^n$, then there exists a vector **b** in $\mathbb{R}^n$ such that $Ax = b$ does not have a solution.

    Ans: True

16. **True** or **False**: If the columns of a matrix A with n rows and m columns spans $\mathbb{R}^n$, then $m \geq n$.

    Ans: True

2.3 Linear Independence

1. Determine if the given vectors are linearly independent.

   $$\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

   Ans: Linearly independent

2. Determine if the given vectors are linearly independent.

   $$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -4 \\ -8 \end{bmatrix}$$

   Ans: Not linearly independent

3. Determine if the given vectors are linearly independent.

   $$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \end{bmatrix}$$

   Ans: Linearly independent

4. Determine if the columns of the given matrix are linearly independent.

   $$\begin{bmatrix} 3 & 2 \\ -2 & -3 \end{bmatrix}$$

   Ans: Linearly independent

5. Determine if the columns of the given matrix are linearly independent.

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 0 \\ 3 & 2 & 2 \end{bmatrix}$$

Ans: Linearly independent

6. Determine if the columns of the given matrix are linearly independent.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Ans: Not linearly independent

7. Determine if the homogeneous system $\mathrm{Ax} = 0$ has any nontrivial solutions, where

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 2 \\ 0 & 1 \end{bmatrix}.\backslash$$

Ans: $\mathrm{Ax} = 0$ has only the trivial solution.

8. Determine if the homogeneous system $\mathrm{Ax} = 0$ has any nontrivial solutions, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Ans: $\mathrm{Ax} = 0$ has nontrivial solutions.

9. Determine by inspection (that is, with only minimal calculations) if the given vectors form a linearly dependent or linearly independent set. Justify your answer.

$$\mathbf{u} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 20 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Ans: Linearly dependent, by Theorem 2.14

10. Determine if one of the given vectors is in the span of the other vectors.

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

Ans: Yes, since $\mathrm{w} = -\mathrm{u} + \mathrm{v}$.

11. **True** or **False**: Suppose matrix A has n rows and m columns, with $\mathrm{n} < \mathrm{m}$. Then the columns of A are linearly dependent.

Ans: True

12. **True** or **False**: Suppose a matrix A has n rows and m columns, with $\mathrm{n} \geq \mathrm{m}$. Then the columns of A are linearly independent.

Ans: False

13. **True** or **False**: Suppose there exists a vector $\mathbf{x}$ such that $Ax = b$. Then the columns of A are linearly independent.

    Ans: False

14. **True** or **False**: If $Ax \neq 0$ for every $x \neq 0$, then the columns of A are linearly independent.

    Ans: True

15. **True** or **False**: If $\{u_1, u_2\}$, $\{u_1, u_3\}$, and $\{u_2, u_3\}$ are all linearly independent, then $\{u_1, u_2, u_3\}$ is linearly independent.

    Ans: False