CHAPTER 2:
Mathematics for Microeconomics

The problems in this chapter are primarily mathematical. They are intended to give students some practice with the concepts introduced in Chapter 2, but the problems in themselves offer few economic insights. Consequently, no commentary is provided. Results from some of the analytical problems are used in later chapters, however, and in those cases the student will be directed to here.

Solutions

2.1 $U(x, y) = 4x^2 + 3y^2$.

a. $U_x = 8x,\ U_y = 6y.$

b. $U_x = 8, U_y = 6.$

c. $dU = 8x\,dx + 6y\,dy.$

d. $\frac{dy}{dx} = -\frac{U_x}{U_y} = -\frac{4x}{3y}.$

e. $4 \cdot 1^2 + 3 \cdot 2^2 = 16.$

f. $\frac{dy}{dx} = \frac{-4 \cdot 1}{3 \cdot 2} = -\frac{2}{3}.$

g. The $U = 16$ contour line is an ellipse centered at the origin. The slope of the line at any point is given by $dy/dx = -4x/3y$.

2.2 a. Profits are given by $\pi = R - C = -2q^2 + 40q - 100$. The maximum value is found by setting the derivative equal to 0:

$$\frac{d\pi}{dq} = -4q + 40 = 0$$

implies $q^* = 10$ and $\pi^* = 100$.

b. Since $d^2\pi/dq^2 = -4 < 0$, this is a global maximum.

c. $MR = dR/dq = 70 - 2q$. $MC = dC/dq = 2q + 30$. So $q^* = 10$ obeys $MR = MC = 50$.

2.3 First, use the substitution method. Substituting $y = 1 - x$ yields $\tilde{f}(x) = f(x, 1-x) = x(1-x) = x - x^2$. Taking the first-order condition, $\tilde{f}'(x) = 1 - 2x = 0$, and solving yields $x^* = 0.5$, $y^* = 0.5$, and $\tilde{f}(x^*) = f(x^*, y^*) = 0.25$. Since $\tilde{f}''(x^*) = -2 < 0$, this is a local and global maximum.

Next, use the Lagrange method. The Lagrangian is $L = xy + \lambda(1 - x - y)$. The first-order conditions are

$$L_x = y - \lambda = 0$$
$$L_y = x - \lambda = 0$$
$$L_\lambda = 1 - x - y = 0.$$

Solving simultaneously, $x = y$. Using the constraint gives $x^* = y^* = 0.5$, $\lambda = 0.5$, and $x^* y^* = 0.25$.

2.4 Setting up the Lagrangian, $L = x + y + \lambda(0.25 - xy)$. The first-order conditions are

$$L_x = 1 - \lambda y$$
$$L_y = 1 - \lambda x$$
$$L_\lambda = 0.25 - xy = 0.$$

So $x = y$. Using the constraint $(xy = x^2 = 0.25)$ gives $x^* = y^* = 0.5$ and $\lambda = 2$. Note that the solution is the same here as in Problem 2.3, but here the value for the Lagrangian multiplier is the reciprocal of the value in Problem 2.3.

2.5 a. The height of the ball is given by $f(t) = -0.5gt^2 + 40t$. The value of t for which height is maximized is found by taking the first-order condition: $df/dt = -gt + 40 = 0$, implying $t^* = 40/g$.

b. Substituting for t^*,

$$f(t^*) = -0.5g\left(\frac{40}{g}\right)^2 + 40\left(\frac{40}{g}\right) = \frac{800}{g}.$$

Hence,

$$\frac{\partial f(t^*)}{\partial g} = -\frac{800}{g^2}.$$

c. Differentiation of the original function at its optimal value yields

$$\frac{\partial f(t^*)}{\partial g} = -0.5(t^*)^2.$$

Because the optimal value of t depends on g,

$$\frac{\partial f(t^*)}{\partial g} = -0.5(t^*)^2 = -0.5\left(\frac{40}{g}\right)^2 = \frac{-800}{g^2},$$

as was also shown in part c.

d. If $g = 32$, $t^* = 5/4$. Maximum height is $800/32 = 25$. If $g = 32.1$, maximum height is $800/32.1 = 24.92$, a reduction of 0.08. This could have been predicted from the envelope theorem, since

$$df(t^*) = \left(\frac{-800}{32^2}\right)dg = \left(\frac{-25}{32}\right)(.01) \approx -0.08.$$

2.6 a. This is the volume of a rectangular solid made from a piece of metal which is x by $3x$ with the defined corner squares removed.

b. The first order condition for maximum volume is given by

$$\frac{\partial V}{\partial t} = 3x^2 - 16xt + 12t^2 = 0.$$

Applying the quadratic formula to this expression yields

$$t = \frac{16x \pm \sqrt{256x^2 - 144x^2}}{24} = \frac{16x \pm 10.6x}{24} = 0.225x.$$

The second value given by the quadratic $(1.11x)$ is obviously extraneous.

c. If $t = 0.225x$,

$$V \approx 0.67x^3 - .04x^3 + .05x^3 \approx 0.68x^3.$$

So volume increases without limit.

d. This would require a solution using the Lagrangian method. The optimal solution requires solving three non-linear simultaneous equations, a task not undertaken here. But it seems clear that the solution would involve a different relationship between t and x than in parts a–c.

2.7 a. Set up the Lagrangian: $L = x_1 + 5\ln x_2 + \lambda(k - x_1 - x_2)$. The first-order conditions are

$$L_{x_1} = 1 - \lambda = 0$$

$$L_{x_2} = \frac{5}{x^2} - \lambda = 0$$

$$L_{\lambda} = k - x_1 - x_2 = 0.$$

Hence, $\lambda = 1 = 5/x_2$. With $k = 10$, the optimal solution is $x_1^* = x_2^* = 5$.

b. With $k = 4$, solving the first order conditions yields $x_1^* = -1$ and $x_2^* = 5$.

c. If all variables must be non-negative, it is clear that any positive value for x_1 reduces y. Hence, the optimal solution is $x_1^* = 0$, $x_2^* = 4$, and $y^* = 5\ln 4$.

d. If $k = 20$, optimal solution is $x_1^* = 15$, $x_2^* = 5$, $y^* = 15 + 5\ln 5$. Because x_2 provides a diminishing marginal increment to y, whereas x_1 does not, all optimal solutions require that, once x_2 reaches 5, any extra amounts be devoted entirely to x_2. In consumer theory this function can be used to illustrate how diminishing marginal usefulness can lead to a ceiling in purchases of certain goods.

2.8 a. Because MC is the derivative of TC, TC is an antiderivative of MC. By the fundamental theorem of calculus,

$$\int_0^q MC(x)\,dx = TC(q) - TC(0),$$

where $TC(0)$ is the fixed cost, which we will denote $TC(0) = K$ for short. Rearranging,

$$\begin{aligned} TC(q) &= \int_0^q MC(x)\,dx + K \\ &= \int_0^q (x+1)dx + K \\ &= \left(\frac{x^2}{2} + x\right)_{x=0}^{x=q} + K \\ &= \frac{q^2}{2} + q + K. \end{aligned}$$

b. By profit maximization, $p = MC(q) = q + 1$, implying $q = p - 1$. But $p = 15$ implies $q = 14$. Profit equals

$$\begin{aligned} TR - TC &= pq - TC(q) \\ &= 15 \cdot 14 - \left(\frac{14^2}{2} + 14 + K\right) \\ &= 98 - K. \end{aligned}$$

If the firm is just breaking even, profit equals 0, implying fixed cost is $K = 98$.

c. When $p = 20$ and $q = 19$, follow the same steps as in part b, substituting fixed cost $K = 98$. Profit equals

$$\begin{aligned} TR - TC &= pq - TC(q) \\ &= 20 \cdot 19 - \left(\frac{19^2}{2} + 19 + K \right) \\ &= 180.5 - 98 \\ &= 82.5. \end{aligned}$$

d. Assuming profit maximization, we have

$$\begin{aligned} \pi(p) &= pq - TC(q) \\ &= p(p-1) - \left[\frac{(p-1)^2}{2} + (p-1) + 98 \right] \\ &= \frac{(p-1)^2}{2} - 98. \end{aligned}$$

e.

i. Using the above equation, $\pi(p = 20) - \pi(p = 15) = 82.5 - 0 = 82.5.$

ii. The envelope theorem states that $\partial\pi/\partial p = q^*(p)$. That is, the derivative of the profit function yields this firm's supply function. Integrating over p shows the change in profits by the fundamental theorem of calculus:

$$\begin{aligned} \pi(20) - \pi(15) &= \int_{15}^{20} \frac{\partial \pi}{\partial p} dp \\ &= \int_{15}^{20} (p-1) dp \\ &= \left(\frac{p^2}{2} - p \right)_{p=15}^{p=20} \\ &= 180 - 97.5 \\ &= 82.5. \end{aligned}$$

Analytical Problems

2.9 Concave and Quasiconcave Functions

The proof is most easily accomplished through the use of the matrix algebra of quadratic forms. See, for example, Mas Colell *et al*., pp. 937–939. Intuitively, because concave functions lie below any tangent plane, their level curves must also be convex. But the converse is not true. Quasi-concave functions may exhibit "increasing returns to scale"; even though their level curves are convex, they may rise above the tangent plane when all variables are increased together.

A counter example would be the Cobb-Douglas function which is always quasi-concave, but convex when $\alpha + \beta > 1$.

2.10 The Cobb-Douglas Function

a. $f_1 = \alpha x_1^{\alpha-1} x_2^{\beta} > 0$

$f_2 = \beta x_1^{\alpha} x_2^{\beta-1} > 0$

$f_{11} = \alpha(\alpha - 1) x_1^{\alpha-2} x_1^{\beta} < 0$

$f_{22} = \beta(\beta - 1) x_1^{\alpha} x_2^{\beta-2} < 0$

$f_{12} = f_{21} = \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} > 0.$

Clearly, all the terms in Equation 2.114 are negative.

b. A contour line is found by setting the function equal to a constant: $y = c = x_1^{\alpha} x_2^{\beta}$, implying $x_2 = c^{1/\beta} x_1^{-\alpha/\beta}$. Hence,

$$\frac{dx_2}{dx_1} < 0.$$

Further,

$$\frac{d^2 x_2}{dx_1^2} < 0,$$

implying the countour line is convex.

c. Using Equation 2.98, $f_{11} f_{22} - f_{12}^2 = \alpha\beta(1 - \beta - \alpha) x_1^{2\alpha-2} x_2^{2\beta-2}$, which is negative for $\alpha + \beta > 1$.

2.11 The Power Function

a. Since $y' > 0$ and $y'' < 0$, the function is concave.

b. Because $f_{11}, f_{22} < 0$ and $f_{12} = f_{21} = 0$, Equation 2.98 is satisfied; and the function is concave. Because $f_1, f_2 > 0$, Equation 2.114 is also satisfied; so the function is quasi-concave.

c. y is quasi-concave as is y^{γ}. However, y is not concave for $\gamma\delta > 1$. This can be shown most easily by $f(2x_1, 2x_2) = 2^{\gamma\delta} f(x_1, x_2)$.

2.12 Proof of Envelope Theorem

a. The Lagrangian for this problem is

$$\mathrm{L}(x_1, x_2, a) = f(x_1, x_2, a) + \lambda g(x_1, x_2, a).$$

The first-order conditions are

$$\mathrm{L}_1 = f_1 + \lambda g_1 = 0$$

$$\mathrm{L}_2 = f_2 + \lambda g_2 = 0$$

$$\mathrm{L}_\lambda = g = 0.$$

b.,c. Multiplication of each first order condition by the appropriate deriviative yields

$$f_1 \frac{dx_1}{da} + f_2 \frac{dx_2}{da} + \lambda\left(g_1 \frac{dx_1}{da} + g_2 \frac{dx_2}{da} \right) = 0.$$

d. The optimal value of f is given by $f\left(x_1(a), x_2(a), a\right)$. Differentiation of this with respect to a shows how this optimal value changes with a:

$$\frac{df^*}{da} = f_1 \frac{dx_1}{da} + f_2 \frac{dx_2}{da} + f_a.$$

e. Differentiation of the constraint $g\left(x_1(a), x_2(a), a\right) = 0$ yields

$$\frac{dg}{da} = 0 = g_1 \frac{dx_1}{da} + g_2 \frac{dx_2}{da} + g_a.$$

f. Multiplying the results from part e by λ and using parts b and c yields

$$\frac{df^*}{da} = f_a + \lambda g_a = \mathrm{L}_a.$$

This proves the envelope theorem.

g. In Example 2.8 we showed that $\lambda = P/8.$ and that this hsows how much an extra unit of perimeter would raise the enclosed area. Direct differentiation of the Lagrangian in Equation 2.62 shows also that

$$\frac{dA^*}{dP} = \mathrm{L}_P = \lambda.$$

This shows that the Lagrange multiplier does indeed show this incremental gain in this problem.

2.13 Taylor Approximations

a. From Equation 2.85, a function in one variable is concave if $f''(x) < 0.$ Using the quadratic Taylor formula to approximate this function at point a:

$$f(x) \approx f(a) + f'(a)(x-a) + 0.5f''(a)(x-a)^2$$
$$\leq f(a) + f'(a)(x-a).$$

The inequality holds because $f''(a) < 0$. But the right hand side of this equation is the equation for the tangent to the function at point a. So we have shown that any concave function must lie on or below the tangent to the function at that point.

b. From Equation 2.98, a function in two variables is concave if $f_{11}f_{22} - f_{12}^2 > 0$. Hence the quadratic form $(f_{11}dx^2 + 2f_{12}dx\,dy + f_{22}dy^2)$ will also be negative. But this says that the final portion of the Taylor expansion will be negative (by setting $dx = x - a$ and $dy = y - b$) and hence the function will be below its tangent plane.

2.14 More on Expected Value

a. The tangent to $g(x)$ at the point $E(x)$ will have the form $c + dx \geq g(x)$ for all values of x and $c + dE(x) = g(E(x))$. But, because the line $c + dx$ is above the function $g(x)$ we know

$$E(g(x)) \leq E(c + dx) = c + dE(x) = g(E(x)).$$

This proves Jensen's inequality.

b. Use the same procedure as in part a, but reverse the inequalities.

c. Let $u = 1 - F(x),\ du = -f(x),\ x = v,\ dx = dv.$

$$\int_0^\infty [1 - F(x)]dx = [(1 - F(x))x]_{x=0}^{x=\infty} - \int_0^\infty [-f(x)]xdx$$
$$= 0 + E(x)$$
$$= E(x).$$

d. Use the hint to break up the integral defining expected value:

$$\frac{E(x)}{t} = t^{-1}\left[\int_0^t xf(x)dx + \int_t^\infty xf(x)dx\right]$$
$$\geq t^{-1}\int_t^\infty xf(x)dx$$
$$\geq t^{-1}\int_t^\infty tf(x)dx$$
$$= \int_t^\infty f(x)dx$$
$$= P(x \geq t).$$

e. 1. Show that this function integrates to 1:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{1}^{\infty} 2x^{-3}dx = -x^{-2}\Big|_{x=1}^{x=\infty} = 1.$$

2. Calculate the cumulative distribution function:

$$F(x) = \int_{1}^{t} 2t^{-3}dt = -t^{-2}\Big|_{t=1}^{t=x} = 1 - x^{-2}.$$

3. Using the result from part c:

$$E(x) = \int_{1}^{\infty} [1 - F(x)]dx = \int_{1}^{\infty} x^{-2}dx = -x^{-1}\Big|_{x=1}^{x=\infty} = 1.$$

4. To show Markov's inequality use

$$P(x \geq t) = 1 - F(t) = t^{-2} < t^{-1} = \frac{E(x)}{t}.$$

f. 1. Show that the PDF integrates to 1:

$$\int_{-1}^{2} \frac{x^2}{3}dx = \frac{x^3}{9}\Bigg|_{x=-1}^{x=2} = \frac{8}{9} - \left(-\frac{1}{9}\right) = 1.$$

2. Calculate the expected value:

$$E(x) = \int_{-1}^{2} \frac{x^3}{3}dx = \frac{x^4}{12}\Bigg|_{x=-1}^{x=2} = \frac{15}{12} = \frac{5}{4}.$$

3. Calculate $P(-1 \leq x \leq 0)$:

$$\int_{-1}^{0} \frac{x^2}{3}dx = \frac{x^3}{9}\Bigg|_{x=-1}^{x=0} = \frac{1}{9}.$$

4. All we must do is adjust the PDF so that it now sums to 1 over the new, smaller interval. Since $P(A) = 8/9$,

$$f(x \mid A) = \frac{f(x)}{8/9} = \frac{3x^2}{8} \quad \text{defined on } 0 \leq x \leq 2.$$

5. The expected value is again found through integration:

$$E(x \mid A) = \int_{0}^{2} \frac{3x^3}{8}dx = \frac{3x^4}{32}\Bigg|_{x=0}^{x=2} = \frac{3}{2}.$$

6. Eliminating the lowest values for x increases the expected value of the remaining values.

2.15 More on Variances

a. This is just an application of the definition of variance:

$$\begin{aligned}\text{Var}(x) &= E\left[x - E(x)\right]^2 \\ &= E\left[x^2 - 2xE(x) + [E(x)]^2\right] \\ &= E(x^2) - 2[E(x)]^2 + [E(x)]^2 \\ &= E(x^2) - [E(x)]^2.\end{aligned}$$

b. Here we let $y = x - \mu_x$ and apply Markov's inequality to y and remember that x can only take on positive values.

$$P(y \geq k) = P(y^2 \geq k^2) \leq \frac{E(y^2)}{k^2} = \frac{\sigma_x^2}{k^2}.$$

c. Let x_i, $i = 1, \ldots, n$, be n independent random variables each with expected value μ and variance σ^2.

$$E\left(\sum_{i=1}^{n} x_i\right) = \mu + \ldots + \mu = n\mu.$$

$$\text{Var}\left(\sum_{i=1}^{n} x_i\right) = \sigma^2 + \ldots + \sigma^2 = n\sigma^2.$$

Now, let $\bar{x} = \sum_{i=1}^{n}(x_i/n)$.

$$E(\bar{x}) = \frac{n\mu}{n} = \mu.$$

$$\text{Var}(\bar{x}) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

d. Let $X = kx_1 + (1-k)x_2$ and $E(X) = k\mu + (1-k)\mu = \mu.$

$$\text{Var}(X) = k^2\sigma^2 + (1-k)^2\sigma^2 = (2k^2 - 2k + 1)\sigma^2.$$

$$\frac{d\text{Var}(X)}{dk} = (4k - 2)\sigma^2 = 0.$$

Hence, variance in minimized for $k = 0.5$. In this case, $\text{Var}(X) = 0.5\sigma^2$. If $k = 0.7$, $\text{Var}(X) = 0.58\sigma^2$ (not much of an increase).

e. Suppose that $r(x_1) = \sigma^2$ and $\text{Var}(x_2) = r\sigma^2$. Now

$$\text{Var}(X) = k^2\sigma^2 + (1-k)^2 r\sigma^2 = \left[(1+r)k^2 - 2kr + r\right]\sigma^2.$$

$$\frac{d\text{Var}(X)}{dk} = \left[2(1+r)k - 2r\right]\sigma^2 = 0.$$

$$k = \frac{r}{1+r}.$$

For example, if $r = 2$, then $k = 2/3$ and optimal diversification requires that the lower risk asset constitute two-thirds of the portfolio. Note, however, that it is still optimal to have some of the higher risk asset because asset returns are independent.

2.16 More on Covariances

a. This is a direct result of the definition of covariance:

$$\begin{aligned}\text{Cov}(x,y) &= E\left[(x-E(x))(y-E(y))\right] \\ &= E[xy - xE(y) - yE(x) + E(x)E(y)] \\ &= E(xy) - E(x)E(y) - E(y)E(x) + E(x)E(y) \\ &= E(xy) - E(x)E(y).\end{aligned}$$

b.
$$\begin{aligned}\text{Var}(ax \pm by) &= E[(ax \pm by)^2] - [E(ax \pm by)]^2 \\ &= a^2E(x^2) \pm 2abE(xy) + b^2E(y^2) - a^2[E(x)]^2 \\ &\qquad \pm 2ab\text{E}(x)\text{E}(y) - b^2[\text{E}(y)]^2 \\ &= a^2\text{Var}(x) + b^2\text{Var}(y) \pm 2ab\text{Cov}(x,y).\end{aligned}$$

The final line is a result of Problems 2.15a and 2.16a.

c. The presence of the covariance term in the result of Problem 2.16b suggests that the results would differ. In the two variable case, however, this is not necessarily the situation. For example, suppose that x and y are identically distributed and that $\text{Cov}(x,y) = r\sigma^2$. Using the prior notation,

$$\text{Var}(X) = k^2\sigma^2 + (1-k)^2\sigma^2 + 2k(1-k)r\sigma^2.$$

The first order condition for a minimum is

$$(4k - 2 + 2r - 4rk)\sigma^2 = 0,$$

implying

$$k^* = \frac{2-2r}{4-4r} = 0.5.$$

Regardless of the value of r. With more than two random variables, however, covariances may indeed affect optimal weightings.

d. If $x_1 = kx_2$, the correlation coefficient will be either $+1$ (if k is positive) or -1 (if k is negative) since k will factor out of the definition leaving only the ratio of the common variance of the two variables. With less than a perfect linear relationship $|\text{Cov}(x,y)| < \left[\text{Var}(x)\text{Var}(y)\right]^{0.5}$.

e. If $y = \alpha + \beta x$,

$$\begin{aligned}\text{Cov}(x, y) &= E\left[(x - E(x))(y - E(y))\right] \\ &= E[(x - E(x))(\alpha + \beta x - \alpha - \beta E(x))] \\ &= \beta \text{Var}(x).\end{aligned}$$

Hence,

$$\beta = \frac{\text{Cov}(x, y)}{\text{Var}(x)}.$$