Problems

2.1 Estimate the kinetic energy of the ocean liner *Queen Mary 2*, with mass (displacement) 76 000 tons, moving at a cruising speed of 26 knots. Compare with the change in potential energy of the *Queen Mary 2* when lifted by a three foot tide.

The knot is a nautical unit of speed, 1 knot = 0.5144 m/s, so 26 knots = 13.4 m/s. The Queen Mary 2's kinetic energy is then

$$E_{kin} = \frac{1}{2}mv^2 = 0.5(76 \times 10^6 \text{ kg})\,(13.4 \text{ m/s})^2 = 6.82 \text{ GJ}\,.$$

The change in potential energy when lifted by a three foot tide is

$$V = mgh = (76 \times 10^6 \text{ kg})(9.81 \text{ m/s}^2)(3 \text{ ft})\left(\frac{0.3048 \text{ m}}{1 \text{ ft}}\right) = 663 \text{ MJ}\,,$$

or about 10% of the kinetic energy.

2.2 [T] A mass m moves under the influence of a force derived from the potential $V(x) = V_0 \cosh ax$, where the properties of cosh x and other hyperbolic functions are given in Appendix B.4.2. What is the force on the mass? What is the frequency of small oscillations about the origin? As the magnitude of the oscillations grows, does their frequency increase, decrease, or stay the same?

The force on the mass is $F(x) = -dV/dx = -V_0 a \sinh ax$, with $\sinh x = \frac{1}{2}(e^x - e^{-x})$. For small x, the exponential function is approximated using a Taylor expansion by

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \ldots,$$

so $\cosh y = 1 + y^2/2 + \ldots$ and $\sinh y = y + y^3/6 + \ldots$. For small oscillations, i.e. when $ax \ll 1$, so we only need keep the first term in the expansion of sinh ax, the force is $F(x) = -V_0 a^2 x$. The equation of motion for small oscillations is

$$m\ddot{x} = -V_0 a^2 x,$$

which is the equation for a harmonic oscillator (see eq. (2.10)) with frequency $\omega = a\sqrt{V_0/m}$.

This frequency is independent of magnitude of oscillations. If, however, the amplitude of the oscillation becomes large compared to $1/a$, then the non-linear terms in the expansion of sinh x cannot be ignored and the equation of motion becomes

$$m\ddot{x} = -V_0 a\left(ax + \frac{1}{6}a^3 x^3 + \ldots\right).$$

All of the terms in the expansion have the same sign, so as the amplitude of oscillations increases, the magnitude of the force increases compared to just a simple harmonic oscillator and the frequency of the oscillations increases.

2.3 [T] An object of mass m is held near the origin by a spring, $F = -kx$. What is its potential energy? Show that $x(t) = (x_0, y_0, z_0)\cos \omega t$ is a possible classical motion for the mass. What is the energy of this solution? Is this the most general motion? If not, give an example of another solution.

According to eq. (2.18), the force is the negative gradient of the potential energy. Setting $F = -kx = -\nabla V$, the solution is

$$V(x) = \frac{1}{2}kx^2 + C,$$

which can be checked by differentiating with respect to each coordinate. We set the integration constant $C = 0$ so that the potential energy is zero at $x = 0$.

The equation of motion is $m\ddot{x} = F = -kx$, and suggested solution can be written $x(t) = x_0 \cos \omega t$, where $x_0 = (x_0, y_0, z_0)$. Differentiating twice with respect to t we see $\ddot{x} = -\omega^2 x$, which satisfies the equation of motion if $\omega = \sqrt{k/m}$.

The total energy is the sum of kinetic, $E_{kin} = \frac{1}{2}m\dot{x}^2$, and potential, $V(x)$.

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$
$$= \frac{1}{2}m\omega^2 x_0^2 \sin^2 \omega t + \frac{1}{2}kx_0^2 \cos^2 \omega t = \frac{1}{2}kx_0^2\,,$$

where we have used $m\omega^2 = k$ and $\sin^2 + \cos^2 = 1$.

This solution is not the most general since it requires the oscillator to be maximally stretched at $t = 0$. It is easy to show that $x(t) = x_0 \cos \omega(t - t_0)$ is a solution for any t_0. Using the identity $\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$, it is clear that this solution is a sum of sine and cosine functions.

2.4 Estimate the potential energy of the international space station (mass 370 000 kg) relative to Earth's surface when in orbit at a height of 350 km. Compute the velocity of the space station in a circular orbit and compare the kinetic and potential energy.

From eq. (2.20), the potential energy relative to the surface is

$$V(h) = -\frac{GmM_\oplus}{R_\oplus + h} + G\frac{mM_\oplus}{R_\oplus}$$
$$= (6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2)\,(3.7 \times 10^5 \text{ kg})\,(6 \times 10^{24} \text{ kg}) \times \left(\frac{1}{6.37 \times 10^6 \text{ m}} - \frac{1}{6.37 \times 10^6 + 3.5 \times 10^5 \text{ m}}\right) = 1.21 \text{ TJ}.$$

In a circular orbit, the gravitational force is equal to the mass times the centripetal acceleration,

$$\frac{mv^2}{r} = G\frac{M_\oplus m}{r^2}.$$

Solving for velocity,

$$v = \sqrt{\frac{GM_\oplus}{r}} = \sqrt{\frac{(6.67 \times 10^{-11})(6 \times 10^{24})}{(6.37 \times 10^6 + 3.5 \times 10^5)}} \text{ m/s} = 7.7 \text{ km/s}.$$

The kinetic energy is equal to

$$E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}(3.7 \times 10^5 \text{ kg})\,(7.7 \times 10^3 \text{ m/s})^2 = 11 \text{ TJ}.$$

Thus, the kinetic energy is 9.1 times the potential energy relative to Earth's surface.

2.5 [T] Relate eq. (2.20) to eq. (2.9) and compute g in terms of G, $M_\oplus$, $R_\oplus$ as described in the text.

The potential yielding the r^{-2} force law is $V(r) = -GMm/r$. In the approximation that the gravitational force is constant, the potential is $V(z) = mgz$, where $z = r - R_\oplus$ is the height from Earth's surface, $R_\oplus$ is Earth's radius and $z \ll R_\oplus$. By definition,

$$V(z) - V(0) = -\frac{GM_\oplus m}{R_\oplus + z} + G\frac{M_\oplus m}{R_\oplus}.$$

Use a series expansion for $1/(R_\oplus + z)$, (see eq. (B.63)),

$$\frac{1}{R_\oplus + z} = \frac{1}{R_\oplus} - \frac{z}{R_\oplus^2} + \ldots,$$

giving

$$V(z) - V(0) = \frac{GM_\oplus m}{R_\oplus^2}\left(z + O(z^2/R_\oplus^3)\right)$$

Comparing this to $V(z) = mgz$, and setting $V(0) = 0$, we get to leading order in $z/R_\oplus$,

$$g = \frac{GM_\oplus}{R_\oplus^2} = \frac{(6.67\times10^{-11}\ \mathrm{N\,m^2/kg^2})(5.97\times10^{24}\ \mathrm{kg})}{(6.37\times10^{6}\ \mathrm{m})^2}$$

$$9.81\ \mathrm{m/s^2}.$$

2.6 Make a rough estimate of the maximum hydropower available from rainfall in the US state of Colorado. Look up the average yearly rainfall and average elevation of Colorado and estimate the potential energy (with respect to sea level) of all the water that falls on Colorado over a year. How does this compare with the US yearly total energy consumption?

Colorado has an average yearly precipitation of ~0.4 m. Suppose this is typical of the whole state of Colorado. Colorado has a mean elevation of approximately 2 070 m and a total surface area of 270 000 km^2. All the precipitation falling on the state in one year then has a total potential energy of

$$mgh \approx \rho Vgh \quad (1000\ \mathrm{kg/m^3})(0.4\ \mathrm{m})(2.7\times10^{11}\ \mathrm{m^2})$$

$$\times (9.8\ \mathrm{m/s^2})(2070\ \mathrm{m}) \quad 2.2\times10^{18}\ \mathrm{J} \quad 2.2\ \mathrm{EJ}.$$

This is about 2% of the total energy consumed in the US each year.

2.7 Choose your favorite local mountain. Estimate how much energy it takes to hike to the top of the mountain (ignoring all the local ups and downs of a typical trail). How does your result compare to a typical day's food energy intake of 10 MJ, taking into account the fact that human muscles are not 100% efficient at converting energy into work?

Mount Washington is the tallest mountain in the State of New Hampshire? The peak of Mount Washington is at 1 917 meters (6288 feet) above sea level. A typical route, however, begins at about 600 meters above sea level, so the elevation gain is roughly 1 300 m. For a hiker of 80 kg (including food, water, extra clothing, etc.), the potential energy gain on the hike is

$$V = mgh \quad (80\ \mathrm{kg})(9.8\ \mathrm{m/s^2})(1300\ \mathrm{m}) \quad 1\ \mathrm{MJ}. \qquad (2.45)$$

At 20-30% muscle efficiency, this represents about one third to one half of the useful energy output from the typical hiker's 10 MJ/day food energy intake. This is computed just from the elevation gain, without including local ups and downs of the trail, horizontal distance covered, or the hike down. So don't feel bad about consuming a lot of high-calorie trail food next time you are on a strenuous hike!

2.8 Use any means at your disposal (including the internet) to justify estimates of the following (an order-of-magnitude estimate is sufficient): (a) The kinetic energy of the Moon in its motion around Earth. (b) The kinetic energy of a raindrop before it hits the ground. (c) The potential energy of the water in Lake Powell (relative to the level of the Colorado river directly beneath the dam). (d) Energy needed for you to power a bicycle 15km at 15 km/h on flat terrain. (e) The energy lost to air resistance by an Olympic athlete running a 100 m race. (f) The gravitational potential energy of a climber at the top of Mount Everest.

(a) The moon orbits the earth about once a month and has an orbital radius of about 380 000 km, so its speed is $v = 2\pi R/T$ 0.91 km/s. The moon has a mass of 7.3×10^{22} kg, so this corresponds to a kinetic energy of

$$E_{kin} = \frac{1}{2}mv^2 = 0.5(7.3\times10^{22}\ \mathrm{kg})(0.91\times10^{3}\ \mathrm{m/s})^2 \quad 3.0\times10^{28}\ \mathrm{J}.$$

(b) Large raindrops are about 5 mm in diameter and fall at ~9 m/s, so they have a kinetic energy of

$$E_{kin} \quad \frac{1}{2}\left(\frac{4\pi\rho}{3}\right)\left(\frac{d}{2}\right)^3 v^2$$

$$0.5(1000\ \mathrm{kg/m^3})\left(\frac{4\pi}{3}\right)(2.5\times10^{-3}\ \mathrm{m})^3)(9\ \mathrm{m/s})^2$$

$$\sim 3\ \mathrm{mJ}.$$

(c) Lake Powell has a volume of approximately 33 km^3 and the Glen Canyon Dam has a height of $Z = 220$ m. Treating the lake as a box of area A and volume $V = ZA$, its potential energy is

$$V = \rho gA\int_0^Z dz\,z = \rho gAZ^2/2 = (1/2)\rho gVZ$$

$$\sim (0.5)(1000\ \mathrm{kg/m^3})(9.8\ \mathrm{m/s^2})(33\times10^{9}\ \mathrm{m^3})(220\ \mathrm{m})$$

$$\sim 36\ \mathrm{PJ}.$$

(d) The cross sectional area of a person on a bike is around 0.5 m^2 with a drag coefficient around 1. The energy output is mostly to counteract air resistance. The energy lost is

$$\frac{1}{2}\rho_{air} c_d A v^2 d = 0.5(1.2\ \mathrm{kg/m^3})(0.5\ \mathrm{m^2})$$

$$\times (4.2\ \mathrm{m/s})^2(1.5\times10^{4}\ \mathrm{m}) \quad 79\ \mathrm{kJ}.$$

Assuming an efficiency of ~25% for the human body, around 340 kJ will will have to be expended

(e) Assume the runner has a cross-sectional area of approximately 0.5 m^3 and a drag coefficient around 1. Good runners finish in around 10 s, so they have a speed of 10 m/s. The total energy lost to air resistance will be $(1/2)c_d\rho_{air}Av^2d \sim 3$ kJ.

(f) Estimate that a climber and gear have a mass of approximately 90 kg. The top of Mt. Everest is 8 848 m above sea level, so the climber will have

$$mgh \quad (90\ \mathrm{kg})(9.8\ \mathrm{m/s^2})(8.9\times10^{3}\ \mathrm{m}) \quad 7.8 \quad \mathrm{MJ}$$

of gravitational potential energy with respect to sea level.

2.9 Verify the claim that conversion of the potential energy at the top of a 15 m hill to kinetic energy would increase the speed of the Toyota Camry from 62 to 74 mph.

A Camry has a mass of 1800 kg, so it gains 265 kJ of kinetic energy when its elevation falls by 15 m. Conservation of energy requires that

$$\frac{1}{2}mv^2 + mgh = \text{constant}.$$

Using this, the final velocity if it starts at 62 mi/hr is

$$v_f = \sqrt{v_i^2 + 2gh} = 32.6\ \mathrm{m/s} = 73\ \mathrm{mi/hr}.$$

2.10 Consider a collision between two objects of different masses M, m moving in one dimension with initial speeds V, $-v$. In the center-of-mass frame, the total momentum is zero before and after the collision. In an elastic collision both energy and momentum are conserved. Compute the velocities of the two objects after an elastic collision in the center-of-mass frame. Show that in the limit where $m/M \to 0$, the more massive object remains at rest in the center-of-mass frame (V = 0 before and after the collision), and the less massive object simply bounces off the more massive object (like a tennis ball off a concrete wall).

In the center-of-mass frame the objects have velocities $V' = V + u$ and $v' = v + u$, where $-u$ is the velocity of the center-of-mass viewed from the original frame. The total momentum of the two objects in the center-of-mass vanishes: $p_{tot} = M(V+u)+m(-v+u) = 0$, which determines the center-of-mass velocity

$$u = \frac{mv - MV}{m+M}$$

and

$$V' = V + u = \frac{m}{m+M}(v + V)$$
$$v' = -v + u = -\frac{M}{m+M}(v + V)$$

are the objects' velocities before the collision in the center-of-mass. If energy and momentum are both conserved in the collision, the objects' must reverse their center-of-mass velocities after the collision,

$$V'_f = -V' = -\frac{m}{m+M}(v + V)$$
$$v'_f = -v' = \frac{M}{m+M}(v + V)\,.$$

In the limit $m/M \to 0$, $V' \to 0$ and $V'_{f,} \to 0$, the heavier object begins and remains at rest, while $v'_f = -v' = (v + V)$, the lighter object bounces off.

2.11 As we explain in §34.2.1, the density of air decreases with altitude roughly as $\rho \sim \rho_0 e^{-z/H}$, where z is the height above the Earth's surface and H 8.5 km near the surface. Compute the ratio of air resistance losses for an airplane traveling at 750 km/h at an altitude of 12000 m compared to an altitude of 2000 m. What happens to this ratio as the speed of the airplane changes? How do automobile air resistance losses at 2000 m compare to losses at sea level?

Air resistance losses are linearly proportional to the density of air, so the ratio of loses at 12 000 m compared to 2000 m is $r = e^{-\Delta z/H}$ $e^{-10/8.5}$ 31%. Although the losses (dE_{air}/dt) are proportional to speed cubed, the speed cancels out in the ratio of losses at two different air densities, so the ratio is independent of the airplane's speed. Similarly for a car at 2000 m compared to sea level r $e^{-2/8.5}$ 79%.

2.12 Consider an airplane with mass 70000 kg, cross-sectional area 12 m^2, and drag coefficient 0.03. Estimate the energy needed to get the plane moving at 800 km/h and lift the plane to 10 000 m, and estimate air-resistance losses for a flight of 2000 km using the formula in the previous problem. Do a rough comparison of the energy used per person to do a similar trip in an automobile, assuming that the plane carries 50 passengers and the automobile carries two people.

For simplicity, assume that acceleration occurs quickly so that air resistance during acceleration can be neglected. The plane requires a total kinetic energy of

$$E_{kin} = \frac{1}{2}mv^2 = 0.5(70000\ \text{kg})(800\times 10^3\ \text{m}/3600\ \text{s})^2 = 1.73\ \text{GJ}\,,$$

and at 10 000 m has a potential energy of

$$V = mgh = (70000\ \text{kg})(9.8\ \text{m/s}^2)(10000\ \text{m}) = 6.86\ \text{GJ}.$$

The density of air at 10 000 m is approximately $e^{-10/8.5} = 31\%$ of the density at sea level $\rho_{air} = 0.31(1.17\ \text{kg/m}^3)$ 0.36 kg/m^3. Traveling at $v = 800$ km/h for $d = 2\,000$ km at 10 000 m, the plane loses

$$E_{air} \quad \frac{1}{2}\rho_{air} c_d A v^2 d \quad 0.5(0.36\ \text{kg/m}^3)(0.03)$$
$$\times (12\ \text{m}^2)(222\ \text{m/s})^2(2000\times 10^3\ \text{m}) \quad 6.4\ \text{GJ}\,.$$

due to air resistance. The total energy used by the plane over the flight is approximately 15.3 GJ. This is 306 MJ/passenger or 153 kJ/passenger-km. A typical car uses around 210 MJ of mechanical energy over the 340 km trip between New York and Boston. With 2 passengers, this is 310 kJ/passenger-km, or nearly twice the energy usage of the airplane. So, before engine efficiencies are considered, automobiles use roughly twice as much energy over a long trip.

2.13 In the American game of baseball, a pitcher throws a baseball, which is a round sphere of diameter b = 0.075 m, a distance of 18.4 m (60.5 feet), to a batter, who tries to hit the ball as far as he can. A baseball has a mass close to 0.15 kg. A radar gun measures the speed of a baseball at the time it reaches the batter at 44.7 m/s (100 mph). The drag coefficient c_d of a baseball is about 0.3. Give a semi-quantitative estimate of the speed of the ball when it left the pitcher's hand

by (a) assuming that the ball's speed is never too different from 100 mph to compute roughly how long it takes to go from the pitcher to the batter, (b) using (a) to estimate the energy lost to air resistance, and (c) using (b) to estimate the original kinetic energy and velocity.

The ball has an effective area of $A = \pi R^2 = 0.00442\ \text{m}^2$. For air at sea level and 25°C, $\rho_{air} = 1.17\ \text{kg/m}^3$. The energy loss rate is

$$\frac{dE}{dt} = -\frac{1}{2}c_d A \rho_{air} v^3 \quad -0.5(0.3)(0.00442\ \text{m}^2)$$
$$\times (1.17\ \text{kg/m}^3)(44.7\ \text{m/s})^3 \quad -69\ \text{J/s}.$$

If the ball's speed does not differ significantly from 100 mph, it takes (a) $t = (18.4\ \text{m})/(44.7\ \text{m/s})$ 0.41 sec for the ball to reach the batter. The total amount of energy lost is (b) $\Delta E = \frac{dE}{dt}\times t$ −28 J.

At 100 mph, the ball has a kinetic energy of

$$E_{kin\,f} = \frac{1}{2}mv^2 \quad 150\ \text{J}.$$

(c) The initial kinetic energy of the ball is just the difference of this and the energy loss,

$$E_{kin_i} = 150\ \text{J} + 28\ \text{J} = 178\ \text{J}.$$

This corresponds to a velocity of

$$v = \sqrt{\frac{E_{kin_i}}{m}} \quad \sqrt{\frac{2(178\ \text{J})}{(0.15\ \text{kg})}}\ \text{m/s} \quad 49\ \text{m/s} \quad 110\ \text{mph}.$$

2.14 Estimate the power output of an elite cyclist pedaling a bicycle on a flat road at 14 m/s. Assume all energy is lost to air resistance, the cross-sectional area of the cyclist and the bicycle is 0.4 m^2, and the drag coefficient is 0.75. Now estimate the power output of the same elite cyclist pedaling a bicycle up a hill with slope 8% at 5 m/s. Compute the air resistance assuming the drag coefficient times cross-sectional area is $c_d A$ = 0.45 m^2 (rider is in a less aerodynamic position). Compute the ratio of the power output to potential energy gain. Assume that the mass of the rider plus bicycle is 90 kg.

On the flat road, the power output of the cyclist is equal to the power lost due to air resistance,

$$P = \frac{1}{2}\rho c_d A v^3 \quad (0.5)(1.2\ \text{kg/m}^3)(0.75)(0.4\ \text{m}^2)(14\ \text{m/s})^3 \quad 490\ \text{W}.$$

The power output of the cyclist on the slope is equal to the power lost due to air resistance and due to the rate of change of potential energy,

$$P = P_{air} + P_g.$$

The contribution from air resistance is

$$P_{air} = \frac{1}{2}\rho c_d A v^3 = (0.5)(1.2\ \text{kg/m}^3)(0.45\ \text{m}^2)(5\ \text{m/s})^3 = 34\ \text{W}.$$

The contribution from gravitational potential energy

$$P_g = mg\frac{dz}{dt} \quad (90\ \text{kg})(9.8\ \text{m/s})(5\ \text{m/s})\sin(\arctan(8/100)) \quad 350\ \text{W}$$

so the total power output of the cyclist on the slope is

$$P = 380\ \text{W}.$$

2.15 Compare the rate of power lost to air resistance for the following two vehicles at 60 km/h and 120 km/h: (a) General Motors EV1 with $c_d A$ 0.37 m^2, (b) Hummer H2 with $c_d A$ 2.45 m^2.

(a). At 60 km/h (16.67 m/s)

$$\frac{dE_{loss}}{dt} = \frac{1}{2}\rho c_d A v^3$$
$$(0.5)(1.2\ \text{kg/m}^3)(0.37\ \text{m}^2)(16.7\ \text{m/s})^3 = 1.03\ \text{kW}$$

At 120 km/h, the power is 8 times as high, or 8.2 kW.

(b). The power for the Hummer H2 scales with the $c_d A$, so the H2 loses energy at a rate of (2.45/.37)(1.03 kW) = 6.8 kW at 60 km/h and (2.45/.37)(8.2 kW) = 54 kW at 120 km/h. In both cases, the H2 loses energy at a rate 6.6 times that of the EV1.

2.16 [T] Consider an idealized cylinder of cross-sectional area A moving along its axis through an idealized diffuse gas of air molecules with vanishing initial velocity. Assume that the air molecules are pointlike and do not interact with one another. Compute the velocity that each air molecule acquires after a collision with the cylinder, assuming that the cylinder is much more massive than the air molecules. [Hint: assume that energy is conserved in the reference frame of the moving cylinder and use the result of Problem 2.10.] Use this result to show that the drag coefficient of the cylinder in this idealized approximation is $c_d = 4$.

Suppose the cylinder is traveling at a velocity v. In the cylinder's rest frame, the air molecules are traveling at a velocity $-v$ perpendicular to the surface of the cylinder. Taking the limit that the cylinder is infinitely more massive than the air molecules, this is the center of mass frame. In this limit, a collision may change the direction of the air molecule but not the speed. Since the velocity the air molecules is perpendicular to the end of the cylinder, the final velocity must be $+v$ in the cylinder rest frame. Going back to the lab frame, we add a velocity $+v$ to everything, so that the cylinder once again is traveling at a velocity of $+v$ and the air molecule has a velocity of $+2v$.

To find c_d, we must now compute the rate of energy loss of the cylinder. In time dt, the front end of the cylinder passes through a volume $Avdt$ of air, corresponding to a mass of $dm = A\rho v dt$. This mass is accelerated from rest to a velocity of $2v$. So, the column of air gains an energy of

$$dE_{air} = \frac{1}{2}dm v_{air}^2 = 2A\rho v^3 dt.$$

From conservation of energy, the cylinder loses an equal amount of energy, so the power output due to air resistance is

$$\left(\frac{dE}{dt}\right)_{cyl} = -2A\rho v^3 = -\frac{1}{2}c_d A\rho v^3,$$

where $c_d = 4$.

2.17 One way to estimate the effective area (see eq. (2.31)) of an object is to measure its limiting velocity v_∞ falling in air. Explain how this works and find the expression for A_{eff} as a function of m (the mass of the object), v_∞, g, and the density of air. The drag coefficient of a soccer ball (radius 11 cm, mass 0.43 kg) is $c_d \approx 0.25$. What is its limiting velocity?

Because the force due to air resistance increases with velocity, if an object is allowed to fall in air, there is some velocity at which the upward force due to air resistance is equal in magnitude to the downward gravitational force. When this happens, there is no net force on the object, so the object continues falling at this velocity, which is v_∞. Thus,

$$mg = \frac{1}{2}\rho_{air} A_{eff} v_\infty^2.$$

Solving for A_{eff},

$$A_{eff} = \frac{2mg}{\rho_{air} v_\infty^2}.$$

The effective area of the soccer ball is $A_{eff} = c_d A = c_d \pi r^2$ = 0.0095 m^2. The limiting velocity is

$$v_\infty = \sqrt{\frac{2mg}{\rho_{air}A_{eff}}} \quad \sqrt{\frac{2(0.43\ \text{kg})(9.8\ \text{m/s}^2)}{(1.2\ \text{kg/m}^3)(0.0095\ \text{m}^2)}} \quad 27\ \text{m/s}.$$

2.18 If the vehicle used as an example in this chapter accelerates to 50 km/h between each stoplight, find the maximum distance between stoplights for which the energy used to accelerate the vehicle exceeds the energy lost to air resistance. (You may assume that the time for acceleration is negligible in this calculation.) How does the result change if the vehicle travels at 100 km/h between lights? Why?

The vehicle in this chapter has $A = 2.7\ m^2$, $c_d = 1/3$, and m = 1800 kg. The total energy lost to friction after a distance d at a velocity v is

$$E_{lost} = \frac{1}{2}\rho c_d A v^2 d.$$

Setting this equal to the kinetic energy and solving for d to get the maximum distance for which the kinetic energy exceeds the energy lost due to air resistance,

$$\frac{1}{2}mv^2 = \frac{1}{2}\rho c_d A v^2 d$$

$$d = \frac{m}{\rho c_d A} \quad \frac{1\,800\ \text{kg}}{(1.2\ \text{kg/m}^3)(1/3)(2.7\ \text{m}^2)} \quad 1.67\ \text{km}.$$

The result is the same if the vehicle travels at 100 km/h. Both the kinetic energy and the energy loss are proportional to v^2 so the distance at which they are equal is independent of the velocity.

2.19 Estimate the rotational kinetic energy in a spinning yo-yo (a plastic toy that you can assume is a cylinder of diameter 5.7 cm and mass 52 g, which rotates at 100 Hz). Compare to the gravitational potential energy of the yo-yo at height of 0.75 m.

The moment of inertia for a cylinder of mass m, length z and radius R rotating around its axis is (see Example 2.4),

$$I_{cylinder} = \frac{1}{2}\rho\pi z R^4 = \frac{1}{2} m R^2 . \tag{2.46}$$

The energy of the rotating yo-yo is $E = \frac{1}{2} I \omega^2$ and $\omega = 2\pi\nu = (2\pi\ s^{-1}/\text{Hz})(100\ \text{Hz}) \approx 630\ s^{-1}$, so

$$E_{rot} = \frac{1}{2} I \omega^2 \approx (0.25)(0.052\ \text{kg})(0.0285\ \text{m})^2 (630\ s^{-1})^2 \approx 4.2\ \text{J} . \tag{2.47}$$

Its gravitational potential energy at a height of 0.75 m is $mgh \approx 0.38$ J, almost a factor of ten smaller than its rotational kinetic energy at 10 Hz. (Most of the energy is imparted in the initial "throw," and not from gravitational potential energy.)

2.20 Verify the assertion (see Example 2.3) that $E_{kin} = -\frac{1}{2}V$ for the Moon in a circular orbit around Earth. [Hint: the magnitude of the centripetal acceleration for circular motion (2.35) can be rewritten $a = v^2/r$.]

Assume throughout that $M_\oplus \gg m_{moon}$, so we may take Earth to be at rest. For a circular orbit, the centripetal acceleration is $a = v^2/r$. The acceleration due to gravity is $GM_\oplus/r^2$ where $M_\oplus$ is Earth's mass, which is much greater than that of the moon. The gravitational potential energy of the moon is $V = -GM_\oplus m/r$

The centripetal acceleration and gravitational acceleration are identical, so

$$\frac{v^2}{r} = G\frac{M_\oplus}{r^2} \text{ or } v^2 = \frac{GM_\oplus}{r}, \text{ so}$$

$$E_{kin} = \frac{1}{2} m v^2 = -\frac{1}{2}\left(-\frac{GM_\oplus m}{r}\right) = -\frac{1}{2} V .$$

2.21 Estimate Earth's kinetic energy of rotation (the moment of inertia of a uniform sphere is $\frac{2}{5} M R^2$).

Earth has a mass of 5.972×10^{24} kg and a mean radius of 6 371 km. It rotates about its axis once per sidereal day (23.9345 h), giving it an angular velocity of $\omega = 2\pi/(1\ \text{sidereal day}) = 7.292 \times 10^{-5}\ s^{-1}$. Its kinetic energy of rotation is

$$E = \frac{MR^2}{5}\omega^2 = (0.2)(5.972 \times 10^{24}\ \text{kg})(6\,371 \times 10^3\ \text{m})^2 \times (7.292 \times 10^{-5}\ \text{rad/s})^2 = 2.578 \times 10^{29}\ \text{J} .$$