CHAPTER 2
Analytic Trigonometry

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Section 2.1 Using Fundamental Identities

1. $\tan u$

2. $\csc u$

3. $\cot u$

4. $\csc u$

5. 1

6. $-\sin u$

7. $\sec x = -\frac{5}{2}, \tan x < 0 \Rightarrow x$ is in Quadrant II.

$$\cos x = \frac{1}{\sec x} = \frac{1}{-\frac{5}{2}} = -\frac{2}{5}$$

$$\sin x = \sqrt{1 - \left(-\frac{2}{5}\right)^2} = \sqrt{1 - \frac{4}{25}} = \frac{\sqrt{21}}{5}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{\sqrt{21}}{5}}{-\frac{2}{5}} = -\frac{\sqrt{21}}{2}$$

$$\csc x = \frac{1}{\sin x} = \frac{5}{\sqrt{21}} = \frac{5\sqrt{21}}{21}$$

$$\cot x = \frac{1}{\tan x} = -\frac{2}{\sqrt{21}} = -\frac{2\sqrt{21}}{21}$$

8. $\csc x = -\frac{7}{6}, \tan x > 0 \Rightarrow x$ is in Quadrant III.

$$\sin x = \frac{1}{\csc x} = \frac{1}{-\frac{7}{6}} = -\frac{6}{7}$$

$$\cos x = -\sqrt{1 - \left(-\frac{6}{7}\right)^2} = -\sqrt{1 - \frac{36}{49}} = -\frac{\sqrt{13}}{7}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{6}{7}}{-\frac{\sqrt{13}}{7}} = \frac{6}{\sqrt{13}} = \frac{6\sqrt{13}}{13}$$

$$\sec x = \frac{1}{\cos x} = \frac{1}{-\frac{\sqrt{13}}{7}} = -\frac{7}{\sqrt{13}} = -\frac{7\sqrt{13}}{13}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\frac{6}{\sqrt{13}}} = \frac{\sqrt{13}}{6}$$

9. $\sin\theta = -\frac{3}{4}, \cos\theta > 0 \Rightarrow \theta$ is in Quadrant IV.

$$\cos\theta = \sqrt{1 - \left(-\frac{3}{4}\right)^2} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{3}{4}}{\frac{\sqrt{7}}{4}} = -\frac{3}{\sqrt{7}} = -\frac{3\sqrt{7}}{7}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{\sqrt{7}}{4}} = \frac{4}{\sqrt{7}} = \frac{4\sqrt{7}}{7}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\frac{3}{\sqrt{7}}} = -\frac{\sqrt{7}}{3}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{3}{4}} = -\frac{4}{3}$$

10. $\cos\theta = \frac{2}{3}, \sin\theta < 0 \Rightarrow \theta$ is in Quadrant IV.

$$\sin\theta = -\sqrt{1 - \left(\frac{2}{3}\right)^2} = -\sqrt{1 - \frac{4}{9}} = -\frac{\sqrt{5}}{3}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{\sqrt{5}}{2}}{\frac{2}{3}} = -\frac{\sqrt{5}}{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\frac{\sqrt{5}}{2}} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{\sqrt{5}}{3}} = -\frac{3}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

11. $\tan x = \dfrac{2}{3}, \cos x > 0 \Rightarrow x$ is in Quadrant I.

$$\cot x = \frac{1}{\tan x} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

$$\sec x = \sqrt{1 + \left(\frac{2}{3}\right)^2} = \sqrt{1 + \frac{4}{9}} = \frac{\sqrt{13}}{3}$$

$$\csc x = \sqrt{1 + \left(\frac{3}{2}\right)^2} = \sqrt{1 + \frac{9}{4}} = \frac{\sqrt{13}}{2}$$

$$\sin x = \frac{1}{\csc x} = \frac{1}{\frac{\sqrt{13}}{2}} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

$$\cos x = \frac{1}{\sec x} = \frac{1}{\frac{\sqrt{13}}{3}} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

12. $\cot x = \dfrac{7}{4}, \sin x < 0 \Rightarrow x$ is in Quadrant III.

$$\tan x = \frac{1}{\cot x} = \frac{1}{\frac{7}{4}} = \frac{4}{7}$$

$$\sec x = -\sqrt{1 + \left(\frac{4}{7}\right)^2} = -\sqrt{1 + \frac{16}{49}} = -\frac{\sqrt{65}}{7}$$

$$\csc x = -\sqrt{1 + \left(\frac{7}{4}\right)^2} = -\sqrt{1 + \frac{49}{16}} = -\frac{\sqrt{65}}{4}$$

$$\sin x = \frac{1}{\csc x} = \frac{1}{-\frac{\sqrt{65}}{4}} = -\frac{4}{\sqrt{65}} = -\frac{4\sqrt{65}}{65}$$

$$\cos x = \frac{1}{\sec x} = \frac{1}{-\frac{\sqrt{65}}{7}} = -\frac{7}{\sqrt{65}} = -\frac{7\sqrt{65}}{65}$$

13. $\sec x \cos x = \left(\dfrac{1}{\cancel{\cos x}}\right)\cancel{\cos x}$

$= 1$

Matches (c).

14. $\cot^2 x - \csc^2 x = (\csc^2 x - 1) - \csc^2 x$

$= -1$

Matches (b).

15. $\cos x(1 + \tan^2 x) = \cos x(\sec^2 x)$

$$= \cos x\left(\frac{1}{\cos^2 x}\right)$$

$$= \frac{1}{\cos x}$$

$$= \sec x$$

Matches (f).

16. $\cot x \sec x = \dfrac{\cos x}{\sin x} \cdot \dfrac{1}{\cos x} = \dfrac{1}{\sin x} = \csc x$

Matches (a).

17. $\dfrac{\sec^2 x - 1}{\sin^2 x} = \dfrac{\tan^2 x}{\sin^2 x} = \dfrac{\sin^2 x}{\cos^2 x} \cdot \dfrac{1}{\sin^2 x} = \sec^2 x$

Matches (e).

18. $\dfrac{\cos^2[(\pi/2) - x]}{\cos x} = \dfrac{\sin^2 x}{\cos x} = \dfrac{\sin x}{\cos x}\sin x = \tan x \sin x$

Matches (d).

19. $\dfrac{\tan\theta \cot\theta}{\sec\theta} = \dfrac{\tan\theta\left(\frac{1}{\tan\theta}\right)}{\frac{1}{\cos\theta}}$

$$= \frac{1}{\frac{1}{\cos\theta}}$$

$$= \cos\theta$$

20. $\cos\left(\dfrac{\pi}{2} - x\right)\sec x = \sin x \sec x$

$$= \sin x\left(\frac{1}{\cos x}\right)$$

$$= \tan x$$

21. $\tan^2 x - \tan^2 x \sin^2 x = \tan^2 x(1 - \sin^2 x)$

$$= \tan^2 x \cos^2 x$$

$$= \frac{\sin^2 x}{\cos^2 x} \cdot \cos^2 x$$

$$= \sin^2 x$$

22. $\sin^2 x \sec^2 x - \sin^2 x = \sin^2 x(\sec^2 x - 1)$

$$= \sin^2 x \tan^2 x$$

23. $\dfrac{\sec^2 x - 1}{\sec x - 1} = \dfrac{(\sec x + 1)(\sec x - 1)}{\sec x - 1}$

$$= \sec x + 1$$

24. $\dfrac{\cos x - 2}{\cos^2 x - 4} = \dfrac{\cos x - 2}{(\cos x + 2)(\cos x - 2)}$

$$= \frac{1}{\cos x + 2}$$

25. $1 - 2\cos^2 x + \cos^4 x = (1 - \cos^2 x)^2$
$= (\sin^2 x)^2$
$= \sin^4 x$

26. $\sec^4 x - \tan^4 x = (\sec^2 x + \tan^2 x)(\sec^2 x - \tan^2 x)$
$= (\sec^2 x + \tan^2 x)(1)$
$= \sec^2 x + \tan^2 x$

27. $\cot^3 x + \cot^2 x + \cot x + 1 = \cot^2 x(\cot x + 1) + (\cot x + 1)$
$= (\cot x + 1)(\cot^2 x + 1)$
$= (\cot x + 1)\csc^2 x$

28. $\sec^3 x - \sec^2 x - \sec x + 1 = \sec^2 x(\sec x - 1) - (\sec x - 1)$
$= (\sec^2 x - 1)(\sec x - 1)$
$= \tan^2 x(\sec x - 1)$

29. $3\sin^2 x - 5\sin x - 2 = (3\sin x + 1)(\sin x - 2)$

30. $6\cos^2 x + 5\cos x - 6 = (3\cos x - 2)(2\cos x + 3)$

31. $\cot^2 x + \csc x - 1 = (\csc^2 x - 1) + \csc x - 1$
$= \csc^2 x + \csc x - 2$
$= (\csc x - 1)(\csc x + 2)$

32. $\sin^2 x + 3\cos x + 3 = (1 - \cos^2 x) + 3\cos x + 3$
$= -\cos^2 x + 3\cos x + 4$
$= -(\cos^2 x - 3\cos x - 4)$
$= -(\cos x + 1)(\cos x - 4)$

33. $\tan\theta\csc\theta = \dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{1}{\sin\theta} = \dfrac{1}{\cos\theta} = \sec\theta$

34. $\tan(-x)\cos x = -\tan x\cos x$
$= -\dfrac{\sin x}{\cos x}\cdot\cos x$
$= -\sin x$

35. $\sin\phi(\csc\phi - \sin\phi) = (\sin\phi)\dfrac{1}{\sin\phi} - \sin^2\phi$
$= 1 - \sin^2\phi = \cos^2\phi$

36. $\cos x(\sec x - \cos x) = \cos x\left(\dfrac{1}{\cos x} - \cos x\right)$
$= 1 - \cos^2 x$
$= \sin^2 x$

37. $\sin\beta\tan\beta + \cos\beta = (\sin\beta)\dfrac{\sin\beta}{\cos\beta} + \cos\beta$
$= \dfrac{\sin^2\beta}{\cos\beta} + \dfrac{\cos^2\beta}{\cos\beta}$
$= \dfrac{\sin^2\beta + \cos^2\beta}{\cos\beta}$
$= \dfrac{1}{\cos\beta}$
$= \sec\beta$

38. $\cot u\sin u + \tan u\cos u = \dfrac{\cos u}{\sin u}(\sin u) + \dfrac{\sin u}{\cos u}(\cos u)$
$= \cos u + \sin u$

39. $\dfrac{1 - \sin^2 x}{\csc^2 x - 1} = \dfrac{\cos^2 x}{\cot^2 x} = \cos^2 x\tan^2 x = (\cos^2 x)\dfrac{\sin^2 x}{\cos^2 x}$
$= \sin^2 x$

40. $\dfrac{\cos^2 y}{1 - \sin y} = \dfrac{1 - \sin^2 y}{1 - \sin y}$
$= \dfrac{(1 + \sin y)(1 - \sin y)}{1 - \sin y} = 1 + \sin y$

41. $(\sin x + \cos x)^2 = \sin^2 x + 2\sin x\cos x + \cos^2 x$
$= (\sin^2 x + \cos^2 x) + 2\sin x\cos x$
$= 1 + 2\sin x\cos x$

42. $(2\csc x + 2)(2\csc x - 2) = 4\csc^2 x - 4$
$= 4(\csc^2 x - 1)$
$= 4\cot^2 x$

43. $\dfrac{1}{1 + \cos x} + \dfrac{1}{1 - \cos x} = \dfrac{1 - \cos x + 1 + \cos x}{(1 + \cos x)(1 - \cos x)}$
$= \dfrac{2}{1 - \cos^2 x}$
$= \dfrac{2}{\sin^2 x}$
$= 2\csc^2 x$

44. $\dfrac{1}{\sec x + 1} - \dfrac{1}{\sec x - 1} = \dfrac{\sec x - 1 - (\sec x + 1)}{(\sec x + 1)(\sec x - 1)}$
$= \dfrac{\sec x - 1 - \sec x - 1}{\sec^2 x - 1}$
$= \dfrac{-2}{\tan^2 x}$
$= -2\left(\dfrac{1}{\tan^2 x}\right)$
$= -2\cot^2 x$

45. $$\frac{\cos x}{1+\sin x} - \frac{\cos x}{1-\sin x} = \frac{\cos x(1-\sin x) - \cos x(1+\sin x)}{(1+\sin x)(1-\sin x)}$$
$$= \frac{\cos x - \sin x\cos x - \cos x - \sin x\cos x}{(1+\sin x)(1-\sin x)}$$
$$= \frac{-2\sin x\cos x}{1-\sin^2 x}$$
$$= \frac{-2\sin x\cos x}{\cos^2 x}$$
$$= \frac{-2\sin x}{\cos x}$$
$$= -2\tan x$$

46. $$\frac{\sin x}{1+\cos x} + \frac{\sin x}{1-\cos x} = \frac{\sin x(1-\cos x) + \sin x(1+\cos x)}{(1+\cos x)(1-\cos x)}$$
$$= \frac{\sin x - \sin x\cos x + \sin x + \sin x\cos x}{(1+\cos x)(1-\cos x)}$$
$$= \frac{2\sin x}{1-\cos^2 x}$$
$$= \frac{2\sin x}{\sin^2 x}$$
$$= \frac{2}{\sin x}$$
$$= 2\csc x$$

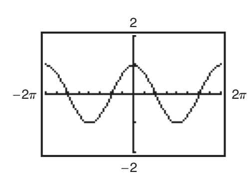
47. $$\tan x - \frac{\sec^2 x}{\tan x} = \frac{\tan^2 x - \sec^2 x}{\tan x}$$
$$= \frac{-1}{\tan x} = -\cot x$$

48. $$\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} = \frac{\cos^2 x + (1+\sin x)^2}{\cos x(1+\sin x)}$$
$$= \frac{\cos^2 x + 1 + 2\sin x + \sin^2 x}{\cos x(1+\sin x)}$$
$$= \frac{2+2\sin x}{\cos x(1+\sin x)}$$
$$= \frac{2(1+\sin x)}{\cos x(1+\sin x)}$$
$$= \frac{2}{\cos x}$$
$$= 2\sec x$$

49. $$\frac{\sin^2 y}{1-\cos y} = \frac{1-\cos^2 y}{1-\cos y}$$
$$= \frac{(1+\cos y)(1-\cos y)}{1-\cos y} = 1+\cos y$$

50. $$\frac{5}{\tan x + \sec x} \cdot \frac{\tan x - \sec x}{\tan x - \sec x} = \frac{5(\tan x - \sec x)}{\tan^2 x - \sec^2 x}$$
$$= \frac{5(\tan x - \sec x)}{-1}$$
$$= 5(\sec x - \tan x)$$

51. $$y_1 = \frac{1}{2}(\sin x\cot x + \cos x)$$
$$= \frac{1}{2}\left(\sin x\left(\frac{\cos x}{\sin x}\right) + \cos x\right)$$
$$= \frac{1}{2}(\cos x + \cos x)$$
$$= \cos x$$



52. $y_1 = \sec x \csc x - \tan x$

$$= \frac{1}{\cos x}\left(\frac{1}{\sin x}\right) - \frac{\sin x}{\cos x}$$

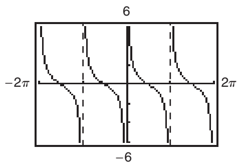
$$= \frac{1}{\cos x \sin x} - \frac{\sin^2 x}{\cos x \sin x}$$

$$= \frac{1 - \sin^2 x}{\cos x \sin x}$$

$$= \frac{\cos^2 x}{\cos x \sin x}$$

$$= \frac{\cos x}{\sin x}$$

$$= \cot x$$



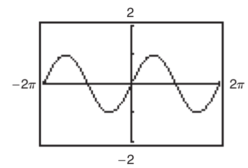
53. $y_1 = \dfrac{\tan x + 1}{\sec x + \csc x}$

$$= \frac{\dfrac{\sin x}{\cos x} + 1}{\dfrac{1}{\cos x} + \dfrac{1}{\sin x}}$$

$$= \frac{\dfrac{\sin x + \cos x}{\cos x}}{\dfrac{\sin x + \cos x}{\sin x \cos x}}$$

$$= \left(\frac{\sin x + \cos x}{\cos x}\right)\left(\frac{\sin x \cos x}{\sin x + \cos x}\right)$$

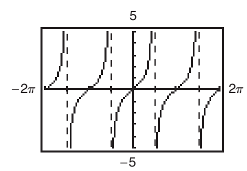
$$= \sin x$$



54. $y_1 = \dfrac{1}{\sin x}\left(\dfrac{1}{\cos x} - \cos x\right) = \tan x$

$$\frac{1}{\sin x}\left(\frac{1}{\cos x} - \cos x\right) = \frac{1}{\sin x \cos x} - \frac{\cos x}{\sin x}$$

$$= \frac{1 - \cos^2 x}{\sin x \cos x} = \frac{\sin^2 x}{\sin x \cos x} = \frac{\sin x}{\cos x} = \tan x$$



55. Let $x = 3\cos\theta$.

$$\sqrt{9 - x^2} = \sqrt{9 - (3\cos\theta)^2}$$

$$= \sqrt{9 - 9\cos^2\theta}$$

$$= \sqrt{9(1 - \cos^2\theta)}$$

$$= \sqrt{9\sin^2\theta} = 3\sin\theta$$

56. Let $x = 7\sin\theta$.

$$\sqrt{49 - x^2} = \sqrt{49 - (7\sin\theta)^2}$$

$$= \sqrt{49 - 49\sin^2\theta}$$

$$= \sqrt{49(1 - \sin^2\theta)}$$

$$= \sqrt{49\cos^2\theta}$$

$$= 7\cos\theta$$

57. Let $x = 2\sec\theta$.

$$\sqrt{x^2 - 4} = \sqrt{(2\sec\theta)^2 - 4}$$

$$= \sqrt{4(\sec^2\theta - 1)}$$

$$= \sqrt{4\tan^2\theta}$$

$$= 2\tan\theta$$

58. Let $3x = 5\tan\theta$.

$$\sqrt{9x^2 + 25} = \sqrt{(3x)^2 + 25}$$

$$= \sqrt{(5\tan\theta)^2 + 25}$$

$$= \sqrt{25\tan^2\theta + 25}$$

$$= \sqrt{25(\tan^2\theta + 1)}$$

$$= \sqrt{25\sec^2\theta}$$

$$= 5\sec\theta$$

59. Let $x = 2\sin\theta$.

$$\sqrt{4 - x^2} = \sqrt{2}$$
$$\sqrt{4 - (2\sin\theta)^2} = \sqrt{2}$$
$$\sqrt{4 - 4\sin^2\theta} = \sqrt{2}$$
$$\sqrt{4(1 - \sin^2\theta)} = \sqrt{2}$$
$$\sqrt{4\cos^2\theta} = \sqrt{2}$$
$$2\cos\theta = \sqrt{2}$$
$$\cos\theta = \frac{\sqrt{2}}{2}$$
$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2} = \pm\frac{\sqrt{2}}{2}$$

60. Let $x = 2\cos\theta$.

$$\sqrt{16 - 4x^2} = 2\sqrt{2}$$
$$\sqrt{16 - 4(2\cos\theta)^2} = 2\sqrt{2}$$
$$\sqrt{16 - 16\cos^2\theta} = 2\sqrt{2}$$
$$\sqrt{16(1 - \cos^2\theta)} = 2\sqrt{2}$$
$$\sqrt{16\sin^2\theta} = 2\sqrt{2}$$
$$4\sin\theta = \pm 2\sqrt{2}$$
$$\sin\theta = \pm\frac{\sqrt{2}}{2}$$
$$\cos\theta = \sqrt{1 - \sin^2\theta}$$
$$= \sqrt{1 - \frac{1}{2}}$$
$$= \sqrt{\frac{1}{2}}$$
$$= \frac{\sqrt{2}}{2}$$

61. $x = 6\sin\theta$

$$3 = \sqrt{36 - x^2}$$
$$= \sqrt{36 - (6\sin\theta)^2}$$
$$= \sqrt{36(1 - \sin^2\theta)}$$
$$= \sqrt{36\cos^2\theta}$$
$$= 6\cos\theta$$
$$\cos\theta = \frac{3}{6} = \frac{1}{2}$$
$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$
$$= \pm\sqrt{1 - \left(\frac{1}{2}\right)^2}$$
$$= \pm\sqrt{\frac{3}{4}}$$
$$= \pm\frac{\sqrt{3}}{2}$$

62. $x = 10\cos\theta$

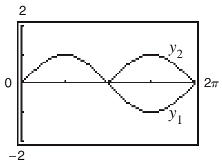
$$5\sqrt{3} = \sqrt{100 - x^2}$$
$$5\sqrt{3} = \sqrt{100 - (10\cos\theta)^2}$$
$$5\sqrt{3} = \sqrt{100(1 - \cos^2\theta)}$$
$$5\sqrt{3} = \sqrt{100\sin^2\theta}$$
$$5\sqrt{3} = 10\sin\theta$$
$$\sin\theta = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}$$
$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}$$

63. $\sin\theta = \sqrt{1 - \cos^2\theta}$

Let $y_1 = \sin x$ and $y_2 = \sqrt{1 - \cos^2 x}, 0 \le x \le 2\pi$.

$y_1 = y_2$ for $0 \le x \le \pi$.

So, $\sin\theta = \sqrt{1 - \cos^2\theta}$ for $0 \le \theta \le \pi$.

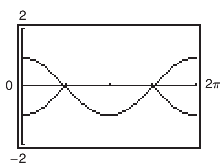


64. $\cos\theta = -\sqrt{1 - \sin^2\theta}$

Let $y_1 = \cos x$ and $y_2 = -\sqrt{1 - \sin^2 x}, 0 \le x \le 2\pi$.

$y_1 = y_2$ for $\dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}$.

So, $\cos\theta = -\sqrt{1 - \sin^2\theta}$ for $\dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{2}$.

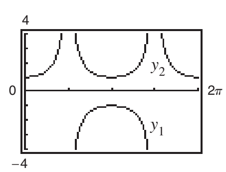


65. $\sec\theta = \sqrt{1 + \tan^2\theta}$

Let $y_1 = \dfrac{1}{\cos x}$ and $y_2 = \sqrt{1 + \tan^2 x}, 0 \le x \le 2\pi$.

$y_1 = y_2$ for $0 \le x < \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2} < x \le 2\pi$.

So, $\sec\theta = \sqrt{1 + \tan^2\theta}$ for $0 \le \theta < \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2} < \theta < 2\pi$.

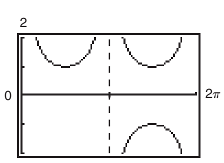


66. $\csc\theta = \sqrt{1 + \cot^2\theta}$

Let $y_1 = \dfrac{1}{\sin x}$ and $y_2 = \sqrt{1 + \cot^2 x}, 0 \le x \le 2\pi$.

$y_1 = y_2$ for $0 < x < \pi$.

So, $\csc\theta = \sqrt{1 + \cot^2\theta}$ for $0 < \theta < \pi$.



67. $\mu W \cos\theta = W \sin\theta$

$$\mu = \frac{W\sin\theta}{W\cos\theta} = \tan\theta$$

68.
$$\begin{aligned}\sec x\tan x - \sin x &= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} - \sin x\\ &= \frac{\sin x}{\cos^2 x} - \sin x\\ &= \frac{\sin x - \sin x\cos^2 x}{\cos^2 x}\\ &= \frac{\sin x\left(1 - \cos^2 x\right)}{\cos^2 x}\\ &= \frac{\sin x\sin^2 x}{\cos^2 x}\\ &= \sin x\tan^2 x\end{aligned}$$

69. True.

$$\tan u = \frac{\sin u}{\cos u}$$
$$\cot u = \frac{\cos u}{\sin u}$$
$$\sec u = \frac{1}{\cos u}$$
$$\csc u = \frac{1}{\sin u}$$

70. False. A cofunction identity can be used to transform a tangent function so that it can be represented by a cotangent function.

71. As $x \to \dfrac{\pi^-}{2}$, $\tan x \to \infty$ and $\cot x \to 0$.

72. As $x \to \pi^+$, $\sin x \to 0$ and $\csc x = \dfrac{1}{\sin x} \to -\infty$.

73. $\cos(-\theta) \ne -\cos\theta$

$\cos(-\theta) = \cos\theta$

The correct identity is $\dfrac{\sin\theta}{\cos(-\theta)} = \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta$

74. Let $u = a\tan\theta$, then

$$\begin{aligned}\sqrt{a^2 + u^2} &= \sqrt{a^2 + (a\tan\theta)^2}\\ &= \sqrt{a^2 + a^2\tan^2\theta}\\ &= \sqrt{a^2\left(1 + \tan^2\theta\right)}\\ &= \sqrt{a^2\sec^2\theta}\\ &= a\sec\theta.\end{aligned}$$

75. Because $\sin^2\theta + \cos^2\theta = 1$, then $\cos^2\theta = 1 - \sin^2\theta$.

$$\cos\theta = \pm\sqrt{1 - \sin\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\pm\sqrt{1 - \sin^2\theta}}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\pm\sqrt{1 - \sin^2\theta}}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\pm\sqrt{1 - \sin^2\theta}}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

76. To derive $\sin^2\theta + \cos^2\theta = 1$, let $\sin\theta = \dfrac{a}{\sqrt{a^2 + b^2}}$ and $\cos\theta = \dfrac{b}{\sqrt{a^2 + b^2}}$.

$$\text{So, } \sin^2\theta + \cos^2\theta = \left(\frac{a}{\sqrt{a^2 + b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2}}\right)^2 = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2}$$
$$= \frac{a^2 + b^2}{a^2 + b^2}$$
$$= 1.$$

To derive $1 + \tan^2\theta = \sec^2\theta$, let $\tan\theta = \dfrac{a}{b}$ and $\sec\theta = \dfrac{\sqrt{a^2 + b^2}}{b}$.

$$\text{So, } 1 + \tan^2\theta = 1 + \left(\frac{a}{b}\right)^2 = 1 + \frac{a^2}{b^2} = \frac{b^2 + a^2}{b^2}$$
$$= \left(\sqrt{\frac{a^2 + b^2}{b^2}}\right)^2 = \left(\frac{\sqrt{a^2 + b^2}}{b}\right)^2$$
$$= \sec^2\theta.$$

To derive $1 + \cot^2\theta = \csc^2\theta$, let $\cot\theta = \dfrac{b}{a}$ and $\csc\theta = \dfrac{\sqrt{a^2 + b^2}}{a}$.

$$\text{So, } 1 + \cot^2\theta = 1 + \left(\frac{b}{a}\right)^2 = 1 + \frac{b^2}{a^2}$$
$$= \frac{a^2 + b^2}{a^2} = \left(\sqrt{\frac{a^2 + b^2}{a^2}}\right)^2$$
$$= \left(\frac{\sqrt{a^2 + b^2}}{a}\right)^2 = \csc^2\theta.$$

Answers will vary.

77. $$\frac{\sec\theta(1+\tan\theta)}{\sec\theta+\csc\theta} = \frac{\left(\dfrac{1}{\cos\theta}\right)\left(1+\dfrac{\sin\theta}{\cos\theta}\right)}{\dfrac{1}{\cos\theta}+\dfrac{1}{\sin\theta}}$$
$$= \frac{\dfrac{\cos\theta+\sin\theta}{\cos^2\theta}}{\dfrac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}}$$
$$= \left(\frac{\sin\theta+\cos\theta}{\cos^2\theta}\right)\left(\frac{\sin\theta\cos\theta}{\sin\theta+\cos\theta}\right)$$
$$= \frac{\sin\theta}{\cos\theta}$$

Section 2.2 Verifying Trigonometric Identities

1. identity

2. conditional equation

3. $\tan u$

4. $\cot u$

5. $\sin u$

6. $\cot^2 u$

7. $-\csc u$

8. $\sec u$

9. $\tan t\cot t = \dfrac{\sin t}{\cos t}\cdot\dfrac{\cos t}{\sin t} = 1$

10. $\dfrac{\tan x\cot x}{\cos x} = \dfrac{1}{\cos x} = \sec x$

11. $(1+\sin\alpha)(1-\sin\alpha) = 1-\sin^2\alpha = \cos^2\alpha$

12. $$\cos^2\beta-\sin^2\beta = \cos^2\beta-(1-\cos^2\beta)$$
$$= 2\cos^2\beta-1$$

13. $$\cos^2\beta-\sin^2\beta = (1-\sin^2\beta)-\sin^2\beta$$
$$= 1-2\sin^2\beta$$

14. $$\sin^2\alpha-\sin^4\alpha = \sin^2\alpha(1-\sin^2\alpha)$$
$$= (1-\cos^2\alpha)(\cos^2\alpha)$$
$$= \cos^2\alpha-\cos^4\alpha$$

15. $$\tan\left(\frac{\pi}{2}-\theta\right)\tan\theta = \cot\theta\tan\theta$$
$$= \left(\frac{1}{\tan\theta}\right)\tan\theta$$
$$= 1$$

16. $\dfrac{\cos[(\pi/2)-x]}{\sin[(\pi/2)-x]} = \dfrac{\sin x}{\cos x} = \tan x$

17. $$\sin t\csc\left(\frac{\pi}{2}-t\right) = \sin t\sec t = \sin t\left(\frac{1}{\cos t}\right)$$
$$= \frac{\sin t}{\cos t} = \tan t$$

18. $\sec^2 y-\cot^2\left(\dfrac{\pi}{2}-y\right) = \sec^2 y-\tan^2 y = 1$

19. $$\frac{1}{\tan x}+\frac{1}{\cot x} = \frac{\cot x+\tan x}{\tan x\cot x}$$
$$= \frac{\cot x+\tan x}{1}$$
$$= \tan x+\cot x$$

20. $$\frac{1}{\sin x}-\frac{1}{\csc x} = \frac{\csc x-\sin x}{\sin x\csc x}$$
$$= \frac{\csc x-\sin x}{1}$$
$$= \csc x-\sin x$$

21. $$\frac{1+\sin\theta}{\cos\theta}+\frac{\cos\theta}{1+\sin\theta} = \frac{(1+\sin\theta)^2+\cos^2\theta}{\cos\theta(1+\sin\theta)}$$
$$= \frac{1+2\sin\theta+\sin^2\theta+\cos^2\theta}{\cos\theta(1+\sin\theta)}$$
$$= \frac{2+2\sin\theta}{\cos\theta(1+\sin\theta)}$$
$$= \frac{2(1+\sin\theta)}{\cos\theta(1+\sin\theta)}$$
$$= \frac{2}{\cos\theta}$$
$$= 2\sec\theta$$

22. $\dfrac{\cos\theta\cot\theta}{1-\sin\theta} - 1 = \dfrac{\cos\theta\cot\theta - (1-\sin\theta)}{1-\sin\theta}$

$= \dfrac{\cos\theta\left(\dfrac{\cos\theta}{\sin\theta}\right) - 1 + \sin\theta}{1-\sin\theta} \cdot \dfrac{\sin\theta}{\sin\theta}$

$= \dfrac{\cos^2\theta - \sin\theta + \sin^2\theta}{\sin\theta(1-\sin\theta)}$

$= \dfrac{1-\sin\theta}{\sin\theta(1-\sin\theta)}$

$= \dfrac{1}{\sin\theta}$

$= \csc\theta$

23. $\dfrac{1}{\cos x + 1} + \dfrac{1}{\cos x - 1} = \dfrac{\cos x - 1 + \cos x + 1}{(\cos x + 1)(\cos x - 1)}$

$= \dfrac{2\cos x}{\cos^2 x - 1}$

$= \dfrac{2\cos x}{-\sin^2 x}$

$= -2 \cdot \dfrac{1}{\sin x} \cdot \dfrac{\cos x}{\sin x}$

$= -2\csc x\cot x$

24. $\cos x - \dfrac{\cos x}{1-\tan x} = \dfrac{\cos x(1-\tan x) - \cos x}{1-\tan x}$

$= \dfrac{-\cos x\tan x}{1-\tan x}$

$= \dfrac{-\cos x(\sin x/\cos x)}{1-(\sin x/\cos x)} \cdot \dfrac{\cos x}{\cos x}$

$= \dfrac{-\sin x\cos x}{\cos x - \sin x}$

$= \dfrac{\sin x\cos x}{\sin x - \cos x}$

25. $\sec y\cos y = \left(\dfrac{1}{\cos y}\right)\cos y = 1$

26. $\cot^2 y(\sec^2 y - 1) = \cot^2 y\tan^2 y = 1$

27. $\dfrac{\tan^2\theta}{\sec\theta} = \dfrac{(\sin\theta/\cos\theta)\tan\theta}{1/\cos\theta} = \sin\theta\tan\theta$

28. $\dfrac{\cot^3 t}{\csc t} = \dfrac{\cot t\cot^2 t}{\csc t}$

$= \dfrac{\cot t(\csc^2 t - 1)}{\csc t}$

$= \dfrac{\dfrac{\cos t}{\sin t}(\csc^2 t - 1)}{\dfrac{1}{\sin t}}$

$= \dfrac{\cos t\sin t}{\sin t}(\csc^2 t - 1)$

$= \cos t(\csc^2 t - 1)$

29. $\dfrac{1}{\tan\beta} + \tan\beta = \dfrac{1+\tan^2\beta}{\tan\beta}$

$= \dfrac{\sec^2\beta}{\tan\beta}$

30. $\dfrac{\sec\theta - 1}{1-\cos\theta} = \dfrac{\sec\theta - 1}{1-(1/\sec\theta)} \cdot \dfrac{\sec\theta}{\sec\theta} = \dfrac{\sec\theta(\sec\theta - 1)}{\sec\theta - 1} = \sec\theta$

31. $\dfrac{\cot^2 t}{\csc t} = \dfrac{\cos^2 t/\sin^2 t}{1/\sin t} = \dfrac{\cos^2 t}{\sin t} = \dfrac{1-\sin^2 t}{\sin t}$

32. $\cos x + \sin x\tan x = \cos x + \sin x\left(\dfrac{\sin x}{\cos x}\right)$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos x}$

$= \dfrac{1}{\cos x}$

$= \sec x$

33. $\sec x - \cos x = \dfrac{1}{\cos x} - \cos x$

$= \dfrac{1-\cos^2 x}{\cos x}$

$= \dfrac{\sin^2 x}{\cos x}$

$= \sin x \cdot \dfrac{\sin x}{\cos x}$

$= \sin x\tan x$

34.
$$\begin{aligned}\cot x - \tan x &= \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}\\ &= \frac{\cos^2 x - \sin^2 x}{\sin x \cos x}\\ &= \frac{1 - \sin^2 x - \sin^2 x}{\sin x \cos x}\\ &= \frac{1 - 2\sin^2 x}{\sin x \cos x}\\ &= \frac{1}{\cos x}\left(\frac{1 - 2\sin^2 x}{\sin x}\right)\\ &= \frac{1}{\cos x}\left(\frac{1}{\sin x} - \frac{2\sin^2 x}{\sin x}\right)\\ &= \sec x(\csc x - 2\sin x)\end{aligned}$$

35. $\dfrac{\cot x}{\sec x} = \dfrac{\cos x/\sin x}{1/\cos x} = \dfrac{\cos^2 x}{\sin x} = \dfrac{1 - \sin^2 x}{\sin x} = \dfrac{1}{\sin x} - \dfrac{\sin^2 x}{\sin x} = \csc x - \sin x$

36.
$$\begin{aligned}\frac{\csc(-x)}{\sec(-x)} &= \frac{1/\sin(-x)}{1/\cos(-x)}\\ &= \frac{\cos(-x)}{\sin(-x)}\\ &= \frac{\cos x}{-\sin x}\\ &= -\cot x\end{aligned}$$

37. $\sin^{1/2} x \cos x - \sin^{5/2} x \cos x = \sin^{1/2} x \cos x(1 - \sin^2 x) = \sin^{1/2} x \cos x \cdot \cos^2 x = \cos^3 x\sqrt{\sin x}$

38. $\sec^6 x(\sec x \tan x) - \sec^4 x(\sec x \tan x) = \sec^4 x(\sec x \tan x)(\sec^2 x - 1) = \sec^4 x(\sec x \tan x)\tan^2 x = \sec^5 x \tan^3 x$

39.
$$\begin{aligned}(1 + \sin y)[1 + \sin(-y)] &= (1 + \sin y)(1 - \sin y)\\ &= 1 - \sin^2 y\\ &= \cos^2 y\end{aligned}$$

40.
$$\begin{aligned}\frac{\tan x + \tan y}{1 - \tan x \tan y} &= \frac{\dfrac{1}{\cot x} + \dfrac{1}{\cot y}}{1 - \dfrac{1}{\cot x}\cdot\dfrac{1}{\cot y}} \cdot \frac{\cot x \cot y}{\cot x \cot y}\\ &= \frac{\cot y + \cot x}{\cot x \cot y - 1}\end{aligned}$$

41.
$$\begin{aligned}\sqrt{\frac{1 + \sin\theta}{1 - \sin\theta}} &= \sqrt{\frac{1 + \sin\theta}{1 - \sin\theta}\cdot\frac{1 + \sin\theta}{1 + \sin\theta}}\\ &= \sqrt{\frac{(1 + \sin\theta)^2}{1 - \sin^2\theta}}\\ &= \sqrt{\frac{(1 + \sin\theta)^2}{\cos^2\theta}}\\ &= \frac{1 + \sin\theta}{|\cos\theta|}\end{aligned}$$

42.
$$\begin{aligned}\frac{\cos x - \cos y}{\sin x + \sin y} + \frac{\sin x - \sin y}{\cos x + \cos y} &= \frac{(\cos x - \cos y)(\cos x + \cos y) + (\sin x - \sin y)(\sin x + \sin y)}{(\sin x + \sin y)(\cos x + \cos y)}\\ &= \frac{\cos^2 x - \cos^2 y + \sin^2 x - \sin^2 y}{(\sin x + \sin y)(\cos x + \cos y)}\\ &= \frac{(\cos^2 x + \sin^2 x) - (\cos^2 y + \sin^2 y)}{(\sin x + \sin y)(\cos x + \cos y)}\\ &= 0\end{aligned}$$

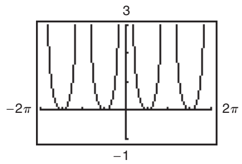
43. $\cot(-x) \neq \cot x$

The correct substitution is $\cot(-x) = -\cot x$.

$\dfrac{1}{\tan x} + \cot(-x) = \cot x - \cot x = 0$

44. The first line claims that $\sec(-\theta) = -\sec\theta$ and $\sin(-\theta) = \sin\theta$. The correct substitutions are $\sec(-\theta) = \sec\theta$ and $\sin(-\theta) = -\sin\theta$.

45. (a)



Identity

(b)

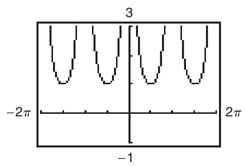
X	Y1	Y2
-3	49.214	49.214
-2	.20945	.20945
-1	.41228	.41228
0	ERROR	ERROR
1	.41228	.41228
2	.20945	.20945
3	49.214	49.214

X=-3

Identity

(c) $(1 + \cot^2 x)(\cos^2 x) = \csc^2 x \cos^2 x = \dfrac{1}{\sin^2 x} \cdot \cos^2 x = \cot^2 x$

46. (a)



Identity

(b)

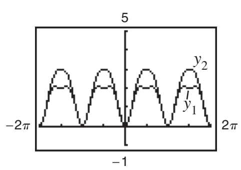
X	Y1	Y2
-3	50.214	50.214
-2	1.2095	1.2095
-1	1.4123	1.4123
0	ERROR	ERROR
1	1.4123	1.4123
2	1.2095	1.2095
3	50.214	50.214

X=-3

Identity

(c)
$$\begin{aligned}\csc x(\csc x - \sin x) + \frac{\sin x - \cos x}{\sin x} + \cot x &= \csc^2 x - \csc x \sin x + 1 - \frac{\cos x}{\sin x} + \cot x\\ &= \csc^2 x - 1 + 1 - \cot x + \cot x\\ &= \csc^2 x\end{aligned}$$

47. (a)



Not an identity

(b)

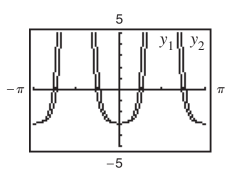
X	Y1	Y2
-4.712	2	3
-3.142	0	0
-1.571	2	3
0	0	0
1.5708	2	3
3.1416	0	0
4.7124	2	3

X=-4.71238898038

Not an identity

(c) $2 + \cos^2 x - 3\cos^4 x = (1 - \cos^2 x)(2 + 3\cos^2 x) = \sin^2 x(2 + 3\cos^2 x) \neq \sin^2 x(3 + 2\cos^2 x)$

48. (a)



Not an identity

(b)

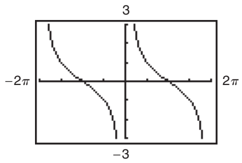
X	Y1	Y2
-3	-2.979	-2.978
-2	24.569	92.954
-1	5.3087	22.958
0	-3	-3
1	5.3087	22.958
2	24.569	92.954
3	-2.979	-2.978

X=-3

Not an identity

(c)
$$\begin{aligned}\tan^4 x + \tan^2 x - 3 &= \frac{\sin^4 x}{\cos^4 x} + \frac{\sin^2 x}{\cos^2 x} - 3\\ &= \frac{1}{\cos^2 x}\left(\frac{\sin^4 x}{\cos^2 x} + \sin^2 x\right) - 3\\ &= \frac{1}{\cos^2 x}\left(\frac{\sin^4 x + \sin^2 x \cos^2 x}{\cos^2 x}\right) - 3\\ &= \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{\cos^2 x}\right)(\sin^2 x + \cos^2 x) - 3\\ &= \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{\cos^2 x} \cdot 1\right) - 3\\ &= \sec^2 x \tan^2 x - 3\\ &\neq \sec^2 x(4\tan^2 x - 3)\end{aligned}$$

49. (a)



Identity

(b)

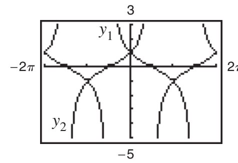
X	Y1	Y2
−3	−.0709	−.0709
−2	−.6421	−.6421
−1	−1.83	−1.83
0	ERROR	ERROR
1	1.8305	1.8305
2	.64209	.64209
3	.07091	.07091

X=−3

Identity

(c) $$\frac{1+\cos x}{\sin x} = \frac{(1+\cos x)(1-\cos x)}{\sin x(1-\cos x)}$$
$$= \frac{1-\cos^2 x}{\sin x(1-\cos x)}$$
$$= \frac{\sin^2 x}{\sin x(1-\cos x)}$$
$$= \frac{\sin x}{1-\cos x}$$

50. (a)



Not an identity

(b)

X	Y1	Y2
−3	−1.153	−.8676
−2	−4.588	−.218
−1	3.4082	.29341
0	ERROR	ERROR
1	.29341	3.4082
2	−.218	−4.588
3	−.8676	−1.153

X=−3

Not an identity

(c) $\dfrac{\cot \alpha}{\csc \alpha + 1}$ is the reciprocal of $\dfrac{\csc \alpha + 1}{\cot \alpha}$.

They will only be equivalent at isolated points in their respective domains. So, not an identity.

51. $$\tan^3 x \sec^2 x - \tan^3 x = \tan^3 x(\sec^2 x - 1)$$
$$= \tan^3 x \tan^2 x$$
$$= \tan^5 x$$

52. $$(\tan^2 x + \tan^4 x)\sec^2 x = \left(\frac{\sin^2 x}{\cos^2 x} + \frac{\sin^4 x}{\cos^4 x}\right)\frac{1}{\cos^2 x}$$
$$= \frac{1}{\cos^4 x}\left(\sin^2 x + \frac{\sin^4 x}{\cos^2 x}\right)$$
$$= \frac{1}{\cos^4 x}\left(\frac{\sin^2 x \cos^2 x + \sin^4 x}{\cos^2 x}\right)$$
$$= \frac{1}{\cos^4 x}\left(\frac{\sin^2 x(\cos^2 x + \sin^2 x)}{\cos^2 x}\right)$$
$$= \frac{1}{\cos^4 x}\left(\frac{\sin^2 x}{\cos^2 x}\cdot 1\right) = \sec^4 x \cdot \tan^2 x$$

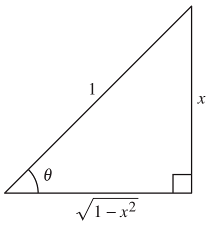
53. $$(\sin^2 x - \sin^4 x)\cos x = \sin^2 x(1 - \sin^2 x)\cos x$$
$$= \sin^2 x \cos^2 x \cos x$$
$$= \sin^2 x \cos^3 x$$

54. $$\sin^4 x + \cos^4 x = \sin^2 x \sin^2 x + \cos^4 x$$
$$= (1 - \cos^2 x)(1 - \cos^2 x) + \cos^4 x$$
$$= 1 - 2\cos^2 x + \cos^4 x + \cos^4 x$$
$$= 1 - 2\cos^2 x + 2\cos^4 x$$

55. $$\sin^2 25° + \sin^2 65° = \sin^2 25° + \cos^2(90° - 65°)$$
$$= \sin^2 25° + \cos^2 25°$$
$$= 1$$

56. $$\tan^2 63° + \cot^2 16° - \sec^2 74° - \csc^2 27° = \tan^2 63° + \cot^2 16° - \csc^2(90° - 74°) - \sec^2(90° - 27°)$$
$$= \tan^2 63° + \cot^2 16° - \csc^2 16° - \sec^2 63°$$
$$= (\tan^2 63° - \sec^2 63°) + (\cot^2 16° - \csc^2 16°)$$
$$= -1 + (-1)$$
$$= -2$$

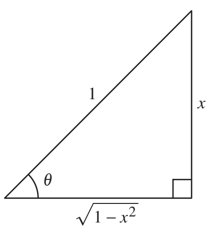
57. Let $\theta = \sin^{-1} x \Rightarrow \sin\theta = x = \frac{x}{1}$.



From the diagram,

$$\tan\left(\sin^{-1} x\right) = \tan\theta = \frac{x}{\sqrt{1 - x^2}}.$$

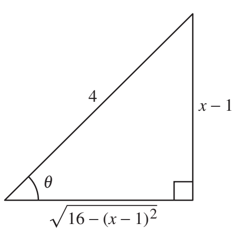
58. Let $\theta = \sin^{-1} x \Rightarrow \sin\theta = x = \frac{x}{1}$.



From the diagram,

$$\cos\left(\sin^{-1} x\right) = \cos\theta = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}.$$

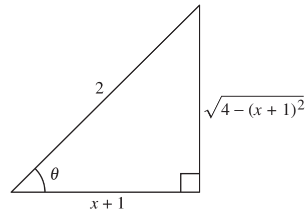
59. Let $\theta = \sin^{-1} \frac{x-1}{4} \Rightarrow \sin\theta = \frac{x-1}{4}$.



From the diagram,

$$\tan\left(\sin^{-1} \frac{x-1}{4}\right) = \tan\theta = \frac{x-1}{\sqrt{16 - (x-1)^2}}.$$

60. Let $\theta = \cos^{-1} \frac{x+1}{2} \Rightarrow \cos\theta = \frac{x+1}{2}$.



From the diagram,

$$\tan\left(\cos^{-1} \frac{x+1}{2}\right) = \tan\theta = \frac{\sqrt{4 - (x+1)^2}}{x+1}.$$

61.
$$\begin{aligned}\cos x - \csc x \cot x &= \cos x - \frac{1}{\sin x}\frac{\cos x}{\sin x}\\ &= \cos x\left(1 - \frac{1}{\sin^2 x}\right)\\ &= \cos x\left(1 - \csc^2 x\right)\\ &= -\cos x\left(\csc^2 x - 1\right)\\ &= -\cos x \cot^2 x\end{aligned}$$

62. (a) $\dfrac{h\sin(90° - \theta)}{\sin\theta} = \dfrac{h\cos\theta}{\sin\theta} = h\cot\theta$

(b)

θ	15°	30°	45°	60°	75°	90°
s	18.66	8.66	5	2.89	1.34	0

(c) Maximum: 15°

Minimum: 90°

(d) Noon

63. False. $\tan x^2 = \tan(x \cdot x)$ and $\tan^2 x = (\tan x)(\tan x)$, $\tan x^2 \neq \tan^2 x$.

64. True. Cosine is an even function,

$$\begin{aligned}\cos\left(\theta - \frac{\pi}{2}\right) &= \cos\left[-\left(\frac{\pi}{2} - \theta\right)\right]\\ &= \cos\left(\frac{\pi}{2} - \theta\right)\\ &= \sin\theta.\end{aligned}$$

65. False. For the equation to be an identity, it must be true for all values of θ in the domain.

66. If $\sin\theta = \dfrac{a}{c}$, $\sec\theta = \dfrac{c}{b}$, and

$a^2 + b^2 = c^2 \Rightarrow a^2 = c^2 - b^2$, then

$$\begin{aligned}\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{\left(\frac{c}{b}\right)^2 - 1}{\left(\frac{c}{b}\right)^2}\\ &= \frac{\frac{c^2}{b^2} - 1}{\frac{c^2}{b^2}}\\ &= \frac{\frac{c^2 - b^2}{b^2}}{\frac{c^2}{b^2}}\\ &= \frac{c^2 - b^2}{b^2}\cdot\frac{b^2}{c^2}\\ &= \frac{c^2 - b^2}{c^2}\\ &= \frac{a^2}{c^2}\\ &= \left(\frac{a}{c}\right)^2\\ &= \sin^2\theta.\end{aligned}$$

67. Because $\sin^2\theta = 1 - \cos^2\theta$, then

$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$; $\sin\theta \neq \sqrt{1 - \cos^2\theta}$ if θ lies in Quadrant III or IV.

One such angle is $\theta = \dfrac{7\pi}{4}$.

68. $\tan\theta = \sqrt{\sec^2\theta - 1}$

True identity: $\tan\theta = \pm\sqrt{\sec^2\theta - 1}$

$\tan\theta = \sqrt{\sec^2\theta - 1}$ is not true for $\pi/2 < \theta < \pi$ or $3\pi/2 < \theta < 2\pi$. So, the equation is not true for $\theta = 3\pi/4$.

69.

$$\begin{aligned}1 - \cos\theta &= \sin\theta\\ (1 - \cos\theta)^2 &= (\sin\theta)^2\\ 1 - 2\cos\theta + \cos^2\theta &= \sin^2\theta\\ 1 - 2\cos\theta + \cos^2\theta &= 1 - \cos^2\theta\\ 2\cos^2\theta - 2\cos\theta &= 0\\ 2\cos\theta(\cos\theta - 1) &= 0\end{aligned}$$

The equation is not an identity because it is only true when $\cos\theta = 0$ or $\cos\theta = 1$. So, one angle for which the equation is not true is $-\dfrac{\pi}{2}$.

70.

$$\begin{aligned}1 + \tan\theta &= \sec\theta\\ (1 + \tan\theta)^2 &= (\sec\theta)^2\\ 1 + 2\tan\theta + \tan^2\theta &= \sec^2\theta\\ 1 + 2\tan\theta + \tan^2\theta &= 1 + \tan^2\theta\\ 2\tan\theta &= 0\\ \tan\theta &= 0\end{aligned}$$

This equation is not an identity because it is only true when $\tan\theta = 0$. So, one angle for which the equation is not true is $\dfrac{\pi}{6}$.

Section 2.3 Solving Trigonometric Equations

1. isolate

2. general

3. quadratic

4. extraneous

5. $\tan x - \sqrt{3} = 0$

(a) $x = \dfrac{\pi}{3}$

$$\tan\frac{\pi}{3} - \sqrt{3} = \sqrt{3} - \sqrt{3} = 0$$

(b) $x = \dfrac{4\pi}{3}$

$$\tan\frac{4\pi}{3} - \sqrt{3} = \sqrt{3} - \sqrt{3} = 0$$

6. $\sec x - 2 = 0$

(a) $x = \dfrac{\pi}{3}$

$$\begin{aligned}\sec\frac{\pi}{3} - 2 &= \frac{1}{\cos(\pi/3)} - 2\\ &= \frac{1}{1/2} - 2 = 2 - 2 = 0\end{aligned}$$

(b) $x = \dfrac{5\pi}{3}$

$$\begin{aligned}\sec\frac{5\pi}{3} - 2 &= \frac{1}{\cos(5\pi/3)} - 2\\ &= \frac{1}{1/2} - 2 = 2 - 2 = 0\end{aligned}$$

7. $3\tan^2 2x - 1 = 0$

(a) $x = \dfrac{\pi}{12}$

$$3\left[\tan 2\left(\frac{\pi}{12}\right)\right]^2 - 1 = 3\tan^2\frac{\pi}{6} - 1 = 3\left(\frac{1}{\sqrt{3}}\right)^2 - 1 = 0$$

(b) $x = \dfrac{5\pi}{12}$

$$3\left[\tan 2\left(\frac{5\pi}{12}\right)\right]^2 - 1 = 3\tan^2\frac{5\pi}{6} - 1 = 3\left(-\frac{1}{\sqrt{3}}\right)^2 - 1 = 0$$

8. $2\cos^2 4x - 1 = 0$

(a) $x = \dfrac{\pi}{16}$

$$2\cos^2\left[4\left(\frac{\pi}{16}\right)\right] - 1 = 2\cos^2\frac{\pi}{4} - 1 = 2\left(\frac{\sqrt{2}}{2}\right)^2 - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$

(b) $x = \dfrac{3\pi}{16}$

$$2\cos^2\left[4\left(\frac{3\pi}{16}\right)\right] - 1 = 2\cos^2\frac{3\pi}{4} - 1 = 2\left(-\frac{\sqrt{2}}{2}\right)^2 - 1 = 2\left(\frac{1}{2}\right) - 1 = 0$$

9. $2\sin^2 x - \sin x - 1 = 0$

(a) $x = \dfrac{\pi}{2}$

$$2\sin^2\frac{\pi}{2} - \sin\frac{\pi}{2} - 1 = 2(1)^2 - 1 - 1 = 0$$

(b) $x = \dfrac{7\pi}{6}$

$$2\sin^2\frac{7\pi}{6} - \sin\frac{7\pi}{6} - 1 = 2\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) - 1 = \frac{1}{2} + \frac{1}{2} - 1 = 0$$

10. $\csc^4 x - 4\csc^2 x = 0$

(a) $x = \dfrac{\pi}{6}$

$$\csc^4\frac{\pi}{6} - 4\csc^2\frac{\pi}{6} = \frac{1}{\sin^4(\pi/6)} - \frac{4}{\sin^2(\pi/6)} = \frac{1}{(1/2)^4} - \frac{4}{(1/2)^2} = 16 - 16 = 0$$

(b) $x = \dfrac{5\pi}{6}$

$$\csc^4\frac{5\pi}{6} - 4\csc\frac{5\pi}{6} = \frac{1}{\sin^4(5\pi/6)} - \frac{4}{\sin^2(5\pi/6)} = \frac{1}{(1/2)^4} - \frac{4}{(1/2)^2} = 16 - 16 = 0$$

11. $\sqrt{3}\csc x - 2 = 0$

$$\sqrt{3}\csc x = 2$$
$$\csc x = \frac{2}{\sqrt{3}}$$
$$x = \frac{\pi}{3} + 2n\pi$$
$$\text{or } x = \frac{2\pi}{3} + 2n\pi$$

12. $\tan x + \sqrt{3} = 0$

$$\tan x = -\sqrt{3}$$
$$x = \frac{2\pi}{3} + n\pi$$

13. $\cos x + 1 = -\cos x$

$$2\cos x + 1 = 0$$
$$\cos x = -\frac{1}{2}$$
$$x = \frac{2\pi}{3} + 2n\pi \text{ or } x = \frac{4\pi}{3} + 2n\pi$$

14. $3\sin x + 1 = \sin x$

$$2\sin x + 1 = 0$$
$$\sin x = -\frac{1}{2}$$
$$x = \frac{7\pi}{6} + 2n\pi \text{ or}$$
$$x = \frac{11\pi}{6} + 2n\pi$$

15. $3\sec^2 x - 4 = 0$

$$\sec^2 x = \frac{4}{3}$$

$$\sec x = \pm\frac{2}{\sqrt{3}}$$

$$x = \frac{\pi}{6} + n\pi$$

$$\text{or } x = \frac{5\pi}{6} + n\pi$$

16. $3\cot^2 x - 1 = 0$

$$\cot^2 x = \frac{1}{3}$$

$$\cot x = \pm\frac{1}{\sqrt{3}}$$

$$x = \frac{\pi}{3} + n\pi$$

$$\text{or } x = \frac{2\pi}{3} + n\pi$$

17. $4\cos^2 x - 1 = 0$

$$\cos^2 x = \frac{1}{4}$$

$$\cos x = \pm\frac{1}{2}$$

$$x = \frac{\pi}{3} + n\pi \quad \text{or} \quad x = \frac{2\pi}{3} + n\pi$$

18. $2 - 4\sin^2 x = 0$

$$\sin^2 x = \frac{1}{2}$$

$$\sin x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

$$x = \frac{\pi}{4} + 2n\pi$$

$$x = \frac{3\pi}{4} + 2n\pi$$

$$x = \frac{5\pi}{4} + 2n\pi$$

$$x = \frac{7\pi}{4} + 2n\pi$$

These answers can be represented as $x = \frac{\pi}{4} + \frac{n\pi}{2}$.

19. $\sin x(\sin x + 1) = 0$

$\sin x = 0$ or $\sin x = -1$

$x = n\pi$ $\quad x = \frac{3\pi}{2} + 2n\pi$

20. $(2\sin^2 x - 1)(\tan^2 x - 3) = 0$

$2\sin^2 x - 1 = 0$ or $\tan^2 x = 3$

$\sin^2 x = \frac{1}{2}$ $\quad \tan x = \pm\sqrt{3}$

$\sin x = \pm\frac{1}{\sqrt{2}}$ $\quad x = \frac{\pi}{3} + n\pi$

$\sin x = \pm\frac{\sqrt{2}}{2}$ $\quad x = \frac{2\pi}{3} + n\pi$

$$x = \frac{\pi}{4} + 2n\pi$$

$$x = \frac{3\pi}{4} + 2n\pi$$

$$x = \frac{5\pi}{4} + 2n\pi$$

$$x = \frac{7\pi}{4} + 2n\pi$$

21. $\cos^3 x - \cos x = 0$

$\cos x(\cos^2 x - 1) = 0$

$\cos x = 0$ or $\cos^2 x - 1 = 0$

$x = \frac{\pi}{2} + n\pi$ $\quad \cos x = \pm 1$

$x = n\pi$

Both of these answers can be represented as $x = \frac{n\pi}{2}$.

22. $\sec^2 x - 1 = 0$

$$\sec^2 x = 1$$

$$\sec x = \pm 1$$

$$x = n\pi$$

23. $3\tan^3 x = \tan x$

$3\tan^3 x - \tan x = 0$

$\tan x(3\tan^2 x - 1) = 0$

$\tan x = 0$ or $3\tan^2 x - 1 = 0$

$x = n\pi$ $\quad \tan x = \pm\frac{\sqrt{3}}{3}$

$x = \frac{\pi}{6} + n\pi, \frac{5\pi}{6} + n\pi$

24. $\sec x \csc x = 2\csc x$

$\sec x \csc x - 2\csc x = 0$

$\csc x(\sec x - 2) = 0$

$\csc x = 0$ or $\sec x - 2 = 0$

No solution $\quad \sec x = 2$

$x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$

25. $2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$2\cos x - 1 = 0$ or $\cos x + 1 = 0$

$\cos x = \frac{1}{2}$ $\qquad$ $\cos x = -1$

$x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$ $\qquad$ $x = \pi + 2n\pi$

26. $2\sin^2 x + 3\sin x + 1 = 0$

$(2\sin x + 1)(\sin x + 1) = 0$

$2\sin x + 1 = 0$ or $\sin x + 1 = 0$

$\sin x = -\frac{1}{2}$ $\qquad$ $\sin x = -1$

$x = \frac{7\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi$ $\qquad$ $x = \frac{3\pi}{2} + 2n\pi$

27. $\sec^2 x - \sec x = 2$

$\sec^2 x - \sec x - 2 = 0$

$(\sec x - 2)(\sec x + 1) = 0$

$\sec x - 2 = 0$ or $\sec x + 1 = 0$

$\sec x = 2$ $\qquad$ $\sec x = -1$

$x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$ $\qquad$ $x = \pi + 2n\pi$

28. $\csc^2 x + \csc x = 2$

$\csc^2 x + \csc x - 2 = 0$

$(\csc x + 2)(\csc x - 1) = 0$

$\csc x + 2 = 0$ or $\csc x - 1 = 0$

$\csc x = -2$ $\qquad$ $\csc x = 1$

$x = \frac{7\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi$ $\qquad$ $x = \frac{\pi}{2} + 2n\pi$

29. $\sin x - 2 = \cos x - 2$

$\sin x = \cos x$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$x = \tan^{-1} 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

30. $\cos x + \sin x \tan x = 2$

$\cos x + \sin x\left(\frac{\sin x}{\cos x}\right) = 2$

$\frac{\cos^2 x + \sin^2 x}{\cos x} = 2$

$\frac{1}{\cos x} = 2$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

31.
$$2\sin^2 x = 2 + \cos x$$
$$2 - 2\cos^2 x = 2 + \cos x$$
$$2\cos^2 x + \cos x = 0$$
$$\cos x(2\cos x + 1) = 0$$
$$\cos x = 0 \quad \text{or} \quad 2\cos x + 1 = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad 2\cos x = -1$$
$$\cos x = -\frac{1}{2}$$
$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

32.
$$\tan^2 x = \sec x - 1$$
$$\sec^2 x - 1 = \sec x - 1$$
$$\sec^2 x - \sec x = 0$$
$$\sec x(\sec x - 1) = 0$$
$$\sec x = 0 \quad \text{or} \quad \sec x - 1 = 0$$
No Solutions
$$\sec x = 1$$
$$x = 0$$

33.
$$\sin^2 x = 3\cos^2 x$$
$$\sin^2 x - 3\cos^2 x = 0$$
$$\sin^2 x - 3(1 - \sin^2 x) = 0$$
$$4\sin^2 x = 3$$
$$\sin x = \pm\frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

34.
$$2\sec^2 x + \tan^2 x - 3 = 0$$
$$2(\tan^2 x + 1) + \tan^2 x - 3 = 0$$
$$3\tan^2 x - 1 = 0$$
$$\tan x = \pm\frac{\sqrt{3}}{3}$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

35.
$$2\sin x + \csc x = 0$$
$$2\sin x + \frac{1}{\sin x} = 0$$
$$2\sin^2 x + 1 = 0$$
$$\sin^2 x = -\frac{1}{2} \Rightarrow \text{No solution}$$

36.
$$3\sec x - 4\cos x = 0$$
$$\frac{3}{\cos x} - 4\cos x = 0$$
$$\frac{3 - 4\cos^2 x}{\cos x} = 0$$
$$3 - 4\cos^2 x = 0$$
$$\cos^2 x = \frac{3}{4}$$
$$\cos x = \pm\frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

37.
$$\csc x + \cot x = 1$$
$$(\csc x + \cot x)^2 = 1^2$$
$$\csc^2 x + 2\csc x\cot x + \cot^2 x = 1$$
$$\cot^2 x + 1 + 2\csc x\cot x + \cot^2 x = 1$$
$$2\cot^2 x + 2\csc x\cot x = 0$$
$$2\cot x(\cot x + \csc x) = 0$$
$$2\cot x = 0 \quad \text{or} \quad \cot x + \csc x = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \frac{\cos x}{\sin x} = -\frac{1}{\sin x}$$
$$\left(\frac{3\pi}{2}\text{ is extraneous.}\right) \qquad \cos x = -1$$
$$x = \pi$$
$$(\pi \text{ is extraneous.})$$
$x = \pi/2$ is the only solution.

38.
$$\sec x + \tan x = 1$$
$$\frac{1}{\cos x} + \frac{\sin x}{\cos x} = 1$$
$$1 + \sin x = \cos x$$
$$(1 + \sin x)^2 = \cos^2 x$$
$$1 + 2\sin x + \sin^2 x = \cos^2 x$$
$$1 + 2\sin x + \sin^2 x = 1 - \sin^2 x$$
$$2\sin^2 x + 2\sin x = 0$$
$$2\sin x(\sin x + 1) = 0$$
$$\sin x = 0 \quad \text{or} \quad \sin x + 1 = 0$$
$$x = 0, \pi \qquad \sin x = -1$$
$$(\pi \text{ is extraneous.}) \qquad x = \frac{3\pi}{2}$$
$$\left(\frac{3\pi}{2}\text{ is extraneous.}\right)$$
$x = 0$ is the only solution.

39. $2\cos 2x - 1 = 0$

$\cos 2x = \dfrac{1}{2}$

$2x = \dfrac{\pi}{3} + 2n\pi$ or $2x = \dfrac{5\pi}{3} + 2n\pi$

$x = \dfrac{\pi}{6} + n\pi$ $\qquad x = \dfrac{5\pi}{6} + n\pi$

40. $2\sin 2x + \sqrt{3} = 0$

$\sin 2x = -\dfrac{\sqrt{3}}{2}$

$2x = \dfrac{4\pi}{3} + 2n\pi$ or $2x = \dfrac{5\pi}{3} + 2n\pi$

$x = \dfrac{2\pi}{3} + n\pi$ $\qquad x = \dfrac{5\pi}{6} + n\pi$

41. $\tan 3x - 1 = 0$

$\tan 3x = 1$

$3x = \dfrac{\pi}{4} + n\pi$

$x = \dfrac{\pi}{12} + \dfrac{n\pi}{3}$

42. $\sec 4x - 2 = 0$

$\sec 4x = 2$

$\cos 4x = \dfrac{1}{2}$

$4x = \dfrac{\pi}{3} + 2n\pi$ or $4x = \dfrac{5\pi}{3} + 2n\pi$

$x = \dfrac{\pi}{12} + \dfrac{n\pi}{2}$ $\qquad x = \dfrac{5\pi}{12} + \dfrac{n\pi}{2}$

43. $2\cos\dfrac{x}{2} = \sqrt{2} = 0$

$\cos\dfrac{x}{2} = \dfrac{\sqrt{2}}{2}$

$\dfrac{x}{2} = \dfrac{\pi}{4} + 2n\pi$ or $\dfrac{x}{2} = \dfrac{7\pi}{4} + 2n\pi$

$x = \dfrac{\pi}{2} + 4n\pi$ $\qquad x = \dfrac{7\pi}{2} + 4n\pi$

44. $2\sin\dfrac{x}{2} = \sqrt{3} = 0$

$\sin\dfrac{x}{2} = -\dfrac{\sqrt{3}}{2}$

$\dfrac{x}{2} = \dfrac{4\pi}{3} + 2n\pi$ or $\dfrac{x}{2} = \dfrac{5\pi}{3} + 2n\pi$

$x = \dfrac{8\pi}{3} + 4n\pi$ $\qquad x = \dfrac{10\pi}{3} + 4n\pi$

45. $3\tan\dfrac{x}{2} - \sqrt{3} = 0$

$\tan\dfrac{x}{2} = \dfrac{\sqrt{3}}{3}$

$\dfrac{x}{2} = \dfrac{\pi}{6} + n\pi \Rightarrow x = \dfrac{\pi}{3} + 2n\pi$

46. $\tan\dfrac{x}{2} + \sqrt{3} = 0$

$\tan\dfrac{x}{2} = -\sqrt{3}$

$\dfrac{x}{2} = \dfrac{2\pi}{3} + n\pi \Rightarrow x = \dfrac{4\pi}{3} + 2n\pi$

47. $y = \sin\dfrac{\pi x}{2} + 1$

$\sin\left(\dfrac{\pi x}{2}\right) + 1 = 0$

$\sin\left(\dfrac{\pi x}{2}\right) = -1$

$\dfrac{\pi x}{2} = \dfrac{3\pi}{2} + 2n\pi$

$x = 3 + 4n$

For $-2 < x < 4$, the intercepts are -1 and 3.

48. $y = \sin \pi x + \cos \pi x$

$\sin \pi x + \cos \pi x = 0$

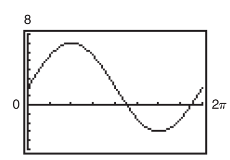
$\sin \pi x = -\cos \pi x$

$\pi x = -\dfrac{\pi}{4} + n\pi$

$x = -\dfrac{1}{4} + n$

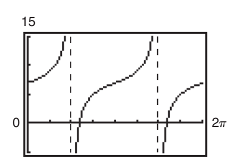
For $-1 < x < 3$, the intercepts are $-\dfrac{1}{4}, \dfrac{3}{4}, \dfrac{7}{4}, \dfrac{11}{4}$.

49. $5\sin x + 2 = 0$



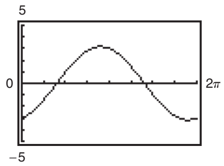
$x \approx 3.553$ and $x \approx 5.872$

50. $2\tan x + 7 = 0$



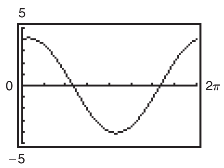
$x \approx 1.849$ and $x \approx 4.991$

51. $\sin x - 3\cos x = 0$



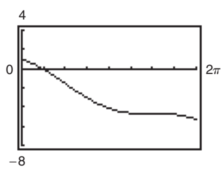
$x \approx 1.249$ and $x \approx 4.391$

52. $\sin x + 4\cos x = 0$



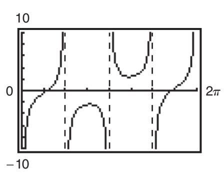
$x \approx 1.816$ and $x \approx 4.957$

53. $\cos x = x$



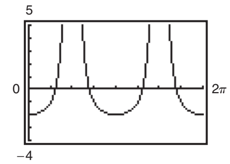
$x \approx 0.739$

54. $\tan x = \csc x$



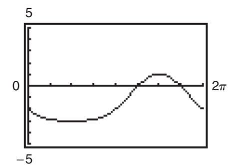
$x \approx 0.905$ and $x \approx 5.379$

55. $\sec^2 x - 3 = 0$



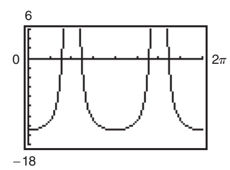
$x \approx 0.955, x \approx 2.186, x \approx 4.097$ and $x \approx 5.328$

56. $\csc^2 x - 5 = 0$



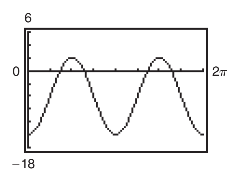
$x \approx 0.464, x \approx 2.678, x = 3.605$ and $x \approx 5.820$

57. $2\tan^2 x = 15$



$x \approx 1.221, x \approx 1.921, x \approx 4.362$ and $x \approx 5.062$

58. $6\sin^2 x = 5$



$x \approx 1.150, x \approx 1.991, x \approx 4.292$ and $x \approx 5.133$

59. $\tan^2 x + \tan x - 12 = 0$

$(\tan x + 4)(\tan x - 3) = 0$

$\tan x + 4 = 0$ or $\tan x - 3 = 0$

$\tan x = -4$ $\qquad$ $\tan x = 3$

$x = \arctan(-4) + n\pi$ $\qquad$ $x = \arctan 3 + n\pi$

60. $\tan^2 x - \tan x - 2 = 0$

$(\tan x + 1)(\tan x - 2) = 0$

$\tan x + 1 = 0$ or $\tan x - 2 = 0$

$\tan x = -1$ $\qquad$ $\tan x = 2$

$x = \dfrac{3\pi}{4} + n\pi$ $\qquad$ $x = \arctan 2 + n\pi$

61. $\sec^2 x - 6\tan x = -4$

$1 + \tan^2 x - 6\tan x + 4 = 0$

$\tan^2 x - 6\tan x + 5 = 0$

$(\tan x - 1)(\tan x - 5) = 0$

$\tan x - 1 = 0 \qquad \tan x - 5 = 0$

$\tan x = 1 \qquad \tan x = 5$

$x = \dfrac{\pi}{4} + n\pi \qquad x = \arctan 5 + n\pi$

62. $\sec^2 x + \tan x - 3 = 0$

$1 + \tan^2 x + \tan x - 3 = 0$

$\tan^2 x + \tan x - 2 = 0$

$(\tan x + 2)(\tan x - 1) = 0$

$\tan x + 2 = 0 \qquad \tan x - 1 = 0$

$\tan x = -2 \qquad \tan x = 1$

$x = \arctan(-2) + n\pi \qquad x = \arctan(1) + n\pi$

$\approx -1.1071 + n\pi \qquad = \dfrac{\pi}{4} + n\pi$

63. $2\sin^2 x + 5\cos x = 4$

$2(1 - \cos^2 x) + 5\cos x - 4 = 0$

$-2\cos^2 x + 5\cos x - 2 = 0$

$-(2\cos x - 1)(\cos x - 2) = 0$

$2\cos x - 1 = 0$ or $\cos x - 2 = 0$

$\cos x = \dfrac{1}{2} \qquad \cos x = 2$

$x = \dfrac{\pi}{3} + 2n\pi, \dfrac{5\pi}{3} + 2n\pi$ No solution

64. $2\cos^2 x + 7\sin x = 5$

$2(1 - \sin^2 x) + 7\sin x - 5 = 0$

$-2\sin^2 x + 7\sin x - 3 = 0$

$-(2\sin x - 1)(\sin x - 3) = 0$

$2\sin x - 1 = 0$ or $\sin x - 3 = 0$

$\sin x = \dfrac{1}{2} \qquad \sin x = 3$

$x = \dfrac{\pi}{6} + 2n\pi, \dfrac{5\pi}{6} + 2n\pi$ No solution

65. $\cot^2 x - 9 = 0$

$\cot^2 x = 9$

$\dfrac{1}{9} = \tan^2 x$

$\pm\dfrac{1}{3} = \tan x$

$x = \arctan \tfrac{1}{3} + n\pi, \arctan\left(-\tfrac{1}{3}\right) + n\pi$

66. $\cot^2 x - 6\cot x + 5 = 0$

$(\cot x - 5)(\cot x - 1) = 0$

$\cot x - 5 = 0$ or $\cot x - 1 = 0$

$\cot x = 5$ $\qquad$ $\cot x = 1$

$\frac{1}{5} = \tan x$ $\qquad$ $1 = \tan x$

$x = \arctan\frac{1}{5} + n\pi$ $\qquad$ $x = \frac{\pi}{4} + n\pi$

67. $\sec^2 x - 4\sec x = 0$

$\sec x(\sec x - 4) = 0$

$\sec x = 0$ $\qquad$ $\sec x - 4 = 0$

No solution $\qquad$ $\sec x = 4$

$\frac{1}{4} = \cos x$

$x = \arccos\frac{1}{4} + 2n\pi, -\arccos\frac{1}{4} + 2n\pi$

68. $\sec^2 x + 2\sec x - 8 = 0$

$(\sec x + 4)(\sec x - 2) = 0$

$\sec x + 4 = 0$ or $\sec x - 2 = 0$

$\sec x = -4$ $\qquad$ $\sec x = 2$

$-\frac{1}{4} = \cos x$ $\qquad$ $\frac{1}{2} = \cos x$

$x = \arccos\left(-\frac{1}{4}\right) + 2n\pi, -\arccos\left(-\frac{1}{4}\right) + 2n\pi$ $\qquad$ $x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$

69. $\csc^2 x + 3\csc x - 4 = 0$

$(\csc x + 4)(\csc x - 1) = 0$

$\csc x + 4 = 0$ or $\csc x - 1 = 0$

$\csc x = -4$ $\qquad$ $\csc x = 1$

$-\frac{1}{4} = \sin x$ $\qquad$ $1 = \sin x$

$x = \arcsin\left(\frac{1}{4}\right) + 2n\pi, \arcsin\left(-\frac{1}{4}\right) + 2n\pi$ $\qquad$ $x = \frac{\pi}{2} + 2n\pi$

70. $\csc^2 x - 5\csc x = 0$

$\csc x(\csc x - 5) = 0$

$\csc x = 0$ or $\csc x - 5 = 0$

No solution $\qquad$ $\csc x = 5$

$\frac{1}{5} = \sin x$

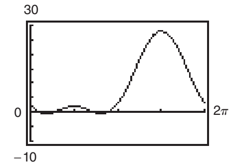
$x = \arcsin\left(\frac{1}{5}\right) + 2n\pi, \arcsin\left(-\frac{1}{5}\right) + 2n\pi$

71. $12\sin^2 x - 13\sin x + 3 = 0$

$$\sin x = \frac{-(-13) \pm \sqrt{(-13)^2 - 4(12)(3)}}{2(12)} = \frac{13 \pm 5}{24}$$

$\sin x = \frac{1}{3}$ or $\sin x = \frac{3}{4}$

$x \approx 0.3398, 2.8018$ $\quad x \approx 0.8481, 2.2935$



The x-intercepts occur at $x \approx 0.3398$, $x \approx 0.8481$, $x \approx 2.2935$, and $x \approx 2.8018$.

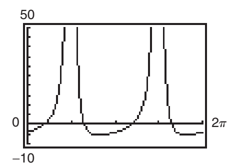
72. $3\tan^2 x + 4\tan x - 4 = 0$

$$\tan x = \frac{-4 \pm \sqrt{4^2 - 4(3)(-4)}}{2(3)} = \frac{-4 \pm \sqrt{64}}{6} = -2, \frac{2}{3}$$

$\tan x = -2$ $\quad \tan x = \frac{2}{3}$

$x = \arctan(-2) + n\pi$ $\quad x = \arctan\left(\frac{2}{3}\right) + n\pi$

$\approx -1.1071 + n\pi$ $\quad \approx 0.5880 + n\pi$



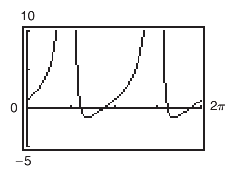
The values of x in $[0, 2\pi)$ are 0.5880, 3.7296, 2.0344, 5.1760.

73. $\tan^2 x + 3\tan x + 1 = 0$

$$\tan x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)} = \frac{-3 \pm \sqrt{5}}{2}$$

$\tan x = \frac{-3 - \sqrt{5}}{2}$ or $\tan x = \frac{-3 + \sqrt{5}}{2}$

$x \approx 1.9357, 5.0773$ $\quad x \approx 2.7767, 5.9183$



The x-intercepts occur at $x \approx 1.9357$, $x \approx 2.7767$, $x \approx 5.0773$, and $x \approx 5.9183$.

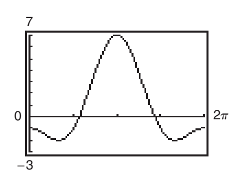
74. $4\cos^2 x - 4\cos x - 1 = 0$

$$\cos x = \frac{4 \pm \sqrt{(-4)^2 - 4(4)(-1)}}{2(4)} = \frac{4 \pm \sqrt{32}}{8} = \frac{1 \pm \sqrt{2}}{2}$$

$\cos x = \frac{1 - \sqrt{2}}{2}$ $\quad \cos x = \frac{1 + \sqrt{2}}{2}$

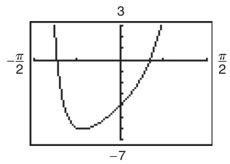
$x = \arccos\left(\frac{1 - \sqrt{2}}{2}\right)$ $\quad$ No solution

≈ 1.7794 $\quad \left(\frac{1 + \sqrt{2}}{2} > 1\right)$



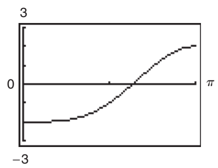
Solutions in $[0, 2\pi)$ are $\arccos\left(\frac{1 - \sqrt{2}}{2}\right)$ and $2\pi - \arccos\left(\frac{1 - \sqrt{2}}{2}\right)$: 1.7794, 4.5038.

75. $3\tan^2 x + 5\tan x - 4 = 0, \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$



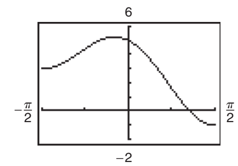
$x \approx -1.154, 0.534$

76. $\cos^2 x - 2\cos x - 1 = 0, [0, \pi]$



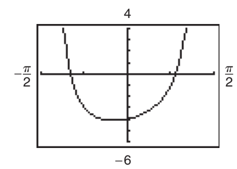
$x \approx 1.998$

77. $4\cos^2 x - 2\sin x + 1 = 0, \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$



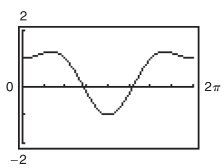
$x \approx 1.110$

78. $2\sec^2 x + \tan x - 6 = 0, \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$



$x \approx -1.035, 0.870$

79. (a) $f(x) = \sin^2 x + \cos x$



Maximum: $(1.0472, 1.25)$
Maximum: $(5.2360, 1.25)$
Minimum: $(0, 1)$
Minimum: $(3.1416, -1)$

(b) $2\sin x\cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

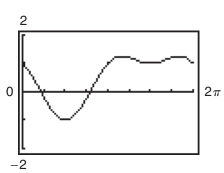
$\sin x = 0$ or $2\cos x - 1 = 0$

$x = 0, \pi \approx 0, 3.1416$

$\cos x = \dfrac{1}{2}$

$x = \dfrac{\pi}{3}, \dfrac{5\pi}{3} \approx 1.0472, 5.2360$

80. (a) $f(x) = \cos^2 x - \sin x$



Maximum: $(3.6652, 1.25)$
Maximum: $(5.7596, 1.25)$
Minimum: $(1.5708, -1)$
Minimum: $(4.7124, 1)$

(b) $-2\sin x\cos x - \cos x = 0$

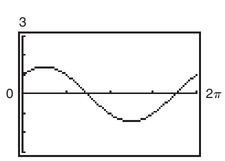
$-\cos x(2\sin x + 1) = 0$

$-\cos x = 0$ $2\sin x + 1 = 0$

$\cos x = 0$ $\sin x = -\dfrac{1}{2}$

$x = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \approx 1.5708, 4.7124$ $x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6} \approx 3.6652, 5.7596$

81. (a) $f(x) = \sin x + \cos x$



Maximum: $(0.7854, 1.4142)$
Minimum: $(3.9270, -1.4142)$

(b) $\cos x - \sin x = 0$

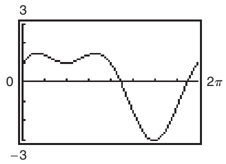
$\cos x = \sin x$

$1 = \dfrac{\sin x}{\cos x}$

$\tan x = 1$

$x = \dfrac{\pi}{4}, \dfrac{5\pi}{4} \approx 0.7854, 3.9270$

82. (a) $f(x) = 2\sin x + \cos 2x$



Maximum: $(0.5236, 1.5)$

Maximum: $(2.6180, 1.5)$

Minimum: $(1.5708, 1.0)$

Minimum: $(4.7124, -3.0)$

(b) $2\cos x - 4\sin x\cos x = 0$

$$2\cos x(1 - 2\sin x) = 0$$

$2\cos x = 0$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\approx 1.5708, 4.7124$$

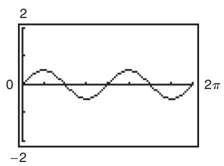
$1 - 2\sin x = 0$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\approx 0.5236, 2.6180$$

83. (a) $f(x) = \sin x\cos x$



Maximum: $(0.7854, 0.5)$

Maximum: $(3.9270, 0.5)$

Minimum: $(2.3562, -0.5)$

Minimum: $(5.4978, -0.5)$

(b)

$$-\sin^2 x + \cos^2 x = 0$$

$$-\sin^2 x + 1 - \sin^2 x = 0$$

$$-2\sin^2 x + 1 = 0$$

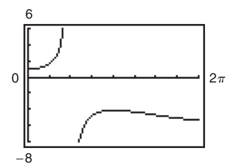
$$\sin^2 x = \frac{1}{2}$$

$$\sin x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\approx 0.7854, 2.3562, 3.9270, 5.4978$$

84. (a) $f(x) = \sec x + \tan x - x$



Maximum: $(3.1416, -4.1416)$

Minimum: $(0, 1)$

(b)

$$\sec x\tan x + \sec^2 x - 1 = 0$$

$$\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} + \frac{1}{\cos^2 x} - 1 = 0$$

$$\frac{\sin x + 1}{\cos^2 x} - 1 = 0$$

$$\frac{\sin x + 1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x} = 0$$

$$\frac{\sin x + 1 - \cos^2 x}{\cos^2 x} = 0$$

$$\frac{\sin x + \sin^2 x}{\cos^2 x} = 0$$

$$\sin x + \sin^2 x = 0$$

$$\sin x(1 + \sin x) = 0$$

$\sin x = 0$ or $1 + \sin x = 0$

$x = 0, \pi$ $\quad$ $\sin x = -1$

$\approx 0, 3.1416$ $\quad$ $x = \frac{3\pi}{2}$

$\frac{3\pi}{2}$ is undefined in original function. So, it is not a solution.

85. The graphs of $y_1 = 2\sin x$ and $y_2 = 3x + 1$ appear to have one point of intersection. This implies there is one solution to the equation $2\sin x = 3x + 1$.

86. The graphs of $y_1 = 2\sin x$ and $y_2 = \frac{1}{2}x + 1$ appear to have three points of intersection. This implies there are three solutions to the equation $2\sin x = \frac{1}{2}x + 1$.

87. $f(x) = \dfrac{\sin x}{x}$

(a) Domain: all real numbers except $x = 0$.

(b) The graph has y-axis symmetry.

(c) As $x \to 0$, $f(x) \to 1$.

(d) $\dfrac{\sin x}{x} = 0$ has four solutions in the interval $[-8, 8]$.

$$\sin x\left(\frac{1}{x}\right) = 0$$
$$\sin x = 0$$
$$x = -2\pi, -\pi, \pi, 2\pi$$

88. $f(x) = \cos\dfrac{1}{x}$

(a) Domain: all real numbers x except $x = 0$.

(b) The graph has y-axis symmetry and a horizontal asymptote at $y = 1$.

(c) As $x \to 0$, $f(x)$ oscillates between -1 and 1.

(d) There are infinitely many solutions in the interval $[-1, 1]$. They occur at $x = \dfrac{2}{(2n + 1)\pi}$ where n is any integer.

(e) The greatest solution appears to occur at $x \approx 0.6366$.

89.
$$y = \frac{1}{12}(\cos 8t - 3\sin 8t)$$
$$\frac{1}{12}(\cos 8t - 3\sin 8t) = 0$$
$$\cos 8t = 3\sin 8t$$
$$\frac{1}{3} = \tan 8t$$
$$8t \approx 0.32175 + n\pi$$
$$t \approx 0.04 + \frac{n\pi}{8}$$

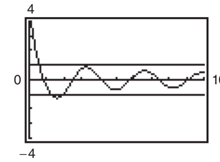
In the interval $0 \le t \le 1$, $t \approx 0.04, 0.43,$ and 0.83.

90. Graph the following equations.

$y_1 = 1.56t^{-1/2}\cos 1.9t$

$y_2 = 1$

$y_3 = -1$



The rightmost point of intersection is at approximately $(1.91, -1)$.

The displacement does not exceed one foot from equilibrium after $t \approx 1.91$ seconds.

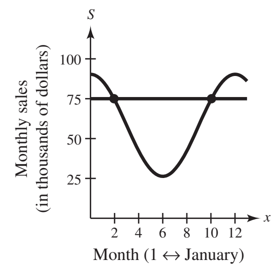
91. Graph $y_1 = 58.3 + 32\cos\left(\dfrac{\pi t}{6}\right)$

$y_2 = 75.$

Left point of intersection: $(1.95, 75)$

Right point of intersection: $(10.05, 75)$

So, sales exceed 7500 in January, November, and December.

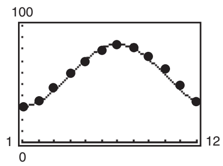


92.
$$\text{Range} = 300 \text{ feet}$$
$$v_0 = 100 \text{ feet per second}$$
$$r = \tfrac{1}{32}v_0^2 \sin 2\theta$$
$$\tfrac{1}{32}(100)^2 \sin 2\theta = 300$$
$$\sin 2\theta = 0.96$$
$$2\theta = \arcsin(0.96) \approx 73.74°$$
$$\theta \approx 36.9°$$

or

$$2\theta = 180° - \arcsin(0.96) \approx 106.26°$$
$$\theta \approx 53.1°$$

93. (a) and (c)



The model fits the data well.

(b) $C = a\cos(bt - c) + d$

$$a = \frac{1}{2}[\text{high} - \text{low}] = \frac{1}{2}[84.1 - 31.0] = 26.55$$

$$p = 2[\text{high time} - \text{low time}] = 2[7 - 1] = 12$$

$$b = \frac{2\pi}{p} = \frac{2\pi}{12} = \frac{\pi}{6}$$

The maximum occurs at 7, so the left end point is

$$\frac{c}{b} = 7 \Rightarrow c = 7\left(\frac{\pi}{6}\right) = \frac{7\pi}{6}$$

$$d = \frac{1}{2}[\text{high} + \text{low}] = \frac{1}{2}[93.6 + 62.3] = 57.55$$

$$C = 26.55\cos\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 57.55$$

(d) The constant term, d, gives the average maximum temperature.

The average maximum temperature in Chicago is 57.55°F.

(e) The average maximum temperature is above 72°F from June through September. The average maximum temperature is below 70°F from October through May.

94. $h(t) = 53 + 50\sin\left(\frac{\pi}{16}t - \frac{\pi}{2}\right)$

(a) $h(t) = 53$ when $50\sin\left(\frac{\pi}{16}t - \frac{\pi}{2}\right) = 0.$

$$\frac{\pi}{16}t - \frac{\pi}{2} = 0 \quad \text{or} \quad \frac{\pi}{16}t - \frac{\pi}{2} = \pi$$

$$\frac{\pi}{16}t = \frac{\pi}{2} \qquad \frac{\pi}{16}t = \frac{3\pi}{2}$$

$$t = 8 \qquad t = 24$$

A person on the Ferris wheel will be 53 feet above ground at 8 seconds and at 24 seconds

(b) The person will be at the top of the Ferris wheel when

$$\sin\left(\frac{\pi}{16}t - \frac{\pi}{2}\right) = 1$$

$$\frac{\pi}{16}t - \frac{\pi}{2} = \frac{\pi}{2}$$

$$\frac{\pi}{16}t = \pi$$

$$t = 16.$$

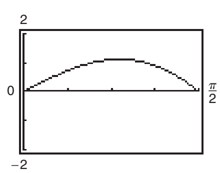
The first time this occurs is after 16 seconds.

The period of this function is $\frac{2\pi}{\pi/16} = 32.$

During 160 seconds, 5 cycles will take place and the person will be at the top of the ride 5 times, spaced 32 seconds apart. The times are: 16 seconds, 48 seconds, 80 seconds, 112 seconds, and 144 seconds.

95. $A = 2x\cos x, 0 < x < \frac{\pi}{2}$

(a)



The maximum area of $A \approx 1.12$ occurs when $x \approx 0.86.$

(b) $A \geq 1$ for $0.6 < x < 1.1$

96. $f(x) = 3\sin(0.6x - 2)$

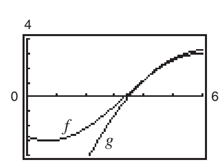
(a) Zero: $\sin(0.6x - 2) = 0$

$$0.6x - 2 = 0$$

$$0.6x = 2$$

$$x = \frac{2}{0.6} = \frac{10}{3}$$

(b) $g(x) = -0.45x^2 + 5.52x - 13.70$



For $3.5 \leq x \leq 6$ the approximation appears to be good.

(c) $-0.45x^2 + 5.52x - 13.70 = 0$

$$x = \frac{-5.52 \pm \sqrt{(5.52)^2 - 4(-0.45)(-13.70)}}{2(-0.45)}$$

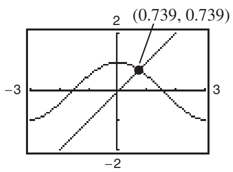
$$x \approx 3.46, 8.81$$

The zero of g on $[0, 6]$ is 3.46. The zero is close to the zero $\frac{10}{3} \approx 3.33$ of f.

97. $f(x) = \tan \frac{\pi x}{4}$

Because $\tan \pi/4 = 1$, $x = 1$ is the smallest nonnegative fixed point.

98. Graph $y = \cos x$ and $y = x$ on the same set of axes. Their point of intersection gives the value of c such that $f(c) = c \Rightarrow \cos c = c$.



$c \approx 0.739$

99. True. The period of $2 \sin 4t - 1$ is $\frac{\pi}{2}$ and the period of $2 \sin t - 1$ is 2π.

In the interval $[0, 2\pi)$ the first equation has four cycles whereas the second equation has only one cycle, so the first equation has four times the x-intercepts (solutions) as the second equation.

100. False.

$\sin x = 3.4$ has no solution because 3.4 is outside the range of sine.

101. $\cot x \cos^2 x = 2 \cot x$

$\cos^2 x = 2$

$\cos x = \pm\sqrt{2}$

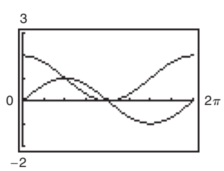
No solution

Because you solved this problem by first dividing by $\cot x$, you do not get the same solution as Example 3.

When solving equations, you do not want to divide each side by a variable expression that will cancel out because you may accidentally remove one of the solutions.

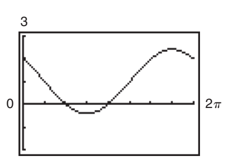
102. The equation $2 \cos x - 1 = 0$ is equivalent to $\cos x = \frac{1}{2}$. So, the points of intersection of $y = \cos x$ and $y = \frac{1}{2}$ represent the solutions of the equation $2 \cos x - 1 = 0$. In the interval $(-2\pi, 2\pi)$ the solutions of the equation are $x = -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}$, and $\frac{5\pi}{3}$.

103. (a)



The graphs intersect when $x = \frac{\pi}{2}$ and $x = \pi$.

(b)



The x-intercepts are $\left(\frac{\pi}{2}, 0\right)$ and $(\pi, 0)$.

(c) Both methods produce the same x-values. Answers will vary on which method is preferred.

Section 2.4 Sum and Difference Formulas

1. $\sin u \cos v - \cos u \sin v$

2. $\cos u \cos v - \sin u \sin v$

3. $\dfrac{\tan u + \tan v}{1 - \tan u \tan v}$

4. $\sin u \cos v + \cos u \sin v$

5. $\cos u \cos v + \sin u \sin v$

6. $\dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

7. (a) $\cos\left(\dfrac{\pi}{4} + \dfrac{\pi}{3}\right) = \cos\dfrac{\pi}{4}\cos\dfrac{\pi}{3} - \sin\dfrac{\pi}{4}\sin\dfrac{\pi}{3}$

$= \dfrac{\sqrt{2}}{2}\cdot\dfrac{1}{2} - \dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{3}}{2}$

$= \dfrac{\sqrt{2} - \sqrt{6}}{4}$

(b) $\cos\dfrac{\pi}{4} + \cos\dfrac{\pi}{3} = \dfrac{\sqrt{2}}{2} + \dfrac{1}{2} = \dfrac{\sqrt{2} + 1}{2}$

8. (a) $\sin\left(\dfrac{7\pi}{6} - \dfrac{\pi}{3}\right) = \sin\dfrac{5\pi}{6} = \sin\dfrac{\pi}{6} = \dfrac{1}{2}$

(b) $\sin\dfrac{7\pi}{6} - \sin\dfrac{\pi}{3} = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2} = \dfrac{-1 - \sqrt{3}}{2}$

9. (a) $\sin(135° - 30°) = \sin 135° \cos 30° - \cos 135° \sin 30°$

$= \left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right) - \left(-\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{\sqrt{6} + \sqrt{2}}{4}$

(b) $\sin 135° - \cos 30° = \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{2} - \sqrt{3}}{2}$

10. (a) $\cos(120° + 45°) = \cos 120° \cos 45° - \sin 120° \sin 45°$

$= \left(-\dfrac{1}{2}\right)\left(\dfrac{\sqrt{2}}{2}\right) - \left(\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{-\sqrt{2} - \sqrt{6}}{4}$

(b) $\cos 120° + \cos 45° = -\dfrac{1}{2} + \dfrac{\sqrt{2}}{2} = \dfrac{-1 + \sqrt{2}}{2}$

11. $\sin\dfrac{11\pi}{12} = \sin\left(\dfrac{3\pi}{4} + \dfrac{\pi}{6}\right)$

$= \sin\dfrac{3\pi}{4}\cos\dfrac{\pi}{6} + \cos\dfrac{3\pi}{4}\sin\dfrac{\pi}{6}$

$= \dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{3}}{2} + \left(-\dfrac{\sqrt{2}}{2}\right)\dfrac{1}{2}$

$= \dfrac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$

$\cos\dfrac{11\pi}{12} = \cos\left(\dfrac{3\pi}{4} + \dfrac{\pi}{6}\right)$

$= \cos\dfrac{3\pi}{4}\cos\dfrac{\pi}{6} - \sin\dfrac{3\pi}{4}\sin\dfrac{\pi}{6}$

$= -\dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{2}}{2}\cdot\dfrac{1}{2} = -\dfrac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$

$\tan\dfrac{11\pi}{4} = \tan\left(\dfrac{3\pi}{4} + \dfrac{\pi}{6}\right)$

$= \dfrac{\tan\dfrac{3\pi}{4} + \tan\dfrac{\pi}{6}}{1 - \tan\dfrac{3\pi}{4}\tan\dfrac{\pi}{6}}$

$= \dfrac{-1 + \dfrac{\sqrt{3}}{3}}{1 - (-1)\dfrac{\sqrt{3}}{3}}$

$= \dfrac{-3 + \sqrt{3}}{3 + \sqrt{3}}\cdot\dfrac{3 - \sqrt{3}}{3 - \sqrt{3}}$

$= \dfrac{-12 + 6\sqrt{3}}{6} = -2 + \sqrt{3}$

12. $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$

$$\sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$
$$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos\frac{\pi}{3}$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$
$$= \frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$$\cos\frac{7\pi}{12} = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$
$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$
$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)$$

$$\tan\frac{7\pi}{12} = \tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$
$$= \frac{\tan\frac{\pi}{3} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{3}\tan\frac{\pi}{4}}$$
$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}}$$
$$= -2 - \sqrt{3}$$

13. $\sin\frac{17\pi}{12} = \sin\left(\frac{9\pi}{4} - \frac{5\pi}{6}\right)$

$$= \sin\frac{9\pi}{4}\cos\frac{5\pi}{6} - \cos\frac{9\pi}{4}\sin\frac{5\pi}{6}$$
$$= \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= -\frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$$\cos\frac{17\pi}{12} = \cos\left(\frac{9\pi}{4} - \frac{5\pi}{6}\right)$$
$$= \cos\frac{9\pi}{4}\cos\frac{5\pi}{6} + \sin\frac{9\pi}{4}\sin\frac{5\pi}{6}$$
$$= \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)$$

$$\tan\frac{17\pi}{12} = \tan\left(\frac{9\pi}{4} - \frac{5\pi}{6}\right)$$
$$= \frac{\tan(9\pi/4) - \tan(5\pi/6)}{1 + \tan(9\pi/4)\tan(5\pi/6)}$$
$$= \frac{1 - \left(-\sqrt{3}/3\right)}{1 + \left(-\sqrt{3}/3\right)}$$
$$= \frac{3 + \sqrt{3}}{3 - \sqrt{3}}\cdot\frac{3 + \sqrt{3}}{3 + \sqrt{3}}$$
$$= \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3}$$

14. $-\frac{\pi}{12} = \frac{\pi}{6} - \frac{\pi}{4}$

$$\sin\left(-\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{6} - \frac{\pi}{4}\right)$$
$$= \sin\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos\frac{\pi}{6}$$
$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)$$

$$\cos\left(-\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{6} - \frac{\pi}{4}\right)$$
$$= \cos\frac{\pi}{6}\cos\frac{\pi}{4} + \sin\frac{\pi}{6}\sin\frac{\pi}{4}$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$$\tan\left(-\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{6} - \frac{\pi}{4}\right)$$
$$= \frac{\tan\frac{\pi}{6} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{6}\tan\frac{\pi}{4}}$$
$$= \frac{\frac{\sqrt{3}}{3} - 1}{1 + \frac{\sqrt{3}}{3}}$$
$$= -2 + \sqrt{3}$$

15. $\sin 105° = \sin(60° + 45°)$
$= \sin 60° \cos 45° + \cos 60° \sin 45°$
$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$
$= \frac{\sqrt{2}}{4}(\sqrt{3} + 1)$

$\cos 105° = \cos(60° + 45°)$
$= \cos 60° \cos 45° - \sin 60° \sin 45°$
$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$
$= \frac{\sqrt{2}}{4}(1 - \sqrt{3})$

$\tan 105° = \tan(60° + 45°)$
$= \frac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°}$
$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}} = \frac{\sqrt{3} + 1}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}}$
$= \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$

16. $165° = 135° + 30°$

$\sin 165° = \sin(135° + 30°)$
$= \sin 135° \cos 30° + \sin 30° \cos 135°$
$= \sin 45° \cos 30° - \sin 30° \cos 45°$
$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$
$= \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$

$\cos 165° = \cos(135° + 30°)$
$= \cos 135° \cos 30° - \sin 135° \sin 30°$
$= -\cos 45° \cos 30° - \sin 45° \cos 30°$
$= -\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$
$= -\frac{\sqrt{2}}{4}(\sqrt{3} + 1)$

$\tan 165° = \tan(135° + 30°)$
$= \frac{\tan 135° + \tan 30°}{1 - \tan 135° \tan 30°}$
$= \frac{-\tan 45° + \tan 30°}{1 + \tan 45° \tan 30°}$
$= \frac{-1 + \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}$
$= -2 + \sqrt{3}$

17. $\sin(-195°) = \sin(30° - 225°)$
$= \sin 30° \cos 225° - \cos 30° \sin 225°$
$= \sin 30°(-\cos 45°) - \cos 30°(-\sin 45°)$
$= \frac{1}{2}\left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right)$
$= -\frac{\sqrt{2}}{4}(1 - \sqrt{3})$
$= \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$

$\cos(-195°) = \cos(30° - 225°)$
$= \cos 30° \cos 225° + \sin 30° \sin 225°$
$= \cos 30°(-\cos 45°) + \sin 30°(-45°)$
$= \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\left(-\frac{\sqrt{2}}{2}\right)$
$= -\frac{\sqrt{2}}{4}(\sqrt{3} + 1)$

$\tan(-195°) = \tan(30° - 225°)$
$= \frac{\tan 30° - \tan 225°}{1 + \tan 30° \tan 225°}$
$= \frac{\tan 30° - \tan 45°}{1 + \tan 30° \tan 45°}$
$= \frac{\left(\frac{\sqrt{3}}{3}\right) - 1}{1 + \left(\frac{\sqrt{3}}{3}\right)} = \frac{\sqrt{3} - 3}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}}$
$= \frac{-12 + 6\sqrt{3}}{6} = -2 + \sqrt{3}$

18. $225° = 300° - 45°$

$\sin 255° = \sin(300° - 45°)$
$= \sin 300° \cos 45° - \sin 45° \cos 300°$
$= -\sin 60° \cos 45° - \sin 45° \cos 60°$
$= -\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = -\frac{\sqrt{2}}{4}(\sqrt{3} + 1)$

$\cos 255° = \cos(300° - 45°)$
$= \cos 300° \cos 45° + \sin 300° \sin 45°$
$= \cos 60° \cos 45° - \sin 60° \sin 45°$
$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}(1 - \sqrt{3})$

$\tan 255° = \tan(300° - 45°)$
$= \frac{\tan 300° - \tan 45°}{1 + \tan 300° \tan 45°}$
$= \frac{-\tan 60° - \tan 45°}{1 - \tan 60° \tan 45°}$
$= \frac{-\sqrt{3} - 1}{1 - \sqrt{3}} = 2 + \sqrt{3}$

19. $\frac{13\pi}{12} = \frac{3\pi}{4} + \frac{\pi}{3}$

$$\sin\frac{13\pi}{12} = \sin\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)$$
$$= \sin\frac{3\pi}{4}\cos\frac{\pi}{3} + \cos\frac{3\pi}{4}\sin\frac{\pi}{3}$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)$$

$$\cos\frac{13\pi}{12} = \cos\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)$$
$$= \cos\frac{3\pi}{4}\cos\frac{\pi}{3} - \sin\frac{3\pi}{4}\sin\frac{\pi}{3}$$
$$= -\frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = -\frac{\sqrt{2}}{4}\left(1 + \sqrt{3}\right)$$

$$\tan\frac{13\pi}{12} = \tan\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)$$
$$= \frac{\tan\left(\frac{3\pi}{4}\right) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan\left(\frac{3\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)}$$
$$= \frac{-1 + \sqrt{3}}{1 - (-1)\left(\sqrt{3}\right)}$$
$$= -\frac{1 - \sqrt{3}}{1 + \sqrt{3}}\cdot\frac{1 - \sqrt{3}}{1 - \sqrt{3}}$$
$$= -\frac{4 - 2\sqrt{3}}{-2}$$
$$= 2 - \sqrt{3}$$

20. $\frac{19\pi}{12} = \frac{\pi}{3} + \frac{5\pi}{4}$

$$\sin\frac{19\pi}{12} = \sin\left(\frac{\pi}{3} + \frac{5\pi}{4}\right)$$
$$= \sin\frac{\pi}{3}\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4}\cos\frac{\pi}{3}$$
$$= \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{1}{2}$$
$$= -\frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$$\cos\frac{19\pi}{12} = \cos\left(\frac{\pi}{3} + \frac{5\pi}{4}\right)$$
$$= \cos\frac{\pi}{3}\cos\frac{5\pi}{4} - \sin\frac{\pi}{3}\sin\frac{5\pi}{4}$$
$$= \frac{1}{2}\left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{4}\left(-1 + \sqrt{3}\right)$$

$$\tan\frac{19\pi}{12} = \tan\left(\frac{\pi}{3} + \frac{5\pi}{4}\right)$$
$$= \frac{\tan\frac{\pi}{3} + \tan\frac{5\pi}{4}}{1 - \tan\frac{\pi}{3}\tan\frac{5\pi}{4}}$$
$$= \frac{\tan\frac{\pi}{3} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{3}\tan\frac{\pi}{4}}$$
$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}}\cdot\frac{1 + \sqrt{3}}{1 + \sqrt{3}}$$
$$= \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$$

21. $-\frac{5\pi}{12} = -\frac{\pi}{4} - \frac{\pi}{6}$

$$\sin\left(-\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{4}\right)\cos\frac{\pi}{6} - \cos\left(-\frac{\pi}{4}\right)\sin\frac{\pi}{6}$$

$$= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = -\frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$$\cos\left(-\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\left(-\frac{\pi}{4}\right)\cos\frac{\pi}{6} + \sin\left(-\frac{\pi}{4}\right)\sin\frac{\pi}{6}$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

$$\tan\left(-\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\tan\left(-\frac{\pi}{4}\right) - \tan\frac{\pi}{6}}{1 + \tan\left(-\frac{\pi}{4}\right)\tan\frac{\pi}{6}}$$

$$= \frac{-1 - \frac{\sqrt{3}}{3}}{1 + (-1)\left(\frac{\sqrt{3}}{3}\right)} = \frac{-3 - \sqrt{3}}{3 - \sqrt{3}}$$

$$= \frac{-3 - \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}}$$

$$= \frac{-12 - 6\sqrt{3}}{6} = -2 - \sqrt{3}$$

22. $-\frac{7\pi}{12} = -\frac{\pi}{3} - \frac{\pi}{4}$

$$\sin\left(-\frac{7\pi}{12}\right) = \sin\left(-\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\left(-\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(-\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$$\cos\left(-\frac{7\pi}{12}\right) = \cos\left(-\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left(-\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)$$

$$\tan\left(-\frac{7\pi}{12}\right) = \tan\left(-\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\left(-\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\left(-\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)} = \frac{-\sqrt{3} - 1}{1 + \left(-\sqrt{3}\right)(1)} = 2 + \sqrt{3}$$

23. $285° = 225° + 60°$

$$\sin 285° = \sin(225° + 60°) = \sin 225° \cos 60° + \cos 225° \sin 60°$$

$$= -\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$$\cos 285° = \cos(225° + 60°) = \cos 225° \cos 60° - \sin 225° \sin 60°$$

$$= -\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

$$\tan 285° = \tan(225° + 60°) = \frac{\tan 225° + \tan 60°}{1 - \tan 225° \tan 60°}$$

$$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}} = \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3} = -\left(2 + \sqrt{3}\right)$$

24. $15° = 45° - 30°$

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{2}\left(\sqrt{3} - 1\right)}{4} = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

$$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{2}\left(\sqrt{3} + 1\right)}{4} = \frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$$

$$\tan 15° = \tan(45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + (1)\left(\frac{\sqrt{3}}{3}\right)} = \frac{\frac{3 - \sqrt{3}}{3}}{\frac{3 + \sqrt{3}}{3}} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}} = \frac{12 - 6\sqrt{3}}{6} = 2 - \sqrt{3}$$

25. $-165° = -(120° + 45°)$

$$\sin(-165°) = \sin\left[-(120° + 45°)\right] = -\sin(120° + 45°) = -\left[\sin 120° \cos 45° + \cos 120° \sin 45°\right]$$

$$= -\left[\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}\right] = -\frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

$$\cos(-165°) = \cos\left[-(120° + 45°)\right] = \cos(120° + 45°) = \cos 120° \cos 45° - \sin 120° \sin 45°$$

$$= -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{4}\left(1 + \sqrt{3}\right)$$

$$\tan(-165°) = \tan\left[-(120° + 45°)\right] = -\tan(120° + \tan 45°) = -\frac{\tan 120° + \tan 45°}{1 - \tan 120° \tan 45°}$$

$$= -\frac{-\sqrt{3} + 1}{1 - \left(-\sqrt{3}\right)(1)} = -\frac{1 - \sqrt{3}}{1 + \sqrt{3}} \cdot \frac{1 - \sqrt{3}}{1 - \sqrt{3}} = -\frac{4 - 2\sqrt{3}}{-2} = 2 - \sqrt{3}$$

26. $-105 = 30° - 135°$

$\sin(30° - 135°) = \sin 30° \cos 135° - \cos 30° \sin 135° = \sin 30°(-\cos 45°) - \cos 30° \sin 45°$

$$= \left(\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{4}\left(1 + \sqrt{3}\right)$$

$\cos(30° - 135°) = \cos 30° \cos 135° + \sin 30° \sin 135° = \cos 30°(-\cos 45°) + \sin 30° \sin 45°$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}\left(1 - \sqrt{3}\right)$$

$$\tan(30° - 135°) = \frac{\tan 30° - \tan 135°}{1 + \tan 30° \tan 135°} = \frac{\tan 30° - (-\tan 45°)}{1 + \tan 30°(-\tan 45°)}$$

$$= \frac{\frac{\sqrt{3}}{3} - (-1)}{1 + \left(\frac{\sqrt{3}}{3}\right)(-1)} = 2 + \sqrt{3}$$

27. $\sin 3 \cos 1.2 - \cos 3 \sin 1.2 = \sin(3 - 1.2) = \sin 1.8$

28. $\cos \frac{\pi}{7} \cos \frac{\pi}{5} - \sin \frac{\pi}{7} \sin \frac{\pi}{5} = \cos\left(\frac{\pi}{7} + \frac{\pi}{5}\right)$

$$= \cos\frac{12\pi}{35}$$

29. $\sin 60° \cos 15° + \cos 60° \sin 15° = \sin(60° + 15°)$

$$= \sin 75°$$

30. $\cos 130° \cos 40° - \sin 130° \sin 40° = \cos(130° + 40°)$

$$= \cos 170°$$

31. $\dfrac{\tan(\pi/15) + \tan(2\pi/5)}{1 - \tan(\pi/15)\tan(2\pi/5)} = \tan(\pi/15 + 2\pi/5)$

$$= \tan(7\pi/15)$$

32. $\dfrac{\tan 1.1 - \tan 4.6}{1 + \tan 1.1 \tan 4.6} = \tan(1.1 - 4.6) = \tan(-3.5)$

33. $\cos 3x \cos 2y + \sin 3x \sin 2y = \cos(3x - 2y)$

34. $\sin x \cos 2x + \cos x \sin 2x = \sin(x + 2x) = \sin(3x)$

35. $\sin \frac{\pi}{12} \cos \frac{\pi}{4} + \cos \frac{\pi}{12} \sin \frac{\pi}{4} = \sin\left(\frac{\pi}{12} + \frac{\pi}{4}\right)$

$$= \sin \frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2}$$

36. $\cos \frac{\pi}{16} \cos \frac{3\pi}{16} - \sin \frac{\pi}{16} \sin \frac{3\pi}{16} = \cos\left(\frac{\pi}{16} + \frac{3\pi}{16}\right)$

$$= \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

37. $\cos 130° \cos 10° + \sin 130° \sin 10° = \cos(130° - 10°)$

$$= \cos 120°$$

$$= -\frac{1}{2}$$

38. $\sin 100° \cos 40° - \cos 100° \sin 40° = \sin(100° - 40°)$

$$= \sin 60°$$

$$= \frac{\sqrt{3}}{2}$$

39. $\dfrac{\tan(9\pi/8) - \tan(\pi/8)}{1 + \tan(9\pi/8)\tan(\pi/8)} = \tan\left(\frac{9\pi}{8} - \frac{\pi}{8}\right)$

$$= \tan \pi$$

$$= 0$$

40. $\dfrac{\tan 25° + \tan 110°}{1 - \tan 25° \tan 110°} = \tan(25° + 110°)$

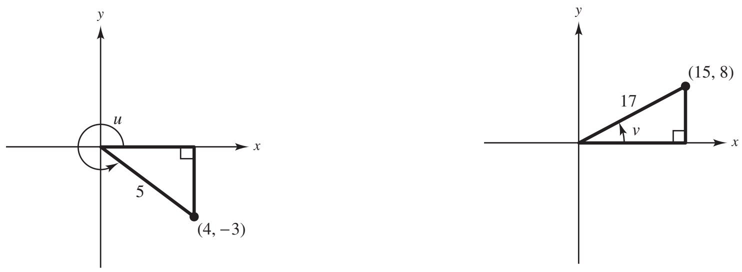
$$= \tan 135°$$

$$= -1$$

For Exercises 41–46, you have:

$\sin u = -\frac{3}{5}$, u in Quadrant IV $\Rightarrow \cos u = \frac{4}{5}$, $\tan u = -\frac{4}{3}$

$\cos v = \frac{15}{17}$, v in Quadrant I $\Rightarrow \sin v = \frac{8}{17}$, $\tan v = \frac{8}{15}$



Figures for Exercises 41–46

41. $\sin(u+v) = \sin u \cos v + \cos u \sin v$

$$= \left(-\frac{3}{5}\right)\left(\frac{15}{17}\right) + \left(\frac{4}{5}\right)\left(\frac{8}{17}\right)$$

$$= -\frac{13}{85}$$

42. $\cos(u-v) = \cos u \cos v + \sin u \sin v$

$$= \left(\frac{4}{5}\right)\left(\frac{15}{17}\right) + \left(-\frac{3}{5}\right)\left(\frac{8}{17}\right)$$

$$= \frac{60}{85} + \frac{-24}{85} = \frac{36}{85}$$

43. $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v} = \dfrac{-\frac{3}{4} + \left(\frac{8}{15}\right)}{1 - \left(-\frac{3}{4}\right)\left(\frac{8}{15}\right)}$

$$= \frac{-\frac{13}{60}}{1 + \frac{32}{60}} = \left(-\frac{13}{60}\right)\left(\frac{5}{7}\right) = -\frac{13}{84}$$

44. $\csc(u-v) = \dfrac{1}{\sin(u-v)} = \dfrac{1}{\sin u \cos v - \cos u \sin v}$

$$= \frac{1}{\left(-\frac{3}{5}\right)\left(\frac{15}{17}\right) - \left(\frac{4}{5}\right)\left(\frac{8}{17}\right)}$$

$$= \frac{1}{-\frac{77}{85}} = -\frac{85}{77}$$

45. $\sec(v-u) = \dfrac{1}{\cos(v-u)} = \dfrac{1}{\cos v \cos u + \sin v \sin u}$

$$= \frac{1}{\left(\frac{15}{17}\right)\left(\frac{4}{5}\right) + \left(\frac{8}{17}\right)\left(-\frac{3}{5}\right)} = \frac{1}{\left(\frac{60}{85}\right) + \left(-\frac{24}{85}\right)}$$

$$= \frac{1}{\frac{36}{85}} = \frac{85}{36}$$

46. $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v} = \dfrac{\left(-\frac{3}{4}\right) + \left(\frac{8}{15}\right)}{1 - \left(-\frac{3}{4}\right)\left(\frac{8}{15}\right)}$

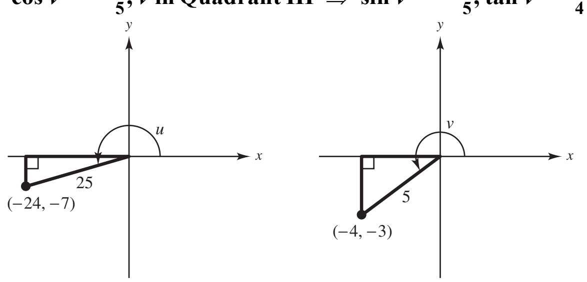
$$= \frac{-\frac{13}{60}}{\frac{7}{5}} = -\frac{13}{84}$$

$$\cot(u+v) = \frac{1}{\tan(u+v)} = \frac{1}{-\frac{13}{84}} = -\frac{84}{13}$$

For Exercises 47–52, you have:

$\sin u = -\frac{7}{25}$, u in Quadrant III $\Rightarrow \cos u = -\frac{24}{25}$, $\tan u = \frac{7}{24}$

$\cos v = -\frac{4}{5}$, v in Quadrant III $\Rightarrow \sin v = -\frac{3}{5}$, $\tan v = \frac{3}{4}$



Figures for Exercises 47–52

47. $\cos(u+v) = \cos u \cos v - \sin u \sin v$

$= \left(-\frac{24}{25}\right)\left(-\frac{4}{5}\right) - \left(-\frac{7}{25}\right)\left(-\frac{3}{5}\right)$

$= \frac{3}{5}$

48. $\sin(u+v) = \sin u \cos v + \cos u \sin v$

$= \left(-\frac{7}{25}\right)\left(-\frac{4}{5}\right) + \left(-\frac{24}{25}\right)\left(-\frac{3}{5}\right)$

$= \frac{28}{125} + \frac{72}{125} = \frac{100}{125} = \frac{4}{5}$

49. $\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

$= \dfrac{\frac{7}{24} - \frac{3}{4}}{1 + \left(\frac{7}{24}\right)\left(\frac{3}{4}\right)} = \dfrac{-\frac{11}{24}}{\frac{39}{32}} = -\dfrac{44}{117}$

50. $\tan(v-u) = \dfrac{\tan v - \tan u}{1 + \tan v \tan u} = \dfrac{\left(\frac{3}{4}\right) - \left(\frac{7}{24}\right)}{1 + \left(\frac{3}{4}\right)\left(\frac{7}{24}\right)}$

$= \dfrac{\frac{11}{24}}{\frac{39}{32}} = \dfrac{44}{117}$

$\cot(v-u) = \dfrac{1}{\tan(v-u)} = \dfrac{1}{\frac{44}{117}} = \dfrac{117}{44}$

51. $\csc(u-v) = \dfrac{1}{\sin(u-v)} = \dfrac{1}{\sin u \cos v - \cos u \sin v}$

$= \dfrac{1}{\left(-\frac{7}{25}\right)\left(-\frac{4}{5}\right) - \left(-\frac{24}{25}\right)\left(-\frac{3}{5}\right)}$

$= \dfrac{1}{-\frac{44}{125}}$

$= -\dfrac{125}{44}$

52. $\sec(v-u) = \dfrac{1}{\cos(v-u)}$

$= \dfrac{1}{\cos v \cos u + \sin v \sin u}$

$= \dfrac{1}{\left(-\frac{4}{5}\right)\left(-\frac{24}{25}\right) + \left(-\frac{3}{5}\right)\left(-\frac{7}{25}\right)}$

$= \dfrac{1}{\frac{117}{125}}$

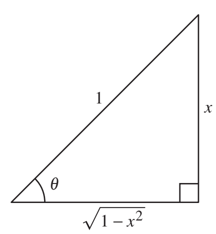
$= \dfrac{125}{117}$

53. $\sin(\arcsin x + \arccos x) = \sin(\arcsin x)\cos(\arccos x) + \sin(\arccos x)\cos(\arcsin x)$

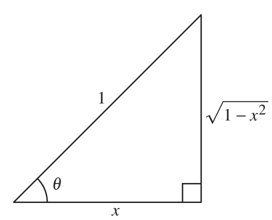
$= x \cdot x + \sqrt{1-x^2} \cdot \sqrt{1-x^2}$

$= x^2 + 1 - x^2$

$= 1$



$\theta = \arcsin x$



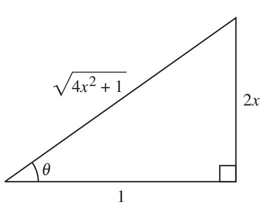
$\theta = \arccos x$

54. $\sin(\arctan 2x - \arccos x) = \sin(\arctan 2x - \arccos x)$

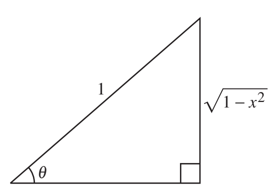
$= \sin(\arctan 2x)\cos(\arccos x) - \cos(\arctan 2x)\sin(\arccos x)$

$= \dfrac{2x}{\sqrt{4x^2+1}}(x) - \dfrac{1}{\sqrt{4x^2+1}}\left(\sqrt{1-x^2}\right)$

$= \dfrac{2x^2 - \sqrt{1-x^2}}{\sqrt{4x^2+1}}$



$\theta = \arctan 2x$



$\theta = \arccos x$

55. $\cos(\arccos x + \arcsin x) = \cos(\arccos x)\cos(\arcsin x) - \sin(\arccos x)\sin(\arcsin x)$

$$= x \cdot \sqrt{1 - x^2} - \sqrt{1 - x^2} \cdot x$$

$$= 0$$

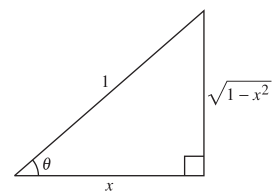
(Use the triangles in Exercise 53.)

56. $\cos(\arccos x - \arctan x) = \cos(\arccos x - \arctan x)$

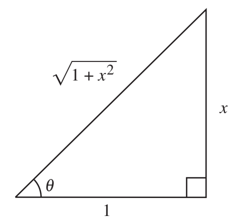
$$= \cos(\arccos x)\cos(\arctan x) + \sin(\arccos x)\sin(\arctan x)$$

$$= (x)\left(\frac{1}{\sqrt{1 + x^2}}\right) + \left(\sqrt{1 - x^2}\right)\left(\frac{x}{\sqrt{1 + x^2}}\right)$$

$$= \frac{x + x\sqrt{1 - x^2}}{\sqrt{1 + x^2}}$$



$\theta = \arccos x$



$\theta = \arctan x$

57. $\sin\left(\frac{\pi}{2} - x\right) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$

$$= (1)(\cos x) - (0)(\sin x)$$

$$= \cos x$$

58. $\sin\left(\frac{\pi}{2} + x\right) = \sin\frac{\pi}{2}\cos x + \sin x \cos\frac{\pi}{2}$

$$= (1)(\cos x) + (\sin x)(0)$$

$$= \cos x$$

59. $\sin\left(\frac{\pi}{6} + x\right) = \sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x$

$$= \frac{1}{2}\left(\cos x + \sqrt{3}\sin x\right)$$

60. $\cos\left(\frac{5\pi}{4} - x\right) = \cos\frac{5\pi}{4}\cos x + \sin\frac{5\pi}{4}\sin x$

$$= -\frac{\sqrt{2}}{2}(\cos x + \sin x)$$

61. $\tan(\theta + \pi) = \dfrac{\tan\theta + \tan\pi}{1 - \tan\theta\tan\pi} = \dfrac{\tan\theta + 0}{1 - (\tan\theta)(0)} = \dfrac{\tan\theta}{1} = \tan\theta$

62. $\tan\left(\frac{\pi}{4} - \theta\right) = \dfrac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta} = \dfrac{1 - \tan\theta}{1 + \tan\theta}$

63. $\cos(\pi - \theta) + \sin\left(\frac{\pi}{2} + \theta\right) = \cos\pi\cos\theta + \sin\pi\sin\theta + \sin\frac{\pi}{2}\cos\theta + \cos\frac{\pi}{2}\sin\theta$

$$= (-1)(\cos\theta) + (0)(\sin\theta) + (1)(\cos\theta) + (\sin\theta)(0)$$

$$= -\cos\theta + \cos\theta$$

$$= 0$$

64. $\cos(x + y)\cos(x - y) = (\cos x\cos y - \sin x\sin y)(\cos x\cos y + \sin x\sin y)$

$$= \cos^2 x\cos^2 y - \sin^2 x\sin^2 y$$

$$= \cos^2 x\left(1 - \sin^2 y\right) - \sin^2 x\sin^2 y$$

$$= \cos^2 x - \cos^2 x\sin^2 y - \sin^2 x\sin^2 y$$

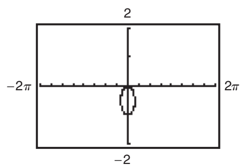
$$= \cos^2 x - \sin^2 y\left(\cos^2 x + \sin^2 x\right)$$

$$= \cos^2 x - \sin^2 y$$

65. $\cos\left(\frac{3\pi}{2} - \theta\right) = \cos\frac{3\pi}{2}\cos\theta + \sin\frac{3\pi}{2}\sin\theta$

$= (0)(\cos\theta) + (-1)(\sin\theta)$

$= -\sin\theta$

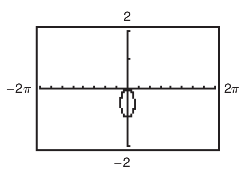


The graphs appear to coincide, so $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta.$

66. $\sin(\pi + \theta) = \sin\pi\cos\theta + \cos\pi\sin\theta$

$= (0)\cos\theta + (-1)\sin\theta$

$= -\sin\theta$



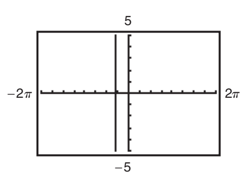
The graphs appear to coincide, so $\sin(\pi + \theta) = -\sin(\theta).$

67. $\sin\left(\frac{3\pi}{2} + \theta\right) = \sin\frac{3\pi}{2}\cos\theta + \cos\frac{3\pi}{2}\sin\theta$

$= (-1)(\cos\theta) + (0)(\sin\theta)$

$= -\cos\theta$

$\csc\left(\frac{3\pi}{2} + \theta\right) = \frac{1}{\sin\left(\frac{3\pi}{2} + \theta\right)} = \frac{1}{-\cos\theta} = -\sec\theta$



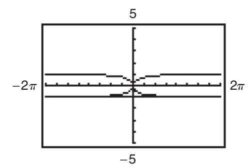
The graphs appear to coincide, so $\csc\left(\frac{3\pi}{2} + \theta\right) = -\sec\theta.$

68. $\tan(\pi + \theta) = \frac{\tan\pi + \tan\theta}{1 - \tan\pi\tan\theta}$

$= \frac{0 + \tan\theta}{1 - (0)\tan\theta}$

$= \tan\theta$

$\cot(\pi + \theta) = \frac{1}{\tan(\pi + \theta)} = \frac{1}{\tan\theta} = \cot\theta$



The graphs appear to coincide, so $\cot(\pi + \theta) = \cot\theta$

69.

$$\begin{aligned}
\sin(x + \pi) - \sin x + 1 &= 0 \\
\sin x\cos\pi + \cos x\sin\pi - \sin x + 1 &= 0 \\
(\sin x)(-1) + (\cos x)(0) - \sin x + 1 &= 0 \\
-2\sin x + 1 &= 0 \\
\sin x &= \frac{1}{2} \\
x &= \frac{\pi}{6}, \frac{5\pi}{6}
\end{aligned}$$

70.

$$\begin{aligned}
\cos(x + \pi) - \cos x - 1 &= 0 \\
\cos x\cos\pi - \sin x\sin\pi - \cos x - 1 &= 0 \\
(\cos x)(-1) - (\sin x)(0) - \cos x - 1 &= 0 \\
-2\cos x - 1 &= 0 \\
\cos x &= -\frac{1}{2} \\
x &= \frac{2\pi}{3}, \frac{4\pi}{3}
\end{aligned}$$

71.

$$\cos\left(x + \frac{\pi}{4}\right) - \cos\left(x - \frac{\pi}{4}\right) = 1$$

$$\cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4} - \left(\cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4}\right) = 1$$

$$-2 \sin x\left(\frac{\sqrt{2}}{2}\right) = 1$$

$$-\sqrt{2} \sin x = 1$$

$$\sin x = -\frac{1}{\sqrt{2}}$$

$$\sin x = -\frac{\sqrt{2}}{2}$$

$$x = \frac{5\pi}{4}, \frac{7\pi}{4}$$

72.

$$\sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{7\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\sin x \cos \frac{\pi}{6} + \cos x \sin \frac{\pi}{6} - \left(\sin x \cos \frac{7\pi}{6} - \cos x \sin \frac{7\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$(\sin x)\left(\frac{\sqrt{3}}{2}\right) + (\cos x)\left(\frac{1}{2}\right) - (\sin x)\left(-\frac{\sqrt{3}}{2}\right) + (\cos x)\left(-\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$$

$$\sqrt{3} \sin x = \frac{\sqrt{3}}{2}$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

73.

$$\tan(x + \pi) + 2 \sin(x + \pi) = 0$$

$$\frac{\tan x + \tan \pi}{1 - \tan x \tan \pi} + 2(\sin x \cos \pi + \cos x \sin \pi) = 0$$

$$\frac{\tan x + 0}{1 - \tan x(0)} + 2\left[\sin x(-1) + \cos x(0)\right] = 0$$

$$\frac{\tan x}{1} - 2 \sin x = 0$$

$$\frac{\sin x}{\cos x} = 2 \sin x$$

$$\sin x = 2 \sin x \cos x$$

$$\sin x(1 - 2 \cos x) = 0$$

$$\sin x = 0 \quad \text{or} \quad \cos x = \frac{1}{2}$$

$$x = 0, \pi \qquad x = \frac{\pi}{3}, \frac{5\pi}{3}$$

74. $$\sin\left(x + \frac{\pi}{2}\right) - \cos^2 x = 0$$

$$\sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2} - \cos^2 x = 0$$

$$(\sin x)(0) + (\cos x)(1) - \cos^2 x = 0$$

$$\cos x - \cos^2 x = 0$$

$$\cos x(1 - \cos x) = 0$$

$\cos x = 0$ or $1 - \cos x = 0$

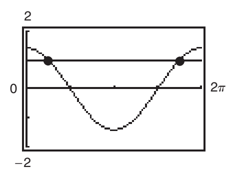
$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad \cos x = 1$

$x = 0$

75. $\cos\left(x + \frac{\pi}{4}\right) + \cos\left(x - \frac{\pi}{4}\right) = 1$

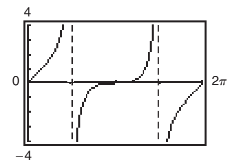
Graph $y_1 = \cos\left(x + \frac{\pi}{4}\right) + \cos\left(x - \frac{\pi}{4}\right)$ and $y_2 = 1$.

$x = \frac{\pi}{4}, \frac{7\pi}{4}$

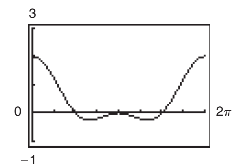


76. $\tan(x + \pi) - \cos\left(x + \frac{\pi}{2}\right) = 0$

$x = 0, \pi$

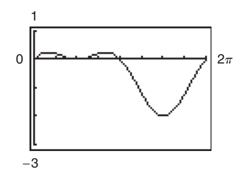


77. $\sin\left(x + \frac{\pi}{2}\right) + \cos^2 x = 0$



$x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

78. $\cos\left(x - \frac{\pi}{2}\right) - \sin^2 x = 0$



$x = 0, \frac{\pi}{2}, \pi$

79. $y = \frac{1}{3}\sin 2t + \frac{1}{4}\cos 2t$

(a) $a = \frac{1}{3}, b = \frac{1}{4}, B = 2$

$$C = \arctan\frac{b}{a} = \arctan\frac{3}{4} \approx 0.6435$$

$$y \approx \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2}\sin(2t + 0.6435) = \frac{5}{12}\sin(2t + 0.6435)$$

(b) Amplitude: $\frac{5}{12}$ feet

(c) Frequency: $\frac{1}{\text{period}} = \frac{B}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}$ cycle per second

80.
$$y_1 = A\cos 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$
$$y_2 = A\cos 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$
$$y_1 + y_2 = A\cos 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + A\cos 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$
$$y_1 + y_2 = A\left[\cos 2\pi\frac{t}{T}\cos 2\pi\frac{x}{\lambda} + \sin 2\pi\frac{t}{T}\sin 2\pi\frac{x}{\lambda}\right] + A\left[\cos 2\pi\frac{t}{T}\cos 2\pi\frac{x}{\lambda} - \sin 2\pi\frac{t}{T}\sin 2\pi\frac{x}{\lambda}\right] = 2A\cos 2\pi\frac{t}{T}\cos 2\pi\frac{x}{\lambda}$$

81. True.

$\sin(u + v) = \sin u\cos v + \cos u\sin v$

$\sin(u - v) = \sin u\cos v - \cos u\sin v$

So, $\sin(u \pm v) = \sin u\cos v \pm \cos u\sin v$.

82. False.

$\cos(u + v) = \cos u\cos v - \sin u\sin v$

$\cos(u - v) = \cos u\cos v + \sin u\sin v$

So, $\cos(u \pm v) = \cos u\cos v \mp \sin u\sin v$.

83.
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha = 0$$
$$\sin\alpha\cos\beta + \sin\beta\cos\alpha = 0$$
$$\sin\alpha\cos\beta = -\sin\beta\cos\alpha$$

False. When α and β are supplementary, $\sin\alpha\cos\beta = -\cos\alpha\sin\beta$.

84.
$$\begin{aligned}\cos(A + B) &= \cos(180° - C)\\ &= \cos(180°)\cos(C) + \sin(180°)\sin(C)\\ &= (-1)\cos(C) + (0)\sin(C)\\ &= -\cos(C)\end{aligned}$$

True. $\cos(A + B) = -\cos C$. When A, B and C form ΔABC, $A + B + C = 180°$, so $A + B = 180° - C$.

85. The denominator should be $1 + \tan x\tan(\pi/4)$.
$$\tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - \tan(\pi/4)}{1 + \tan x\tan(\pi/4)} = \frac{\tan x - 1}{1 + \tan x}$$

86. (a) Using the graph, $\sin(u + v) \approx 0$ and $\sin u + \sin v \approx 0.7 + 0.7 = 1.4$. Because $0 \ne 1.4$, $\sin(u + v) \ne \sin u + \sin v$.

(b) Using the graph, $\sin(u - v) \approx -1$ and $\sin u - \sin v \approx 0.7 - 0.7 = 0$. Because $-1 \ne 0$, $\sin(u - v) \ne \sin u - \sin v$.

87.
$$\begin{aligned}\cos(n\pi + \theta) &= \cos n\pi\cos\theta - \sin n\pi\sin\theta\\ &= (-1)^n(\cos\theta) - (0)(\sin\theta)\\ &= (-1)^n(\cos\theta),\text{ where } n \text{ is an integer.}\end{aligned}$$

88.
$$\begin{aligned}\sin(n\pi + \theta) &= \sin n\pi\cos\theta + \sin\theta\cos n\pi\\ &= (0)(\cos\theta) + (\sin\theta)(-1)^n\\ &= (-1)^n(\sin\theta),\text{ where } n \text{ is an integer.}\end{aligned}$$

89. $C = \arctan\dfrac{b}{a} \Rightarrow \sin C = \dfrac{b}{\sqrt{a^2 + b^2}}, \cos C = \dfrac{a}{\sqrt{a^2 + b^2}}$
$$\sqrt{a^2 + b^2}\sin(B\theta + C) = \sqrt{a^2 + b^2}\left(\sin B\theta\cdot\frac{a}{\sqrt{a^2 + b^2}} + \frac{b}{\sqrt{a^2 + b^2}}\cdot\cos B\theta\right) = a\sin B\theta + b\cos B\theta$$

90. $C = \arctan\dfrac{a}{b} \Rightarrow \sin C = \dfrac{a}{\sqrt{a^2 + b^2}}, \cos C = \dfrac{b}{\sqrt{a^2 + b^2}}$
$$\begin{aligned}\sqrt{a^2 + b^2}\cos(B\theta - C) &= \sqrt{a^2 + b^2}\left(\cos B\theta\cdot\frac{b}{\sqrt{a^2 + b^2}} + \sin B\theta\cdot\frac{a}{\sqrt{a^2 + b^2}}\right)\\ &= b\cos B\theta + a\sin B\theta = a\sin B\theta + b\cos B\theta\end{aligned}$$

91. $\sin\theta + \cos\theta$

$a = 1, b = 1, B = 1$

(a) $C = \arctan\frac{b}{a} = \arctan 1 = \frac{\pi}{4}$

$$\sin\theta + \cos\theta = \sqrt{a^2 + b^2}\sin(B\theta + C) = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$$

(b) $C = \arctan\frac{a}{b} = \arctan 1 = \frac{\pi}{4}$

$$\sin\theta + \cos\theta = \sqrt{a^2 + b^2}\cos(B\theta - C) = \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$$

92. $3\sin 2\theta + 4\cos 2\theta$

$a = 3, b = 4, B = 2$

(a) $C = \arctan\frac{b}{a} = \arctan\frac{4}{3} \approx 0.9273$

$$3\sin 2\theta + 4\cos 2\theta = \sqrt{a^2 + b^2}\sin(B\theta + C) \approx 5\sin(2\theta + 0.9273)$$

(b) $C = \arctan\frac{a}{b} = \arctan\frac{3}{4} \approx 0.6435$

$$3\sin 2\theta + 4\cos 2\theta = \sqrt{a^2 + b^2}\cos(B\theta - C) \approx 5\cos(2\theta - 0.6435)$$

93. $12\sin 3\theta + 5\cos 3\theta$

$a = 12, b = 5, B = 3$

(a) $C = \arctan\frac{b}{a} = \arctan\frac{5}{12} \approx 0.3948$

$$12\sin 3\theta + 5\cos 3\theta = \sqrt{a^2 + b^2}\sin(B\theta + C) \approx 13\sin(3\theta + 0.3948)$$

(b) $C = \arctan\frac{a}{b} = \arctan\frac{12}{5} \approx 1.1760$

$$12\sin 3\theta + 5\cos 3\theta = \sqrt{a^2 + b^2}\cos(B\theta - C) \approx 13\cos(3\theta - 1.1760)$$

94. $\sin 2\theta + \cos 2\theta$

$a = 1, b = 1, B = 2$

(a) $C = \arctan\frac{b}{a} = \arctan(1) = \frac{\pi}{4}$

$$\sin 2\theta + \cos 2\theta = \sqrt{a^2 + b^2}\sin(B\theta + C) = \sqrt{2}\sin\left(2\theta + \frac{\pi}{4}\right)$$

(b) $C = \arctan\frac{a}{b} = \arctan(1) = \frac{\pi}{4}$

$$\sin 2\theta + \cos 2\theta = \sqrt{a^2 + b^2}\cos(B\theta - C) = \sqrt{2}\cos\left(2\theta - \frac{\pi}{4}\right)$$

95. $C = \arctan\frac{b}{a} = \frac{\pi}{4} \Rightarrow a = b, a > 0, b > 0$

$\sqrt{a^2 + b^2} = 2 \Rightarrow a = b = \sqrt{2}$

$B = 1$

$2\sin\left(\theta + \frac{\pi}{4}\right) = \sqrt{2}\sin\theta + \sqrt{2}\cos\theta$

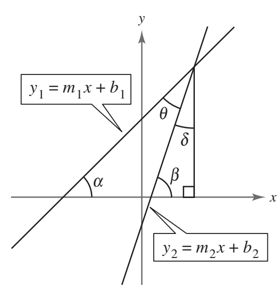
96. $C = \arctan\frac{b}{a} = \frac{\pi}{4} \Rightarrow a = b, a > 0, b > 0$

$\sqrt{a^2 + b^2} = 5 \Rightarrow a = b = \frac{5\sqrt{2}}{2}$

$B = 1$

$5\cos\left(\theta - \frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}\sin\theta + \frac{5\sqrt{2}}{2}\cos\theta$

97.



$m_1 = \tan\alpha$ and $m_2 = \tan\beta$

$\beta + \delta = 90° \Rightarrow \delta = 90° - \beta$

$$\alpha + \theta + \delta = 90° \Rightarrow \alpha + \theta + (90° - \beta) = 90° \Rightarrow \theta = \beta - \alpha$$

So, $\theta = \arctan m_2 - \arctan m_1$. For $y = x$ and $y = \sqrt{3}x$ you have $m_1 = 1$ and $m_2 = \sqrt{3}$.

$\theta = \arctan\sqrt{3} - \arctan 1 = 60° - 45° = 15°$

98. For $m_2 > m_1 > 0$, the angle θ between the lines is:

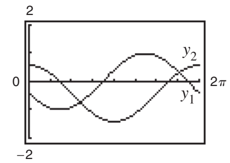
$$\theta = \arctan\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

$$m_2 = 1$$

$$m_1 = \frac{1}{\sqrt{3}}$$

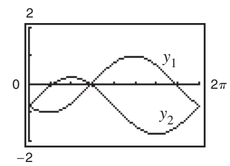
$$\theta = \arctan\left(\frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}\right) = \arctan\left(2 - \sqrt{3}\right) = 15°$$

99. $y_1 = \cos(x + 2),\ y_2 = \cos x + \cos 2$



No, $y_1 \neq y_2$ because their graphs are different.

100. $y_1 = \sin(x + 4),\ y_2 = \sin x + \sin 4$



No, $y_1 \neq y_2$ because their graphs are different.

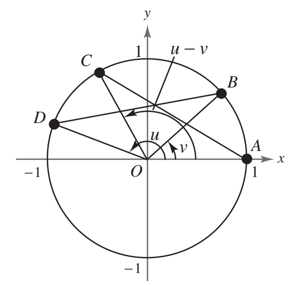
101. (a) To prove the identity for $\sin(u + v)$ you first need to prove the identity for $\cos(u - v)$.

Assume $0 < v < u < 2\pi$ and locate u, v, and $u - v$ on the unit circle.

The coordinates of the points on the circle are:

$A = (1, 0)$, $B = (\cos v, \sin v)$, $C = (\cos(u - v), \sin(u - v))$, and $D = (\cos u, \sin u)$.

Because $\angle DOB = \angle COA$, chords AC and BD are equal. By the Distance Formula:



$$\sqrt{\left[\cos(u - v) - 1\right]^2 + \left[\sin(u - v) - 0\right]^2} = \sqrt{(\cos u - \cos v)^2 + (\sin u - \sin v)^2}$$

$$\cos^2(u - v) - 2\cos(u - v) + 1 + \sin^2(u - v) = \cos^2 u - 2\cos u \cos v + \cos^2 v + \sin^2 u - 2\sin u \sin v + \sin^2 v$$

$$\left[\cos^2(u - v) + \sin^2(u - v)\right] + 1 - 2\cos(u - v) = \left(\cos^2 u + \sin^2 u\right) + \left(\cos^2 v + \sin^2 v\right) - 2\cos u \cos v - 2\sin u \sin v$$

$$2 - 2\cos(u - v) = 2 - 2\cos u \cos v - 2\sin u \sin v$$

$$-2\cos(u - v) = -2(\cos u \cos v + \sin u \sin v)$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

Now, to prove the identity for $\sin(u + v)$, use cofunction identities.

$$\sin(u + v) = \cos\left[\frac{\pi}{2} - (u + v)\right] = \cos\left[\left(\frac{\pi}{2} - u\right) - v\right]$$

$$= \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v$$

$$= \sin u \cos v + \cos u \sin v$$

(b) First, prove $\cos(u - v) = \cos u \cos v + \sin u \sin v$ using the figure containing points

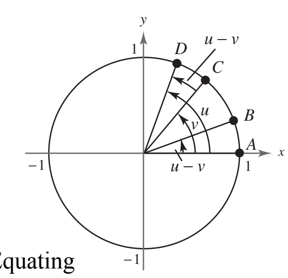
$A(1, 0)$

$B(\cos(u - v), \sin(u - v))$

$C(\cos v, \sin v)$

$D(\cos u, \sin u)$

on the unit circle.



Because chords AB and CD are each subtended by angle $u - v$, their lengths are equal. Equating $\left[d(A, B)\right]^2 = \left[d(C, D)\right]^2$ you have $(\cos(u - v) - 1)^2 + \sin^2(u - v) = (\cos u - \cos v)^2 + (\sin u - \sin v)^2$.

Simplifying and solving for $\cos(u - v)$, you have $\cos(u - v) = \cos u \cos v + \sin u \sin v$.

Using $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$,

$$\sin(u - v) = \cos\left[\frac{\pi}{2} - (u - v)\right] = \cos\left[\left(\frac{\pi}{2} - u\right) - (-v)\right] = \cos\left(\frac{\pi}{2} - u\right)\cos(-v) + \sin\left(\frac{\pi}{2} - u\right)\sin(-v)$$

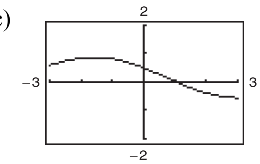
$$= \sin u \cos v - \cos u \sin v$$

102. (a) The domains of f and g are the same, all real numbers h, except $h = 0$.

(b)

h	0.5	0.2	0.1	0.05	0.02	0.01
$f(h)$	0.267	0.410	0.456	0.478	0.491	0.496
$g(h)$	0.267	0.410	0.456	0.478	0.491	0.496

(c)



(d) As $h \to 0^*$, $f \to 0.5$ and $g \to 0.5$.

Section 2.5 Multiple-Angle and Product-to-Sum Formulas

1. $2 \sin u \cos u$

2. $\cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$

3. $\frac{1}{2}\left[\sin(u + v) + \sin(u - v)\right]$

4. $\tan^2 u$

5. $\pm\sqrt{\frac{1 - \cos u}{2}}$

6. $-2 \sin\left(\frac{u + v}{2}\right)\sin\left(\frac{u - v}{2}\right)$

7.

$$\sin 2x - \sin x = 0$$
$$2 \sin x \cos x - \sin x = 0$$
$$\sin x(2 \cos x - 1) = 0$$
$$\sin x = 0 \quad \text{or} \quad 2\cos x - 1 = 0$$
$$x = n\pi \qquad \cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$$

8.

$$\sin 2x \sin x = \cos x$$
$$2 \sin x \cos x \sin x - \cos x = 0$$
$$\cos x(2\sin^2 x - 1) = 0$$
$$\cos x = 0 \quad \text{or} \quad 2\sin^2 x - 1 = 0$$
$$x = \frac{\pi}{2} + 2n\pi \qquad \sin^2 x = \frac{1}{2}$$
$$\sin x = \pm\frac{\sqrt{2}}{2}$$
$$x = \frac{\pi}{4} + \frac{n\pi}{2}$$

9.

$$\cos 2x - \cos x = 0$$
$$\cos 2x = \cos x$$
$$\cos^2 x - \sin^2 x = \cos x$$
$$\cos^2 x - (1 - \cos^2 x) - \cos x = 0$$
$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$
$$2\cos x + 1 = 0 \quad \text{or} \quad \cos x - 1 = 0$$
$$\cos x = -\frac{1}{2} \qquad \cos x = 1$$
$$x = \frac{2n\pi}{3} \qquad x = 0$$

10.

$$\cos 2x + \sin x = 0$$
$$1 - 2\sin^2 x + \sin x = 0$$
$$2\sin^2 x - \sin x - 1 = 0$$
$$(2\sin x + 1)(\sin x - 1) = 0$$
$$2\sin x + 1 = 0 \quad \text{or} \quad \sin x - 1 = 0$$
$$\sin x = -\frac{1}{2} \qquad \sin x = 1$$
$$x = \frac{7\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi \qquad x = \frac{\pi}{2} + 2n\pi$$

11.

$$\begin{aligned}
\sin 4x &= -2\sin 2x\\
\sin 4x + 2\sin 2x &= 0\\
2\sin 2x\cos 2x + 2\sin 2x &= 0\\
2\sin 2x(\cos 2x + 1) &= 0
\end{aligned}$$

$$\begin{aligned}
2\sin 2x &= 0 &\quad\text{or}\quad \cos 2x + 1 &= 0\\
\sin 2x &= 0 & \cos 2x &= -1\\
2x &= n\pi & 2x &= \pi + 2n\pi\\
x &= \frac{n}{2}\pi & x &= \frac{\pi}{2} + n\pi
\end{aligned}$$

12.

$$\begin{aligned}
(\sin 2x + \cos 2x)^2 &= 1\\
\sin^2 2x + 2\sin 2x\cos 2x + \cos^2 2x &= 1\\
2\sin 2x\cos 2x &= 0\\
\sin 4x &= 0\\
4x &= n\pi\\
x &= \frac{n\pi}{4}
\end{aligned}$$

13.

$$\begin{aligned}
\tan 2x - \cot x &= 0\\
\frac{2\tan x}{1-\tan^2 x} &= \cot x\\
2\tan x &= \cot x(1-\tan^2 x)\\
2\tan x &= \cot x - \cot x\tan^2 x\\
2\tan x &= \cot x - \tan x\\
3\tan x &= \cot x\\
3\tan x - \cot x &= 0\\
3\tan x - \frac{1}{\tan x} &= 0\\
\frac{3\tan^2 x - 1}{\tan x} &= 0\\
\frac{1}{\tan x}(3\tan^2 x - 1) &= 0\\
\cot x(3\tan^2 x - 1) &= 0
\end{aligned}$$

$$\begin{aligned}
\cot x &= 0 &\quad\text{or}\quad 3\tan^2 x - 1 &= 0\\
x &= \frac{\pi}{2} + n\pi & \tan^2 x &= \frac{1}{3}\\
& & \tan x &= \pm\frac{\sqrt{3}}{3}\\
& & x &= \frac{\pi}{6} + n\pi, \frac{5\pi}{6} + n\pi
\end{aligned}$$

14.

$$\tan 2x - 2\cos x = 0$$
$$\frac{2\tan x}{1 - \tan^2 x} = 2\cos x$$
$$2\tan x = 2\cos x\left(1 - \tan^2 x\right)$$
$$2\tan x = 2\cos x - 2\cos x\tan^2 x$$
$$2\tan x = 2\cos x - 2\cos x\frac{\sin^2 x}{\cos^2 x}$$
$$2\tan x = 2\cos x - 2\frac{\sin^2 x}{\cos x}$$
$$\tan x = \cos x - \frac{\sin^2 x}{\cos x}$$
$$\frac{\sin x}{\cos x} = \cos x - \frac{\sin^2 x}{\cos x}$$
$$\frac{\sin x}{\cos x} + \frac{\sin^2 x}{\cos x} - \cos x = 0$$
$$\frac{\sin x + \sin^2 x - \cos^2 x}{\cos x} = 0$$
$$\frac{1}{\cos x}\left[\sin x + \sin^2 x - \left(1 - \sin^2 x\right)\right] = 0$$
$$\sec x\left[2\sin^2 x + \sin x - 1\right] = 0$$
$$\sec x(2\sin x - 1)(\sin x + 1) = 0$$

$\sec x = 0$ or $2\sin x - 1 = 0$ or $\sin x + 1 = 0$

No solution; $\sin x = \frac{1}{2}$, $x = \frac{\pi}{6}, \frac{5\pi}{6}$; $\sin x = -1$, $x = \frac{3\pi}{2}$

Also, values for which $\cos x = 0$ need to be checked.

$\frac{\pi}{2}, \frac{3\pi}{2}$ are solutions.

$$x = \frac{\pi}{6} + 2n\pi, \frac{\pi}{2} + n\pi, \frac{5\pi}{6} + 2n\pi$$

15. $6\sin x\cos x = 3(2\sin x\cos x)$
$= 3\sin 2x$

16. $\sin x\cos x = \frac{1}{2}(2\sin x\cos x)$
$= \frac{1}{2}\sin 2x$

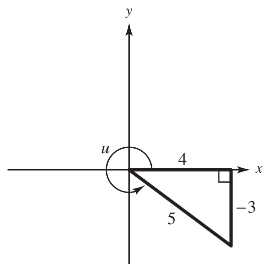
17. $6\cos^2 x - 3 = 3\left(2\cos^2 x - 1\right)$
$= 3\cos 2x$

18. $\cos^2 x - \frac{1}{2} = \frac{1}{2}\left[2\left(\cos^2 x - \frac{1}{2}\right)\right]$
$= \frac{1}{2}\left(2\cos^2 x - 1\right)$
$= \frac{1}{2}\cos 2x$

19. $4 - 8\sin^2 x = 4\left(1 - 2\sin^2 x\right)$
$= 4\cos 2x$

20. $10\sin^2 x - 5 = 5\left(2\sin^2 x - 1\right)$
$= -5\left(1 - 2\sin^2 x\right)$
$= -5\cos 2x$

21. $\sin u = -\frac{3}{5}, \frac{3\pi}{2} < u < 2\pi$

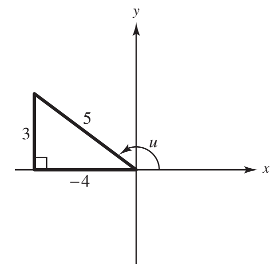


$$\sin 2u = 2\sin u\cos u = 2\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right) = -\frac{24}{25}$$

$$\cos 2u = \cos^2 u - \sin^2 u = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u} = \frac{2\left(-\frac{3}{4}\right)}{1 - \frac{9}{16}} = -\frac{3}{2}\left(\frac{16}{7}\right) = -\frac{24}{7}$$

22. $\cos u = -\frac{4}{5}, \frac{\pi}{2} < u < \pi$

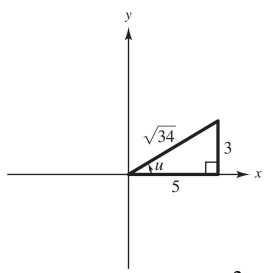


$$\sin 2u = 2\sin u\cos u = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}$$

$$\cos 2u = \cos^2 u - \sin^2 u = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u} = \frac{2\left(-\frac{3}{4}\right)}{1 - \frac{9}{16}} = -\frac{3}{2}\left(\frac{16}{7}\right) = -\frac{24}{7}$$

23. $\tan u = \frac{3}{5}, 0 < u < \frac{\pi}{2}$

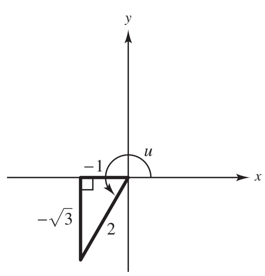


$$\sin 2u = 2\sin u\cos u = 2\left(\frac{3}{\sqrt{34}}\right)\left(\frac{5}{\sqrt{34}}\right) = \frac{15}{17}$$

$$\cos 2u = \cos^2 u - \sin^2 u = \frac{25}{34} - \frac{9}{34} = \frac{8}{17}$$

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u} = \frac{2\left(\frac{3}{5}\right)}{1 - \frac{9}{25}} = \frac{6}{5}\left(\frac{25}{16}\right) = \frac{15}{8}$$

24. $\sec u = -2, \pi < u < \frac{3\pi}{2}$



$$\sin 2u = 2\sin u\cos u = 2\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$$

$$\cos 2u = \cos^2 u - \sin^2 u = \left(-\frac{1}{2}\right)^2 - \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$$

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u} = \frac{2\left(\frac{\sqrt{3}}{1}\right)}{1 - \left(\frac{\sqrt{3}}{1}\right)^2} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$

25.
$$\begin{aligned}\cos 4x &= \cos(2x + 2x)\\ &= \cos 2x\cos 2x - \sin 2x\sin 2x\\ &= \cos^2 2x - \sin^2 2x\\ &= \cos^2 2x - \left(1 - \cos^2 2x\right)\\ &= 2\cos^2 2x - 1\\ &= 2(\cos 2x)^2 - 1\\ &= 2\left(2\cos^2 x - 1\right)^2 - 1\\ &= 2\left(4\cos^4 x - 4\cos x + 1\right) - 1\\ &= 8\cos^4 x - 8\cos x + 1\end{aligned}$$

26.
$$\begin{aligned}\tan 3x &= \tan(2x + x)\\ &= \frac{\tan 2x + \tan x}{1 - \tan 2x\tan x}\\ &= \frac{\frac{2\tan x}{1 - \tan^2 x} + \tan x}{1 - \left(\frac{2\tan x}{1 - \tan^2 x}\right)(\tan x)}\\ &= \frac{\frac{2\tan x + \tan x - \tan^3 x}{1 - \tan^2 x}}{\frac{1 - \tan^2 x - 2\tan^2 x}{1 - \tan^2 x}}\\ &= \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\end{aligned}$$

27. $$\cos^4 x = (\cos^2 x)(\cos^2 x) = \left(\frac{1 + \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) = \frac{1 + 2\cos 2x + \cos^2 2x}{4}$$

$$= \frac{1 + 2\cos 2x + \dfrac{1 + \cos 4x}{2}}{4}$$

$$= \frac{2 + 4\cos 2x + 1 + \cos 4x}{8}$$

$$= \frac{3 + 4\cos 2x + \cos 4x}{8}$$

$$= \frac{1}{8}(3 + 4\cos 2x + \cos 4x)$$

28. $$\sin^8 x = (\sin^4 x)(\sin^4 x) = (\sin^2 x)^2 (\sin^2 x)^2$$

$$= \left(\frac{1 - \cos 2x}{2}\right)^2\left(\frac{1 - \cos 2x}{2}\right)^2$$

$$= \left(\frac{1 - 2\cos 2x + \cos^2 2x}{4}\right)\left(\frac{1 - 2\cos 2x + \cos^2 2x}{4}\right)$$

$$= \frac{1 - 2\cos 2x + \cos^2 2x - 2\cos 2x + 4\cos^2 2x - 2\cos^3 2x + \cos^2 2x - 2\cos^3 2x + \cos^4 2x}{16}$$

$$= \frac{1 - 4\cos 2x + 6\cos^2 2x - 4\cos^3 2x + \cos^4 2x}{16}$$

$$= \frac{1 - 4\cos 2x + 6\cos^2 2x - 4\cos^3 2x + (\cos^2 2x)^2}{16}$$

$$= \frac{1 - 4\cos 2x + 6\left(\dfrac{1 + \cos 4x}{2}\right) - 4\cos^3 2x + \left(\dfrac{1 + \cos 4x}{2}\right)^2}{16}$$

$$= \frac{1 - 4\cos 2x + 3 + 3\cos 4x - 4\cos^3 2x + \left(\dfrac{1 + 2\cos 4x + \cos^2 4x}{4}\right)}{16}$$

$$= \frac{4 - 16\cos 2x + 12 + 12\cos 4x - 16\cos^3 2x + 1 + 2\cos 4x + \cos^2 4x}{64}$$

$$= \frac{17 - 16\cos 2x + 14\cos 4x - 16\cos^3 2x + \left(\dfrac{1 + \cos 8x}{2}\right)}{64}$$

$$= \frac{34 - 32\cos 2x + 28\cos 4x - 32\cos^3 2x + 1 + \cos 8x}{128}$$

$$= \frac{35 - 32\cos 2x + 28\cos 4x - 32\cos^3 2x + \cos 8x}{128}$$

$$= \frac{35 - 32\cos 2x + 28\cos 4x - 32\cos^2 2x\cos 2x + \cos 8x}{128}$$

$$= \frac{35 - 32\cos 2x + 28\cos 4x - 32\left(\dfrac{1 + \cos 4x}{2}\right)\cos 2x + \cos 8x}{128}$$

$$= \frac{35 - 32\cos 2x + 28\cos 4x - 16\cos 2x - 16\cos 4x\cos 2x + \cos 8x}{128}$$

$$= \frac{35 - 48\cos 2x + 28\cos 4x - 16\cos 4x\cos 2x + \cos 8x}{128}$$

$$= \frac{1}{128}(35 - 48\cos 2x + 28\cos 4x + \cos 8x - 16\cos 2x\cos 4x)$$

29. $$\begin{aligned}\sin^4 2x &= (\sin^2 2x)^2\\ &= \left(\frac{1 - \cos 4x}{2}\right)^2\\ &= \frac{1}{4}(1 - 2\cos 4x + \cos^2 4x)\\ &= \frac{1}{4}\left(1 - 2\cos 4x + \frac{1 + \cos 8x}{2}\right)\\ &= \frac{1}{4} - \frac{1}{2}\cos 4x + \frac{1}{8} + \frac{1}{8}\cos 8x\\ &= \frac{3}{8} - \frac{1}{2}\cos 4x + \frac{1}{8}\cos 8x\\ &= \frac{1}{8}(3 - 4\cos 4x + \cos 8x)\end{aligned}$$

30. $$\begin{aligned}\cos^4 2x &= (\cos^2 2x)^2\\ &= \left(\frac{1 + \cos 4x}{2}\right)^2\\ &= \frac{1}{4}(1 + 2\cos 4x + \cos^2 4x)\\ &= \frac{1}{4}\left(1 + 2\cos 4x + \frac{1 + \cos 8x}{2}\right)\\ &= \frac{1}{4} + \frac{1}{2}\cos 4x + \frac{1}{8} + \frac{1}{8}\cos 8x\\ &= \frac{3}{8} + \frac{1}{2}\cos 4x + \frac{1}{8}\cos 8x\\ &= \frac{1}{8}(3 + 4\cos 4x + \cos 8x)\end{aligned}$$

31. $$\begin{aligned}\tan^4 2x &= (\tan^2 2x)^2\\ &= \left(\frac{1 - \cos 4x}{1 + \cos 4x}\right)^2\\ &= \frac{1 - 2\cos 4x + \cos^2 4x}{1 + 2\cos 4x + \cos^2 4x}\\ &= \frac{1 - 2\cos 4x + \dfrac{1 + \cos 8x}{2}}{1 + 2\cos 4x + \dfrac{1 + \cos 8x}{2}}\\ &= \frac{\frac{1}{2}(2 - 4\cos 4x + 1 + \cos 8x)}{\frac{1}{2}(2 + 4\cos 4x + 1 + \cos 8x)}\\ &= \frac{3 - 4\cos 4x + \cos 8x}{3 + 4\cos 4x + \cos 8x}\end{aligned}$$

32. $$\begin{aligned}\tan^2 2x \cos^4 2x &= \left(\frac{1 - \cos 4x}{1 + \cos 4x}\right)(\cos^2 2x)^2\\ &= \left(\frac{1 - \cos 4x}{1 + \cos 4x}\right)\left(\frac{1 + \cos 4x}{2}\right)^2\\ &= \frac{(1 - \cos 4x)(1 + \cos 4x)(1 + \cos 4x)}{4(1 + \cos 4x)}\\ &= \frac{(1 - \cos 4x)(1 + \cos 4x)}{4}\\ &= \frac{1}{4}(1 - \cos^2 4x)\\ &= \frac{1}{4}\left(1 - \frac{1 + \cos 8x}{2}\right)\\ &= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos 8x\\ &= \frac{1}{8} - \frac{1}{8}\cos 8x\\ &= \frac{1}{8}(1 - \cos 8x)\end{aligned}$$

33. $$\begin{aligned}\sin^2 2x \cos^2 2x &= \left(\frac{1 - \cos 4x}{2}\right)\left(\frac{1 + \cos 4x}{2}\right)\\ &= \frac{1}{4}(1 - \cos^2 4x)\\ &= \frac{1}{4}\left(1 - \frac{1 + \cos 8x}{2}\right)\\ &= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos 8x\\ &= \frac{1}{8} - \frac{1}{8}\cos 8x\\ &= \frac{1}{8}(1 - \cos 8x)\end{aligned}$$

34. $\sin^4 x \cos^2 x = \sin^2 x \sin^2 x \cos^2 x$

$$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{8}(1-\cos 2x)(1-\cos^2 2x)$$

$$= \frac{1}{8}(1-\cos 2x-\cos^2 2x+\cos^3 2x)$$

$$= \frac{1}{8}\left[1-\cos 2x-\left(\frac{1+\cos 4x}{2}\right)+\cos 2x\left(\frac{1+\cos 4x}{2}\right)\right]$$

$$= \frac{1}{16}[2-2\cos 2x-1-\cos 4x+\cos 2x+\cos 2x\cos 4x]$$

$$= \frac{1}{16}[1-\cos 2x-\cos 4x+\cos 2x\cos 4x]$$

35. $\sin 75° = \sin\left(\frac{1}{2}\cdot 150°\right) = \sqrt{\frac{1-\cos 150°}{2}} = \sqrt{\frac{1+\left(\sqrt{3}/2\right)}{2}}$

$$= \frac{1}{2}\sqrt{2+\sqrt{3}}$$

$\cos 75° = \cos\left(\frac{1}{2}\cdot 150°\right) = \sqrt{\frac{1+\cos 150°}{2}} = \sqrt{\frac{1-\left(\sqrt{3}/2\right)}{2}}$

$$= \frac{1}{2}\sqrt{2-\sqrt{3}}$$

$\tan 75° = \tan\left(\frac{1}{2}\cdot 150°\right) = \frac{\sin 150°}{1+\cos 150°} = \frac{1/2}{1-\left(\sqrt{3}/2\right)}$

$$= \frac{1}{2-\sqrt{3}}\cdot\frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}$$

36. $\sin 165° = \sin\left(\frac{1}{2}\cdot 330°\right) = \sqrt{\frac{1-\cos 330°}{2}} = \sqrt{\frac{1-\left(\sqrt{3}/2\right)}{2}}$

$$= \frac{1}{2}\sqrt{2-\sqrt{3}}$$

$\cos 165° = \cos\left(\frac{1}{2}\cdot 330°\right) = -\sqrt{\frac{1+\cos 330°}{2}} = -\sqrt{\frac{1+\left(\sqrt{3}/2\right)}{2}}$

$$= -\frac{1}{2}\sqrt{2+\sqrt{3}}$$

$\tan 165° = \tan\left(\frac{1}{2}\cdot 330°\right) = \frac{\sin 330°}{1+\cos 330°} = \frac{-1/2}{1+\left(\sqrt{3}/2\right)}$

$$= \frac{-1}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{-2+\sqrt{3}}{4-3} = -2+\sqrt{3}$$

37. $\sin 112° 30' = \sin\left(\frac{1}{2} \cdot 225°\right) = \sqrt{\frac{1 - \cos 225°}{2}} = \sqrt{\frac{1 - \left(-\sqrt{2}/2\right)}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{2}}$

$\cos 112° 30' = \cos\left(\frac{1}{2} \cdot 225°\right) = -\sqrt{\frac{1 + \cos 225°}{2}} = -\sqrt{\frac{1 + \left(-\sqrt{2}/2\right)}{2}} = \frac{1}{2} - \sqrt{2 - 2}$

$\tan 112° 30' = \tan\left(\frac{1}{2} \cdot 225°\right) = \frac{\sin 225°}{1 + \cos 225°} = \frac{-\sqrt{2}/2}{1 + \left(-\sqrt{2}/2\right)} = \frac{-\sqrt{2}}{2 - \sqrt{2}} \cdot \frac{2 + \sqrt{2}}{2 + \sqrt{2}} = \frac{-2\sqrt{2} - 2}{2} = -1 - \sqrt{2}$

38. $\sin 67° 30' = \sin\left(\frac{1}{2} \cdot 135°\right) = \sqrt{\frac{1 - \cos 135°}{2}} = \sqrt{\frac{1 + \left(\sqrt{2}/2\right)}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{2}}$

$\cos 67° 30' = \cos\left(\frac{1}{2} \cdot 135°\right) = \sqrt{\frac{1 + \cos 135°}{2}} = \sqrt{\frac{1 - \left(\sqrt{2}/2\right)}{2}} = \frac{1}{2}\sqrt{2 - \sqrt{2}}$

$\tan 67° 30' = \tan\left(\frac{1}{2} \cdot 135°\right) = \frac{\sin 135°}{1 + \cos 135°} = \frac{\sqrt{2}/2}{1 - \left(\sqrt{2}/2\right)} = 1 + \sqrt{2}$

39. $\sin \frac{\pi}{8} = \sin\left[\frac{1}{2}\left(\frac{\pi}{4}\right)\right] = \sqrt{\frac{1 - \cos \frac{\pi}{4}}{2}} = \frac{1}{2}\sqrt{2 - \sqrt{2}}$

$\cos \frac{\pi}{8} = \cos\left[\frac{1}{2}\left(\frac{\pi}{4}\right)\right] = \sqrt{\frac{1 + \cos \frac{\pi}{4}}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{2}}$

$\tan \frac{\pi}{8} = \tan\left[\frac{1}{2}\left(\frac{\pi}{4}\right)\right] = \frac{\sin \frac{\pi}{4}}{1 + \cos \frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = \sqrt{2} - 1$

40. $\sin \frac{7\pi}{12} = \sin\left[\frac{1}{2}\left(\frac{7\pi}{6}\right)\right] = \sqrt{\frac{1 - \cos \frac{7\pi}{6}}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{3}}$

$\cos \frac{7\pi}{12} = \cos\left[\frac{1}{2}\left(\frac{7\pi}{6}\right)\right] = -\sqrt{\frac{1 + \cos \frac{7\pi}{6}}{2}} = -\sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = -\frac{1}{2}\sqrt{2 - \sqrt{3}}$

$\tan \frac{7\pi}{12} = \tan\left[\frac{1}{2}\left(\frac{7\pi}{6}\right)\right] = \frac{\sin \frac{7\pi}{6}}{1 + \cos \frac{7\pi}{6}} = \frac{-\frac{1}{2}}{1 - \frac{\sqrt{3}}{2}} = -2 - \sqrt{3}$

41. $\cos u = \frac{7}{25}, 0 < u < \frac{\pi}{2}$

(a) Because u is in Quadrant I, $\frac{u}{2}$ is also in Quadrant I.

(b) $\sin\frac{u}{2} = \sqrt{\frac{1 - \cos u}{2}} = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$\cos\frac{u}{2} = \sqrt{\frac{1 + \cos u}{2}} = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\tan\frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{1 - \frac{7}{25}}{\frac{24}{25}} = \frac{3}{4}$

42. $\sin u = \frac{5}{13}, \frac{\pi}{2} < u < \pi \Rightarrow \cos u = -\frac{12}{13}$

(a) Because u is in Quadrant II, $\frac{u}{2}$ is in Quadrant I.

(b) $\sin\left(\frac{u}{2}\right) = \sqrt{\frac{1 - \cos u}{2}} = \sqrt{\frac{1 + \frac{12}{13}}{2}} = \frac{5\sqrt{26}}{26}$

$\cos\left(\frac{u}{2}\right) = \sqrt{\frac{1 + \cos u}{2}} = \sqrt{\frac{1 - \frac{12}{13}}{2}} = \frac{\sqrt{26}}{26}$

$\tan\left(\frac{u}{2}\right) = \frac{\sin u}{1 + \cos u} = \frac{\frac{5}{13}}{1 - \frac{12}{13}} = 5$

43. $\tan u = -\frac{5}{12}, \frac{3\pi}{2} < u < 2\pi$

(a) Because u is in Quadrant IV, $\frac{u}{2}$ is in Quadrant II.

(b) $\sin\frac{u}{2} = \sqrt{\frac{1 - \cos u}{2}} = \sqrt{\frac{1 - \frac{12}{13}}{2}} = \sqrt{\frac{1}{26}} = \frac{\sqrt{26}}{26}$

$\cos\frac{u}{2} = -\sqrt{\frac{1 + \cos u}{2}} = -\sqrt{\frac{1 + \frac{12}{13}}{2}} = -\sqrt{\frac{25}{26}} = -\frac{5\sqrt{26}}{26}$

$\tan\frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{1 - \frac{12}{13}}{\left(-\frac{5}{13}\right)} = -\frac{1}{5}$

44. $\cot u = 3, \pi < u < \frac{3\pi}{2}$

(a) Because u is in Quadrant III, $\frac{u}{2}$ is in Quadrant II.

(b) $\sin\left(\frac{u}{2}\right) = \sqrt{\frac{1 - \cos u}{2}} = \sqrt{\frac{1 + \frac{3}{\sqrt{10}}}{2}} = \sqrt{\frac{10 + 3\sqrt{10}}{20}} = \frac{1}{2}\sqrt{\frac{10 + 3\sqrt{10}}{5}}$

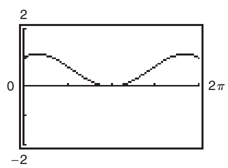
$\cos\left(\frac{u}{2}\right) = -\sqrt{\frac{1 + \cos u}{2}} = -\sqrt{\frac{1 - \frac{3}{\sqrt{10}}}{2}} = -\sqrt{\frac{10 - 3\sqrt{10}}{20}} = -\frac{1}{2}\sqrt{\frac{10 - 3\sqrt{10}}{5}}$

$\tan\left(\frac{u}{2}\right) = \frac{1 - \cos u}{\sin u} = \frac{1 + \frac{3}{\sqrt{10}}}{-\frac{1}{\sqrt{10}}} = -\sqrt{10} - 3$

45.
$$\begin{aligned} \sin\frac{x}{2} + \cos x &= 0 \\ \pm\sqrt{\frac{1-\cos x}{2}} &= -\cos x \\ \frac{1-\cos x}{2} &= \cos^2 x \\ 0 &= 2\cos^2 x + \cos x - 1 \\ &= (2\cos x - 1)(\cos x + 1) \end{aligned}$$

$\cos x = \frac{1}{2}$ or $\cos x = -1$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ $\qquad x = \pi$



By checking these values in the original equation, $x = \pi/3$ and $x = 5\pi/3$ are extraneous, and $x = \pi$ is the only solution.

46.
$$h(x) = \sin\frac{x}{2} + \cos x - 1$$

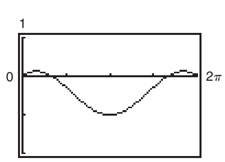
$$\begin{aligned} \sin\frac{x}{2} + \cos x - 1 &= 0 \\ \pm\sqrt{\frac{1-\cos x}{2}} &= 1 - \cos x \\ \frac{1-\cos x}{2} &= 1 - 2\cos x + \cos^2 x \\ 1 - \cos x &= 2 - 4\cos x + 2\cos^2 x \\ 2\cos^2 x - 3\cos x + 1 &= 0 \\ (2\cos x - 1)(\cos x - 1) &= 0 \end{aligned}$$

$2\cos x - 1 = 0$ or $\cos x - 1 = 0$

$\cos x = \frac{1}{2}$ $\qquad \cos x = 1$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ $\qquad x = 0$

$0, \frac{\pi}{3},$ and $\frac{5\pi}{3}$ are all solutions to the equation.



47.
$$\begin{aligned} \cos\frac{x}{2} - \sin x &= 0 \\ \pm\sqrt{\frac{1+\cos x}{2}} &= \sin x \\ \frac{1+\cos x}{2} &= \sin^2 x \\ 1 + \cos x &= 2\sin^2 x \\ 1 + \cos x &= 2 - 2\cos^2 x \\ 2\cos^2 x + \cos x - 1 &= 0 \\ (2\cos x - 1)(\cos x + 1) &= 0 \end{aligned}$$

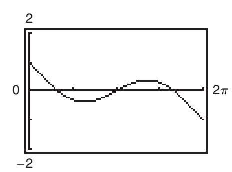
$2\cos x - 1 = 0$ or $\cos x + 1 = 0$

$\cos x = \frac{1}{2}$ $\qquad \cos x = -1$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ $\qquad x = \pi$

$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

$\pi/3, \pi,$ and $5\pi/3$ are all solutions to the equation.



48.
$$g(x) = \tan\frac{x}{2} - \sin x$$

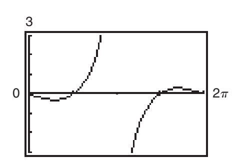
$$\begin{aligned} \tan\frac{x}{2} - \sin x &= 0 \\ \frac{1-\cos x}{\sin x} &= \sin x \\ 1 - \cos x &= \sin^2 x \\ 1 - \cos x &= 1 - \cos^2 x \\ \cos^2 x - \cos x &= 0 \\ \cos x(\cos x - 1) &= 0 \end{aligned}$$

$\cos x = 0$ or $\cos x - 1 = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad \cos x = 1$

$x = 0$

$0, \frac{\pi}{2},$ and $\frac{3\pi}{2}$ are all solutions to the equation.



49. $\sin 5\theta \sin 3\theta = \frac{1}{2}\left[\cos(5\theta - 3\theta) - \cos(5\theta + 3\theta)\right] = \frac{1}{2}(\cos 2\theta - \cos 8\theta)$

50. $7\cos(-5\beta)\sin 3\beta = 7 \cdot \frac{1}{2}\left[\sin(-5\beta + 3\beta) - \sin(-5\beta - 3\beta)\right] = \frac{7}{2}(\sin(-2\beta) - \sin(-8\beta))$

51. $\cos 2\theta \cos 4\theta = \frac{1}{2}\left[\cos(2\theta - 4\theta) + \cos(2\theta + 4\theta)\right] = \frac{1}{2}\left[\cos(-2\theta) + \cos 6\theta\right]$

52. $\sin(x + y)\cos(x - y) = \frac{1}{2}(\sin 2x + \sin 2y)$

53. $\sin 5\theta - \sin 3\theta = 2\cos\left(\frac{5\theta + 3\theta}{2}\right)\sin\left(\frac{5\theta - 3\theta}{2}\right)$

$= 2\cos 4\theta \sin\theta$

54. $\sin 3\theta + \sin\theta = 2\sin\left(\frac{3\theta + \theta}{2}\right)\cos\left(\frac{3\theta - \theta}{2}\right)$

$= 2\sin 2\theta \cos\theta$

55. $\cos 6x + \cos 2x = 2\cos\left(\frac{6x + 2x}{2}\right)\cos\left(\frac{6x - 2x}{2}\right)$

$= 2\cos 4x \cos 2x$

56. $\cos x + \cos 4x = 2\cos\left(\frac{x + 4x}{2}\right)\cos\left(\frac{x - 4x}{2}\right)$

$= 2\cos\left(\frac{5x}{2}\right)\cos\left(\frac{-3x}{2}\right)$

57. $\sin 75° + \sin 15° = 2\sin\left(\frac{75° + 15°}{2}\right)\cos\left(\frac{75° - 15°}{2}\right) = 2\sin 45° \cos 30° = 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}}{2}$

58. $\cos 120° + \cos 60° = 2\cos\left(\frac{120° + 60°}{2}\right)\cos\left(\frac{120° - 60°}{2}\right) = 2\cos 90° \cos 30° = 2(0)\left(\frac{\sqrt{3}}{2}\right) = 0$

59. $\cos\frac{3\pi}{4} - \cos\frac{\pi}{4} = -2\sin\left(\frac{\frac{3\pi}{4} + \frac{\pi}{4}}{2}\right)\sin\left(\frac{\frac{3\pi}{4} - \frac{\pi}{4}}{2}\right) = -2\sin\frac{\pi}{2}\sin\frac{\pi}{4}$

$\cos\frac{3\pi}{4} - \cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

60. $\sin\frac{5\pi}{4} - \sin\frac{3\pi}{4} = 2\cos\left(\frac{\frac{5\pi}{4} + \frac{3\pi}{4}}{2}\right)\sin\left(\frac{\frac{5\pi}{4} - \frac{3\pi}{4}}{2}\right) = 2\cos\pi\sin\frac{\pi}{4}$

$\sin\frac{5\pi}{4} - \sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

61. $\sin 6x + \sin 2x = 0$

$2\sin\left(\frac{6x + 2x}{2}\right)\cos\left(\frac{6x - 2x}{2}\right) = 0$

$2(\sin 4x)\cos 2x = 0$

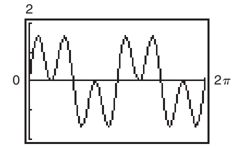
$\sin 4x = 0$ or $\cos 2x = 0$

$4x = n\pi \qquad 2x = \frac{\pi}{2} + n\pi$

$x = \frac{n\pi}{4} \qquad x = \frac{\pi}{4} + \frac{n\pi}{2}$

In the interval $[0, 2\pi)$

$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}.$



62. $h(x) = \cos 2x - \cos 6x$

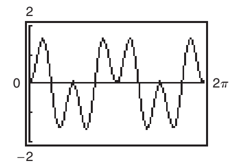
$$\cos 2x - \cos 6x = 0$$
$$-2 \sin 4x \sin(-2x) = 0$$
$$2 \sin 4x \sin 2x = 0$$

$\sin 4x = 0$ or $\sin 2x = 0$

$4x = n\pi$ $2x = n\pi$

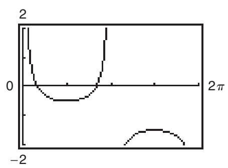
$x = \dfrac{n\pi}{4}$ $x = \dfrac{n\pi}{2}$

$x = 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \pi, \dfrac{5\pi}{4}, \dfrac{3\pi}{2}, \dfrac{7\pi}{4}$ $x = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$



63. $\dfrac{\cos 2x}{\sin 3x - \sin x} - 1 = 0$

$$\frac{\cos 2x}{\sin 3x - \sin x} = 1$$
$$\frac{\cos 2x}{2 \cos 2x \sin x} = 1$$
$$2 \sin x = 1$$
$$\sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$



64. $f(x) = \sin^2 3x - \sin^2 x$

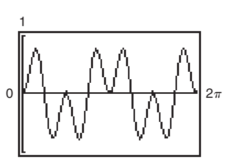
$$\sin^2 3x - \sin^2 x = 0$$
$$(\sin 3x + \sin x)(\sin 3x - \sin x) = 0$$
$$(2 \sin 2x \cos x)(2 \cos 2x \sin x) = 0$$

$\sin 2x = 0 \Rightarrow x = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$ or

$\cos x = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ or

$\cos 2x = 0 \Rightarrow x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ or

$\sin x = 0 \Rightarrow x = 0, \pi$



65. $\csc 2\theta = \dfrac{1}{\sin 2\theta}$

$$= \frac{1}{2 \sin \theta \cos \theta}$$
$$= \frac{1}{\sin \theta} \cdot \frac{1}{2 \cos \theta}$$
$$= \frac{\csc \theta}{2 \cos \theta}$$

66. $\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$

$$= (\cos 2x)(1)$$
$$= \cos 2x$$

67. $(\sin x + \cos x)^2 = \sin^2 x + 2 \sin x \cos x + \cos^2 x$

$$= (\sin^2 x + \cos^2 x) + 2 \sin x \cos x$$
$$= 1 + \sin 2x$$

68. $\tan \dfrac{u}{2} = \dfrac{1 - \cos u}{\sin u}$

$$= \frac{1}{\sin u} - \frac{\cos u}{\sin u}$$
$$= \csc u - \cot u$$

69. $\dfrac{\sin x \pm \sin y}{\cos x + \cos y} = \dfrac{2 \sin\left(\dfrac{x \pm y}{2}\right) \cos\left(\dfrac{x \mp y}{2}\right)}{2 \cos\left(\dfrac{x + y}{2}\right) \cos\left(\dfrac{x - y}{2}\right)}$

$$= \tan\left(\frac{x \pm y}{2}\right)$$

70. $\cos\left(\dfrac{\pi}{3} + x\right) + \cos\left(\dfrac{\pi}{3} - x\right) = 2 \cos\left(\dfrac{\dfrac{\pi}{3} + x + \dfrac{\pi}{3} - x}{2}\right) \cos\left(\dfrac{\dfrac{\pi}{3} + x - \left(\dfrac{\pi}{3} - x\right)}{2}\right)$

$$= 2 \cos\left(\frac{\pi}{3}\right) \cos(x)$$
$$= 2\left(\frac{1}{2}\right) \cos x = \cos x$$

71. (a)
$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}} = \frac{1}{M}$$
$$\left(\pm\sqrt{\frac{1-\cos\theta}{2}}\right)^2 = \left(\frac{1}{M}\right)^2$$
$$\frac{1-\cos\theta}{2} = \frac{1}{M^2}$$
$$M^2(1-\cos\theta) = 2$$
$$1-\cos\theta = \frac{2}{M^2}$$
$$-\cos\theta = \frac{2}{M^2} - 1$$
$$\cos\theta = 1 - \frac{2}{M^2}$$
$$\cos\theta = \frac{M^2-2}{M^2}$$

(b) When $M = 2$, $\cos\theta = \frac{2^2-2}{2^2} = \frac{1}{2}$. So, $\theta = \frac{\pi}{3}$.

(c) When $M = 4.5$, $\cos\theta = \frac{(4.5)^2 - 2}{(4.5)^2}$
$$\cos\theta \approx 0.901235.$$
So, $\theta \approx 0.4482$ radian.

(d) When $M = 2$, $\frac{\text{speed of object}}{\text{speed of sound}} = M$
$$\frac{\text{speed of object}}{760\text{ mph}} = 2$$
$$\text{speed of object} = 1520\text{ mph.}$$

When $M = 4.5$, $\frac{\text{speed of object}}{\text{speed of sound}} = M$
$$\frac{\text{speed of object}}{760\text{ mph}} = 4.5$$
$$\text{speed of object} = 3420\text{ mph.}$$

72. $\frac{1}{32}(75)^2 \sin 2\theta = 130$
$$\sin 2\theta = \frac{130(32)}{75^2}$$
$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{130(32)}{75^2}\right)$$
$$\theta \approx 23.85°$$

73. $\frac{x}{2} = 2r\sin^2\frac{\theta}{2} = 2r\left(\frac{1-\cos\theta}{2}\right)$
$$= r(1-\cos\theta)$$
So, $x = 2r(1-\cos\theta)$.

74. (a) Using the graph, $\sin 2u \approx 1$ and $2\sin u\cos u \approx 2(0.7)(0.7) \approx 1$.
Because $1 = 1$, $\sin 2u = 2\sin u\cos u$.

(b) Using the graph, $\cos 2u \approx 0$ and $\cos^2 u - \sin^2 u \approx (0.7)^2 - (0.7)^2 = 0$.
Because $0 = 0$, $\cos 2u = \cos^2 u - \sin^2 u$.

75. True. Using the double angle formula and that sine is an odd function and cosine is an even function,
$$\sin(-2x) = \sin[2(-x)]$$
$$= 2\sin(-x)\cos(-x)$$
$$= 2(-\sin x)\cos x$$
$$= -2\sin x\cos x.$$

76. False. If $90° < u < 180°$,
$\frac{u}{2}$ is in the first quadrant and
$$\sin\frac{u}{2} = \sqrt{\frac{1-\cos u}{2}}.$$

77. Because ϕ and θ are complementary angles, $\sin\phi = \cos\theta$ and $\cos\phi = \sin\theta$.

(a)
$$\sin(\phi-\theta) = \sin\phi\cos\theta - \sin\theta\cos\phi$$
$$= (\cos\theta)(\cos\theta) - (\sin\theta)(\sin\theta)$$
$$= \cos^2\theta - \sin^2\theta$$
$$= \cos 2\theta$$

(b)
$$\cos(\phi-\theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$$
$$= (\sin\theta)(\cos\theta) + (\cos\theta)(\sin\theta)$$
$$= 2\sin\theta\cos\theta$$
$$= \sin 2\theta$$

Review Exercises for Chapter 2

1. $\cot x$

2. $\sec x$

3. $\cos x$

4. $\sqrt{\cot^2 x + 1} = \sqrt{\csc^2 x} = |\csc x|$

5. $\cos\theta = -\frac{2}{5}$, $\tan\theta > 0$, θ is in Quadrant III.

$$\sec\theta = \frac{1}{\cos\theta} = -\frac{5}{2}$$

$$\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\frac{4}{25}} = -\sqrt{\frac{21}{25}} = -\frac{\sqrt{21}}{5}$$

$$\csc\theta = \frac{1}{\sin\theta} = -\frac{5}{\sqrt{21}} = -\frac{5\sqrt{21}}{21}$$

$$\tan\theta = \frac{\sin}{\cos\theta} = \frac{-\frac{\sqrt{21}}{5}}{-\frac{2}{5}} = \frac{\sqrt{21}}{2}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

6. $\cot x = -\frac{2}{3}$, $\cos x < 0$, x is in Quadrant II.

$$\tan x = \frac{1}{\cot x} = -\frac{3}{2}$$

$$\csc x = \sqrt{1+\cot^2 x} = \sqrt{1+\frac{4}{9}} = \sqrt{\frac{13}{9}} = \frac{\sqrt{13}}{3}$$

$$\sin x = \frac{1}{\csc x} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\cos x = -\sqrt{1-\sin^2 x} = -\sqrt{1-\frac{9}{13}} = -\sqrt{\frac{4}{13}} = -\frac{2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}$$

$$\sec x = \frac{1}{\cos x} = -\frac{\sqrt{13}}{2}$$

7. $$\frac{1}{\cot^2 x + 1} = \frac{1}{\csc^2 x} = \sin^2 x$$

8. $$\frac{\tan\theta}{1-\cos^2\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\sin^2\theta} = \frac{1}{\sin\theta\cos\theta} = \csc\theta\sec\theta$$

9. $$\tan^2 x(\csc^2 x - 1) = \tan^2 x(\cot^2 x) = \tan^2 x\left(\frac{1}{\tan^2 x}\right) = 1$$

10. $$\cot^2 x(\sin^2 x) = \frac{\cos^2 x}{\sin^2 x}\sin^2 x = \cos^2 x$$

11. $$\frac{\cot\left(\frac{\pi}{2} - u\right)}{\cos u} = \frac{\tan u}{\cos u} = \tan u \sec u$$

12. $$\frac{\sec^2(-\theta)}{\csc^2\theta} = \frac{\sec^2\theta}{\csc^2\theta} = \frac{1/\cos^2\theta}{1/\sin^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$

13. $$\begin{aligned}\cos^2 x + \cos^2 x\cot^2 x &= \cos^2 x(1+\cot^2 x)\\ &= \cos^2 x(\csc^2 x)\\ &= \cos^2 x\left(\frac{1}{\sin^2 x}\right)\\ &= \frac{\cos^2 x}{\sin^2 x}\\ &= \cot^2 x\end{aligned}$$

14. $$\begin{aligned}(\tan x + 1)^2\cos x &= (\tan^2 x + 2\tan x + 1)\cos x\\ &= (\sec^2 x + 2\tan x)\cos x\\ &= \sec^2 x\cos x + 2\left(\frac{\sin x}{\cos x}\right)\cos x\\ &= \sec x + 2\sin x\end{aligned}$$

15. $\dfrac{1}{\csc\theta + 1} - \dfrac{1}{\csc\theta - 1} = \dfrac{(\csc\theta - 1) - (\csc\theta + 1)}{(\csc\theta + 1)(\csc\theta - 1)}$

$= \dfrac{-2}{\csc^2\theta - 1}$

$= \dfrac{-2}{\cot^2\theta}$

$= -2\tan^2\theta$

16. $\dfrac{\tan^2 x}{1 + \sec x} = \dfrac{\sec^2 x - 1}{1 + \sec x}$

$= \dfrac{(\sec x + 1)(\sec x - 1)}{\sec x + 1}$

$= \sec x - 1$

17. Let $x = 5\sin\theta$, then

$$\sqrt{25 - x^2} = \sqrt{25 - (5\sin\theta)^2} = \sqrt{25 - 25\sin^2\theta} = \sqrt{25(1 - \sin^2\theta)} = \sqrt{25\cos^2\theta} = 5\cos\theta.$$

18. Let $x = 4\sec\theta$, then

$$\sqrt{x^2 - 16} = \sqrt{(4\sec\theta)^2 - 16} = \sqrt{16\sec^2\theta - 16} = \sqrt{16(\sec^2\theta - 1)} = \sqrt{16\tan^2\theta} = 4\tan\theta.$$

19. $\cos x(\tan^2 x + 1) = \cos x \sec^2 x$

$= \dfrac{1}{\sec x}\sec^2 x$

$= \sec x$

20. $\sec^2 x\cot x - \cot x = \cot x(\sec^2 x - 1)$

$= \cot x\tan^2 x$

$= \left(\dfrac{1}{\tan x}\right)\tan^2 x = \tan x$

21. $\sin\left(\dfrac{\pi}{2} - \theta\right)\tan\theta = \cos\theta\tan\theta$

$= \cos\theta\left(\dfrac{\sin\theta}{\cos\theta}\right)$

$= \sin\theta$

22. $\cot\left(\dfrac{\pi}{2} - \theta\right)\csc\theta = \tan\theta\csc\theta$

$= \left(\dfrac{\sin\theta}{\cos\theta}\right)\left(\dfrac{1}{\sin\theta}\right)$

$= \dfrac{1}{\cos\theta}$

$= \sec\theta$

23. $\dfrac{1}{\tan\theta\csc\theta} = \dfrac{1}{\dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{1}{\sin\theta}} = \cos\theta$

24. $\dfrac{1}{\tan x\csc x\sin x} = \dfrac{1}{(\tan x)\left(\dfrac{1}{\sin x}\right)(\sin x)} = \dfrac{1}{\tan x}$

$= \cot x$

25. $\sin^5 x\cos^2 x = \sin^4 x\cos^2 x\sin x$

$= (1 - \cos^2 x)^2\cos^2 x\sin x$

$= (1 - 2\cos^2 x + \cos^4 x)\cos^2 x\sin x$

$= (\cos^2 x - 2\cos^4 x + \cos^6 x)\sin x$

26. $\cos^3 x\sin^2 x = \cos x\cos^2 x\sin^2 x$

$= \cos x(1 - \sin^2 x)\sin^2 x$

$= \cos x(\sin^2 x - \sin^4 x)$

$= (\sin^2 x - \sin^4 x)\cos x$

27. $\sin x = \sqrt{3} - \sin x$

$\sin x = \dfrac{\sqrt{3}}{2}$

$x = \dfrac{\pi}{3} + 2\pi n, \dfrac{2\pi}{3} + 2\pi n$

28. $4\cos\theta = 1 + 2\cos\theta$

$2\cos\theta = 1$

$\cos\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{3} + 2n\pi \quad \text{or} \quad \dfrac{5\pi}{3} + 2n\pi$

29. $3\sqrt{3}\tan u = 3$

$\tan u = \dfrac{1}{\sqrt{3}}$

$u = \dfrac{\pi}{6} + n\pi$

30. $\frac{1}{2}\sec x - 1 = 0$

$$\frac{1}{2}\sec x = 1$$
$$\sec x = 2$$
$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad \frac{5\pi}{3} + 2n\pi$$

31. $3\csc^2 x = 4$

$$\csc^2 x = \frac{4}{3}$$
$$\sin x = \pm\frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3} + 2\pi n, \frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n$$

These can be combined as:

$$x = \frac{\pi}{3} + n\pi \quad \text{or} \quad x = \frac{2\pi}{3} + n\pi$$

32. $4\tan^2 u - 1 = \tan^2 u$

$$3\tan^2 u - 1 = 0$$
$$\tan^2 u = \frac{1}{3}$$
$$\tan u = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$
$$u = \frac{\pi}{6} + n\pi \quad \text{or} \quad \frac{5\pi}{6} + n\pi$$

33. $\sin^3 x = \sin x$

$$\sin^3 x - \sin x = 0$$
$$\sin x(\sin^2 x - 1) = 0$$
$$\sin x = 0 \Rightarrow x = 0, \pi$$
$$\sin^2 x = 1$$
$$\sin x = \pm 1 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

34. $2\cos^2 x + 3\cos x = 0$

$$\cos x(2\cos x + 3) = 0$$
$$\cos x = 0 \quad \text{or} \quad 2\cos x + 3 = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad 2\cos x = -3$$
$$\cos x = -\frac{3}{2}$$

No solution

35. $\cos^2 x + \sin x = 1$

$$1 - \sin^2 x + \sin x - 1 = 0$$
$$-\sin x(\sin x - 1) = 0$$
$$\sin x = 0 \qquad \sin x - 1 = 0$$
$$x = 0, \pi \qquad \sin x = 1$$
$$x = \frac{\pi}{2}$$

36. $\sin^2 x + 2\cos x = 2$

$$1 - \cos^2 x + 2\cos x = 2$$
$$0 = \cos^2 x - 2\cos x + 1$$
$$0 = (\cos x - 1)^2$$
$$\cos x - 1 = 0$$
$$\cos x = 1$$
$$x = 0$$

37. $2\sin 2x - \sqrt{2} = 0$

$$\sin 2x = \frac{\sqrt{2}}{2}$$
$$2x = \frac{\pi}{4} + 2\pi n, \frac{3\pi}{4} + 2\pi n$$
$$x = \frac{\pi}{8} + \pi n, \frac{3\pi}{8} + \pi n$$
$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$$

38. $2\cos\frac{x}{2} + 1 = 0$

$$\cos\frac{x}{2} = -\frac{1}{2}$$
$$\frac{x}{2} = \frac{2\pi}{3}$$
$$x = \frac{4\pi}{3}$$

39. $3\tan^2\left(\frac{x}{3}\right) - 1 = 0$

$$\tan^2\left(\frac{x}{3}\right) = \frac{1}{3}$$
$$\tan\frac{x}{3} = \pm\sqrt{\frac{1}{3}}$$
$$\tan\frac{x}{3} = \pm\frac{\sqrt{3}}{3}$$
$$\frac{x}{3} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$$
$$x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$\frac{5\pi}{2}$ and $\frac{7\pi}{2}$ are greater than 2π, so they are not solutions. The solution is $x = \frac{\pi}{2}$.

40. $\sqrt{3}\tan 3x = 0$

$\tan 3x = 0$

$3x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$

$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

41. $\cos 4x(\cos x - 1) = 0$

$\cos 4x = 0 \qquad \cos x - 1 = 0$

$4x = \frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n \qquad \cos x = 1$

$x = \frac{\pi}{8} + \frac{\pi}{2}n, \frac{3\pi}{8} + \frac{\pi}{2}n \qquad x = 0$

$x = 0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

42. $3\csc^2 5x = -4$

$\csc^2 5x = -\frac{4}{3}$

$\csc 5x = \pm\sqrt{-\frac{4}{3}}$

No real solution

43. $\tan^2 x - 2\tan x = 0$

$\tan x(\tan x - 2) = 0$

$\tan x = 0 \quad \text{or} \quad \tan x - 2 = 0$

$x = n\pi \qquad \tan x = 2$

$x = \arctan 2 + n\pi$

44. $2\tan^2 x - 3\tan x = -1$

$2\tan^2 x - 3\tan x + 1 = 0$

$(2\tan x - 1)(\tan x - 1) = 0$

$2\tan x - 1 = 0 \quad \text{or} \quad \tan x - 1 = 0$

$2\tan x = 1 \qquad \tan x = 1$

$\tan x = \frac{1}{2} \qquad x = \frac{\pi}{4} + n\pi$

$x = \arctan\left(\frac{1}{2}\right) + n\pi$

45. $\tan^2\theta + \tan\theta - 6 = 0$

$(\tan\theta + 3)(\tan\theta - 2) = 0$

$\tan\theta + 3 = 0 \quad \text{or} \quad \tan\theta - 2 = 0$

$\tan\theta = -3 \qquad \tan\theta = 2$

$\theta = \arctan(-3) + n\pi \qquad \theta = \arctan 2 + n\pi$

46. $\sec^2 x + 6\tan x + 4 = 0$

$1 + \tan^2 x + 6\tan x + 4 = 0$

$\tan^2 x + 6\tan x + 5 = 0$

$(\tan x + 5)(\tan x + 1) = 0$

$\tan x + 5 = 0 \quad \text{or} \quad \tan x + 1 = 0$

$\tan x = -5 \qquad \tan x = -1$

$x = \arctan(-5) + n\pi \qquad x = \frac{3\pi}{4} + n\pi$

47. $\sin 75° = \sin(120° - 45°)$

$= \sin 120° \cos 45° - \cos 120° \sin 45°$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$

$\cos 75° = \cos(120° - 45°)$

$= \cos 120° \cos 45° + \sin 120° \sin 45°$

$= \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$

$\tan 75° = \tan(120° - 45°) = \frac{\tan 120° - \tan 45°}{1 + \tan 120° \tan 45°}$

$= \frac{-\sqrt{3} - 1}{1 + \left(-\sqrt{3}\right)(1)} = \frac{-\sqrt{3} - 1}{1 - \sqrt{3}}$

$= \frac{-\sqrt{3} - 1}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}}$

$= \frac{-4 - 2\sqrt{3}}{-2} = 2 + \sqrt{3}$

48. $\sin(375°) = \sin(135° + 240°)$

$= \sin 135° \cos 240° + \cos 135° \sin 240°$

$= \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$

$\cos(375°) = \cos(135° + 240°)$

$= \cos 135° \cos 240° - \sin 135° \sin 240°$

$= \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{2}}{4}\left(1 + \sqrt{3}\right)$

$\tan(375°) = \tan(135° + 240°)$

$= \frac{\tan 135° + \tan 240°}{1 - \tan 135° \tan 240°}$

$= \frac{-1 + \sqrt{3}}{1 - (-1)\left(\sqrt{3}\right)}$

$= \frac{-1 + \sqrt{3}}{1 + \sqrt{3}} \cdot \frac{1 - \sqrt{3}}{1 - \sqrt{3}} = \frac{-4 + 2\sqrt{3}}{1 - 3} = 2 - \sqrt{3}$

49. $\sin\frac{25\pi}{12} = \sin\left(\frac{11\pi}{6} + \frac{\pi}{4}\right) = \sin\frac{11\pi}{6}\cos\frac{\pi}{4} + \cos\frac{11\pi}{6}\sin\frac{\pi}{4}$

$= \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$

$\cos\frac{25\pi}{12} = \cos\left(\frac{11\pi}{6} + \frac{\pi}{4}\right) = \cos\frac{11\pi}{6}\cos\frac{\pi}{4} - \sin\frac{11\pi}{6}\sin\frac{\pi}{4}$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$

$\tan\frac{25\pi}{12} = \tan\left(\frac{11\pi}{6} + \frac{\pi}{4}\right) = \frac{\tan\frac{11\pi}{6} + \tan\frac{\pi}{4}}{1 - \tan\frac{11\pi}{6}\tan\frac{\pi}{4}}$

$= \frac{\left(-\frac{\sqrt{3}}{3}\right) + 1}{1 - \left(-\frac{\sqrt{3}}{3}\right)(1)} = 2 - \sqrt{3}$

50. $\sin\left(\frac{19\pi}{12}\right) = \sin\left(\frac{11\pi}{6} - \frac{\pi}{4}\right)$

$= \sin\frac{11\pi}{6}\cos\frac{\pi}{4} - \cos\frac{11\pi}{6}\sin\frac{\pi}{4}$

$= -\frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$

$= -\frac{\sqrt{2}}{4}\left(1 + \sqrt{3}\right) = -\frac{\sqrt{2}}{4}\left(\sqrt{3} + 1\right)$

$\cos\left(\frac{19\pi}{12}\right) = \cos\left(\frac{11\pi}{6} - \frac{\pi}{4}\right)$

$= \cos\frac{11\pi}{6}\cos\frac{\pi}{4} + \sin\frac{11\pi}{6}\sin\frac{\pi}{4}$

$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \left(-\frac{1}{2}\right)\frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$

$\tan\left(\frac{19\pi}{12}\right) = \tan\left(\frac{11\pi}{6} - \frac{\pi}{4}\right)$

$= \dfrac{\tan\frac{11\pi}{6} - \tan\frac{\pi}{4}}{1 + \tan\frac{11\pi}{6}\tan\frac{\pi}{4}}$

$= \dfrac{-\frac{\sqrt{3}}{3} - 1}{1 + \left(-\frac{\sqrt{3}}{3}\right)(1)} = \dfrac{-\sqrt{3} - 3}{3 - \sqrt{3}}\cdot\dfrac{3 + \sqrt{3}}{3 + \sqrt{3}}$

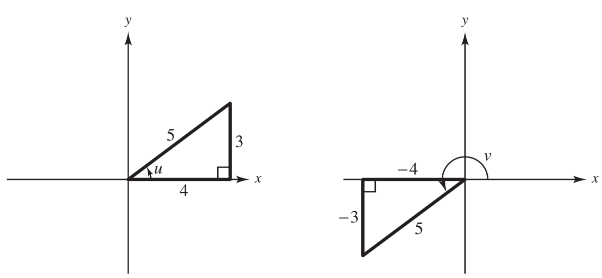
$= \dfrac{-\left(12 + 6\sqrt{3}\right)}{6} = -2 - \sqrt{3}$

51. $\sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ = \sin\left(60^\circ - 45^\circ\right)$

$= \sin 15^\circ$

52. $\dfrac{\tan 68^\circ - \tan 115^\circ}{1 + \tan 68^\circ \tan 115^\circ} = \tan\left(68^\circ - 115^\circ\right)$

$= \tan\left(-47^\circ\right)$



Figures for Exercises 53–56

53. $\sin(u + v) = \sin u \cos v + \cos u \sin v = \frac{3}{5}\left(-\frac{4}{5}\right) + \frac{4}{5}\left(-\frac{3}{5}\right) = -\frac{24}{25}$

54. $\tan(u + v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v} = \dfrac{\frac{3}{4} + \frac{3}{4}}{1 - \frac{3}{4}\left(\frac{3}{4}\right)} = \dfrac{\frac{3}{2}}{\frac{7}{16}} = \frac{3}{2}\left(\frac{16}{7}\right) = \frac{24}{7}$

55. $\cos(u - v) = \cos u \cos v + \sin u \sin v = \frac{4}{5}\left(-\frac{4}{5}\right) + \frac{3}{5}\left(-\frac{3}{5}\right) = -1$

56. $\sin(u - v) = \sin u \cos v - \cos u \sin v = \frac{3}{5}\left(-\frac{4}{5}\right) - \frac{4}{5}\left(-\frac{3}{5}\right) = 0$

57. $\cos\left(x + \frac{\pi}{2}\right) = \cos x \cos\frac{\pi}{2} - \sin x \sin\frac{\pi}{2} = \cos x(0) - \sin x(1) = -\sin x$

58. $\tan\left(x - \dfrac{\pi}{2}\right) = -\tan\left(\dfrac{\pi}{2} - x\right) = -\cot x$

59. $\tan(\pi - x) = \dfrac{\tan \pi - \tan x}{1 - \tan \pi \tan x} = -\tan x$

60. $\sin(x - \pi) = \sin x \cos \pi - \cos x \sin \pi$

$= \sin x(-1) - \cos x(0)$

$= -\sin x$

61. $\sin\left(x + \dfrac{\pi}{4}\right) - \sin\left(x - \dfrac{\pi}{4}\right) = 1$

$2 \cos x \sin \dfrac{\pi}{4} = 1$

$\cos x = \dfrac{\sqrt{2}}{2}$

$x = \dfrac{\pi}{4}, \dfrac{7\pi}{4}$

62. $\cos\left(x + \dfrac{\pi}{6}\right) - \cos\left(x - \dfrac{\pi}{6}\right) = 1$

$\left(\cos x \cos \dfrac{\pi}{6} - \sin x \sin \dfrac{\pi}{6}\right) - \left(\cos x \cos \dfrac{\pi}{6} + \sin x \sin \dfrac{\pi}{6}\right) = 1$

$-2 \sin x \sin \dfrac{\pi}{6} = 1$

$-2 \sin x\left(\dfrac{1}{2}\right) = 1$

$\sin x = -1$

$x = \dfrac{3\pi}{2}$

63. $\sin u = -\dfrac{4}{5}, \pi < u < \dfrac{3\pi}{2}$

$\cos u = -\sqrt{1 - \sin^2 u} = \dfrac{-3}{5}$

$\tan u = \dfrac{\sin u}{\cos u} = \dfrac{4}{3}$

$\sin 2u = 2 \sin u \cos u = 2\left(-\dfrac{4}{5}\right)\left(-\dfrac{3}{5}\right) = \dfrac{24}{25}$

$\cos 2u = \cos^2 u - \sin^2 u = \left(-\dfrac{3}{5}\right)^2 - \left(-\dfrac{4}{5}\right)^2 = -\dfrac{7}{25}$

$\tan 2u = \dfrac{2 \tan u}{1 - \tan^2 u} = \dfrac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2} = -\dfrac{24}{7}$

64. $\cos u = -\dfrac{2}{\sqrt{5}}, \dfrac{\pi}{2} < u < \pi \Rightarrow \sin u = \dfrac{1}{\sqrt{5}}$ and

$\tan u = -\dfrac{1}{2}$

$\sin 2u = 2 \sin u \cos u = 2\left(\dfrac{1}{\sqrt{5}}\right)\left(-\dfrac{2}{\sqrt{5}}\right) = -\dfrac{4}{5}$

$\cos 2u = \cos^2 u - \sin^2 u = \left(-\dfrac{2}{\sqrt{5}}\right)^2 - \left(\dfrac{1}{\sqrt{5}}\right)^2 = \dfrac{3}{5}$

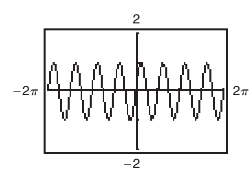
$\tan 2u = \dfrac{2 \tan u}{1 - \tan^2 u} = \dfrac{2\left(-\frac{1}{2}\right)}{1 - \left(-\frac{1}{2}\right)^2} = \dfrac{-1}{\frac{3}{4}} = -\dfrac{4}{3}$

65. $\sin 4x = 2 \sin 2x \cos 2x$

$= 2\left[2 \sin x \cos x\left(\cos^2 x - \sin^2 x\right)\right]$

$= 4 \sin x \cos x\left(2 \cos^2 x - 1\right)$

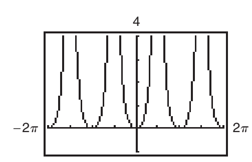
$= 8 \cos^3 x \sin x - 4 \cos x \sin x$



66. $\dfrac{1 - \cos 2x}{1 + \cos 2x} = \dfrac{1 - \left(1 - 2 \sin^2 x\right)}{1 + \left(2 \cos x^2 - 1\right)}$

$= \dfrac{2 \sin^2 x}{2 \cos^2 x}$

$= \tan^2 x$



67. $\tan^2 3x = \dfrac{\sin^2 3x}{\cos^2 3x} = \dfrac{\frac{1 - \cos 6x}{2}}{\frac{1 + \cos 6x}{2}} = \dfrac{1 - \cos 6x}{1 + \cos 6x}$

68. $\sin^2 x \cos^2 x = \left(\dfrac{1 - \cos 2x}{2}\right)\left(\dfrac{1 + \cos 2x}{2}\right)$

$$= \frac{1 - \cos^2 2x}{4}$$

$$= \frac{1 - \left(\dfrac{1 + \cos 4x}{2}\right)}{4}$$

$$= \frac{1 - \cos 4x}{8}$$

69. $\sin(-75°) = -\sqrt{\dfrac{1 - \cos 150°}{2}} = -\sqrt{\dfrac{1 - \left(-\dfrac{\sqrt{3}}{2}\right)}{2}} = -\dfrac{\sqrt{2 + \sqrt{3}}}{2} = -\dfrac{1}{2}\sqrt{2 + \sqrt{3}}$

$\cos(-75°) = -\sqrt{\dfrac{1 + \cos 150°}{2}} = \sqrt{\dfrac{1 + \left(-\dfrac{\sqrt{3}}{2}\right)}{2}} = \dfrac{\sqrt{2 - \sqrt{3}}}{2} = \dfrac{1}{2}\sqrt{2 - \sqrt{3}}$

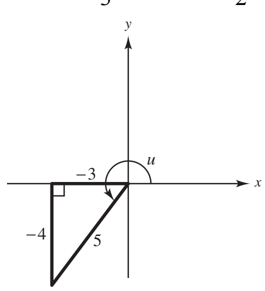
$\tan(-75°) = -\left(\dfrac{1 - \cos 150°}{\sin 150°}\right) = -\left(\dfrac{1 - \left(-\dfrac{\sqrt{3}}{2}\right)}{\dfrac{1}{2}}\right) = -\left(2 + \sqrt{3}\right) = -2 - \sqrt{3}$

70. $\sin\left(\dfrac{5\pi}{12}\right) = \sqrt{\dfrac{1 - \cos\dfrac{5\pi}{6}}{2}} = \sqrt{\dfrac{1 - \left(-\dfrac{\sqrt{3}}{2}\right)}{2}} = \dfrac{\sqrt{2 + \sqrt{3}}}{2} = \dfrac{1}{2}\sqrt{2 + \sqrt{3}}$

$\cos\left(\dfrac{5\pi}{12}\right) = \sqrt{\dfrac{1 + \cos\dfrac{5\pi}{6}}{2}} = \sqrt{\dfrac{1 + \left(-\dfrac{\sqrt{3}}{2}\right)}{2}} = \dfrac{\sqrt{2 - \sqrt{3}}}{2} = \dfrac{1}{2}\sqrt{2 - \sqrt{3}}$

$\tan\left(\dfrac{5\pi}{12}\right) = \dfrac{1 - \cos\dfrac{5\pi}{6}}{\sin\dfrac{5\pi}{6}} = \dfrac{1 - \left(-\dfrac{\sqrt{3}}{2}\right)}{\dfrac{1}{2}} = 2 + \sqrt{3}$

71. $\tan u = \dfrac{4}{3}, \pi < u < \dfrac{3\pi}{2}$



(a) Because u is in Quadrant III, $\dfrac{u}{2}$ is in Quadrant II.

(b) $\sin\dfrac{u}{2} = \sqrt{\dfrac{1 - \cos u}{2}} = \sqrt{\dfrac{1 - \left(-\dfrac{3}{5}\right)}{2}} = \sqrt{\dfrac{4}{5}}$

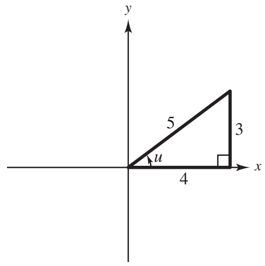
$$= \frac{2\sqrt{5}}{5}$$

$\cos\dfrac{u}{2} = -\sqrt{\dfrac{1 + \cos u}{2}} = -\sqrt{\dfrac{1 + \left(-\dfrac{3}{5}\right)}{2}} = -\sqrt{\dfrac{1}{5}}$

$$= -\frac{\sqrt{5}}{5}$$

$\tan\dfrac{u}{2} = \dfrac{1 - \cos u}{\sin u} = \dfrac{1 - \left(-\dfrac{3}{5}\right)}{\left(-\dfrac{4}{5}\right)} = -2$

72. $\sin u = \frac{3}{5}, 0 < u < \frac{\pi}{2}$



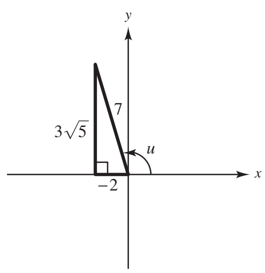
(a) Because u is in Quadrant I, $\frac{u}{2}$ is in Quadrant I.

(b) $\sin\frac{u}{2} = \sqrt{\frac{1-\cos u}{2}} = \sqrt{\frac{1-\left(\frac{4}{5}\right)}{2}} = \sqrt{\frac{1}{10}} = \frac{\sqrt{10}}{10}$

$\cos\frac{u}{2} = \sqrt{\frac{1+\cos u}{2}} = \sqrt{\frac{1+\left(\frac{4}{5}\right)}{2}} = \sqrt{\frac{9}{10}} = \frac{3\sqrt{10}}{10}$

$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{1-\left(\frac{4}{5}\right)}{\frac{3}{5}} = \frac{1}{3}$

73. $\cos u = -\frac{2}{7}, \frac{\pi}{2} < u < \pi$



(a) Because u is in Quadrant II, $\frac{u}{2}$ is in Quadrant I.

(b) $\sin\frac{u}{2} = \sqrt{\frac{1-\cos u}{2}} = \sqrt{\frac{1-\left(-\frac{2}{7}\right)}{2}} = \sqrt{\frac{9}{14}}$

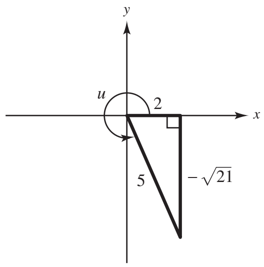
$= \frac{3\sqrt{14}}{14}$

$\cos\frac{u}{2} = \sqrt{\frac{1+\cos u}{2}} = \sqrt{\frac{1+\left(-\frac{2}{7}\right)}{2}} = \sqrt{\frac{5}{14}}$

$= \frac{\sqrt{70}}{14}$

$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{1-\left(-\frac{2}{7}\right)}{\frac{3\sqrt{5}}{7}} = \frac{3\sqrt{5}}{5}$

74. $\tan u = -\dfrac{\sqrt{21}}{2}, \dfrac{3\pi}{2} < u < 2\pi$



(a) Because u is in Quadrant IV, $\dfrac{u}{2}$ is in Quadrant II.

(b) $\sin\dfrac{u}{2} = \sqrt{\dfrac{1-\cos u}{2}} = \sqrt{\dfrac{1-\left(\frac{2}{5}\right)}{2}} = \sqrt{\dfrac{3}{10}} = \dfrac{\sqrt{30}}{10}$

$\cos\dfrac{u}{2} = -\sqrt{\dfrac{1+\cos u}{2}} = -\sqrt{\dfrac{1+\left(\frac{2}{5}\right)}{2}} = -\sqrt{\dfrac{7}{10}} = -\dfrac{\sqrt{70}}{10}$

$\tan\dfrac{u}{2} = \dfrac{1-\cos u}{\sin u} = \dfrac{1-\left(\frac{2}{5}\right)}{\left(-\frac{\sqrt{21}}{5}\right)} = -\dfrac{3}{\sqrt{21}} = -\dfrac{3\sqrt{21}}{21} = -\dfrac{\sqrt{21}}{7}$

75. $\cos 4\theta \sin 6\theta = \frac{1}{2}\left[\sin(4\theta + 6\theta) - \sin(4\theta - 6\theta)\right] = \frac{1}{2}\left[\sin 10\theta - \sin(-2\theta)\right]$

76. $2\sin 7\theta \cos 3\theta = 2 \cdot \frac{1}{2}\left[\sin(7\theta + 3\theta) + \sin(7\theta - 3\theta)\right] = \sin 10\theta + \sin 4\theta$

77. $\cos 6\theta + \cos 5\theta = 2\cos\left(\dfrac{6\theta + 5\theta}{2}\right)\cos\left(\dfrac{6\theta - 5\theta}{2}\right) = 2\cos\dfrac{11\theta}{2}\cos\dfrac{\theta}{2}$

78. $\sin 3x - \sin x = 2\cos\left(\dfrac{3x + x}{2}\right)\sin\left(\dfrac{3x - x}{2}\right)$

$= 2\cos 2x \sin x$

79. $r = \dfrac{1}{32}v_0^2 \sin 2\theta$

range $= 100$ feet

$v_0 = 80$ feet per second

$r = \dfrac{1}{32}(80)^2 \sin 2\theta = 100$

$\sin 2\theta = 0.5$

$2\theta = 30°$

$\theta = 15°$ or $\dfrac{\pi}{12}$

80. Volume V of the trough will be the area A of the isosceles triangle times the length l of the trough.

$V = A \cdot l$

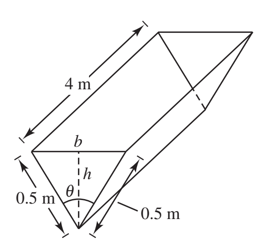
(a) $A = \dfrac{1}{2}bh$

$\cos\dfrac{\theta}{2} = \dfrac{h}{0.5} \Rightarrow h = 0.5\cos\dfrac{\theta}{2}$

$\sin\dfrac{\theta}{2} = \dfrac{\frac{b}{2}}{0.5} \Rightarrow \dfrac{b}{2} = 0.5\sin\dfrac{\theta}{2}$

$A = 0.5\sin\dfrac{\theta}{2}\,0.5\cos\dfrac{\theta}{2} = (0.5)^2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} = 0.25\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$ square meters

$V = (0.25)(4)\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$ cubic meters $= \sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$ cubic meters



Not drawn to scale

(b) $V = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = \frac{1}{2}\sin\theta$ cubic meters

Volume is maximum when $\theta = \frac{\pi}{2}$.

81. False. If $\frac{\pi}{2} < \theta < \pi$, then $\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$, and $\frac{\theta}{2}$ is in Quadrant I. $\cos\frac{\theta}{2} > 0$

82. True. $\cot x \sin^2 x = \left(\frac{\cos x}{\sin x}\right)\sin^2 x = \cos x \sin x$.

83. True. $4\sin(-x)\cos(-x) = 4(-\sin x)\cos x$

$$\begin{aligned} &= -4\sin x\cos x \\ &= -2(2\sin x\cos x) \\ &= -2\sin 2x \end{aligned}$$

84. True. It can be verified using a product-to-sum formula.

$$\begin{aligned} 4\sin 45^\circ\cos 15^\circ &= 4\cdot\frac{1}{2}[\sin 60^\circ + \sin 30^\circ] \\ &= 2\left[\frac{\sqrt{3}}{2} + \frac{1}{2}\right] = \sqrt{3} + 1 \end{aligned}$$

85. Yes. *Sample Answer*. When the domain is all real numbers, the solutions of $\sin x = \frac{1}{2}$ are $x = \frac{\pi}{6} + 2n\pi$ and $x = \frac{5\pi}{6} + 2n\pi$, so there are infinitely many solutions.

Problem Solving for Chapter 2

1. $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{\sqrt{1 - \cos^2\theta}}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} = \pm\frac{1}{\sqrt{1 - \cos^2\theta}}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \pm\frac{\cos\theta}{\sqrt{1 - \cos^2\theta}}$$

You also have the following relationships:

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\theta = \frac{\cos[(\pi/2) - \theta]}{\cos\theta}$$

$$\csc\theta = \frac{1}{\cos[(\pi/2) - \theta]}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\cos[(\pi/2) - \theta]}$$

2. $$\begin{aligned} \cos\left[\frac{(2n+1)\pi}{2}\right] &= \cos\left(\frac{2n\pi + \pi}{2}\right) \\ &= \cos\left(n\pi + \frac{\pi}{2}\right) \\ &= \cos n\pi\cos\frac{\pi}{2} - \sin n\pi\sin\frac{\pi}{2} \\ &= (\pm 1)(0) - (0)(1) \\ &= 0 \end{aligned}$$

So, $\cos\left[\frac{(2n+1)\pi}{2}\right] = 0$ for all integers n.

3. $$\begin{aligned} \sin\left[\frac{(12n+1)\pi}{6}\right] &= \sin\left[\frac{1}{6}(12n\pi + \pi)\right] \\ &= \sin\left(2n\pi + \frac{\pi}{6}\right) \\ &= \sin\frac{\pi}{6} = \frac{1}{2} \end{aligned}$$

So, $\sin\left[\frac{(12n+1)\pi}{6}\right] = \frac{1}{2}$ for all integers n.

4. $p(t) = \dfrac{1}{4\pi}\left[p_1(t) + 30p_2(t) + p_3(t) + p_5(t) + 30p_6(t)\right]$

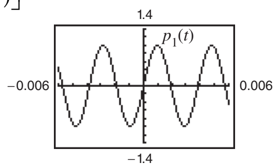
(a) $p_1(t) = \sin(524\pi t)$

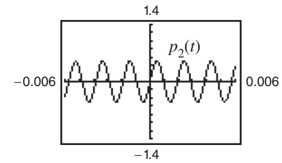
$p_2(t) = \dfrac{1}{2}\sin(1048\pi t)$

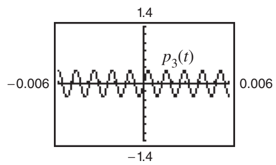
$p_3(t) = \dfrac{1}{3}\sin(1572\pi t)$

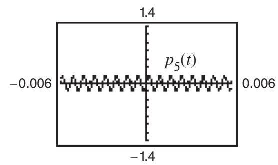
$p_5(t) = \dfrac{1}{5}\sin(2620\pi t)$

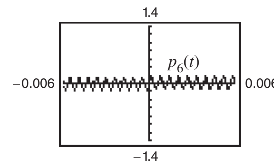
$p_6(t) = \dfrac{1}{6}\sin(3144\pi t)$







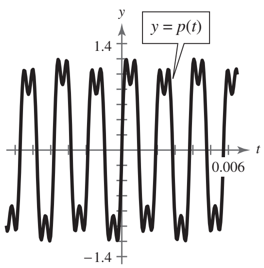




The graph of

$$p(t) = \frac{1}{4\pi}\left[\sin(524\pi t) + 15\sin(1048\pi t) + \frac{1}{3}\sin(1572\pi t) + \frac{1}{5}\sin(2620\pi t) + 5\sin(3144\pi t)\right]$$

yields the graph shown in the text below.

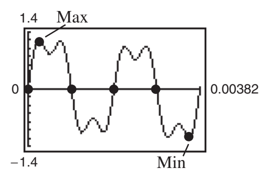


(b)

Function	Period
$p_1(t)$	$\dfrac{2\pi}{524\pi} = \dfrac{1}{262} \approx 0.0038$
$p_2(t)$	$\dfrac{2\pi}{1048\pi} = \dfrac{1}{524} \approx 0.0019$
$p_3(t)$	$\dfrac{2\pi}{1572\pi} = \dfrac{1}{786} \approx 0.0013$
$p_5(t)$	$\dfrac{2\pi}{2620\pi} = \dfrac{1}{1310} \approx 0.0008$
$p_6(t)$	$\dfrac{2\pi}{3144\pi} = \dfrac{1}{1572} \approx 0.0006$

The graph of p appears to be periodic with a period of $\dfrac{1}{262} \approx 0.0038$.

(c)



Over one cycle, $0 \le t < \dfrac{1}{262}$, you have five t-intercepts:

$t = 0, t \approx 0.00096, t \approx 0.00191, t \approx 0.00285,$

$t \approx 0.00382$

(d) The absolute maximum value of p over one cycle is $p \approx 1.1952$, and the absolute minimum value of p over one cycle is $p \approx -1.1952$.

5. From the figure, it appears that $u + v = w$. Assume that u, v, and w are all in Quadrant I.

From the figure:

$$\tan u = \frac{s}{3s} = \frac{1}{3}$$

$$\tan v = \frac{s}{2s} = \frac{1}{2}$$

$$\tan w = \frac{s}{s} = 1$$

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} = \frac{1/3 + 1/2}{1 - (1/3)(1/2)} = \frac{5/6}{1 - (1/6)} = 1 = \tan w.$$

So, $\tan(u + v) = \tan w$. Because u, v, and w are all in Quadrant I, you have

$\arctan[\tan(u + v)] = \arctan[\tan w]u + v = w.$

6. $y = -\frac{16}{v_0^2 \cos^2 \theta}x^2 + (\tan \theta)x + h_0$

Let $h_0 = 0$ and take half of the horizontal distance:

$$\frac{1}{2}\left(\frac{1}{32}v_0^2 \sin 2\theta\right) = \frac{1}{64}v_0^2(2 \sin \theta \cos \theta) = \frac{1}{32}v_0^2 \sin \theta \cos \theta$$

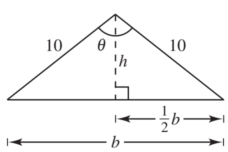
Substitute this expression for x in the model.

$$y = -\frac{16}{v_0^2 \cos^2 \theta}\left(\frac{1}{32}v_0^2 \sin \theta \cos \theta\right)^2 + \left(\frac{\sin \theta}{\cos \theta}\right)\left(\frac{1}{32}v_0^2 \sin \theta \cos \theta\right)$$

$$= -\frac{1}{64}v_0^2 \sin^2 \theta + \frac{1}{32}v_0^2 \sin^2 \theta$$

$$= \frac{1}{64}v_0^2 \sin^2 \theta$$

7. (a)



$$\sin \frac{\theta}{2} = \frac{\frac{1}{2}b}{10} \quad \text{and} \quad \cos \frac{\theta}{2} = \frac{h}{10}$$

$$b = 20 \sin \frac{\theta}{2} \qquad h = 10 \cos \frac{\theta}{2}$$

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}\left(20 \sin \frac{\theta}{2}\right)\left(10 \cos \frac{\theta}{2}\right)$$

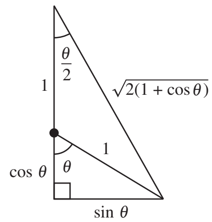
$$= 100 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

(b) $A = 50\left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right)$

$$= 50 \sin\left(2\left(\frac{\theta}{2}\right)\right)$$

$$= 50 \sin \theta$$

Because $\sin \frac{\pi}{2} = 1$ is a maximum, $\theta = \frac{\pi}{2}$. So, the area is a maximum at $A = 50 \sin \frac{\pi}{2} = 50$ square meters.

8. 

The hypotenuse of the larger right triangle is:

$$\sqrt{\sin^2\theta + (1+\cos\theta)^2} = \sqrt{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}$$
$$= \sqrt{2 + 2\cos\theta}$$
$$= \sqrt{2(1+\cos\theta)}$$

$$\sin\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{\sqrt{2(1+\cos\theta)}} = \frac{\sin\theta}{\sqrt{2(1+\cos\theta)}}\cdot\frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}} = \frac{\sin\theta\sqrt{1-\cos\theta}}{\sqrt{2(1-\cos^2\theta)}} = \frac{\sin\theta\sqrt{1-\cos\theta}}{\sqrt{2}\sin\theta} = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{\sqrt{2(1+\cos\theta)}} = \frac{\sqrt{(1+\cos\theta)^2}}{\sqrt{2(1+\cos\theta)}} = \sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta}$$

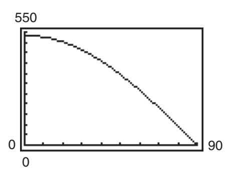
9. $F = \dfrac{0.6W\sin(\theta + 90°)}{\sin 12°}$

(a) $F = \dfrac{0.6W(\sin\theta\cos 90° + \cos\theta\sin 90°)}{\sin 12°}$

$= \dfrac{0.6W[(\sin\theta)(0) + (\cos\theta)(1)]}{\sin 12°}$

$= \dfrac{0.6W\cos\theta}{\sin 12°}$

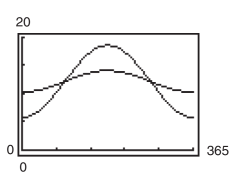
(b) Let $y_1 = \dfrac{0.6(185)\cos x}{\sin 12°}$.



(c) The force is maximum (533.88 pounds) when $\theta = 0°$.
The force is minimum (0 pounds) when $\theta = 90°$.

10. Seward: $D = 12.2 - 6.4\cos\left[\dfrac{\pi(t+0.2)}{182.6}\right]$

New Orleans: $D = 12.2 - 1.9\cos\left[\dfrac{\pi(t+0.2)}{182.6}\right]$

(a) 

(b) The graphs intersect when $t \approx 91$ and when $t \approx 274$. These values correspond to April 1 and October 1, the spring equinox and the fall equinox.

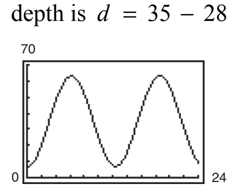
(c) Seward has the greater variation in the number of daylight hours. This is determined by the amplitudes, 6.4 and 1.9.

(d) Period: $\dfrac{2\pi}{\pi/182.6} = 365.2$ days

11. $d = 35 - 28\cos\dfrac{\pi}{6.2}t$ when $t = 0$ corresponds to 12:00 A.M.

(a) The high tides occur when $\cos\dfrac{\pi}{6.2}t = -1$. Solving yields $t = 6.2$ or $t = 18.6$.
These t-values correspond to 6:12 A.M. and 6:36 P.M.
The low tide occurs when $\cos\dfrac{\pi}{6.2}t = 1$. Solving yields $t = 0$ and $t = 12.4$ which corresponds to 12:00 A.M. and 12:24 P.M.

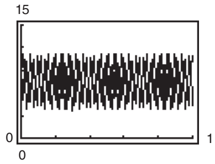
(b) The water depth is never 3.5 feet. At low tide, the depth is $d = 35 - 28 = 7$ feet.

(c) 

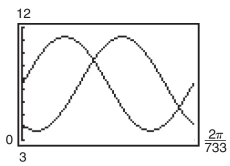
12. $h_1 = 3.75 \sin 733t + 7.5$

$h_2 = 3.75 \sin 733\left(t + \frac{4\pi}{3}\right) + 7.5$

(a)



(b) The period for h_1 and h_2 is $\frac{2\pi}{733} \approx 0.0086$.



The graphs intersect twice per cycle.

There are $\frac{1}{2\pi/733} \approx 116.66$ cycles in the interval $[0, 1]$, so the graphs intersect approximately 233.3 times.

13. (a) $$n = \frac{\sin\left(\frac{\theta}{2} + \frac{\alpha}{2}\right)}{\sin \frac{\theta}{2}}$$

$$= \frac{\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\alpha}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

$$= \cos\left(\frac{\alpha}{2}\right) + \cot\left(\frac{\theta}{2}\right)\sin\left(\frac{\alpha}{2}\right)$$

For $\alpha = 60°$, $n = \cos 30° + \cot\left(\frac{\theta}{2}\right)\sin 30°$

$$n = \frac{\sqrt{3}}{2} + \frac{1}{2}\cot\left(\frac{\theta}{2}\right).$$

(b) For glass, $n = 1.50$.

$$1.50 = \frac{\sqrt{3}}{2} + \frac{1}{2}\cot\left(\frac{\theta}{2}\right)$$

$$2\left(1.50 - \frac{\sqrt{3}}{2}\right) = \cot\left(\frac{\theta}{2}\right)$$

$$\frac{1}{3 - \sqrt{3}} = \tan\left(\frac{\theta}{2}\right)$$

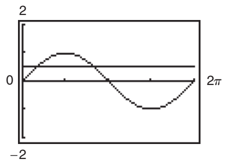
$$\theta = 2\tan^{-1}\left(\frac{1}{3 - \sqrt{3}}\right)$$

$$\theta \approx 76.5°$$

14. (a) $$\begin{aligned}\sin(u + v + w) &= \sin\left[(u + v) + w\right] \\ &= \sin(u + v)\cos w + \cos(u + v)\sin w \\ &= \left[\sin u \cos v + \cos u \sin v\right]\cos w + \left[\cos u \cos v - \sin u \sin v\right]\sin w \\ &= \sin u \cos v \cos w + \cos u \sin v \cos w + \cos u \cos v \sin w - \sin u \sin v \sin w\end{aligned}$$

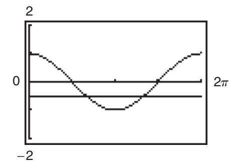
(b) $$\begin{aligned}\tan(u + v + w) &= \tan\left[(u + v) + w\right] \\ &= \frac{\tan(u + v) + \tan w}{1 - \tan(u + v)\tan w} \\ &= \frac{\left[\frac{\tan u + \tan v}{1 - \tan u \tan v}\right] + \tan w}{1 - \left[\frac{\tan u + \tan v}{1 - \tan u \tan v}\right]\tan w} \cdot \frac{(1 - \tan u \tan v)}{(1 - \tan u \tan v)} \\ &= \frac{\tan u + \tan v + (1 - \tan u \tan v)\tan w}{(1 - \tan u \tan v) - (\tan u + \tan v)\tan w} \\ &= \frac{\tan u + \tan v + \tan w - \tan u \tan v \tan w}{1 - \tan u \tan v - \tan u \tan w - \tan v \tan w}\end{aligned}$$

15. (a) Let $y_1 = \sin x$ and $y_2 = 0.5$.



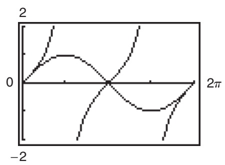
$\sin x \geq 0.5$ on the interval $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$.

(b) Let $y_1 = \cos x$ and $y_2 = -0.5$.



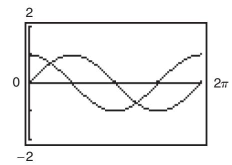
$\cos x \leq -0.5$ on the interval $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$.

(c) Let $y_1 = \tan x$ and $y_2 = \sin x$.



$\tan x < \sin x$ on the intervals $\left(\frac{\pi}{2}, \pi\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$.

(d) Let $y_1 = \cos x$ and $y_2 = \sin x$.



$\cos x \geq \sin x$ on the intervals $\left[0, \frac{\pi}{4}\right]$ and $\left[\frac{5\pi}{4}, 2\pi\right)$.

16. (a)

$$\begin{aligned} f(x) &= \sin^4 x + \cos^4 x \\ &= \left(\sin^2 x\right)^2 + \left(\cos^2 x\right)^2 \\ &= \left(\frac{1 - \cos 2x}{2}\right)^2 + \left(\frac{1 + \cos 2x}{2}\right)^2 \\ &= \frac{1}{4}\left[\left(1 - 2\cos 2x + \cos^2 2x\right) + \left(1 + 2\cos 2x + \cos^2 2x\right)\right] \\ &= \frac{1}{4}\left(2 + 2\cos^2 2x\right) \\ &= \frac{1}{2}\left(1 + \cos^2 2x\right) \\ &= \frac{1}{2}\left(1 + \frac{\cos 4x}{2}\right) \\ &= \frac{1}{4}\left(3 + \cos 4x\right) \end{aligned}$$

(b)

$$\begin{aligned} f(x) &= \sin^4 x + \cos^4 x \\ &= \left(\sin^2 x\right)^2 + \cos^4 x \\ &= \left(1 - \cos^2 x\right)^2 + \cos^4 x \\ &= 1 - 2\cos^2 2x + \cos^4 x + \cos^4 x \\ &= 2\cos^4 x - 2\cos^2 x + 1 \end{aligned}$$

(c)

$$\begin{aligned} f(x) &= \sin^4 x + \cos^4 x \\ &= \sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x - 2\sin^2 x\cos^2 x \\ &= \left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x\cos^2 x \\ &= 1 - 2\sin^2 x\cos^2 x \end{aligned}$$

(d) $$\begin{aligned} f(x) &= 1 - 2\sin^2 x\cos^2 x \\ &= 1 - 2\left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) \\ &= 1 - \frac{1}{2}\left(1 - \cos^2 2x\right) \\ &= \frac{1}{2} + \frac{1}{2}\cos^2 2x \\ &= \frac{1}{2} + \frac{1}{2}\left(1 - \sin^2 2x\right) \\ &= 1 - \frac{1}{2}\sin^2 2x \end{aligned}$$

(e) No; there is often more than one way to rewrite a trigonometric expression, so your result and your friend's result could both be correct.

Practice Test for Chapter 2

1. Find the value of the other five trigonometric functions, given $\tan x = \frac{4}{11}$, $\sec x < 0$.

2. Simplify $\dfrac{\sec^2 x + \csc^2 x}{\csc^2 x(1 + \tan^2 x)}$.

3. Rewrite as a single logarithm and simplify $\ln|\tan\theta| - \ln|\cot\theta|$.

4. True or false:

$$\cos\left(\frac{\pi}{2} - x\right) = \frac{1}{\csc x}$$

5. Factor and simplify: $\sin^4 x + (\sin^2 x)\cos^2 x$

6. Multiply and simplify: $(\csc x + 1)(\csc x - 1)$

7. Rationalize the denominator and simplify:

$$\frac{\cos^2 x}{1 - \sin x}$$

8. Verify:

$$\frac{1 + \cos\theta}{\sin\theta} + \frac{\sin\theta}{1 + \cos\theta} = 2\csc\theta$$

9. Verify:

$$\tan^4 x + 2\tan^2 x + 1 = \sec^4 x$$

10. Use the sum or difference formulas to determine:

(a) $\sin 105^\circ$

(b) $\tan 15^\circ$

11. Simplify: $(\sin 42^\circ)\cos 38^\circ - (\cos 42^\circ)\sin 38^\circ$

12. Verify $\tan\left(\theta + \dfrac{\pi}{4}\right) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$.

13. Write $\sin(\arcsin x - \arccos x)$ as an algebraic expression in x.

14. Use the double-angle formulas to determine:

(a) $\cos 120^\circ$

(b) $\tan 300^\circ$

15. Use the half-angle formulas to determine:

(a) $\sin 22.5^\circ$

(b) $\tan \dfrac{\pi}{12}$

16. Given $\sin\theta = 4/5$, θ lies in Quadrant II, find $\cos(\theta/2)$.

17. Use the power-reducing identities to write $(\sin^2 x)\cos^2 x$ in terms of the first power of cosine.

18. Rewrite as a sum: $6(\sin 5\theta)\cos 2\theta$.

19. Rewrite as a product: $\sin(x + \pi) + \sin(x - \pi)$.

20. Verify $\dfrac{\sin 9x + \sin 5x}{\cos 9x - \cos 5x} = -\cot 2x$.

21. Verify:

$(\cos u)\sin v = \frac{1}{2}\left[\sin(u + v) - \sin(u - v)\right]$.

22. Find all solutions in the interval $[0, 2\pi)$:

$4\sin^2 x = 1$

23. Find all solutions in the interval $[0, 2\pi)$:

$\tan^2 \theta + \left(\sqrt{3} - 1\right)\tan \theta - \sqrt{3} = 0$

24. Find all solutions in the interval $[0, 2\pi)$:

$\sin 2x = \cos x$

25. Use the quadratic formula to find all solutions in the interval $[0, 2\pi)$:

$\tan^2 x - 6\tan x + 4 = 0$

2 Analytic Trigonometry

2.1 Using Fundamental Identities

Objectives

- Recognize and write the fundamental trigonometric identities.

- Use the fundamental trigonometric identities to evaluate trigonometric functions, simplify trigonometric expressions, and rewrite trigonometric expressions.

Introduction

Introduction

You will learn how to use the fundamental identities to do the following.

1. Evaluate trigonometric functions.

2. Simplify trigonometric expressions.

3. Develop additional trigonometric identities.

4. Solve trigonometric equations.

Introduction

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin u = \frac{1}{\csc u} \qquad \cos u = \frac{1}{\sec u} \qquad \tan u = \frac{1}{\cot u}$$

$$\csc u = \frac{1}{\sin u} \qquad \sec u = \frac{1}{\cos u} \qquad \cot u = \frac{1}{\tan u}$$

Quotient Identities

$$\tan u = \frac{\sin u}{\cos u} \qquad \cot u = \frac{\cos u}{\sin u}$$

Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1 \qquad 1 + \tan^2 u = \sec^2 u \qquad 1 + \cot^2 u = \csc^2 u$$

Introduction

cont'd

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \cos\left(\frac{\pi}{2} - u\right) = \sin u$$

$$\tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u$$

$$\sec\left(\frac{\pi}{2} - u\right) = \csc u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

Even/Odd Identities

$$\sin(-u) = -\sin u \qquad \cos(-u) = \cos u \qquad \tan(-u) = -\tan u$$

$$\csc(-u) = -\csc u \qquad \sec(-u) = \sec u \qquad \cot(-u) = -\cot u$$

Introduction

Pythagorean identities are sometimes used in radical form such as

$$\sin u = \pm\sqrt{1 - \cos^2 u}$$

or

$$\tan u = \pm\sqrt{\sec^2 u - 1}$$

where the sign depends on the choice of *u*.

Using the Fundamental Identities

Using the Fundamental Identities

One common application of trigonometric identities is to use given values of trigonometric functions to evaluate other trigonometric functions.

Example 1 – *Using Identities to Evaluate a Function*

Use the values $\sec u = -\frac{3}{2}$ and tan $u > 0$ to find the values of all six trigonometric functions.

Solution:

Using a reciprocal identity, you have

$$\cos u = \frac{1}{\sec u} = \frac{1}{-3/2} = -\frac{2}{3}.$$

Using a Pythagorean identity, you have

$$\sin^2 u = 1 - \cos^2 u$$

Pythagorean identity

Example 1 – *Solution*

cont'd

$$= 1 - \left(-\frac{2}{3}\right)^2$$ Substitute $-\frac{2}{3}$ for cos *u*.

$$= \frac{5}{9}.$$ Simplify.

Because sec $u < 0$ and tan $u > 0$, it follows that u lies in Quadrant III.

Moreover, because sin u is negative when u is in Quadrant III, choose the negative root and obtain

$$\sin u = -\sqrt{5}/3.$$

Example 1 – *Solution*

cont'd

Knowing the values of the sine and cosine enables you to find the values of all six trigonometric functions.

$$\sin u = -\frac{\sqrt{5}}{3}$$

$$\cos u = -\frac{2}{3}$$

$$\tan u = \frac{\sin u}{\cos u} = \frac{-\sqrt{5}/3}{-2/3} = \frac{\sqrt{5}}{2}$$

Example 1 – *Solution*

cont'd

$$\csc u = \frac{1}{\sin u} = -\frac{3}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

$$\sec u = \frac{1}{\cos u} = -\frac{3}{2}$$

$$\cot u = \frac{1}{\tan u} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

Example 2 – *Simplifying a Trigonometric Expression*

Simplify

$$\sin x \cos^2 x - \sin x.$$

Solution:

First factor out a common monomial factor and then use a fundamental identity.

$\sin x \cos^2 x - \sin x = \sin x(\cos^2 x - 1)$ — Factor out common monomial factor.

$= -\sin x(1 - \cos^2 x)$ — Factor out –1.

$= -\sin x(\sin^2 x)$ — Pythagorean identity

$= -\sin^3 x$ — Multiply.

Using the Fundamental Identities

When factoring trigonometric expressions, it is helpful to find a special polynomial factoring form that fits the expression.

On occasion, factoring or simplifying can best be done by first rewriting the expression in terms of just *one* trigonometric function or in terms of *sine and cosine only*.

Example 7 – *Rewriting a Trigonometric Expression*

Rewrite $\dfrac{1}{1 + \sin x}$ so that it is *not* in fractional form.

Solution:

From the Pythagorean identity
$\cos^2 x = 1 - \sin^2 x = (1 - \sin x)(1 + \sin x)$
multiplying both the numerator and the denominator by $(1 - \sin x)$ will produce a monomial denominator.

$$\frac{1}{1 + \sin x} = \frac{1}{1 + \sin x} \cdot \frac{1 - \sin x}{1 - \sin x}$$

Multiply numerator and denominator by $(1 - \sin x)$.

$$= \frac{1 - \sin x}{1 - \sin^2 x}$$

Multiply.

Example 7 – *Solution*

cont'd

$$= \frac{1 - \sin x}{\cos^2 x}$$

Pythagorean identity

$$= \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}$$

Write as separate fractions.

$$= \frac{1}{\cos^2 x} - \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

Product of fractions

$$= \sec^2 x - \tan x \sec x$$

Reciprocal and quotient identities

2 Analytic Trigonometry

2.2

Verifying Trigonometric Identities

Objective

- Verify trigonometric identities.

Introduction

Introduction

In this section, you will study techniques for verifying trigonometric identities.

Remember that a *conditional equation* is an equation that is true for only some of the values in its domain.
For example, the conditional equation

$\sin x = 0$ Conditional equation

is true only for

$x = n\pi$

where *n* is an integer. When you find these values, you are *solving* the equation.

Introduction

On the other hand, an equation that is true for all real values in the domain of the variable is an *identity*. For example, the familiar equation

$\sin^2 x = 1 - \cos^2 x$ Identity

is true for all real numbers x. So, it is an identity.

Verifying Trigonometric Identities

Verifying Trigonometric Identities

Although there are similarities, verifying that a trigonometric equation is an identity is quite different from solving an equation. There is no well-defined set of rules to follow in verifying trigonometric identities, and it is best to learn the process by practicing.

Guidelines for Verifying Trigonometric Identities

1. Work with one side of the equation at a time. It is often better to work with the more complicated side first.
2. Look for opportunities to factor an expression, add fractions, square a binomial, or create a monomial denominator.
3. Look for opportunities to use the fundamental identities. Note which functions are in the final expression you want. Sines and cosines pair up well, as do secants and tangents, and cosecants and cotangents.
4. If the preceding guidelines do not help, then try converting all terms to sines and cosines.
5. Always try *something*. Even making an attempt that leads to a dead end can provide insight.

Verifying Trigonometric Identities

Verifying trigonometric identities is a useful process when you need to convert a trigonometric expression into a form that is more useful algebraically.

When you verify an identity, you cannot *assume* that the two sides of the equation are equal because you are trying to verify that they *are* equal.

As a result, when verifying identities, you cannot use operations such as adding the same quantity to each side of the equation or cross multiplication.

Example 1 – *Verifying a Trigonometric Identity*

Verify the identity $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta.$

Solution:

Start with the left side because it is more complicated.

$$\frac{\sec^2\theta - 1}{\sec^2\theta} = \frac{(\tan^2\theta + 1) - 1}{\sec^2\theta}$$ Pythagorean identity

$$= \frac{\tan^2\theta}{\sec^2\theta}$$ Simplify.

$$= \tan^2\theta(\cos^2\theta)$$ Reciprocal identity

$$= \frac{\sin^2\theta}{\cancel{(\cos^2\theta)}}\cancel{(\cos^2\theta)}$$ Quotient identity

Example 1 – *Solution*

cont'd

$$= \sin^2 \theta \qquad \text{Simplify.}$$

Notice that you verify the identity by starting with the left side of the equation (the more complicated side) and using the fundamental trigonometric identities to simplify it until you obtain the right side.

2 Analytic Trigonometry

2.3 Solving Trigonometric Equations

Objectives

- Use standard algebraic techniques to solve trigonometric equations.
- Solve trigonometric equations of quadratic type.
- Solve trigonometric equations involving multiple angles.
- Use inverse trigonometric functions to solve trigonometric equations.

Introduction

Introduction

To solve a trigonometric equation, use standard algebraic techniques (when possible) such as collecting like terms and factoring.

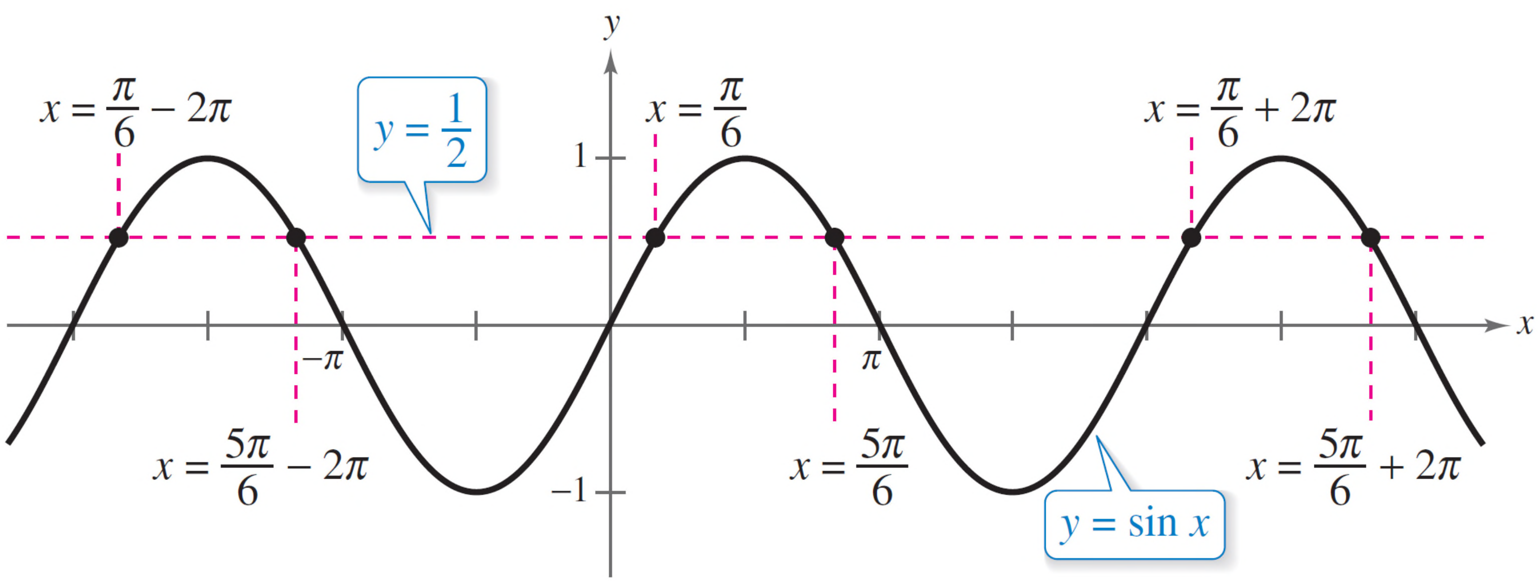
Your preliminary goal in solving a trigonometric equation is to *isolate* the trigonometric function on one side of the equation.

For example, to solve the equation 2 sin x = 1, divide each side by 2 to obtain

$$\sin x = \frac{1}{2}.$$

Introduction

To solve for *x*, note in the figure below that the equation $\sin x = \frac{1}{2}$ has solutions $x = \pi/6$ and $x = 5\pi/6$ in the interval $[0, 2\pi)$.



Introduction

Moreover, because sin *x* has a period of 2π, there are infinitely many other solutions, which can be written as

$$x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi$$

General solution

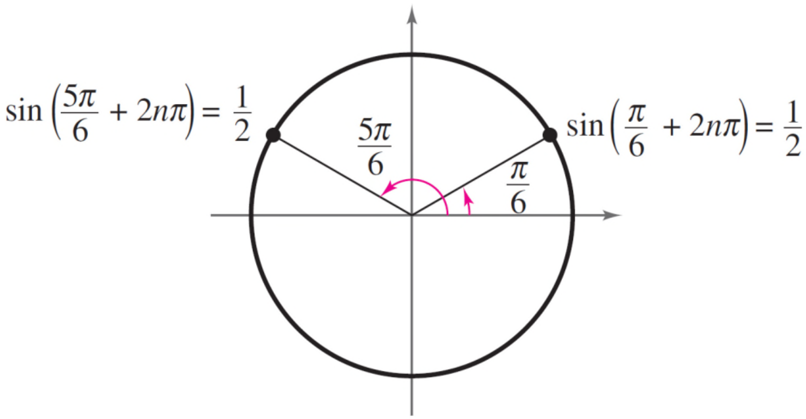
where *n* is an integer, as shown above.

Introduction

The figure below illustrates another way to show that the equation $\sin x = \frac{1}{2}$ has infinitely many solutions.

Any angles that are coterminal with $\pi/6$ or $5\pi/6$ will also be solutions of the equation.

When solving trigonometric equations, you should write your answer(s) using exact values rather than decimal approximations.



Example 1 – *Collecting Like Terms*

Solve $\sin x + \sqrt{2} = -\sin x.$

Solution:

Begin by isolating sin *x* on one side of the equation.

$$\sin x + \sqrt{2} = -\sin x$$ Write original equation.

$$\sin x + \sin x + \sqrt{2} = 0$$ Add sin *x* to each side.

$$\sin x + \sin x = -\sqrt{2}$$ Subtract $\sqrt{2}$ from each side.

Example 1 – *Solution*

cont'd

$$2 \sin x = -\sqrt{2}$$ Combine like terms.

$$\sin x = -\frac{\sqrt{2}}{2}$$ Divide each side by 2.

Because sin *x* has a period of 2π, first find all solutions in the interval $[0, 2\pi)$.

These solutions are $x = 5\pi/4$ and $x = 7\pi/4$. Finally, add multiples of 2π to each of these solutions to obtain the general form

$$x = \frac{5\pi}{4} + 2n\pi \text{ and } x = \frac{7\pi}{4} + 2n\pi$$ General solution

where *n* is an integer.

Equations of Quadratic Type

Equations of Quadratic Type

Many trigonometric equations are of quadratic type $ax^2 + bx + c = 0$, as shown below.

Quadratic in sin *x*

$2 \sin^2 x - \sin x - 1 = 0$

$2(\sin x)^2 - \sin x - 1 = 0$

Quadratic in sec *x*

$\sec^2 x - 3 \sec x - 2 = 0$

$(\sec x)^2 - 3(\sec x) - 2 = 0$

To solve equations of this type, factor the quadratic or, when this is not possible, use the Quadratic Formula.

Example 4 – *Factoring an Equation of Quadratic Type*

Find all solutions of $2\sin^2 x - \sin x - 1 = 0$ in the interval $[0, 2\pi)$.

Solution:

Treat the equation as a quadratic in $\sin x$ and factor.

$2\sin^2 x - \sin x - 1 = 0$ Write original equation.

$(2\sin x + 1)(\sin x - 1) = 0$ Factor.

Example 4 – *Solution*

cont'd

Setting each factor equal to zero, you obtain the following solutions in the interval $[0, 2\pi)$.

$2 \sin x + 1 = 0$ and $\sin x - 1 = 0$

$$\sin x = -\frac{1}{2} \qquad\qquad \sin x = 1$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6} \qquad\qquad x = \frac{\pi}{2}$$

Functions Involving Multiple Angles

Functions Involving Multiple Angles

The next example involve trigonometric functions of multiple angles of the forms cos *ku* and tan *ku*.

To solve equations of these forms, first solve the equation for *ku*, and then divide your result by *k*.

Example 7 – *Solving a Multiple-Angle Equation*

Solve $2 \cos 3t - 1 = 0$.

Solution:

$2 \cos 3t - 1 = 0$	Write original equation.
$2 \cos 3t = 1$	Add 1 to each side.
$\cos 3t = \frac{1}{2}$	Divide each side by 2.

Example 7 – *Solution*

cont'd

In the interval $[0, 2\pi)$, you know that $3t = \pi/3$ and $3t = 5\pi/3$ are the only solutions, so, in general, you have

$$3t = \frac{\pi}{3} + 2n\pi \quad \text{and} \quad 3t = \frac{5\pi}{3} + 2n\pi.$$

Dividing these results by 3, you obtain the general solution

$$t = \frac{\pi}{9} + \frac{2n\pi}{3} \quad \text{and} \quad t = \frac{5\pi}{9} + \frac{2n\pi}{3}$$

General solution

where *n* is an integer.

Using Inverse Functions

Example 9 – *Using Inverse Functions*

$\sec^2 x - 2 \tan x = 4$	Original equation
$1 + \tan^2 x - 2 \tan x - 4 = 0$	Pythagorean identity
$\tan^2 x - 2 \tan x - 3 = 0$	Combine like terms.
$(\tan x - 3)(\tan x + 1) = 0$	Factor.

Setting each factor equal to zero, you obtain two solutions in the interval $(-\pi/2, \pi/2)$. [We know that the range of the inverse tangent function is $(-\pi/2, \pi/2)$.]

$x = \arctan 3$ and $x = \arctan(-1) = -\pi/4$

Example 9 – *Using Inverse Functions* cont'd

Finally, because tan x has a period of π, you add multiples of π to obtain

$$x = \arctan 3 + n\pi \quad \text{and} \quad x = (-\pi/4) + n\pi$$ General solution

where n is an integer.

You can use a calculator to approximate the value of arctan 3.

2 Analytic Trigonometry

2.4 Sum and Difference Formulas

Objective

- Use sum and difference formulas to evaluate trigonometric functions, verify identities, and solve trigonometric equations.

Using Sum and Difference Formulas

Using Sum and Difference Formulas

In this and the following section, you will study the uses of several trigonometric identities and formulas.

Sum and Difference Formulas

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$

$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \qquad \tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Example 1 shows how **sum and difference formulas** can enable you to find exact values of trigonometric functions involving sums or differences of special angles.

Example 1 – *Evaluating a Trigonometric Function*

Find the exact value of $\sin \frac{\pi}{12}$.

Solution:

To find the *exact* value of sin π /12, use the fact that

$$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}.$$

Consequently, the formula for sin(*u* – *v*) yields

$$\sin \frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

Example 1 – *Solution*

cont'd

$$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}.$$

Try checking this result on your calculator. You will find that $\sin \pi/12 \approx 0.259$.

Example 5 – *Proving a Cofunction identity*

Use a difference formula to prove the cofunction identity

$\cos\left(\frac{\pi}{2} - x\right)$ = sin *x*.

Solution:

Using the formula for cos(*u* – *v*), you have

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$$

$$= (0)(\cos x) + (1)(\sin x)$$

$$= \sin x.$$

Using Sum and Difference Formulas

Sum and difference formulas can be used to rewrite expressions such as

$\sin\left(\theta + \frac{n\pi}{2}\right)$ and $\cos\left(\theta + \frac{n\pi}{2}\right)$, where *n* is an integer

as expressions involving only sin θ or cos θ.

The resulting formulas are called **reduction formulas**.

Example 7 – *Solving a Trigonometric Equation*

Find all solutions of $\sin[x + (\pi/4)] + \sin[x - (\pi/4)] = -1$ in the interval $[0, 2\pi)$.

Solution:

Algebraic Solution

Using sum and difference formulas, rewrite the equation as

$$\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4} + \sin x \cos\frac{\pi}{4} - \cos x \sin\frac{\pi}{4} = -1$$

$$2 \sin x \cos\frac{\pi}{4} = -1$$

Example 7 – *Solution*

cont'd

$$2(\sin x)\left(\frac{\sqrt{2}}{2}\right) = -1$$

$$\sin x = -\frac{1}{\sqrt{2}}$$

$$\sin x = -\frac{\sqrt{2}}{2}.$$

So, the only solutions in the interval $[0, 2\pi)$ are $x = 5\pi/4$ and $x = 7\pi/4$.

2 Analytic Trigonometry

2.5 Multiple-Angle and Product-to-Sum Formulas

Objectives

- Use multiple-angle formulas to rewrite and evaluate trigonometric functions.
- Use power-reducing formulas to rewrite and evaluate trigonometric functions.
- Use half-angle formulas to rewrite and evaluate trigonometric functions.

Objectives

- Use product-to-sum and sum-to-product formulas to rewrite and evaluate trigonometric functions.

- Use trigonometric formulas to rewrite real-life models.

Multiple-Angle Formulas

Multiple-Angle Formulas

In this section, you will study four other categories of trigonometric identities.

1. The first category involves *functions of multiple angles* such as $\sin ku$ and $\cos ku$.

2. The second category involves *squares of trigonometric functions* such as $\sin^2 u$.

3. The third category involves *functions of half-angles* such as $\sin(u/2)$.

4. The fourth category involves *products of trigonometric functions* such as $\sin u \cos v$.

Multiple-Angle Formulas

You should learn the **double-angle formulas** because they are used often in trigonometry and calculus.

Double-Angle Formulas

$$\sin 2u = 2 \sin u \cos u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

$$\cos 2u = \cos^2 u - \sin^2 u$$
$$= 2 \cos^2 u - 1$$
$$= 1 - 2 \sin^2 u$$

Example 1 – *Solving a Multiple-Angle Equation*

Solve $2 \cos x + \sin 2x = 0$.

Solution:

Begin by rewriting the equation so that it involves functions of x (rather than $2x$). Then factor and solve.

$$2 \cos x + \sin 2x = 0$$ Write original equation.

$$2 \cos x + 2 \sin x \cos x = 0$$ Double-angle formula.

$$2 \cos x(1 + \sin x) = 0$$ Factor.

$$2 \cos x = 0 \quad \text{and} \quad 1 + \sin x = 0$$ Set factors equal to zero.

Example 1 – *Solution*

cont'd

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \frac{3\pi}{2}$$

Solutions in [0, 2π)

So, the general solution is

$$x = \frac{\pi}{2} + 2n\pi \quad \text{and} \quad x = \frac{3\pi}{2} + 2n\pi$$

where *n* is an integer. Try verifying these solutions graphically.

Power-Reducing Formulas

Power-Reducing Formulas

The double-angle formulas can be used to obtain the following **power-reducing formulas.**

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Example 4 – *Reducing a Power*

Rewrite $\sin^4 x$ in terms of first powers of the cosines of multiple angles.

Solution:

Note the repeated use of power-reducing formulas.

$$\sin^4 x = (\sin^2 x)^2 \qquad \text{Property of exponents}$$

$$= \left(\frac{1 - \cos 2x}{2}\right)^2 \qquad \text{Power-reducing formula}$$

$$= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) \qquad \text{Expand.}$$

Example 4 – *Solution*

cont'd

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right)$$

Power-reducing formula

$$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x$$

Distributive Property

$$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

Simplify.

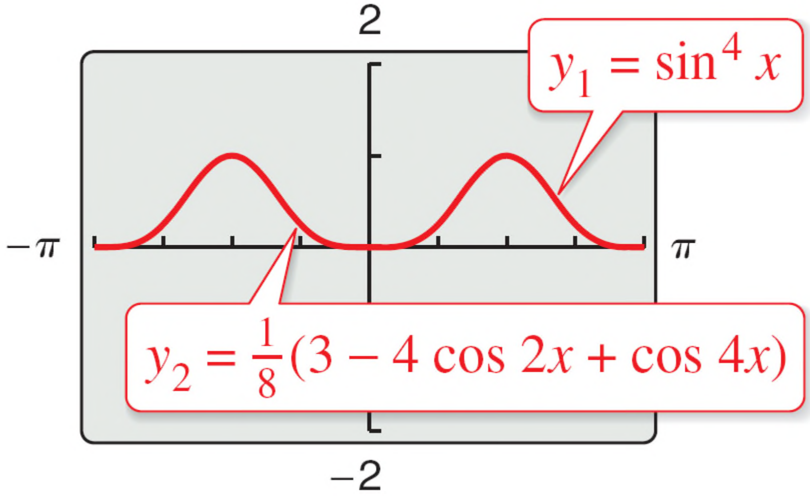
$$= \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$$

Factor out common factor.

Example 4 – *Solution*

cont'd

You can use a graphing utility to check this result, as shown below. Notice that the graphs coincide.



Half-Angle Formulas

Half-Angle Formulas

You can derive some useful alternative forms of the power-reducing formulas by replacing u with $u/2$. The results are called **half-angle formulas.**

Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The signs of $\sin\frac{u}{2}$ and $\cos\frac{u}{2}$ depend on the quadrant in which $\frac{u}{2}$ lies.

Example 5 – *Using a Half-Angle Formula*

Find the exact value of sin 105°.

Solution:

Begin by noting that 105° is half of 210°. Then, using the half-angle formula for sin(u/2) and the fact that 105° lies in Quadrant II, you have

$$\sin 105^\circ = \sqrt{\frac{1 - \cos 210^\circ}{2}}$$

$$= \sqrt{\frac{1 + \left(\sqrt{3}/2\right)}{2}}$$

Example 5 – *Solution*

cont'd

$$= \frac{\sqrt{2 + \sqrt{3}}}{2}.$$

The positive square root is chosen because $\sin \theta$ is positive in Quadrant II.

Product-to-Sum Formulas

Product-to-Sum Formulas

Each of the following **product-to-sum formulas** can be verified using the sum and difference formulas.

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$

Product-to-sum formulas are used in calculus to solve problems involving the products of sines and cosines of two different angles.

Example 7 – *Writing Products as Sums*

Rewrite the product cos 5x sin 4x as a sum or difference.

Solution:

Using the appropriate product-to-sum formula, you obtain

$$\cos 5x \sin 4x = \tfrac{1}{2}[\sin(5x + 4x) - \sin(5x - 4x)]$$

$$= \tfrac{1}{2}\sin 9x - \tfrac{1}{2}\sin x.$$

Product-to-Sum Formulas

Occasionally, it is useful to reverse the procedure and write a sum of trigonometric functions as a product. This can be accomplished with the following **sum-to-product formulas.**

Sum-to-Product Formulas

$$\sin u + \sin v = 2 \sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2 \cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2 \cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2 \sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Application

Example 10 – *Projectile Motion*

Ignoring air resistance, the range of a projectile fired at an angle θ with the horizontal and with an initial velocity of v_0 feet per second is given by

$$r = \frac{1}{16} v_0^2 \sin \theta \cos \theta$$

where r is the horizontal distance (in feet) that the projectile travels.

Example 10 – *Projectile Motion* cont'd

A football player can kick a football from ground level with an initial velocity of 80 feet per second

a. Write the projectile motion model in a simpler form.

b. At what angle must the player kick the football so that the football travels 200 feet?

Example 10 – *Solution*

a. You can use a double-angle formula to rewrite the projectile motion model as

$$r = \frac{1}{32} v_0^2 \, (2 \sin \theta \cos \theta)$$ Rewrite original projectile motion model.

$$= \frac{1}{32} v_0^2 \sin 2\theta.$$ Rewrite model using a double-angle formula.

b. $$r = \frac{1}{32} v_0^2 \sin 2\theta$$ Write projectile motion model.

$$200 = \frac{1}{32} (80)^2 \sin 2\theta$$ Substitute 200 for r and 80 for v_0.

Example 10 – *Solution*

cont'd

$200 = 200 \sin 2\theta$ Simplify.

$1 = \sin 2\theta$ Divide each side by 200.

You know that $2\theta = \pi/2$, so dividing this result by 2 produces $\theta = \pi/4$.

Because $\pi/4 = 45°$, the player must kick the football at an angle of 45° so that the football travels 200 feet.