PREFACE

This solutions manual is designed to accompany the eighth edition of *Linear Algebra with Applications* by Steven J. Leon. The answers in this manual supplement those given in the answer key of the textbook. In addition this manual contains the complete solutions to all of the nonroutine exercises in the book.

At the end of each chapter of the textbook there are two chapter tests (A and B) and a section of computer exercises to be solved using MATLAB. The questions in each Chapter Test A are to be answered as either *true* or *false*. Although the true-false answers are given in the Answer Section of the textbook, students are required to explain or prove their answers. This manual includes explanations, proofs, and counterexamples for all Chapter Test A questions. The chapter tests labeled B contain problems similar to the exercises in the chapter. The answers to these problems are not given in the Answers to Selected Exercises Section of the textbook, however, they are provided in this manual. Complete solutions are given for all of the nonroutine Chapter Test B exercises.

In the MATLAB exercises most of the computations are straightforward. Consequently they have not been included in this solutions manual. On the other hand, the text also includes questions related to the computations. The purpose of the questions is to emphasize the significance of the computations. The solutions manual does provide the answers to most of these questions. There are some questions for which it is not possible to provide a single answer. For example, some exercises involve randomly generated matrices. In these cases the answers may depend on the particular random matrices that were generated.

Steven J. Leon
sleon@umassd.edu

Contents

Chapter 1

Matrices and Systems of Equations

1 SYSTEMS OF LINEAR EQUATIONS

2. (d) $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 & 1 \\ 0 & 0 & 4 & 1 & -2 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

5. (a) $\begin{aligned} 3x_1 + 2x_2 &= 8 \\ x_1 + 5x_2 &= 7 \end{aligned}$

(b) $$\begin{aligned} 5x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + 3x_2 - 4x_3 &= 0 \end{aligned}$$

(c) $$\begin{aligned} 2x_1 + x_2 + 4x_3 &= -1 \\ 4x_1 - 2x_2 + 3x_3 &= 4 \\ 5x_1 + 2x_2 + 6x_2 &= -1 \end{aligned}$$

(d) $$\begin{aligned} 4x_1 - 3x_2 + x_3 + 2x_4 &= 4 \\ 3x_1 + x_2 - 5x_3 + 6x_4 &= 5 \\ x_1 + x_2 + 2x_3 + 4x_4 &= 8 \\ 5x_1 + x_2 + 3x_3 - 2x_4 &= 7 \end{aligned}$$

9. Given the system

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_2x_1 + x_2 &= b_2 \end{aligned}$$

one can eliminate the variable x_2 by subtracting the first row from the second. One then obtains the equivalent system

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ (m_1 - m_2)x_1 &= b_2 - b_1 \end{aligned}$$

(a) If $m_1 \neq m_2$, then one can solve the second equation for x_1

$$x_1 = \frac{b_2 - b_1}{m_1 - m_2}$$

One can then plug this value of x_1 into the first equation and solve for x_2. Thus, if $m_1 \neq m_2$, there will be a unique ordered pair (x_1, x_2) that satisfies the two equations.

(b) If $m_1 = m_2$, then the x_1 term drops out in the second equation

$$0 = b_2 - b_1$$

This is possible if and only if $b_1 = b_2$.

(c) If $m_1 \neq m_2$, then the two equations represent lines in the plane with different slopes. Two nonparallel lines intersect in a point. That point will be the unique solution to the system. If $m_1 = m_2$ and $b_1 = b_2$, then both equations represent the same line and consequently every point on that line will satisfy both equations. If $m_1 = m_2$ and $b_1 \neq b_2$, then the equations represent parallel lines. Since parallel lines do not intersect, there is no point on both lines and hence no solution to the system.

10. The system must be consistent since $(0, 0)$ is a solution.

11. A linear equation in 3 unknowns represents a plane in three space. The solution set to a 3×3 linear system would be the set of all points that lie on all three planes. If the planes are parallel or one plane is parallel to the line of intersection of the other two, then the solution set will be empty. The three equations could represent the same plane or the three planes could all intersect in a line. In either case the solution set will contain infinitely many points. If the three planes intersect in a point then the solution set will contain only that point.

2 ROW ECHELON FORM

2. (b) The system is consistent with a unique solution $(4, -1)$.

4. (b) x_1 and x_3 are lead variables and x_2 is a free variable.
(d) x_1 and x_3 are lead variables and x_2 and x_4 are free variables.
(f) x_2 and x_3 are lead variables and x_1 is a free variable.

5. (l) The solution is $(0, -1.5, -3.5)$.

6. (c) The solution set consists of all ordered triples of the form $(0, -\alpha, \alpha)$.

7. A homogeneous linear equation in 3 unknowns corresponds to a plane that passes through the origin in 3-space. Two such equations would correspond to two planes through the origin. If one equation is a multiple of the other, then both represent the same plane through the origin and every point on that plane will be a solution to the system. If one equation is not a multiple of the other, then we have two distinct planes that intersect in a line through the origin. Every point on the line of intersection will be a solution to the linear system. So in either case the system must have infinitely many solutions.

In the case of a nonhomogeneous 2×3 linear system, the equations correspond to planes that do not both pass through the origin. If one equation is a multiple of the other, then both represent the same plane and there are infinitely many solutions. If the equations represent planes that are parallel, then they do not intersect and hence the system will not have any solutions. If the equations represent distinct planes that are not parallel, then they must intersect in a line and hence there will be infinitely many solutions. So the only possibilities for a nonhomogeneous 2×3 linear system are 0 or infinitely many solutions.

9. (a) Since the system is homogeneous it must be consistent.

13. A homogeneous system is always consistent since it has the trivial solution $(0, \ldots, 0)$. If the reduced row echelon form of the coefficient matrix involves free variables, then there will be infinitely many solutions. If there are no free variables, then the trivial solution will be the only soltion.

14. A nonhomogeneous system could be inconsistent in which case there would be no solutions. If the system is consistent and underdetermined, then there will be free variables and this would imply that we will have infinitely many solutions.

16. At each intersection the number of vehicles entering must equal the number of vehicles leaving in order for the traffic to flow. This condition leads to the following system of equations

$$\begin{aligned}
x_1 + a_1 &= x_2 + b_1 \\
x_2 + a_2 &= x_3 + b_2 \\
x_3 + a_3 &= x_4 + b_3 \\
x_4 + a_4 &= x_1 + b_4
\end{aligned}$$

If we add all four equations we get

$$x_1 + x_2 + x_3 + x_4 + a_1 + a_2 + a_3 + a_4 = x_1 + x_2 + x_3 + x_4 + b_1 + b_2 + b_3 + b_4$$

and hence

$$a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$$

17. If (c_1, c_2) is a solution, then

$$\begin{aligned} a_{11}c_1 + a_{12}c_2 &= 0 \\ a_{21}c_1 + a_{22}c_2 &= 0 \end{aligned}$$

Multiplying both equations through by α, one obtains

$$\begin{aligned} a_{11}(\alpha c_1) + a_{12}(\alpha c_2) &= \alpha \cdot 0 = 0 \\ a_{21}(\alpha c_1) + a_{22}(\alpha c_2) &= \alpha \cdot 0 = 0 \end{aligned}$$

Thus $(\alpha c_1, \alpha c_2)$ is also a solution.

18. (a) If $x_4 = 0$ then x_1, x_2, and x_3 will all be 0. Thus if no glucose is produced then there is no reaction. $(0, 0, 0, 0)$ is the trivial solution in the sense that if there are no molecules of carbon dioxide and water, then there will be no reaction.
(b) If we choose another value of x_4, say $x_4 = 2$, then we end up with solution $x_1 = 12$, $x_2 = 12$, $x_3 = 12$, $x_4 = 2$. Note the ratios are still 6:6:6:1.

3 MATRIX ARITHMETIC

1. (e) $\begin{pmatrix} 8 & -15 & 11 \\ 0 & -4 & -3 \\ -1 & -6 & 6 \end{pmatrix}$

(g) $\begin{pmatrix} 5 & -10 & 15 \\ 5 & -1 & 4 \\ 8 & -9 & 6 \end{pmatrix}$

2. (d) $\begin{pmatrix} 36 & 10 & 56 \\ 10 & 3 & 16 \end{pmatrix}$

5. (a) $5A = \begin{pmatrix} 15 & 20 \\ 5 & 5 \\ 10 & 35 \end{pmatrix}$

$$2A + 3A = \begin{pmatrix} 6 & 8 \\ 2 & 2 \\ 4 & 14 \end{pmatrix} + \begin{pmatrix} 9 & 12 \\ 3 & 3 \\ 6 & 21 \end{pmatrix} = \begin{pmatrix} 15 & 20 \\ 5 & 5 \\ 10 & 35 \end{pmatrix}$$

(b) $6A = \begin{pmatrix} 18 & 24 \\ 6 & 6 \\ 12 & 42 \end{pmatrix}$

$$3(2A) = 3\begin{pmatrix} 6 & 8 \\ 2 & 2 \\ 4 & 14 \end{pmatrix} = \begin{pmatrix} 18 & 24 \\ 6 & 6 \\ 12 & 42 \end{pmatrix}$$

(c) $A^T = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 1 & 7 \end{pmatrix}$

$$(A^T)^T = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 1 & 7 \end{pmatrix}^T = \begin{pmatrix} 3 & 4 \\ 1 & 1 \\ 2 & 7 \end{pmatrix} = A$$

6. (a) $A + B = \begin{pmatrix} 5 & 4 & 6 \\ 0 & 5 & 1 \end{pmatrix} = B + A$

(b) $3(A+B) = 3\begin{pmatrix} 5 & 4 & 6 \\ 0 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 12 & 18 \\ 0 & 15 & 3 \end{pmatrix}$

$$3A + 3B = \begin{pmatrix} 12 & 3 & 18 \\ 6 & 9 & 15 \end{pmatrix} + \begin{pmatrix} 3 & 9 & 0 \\ -6 & 6 & -12 \end{pmatrix}$$

$$= \begin{pmatrix} 15 & 12 & 18 \\ 0 & 15 & 3 \end{pmatrix}$$

(c) $(A+B)^T = \begin{pmatrix} 5 & 4 & 6 \\ 0 & 5 & 1 \end{pmatrix}^T = \begin{pmatrix} 5 & 0 \\ 4 & 5 \\ 6 & 1 \end{pmatrix}$

$$A^T + B^T = \begin{pmatrix} 4 & 2 \\ 1 & 3 \\ 6 & 5 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 3 & 2 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 4 & 5 \\ 6 & 1 \end{pmatrix}$$

7. (a) $3(AB) = 3\begin{pmatrix} 5 & 14 \\ 15 & 42 \\ 0 & 16 \end{pmatrix} = \begin{pmatrix} 15 & 42 \\ 45 & 126 \\ 0 & 48 \end{pmatrix}$

$$(3A)B = \begin{pmatrix} 6 & 3 \\ 18 & 9 \\ -6 & 12 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 15 & 42 \\ 45 & 126 \\ 0 & 48 \end{pmatrix}$$

$$A(3B) = \begin{pmatrix} 2 & 1 \\ 6 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 6 & 12 \\ 3 & 18 \end{pmatrix} = \begin{pmatrix} 15 & 42 \\ 45 & 126 \\ 0 & 48 \end{pmatrix}$$

(b) $(AB)^T = \begin{pmatrix} 5 & 14 \\ 15 & 42 \\ 0 & 16 \end{pmatrix}^T = \begin{pmatrix} 5 & 15 & 0 \\ 14 & 42 & 16 \end{pmatrix}$

$$B^T A^T = \begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 2 & 6 & -2 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 15 & 0 \\ 14 & 42 & 16 \end{pmatrix}$$

8. (a) $(A+B)+C = \begin{pmatrix} 0 & 5 \\ 1 & 7 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 3 & 8 \end{pmatrix}$

$$A + (B+C) = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 3 & 8 \end{pmatrix}$$

(b) $(AB)C = \begin{pmatrix} -4 & 18 \\ -2 & 13 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 24 & 14 \\ 20 & 11 \end{pmatrix}$

$$A(BC) = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -4 & -1 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 24 & 14 \\ 20 & 11 \end{pmatrix}$$

(c) $A(B+C) = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 24 \\ 7 & 17 \end{pmatrix}$

$$AB + AC = \begin{pmatrix} -4 & 18 \\ -2 & 13 \end{pmatrix} + \begin{pmatrix} 14 & 6 \\ 9 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 24 \\ 7 & 17 \end{pmatrix}$$

(d) $(A + B)C = \begin{pmatrix} 0 & 5 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 5 \\ 17 & 8 \end{pmatrix}$

$$AC + BC = \begin{pmatrix} 14 & 6 \\ 9 & 4 \end{pmatrix} + \begin{pmatrix} -4 & -1 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 5 \\ 17 & 8 \end{pmatrix}$$

9. (b) $\mathbf{x} = (2, 1)^T$ is a solution since $\mathbf{b} = 2\mathbf{a}_1 + \mathbf{a}_2$. There are no other solutions since the echelon form of A is strictly triangular.
(c) The solution to $A\mathbf{x} = \mathbf{c}$ is $\mathbf{x} = (-\frac{5}{2}, -\frac{1}{4})^T$. Therefore $\mathbf{c} = -\frac{5}{2}\mathbf{a}_1 - \frac{1}{4}\mathbf{a}_2$.

11. The given information implies that

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

are both solutions to the system. So the system is consistent and since there is more than one solution the row echelon form of A must involve a free variable. A consistent system with a free variable has infinitely many solutions.

12. The system is consistent since $\mathbf{x} = (1, 1, 1, 1)^T$ is a solution. The system can have at most 3 lead variables since A only has 3 rows. Therefore there must be at least one free variable. A consistent system with a free variable has infinitely many solutions.

13. (a) It follows from the reduced row echelon form that the free variables are x_2, x_4, x_5. If we set $x_2 = a$, $x_4 = b$, $x_5 = c$, then

$$\begin{aligned} x_1 &= -2 - 2a - 3b - c \\ x_3 &= 5 - 2b - 4c \end{aligned}$$

and hence the solution consists of all vectors of the form

$$\mathbf{x} = (-2 - 2a - 3b - c,\ a,\ 5 - 2b - 4c,\ b,\ c)^T$$

(b) If we set the free variables equal to 0, then $\mathbf{x}_0 = (-2, 0, 5, 0, 0)^T$ is a solution to $A\mathbf{x} = \mathbf{b}$ and hence

$$\mathbf{b} = A\mathbf{x}_0 = -2\mathbf{a}_1 + 5\mathbf{a}_3 = (8, -7, -1, 7)^T$$

14. A^T is an $n \times m$ matrix. Since A^T has m columns and A has m rows, the multiplication $A^T A$ is possible. The multiplication AA^T is possible since A has n columns and A^T has n rows.

15. If A is skew-symmetric then $A^T = -A$. Since the (j, j) entry of A^T is a_{jj} and the (j, j) entry of $-A$ is $-a_{jj}$, it follows that is $a_{jj} = -a_{jj}$ for each j and hence the diagonal entries of A must all be 0.

16. The search vector is $\mathbf{x} = (1, 0, 1, 0, 1, 0)^T$. The search result is given by the vector

$$\mathbf{y} = A^T\mathbf{x} = (1, 2, 2, 1, 1, 2, 1)^T$$

The ith entry of $\mathbf{y}$ is equal to the number of search words in the title of the ith book.

17. If $\alpha = a_{21}/a_{11}$, then

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & b \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ \alpha a_{11} & \alpha a_{12} + b \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & \alpha a_{12} + b \end{pmatrix}$$

The product will equal A provided

$$\alpha a_{12} + b = a_{22}$$

Thus we must choose

$$b = a_{22} - \alpha a_{12} = a_{22} - \frac{a_{21}a_{12}}{a_{11}}$$

4 MATRIX ALGEBRA

1. (a) $(A+B)^2 = (A+B)(A+B) = (A+B)A + (A+B)B = A^2 + BA + AB + B^2$ In the case of real numbers $ab + ba = 2ab$, however, with matrices $AB + BA$ is generally not equal to $2AB$.

(b)

$$\begin{aligned} (A+B)(A-B) &= (A+B)(A-B) \\ &= (A+B)A - (A+B)B \\ &= A^2 + BA - AB - B^2 \end{aligned}$$

In the case of real numbers $ab - ba = 0$, however, with matrices $AB - BA$ is generally not equal to O.

2. If we replace a by A and b by the identity matrix, I, then both rules will work, since

$$(A+I)^2 = A^2 + IA + AI + B^2 = A^2 + AI + AI + B^2 = A^2 + 2AI + B^2$$

and

$$(A+I)(A-I) = A^2 + IA - AI - I^2 = A^2 + A - A - I^2 = A^2 - I^2$$

3. There are many possible choices for A and B. For example, one could choose

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

More generally if

$$A = \begin{pmatrix} a & b \\ ca & cb \end{pmatrix} \qquad B = \begin{pmatrix} db & eb \\ -da & -ea \end{pmatrix}$$

then $AB = O$ for any choice of the scalars a, b, c, d, e.

4. To construct nonzero matrices A, B, C with the desired properties, first find nonzero matrices C and D such that $DC = O$ (see Exercise 3). Next, for any nonzero matrix A, set $B = A + D$. It follows that

$$BC = (A+D)C = AC + DC = AC + O = AC$$

5. A 2×2 symmetric matrix is one of the form

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Thus

$$A^2 = \begin{pmatrix} a^2 + b^2 & ab + bc \\ ab + bc & b^2 + c^2 \end{pmatrix}$$

If $A^2 = O$, then its diagonal entries must be 0.

$$a^2 + b^2 = 0 \qquad \text{and} \qquad b^2 + c^2 = 0$$

Thus $a = b = c = 0$ and hence $A = O$.

6. Let

$$D = (AB)C = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

It follows that

$$\begin{aligned}
d_{11} &= (a_{11}b_{11} + a_{12}b_{21})c_{11} + (a_{11}b_{12} + a_{12}b_{22})c_{21} \\
&= a_{11}b_{11}c_{11} + a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{22}c_{21} \\
d_{12} &= (a_{11}b_{11} + a_{12}b_{21})c_{12} + (a_{11}b_{12} + a_{12}b_{22})c_{22} \\
&= a_{11}b_{11}c_{12} + a_{12}b_{21}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{22}c_{22} \\
d_{21} &= (a_{21}b_{11} + a_{22}b_{21})c_{11} + (a_{21}b_{12} + a_{22}b_{22})c_{21} \\
&= a_{21}b_{11}c_{11} + a_{22}b_{21}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{22}c_{21} \\
d_{22} &= (a_{21}b_{11} + a_{22}b_{21})c_{12} + (a_{21}b_{12} + a_{22}b_{22})c_{22} \\
&= a_{21}b_{11}c_{12} + a_{22}b_{21}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{22}c_{22}
\end{aligned}$$

If we set

$$E = A(BC) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22} \end{pmatrix}$$

then it follows that

$$\begin{aligned}
e_{11} &= a_{11}(b_{11}c_{11} + b_{12}c_{21}) + a_{12}(b_{21}c_{11} + b_{22}c_{21}) \\
&= a_{11}b_{11}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{21}c_{11} + a_{12}b_{22}c_{21} \\
e_{12} &= a_{11}(b_{11}c_{12} + b_{12}c_{22}) + a_{12}(b_{21}c_{12} + b_{22}c_{22}) \\
&= a_{11}b_{11}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{21}c_{12} + a_{12}b_{22}c_{22} \\
e_{21} &= a_{21}(b_{11}c_{11} + b_{12}c_{21}) + a_{22}(b_{21}c_{11} + b_{22}c_{21}) \\
&= a_{21}b_{11}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{21}c_{11} + a_{22}b_{22}c_{21} \\
e_{22} &= a_{21}(b_{11}c_{12} + b_{12}c_{22}) + a_{22}(b_{21}c_{12} + b_{22}c_{22}) \\
&= a_{21}b_{11}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{21}c_{12} + a_{22}b_{22}c_{22}
\end{aligned}$$

Thus

$$d_{11} = e_{11} \qquad d_{12} = e_{12} \qquad d_{21} = e_{21} \qquad d_{22} = e_{22}$$

and hence

$$(AB)C = D = E = A(BC)$$

9.

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $A^4 = O$. If $n > 4$, then

$$A^n = A^{n-4}A^4 = A^{n-4}O = O$$

10. (a) The matrix C is symmetric since

$$C^T = (A + B)^T = A^T + B^T = A + B = C$$

(b) The matrix D is symmetric since

$$D^T = (AA)^T = A^T A^T = A^2 = D$$

(c) The matrix $E = AB$ is not symmetric since

$$E^T = (AB)^T = B^T A^T = BA$$

and in general $AB \neq BA$.

(d) The matrix F is symmetric since

$$F^T = (ABA)^T = A^T B^T A^T = ABA = F$$

(e) The matrix G is symmetric since

$$G^T = (AB+BA)^T = (AB)^T + (BA)^T = B^T A^T + A^T B^T = BA + AB = G$$

(f) The matrix H is not symmetric since

$$H^T = (AB-BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB = -H$$

11. (a) The matrix A is symmetric since

$$A^T = (C + C^T)^T = C^T + (C^T)^T = C^T + C = A$$

(b) The matrix B is not symmetric since

$$B^T = (C - C^T)^T = C^T - (C^T)^T = C^T - C = -B$$

(c) The matrix D is symmetric since

$$A^T = (C^T C)^T = C^T (C^T)^T = C^T C = D$$

(d) The matrix E is symmetric since

$$\begin{aligned} E^T &= (C^T C - CC^T)^T = (C^T C)^T - (CC^T)^T \\ &= C^T (C^T)^T - (C^T)^T C^T = C^T C - CC^T = E \end{aligned}$$

(e) The matrix F is symmetric since

$$F^T = ((I + C)(I + C^T))^T = (I + C^T)^T (I + C)^T = (I + C)(I + C^T) = F$$

(e) The matrix G is not symmetric.

$$\begin{aligned} F &= (I+C)(I-C^T) = I + C - C^T - CC^T \\ F^T &= ((I+C)(I-C^T))^T = (I-C^T)^T(I+C)^T \\ &= (I-C)(I+C^T) = I - C + C^T - CC^T \end{aligned}$$

F and F^T are not the same. The two middle terms $C - C^T$ and $-C + C^T$ do not agree.

12. If $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$ then

$$\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \begin{bmatrix} \dfrac{a_{11}a_{22} - a_{12}a_{21}}{d} & 0 \\ 0 & \dfrac{a_{11}a_{22} - a_{12}a_{21}}{d} \end{bmatrix} = I$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\left[\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\right]$$

$$= \begin{bmatrix} \dfrac{a_{11}a_{22} - a_{12}a_{21}}{d} & 0 \\ 0 & \dfrac{a_{11}a_{22} - a_{12}a_{21}}{d} \end{bmatrix} = I$$

Therefore

$$\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = A^{-1}$$

13. (b) $\begin{pmatrix} -3 & 5 \\ 2 & -3 \end{pmatrix}$

14. If A were nonsingular and $AB = A$, then it would follow that $A^{-1}AB = A^{-1}A$ and hence that $B = I$. So if $B \neq I$, then A must be singular.

15. Since

$$A^{-1}A = AA^{-1} = I$$

it follows from the definition that A^{-1} is nonsingular and its inverse is A.

16. Since

$$\begin{aligned} A^T(A^{-1})^T &= (A^{-1}A)^T = I \\ (A^{-1})^T A^T &= (AA^{-1})^T = I \end{aligned}$$

it follows that

$$(A^{-1})^T = (A^T)^{-1}$$

17. If $A\mathbf{x} = A\mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, then A must be singular, for if A were nonsingular then we could multiply by A^{-1} and get

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}A\mathbf{y} \\ \mathbf{x} &= \mathbf{y} \end{aligned}$$

18. For $m = 1$,
$$(A^1)^{-1} = A^{-1} = (A^{-1})^1$$
Assume the result holds in the case $m = k$, that is,
$$(A^k)^{-1} = (A^{-1})^k$$
It follows that
$$(A^{-1})^{k+1}A^{k+1} = A^{-1}(A^{-1})^k A^k A = A^{-1}A = I$$
and
$$A^{k+1}(A^{-1})^{k+1} = AA^k(A^{-1})^k A^{-1} = AA^{-1} = I$$
Therefore
$$(A^{-1})^{k+1} = (A^{k+1})^{-1}$$
and the result follows by mathematical induction.

19. If $A^2 = O$, then
$$(I + A)(I - A) = I + A - A + A^2 = I$$
and
$$(I - A)(I + A) = I - A + A + A^2 = I$$
Therefore $I - A$ is nonsingular and $(I - A)^{-1} = I + A$.

20. If $A^{k+1} = O$ then
$$\begin{aligned}(I + A + \cdots + A^k)(I - A) &= (I + A + \cdots + A^k) - (A + A^2 + \cdots + A^{k+1})\\ &= I - A^{k+1} = I\end{aligned}$$
and
$$\begin{aligned}(I - A)(I + A + \cdots + A^k) &= (I + A + \cdots + A^k) - (A + A^2 + \cdots + A^{k+1})\\ &= I - A^{k+1} = I\end{aligned}$$
Therefore $I - A$ is nonsingular and $(I - A)^{-1} = I + A + A^2 + \cdots + A^k$.

21. Since
$$R^T R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
and
$$RR^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
it follows that R is nonsingular and $R^{-1} = R^T$

22.
$$G^2 = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = I$$

23.
$$\begin{aligned}H^2 = (I - 2\mathbf{u}\mathbf{u}^T)^2 &= I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T\\ &= I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T\\ &= I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T = I \quad (\text{since } \mathbf{u}^T\mathbf{u} = 1)\end{aligned}$$

24. In each case if you square the given matrix you will end up with the same matrix.

25. (a) If $A^2 = A$ then

$$(I - A)^2 = I - 2A + A^2 = I - 2A + A = I - A$$

(b) If $A^2 = A$ then

$$(I - \frac{1}{2}A)(I + A) = I - \frac{1}{2}A + A - \frac{1}{2}A^2 = I - \frac{1}{2}A + A - \frac{1}{2}A = I$$

and

$$(I + A)(I - \frac{1}{2}A) = I + A - \frac{1}{2}A - \frac{1}{2}A^2 = I + A - \frac{1}{2}A - \frac{1}{2}A = I$$

Therefore $I + A$ is nonsingular and $(I + A)^{-1} = I - \frac{1}{2}A$.

26. (a)

$$D^2 = \begin{Bmatrix} d_{11}^2 & 0 & \cdots & 0 \\ 0 & d_{22}^2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & d_{nn}^2 \end{Bmatrix}$$

Since each diagonal entry of D is equal to either 0 or 1, it follows that $d_{jj}^2 = d_{jj}$, for $j = 1, \ldots, n$ and hence $D^2 = D$.

(b) If $A = XDX^{-1}$, then

$$A^2 = (XDX^{-1})(XDX^{-1}) = XD(X^{-1}X)DX^{-1} = XDX^{-1} = A$$

27. If A is an involution then $A^2 = I$ and it follows that

$$\begin{aligned} B^2 &= \frac{1}{4}(I + A)^2 = \frac{1}{4}(I + 2A + A^2) = \frac{1}{4}(2I + 2A) = \frac{1}{2}(I + A) = B \\ C^2 &= \frac{1}{4}(I - A)^2 = \frac{1}{4}(I - 2A + A^2) = \frac{1}{4}(2I - 2A) = \frac{1}{2}(I - A) = C \end{aligned}$$

So B and C are both idempotent.

$$BC = \frac{1}{4}(I + A)(I - A) = \frac{1}{4}(I + A - A - A^2) = \frac{1}{4}(I + A - A - I) = O$$

28. $(A^TA)^T = A^T(A^T)^T = A^TA$
$(AA^T)^T = (A^T)^TA^T = AA^T$

29. Let A and B be symmetric $n \times n$ matrices. If $(AB)^T = AB$ then

$$BA = B^TA^T = (AB)^T = AB$$

Conversely if $BA = AB$ then

$$(AB)^T = B^TA^T = BA = AB$$

30. (a)

$$\begin{aligned} B^T &= (A + A^T)^T = A^T + (A^T)^T = A^T + A = B \\ C^T &= (A - A^T)^T = A^T - (A^T)^T = A^T - A = -C \end{aligned}$$

(b) $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$