**SOLUTIONS TO PROBLEMS – Instructors' version (all problems)**

**PROBLEMS 1**

**1.1**

(a) $\bar{\nu}_e + e^+ \to \bar{\nu}_e + e^+$; (b) $p+n \to p+n$; (c) $p + \bar{p} \to e^+ + e^- + \gamma$.

**1.2** From (A),

$$\alpha_i\beta = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -\sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix},$$

and similarly

$$\beta\alpha_i = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \sigma_i \\ -\sigma_i & \mathbf{0} \end{pmatrix}.$$

Hence $\left[\alpha_i, \beta\right] \equiv \alpha_i\beta + \beta\alpha_i = 0$, for $i = 1,2,3$. Using the same relations,

$$\alpha_i\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} = \sigma_i\sigma_j\mathbf{1}$$

and likewise $\alpha_j\alpha_i = \sigma_j\sigma_i\mathbf{1}$. But it is straightforward to verify from (B) that $\sigma_i\sigma_j = -\sigma_j\sigma_i$ for $i \neq j$. So, $\left[\alpha_i, \alpha_j\right] = 0$, for $i \neq j = 1,2,3$.

**1.3** Substituting the explicit forms of the Dirac matrices into (1.13) gives

$$\begin{aligned}
&(E - mc^2)u_1 - cp_z u_3 - c(p_x - ip_y)u_4 = 0, \\
&(E - mc^2)u_2 - c(p_x + ip_y)u_3 + cp_z u_4 = 0, \\
&(E + mc^2)u_3 - cp_z u_1 - c(p_x - ip_y)u_2 = 0, \\
&(E + mc^2)u_4 - c(p_x + ip_y)u_1 + cp_z u_2 = 0.
\end{aligned} \tag{A}$$

*Case 1*: $u_1 = 1,\ u_2 = 0,\ u_3 = a_1,\ u_4 = b_1$

In this case, the third and fourth of equations (A) give the values

$$u_3 = a_1 = \frac{cp_z}{E + mc^2}, \quad u_4 = b_1 = \frac{c(p_x + ip_y)}{E + mc^2}. \tag{B}$$

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For $\mathbf{p} = \mathbf{0}$, the first equation in (A) with $u_1 = 1$ clearly requires $E = mc^2$, while substituting the relations (B) in the first equation of (A) leads to the requirement

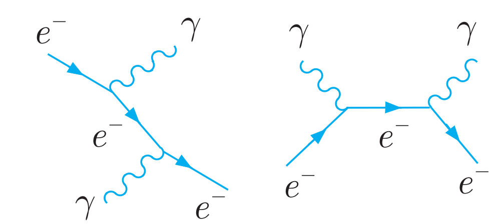
$$E - mc^2 = \frac{c^2\mathbf{p}^2}{E + mc^2} \ .$$

Taken together, these are satisfied by $E = E_+ \equiv (c^2p^2 + m^2c^4)^{1/2}$, but not by $E = E_- \equiv -(c^2p^2 + m^2c^4)^{1/2}$.

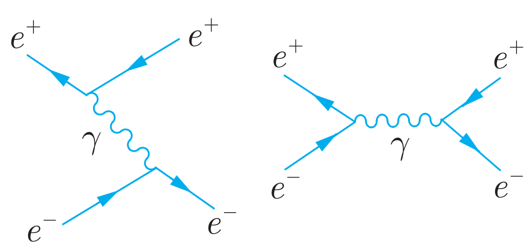
The other solutions are dealt with in a similar way, where the first two require $E = E_+ > 0$ and the second two require $E = E_- < 0$. All four solutions are tabulated below.

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{(E_- - mc^2)} \\ \dfrac{c(p_x - ip_y)}{(E_- - mc^2)} \end{pmatrix} &
\begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x + ip_y)}{(E_- - mc^2)} \\ \dfrac{-cp_z}{(E_- - mc^2)} \end{pmatrix} &
\begin{pmatrix} \dfrac{cp_z}{(E_+ + mc^2)} \\ \dfrac{c(p_x - ip_y)}{(E_+ + mc^2)} \\ 1 \\ 0 \end{pmatrix} &
\begin{pmatrix} \dfrac{c(p_x + ip_y)}{(E_+ + mc^2)} \\ \dfrac{-cp_z}{(E_+ + mc^2)} \\ 0 \\ 1 \end{pmatrix}
\end{array}$$

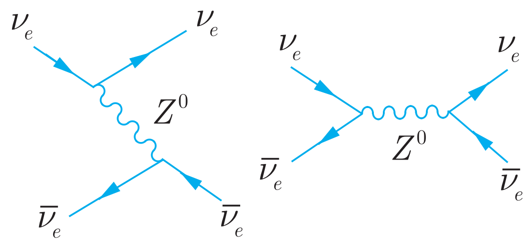
**1.4** The topologically distinct diagrams for reactions (a) are given below.



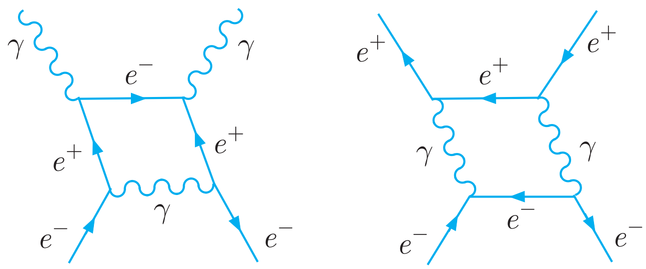

Those for (b) are




and those for (c) are shown below.




**1.5** Two possible diagrams are shown in below. There are others.




**1.6** For a spherically symmetric static solution we can set $\Psi(\mathbf{r},t) = \phi(r)$, where $r = |\mathbf{r}|$, and use

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r},$$

giving

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} = \left(\frac{mc}{\hbar}\right)^2 \phi\,.$$

Substituting $\phi = u(r)/r$ gives

$$\frac{\mathrm{d}^2 u(r)}{\mathrm{d}r^2} = \left(\frac{mc}{\hbar}\right)^2 u(r)$$

and the result follows by solving for $u$, and imposing $\phi \to 0$ as $r \to \infty$.

**1.7** Using spherical polar co-ordinates, we have

$$\mathbf{q}\cdot\mathbf{r} = |\mathbf{q}|\, r\cos\theta \quad \text{and} \quad \mathrm{d}^3\mathbf{r} = r^2 \mathrm{d}r\,\mathrm{d}\cos\theta\,\mathrm{d}\phi\,.$$

Thus, from (1.32),

$$\mathcal{M}(q^2) = \frac{-g^2}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty}\mathrm{d}r\,r^2\frac{e^{-r/R}}{r}\int_{-1}^{+1}\mathrm{d}\cos\theta\exp(iqr\cos\theta/\hbar) = \frac{-g^2\hbar^2}{|\mathbf{q}|^2 + M_X^2c^2},$$

where we have used (1.26).

**1.8** If we impose momentum conservation and neglect the momenta of the initial $e^{\pm}$, we have

$$e^{+}(m,\mathbf{0}) + e^{-}(m,\mathbf{0}) \to \gamma(E_k,\mathbf{k}) + \gamma(E_k,-\mathbf{k}) ,$$

where $E_k = |\mathbf{k}| = m$. Hence in Figure 1.9($a$), the initial virtual process is

$$e^{-}(m,\mathbf{0}) \to e^{-}(E,\mathbf{k}) + \gamma(m,-\mathbf{k}) ,$$

where the energy of the virtual electron is

$$E = (k^2 + m^2)^{1/2} \approx \sqrt{2}m \approx m .$$

Hence $\Delta E = E \approx m$ and from the uncertainty principle, $r \approx 1/\Delta E \approx 1/m$. Restoring factors of and $c$ gives $r \approx \hbar c/mc^2 = 368$ fm .

**1.9** The distance between the two vertices is given by

$$r \approx c\tau \approx \hbar c/\Delta E = 1/\Delta E ,$$

where $\Delta E$ is the energy violation at the first vertex. On evaluating $\Delta E$ in each case, one obtains (a) $r \approx m^{-1}$ and (b) $r' \approx E^{-1}$, respectively, where $E$ is the initial electron energy in the centre-of-mass frame. These are related by a Lorentz contraction $r' = r/\gamma$, where $\gamma = E/m$. (A resumé of special relativity is given in Appendix A.)

**1.10** Set $\tau = (2/m\alpha^5)\hbar^a c^b$ and demand that $\tau$ has the dimensions of time. This gives $a = 1$ and $b = -2$ and hence $\tau = 1.245\times 10^{-10}$ s .

**1.11** Substituting the values for $E$ and $G_F$, gives $\sigma = 3.60\times 10^{-10}\,\mathrm{GeV}^{-2}$ , and using the conversion factor $1\,\mathrm{GeV}^{-2} = 0.389\,\mathrm{m}$, given on the inside of the back cover, to convert from nu to SI units, we obtain $\sigma = 1.40\times 10^{-10}\,\mathrm{mb} = 0.14\,\mathrm{pb}$.

**1.12** (a) In natural units $\alpha = e^2/4\pi\varepsilon_0$ , so we can write $r_c = \alpha/m_e$ . Restoring factors of $\hbar$ and $c$, we have

$$r_c = \alpha\frac{\hbar c}{m_e c^2} = \alpha\frac{1.973\times 10^{-13}\,\mathrm{MeV\,m}}{0.51\,\mathrm{MeV}} = 0.0282\times 10^{-13}\,\mathrm{m} = 2.8\,\mathrm{fm}.$$

(b) The electrostatic energy is

$$E = \tfrac{1}{2}\int \rho\phi \, \mathrm{d}v \, ,$$

where $\phi$ is the potential energy and the charge density for $r < R$ and zero for $r > R$. On a spherical shell of radius $r < R$, the potential energy is just the electrostatic potential due to the enclosed charge $Q$. Hence

$$\phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{er^2}{R^3}$$

and

$$E = \frac{1}{2}\frac{3e}{4\pi R^3}\frac{1}{4\pi\varepsilon_0}\int_0^R \frac{er^2}{R^3} 4\pi r^2 \, \mathrm{d}r = \frac{3}{10}\frac{e^2}{4\pi\varepsilon_0 R} = \frac{3}{10}\frac{\alpha}{R} \text{ in nu.}$$

Setting $E = m$ then gives

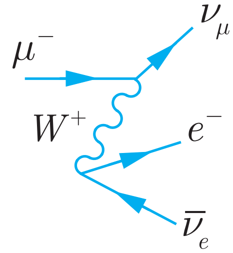
$$R = \frac{3}{10}\frac{\alpha}{m} = \frac{3r_c}{m},$$

where $r_c$ is the 'classical radius' defined in (a) above.

**PROBLEMS 2**

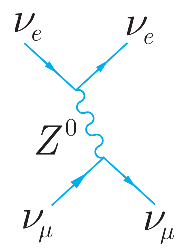
**2.1** Weak interactions conserve baryon number $B$, charge $Q$ and lepton numbers $L_e$, $L_\mu$ and $L_\tau$. They need not conserve the quantum numbers $S$, $C$ or $\widetilde{B}$. Of the decays given, (a) violates $L_\mu$ conservation and (d) violates both $L_\mu$ and $L_\tau$ conservation. They are therefore both forbidden. Reactions (b) and (c) satisfy all the conservation laws and are allowed.

**2.2** The basic processes are $\mu^- \to W^-\nu_\mu$ and $W^- \to e^-\overline{\nu}_e$. The other time-ordered diagram is shown below,

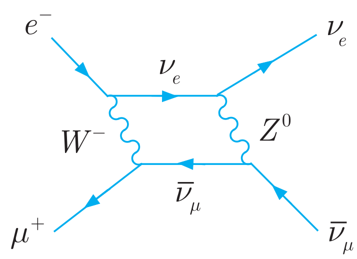


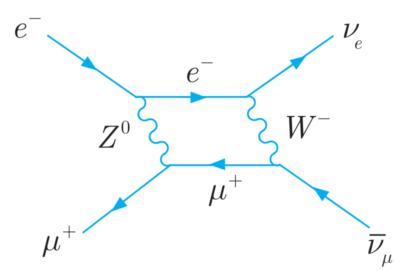

and the basic processes are $\text{vacuum} \to W^+ e^- \overline{\nu}_\mu$ and $\mu^- W^+ \to \overline{\nu}_\mu$.

**2.3** The lowest-order diagram involves the exchange of a single $Z^0$ boson, as shown below.

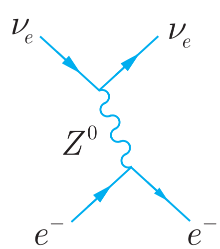


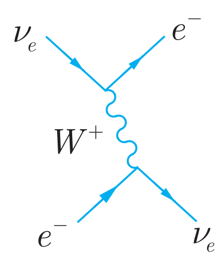

**2.4** The two diagrams are shown below.



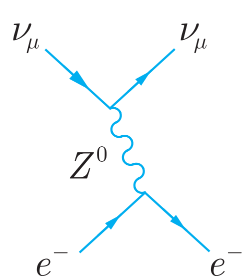




**2.5** The electron neutrino may interact with electrons via both $Z^0$ and $W^-$ exchange:




whereas, because of lepton number conservation, the muon neutrino can only interact via $Z^0$ exchange:




**2.6** Firstly restore factors of $\hbar$ and $c$ by writing $L_0 = 4E\hbar^a c^b \big/ \Delta m_{ij}^2$ and find $a$ and $b$ by demanding that the right-hand side has the dimensions of a length. This gives $a = 1$ and $b = -3$, so that $L_0 = 4E(\hbar c) \big/ \Delta m_{ij}^2 c^4$. Then if $L_0$ is expressed in km, $E$ in GeV and $\Delta m_{ij}^2$ in $(\mathrm{eV/c^2})^2$, we have

$$L_0 = \frac{4E \times (1.97 \times 10^{-13}) \times 10^{18}}{\Delta m_{ij}^2} = \frac{E}{1.27 \Delta m_{ij}^2} \ \text{km}\,.$$

**2.7** By energy conservation

$$E_+ = E_\nu + m_p - m_n - K_n = 1.71\,\text{MeV} - K_n,$$

where $K_n = E_n - m_n$ is the kinetic energy of the recoil neutron. Since $K_n > 0$, this implies $E_+ \leq 1.71\,\text{MeV}$ and hence $|\mathbf{p}_+| < 1.63\,\text{MeV}$, using $E^2 = p^2 + m^2$. By momentum conservation $\mathbf{p}_n = \mathbf{p}_\nu - \mathbf{p}_+$, so that

$$|\mathbf{p}_n| \leq |\mathbf{p}_\nu| + |\mathbf{p}_+| = 4.63\,\text{MeV}\,,$$

neglecting the neutrino mass, and $K_n = \mathbf{p}_n^2 / 2m_n \leq 0.011\,\text{MeV}$. Hence $E_+ = E_\nu + m_p - m_n = 1.71\,\text{MeV}$ to within 0.01 MeV, which is less than 1%.

**2.8** From (2.30) and (2.31), the necessary condition is

$$\cos(2\theta_{12}) = A = -\frac{2\sqrt{2}G_F N_e E}{\Delta m_{12}^2}\ .$$

Since $N_e \leq N_0$, this requires

$$E \geq -\frac{\Delta m_{12}^2 \cos(2\theta_{12})}{2\sqrt{2}G_F N_0}\,. \tag{1}$$

Using the conversion factor $1\text{GeV}^{-1} = 0.197\text{fm}$, gives $1\text{m}^{-1} = 0.197 \times 10^{-12}\text{MeV}$ and hence $N_0 = 2.3 \times 10^{-7}\text{MeV}^3$ in natural units. In these units, $\Delta m_{12}^2 = 7.5 \times 10^{-17}\text{MeV}^2, G_F = 1.17 \times 10^{-11}\text{MeV}^{-2}$, and $\tan^2\theta_{12} = 0.444$ gives $\cos 2\theta_{12} = 0.385$. Substituting into (1) then gives $E \geq 3.8\,\text{MeV}$.

**2.9** Using the results of Problem 2.6, together with the squared mass differences

$$|\Delta m_{13}|^2 \approx 2 \times 10^{-3} \left(\text{eV}/c^2\right)^2 \ \text{and}\ |\Delta m_{21}|^2 \approx 7.5 \times 10^{-5} \left(\text{eV}/c^2\right)^2\ ,$$

we have

$$L_{13} \approx 1.2\,\text{km}, \quad L_{12} \approx 31.5\,\text{km}.$$

Hence the magnitudes of amplitudes of the associated transitions (2.30a) are

$$P_{13} \approx 0.09\sin^2(L/1.2) \quad \text{and} \quad P_{12} \approx 0.9\sin^2(L/32),$$

where $L$ is in km and we have used $\sin^2(2\theta_{12}) \approx 0.08$ and $\sin^2(2\theta_{13}) \approx 0.9$.
For $L$ of order 150–200 km the amplitude of the oscillations of $P_{13}$ is ten times smaller than those of $P_{12}$, so that the former may be neglected to a first approximation. For an $L$ of 1km, $P_{12} \approx 0.9\times10^{-3}$, which is small compared to the amplitude of the oscillation of $P_{13}$, so that $P_{12}$ can, to a good approximation, be neglected. For an $L$ of 10 km, $P_{12} \approx 0.09$, which is comparable to the amplitude of the oscillation of $P_{13}$, so that a two-component mixing model would fail badly.

**2.10** From the solution to Problem 2.6, we have, for maximal mixing ($\theta = \pi/4$), $P(\bar{\nu}_e \to \bar{\nu}_x) = \sin^2[1.27\,\Delta(m^2c^4)L/E]$ where $L$ is measured in metres, $E$ in MeV, $\Delta(m^2c^4)$ in $(\text{eV})^2$ and $\Delta m^2 \equiv m^2(\bar{v}_e) - m^2(\bar{v}_x)$. If $P(\bar{\nu}_e \to \bar{\nu}_e) = 0.9 \pm 0.1$, then at 95% confidence level, $0.3 \ge P(\bar{\nu}_e \to \bar{\nu}_x) \ge 0$ and hence $0 \le \Delta(m^2c^4) \le 3.8\times10^{-3}\ (\text{eV})^2$.

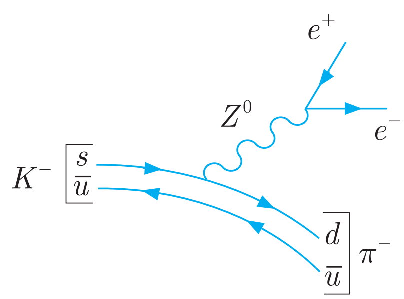
**2.11** From the data given, the total number of nucleons is given by

$$N = \frac{2\times10^{30}}{(0.938\times10^{9})(1.78\times10^{-36})} = 1.2\times10^{57}$$

and hence $n = 8.4\times10^{38}\,\text{km}^{-3}$. Also the mean energy of the neutrinos from reaction (2.35) is 0.26 MeV, so the cross-section is $\sigma = 1.8\times10^{-46}\ \text{m}^2$. Thus, finally, $\lambda \approx 6.6\times10^{12}$ km, i.e. about $10^7$ times the solar radius.

**PROBLEMS 3**

**3.1** (a) forbidden – violates baryon number conservation; (b) forbidden – violates charge conservation; (c) forbidden – violates muon lepton number conservation; (d) allowed by the diagram:




**3.2** (a) Involves neutrinos and is therefore a weak interaction. (b) Involves photons and is therefore electromagnetic. (c) Conserves all quantum numbers and is therefore a strong interaction. (d) Violates both strangeness and charm and is therefore a weak interaction. (e) Conserves all quantum numbers and is a strong interaction. (f) Involves electrons and positrons and is therefore an electromagnetic interaction.

**3.3** The final-state pions have equal energies $E_\pi = E_K/2 = 247\,\text{MeV}$ and $\gamma = E_\pi/m_\pi c^2 = 1.76$. Hence the lifetime in flight is $\tau = \gamma\tau_0 = 4.6\times10^{-8}\text{s}$ and the $\pi^+$ velocity is $v = c(1-1/\gamma^2)^{-1/2} = 1.21c$. The mean distance travelled is thus $\tau v = 17\,\text{m}$.

**3.4** The quantum numbers are:

$$X^0:\ B=1,\ S=-1,\ C=0,\ \widetilde{B}=0\ ;\ Y^-:\ B=1,\ S=-2,\ C=0,\ \widetilde{B}=0\,.$$

From their charges, and the definitions of $\widetilde{B}$, $S$, $C$ and $B$, it follows that $X^0 = uds$ and $Y^- = dss$. The decay $Y^- \to \Lambda + \pi^-$ violates strangeness conservation and is a weak interaction, so we expect $\tau = 10^{-6} - 10^{-13}$ s. (The $Y^-$ is in fact the so-called $\Xi^-(1321)$ state with a lifetime $1.6\times10^{-10}\text{s}$)

**3.5** The quantum number combination (2, 1, 0, 1, 0) corresponds to a baryon *qqq*. It has $S=0$, $C=1$ and $B=0$, so must be of the form *cxy,* where $x$ and $y$ are $u$ or $d$ quarks. The charge $Q=2$ requires both $x$ and $y$ to be $u$ quarks, i.e. *cuu*. The others are established by similar arguments and so the full set is:

$$\begin{array}{lll}(2,1,0,1,0) = cuu, & (0,1,-2,1,0) = css, & (0,0,1,0,-1) = b\bar{s}, \\ (0,-1,1,0,0) = \bar{s}\bar{d}\bar{u}, & (0,1,-1,1,0) = csd, & (-1,1,-3,0,0) = sss.\end{array}$$

They are called $\Sigma_c^{++}$, $\Omega_c^0$, $\bar{B}_s^0$, $\bar{\Lambda}$, $\Sigma_c^0$ and $\Omega^-$, respectively.

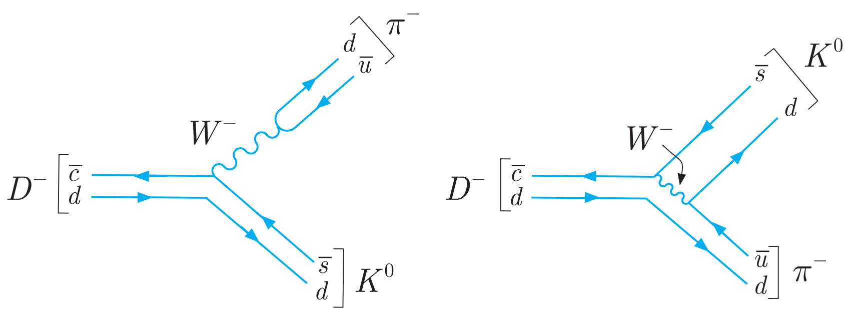
**3.6** From their quark compositions, the $\Sigma^+$, $\Sigma^0$ and $\Sigma^-$ have $B=1, S=-1$. We need to check the reactions for $B, Q$ and $S$ conservation. Reactions (b) and (d) conserve all three quantum numbers and so are allowed in strong interactions. Reactions (a) and (c) violate $S$ and $Q$ conservation respectively, and are forbidden in strong interactions.

**3.7** The $\Sigma^0(1193)$ has $B=1, Q=0, S=-1$, which must all be conserved in a strong interaction. The lightest two-body hadron state with these quantum numbers is $\Lambda\pi^0$, but $m(\Sigma^0) < m(\Lambda) + m(\pi^0)$. Hence strong interaction decays are forbidden by energy conservation. The decay $\Sigma^0 \to \Lambda\gamma$ conserves all three quantum numbers and is first-order in the electromagnetic interaction. Other electromagnetic decays, such as $\Sigma^0 \to \Lambda e^+ e^-$, are higher order , and suppressed by higher powers of $\alpha$. For strong decays $\tau \sim 10^{-22} - 10^{-24}\text{s}$ from Table 3.4. Hence for the decay $\Sigma^0 \to \Lambda\gamma$,

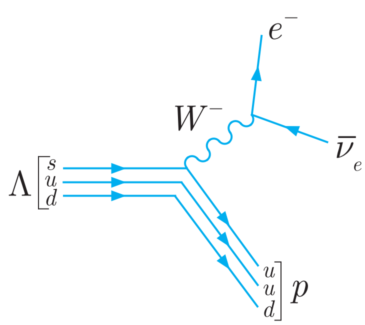
$\tau(\Lambda\gamma) \sim \tau/\alpha \sim 10^{-19} - 10^{-22}\,\mathrm{s}$, with $\alpha \sim 10^{-2} - 10^{-3}$. (The measured value is $7.4\times10^{-20}\mathrm{s}$.)

**3.8** The $\Sigma^+(1189)$ has $B = 1, Q = 1$, which must be conserved in all interactions. The only possible hadron final states with these quantum numbers are $n\pi^+$, $p\pi^0$, i.e the decay modes are $\Sigma^+ \to n\pi^+, p\pi^0$. Since $\Sigma^+(1189)$ has $S = -1$ and the final states have $S = 0$, strangeness is violated and the decays must be via the weak interaction. From Table 3.4, the lifetime is expected to be in the range $(10^{-7} - 10^{-13})\mathrm{s}$. (The actual value is $8\times10^{-11}\,\mathrm{s}$.)

**3.9** (a) The quark compositions are: $D^- = d\bar{c}$ ; $K^0 = d\bar{s}$ ; $\pi^- = d\bar{u}$ and since the dominant decay of a $c$-quark is $c \to s$, we can have either of the diagrams shown below.

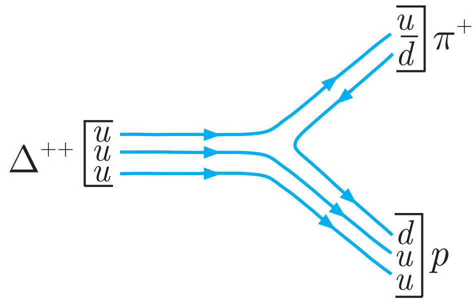



(b) The quark compositions are: $\Lambda = sud$ ; $p = uud$ and since the dominant decay of an $s$-quark is $s \to u$, we have the diagram:

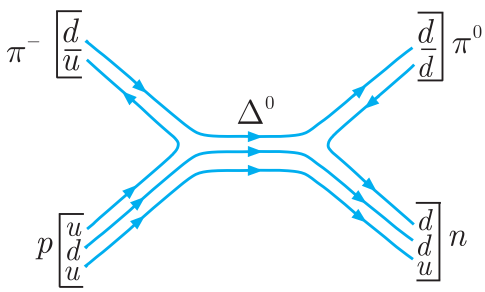


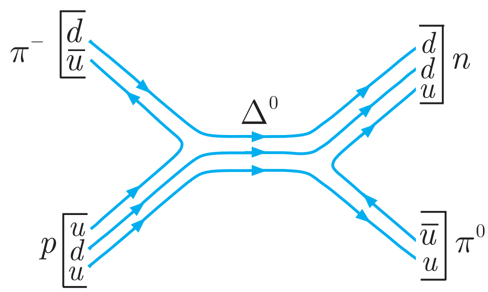

**3.10** For $\gamma \approx 10$, $\tau \approx c$ and the average distance $d \approx c\gamma\tau \approx 3\times10^{-14}$ m $= 30$ fm, if we assume a lifetime for the particle at rest of $10^{-23}$ s. This is much smaller than the best experimental resolution given, but larger than the range of the strong interaction. This is important if the decay of, for example, the $X^-$ produced in reaction (3.26a) is to be treated as the decay of a free particle, since this requires that it should be sufficiently far from any other hadrons that are present (e.g. the proton in (3.26a)).

**3.11** (a) The decay $\Delta^{++} \to \pi^+ p$ in terms of quarks is $uuu \to u\overline{d} + uud$, so a $d\overline{d}$ is created and the quark diagram is




(b) The corresponding diagrams for $\pi^- p \to \Delta^0 \to \pi^0 n$ are




**3.12** An argument similar to that given in Section 3.6 gives the following allowed combinations:

| *Baryons* | | *Mesons* | |
|---|---|---|---|
| $C$ | $Q$ | $C$ | $Q$ |
| 3 | 2 | 1 | 1, 0 |
| 2 | 2, 1 | 0 | 1, 0, −1 |
| 1 | 2, 1, 0 | −1 | 0, −1 |
| 0 | 2, 1, 0, −1 | | |

**3.13** The first two quantum number combinations are compatible with the assignments

$$(1,0,0,1,1) = c\overline{b}, \quad (-1,1,-2,0,-1) = ssb$$

and can exist within the simple quark model. There are no combinations $q\overline{q}$ or $qqq$ which are compatible with the second two combinations, so these cannot exist within the quark model. The combination $(0,0,1,0,1)$ must be a meson $q\overline{q}$ because $B = 0$, but must contain both an $\overline{s}$ antiquark and a $\overline{b}$ antiquark since $S = \widetilde{B} = 1$. These are incompatible requirements. The combination $(-1,1,0,1,-1)$ must be a baryon $qqq$ of the form $xcb$ (where $x = u$ or $d$) since $B = 1$, $S = 0$, $C = 1$ and $\widetilde{B} = -1$. The