Correction of the exercises from the book *A Wavelet Tour of Signal Processing*

Gabriel Peyré
Ceremade
Université Paris-Dauphine
gabriel.peyre@ceremade.dauphine.fr

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Abstract

These corrections refer to the 3rd edition of the book *A Wavelet Tour of Signal Processing – The Sparse Way* by Stéphane Mallat, published in December 2008 by Elsevier. If you find mistakes or imprecisions in these corrections, please send an email to Gabriel Peyré (gabriel.peyre@ceremade.dauphine.fr). More information about the book, including how to order it, numerical simulations, and much more, can be find online at wavelet-tour.com.

1 Chapter 2

Exercise 2.1. For all t, the function $\omega \mapsto e^{-i\omega t} f(t)$ is continuous. If $f \in \mathrm{L}^1(\mathbb{R})$, then for all ω, $|e^{-i\omega t} f(t)| \leqslant |f(t)|$ which is integrable. One can thus apply the theorem of continuity under the integral sign $\int$ which proves that $\hat{f}$ is continuous.

If $\hat{f} \in \mathrm{L}^1(\mathbb{R})$, using the inverse Fourier formula (2.8) and a similar argument, one proves that f is continuous.

Exercise 2.2. If $\int |h| = +\infty$, for all $A > 0$ there exists $B > 0$ such that $\int_{-B}^{B} |h| > A$. Taking $f(x) = 1_{[-A,A]} \operatorname{sign}(h(-x))$ which is integrable and bounded by 1 shows that

$$f \star h(0) = \int_{-B}^{B} \operatorname{sign}(h(t)) h(t) \mathrm{d}t > A.$$

This shows that the operator $f \mapsto f \star h$ is not bounded on L^∞, and thus the filter h is unstable.

Exercise 2.3. Let $f_u(t) = f(t-u)$, by change of variable $t - u \to t$, one gets

$$\hat{f}_u(\omega) = \int f(t-u) e^{-i\omega t} \mathrm{d}t = \int f(t) e^{-i\omega(t+u)} \mathrm{d}t = e^{-i\omega u} \hat{f}(\omega).$$

Let $f_s(t) = f(t/s)$, with $s > 0$, by change of variable $t/s \mapsto t$, one get

$$\hat{f}_s(\omega) = \int f(t/s) e^{-i\omega t} \mathrm{d}t = \int f(t) e^{-i\omega s t} |s| \mathrm{d}t = |s| \hat{f}(s\omega).$$

Let f by C^1 and $g = f'$, the by integration by parts, since $f(t) \to 0$ where $|t| \to +\infty$,

$$\hat{g}(\omega) = \int f'(t)e^{-i\omega t}\mathrm{d}t = -\int f(t)(-i\omega)e^{-i\omega t}\mathrm{d}t = (i\omega)\hat{f}(\omega).$$

Exercise 2.4. One has

$$f_r(t) = \mathrm{Re}[f(t)] = [f(t) + f^*(t)]/2 \quad \text{and} \quad f_i(t) = \mathrm{Ima}[f(t)] = [f(t) - f^*(t)]/2$$

so that

$$\begin{aligned}\hat{f}_r(\omega) &= \int \frac{f(t) + f^*(t)}{2} e^{-i\omega t}\mathrm{d}t = \hat{f}(\omega)/2 + \mathrm{Conj}\left(\int f(t)e^{i\omega t}\mathrm{d}t\right)/2 \\ &= [\hat{f}(\omega) + \hat{f}^*(-\omega)]/2,\end{aligned}$$

where $\mathrm{Conj}(a) = a^*$ is the complex conjugate. The same computation leads to

$$\hat{f}_i(\omega) = [\hat{f}(\omega) - \hat{f}^*(-\omega)]/2.$$

Exercise 2.5. One has

$$\hat{f}(0) = \int f(t)\mathrm{d}t = 0.$$

If $f \in \mathrm{L}^1(\mathbb{R})$, one can apply the theorem of derivation under the integral sign $\int$ and get

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\hat{f}(\omega) = \int -itf(t)e^{-i\omega t}\mathrm{d}t \quad \Longrightarrow \quad \hat{f}'(0) = -i\int tf(t)\mathrm{d}t = 0.$$

Exercise 2.6. If $f = 1_{[-\pi,\pi]}$ then one can verify that

$$\hat{f}(\omega) = \frac{2\sin(\pi\omega)}{\omega}.$$

It result that

$$\int \frac{\sin(\pi\omega)}{\pi\omega} = \frac{1}{2\pi}\int \hat{f}(\omega)\mathrm{d}\omega = f(0) = 1.$$

If $g = 1_{[-1,1]}$ then $\hat{g}(\omega)/2 = \sin(\omega)/\omega$. The inverse Fourier transform of $\hat{g}(\omega)^3$ is $g \star g \star g(t)$ so

$$\int \frac{\sin^3(\omega)}{\omega^3}\mathrm{d}\omega = \frac{1}{8}\int \hat{g}(\omega)^3\mathrm{d}\omega = \frac{2\pi}{8} g \star g \star g(0) = \frac{3\pi}{4},$$

where we used the fact that

$$g \star g \star g(0) = \int_{-1}^{1} h(t)\mathrm{d}t = 3$$

where h is a piecewise linear hat function with $h(0) = 2$.

Exercise 2.7. Writing $u = a - ib$, and differentiating under the integral sign $\int$, one has

$$f'(\omega) = \int -ite^{-ut^2}e^{-i\omega t}\mathrm{d}t.$$

By integration by parts, one gets an ordinary differential equation

$$f'(\omega) = \frac{-\omega}{2u}\hat{f}(\omega)$$

whose solution is

$$f(\omega) = Ke^{-\frac{\omega^2}{4u}}$$

for some constant $K = \hat{f}(0)$. Using a switch from Euclidean coordinates to polar coordinates $(x,y) \to (r,\theta)$ which satisfies $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$, one gets

$$\begin{aligned} K^2 &= \int e^{-ux^2}\mathrm{d}x \int e^{-uy^2}\mathrm{d}y = \iint e^{-u(x^2+y^2)}\mathrm{d}x\mathrm{d}y \\ &= \int_0^{2\pi}\int_0^{+\infty} e^{-ur^2} r\mathrm{d}r\mathrm{d}\theta = 2\pi \int_0^{+\infty} re^{-ur^2}\mathrm{d}r = \frac{\pi}{u}, \end{aligned}$$

which gives the result.

Exercise 2.8. If f is $\mathbf{C}^1$ with a compact support, with an integration by parts we get

$$\hat{f}(\omega) = \frac{1}{i\omega}\int f'(t)e^{-i\omega t}\mathrm{d}t$$

so that

$$|\hat{f}(\omega)| \leqslant \frac{C}{\omega} \quad \text{with} \quad C = \int |f'(t)|\mathrm{d}t < +\infty,$$

which proves that $f(\omega) \to 0$ when $|\omega| \to +\infty$.

Let $f \in \mathbf{L}^1(\mathbb{R})$ and $\varepsilon > 0$. Since $\mathbf{C}^1$ functions are dense in $\mathbf{L^1}(\mathbb{R})$, one can find g such that $\int |f-g| \leqslant \varepsilon/2$. Since $\hat{g}(\omega) \to 0$ when $|\omega| \to +\infty$, there exists A such that $|\hat{g}(\omega)| \leqslant \varepsilon/2$ when $|\omega| > A$. Moreover, the Fourier integral definition implies that

$$|\hat{f}(\omega) - \hat{g}(\omega)| \leqslant \int |f(t) - g(t)|\, dt$$

so for all $|\omega| > A$ we have $|\hat{f}(\omega)| \leqslant \varepsilon$ which proves that $f(\omega) \to 0$ when $|\omega| \to +\infty$.

Exercise 2.9. a) For $f_0(t) = 1_{[0,+\infty)}(t)e^{pt}$, we get

$$\hat{f}_0(\omega) = \int_0^{+\infty} e^{(p-i\omega)t}\mathrm{d}t = \frac{1}{i\omega - p}.$$

For $f_n(t) = t^n 1_{[0,+\infty)}(t)e^{pt}$, an integration by parts gives

$$\hat{f}_n(\omega) = \int_0^{+\infty} t^n e^{(p-i\omega)t}\mathrm{d}t = \frac{n}{i\omega - p}\hat{f}_{n-1}(\omega),$$

so that

$$\hat{f}_n(\omega) = \frac{n!}{(i\omega - p)^n}.$$

b) Computing the Fourier transform on both sides of the differential equation gives

$$g = f \star h \quad \text{where} \quad \hat{h}(\omega) = \frac{\sum_{k=0}^{K} a_k (i\omega)^k}{\sum_{k=0}^{M} b_k (i\omega)^k}.$$

We denote by $\{p_k\}_{k=0}^{L}$ the poles of the polynomial $\sum_{k=0}^{M} b_k z^k$, with multiplicity n_k. If $K < M$, one can decompose the rational fraction into

$$\hat{h}(\omega) = \sum_{k=0}^{L} \frac{Q_k(i\omega)}{(i\omega - p_k)^{n_k}}$$

where each Q_k is a polynomial of degree strictly smaller than n_k. It results that $h(t)$ is a sum of derivatives up to a degree strictly smaller than n_k of the inverse Fourier transform of

$$\hat{f}_{p_k,n_k}(\omega) = \frac{1}{(i\omega - p_k)^{n_k}}$$

which is

$$f_{p_k,n_k}(t) = \frac{1}{n_k!} t^{n_k} 1_{[0,+\infty)}(t) e^{p_k t}.$$

Each filter f_{p_k,n_k} is causal, stable and n_k times differentiable. It results that that h is causal and stable.

If, there exists l with $\mathrm{Re}(p_l) = 0$ then for the frequency $\omega = -ip_l$ we have $|\hat{h}(\omega)| = +\infty$ so h can not be stable.

If, there exists l with $\mathrm{Re}(p_l) > 0$ then by observing that $\hat{f}_{p_l,n_l}(-\omega) = (-1)^{n_l}(i\omega + p_l)^{-n_l}$ and by applying the result in a) we get

$$f_{p_l,n_{k_l}}(t) = \frac{1}{n_l!} t^{n_l} 1_{(-\infty,0]}(t) e^{-p_l t}$$

which is anticausal. We thus derive that h is not causal.
c) Denoting $\alpha = e^{i\pi/3}$, one can write

$$|\hat{h}(\omega)|^2 = \frac{1}{1 - (i\omega/\omega_0)^6}$$

with

$$1/\hat{h}(\omega) = (i\omega/\omega_0 + 1)(i\omega/\omega_0 + \alpha)(i\omega/\omega_0 + \alpha^*) = P(i\omega).$$

Since the zeros of $P(z)$ have all a strictly negative real part, h is stable and causal. To compute $h(t)$ we decompose

$$\hat{h}(\omega) = \frac{a_1}{i\omega/\omega_0 + 1} + \frac{a_2}{i\omega/\omega_0 + \alpha} + \frac{a_3}{i\omega/\omega_0 + \alpha^*},$$

we compute a_1, a_2 and a_3 and by applying the result in (a) we derive that

$$\hat{h}(t) = \omega_0 (a_1\ 1_{[0,+\infty)}(t)\, e^{-t\omega_0} + a_2\ 1_{[0,+\infty)}(t)\, e^{-t\alpha\omega_0} + a_3\ 1_{[0,+\infty)}(t)\, e^{-t\alpha^*\omega_0})\ .$$

Exercise 2.10. For $a > 0$ and $u > 0$ and g a Gaussian function, define

$$f_{a,u}(t) = e^{iat} g(t-u) + e^{-iat} g(t+u).$$

We verify that $\sigma_\omega(f_{a,u})$ increases proportionally to u. Its Fourier transform is

$$\hat{f}_{a,u}(\omega) = e^{-iu\omega} \hat{g}(\omega - a) + e^{iu\omega} \hat{g}(\omega + a)$$

so $\sigma_\omega(f_{a,u})$ increases proportionally to a. For a and u sufficiently large we get the the result.

Exercise 2.11. Since $f(t) \geqslant 0$

$$|\hat{f}(\omega)| = |\int f(t)\, e^{-i\omega t}\, dt| \leqslant \int f(t)\, dt = \hat{f}(0)\ .$$

Exercise 2.12. a) Denoting $u(t) = |\sin(t)|$, one has $g(t) = a(t) u(\omega_0 t)$ so that

$$\hat{g}(\omega) = \frac{1}{2\pi} \hat{a}(\omega) \star \hat{u}(\omega/\omega_0)$$

where $\hat{u}(\omega)$ is a distribution

$$\hat{u}(\omega) = \sum_n c_n \delta(\omega - n)$$

and c_n is the Fourier coefficient

$$c_n = \int_{-\pi}^{\pi} |\sin(t)| e^{-int} \mathrm{d}t = -\int_{-\pi}^{0} \sin(t) e^{-int} \mathrm{d}t + \int_{0}^{\pi} \sin(t) e^{-int} \mathrm{d}t.$$

The change of variable $t \to t + \pi$ in the first integral shows that $c_{2k+1} = 0$ and for $n = 2k$,

$$c_{2k} = 2 \int_0^{\pi} \sin(t) e^{-i2kt} \mathrm{d}t = \frac{4}{1 - 4k^2}.$$

One thus has

$$\hat{u}(\omega) = \frac{1}{2\pi} \sum_n c_n \hat{a}(\omega - n\omega_0) = \frac{2}{\pi} \sum_k \frac{\hat{a}(\omega - 2k\omega_0)}{1 - 4k^2}.$$

b) If $\hat{a}(\omega) = 0$ for $|\omega| > \omega_0$, then h defined by $\hat{h}(\omega) = \frac{\pi}{2} 1_{[-\omega_0, \omega_0]}$ guarantees that $\hat{g}\hat{h} = \hat{a}$ and hence $a = g \star h$.

Exercise 2.13. One has

$$\hat{g}(\omega) = \frac{1}{2} \sum_n \hat{f}_n(\omega) \star [\delta(\omega - 2n\omega_0) + \delta(\omega + 2n\omega_0)] = \frac{1}{2} \sum_n [\hat{f}_n(\omega - 2n\omega_0) + \hat{f}_n(\omega + 2n\omega_0)].$$

Each $\hat{f}_n(\omega \pm 2n\omega_0)$ is supported in $[(-1 \pm 2n)\omega_0, (1 \pm 2n)\omega_0]$, and thus $\hat{g}$ is supported in $[-2N\omega_0, 2N\omega_0]$.

Since the intervals $[(-1 \pm 2n)\omega_0, (1 \pm 2n)\omega_0]$ are disjoint, one has

$$\hat{f}_n(\omega \pm 2n\omega_0) = 2\hat{g}(\omega) 1_{[(-1\pm 2n)\omega_0, (1\pm 2n)\omega_0]}(\omega).$$

The change of variable $\omega \pm 2n\omega_0 \to \omega$ and summing for n and $-n$ gives

$$\hat{f}_n(\omega) = [\hat{g}(\omega - 2n\omega_0) + \hat{g}(\omega + 2n\omega_0)] \hat{h}(\omega),$$

where $\hat{h}(\omega) = 1_{[-\omega_0, \omega_0]}(\omega)$. Denoting $g_n(t) = 2g(t)\cos(2n\omega_0 t)$, one sees that f_n is recovered as

$$f_n = g_n \star h.$$

Exercise 2.14. The function $\phi(t) = \sin(\pi t)/(\pi t)$ is monotone on $[-3/2, 0]$ and $[0, 3/2]$ on which is variation is $1 + \frac{2}{3\pi}$. For each $k \in \mathbb{N}^*$, it is also monotone on each interval $[k + 1/2, k + 3/2]$ on which the variation is $\frac{1}{\pi}[(k + 1/2)^{-1} + (k + 3/2)^{-1}]$. One thus has

$$\|\phi\|_V = 2(1 + \frac{2}{3\pi}) + \frac{2}{\pi} \sum_{k \geqslant 1} [(k + 1/2)^{-1} + (k + 3/2)^{-1}] = +\infty.$$

For $\phi = \lambda 1_{[a,b]}$, $|\phi'| = \lambda\delta_a + \lambda\delta_b$ and hence $\|\phi\|_V = 2\lambda$.

Exercise 2.16. Let

$$f(x) = 1_{[0,1]^2}(x_1, x_2) = f_0(x_1) f_0(x_2) \quad \text{where} \quad f_0(x_1) = 1_{[0,1]}(x_1).$$

One has

$$\hat{f}(\omega_1, \omega_2) = \hat{f}_0(\omega_1)\hat{f}_0(\omega_2) = \frac{(e^{i\omega_1} - 1)(e^{i\omega_2} - 1)}{\omega_1 \omega_2}.$$

Let

$$f(x) = e^{-x_1^2 - x_2^2} = f_0(x_1) f_0(x_2) \quad \text{where} \quad f_0(x_1) = e^{-x_1^2}.$$

One has

$$\hat{f}(\omega_1, \omega_2) = \hat{f}_0(\omega_1)\hat{f}_0(\omega_2) = \pi e^{-(\omega_1^2 + \omega_2^2)/4}.$$

Exercise 2.17. If $|t| > 1$, the ray $\Delta_{t,\theta}$ does not intersect the unit disc, and thus $p_\theta(t) = 0$. For $|t| < 1$, the Radon transform is computed as the length of a cross section of a disc

$$p_\theta(t) = 2\sqrt{1 - t^2}.$$

Exercise 2.18. We prove that the Gibbs oscillation amplitude is independent of the angle θ and is equal to a one-dimensional Gibbs oscillation. Let us decompose $f(x)$ into a continuous part $f_0(x)$ and a discontinuity of constant amplitude A:

$$f(x) = f_0(x) + A\, u(\cos(\theta) x_1 + \sin(\theta) x_2)$$

where $u(t) = 1_{[0,+\infty)}(t)$ is the one-dimensional Heaviside function. The filter satisfies $h_\xi(x_1, x_2) = g_\xi(x_1)\, g_\xi(x_2)$ with $g_\xi(t) = \sin(\xi t)/(\pi t)$. The Gibbs phenomena is produced by the discontinuity corresponding to the Heaviside function so we can consider that $f_0 = 0$. Let us suppose that $|\theta| \leqslant \pi/4$, with no loss of generality. We first prove that

$$f \star h_\xi(x) = f \star g_\xi(x) \tag{1}$$

where $\hat{g}_\xi(\omega_1, \omega_2) = 1_{[-\xi,\xi]}(\omega_2)$. Indeed $f(x)$ is constant along any line of angle θ, one can thus verify that its Fourier transform has a support located on the line in the Fourier plane, of angle $\theta + \pi/2$ which goes through 0. It results that $\hat{f}(\omega)\hat{h}_\xi(\omega) = \hat{f}(\omega)\hat{g}_\xi(\omega)$ because the filtering limits the support of $\hat{f}$ to $|\omega_2| \leqslant \xi$. But $g_\xi(x_1, x_2) = \delta(x_1)\, \sin(\xi x_2)/(\pi x_2)$. The convolution (1) is thus a one-dimensional convolution along the x_2 variable, which is computed in the Gibbs Theorem 2.8. The resulting one-dimensional Gibbs oscillations are of the order of $A \times 0.045$.